Project 2 - Practice Test 2

Luke Greenawalt

October 2024

1. Find the sum of the given vectors **(4.1)**:

$$\boldsymbol{v} = (4, -2, 10) \quad \boldsymbol{u} = (2, 6, -4)$$

2. Write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, and $\boldsymbol{u_3}$ **(4.1)**:

$$\boldsymbol{v} = (10, 1, 4) \quad \boldsymbol{u_1} = (2, 3, 5) \quad \boldsymbol{u_2} = (1, 2, 4) \quad \boldsymbol{u_3} = (-2, 2, 3)$$

3. Use axioms to prove that M_{33} is a vector space **(4.2)**
4. Prove that the set of third degree polynomials is not a vector space **(4.2)**
5. Determine whether the subset S of $\mathbb{R}^3$ consisting of vectors of the form (a, a^3, b) form a subspace **(4.3)**
6. Determine whether the set U of 3×3 diagonal matrices is a subspace of the vector space M_{33} **(4.3)**
7. Determine whether the set S is linearly independent or linearly dependent **(4.4)**:

$$S = \{2 + x - 3x^2, 1 + x + x^2, 3 - 4x^2\}$$

8. Determine whether the set S is linearly independent or linearly dependent **(4.4)**:

$$S = \{(0, 0, 1), (0, 2, -2), (1, -2, 1), (4, 2, 3)\}$$

9. Determine whether the set S is a basis for the vector space $\mathbb{R}^3$ **(4.5)**:

$$S = \{(1, 2, 1), (2, 3, 1), (0, 0, 1)\}$$

10. Determine whether the set S is a basis for the vector space P^2 **(4.5)**:

$$S = \{3 - t^2, 4t, t^3\}$$

11. Find a basis for the subspace of $\mathbb{R}^4$ spanned by S **(4.6)**:

$$S = \{(2, 5, -3, -2), (-2, -3, 2, -4), (1, 3, 1, 4), (0, -3, 0, 15)\}$$

12. Find the nullspace of the matrix **(4.6)**:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}$$

13. Find the coordinate matrix of x in $\mathbb{R}^n$ relative to the basis B' **(4.7)**:

$$B' = \{(2,1,3),(-4,5,1),(0,3,2)\} \quad \boldsymbol{x} = (10,-7,8)$$

14. Find the transition matrix from B to B' **(4.7)**:

$$B = \{(2,3),(5,-4)\} \quad B' = \{(4,2),(5,-6)\}$$

15. Find the Wronskian of the set of functions **(4.8)**:

$$\{x^2, \cos x, \sin x\}$$

16. (a) Verify that each solution satisfies the differential equation. (b) Test the set of solutions for linear independence. (c) if the set is linearly independent, write the general solution of the differential equation **(4.8)**:

$$y'' + 25y = 0 \quad \{\sin 5x, \cos 5x\}$$

17. Find the angle θ between the vectors **(5.1)**

$$\boldsymbol{u} = (1,0,0,1) \qquad \boldsymbol{v} = (0,1,1,1)$$

18. Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, parallel, or neither **(5.1)**

$$\boldsymbol{u} = (1,0,0) \qquad \boldsymbol{v} = (1,0,-2)$$

19. Find (a) $\langle A, B\rangle$, (b) $\|A\|$, (c) $\|B\|$, and (d) $d(A,B)$ for the matrices in M_{22} using the inner product **(5.2)**

$$\langle A, B\rangle = a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + a_{22}b_{22}$$

$$A = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

20. Show that the function defines an inner product on $\mathbb{R}^3$, where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ **(5.2)**.

