Practice Test 1

Luke Greenawalt

Math 257

1. Use back substitution to solve the system **(1.1)**:

$$x - 4y + z = 28$$
$$y + 4z = 14$$
$$z = 3$$

2. Solve the system. If there are infinitely many solutions, parameterize them using t **(1.1)**

$$2x_1 + 3x_2 - 4x_3 = 16$$
$$x_1 - 3x_2 + 16x_3 = 32$$

3. Find the solution set of the system of linear equations represented by the augmented matrix **(1.2)**

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination **(1.2)**

$$x + 3y = -4$$
$$-x - 3y = 4$$

5. Determine the polynomial that passes through the points $(0,0)$, $(1,-1)$ $(5,0)$ **(1.3)**

6. Solve the partial fraction decomposition using matrices **(1.3)**

$$\frac{x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

7. Find AB given matrix A and B **(2.1)**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \; B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

8. Solve the equation $Ax = b$ for x using rref. **DON'T USE** A^{-1} **2.1**

$$x_1 + 2x_2 - 3x_3 = -1$$
$$-x_1 + x_2 + x_3 = 4$$
$$x_1 - x_2 - 2x_3 = 2$$

9. Show that AB and BA are not equal for the given matrices **2.2**

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} B = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$$

10. Find the transpose of matrix A **2.2**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

11. Find the inverse of matrix A **2.3**

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & 5 \\ 2 & 5 & 0 \end{bmatrix}$$

12. Use an inverse matrix to solve the system by hand **2.3**

$$3x - 2y = 4$$
$$3x + 2y = 12$$

13. Find a sequence of elementary matrices whose product is the given nonsingular matrix **2.4**

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

14. Find the inverse of the elementary matrix **2.4**

$$\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

15. Professor Venn has 15 tests in drawer 1, 6 tests in drawer 2, and 4 tests in drawer 3. During a semester, 30% of the tests in drawer 1 sneak over to drawer 2 and 20% sneak to drawer 3. Of the tests in drawer 2, 20% sneak to drawer 1 and 60% sneak to drawer 3. Of the tests in drawer 3, 10% sneak to drawer 1 and 35% sneak to drawer 2. How many tests are in each drawer after one semester? **2.5**

16. The math department at CCBC determines that 10% of the students who purchase the textbook during any semester will not purchase it the next semester. On the other hand, 10% of the people who do not purchase the textbook during any semester will purchase it the next semester. In a population of 500 students, 50 students purchased it this month. How many will purchase the textbook next month? **2.5**

17. Write the encoded row matrices for the message. Then encode the message using the matrix A. The message is: I HOPE I PASS. The row matrix size is 1×3. The encoding matrix A is **2.6**:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

18. Use A^{-1} to decode the cryptogram **2.6**:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 61 & 55 \end{bmatrix} \begin{bmatrix} 84 & 93 \end{bmatrix} \begin{bmatrix} 77 & 30 \end{bmatrix}$$

19. Find the determinant of matrix A **3.1**

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

20. Use expansion by cofactor to find the determinant of the matrix **3.1**

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

21. Determine which property of the determinants the equation illustrates **3.2**

$$\begin{bmatrix} -4 & 3 & 1 \\ 5 & 0 & 0 \\ -4 & 3 & 1 \end{bmatrix}$$

22. Use elementary row or column operations to find the determinant **3.2**

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

23. Use $|cA| = c^n|A|$ to evaluate the determinant of the matrix **3.3**

$$\begin{bmatrix} 3 & 6 & 9 \\ 3 & 3 & -3 \\ 6 & 3 & 3 \end{bmatrix}$$

24. Use the determinant of the coefficient matrix to determine whether or not the system has a unique solution **3.3**.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 4 \\ -x_1 - x_2 + 3x_3 &= 6 \\ 2x_1 + 3x_2 - 5x_3 &= 2 \end{aligned}$$

