Linear Algebra Notes

Luke Greenawalt

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Contents

Chapter 1

Unit 1

1.1 Gaussian Elimination and Gauss-Jordan Elimination

1.1.1 All the Basic Theory

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

in which entry, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

Elementary Row Operations

You can think of elementary row operations in the same way that you can think of operations on a systems of equations. You can:

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of a row to another row

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the properties below:

1. Any row consisting entirely of zeros occur at the bottom of the matrix
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading one**
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Gaussian Elimination with Back Substitution

1. Write the augmented matrix of the system of linear equations
2. Use elementary row operations to rewrite the system in row-echelon form
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution

Note: In Gaussian elimination, you apply elementary row operations to obtain row-echelon form while Gauss-Jordan elimination is when you continue until you have reduced row-echelon form.

Definition of Homogeneous System of Equations

These are systems in which each constant term is zero. For instance:

$$3x + 4y = 0$$

$$4x - 7y = 0$$

In a homogeneous system of three variables, x_1, x_2, and x_3 has the trivial solution $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

1.1.2 Examples

Use back substitution to solve the system:

$$x - 4y + z = 28$$

$$y + 4z = 14$$

$$z = 3$$

Solve the system. If there are infinitely many solutions, parameterize them using t

$$2x_1 + 3x_2 - 4x_3 = 16$$

$$x_1 - 3x_2 + 16x_3 = 32$$

Find the solution set of the system of linear equations represented by the augmented matrix

$$\begin{bmatrix} 2 & 3 & 2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

$$x + 3y = -4$$

$$-x - 3y = 4$$

Determine the polynomial that passes through the points $(0,0)$, $(1,-1)$ $(5,0)$

Solve the partial fraction decomposition using matrices

$$\frac{x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Chapter 2

Unit 2

2.1 Operations with Matrices

2.1.1 Theory

Definition of Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same size $(m \times n)$ and $[a_{ij}] = [b_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix $A + B = [a_{ij} + bij]$

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix $cA = [ca_{ij}]$

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj}$$

2.1.2 Systems of Linear Equations

One practical application of matrix multiplication is representing a system of linear equations. Note how the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written as the matrix equation $Ax = b$, where A is the coefficient matrix of the system, and x and b are column matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2.1.3 Linear Combinations of Column Vectors

The matrix Ax is a linear combination of of the column vectors $a_1, a_2, a_3, \dots, a_n$ that form the coefficient matrix A.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Furthermore, the system $Ax = b$ is consistent if and only if b can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

2.2 Properties of Matrix Operations

2.2.1 Theory

Properties of Matrix Addition and Scalar Multiplication If A, B, and C are $m \times n$ matrices, and c and d are scalars, then the properties below are true.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $(cd)A = c(dA)$
4. $1A = A$
5. $c(A + B) = cA + cB$
6. $(c + d)A = cA + dA$

Properties of Zero Matrices If A is an $m \times n$ matrix and c is a scalar, then the properties below are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$

Properties of Matrix Multiplication If A, B, and C are matrices (with sizes such that the matrices are defined), and c is a scalar, then the properties below are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + BC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

Properties of the Identity Matrix If A is a matrix of size $m \times n$, then the properties below are true.

1. $AI_n = A$
2. $I_m A = A$

Number of Solutions of a Linear System For a system of linear equations, precisely one of the statements below is true.

1. The system has exactly one solution
2. The system has infinitely many solutions
3. The system has no solution

Transpose of a Matrix and Symmetric Matrices The **transpose** of a matrix changes all rows into columns and vice versa.

Properties of Transposes If A and B are matrices (with sizes such that the matrix operations are defined) and c is a scalar, then the properties below are true.

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = c\left(A^T\right)$
4. $(AB)^T = B^T A^T$

2.3 The Inverse of a Matrix

2.3.1 Theory

Definition of an Inverse Matrix An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is the (multiplicative) **inverse** of A. A matrix that does not have an inverse is **noninvertible** (or **singular**).