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 4u_1v_1 + 5u_2v_2 + u_3v_3$$

21. (a) Determine whether the set of vectors is orthogonal, (b) if the set is orthogonal, then determine whether it is orthonormal, and (c) determine whether set is a basis for $\mathbb{R}^n$ **(5.3)**.

$$\{(3,0,-2),(1,-4,5),(2,3,1)\}$$

22. Apply the Gram-Schmidt orthonormalization process to transform the given basis for $\mathbb{R}^n$ into an orthonormal basis. Use the vectors in the order in which they are given **(5.3)**.

$$B = \{(1,-1,1),(1,0,1),(1,1,2)\}$$

23. Find (a) the orthogonal complement and (b) the direct sum **(5.4)**:

$$S = span\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}\right\}$$

24. Find bases for the four fundamental subspaces of the matrix A **(5.4)**.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

25. Find $\boldsymbol{u} \times \boldsymbol{v}$, and show that it is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$ **(5.5)**.

$$\boldsymbol{u} = (2, -1, 3) \qquad \boldsymbol{v} = (-1, 2, 0)$$

26. Find the area of the parallelogram that has the vectors as adjacent sides **(5.5)**:

$$\boldsymbol{u} = (4, -1, 2) \qquad \boldsymbol{v} = (-2, 3, 1)$$

Project 2 Answer Key

Luke Greenawalt

October 2024

1. To sum $\boldsymbol{v}$ and $\boldsymbol{u}$, we must just add the components:

$$\boldsymbol{v}+\boldsymbol{u} = (4+2, -2+6, 10+-4) = \boxed{(6,4,6)}$$

2. Mathematically, the question is asking us to solve for $c_1, c_2, \ldots, c_n$ such that

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \ldots c_n\boldsymbol{u_n} = \boldsymbol{v}$$

Rewriting this

$$c_1\begin{bmatrix}2\\3\\5\end{bmatrix} + c_2\begin{bmatrix}1\\2\\4\end{bmatrix} + c_3\begin{bmatrix}-2\\2\\3\end{bmatrix} = \begin{bmatrix}10\\1\\4\end{bmatrix}$$

Now, it is clear that we can for c_1, c_2, c_3 using an augmented matrix:

$$\begin{bmatrix}2 & 1 & -2 & 10\\3 & 2 & 2 & 1\\5 & 4 & 3 & 4\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -3\end{bmatrix}$$

Therefore, $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$ and

$$1\begin{bmatrix}2\\3\\5\end{bmatrix} + 2\begin{bmatrix}1\\2\\4\end{bmatrix} - 3\begin{bmatrix}-2\\2\\3\end{bmatrix} = \begin{bmatrix}10\\1\\4\end{bmatrix} \implies \boxed{1\boldsymbol{u_1} + 2\boldsymbol{u_2} - 3\boldsymbol{u_3} = \boldsymbol{v}}$$

3. Let's first let $\boldsymbol{u} = \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix} \in M_{33}$, $\boldsymbol{v} = \begin{bmatrix}j & k & l\\m & n & o\\p & q & r\end{bmatrix} \in M_{33}$, $w = \begin{bmatrix}s & t & u\\v & w & x\\y & z & \phi\end{bmatrix} \in M_{33}$, and let α and β be scalars. Now, let's prove each axiom:

(a) $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix} + \begin{bmatrix}j & k & l\\m & n & o\\p & q & r\end{bmatrix} = \begin{bmatrix}a+j & b+k & c+l\\d+m & e+n & f+o\\g+p & h+q & i+r\end{bmatrix} \in M_{33}$

(b) $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix}a+j & b+k & c+l\\d+m & e+n & f+o\\g+p & h+q & i+r\end{bmatrix} = \begin{bmatrix}j+a & k+b & l+c\\m+d & n+e & o+f\\p+g & q+h & r+i\end{bmatrix} \begin{bmatrix}j & k & l\\m & n & o\\p & q & r\end{bmatrix} + \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix} = \boldsymbol{v}+\boldsymbol{u}$

(c) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \left(\begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} + \begin{bmatrix} s & t & u \\ v & w & x \\ y & z & \phi \end{bmatrix} \right) = \begin{bmatrix} a+k+s & b+k+t & c+l+u \\ d+m+v & e+n+w & f+o+x \\ g+p+y & h+q+z & i+r+\phi \end{bmatrix} =$
$\begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix} + \begin{bmatrix} s & t & u \\ v & w & x \\ y & z & \phi \end{bmatrix} = (\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$