25. Find an equation of a line passing through the points $(1, 4)$ and $(6, 2)$ **3.4**

26. Use Cramer's rule to solve the system **3.4**

$$x - 4y + 2z = -2$$

$$x + 2y - 2z = -3$$

$$x - y + z = 4$$

Practice Test 1 Answer Key

Luke Greenawalt

September 2024

1. Using $z = 3$:

$$y + 4(3) = 14$$
$$y = 2$$

$$x - 4(2) + 3 = 28$$

$$x = 33$$

$$(x, y, z) = \boxed{(33, 2, 3)}$$

2. Adding both equations, we get:

$$3x_1 + 12x_3 = 48$$

Dividing by a factor of 3, we get:

$$x_1 + 4x_3 = 16$$

$$x_1 = 16 - 4x_3$$

With this in mind, we can parameterize $x_3 = t$ to get:

$$x_1 = 16 - 4t$$

$$2(16 - 4t) + 3x_2 - 4t = 16$$

$$3x_2 = 12t - 16$$

$$x_2 = 4t - \frac{16}{3}$$

$$(x_1, x_2, x_3) = \boxed{(16 - 4t, 12t - 16, t)}$$

3. Converting the augmented matrix into a system of equations, we have:

$$2x_1 + 3x_2 + 2x_3 = 0$$

$$x_3 = -2$$

and then obviously the last row won't produce an equation of any use. Since we have three variables and only two equations, we will have infinitely many solutions. It'll help to parameterize $x_2 = t$. Doing this and substituting the value for x_3, we have

$$2x_1 + 3t - 4 = 0$$

$$2x_1 = 4 - 3t$$

$$x_1 = 2 - \frac{3t}{2}$$

Therefore, the solution set in terms of the parameter t is $(x_1, x_2, x_3) = \boxed{\left(2 - \frac{3t}{2}, t, -2\right)}$

4. The solution should be trivial by simply looking at the equations, but for the sake of this answer key, I will use Gauss-Jordan Elimination. Turning this system of linear equations into an augmented matrix, we have:

$$\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \end{bmatrix}$$

Using the row operation $R_1 + R_2 \implies R_2$, we have

$$\begin{bmatrix} 1 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Using this to create a single linear equation, we have

$$x + 3y = -4$$

Using $y = t$ as the parameter

$$x = -3t - 4$$

Therefore, the solution set is $(x, y) = \boxed{(-3t - 4, t)}$

5. Using the general equation $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1}$, we can set up a linear equation for each point:

$$a_0 + (0)a_1 + (0^2)a_2 = 0$$

$$a_0 + (1)a_1 + (1^2)a_2 = -1$$

$$a_0 + (5)a_1 + (5^2)a_2 = 0$$

From these equations, an augmented matrix can be constructed:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 5 & 25 & 0 \end{bmatrix}$$

From here, we can use elementary row operations to put this matrix in reduced row echelon form

$$R_3 - R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 1 \\ 1 & 5 & 25 & 0 \end{bmatrix}$$

$$R_3 - R_1 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 1 \\ 0 & 5 & 25 & 0 \end{bmatrix}$$

$$\frac{1}{5}R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 1 \\ 0 & 1 & 5 & 0 \end{bmatrix}$$

$$4R_3 - R_2 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 24 & 1 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$6R_3 + R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -5 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$\frac{1}{4}R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$-\frac{1}{4}R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{bmatrix}$$

From this, we can get that $a_0 = 0$, $a_1 = -\frac{5}{4}$, and $a_2 = \frac{1}{4}$. Substituting these constants into the original equation, we are left with the polynomial

$$p(x) = \boxed{-\frac{5}{4}x + \frac{1}{4}x^2}$$

6. Getting the components of the partial fraction to have the same denominator, we have:

$$\frac{x+2}{x(x-1)^2} = \frac{A(x-1)^2}{x(x-1)^2} + \frac{B(x)(x-1)}{x(x-1)^2} + \frac{Cx}{x(x-1)^2}$$