Uniqueness of an Inverse Matrix If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}

Find the Inverse of a Matrix by Gauss-Jordan Elimination Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of A on the left and the $n \times n$ identity matrix I on the right to obtain $[A, I]$. This process is **adjoining** matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A, I]$. The result will be the matrix $[IA^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying $AA^{-1} = I = A^{-1}A$

Shortcut for finding the inverse of a 2×2 matrix If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Properties of Inverse Matrices If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the statements below are true.

1. $(A^{-1})^{-1} = A$
2. $\left(A^k\right)^{-1} = A^{-1}A^{-1}A^{-1} \dots A^{-1} = \left(A^{-1}\right)^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

The Inverse of a Product If A and B are the invertible matrices of order n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Cancellation Properties If C is an invertible matrix, then the properties below are true.

1. If $AC = BC$, then $A = B \implies$ Right cancellation property
2. If $CA = CB$, then $A = B \implies$ Left cancellation property

Systems of Equations with Unique Solutions If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution $x = A^{-1}b$

2.4 Elementary Row Operations

2.4.1 Theory

Definition of an Elementary Matrix An $n \times n$ matrix is an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation

Remark The identity matrix I_n is elementary by this definition because it can obtained from multiplying any one of its' rows by 1.

Representing Elementary Row Operations Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is the product EA.

Definition of Row Equivalence Let A and B be $m \times n$ matrices. Matrix B is **row-equivalent** to A when there exists a finite number of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} \ldots E_2 E_1 E_A$$

Elementary Matrices are Invertible If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

A Property of Invertible Matrices A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Equivalent Conditions If A is an $n \times n$ matrix, then the statements below are equivalent:

1. A is invertible
2. $Ax = b$ has a unique solution for every $n \times 1$ column matrix b
3. $Ax = 0$ has only the trivial solution
4. A is row-equivalent to I_n
5. A can be written as the product of elementary matrices

2.5 Markov Chains

2.5.1 Theory

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix}$$

P is called the **matrix of transition probabilities** because it gives the probabilities of each possible type of transition (or change) within the population. At each transition, each member in a given state must either stay in that state or change to another state. For probabilities, this means that the sums of the entries in any column of P is 1. For instance, in the first column

$$p_{11} + p_{21} + \ldots + p_{n1} = 1$$

Such a matrix is called **stochastic** (the term ”stochastic” means ”regarding conjecture”). That is, an $n \times n$ matrix P is a **stochastic matrix** when each entry is a number between 0 and 1 inclusive, and the sum of the entries in each column of P is 1.

Finding the Steady State of a Matrix of a Markov Chain

1. Check to see that the matrix of transition probabilities P is a regular matrix
2. Solve the system of linear equations obtained from the matrix equation $P\bar{X} = \bar{X}$ along with the equation $x_1 + x_2 + \ldots + x_n = 1$
3. Check the solution in step 2 in the matrix equation $P\bar{X} = \bar{X}$

Remark If P is not regular, then the corresponding Markov chain may or may not have a unique steady state matrix

Definition of Regular Stochastic Matrix A stochastic matrix P is regular when some power of P has only positive entries

2.6 Cryptography and Regression Analysis

2.6.1 Theory

A **cryptogram** is a message written according to a secret code (the Greek word kryptos means ”hidden”). One method of using matrix multiplication to **encode** and **decode** messages is introduced below. First, start by assigning each number:

$$\begin{array}{ccc} 0 = \_ & 9 = I & 18 = R \\ 1 = A & 10 = J & 19 = S \\ 2 = B & 11 = K & 20 = T \\ 3 = C & 12 = L & 21 = U \\ 4 = D & 13 = M & 22 = V \\ 5 = E & 14 = N & 23 = W \\ 6 = F & 15 = O & 24 = X \\ 7 = G & 16 = P & 25 = Y \\ 8 = H & 17 = Q & 26 = Z \end{array}$$

Then convert the message to numbers and partition it into **uncoded row matrices**, each having n entries.