(d) $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $\quad \boldsymbol{u}+0 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(e) $-\boldsymbol{u} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$; $\quad \boldsymbol{u}+(-\boldsymbol{u}) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

(f) $\alpha\boldsymbol{u} = \alpha \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha i \end{bmatrix} \in M_{33}$

(g) $\alpha(\boldsymbol{u}+\boldsymbol{v}) = \alpha \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix} = \begin{bmatrix} \alpha(a+j) & \alpha(b+k) & \alpha(c+l) \\ \alpha(d+m) & \alpha(e+n) & \alpha(f+o) \\ \alpha(g+p) & \alpha(h+q) & \alpha(i+r) \end{bmatrix} = \begin{bmatrix} \alpha a+\alpha k & \alpha b+\alpha k & \alpha c+\alpha l \\ \alpha d+\alpha m & \alpha e+\alpha n & \alpha f+\alpha o \\ \alpha g+\alpha p & \alpha h+\alpha q & \alpha i+\alpha r \end{bmatrix} =$
$\begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha i \end{bmatrix} + \begin{bmatrix} \alpha j & \alpha k & \alpha l \\ \alpha m & \alpha n & \alpha o \\ \alpha p & \alpha q & \alpha r \end{bmatrix} = \alpha \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \alpha \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \alpha\boldsymbol{u}+\alpha\boldsymbol{v}$

(h) $(\alpha+\beta)\boldsymbol{u} = (\alpha+\beta) \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} (\alpha+\beta)a & (\alpha+\beta)b & (\alpha+\beta)c \\ (\alpha+\beta)d & (\alpha+\beta)e & (\alpha+\beta)f \\ (\alpha+\beta)g & (\alpha+\beta)h & (\alpha+\beta)i \end{bmatrix} = \begin{bmatrix} \alpha a+\beta a & \alpha b+\beta b & \alpha c+\beta c \\ \alpha d+\beta d & \alpha e+\beta e & \alpha f+\beta f \\ \alpha g+\beta g & \alpha h+\beta h & \alpha i+\beta i \end{bmatrix} =$
$\begin{bmatrix} \alpha & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha i \end{bmatrix} + \begin{bmatrix} \beta a & \beta b & \beta c \\ \beta d & \beta e & \beta f \\ \beta g & \beta h & \beta i \end{bmatrix} = \alpha \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \beta \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \alpha\boldsymbol{u}+\beta\boldsymbol{u}$

(i) $\alpha(\beta\boldsymbol{u}) = \alpha \begin{bmatrix} \beta a & \beta b & \beta c \\ \beta d & \beta e & \beta f \\ \beta g & \beta h & \beta i \end{bmatrix} = \begin{bmatrix} \alpha\beta a & \alpha\beta b & \alpha\beta c \\ \alpha\beta d & \alpha\beta e & \alpha\beta f \\ \alpha\beta g & \alpha\beta h & \alpha\beta i \end{bmatrix} = \alpha\beta \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (\alpha\beta)\boldsymbol{u}$

(j) $1\boldsymbol{u} = 1 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boldsymbol{u}$

4. We will prove this using a counterexample. Let $f(x) = 4x^3+3x^2+x+1$ and $g(x) = -4x^3+3x^2+x+1$. Then,

$$(f+g)(x) = f(x)+g(x) = 4x^3+3x^2+x+1-4x^3+3x^2+x+1 = 6x^2+2x+2$$

Which is a second degree polynomial, not a third. Therefore, the set of third degree polynomials is not closed under vector addition. The set is **not a vector space**.