Now, since the denominators are equivalent, we can multiply both sides by the denominator to get

$$x + 2 = A(x-1)^2 + Bx(x-1) + Cx$$

$$x + 2 = A(x^2 - 2x + 1) + Bx^2 - Bx + Cx$$

$$x + 2 = Ax^2 + Bx^2 - 2Ax - Bx + Cx + A$$

$$x + 2 = (A + B)x^2 + (-2A - B + C)x + A$$

Therefore, the system of equations that can be used to solve for A, B, and C is

$$A + B = 0$$

$$-2A - B + C = 1$$

$$A = 2$$

Transforming the system into an augmented matrix,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Using a graphing utility, we can put the augmented matrix in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore, $A = 2$, $B = -2$, and $C = 3$. Substituting this back into the initial partial fraction decomposition, we have

$$\frac{x+2}{x(x-1)^2} = \boxed{\frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}}$$

7. First, let's ensure that you can multiply A and B. Since A is a 3×3 matrix, and B is a 3×2 matrix, you can multiply these two. When multiplying these two matrices together, we will get a 3×2 matrix in return. Let's call this matrix P:

$$\begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 & 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5 & 4 \cdot 2 + 5 \cdot 4 + 6 \cdot 6 \\ 7 \cdot 1 + 8 \cdot 3 + 9 \cdot 5 & 7 \cdot 2 + 8 \cdot 4 + 9 \cdot 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 + 6 + 15 & 2 + 8 + 18 \\ 4 + 15 + 30 & 8 + 20 + 36 \\ 7 + 24 + 45 & 14 + 32 + 54 \end{bmatrix}$$

Therefore, matrix AB is

$$\boxed{\begin{bmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \end{bmatrix}}$$

8. For this problem, if we can't use A^{-1}, the best way to do this is to represent the equation in matrix form.

$$Ax = b \implies \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

When the equation takes this form, it is easiest to set up augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 4 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

From here, we need this to be put into reduced row echelon form. Let's do that with a calculator

$$\begin{bmatrix} 1 & 0 & 0 & -13 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

Therefore, $(x_1, x_2, x_3) = \boxed{(-13, -3, -6)}$

9. First, let's start with AB:

$$AB = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(5) + 2(1) & 1(2) + 2(3) \\ 4(5) + 6(1) & 4(2) + 6(3) \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 26 & 26 \end{bmatrix}$$

Now, BA:

$$BA = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 5(1) + 2(4) & 5(2) + 2(6) \\ 1(1) + 3(4) & 1(2) + 3(6) \end{bmatrix} = \begin{bmatrix} 13 & 22 \\ 13 & 20 \end{bmatrix}$$

Since these two matrices aren't identical, we have shown $AB \neq BA$

10. To find the transpose of a matrix, all we have to do is switch the rows and the columns. With A, this will be

$$\boxed{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}$$

11. To find the inverse of a 3×3 matrix A, we have to set up a separate 3×6 matrix B in the form $\begin{bmatrix} A & I \end{bmatrix}$ that, using elementary row operations, we will transform into a matrix taking the form $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. With matrix A, we can create B as

$$\begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 0 & 5 & 0 & 1 & 0 \\ 2 & 5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

To reduce the left half of the matrix to I, let's use elementary row operations:

$$2R_1 - R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ -3 & 0 & 5 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 & 0 & -1 \end{bmatrix}$$

$$3R_1 + R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 9 & -1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 2 & 0 & -1 \end{bmatrix}$$

$$3R_1 - R_2 \implies R_1$$

$$\begin{bmatrix} 3 & 0 & -5 & 0 & -1 & 0 \\ 0 & 9 & -1 & 3 & 1 & 0 \\ 0 & 1 & -4 & 2 & 0 & -1 \end{bmatrix}$$

$$R_2 - 9R_3 \implies R_3$$

$$\begin{bmatrix} 3 & 0 & -5 & 0 & -1 & 0 \\ 0 & 9 & -1 & 3 & 1 & 0 \\ 0 & 0 & 35 & -15 & 1 & 9 \end{bmatrix}$$

