Quiz 5 - Math 257

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1 Problem 1

Let's first test the solution e^x by finding the derivatives. Using basic calculus, an n-order derivative of e^x will still be e^x. Therefore, we can substitute e^x in for every variable in the differential equation:

$$(e^x) - 2(e^x) + e^x = 0 \implies 2e^x - 2e^x = 0$$
$$0 = 0$$

Therefore, e^x is a solution to the differential equation. Now, let's try xe^x. This time, I will have to calculate the derivatives of xe^x:

$$\frac{d}{dx}(xe^x) = (x)'(e^x) + (x)(e^x)' \implies y' = e^x + xe^x$$

$$\frac{d}{dx}(e^x + xe^x) = \frac{d}{dx}(e^x) + \frac{d}{dx}(xe^x) \implies y'' = e^x + (e^x + xe^x) = 2e^x + xe^x$$

Now, substituting these values into the differential differential equation, I get:

$$2e^x + xe^x - 2(e^x + xe^x) + xe^x = 2e^x + xe^x - 2e^x - 2xe^x + xe^x = 2e^x - 2e^x + 2xe^x - 2xe^x = 0$$

$$0 = 0$$

Therefore, xe^x is also a solution to the differential equation. Thus, $\{e^x, xe^x\}$ is a solution set to the differential equation.

Now, to test for linear dependence using the Wronskian. The Wronskian mustn't be identically equal to zero for the set to be linear independent. I will first set up the Wronskian:

$$W = \begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix}$$

$$W = e^x(e^x + xe^x) - xe^x(e^x) = e^{2x} + xe^{2x} - xe^{2x} \implies W = e^{2x} \neq 0$$

Since $W \neq 0$, the solution set is linearly independent. Therefore, the general solution is:

$$\boxed{y = C_1e^x + C_2xe^x}$$

2 Problem 2

2.1 Part A

For f and g to be orthogonal, $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = 0$. Let's test this:

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \langle x^2, 4x-3 \rangle = \int_0^1 x^2(4x-3)dx = \int_0^1 \left(4x^3 - 3x^2\right) dx$$

$$= \left(x^4 - x^3\right)\Big|_0^1 = \left((1)^4 - (1)^3\right) - \left(0^4 - 0^3\right) \implies 0 = 0$$

Therefore, $\boldsymbol{f}$ and $\boldsymbol{g}$ are orthogonal.

2.2 Part B

To determine the magnitude of $\boldsymbol{f}$ and $\boldsymbol{g}$, I will do the following:

$$\|\boldsymbol{f}\| = \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle} = \sqrt{\langle x^2, x^2 \rangle} = \sqrt{\int_0^1 x^2\left(x^2\right) dx} = \sqrt{\int_0^1 x^4 dx} = \sqrt{\frac{x^5}{5}\Big|_0^1} = \sqrt{\frac{1^5}{5} - \frac{0^5}{5}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

$$\boxed{\|\boldsymbol{f}\| = \frac{\sqrt{5}}{5}}$$

$$\|\boldsymbol{g}\| = \sqrt{\langle \boldsymbol{g}, \boldsymbol{g} \rangle} = \sqrt{\int_0^1 (4x-3)(4x-3)dx} = \sqrt{\int_0^1 \left(16x^2 - 24x + 9\right) dx} = \sqrt{\left(\frac{16x^3}{3} - 12x^2 + 9x\right)\Big|_0^1}$$

$$= \sqrt{\left(\frac{16(1)}{3} - 12(1) + 9(1)\right) - \left(\frac{16(0)}{3} - 12(0) + 9(0)\right)} = \sqrt{\frac{16}{3} - 3} = \sqrt{\frac{16}{3} - \frac{9}{3}} = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$$

$$\boxed{\|\boldsymbol{g}\| = \frac{\sqrt{21}}{3}}$$