5. We will show that S is not closed under vector addition, and therefore, **is not a subspace of** $\mathbb{R}^3$. Let $\boldsymbol{u} = (a_1, a_1^3, b_1)$ and $\boldsymbol{v} = (a_2, a_2^3, b_2)$. Then $\boldsymbol{u}+\boldsymbol{v} = (a_1+a_2, a_1^3+a_2^3, b_1+b_2)$ which is not in S because $(a_1+a_1)^3 \neq a_1^3+a_1^3$.

6. Let $\boldsymbol{u} = \begin{bmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{bmatrix} \in U$, $\boldsymbol{v} = \begin{bmatrix} v_{11} & 0 & 0 \\ 0 & v_{22} & 0 \\ 0 & 0 & v_{33} \end{bmatrix} \in U$, and k be a scalar.Then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_{11} + v_{11} & 0 & 0 \\ 0 & u_{22} + v_{22} & 0 \\ 0 & 0 & u_{33} + v_{33} \end{bmatrix} \in U$$

because it is diagonal. Furthermore,

$$k\boldsymbol{u} = k \begin{bmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{bmatrix} \in U$$

because it is a diagonal matrix. U is nonempty. Therefore, U **is a subspace of** M_{33}

7. For S to be linearly independent, the only solution $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+c_3\boldsymbol{v_3} = 0$ must be $(c_1, c_2, c_3) = (0, 0, 0)$. Let's check this:

$$c_1\left(2 + x - 3x^2\right) + c_2\left(1 + x + x^2\right) + c_3\left(3 - 4x^2\right) = 0$$

$$2c_1 + c_1x - 3c_1x^2 + c_2 + c_2x + c_2x^2 + 3c_3 - 4c_3x^2 = 0$$

$$(2c_1 + c_2 + 3c_3) + (c_1 + c_2 + 0c_3)x + (-3c_1 + c_2 - 4c_3)x^2 = 0$$

$$\begin{cases} 2c_1 + c_2 + 3c_3 = 0 \\ c_1 + c_2 + 0c_3 = 0 \\ -3c_1 + c_2 - 4c_3 = 0 \end{cases} \implies \begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 1 & -4 & 0 \end{bmatrix}$$

Now, to solve for c_1, c_2, and c_3, we must put this matrix in reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 1 & -4 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, $c_1 = c_2 = c_3 = 0$ and S **is linearly independent.**

8. Similar to the previous problems, to check whether the set S is linearly independent or linearly dependent, we must check whether there exists a combination of the elements of S that result in the 0 vector other than the trivial solution:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = 0$$

Let's represent this using an augmented matrix

$$\begin{bmatrix} 0 & 0 & 1 & 4 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 1 & -2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} c_1 + 9c_4 = 0 \implies c_1 = -9c_4 \\ c_2 + 5c_4 = 0 \implies c_2 = -5c_4 \\ c_3 + 4c_4 = 0 \implies c_3 = -4c_4 \end{cases}$$

Parameterizing these solutions with $c_4 = t$, we have

$$\begin{bmatrix} -9t \\ -5t \\ -4t \\ t \end{bmatrix}$$

Since there exists solutions other than the trivial one, the **set S is linearly dependent.**

9. First, we must ensure that S has the right number of vectors. Since we are asked if S is a basis for $\mathbb{R}^3$, and S contains 3 vectors, there are an appropriate amount of vectors. Next, using the same process as the previous problem, we will check for linear independence:

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Using an augmented matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \implies c_1 = c_2 = c_3 = 0$$

Therefore, S is linearly independent. Since S is linearly independent with 3 basis vectors, S **is a basis for $\mathbb{R}^3$.**

10. By observation, we see that t^3 is an element of set S but isn't in P^2. Therefore, S **is not a basis for** P^2

11. To do this, we will represent the elements of S as an augmented matrix and then use RREF form:

$$\begin{bmatrix} 2 & -2 & 1 & 0 \\ 5 & -3 & 3 & -3 \\ -3 & 2 & 1 & 0 \\ -2 & -4 & 4 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the basis for the subspace of $\mathbb{R}^4$ spanned by S is

$$\boxed{B = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}}$$

12. We are trying to find when $A\boldsymbol{x} = 0$, so let's do this using an augmented matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $x_1 = x_2 = 0$ and $\boxed{\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}}$ is the nullspace.

