Quiz 4 - Math 257

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1 Problem 1

To find the basis for the null space, I must first create add a column of zeroes to the matrix A. By doing this, we are solving the equation for $Ax = 0$.

$$\begin{bmatrix} 1 & 4 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -2 & -8 & -4 & -2 & 0 \end{bmatrix}$$

Now, let's put this into reduced row-echelon form using a calculator:

$$\begin{bmatrix} 1 & 0 & -2 & 5 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Representing this as a system of equations, I get:

$$x_1 - 2x_3 + 5x_4 = 0$$

$$x_2 + x_3 - x_4 = 0$$

Let's let $x_3 = t$ and $x_4 = r$:

$$x_1 - 2t + 5r = 0$$

$$x_1 = 2t - 5r$$

$$x_2 + t - r = 0$$

$$x_2 = -t + r$$

Therefore, the solution to the system is

$$\begin{bmatrix} 2t - 5r \\ -t + r \\ t \\ r \end{bmatrix}$$

Which is equal to

$$\begin{bmatrix} 2t \\ -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -5r \\ r \\ 0 \\ r \end{bmatrix}$$

$$t \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the basis for the nullspace of matrix A is

$$\boxed{\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

2 Problem 2

2.1 Part A

For S to be linearly independent, there must not be any linear combination of the elements that satisfy $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v_2} + \ldots + c_k\boldsymbol{v_k} = 0$ other than the trivial solution. Let's test this

$$c_1\left(t^3 - 1\right) + c_2\left(2t^2\right) + c_3\left(t + 3\right) + c_4\left(5 + 2t + 2t^2 + t^3\right) = 0$$

$$c_1t^3 - c_1 + c_22t^2 + c_3t + 3c_3 + 5c_4 + c_42t + 2c_4t^2 + c_4t^3 = 0$$

$$\implies (-c_1 + 3c_3 + 5c_4) + t(c_3 + 2c_4) + t^2(2c_2 + 2c_4) + t^3(c_1 + c_4) = 0 + 0t + 0t^2 + 0t^3$$

Using this large (and unpleasant) equation to create a system of equations, I have

$$-c_1 + 3c_3 + 5c_4 = 0$$

$$c_3 + 2c_4 = 0$$

$$2c_2 + 2c_4 = 0$$

$$c_1 + c_4 = 0$$

Let's represent this system using an augmented matrix and then use a calculator to put it into reduced row-echelon form

$$\begin{bmatrix} -1 & 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewriting this as a system of equations, I get

$$c_1 + c_4 = 0 \implies c_1 = -c_4$$

$$c_2 + c_4 = 0 \implies c_2 = -c_4$$

$$c_3 + 2c_4 = 0 \implies c_3 = -2c_4$$

Since c_4 can take on any value, there are more solutions beyond the trivial one. This means that no, the set S is not linearly independent. It is linearly dependent.

2.2 Part B

For a set S to be a basis of a vector space, it must be linearly independent and span the entire vector space. Since S is linearly dependent, S is not a basis for P_3.