Quiz 6 - Math 257

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1 Problem 1

To find $S^{\perp}$, I will first create a system of equations using the span of S. To do this, I will rewrite S as

$$S = span\left\{\begin{bmatrix}0\\1\\-1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\1\\0\\2\end{bmatrix}\right\} = span\left\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\right\}$$

Therefore, the orthogonal complement of these vectors, say $\boldsymbol{w}$, must satisfy

$$\boldsymbol{w}\cdot\boldsymbol{v_1} = 0 \quad \boldsymbol{w}\cdot\boldsymbol{v_2} = 0 \quad \boldsymbol{w}\cdot\boldsymbol{v_3} = 0$$

$$\begin{cases} 0(\boldsymbol{w_1}) + 1(\boldsymbol{w_2}) - 1(\boldsymbol{w_3}) + 1(\boldsymbol{w_4}) - 1(\boldsymbol{w_5}) = 0 \\ 0(\boldsymbol{w_1}) + 1(\boldsymbol{w_2}) + 0(\boldsymbol{w_3}) + 2(\boldsymbol{w_4}) - 1(\boldsymbol{w_5}) = 0 \\ 2(\boldsymbol{w_1}) + 0(\boldsymbol{w_2}) + 1(\boldsymbol{w_3}) + 0(\boldsymbol{w_4}) + 2(\boldsymbol{w_5}) = 0 \end{cases} \implies \begin{cases} w_2 - w_3 + w_4 - w_5 = 0 \\ w_2 + 2w_4 - w_5 = 0 \\ 2w_1 + w_3 + 2w_5 = 0 \end{cases}$$

Now, I will solve this system using matrices

$$\begin{bmatrix} 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Using this system, I see that

$$w_1 - \frac{1}{2}w_4 + w_5 = 0$$

$$w_2 + 2w_4 - w_5 = 0$$

$$w_3 = -w_4$$

Letting $w_4 = s$ and $w_5 = t$, I have

$$\begin{cases} w_1 - \frac{1}{2}s + t = 0 \implies w_1 = \frac{1}{2}s - t \\ w_2 + 2s - t = 0 \implies w_2 = -2s + t \\ w_3 = -s \\ w_4 = s \\ w_5 = t \end{cases}$$

Therefore, $\boldsymbol{w}$ can be expressed as

$$\boldsymbol{w} = \begin{bmatrix} \frac{1}{2}s - t \\ -2s + t \\ -s \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s \\ -2s \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ t \\ 0 \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally, the orthogonal complement $\boldsymbol{w}$ is

$$\boxed{\boldsymbol{w} = span \left\{ \begin{bmatrix} \frac{1}{2} \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

2 Problem 2

To find the area of the parallelogram with the given sides, I will simply compute $\|\boldsymbol{u} \times \boldsymbol{v}\|$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent sides. First, I will compute $\boldsymbol{u} \times \boldsymbol{v}$

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & 0 \\ -1 & 2 & 0 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= \boldsymbol{i}(-1(0) - 2(0)) - \boldsymbol{j}(2(0) - 0(-1)) + \boldsymbol{k}(2(2) - (-1)(-1)) = 0\boldsymbol{i} - 0\boldsymbol{j} + 3\boldsymbol{k}$$

By observation, I see that $\|\boldsymbol{u} \times \boldsymbol{v}\| = 3$, but let's verify:

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{0^2 + 0^2 + 3^2} = \sqrt{9} = 3$$

Therefore, the area of the parallelogram is $\boxed{3}$ square units