13. $\boldsymbol{x}$ is given in terms of B, so to find $\boldsymbol{x}$ in terms of B', we must find the linear combination of the basis in B' that equals $\boldsymbol{x}$:

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 8 \end{bmatrix}$$

Representing this as an augmented matrix:

$$\begin{bmatrix} 2 & -4 & 0 & 10 \\ 1 & 5 & 3 & -7 \\ 3 & 1 & 2 & 8 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & \frac{41}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & -5 \end{bmatrix}$$

Therefore,

$$\begin{cases} c_1 = \frac{41}{7} \\ c_2 = \frac{3}{7} \\ c_3 = -5 \end{cases} \implies \boxed{[x]_{B'} = \begin{bmatrix} \frac{41}{7} \\ \frac{3}{7} \\ -5 \end{bmatrix}}$$

14. To find the transition matrix from B to B', we set up a matrix in the following format:

$$\begin{bmatrix} B' & B \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 & 5 \\ 2 & -6 & 3 & -4 \end{bmatrix}$$

The goal is to get the matrix into the form $\begin{bmatrix} I_n & P^{-1} \end{bmatrix}$, so we will use reduced row-echelon form:

$$\begin{bmatrix} 4 & 5 & 2 & 5 \\ 2 & -6 & 3 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & \frac{27}{34} & \frac{5}{17} \\ 0 & 1 & -\frac{4}{17} & \frac{13}{17} \end{bmatrix}$$

Therefore, the transition matrix

$$\boxed{P^{-1} = \begin{bmatrix} \frac{27}{34} & \frac{5}{17} \\ -\frac{4}{17} & \frac{13}{17} \end{bmatrix}}$$

15. By definition, we can set up the Wronskian as follows:

$$W = \begin{vmatrix} x^2 & \cos x & \sin x \\ 2x & -\sin x & \cos x \\ 2 & -\cos x & -\sin x \end{vmatrix}$$

$$W = x^2 \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} 2x & \cos x \\ 2 & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} 2x & -\sin x \\ 2 & -\cos x \end{vmatrix}$$

$$W = x^2(\sin^2 x + \cos^2 x) - \cos x(-2x \sin x - 2\cos x) + \sin x(-2x \cos x + 2\sin x)$$

$$W = x^2 + 2x \sin x \cos x + 2\cos^2 x - 2x \sin x \cos x + 2\sin^2 x = x^2 + 2(\sin^2 x + \cos^2 x)$$

$$\boxed{W = x^2 + 2}$$

16. For part (a), we will first find the derivatives of both solutions:

$$y = \sin 5x \implies y' = 5\cos 5x \implies y'' = -25 \sin 5x$$

$$y = \cos 5x \implies y' = -5\sin 5x \implies y'' = -25 \cos 5x$$

Now, let's test each solution using these values

$$y'' + 25y = 0 \implies -25 \sin 5x + 25(\sin 5x)$$
$$= -25 \sin 5x + 25 \sin 5x = 0 \implies 0 = 0$$

$$y'' + 25y = 0 \implies -25 \cos 5x + 25(\cos 5x)$$

$$-25 \cos 5x + 25 \cos 5x = 0 \implies 0 = 0$$

Therefore, both solutions satisfy the differential equation. For part (b), if the solutions are linearly independent, the Wronskian must be nonzero. Let's test this:

$$W = \begin{vmatrix} \sin 5x & \cos 5x \\ 5\cos 5x & -5\sin 5x \end{vmatrix} = -5\sin^2 x - 5\cos^2 x = -5(\sin^2 x + \cos^2 x) = -5 \neq 0$$