Least Squares Regression Analysis For the regression model $Y = XA + E$, the coefficients of the least squares are given by the matrix equation

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \; X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \; E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$A = \left(X^T X\right)^{-1} X^T Y$$

and the sum of squared error is $E^T E$

Chapter 3

Unit 3

3.1 The Determinant of a Matrix

3.1.1 Theory

Definition of the Determinant of a 2×2 Matrix The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is $\det A = |A| = a_{11}a_{22} - a_{21}a_{12}$

Minors and Cofactors of a Square Matrix If A is a square matrix, then the **minor** M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A. The **cofactor** C_{ij} of the entry a_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Sign Patterns for Cofactors

- 3×3 matrix:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- 4×4 matrix:

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Definition of the Determinant of a Square Matrix If A is a square matrix of order $n \geq 2$, then the determinant A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,

$$\det(A) = |A| = \sum_{i=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

Determinant of a Triangular Matrix If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det A = A = a_{11}a_{22}a_{33}\dots a_{nn}$$

3.2 Determinants and Elementary Operations

3.2.1 Theory

Elementary Row Operations and Determinants Let A and B be square matrices:

1. When B is obtained from A by interchanging two rows of A, $\det B = -\det A$
2. When B is obtained from A by adding a multiple of a row of A to another row of A, $\det B = \det A$
3. When B is obtained from A by multiplying a row of A by a nonzero constant c, $\det B = c \det A$

Conditions that Yield a Zero Determinant If A is a square matrix and any one of the conditions below is true, then $\det A = 0$.

1. An entire row (or entire column) consists of zeros.
2. Two rows (or columns) are equal
3. One row (or column) is a multiple of another row (or column)

3.3 Properties of Determinants

Determinant of a Matrix Product If A and B are square matrices of order n, then $\det AB = \det A \det B$

Determinant of a Scalar Multiply of a Matrix If A is a square matrix of order n and c is a scalar, then the determinant of cA is

$$\det cA = c^n \det A$$

Determinant of an Invertible Matrix A square matrix A is invertible (non-singular) if and only if $\det A \neq 0$

Determinant of an Inverse Matrix If A is an $n \times n$ invertible matrix, then $\det A^{-1} = \frac{1}{\det A}$

Equivalent Conditions for a Nonsingular Matrix If A is an $n \times n$ matrix, then the statements below are equivalent.

1. A is invertible
2. $Ax = b$ has a unique solution for every $n \times 1$ column matrix b
3. $Ax = 0$ has only the trivial solution

4. A is row-equivalent to I_n
5. A can be written as the product of elementary matrices
6. $\det A \neq 0$

Determinant of a Transpose If A is a square matrix, then

$$\det A = \det A^T$$

3.4 Applications of Determinants

3.4.1 Theory

Recall that the cofactor C_{ij} of a square matrix is $(-1)^{i+j}$ times the determinant of the matrix obtained by deleting the ith row and jth column of A. The **matrix of cofactors** of A has the form

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

The transpose of this matrix is the **adjoint** of A and is denoted $\operatorname{adj}(A)$. That is,

$$\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

The Inverse of a Matrix Using Its Adjoint If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

Cramer's Rule If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, the the solution of the system is

$$x_1 = \frac{\det A_1}{\det A} \dots x_2 = \frac{\det A_2}{\det A} \dots x_n = \frac{\det A_n}{\det A}$$

where the ith column of A_i is the column of constants in the system of equations

Area of a Triangle in the xy-Plane The area of a triangle with vertices

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

is

$$Area = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_y & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the $\pm$ sign is given to get a positive area

Test for Collinear Points in the xy-Plane Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Two-Point Form of an Equation of a Line An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix}$$

Volume of a Tetrahedron The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$Volume = \pm\frac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign $\pm$ is chosen to give a positive volume

Test for Coplanar Points in Space Four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$$

Three-Point Form of an Equation of a Plane An equation of a plane passing through the distinct point (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

Chapter 4

Unit 4

4.1 Vectors in $\mathbb{R}^n$

4.1.1 Theory

Geometrically, a **vector plane** is represented by a **directed line segment** with its **initial point** at the origin and its **terminal point** at (x_1, x_2).