$$35R_2 + R_3 \implies R_2$$

$$\begin{bmatrix} 3 & 0 & -5 & 0 & -1 & 0 \\ 0 & 315 & 0 & 90 & 36 & 9 \\ 0 & 0 & 35 & -15 & 1 & 9 \end{bmatrix}$$

$$7R_1 + R_3 \implies R_1$$

$$\begin{bmatrix} 21 & 0 & 0 & -15 & -6 & 9 \\ 0 & 315 & 0 & 90 & 36 & 9 \\ 0 & 0 & 35 & -15 & 1 & 9 \end{bmatrix}$$

$$\frac{1}{21}R_1 \implies R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{7} & -\frac{2}{7} & \frac{3}{7} \\ 0 & 315 & 0 & 90 & 36 & 9 \\ 0 & 0 & 35 & -15 & 1 & 9 \end{bmatrix}$$

$$\frac{1}{315}R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{7} & -\frac{2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{35} & \frac{1}{35} \\ 0 & 0 & 35 & -15 & 1 & 9 \end{bmatrix}$$

$$\frac{1}{35}R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{7} & -\frac{2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{4}{35} & \frac{1}{35} \\ 0 & 0 & 1 & -\frac{3}{7} & \frac{1}{35} & \frac{9}{35} \end{bmatrix}$$

Taking just the right side of this matrix, we have our answer

$$\boxed{A^{-1} = \begin{bmatrix} -\frac{5}{7} & -\frac{2}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{4}{35} & \frac{1}{35} \\ -\frac{3}{7} & \frac{1}{35} & \frac{9}{35} \end{bmatrix}}$$

12. We can set up the equation $Ax = b$ where A is the coefficient matrix, x is the variable matrix, and b is the matrix of constants:

$$\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

To solve this, use $x = A^{-1}b$. Let's calculate A^{-1}:

$$A^{-1} = \frac{1}{3(2) - (-2)(3)} \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Multiplying this matrix by matrix b, we will get matrix x

$$A^{-1}b = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}(4) + \frac{1}{6}(12) \\ -\frac{1}{4}(4) + \frac{1}{4}(12) \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$$

Therefore, the solution to the system is $(x, y) = \boxed{\left(\frac{8}{3}, 2\right)}$

13. To do this problem, we will have to use elementary row operations to get this matrix down to the identity matrix I_2

$$R_1 - 3R_2 \implies R_2$$

$$\begin{bmatrix} 3 & 4 \\ 0 & -2 \end{bmatrix}$$

$$R_1 + 2R_2 \implies R_1$$

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\frac{1}{3}R_1 \implies R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$-\frac{1}{2}R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now that we have the identity matrix I_2, we can represent each elementary row operation as a matrix $E_1 \dots E_k$

$$E_1 = R_1 - 3R_2 \implies R_2 = \begin{bmatrix} 1 & 0 \\ 1 & -3 \end{bmatrix}$$

$$E_2 = R_1 + 2R_2 \implies R_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$E_3 = \frac{1}{3}R_1 \implies R_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_4 = -\frac{1}{2}R_2 \implies R_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

Now, we can represent what we did to obtain I_2 algebraically

$$E_4E_3E_2E_1A = I_2$$

Using inverse matrix identities, we can say

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

Let's calculate $E_1^{-1} \dots E_k^{-1}$:

$$E_1^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = 3\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_4^{-1} = -2\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Therefore, we have

$$\boxed{\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}}$$

14. To find the inverse of this elementary matrix, we will use the trick for 2×2 matrix inverses. Since this is an elementary matrix, we should get an elementary matrix when we take the inverse of it. Let's check:

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}}$$

15. To do this problem, we will set up a matrix of transition probabilities P. Using the information given in the problem, we have

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.35 \\ 0.2 & 0.6 & 0.55 \end{bmatrix}$$

Using P, we know that after one semester, the amount of tests in each draw will be PX, where X is the initial amount of tests in each drawer. When calculating PX, we will get a 3×1 matrix b in return. This matrix will be our answer.