Therefore, the solutions are linearly independent. For part (c), we can just say:

$$y = c_1 \sin 5x + c_2 \cos 5x$$

17. Using the dot product definition

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta \implies \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \cos\theta$$

$$\cos\theta = \frac{(1,0,0,1) \cdot ((0,1,1,1)}{\left(\sqrt{1^2+0^2+0^2+1^2}\right)\left(\sqrt{0^1+1^2+1^2+1^2}\right)}$$

$$\cos\theta = \frac{1(0)+0(1)+0(1)+1(1)}{\sqrt{2}\sqrt{3}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \implies \boxed{\cos^{-1}\left(\frac{\sqrt{6}}{6}\right) = \theta}$$

This is the most exact answer, but as an approximate answer, we have

$$\boxed{\theta \approx 1.15 rads}$$

18. For $\boldsymbol{u}$ and $\boldsymbol{v}$ to be orthogonal, the dot product of the two must be equal to 0. For them to be parallel, the angle between the vectors must be $\theta = 0$ or $\theta = \pi$. Let's first test orthogonality:

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1,0,0) \cdot (1,0,-2) = 1(1) + 0(0) + 0(-2) = 1 \neq 0$$

Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal. Now, let's find the angle using the following:

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{(1,0,0) \cdot (1,0,-2)}{\sqrt{1^2+0^2+0^2}\sqrt{1^2+0^2+(-2)^2}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Thus, $\theta = \cos^{-1}\left(\frac{\sqrt{5}}{5}\right)$, which is neither π nor θ. Therefore, $\boldsymbol{u}$ and $\boldsymbol{v}$ are neither parallel nor orthogonal.

19. Let's start with (a):

$$\langle A, B\rangle = a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + a_{22}b_{22} = 1(0) + 2(0)(1) + 2(7)(2) + 1(1) = 1 + 28 = \boxed{29}$$

Now, for (b)

$$\|A\| = \sqrt{\langle A, A\rangle} \implies a_{11}a_{11}+2a_{12}a_{12}+2a_{21}a_{21}+a_{22}a_{22} = 1(1)+2(0)(0)+2(7)(7)+1(1) = 1+98+1 = 100$$

$$\implies \|A\| = \sqrt{100} = \boxed{10}$$

Let's do the same for (c)

$$\|B\| = \sqrt{\langle B, B\rangle} \implies b_{11}b_{11}+2b_{12}b_{12}+2b_{21}b_{21}+b_{22}b_{22} = 0(0)+2(1)(1)+2(2)(2)+1(1) = 2+8+1 = 11$$

$$\implies \|B\| = \boxed{\sqrt{11}}$$

Finally, for (d)

$$d(A,B) = \|A-B\| = \sqrt{\langle A-B, A-B\rangle}$$

$$A-B = \begin{bmatrix}1 & 0\\ 7 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ 5 & 0\end{bmatrix}$$

$$\langle A-B, A-B\rangle = 1(1) + 2(-1)(-1) + 2(5)(5) + 0(0) = 53$$

$$\implies \|A-B\| = \boxed{\sqrt{53}}$$

20. To do this, we will simply prove each property:

    (a) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 4u_1v_1 + 5u_2v_2 + u_3v_3 = 4v_1u_1 + 5v_2u_2 + v_3u_3 = \langle \boldsymbol{v}, \boldsymbol{u}\rangle$

    (b) $\langle \boldsymbol{u}, \boldsymbol{v}+\boldsymbol{w}\rangle = 4u_1(v_1+w_1) + 5u_2(v_2+w_2) + u_3(v_3+w_3) = 4u_1v_1 + 4u_1w_1 + 5u_2v_2 + 5u_2w_2 + u_3v_3 + u_3w_3 = 4u_1v_1 + 5u_2v_2 + u_3v_3 + 4u_1w_1 + 5u_2w_2 + u_3w_3 = \langle \boldsymbol{u}, \boldsymbol{v}\rangle + \langle \boldsymbol{u}, \boldsymbol{w}\rangle$