Definition of Vector Addition and Scalar Multiplication in R^n Let $u = (u_1, u_2, u_3, \dots, u_n)$ and $v = (v_1, v_2, v_3, \dots, v_n)$ be vectors in R^n and let c be a real number. The sum of u and v is the vector

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots, u_n + v_n)$$

and the **scalar multiple** of u by c is the vector

$$cu = (cu_1, cu_2, cu_3, \dots, cu_n)$$

Properties of Vector Addition and Scalar Multiplication in R^n Let u, v, and w be vectors in R^n, and let c and d be scalars

1. $u + v$ is a vector in R^n - Closure under addition
2. $u + v = v + u$ - Commutative Property of Addition
3. $(u + v) + w = u + (v + w)$ - Associative Property of Addition
4. $u + 0 = u$ - Additive Identity Property
5. $u + (-u) = 0$ - Additive Inverse Property
6. cu is a vector in R^n - Closure under scalar multiplication
7. $c(u + v) = cu + cv$ - Distributive Property
8. $(c + d)u = cu + du$ - Distributive Property
9. $c(du) = (cd)u$ - Associative Property of Multiplication
10. $1(u) = u$ - Multiplicative Identity Property

An important type of problem in linear algebra involves writing one vector x as the sum of scalar multiples of other vectors $v_1, v_2, \ldots, v_n$. That is, for scalars $c_1, c_2, \ldots, c_n$,

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

The vector $\boldsymbol{x}$ is called a **linear combination** of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$

4.2 Vector Spaces

4.2.1 Theory

Definition of a Vector Space

Let V be a set on which two operations (**vector addition** and **vector multiplication**) are defined. If the listed axioms are satisfied for every $\boldsymbol{u}$ $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and every scalar (real number) c and d, then V is a **vector space**.

Addition

1. $\boldsymbol{u} + \boldsymbol{v}$ is in V - Closure
2. $u + v = v + u$ - Commutative Property
3. $(u + v) + w = u + (v + w)$ - Associative Property
4. V has a **zero vector 0** such that for every $\boldsymbol{u}$ in V, $\boldsymbol{u} + 0 = \boldsymbol{u}$ - Additive Identity
5. For every $\boldsymbol{u}$ in V, there is a vector in V denoted by $-\boldsymbol{u}$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = 0$ - Additive Inverse

Scalar Multiplication

1. cu is a vector in R^n - Closure under scalar multiplication
2. $c(u + v) = cu + cv$ - Distributive Property
3. $(c + d)u = cu + du$ - Distributive Property
4. $c(du) = (cd)u$ - Associative Property
5. $1(u) = u$ - Scalar Identity Property

Summary of Important Vector Spaces

1. $\mathbb{R}$ - set of all real numbers
2. $\mathbb{R}^2$ - set of all ordered pairs
3. $\mathbb{R}^3$ - set of all ordered triplets
4. $\mathbb{R}^n$ - set of all ordered n-tuples
5. $C(-\infty, \infty)$ - set of all continuous functions defined on the real number line
6. $C[a, b]$ - set of all continuous functions defined on a closed interval $[a, b]$ where $a \neq b$

7. $\mathbb{P}$ - set of all polynomials
8. $\mathbb{P}_n$ - set of all polynomials degree $\leq n$ (together with zero polynomials)
9. $M_{m,n}$ - set of all $m \times n$ matrices
10. $M_{n,n}$ - set of all $n \times n$ square matrices