$$b = PX = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.35 \\ 0.2 & 0.6 & 0.55 \end{bmatrix}\begin{bmatrix} 15 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 9.1 \\ 7.1 \\ 8.8 \end{bmatrix}$$

Let b_1 be the amount of tests in drawer 1, b_2 be the amount of tests in drawer 2, and b_3 be the amount of tests in drawer 3. Therefore, the solution we have is $(b_1, b_2, b_3) = \boxed{(9.1, 7.1, 8.8)}$. Whoever wrote this problem didn't plan ahead and should've picked nicer numbers. Let's hope the students put their names on each page of the test!

16. To solve this problem, we will have to use a 2×2 transition probability matrix P

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

To find how many will purchase a textbook next month, we will multiply P by the matrix x, which we will define as

'

$$x = \begin{bmatrix} 50 \\ 450 \end{bmatrix}$$

Multiplying Px, we get

$$Px = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}\begin{bmatrix} 50 \\ 450 \end{bmatrix} = \begin{bmatrix} 0.9(50) + 0.1(450) \\ 0.1(50) + 0.9(450) \end{bmatrix} = \begin{bmatrix} 90 \\ 410 \end{bmatrix}$$

Therefore, $\boxed{90}$ students will purchase the textbook next month

17. Let's first start by writing the message: I HOPE I PASS using matrices of size 1×3. Using our key, we have:

$$\begin{bmatrix} 9 & 0 & 8 \end{bmatrix} \begin{bmatrix} 15 & 16 & 5 \end{bmatrix} \begin{bmatrix} 0 & 9 & 0 \end{bmatrix} \begin{bmatrix} 16 & 1 & 19 \end{bmatrix} \begin{bmatrix} 19 & 0 & 0 \end{bmatrix}$$

Let's encode each 1×3 matrix by multiplying it by the encoding matrix A

$$\begin{bmatrix} 9 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 16 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 47 & 54 & 31 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 36 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 1 & 19 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 7 & 58 \end{bmatrix}$$

$$\begin{bmatrix} 19 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 19 & -19 & 0 \end{bmatrix}$$

Therefore, the encoded message is $\boxed{9, -1, 24, 47, 54, 31, 18, 36, 9, 18, 7, 58, 19, -19, 0}$

18. For this problems, let's first calculate A^{-1}

$$A^{-1} = \frac{1}{-13} \begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Let's now multiply each 1×2 encoded matrix by A^{-1}

$$\begin{bmatrix} 61 & 55 \end{bmatrix} \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix} = \begin{bmatrix} 8 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 84 & 93 \end{bmatrix} \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix} = \begin{bmatrix} 15 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 77 & 30 \end{bmatrix} \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix} = \begin{bmatrix} 1 & 25 \end{bmatrix}$$

Therefore, the decrypted message is the sequence of letters that correspond to the sequence of numbers $8, 15, 15, 18, 1, 25$. From the key, we can see that this spells out $\boxed{HOORAY}$

19. To find the determinant of a 2×2 matrix, we simply compute $A_{11}A_{22} - A_{21}A_{12}$. This yields

$$3(1) - 2(1) = \boxed{1}$$

20. To find the determinant of a 3×3 matrix, we will perform the following calculations. Note: The sign change in the second term corresponds to the sign pattern for cofactors.

$$|A| = 1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= (1(1) - 0(1)) - 3(2(1) - 0(0) + 2(2(1) - 1(0)) = 1 - 6 + 4 = \boxed{-1}$$

21. This equation illustrates property $\boxed{3.4.2}$, which says that when two rows or columns are equal, then $|A| = 0$. In this case, $R_1 = R_3$.