    (c) $c\langle \boldsymbol{u}, \boldsymbol{v}\rangle = c(4u_1v_1 + 5u_2v_2 + u_3v_3) = 4cu_1v_1 + 5cu_2v_2 + cu_3v_3 = 4(cu_1)v_1 + 5(cu_2)v_2 + (cu_3)v_3 = \langle c\boldsymbol{u}, \boldsymbol{v}\rangle$

    (d) $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 4u_1u_1 + 5u_2u_2 + u_3u_3 \geq 0$ and $\langle \boldsymbol{u}, \boldsymbol{u}\rangle \iff \boldsymbol{u} = 0$

21. (a) For the set to be orthogonal, all combinations of inner products must be equal to zero. Obviously, there will only be 3 combinations since we have 3 vectors. However, when there are more vectors, I think it is good practice to first calculate how many combinations of vectors there are to ensure that the proof of orthogonality is thorough. Let's do this:

    $$\binom{3}{2} = \frac{3!}{2!1!} = 3$$

    Now, we can list the inner products and calculate them:

    - $\langle v_1, v_2\rangle = 3(1) + 0(-4) + -2(5) = 3 - 10 = -7$
    - $\langle v_1, v_3\rangle = 3(2) + 0(3) + -2(1) = 6 - 2 = 4$
    - $\langle v_2, v_3\rangle = 1(2) + -4(3) + 5(1) = -5$

    The inner products of the vectors are not equal to 0, and therefore, the set isn't orthogonal. For part (b), the set isn't orthogonal, so it obviously isn't orthonormal. For part (c), we will test for linear independence:

    $$c_1v_1 + c_2v_2 + c_3v_3 = 0 \implies c_1\begin{bmatrix}3\\ 0\\ -2\end{bmatrix} + c_2\begin{bmatrix}1\\ -4\\ 5\end{bmatrix} + c_3\begin{bmatrix}2\\ 3\\ 1\end{bmatrix} = 0$$

    Using matrices to solve for c_1, c_2, and c_3, we have:

    $$\begin{bmatrix}3 & 1 & 2 & 0\\ 0 & -4 & 3 & 0\\ -2 & 5 & 1 & 0\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \implies c_1 = c_2 = c_3 = 0$$

    Since the set has 3 vectors and is linearly independent, the set is a basis for $\mathbb{R}^3$

22. Let the three vectors $(1,-1,1)$, $(1,0,1)$, and $(1,1,2)$ equal v_1, v_2, and v_3 respectively. Using the formula, we have

$$w_1 = v_1 = (1,-1,1)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1,0,1) - \frac{1(1)+0(-1)+1(1)}{1(1)+(-1)(-1)+1(1)}(1,-1,1)$$

$$(1,0,1) - \frac{2}{3}(1,-1,1) = \left(1-\frac{2}{3}, 0+\frac{2}{3}, 1-\frac{2}{3}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$

$$w_3 = (1,1,2) - \frac{2}{3}(1,-1,1) - \frac{5}{2}\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right) = \left(1-\frac{2}{3}-\frac{5}{6}, 1+\frac{2}{3}-\frac{10}{6}, 2-\frac{2}{3}-\frac{5}{6}\right) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)$$

Now, to normalize the values, we need to divide each vector by its magnitude. Let's let the final vectors be f_1, f_2, and f_3:

$$\|w_1\| = \sqrt{1^2 + -1^2 + 1^2} = \sqrt{3} \implies f_1 = \frac{1}{\sqrt{3}}(1,-1,1) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\|w_2\| = \sqrt{\frac{1}{3}^2 + \frac{2}{3}^2 + \frac{1}{3}^2} = \sqrt{\frac{2}{3}} \implies f_2 = \frac{3}{\sqrt{6}}\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\|w_3\| = \sqrt{-\frac{1}{2}^2 + 0^2 + \frac{1}{2}^2} = \frac{1}{\sqrt{2}} \implies f_3 = \sqrt{2}\left(-\frac{1}{2}, 0, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$$

Finally, we have the orthonormal basis for $\mathbb{R}^3$:

$$\boxed{\left\{\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\}}$$

23. To find the orthogonal complement, the dot product between the given vector and the orthogonal complement $\boldsymbol{x}$ must be equal to zero.