Example Proof 1

Use the axioms to prove M_{22} is a vector space

Let $\boldsymbol{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$, $\boldsymbol{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_{22}, \boldsymbol{w} \in M_{22}$, and α and β be scalars

1. $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in M_{22}$
2. $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{v} + \boldsymbol{u}$
3. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right) = \begin{bmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
4. $\boldsymbol{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \boldsymbol{u} + 0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{u}$
5. $-\boldsymbol{u} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \implies \boldsymbol{u} + (-\boldsymbol{u}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \boldsymbol{0}$
6. $\alpha \boldsymbol{u} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \in M_{22}$
7. $\alpha(\boldsymbol{u}+\boldsymbol{v}) = \alpha \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} \alpha(a+e) & \alpha(b+f) \\ \alpha(c+g) & \alpha(d+h) \end{bmatrix} = \begin{bmatrix} \alpha a + \alpha e & \alpha b + \alpha f \\ \alpha c + \alpha g & \alpha d + \alpha h \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} + \begin{bmatrix} \alpha e & \alpha f \\ \alpha g & \alpha h \end{bmatrix} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \alpha \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \alpha \boldsymbol{u} + \alpha \boldsymbol{v}$
8. $(\alpha+\beta) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & (\alpha+\beta)d \end{bmatrix} = \begin{bmatrix} \alpha a + \beta a & \alpha b + \beta b \\ \alpha c + \beta c & \alpha d + \beta d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} + \begin{bmatrix} \beta a & \beta b \\ \beta c & \beta d \end{bmatrix} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha \boldsymbol{u} + \beta \boldsymbol{u}$
9. $\alpha(\beta \boldsymbol{u}) = \alpha \begin{bmatrix} \beta a & \beta b \\ \beta c & \beta d \end{bmatrix} = \alpha\beta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (\alpha\beta)\boldsymbol{u}$
10. $1\boldsymbol{u} = 1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{u}$

Therefore, M_{22} is a vector space.

4.3 Subspaces of Vector Spaces

4.3.1 Theory

Definition of a Subspace of a Vector Space

A nonempty subset W of a vector space V is a **subspace** of V when W is a vector space under the operations of addition and scalar multiplication defined in V. If W is a subspace of V, then it must be closed under the operations inherited from V.

Test for a Subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the two closure conditions listed below hold.

1. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are in W, then $\boldsymbol{u}+\boldsymbol{v}$ is in W
2. If $\boldsymbol{u}$ is in W and c is any scalar, then $c\boldsymbol{u}$ is in W

The Intersection of Two Subspaces Is A Subspace

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

4.3.2 Examples

Determine whether the subset W of $\mathbb{R}^3$ consisting of vectors of the form (a, a, b) is a subspace.

Proof:

1. W is nonempty, as $(1, 1, 2) \in W$.
2. Let $\boldsymbol{u} = (a_1, a_1, b_1) \in W$ and $\boldsymbol{v} = (a_2, a_2, b_2) \in W$ and let k be a scalar
3. Then $\boldsymbol{u}+\boldsymbol{v} = (a_1 + a_2, a_1 + a_2, b_1 + b_2)$
4. Because the first two components are equal $(\boldsymbol{u}+\boldsymbol{v}) \in W$
5. Now, $k\boldsymbol{u} = (ka_1, ka_1, kb_1) \in W$ because the first two components are equal
6. Therefore, W is a subspace of $\mathbb{R}^3$

Determine whether the subset S of $\mathbb{R}^3$ consisting of vectors of the form (a, a^2, b) form a subspace.

Proof: We will show that S is not closed under vector addition and, therefore, is not a subspace of $\mathbb{R}^3$. Let $\boldsymbol{u} = \left(a_1, a_1^2, b_1\right)$ and $\boldsymbol{v} = \left(a_2, a_2^2, b_2\right)$. Then $\boldsymbol{u}+\boldsymbol{v} = \left(a_1 + a_2, a_1^2 + a_2^2, b_1 + b_2\right)$, which is not in S because $(a_1 + a_2)^2 \neq a_1^2 + a_2^2$.