22. Using elementary row operations, we can reduce this matrix to a simpler matrix and then keep track of the operations we use.

$$R_3 + (-R_1) \implies R_1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & -3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$-\frac{1}{2}R_1 + R_3 \implies R_3$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R_1 + \frac{1}{2}R_2 \implies R_1$$

$$\begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ 0 & -2 & -3 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$R_3 + \frac{1}{4}R_2 \implies R_3$$

$$\begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{5}{4} \end{bmatrix}$$

We now have a triangular matrix. The only elementary row operation we used throughout the entire process was adding a multiple of another row to a row, and therefore, for each step, the determinant stayed the same. This is due to the property that states, when adding a multiple of a row to another row, $\det B = \det A$. With this in mind, the determinant of the triangular matrix is also the determinant of the original matrix. Since the matrix is triangular, we can multiply the numbers on the diagonal to find the determinant:

$$2(-2)\left(-\frac{5}{4}\right) = \boxed{5}$$

23. By observation, we see that the matrix is 3×3. Therefore, whatever value of c we take out from the matrix will be raised to a power of 3. Again, by observation, we can factor out a 3 from the matrix. Therefore, the determinant of the matrix will be

$$27 \det A = 27 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$27 \left(1 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right)$$

$$27\,(1(2) - 2(3) + 3(-1)) = 27(-7) = \boxed{-189}$$

24. To do this, we must check to see if $\det A = 0$. If $\det A = 0$, then there are infinitely many solutions or no solutions. If $\det A \neq 0$, there is a unique solution. Note that this problem only asks us to determine whether there is a unique solution and not to find the unique solution. Let's take the determinant of the coefficient matrix A:

$$\begin{vmatrix} 1 & 1 & -2 \\ -1 & -1 & 3 \\ 2 & 3 & -5 \end{vmatrix} = 1 \begin{vmatrix} -1 & 3 \\ 3 & -5 \end{vmatrix} - 1 \begin{vmatrix} -1 & 3 \\ 2 & -5 \end{vmatrix} - 2 \begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(-4) - 1(-1) - 2(-1) = \boxed{-1}$$

Since $\det A \neq 0$, the system has a unique solution.

25. For this problem, we will use the determinant and set it equal to 0. It's much easier to understand visually, so let's set it up:

$$\det \begin{bmatrix} x & y & 1 \\ 1 & 4 & 1 \\ 6 & 2 & 1 \end{bmatrix} = 0$$

$$x \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 1 \\ 6 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} = 0$$

$$x(4-2) - y(1-6) + 1(2-24) = 0$$

$$\boxed{2x + 5y - 22 = 0}$$

Just to verify our answers:

$$2(1) + 5(4) - 22 = 0$$

$$0 = 0$$

$$2(6) + 5(2) - 22 = 0$$

$$0 = 0$$

26. Using Cramer's Rule, we can set up the following:

$$x = \frac{\det A_1}{\det A} = \frac{\begin{vmatrix} -2 & -4 & 2 \\ -3 & 2 & -2 \\ 4 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & 2 \\ 1 & 2 & -2 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{-2\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} + 4\begin{vmatrix} -3 & -2 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} -3 & 2 \\ 4 & -1 \end{vmatrix}}{1\begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}$$

$$= \frac{-2(0) + 4(5) + 2(-5)}{1(0) + 4(3) + 2(-3)} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

$$y = \frac{\det A_2}{\det A} = \frac{\begin{vmatrix} 1 & -2 & 2 \\ 1 & -3 & -2 \\ 1 & 4 & 1 \end{vmatrix}}{6} = \frac{1\begin{vmatrix} -3 & -2 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix}}{6}$$

$$= \frac{1(5) + 2(3) + 2(7)}{6} = \boxed{\frac{25}{6}}$$

$$z = \frac{\det A_3}{\det A} = \frac{\begin{vmatrix} 1 & -4 & -2 \\ 1 & 2 & -3 \\ 1 & -1 & 4 \end{vmatrix}}{6} = \frac{1\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}{6}$$

$$= \frac{1(5) + 4(7) - 2(-3)}{6} = \frac{39}{6} = \boxed{\frac{13}{2}}$$

Therefore, the solution to the system is $(x, y, z) = \boxed{\left(\frac{5}{3}, \frac{25}{6}, \frac{13}{2}\right)}$