$$(1,2,-1,1)\cdot(x_1,x_2,x_3,x_4) = x_1 + 2x_2 - x_3 + x_4 = 0$$

$$\implies \begin{cases} x_1 = t \\ x_2 = r \\ x_3 = w \\ x_4 = -t-2r+w \end{cases} \implies \begin{bmatrix} t \\ r \\ w \\ -t-2r+w \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \\ -t \end{bmatrix} + \begin{bmatrix} 0 \\ r \\ 0 \\ -2r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w \\ w \end{bmatrix}$$

$$= t \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore,

$$S^{\perp} = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} : t, r, w \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

For part (b):

$$S \oplus S^{\perp} = \mathbb{R}^4$$

24. Let's first find the basis for the nullspace of A:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the nullspace of A is only the trivial solution:

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

For the column space, we can look above and see that the leading zeros are in columns 1, 2, and 3. Therefore, the column space $R(A)$ is:

$$R(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Now, for $N(A^T)$ we must first find A^T and then use reduced row-echelon form:

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 2 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 \end{bmatrix} \implies \begin{cases} x_1 + \frac{1}{2}x_4 = 0 \\ x_2 + \frac{1}{2}x_4 = 0 \\ x_3 + \frac{3}{2}x_4 = 0 \end{cases}$$

Let $x_4 = t$

$$\begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = -\frac{1}{2}t \\ x_3 = -\frac{3}{2}t \\ x_4 = t \end{cases} \implies N(A^T) = \left\{ \begin{bmatrix} -\frac{1}{2}t \\ -\frac{1}{2}t \\ -\frac{3}{2}t \\ t \end{bmatrix} \right\} = \left\{ \frac{1}{2}t \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

Therefore, a basis for $N(A^T)$ is

$$N(A^T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ -3 \\ 2 \end{bmatrix} \right\}$$

Once again, looking at the reduced row-echelon calculation above, we see that the leading ones are in columns 1, 2, and 3, and therefore, the column space $R(A^T)$ is

$$R(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

25. To find $\boldsymbol{u} \times \boldsymbol{v}$, we will use the determinant method:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = \boldsymbol{i}(0-6) - \boldsymbol{j}(0+3) + \boldsymbol{k}(4-1)$$

$$= -6\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k} \implies \boldsymbol{u} \times \boldsymbol{v} = \boxed{(-6, -3, 3)}$$

Now, to check orthogonality, we will use the dot product between the given vectors and the cross product (let's call it $\boldsymbol{w}$):

$$\boldsymbol{u} \cdot \boldsymbol{w} = (2, -1, 3) \cdot (-6, -3, 3) = 2(-6) + -1(-3) + 3(3) = -12 + 3 + 9 = -12 + 12 = 0$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = (-1, 2, 0) \cdot (-6, -3, 3) = -1(-6) + 2(-3) + 0(2) = 6 - 6 = 0$$

Since both given vectors dotted with the cross product equal zero, cross product is orthogonal to both.

26. The area of the parallelogram with the given vectors as adjacent sides will simply be the magnitude of the cross product. Let's compute the cross product first:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -1 & 2 \\ -2 & 3 & 1 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix}$$

$$= \boldsymbol{i}(-1(1) - 2(3)) - \boldsymbol{j}(4(1) - 2(-2)) + \boldsymbol{k}(4(3) - (-2)(-1))$$

$$= -7\boldsymbol{i} - 8\boldsymbol{j} + 10\boldsymbol{j} \implies \boldsymbol{u} \times \boldsymbol{v} = (-7, -8, 10)$$

Now, for $\|\boldsymbol{u} \times \boldsymbol{v}\|$:

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{(-7)^2 + (-8)^2 + (10)^2} = \sqrt{49 + 64 + 100} = \sqrt{213}$$

Therefore, the area of the parallelogram is $\boxed{\sqrt{213}}$