Determine whether the set U of 2×2 diagonal matrices is a subspace of the vector space M_{22}.

Proof: Let $\boldsymbol{u} = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} \in U$, $\boldsymbol{v} = \begin{bmatrix} v_{11} & 0 \\ 0 & v_{22} \end{bmatrix} \in U$, and k be a scalar. Then $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} u_{11}+v_{11} & 0 \\ 0 & u_{22}+v_{22} \end{bmatrix} \in U$ because it is diagonal. Now $k\boldsymbol{u} = k \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & 0 \\ 0 & ku_{22} \end{bmatrix} \in U$ because it is a diagonal matrix. Note that U is nonempty. Therefore, U is a subspace of M_{22}.

4.4 Spanning Sets and Linear Independence

4.4.1 Theory

Definition of a Linear Combination of Vectors

As previously defined, a vector $\boldsymbol{v}$ in a vector space V is a **linear combination** of the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ in V when $\boldsymbol{v}$ can be written in the form

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \ldots + c_k\boldsymbol{u_k}$$

where $c_1, c_2, \ldots, c_k$ are scalars

Definition of a Spanning Set of a Vector Space

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ be a subset of a vector space V. The set S is a **spanning set** of V when every vector in V can be written as a linear combination of vectors in S. In such cases, it is said that S **spans** V.

Definition of the Span of a Set

If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v}_\}$ is a set of vectors in a vector space V, then the **span of** S is the set of all linear combinations of the vectors in S,

$$span(S) = \{c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_k\boldsymbol{v_k} : c_1, c_2, \ldots, c_k \in R\}$$

The span of S is denoted by

$$span(S)$$

or

$$span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$$

When $span(S) = V$, it is said that V is **spanned** by $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$, or that S **spans** V

$Span(S)$ is a Subspace of V

If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is a set of vectors in a vector space V, then $span(S)$ is a subspace of V. Moreover, $span(S)$ is the smallest possible subspace of V that contains S, in the sense that every other subspace of V that contains S must contain $span(S)$.

Definition of Linear Dependence and Linear Independence

A set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ in a vector space V is **linearly independent** when the vector equation

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_k\boldsymbol{v_k} = 0$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0$$

If there are also nontrivial solutions, then S is linearly dependent

Testing for Linear Independence and Linear Dependence

Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, use the steps below.

1. From the vector equation $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_k\boldsymbol{v_k} = 0$, write a system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k
2. Determine whether the system has a unique solution
3. If the system only has the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

A Property of Linearly Dependent Sets

A set $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}, k \geq 2$, is linearly dependent if an only if at least one of the sectors v_i can be written as a linear combination of the other vectors in S.

4.4.2 Examples

Let $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$ be in M_{22}. Determine whether the given matrix is a linear combination of A and B.

$$D = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$$

$$2c_1 + 0 = 6$$

$$-3c_1 + 5c_2 = -19$$

$$4c_1 + c_2 = 10$$

$$c_1 - 2c_2 = 7$$

$$\begin{bmatrix} 2 & 0 & 6 \\ -3 & 5 & -19 \\ 4 & 1 & 10 \\ 1 & -2 & 7 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies D = 3A - 2B$$

Therefore, D is a linear combination of A and B

Determine whether the set S spans $\mathbb{R}^3$.

$$S = \{(1, -2, 0), (0, 0, 1), (-1, 2, 0)\}$$

Let $\boldsymbol{v} = (v_1, v_2, v_3)$ be an arbitrary vector in $\mathbb{R}^3$. Then,

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$c_1 - c_3 = v_1$$

$$-2c_1 + 2c_3 = v_2$$

$$c_2 = v_3$$

Looking here, we can see that the system doesn't have a unique solution and therefore, doesn't span $\mathbb{R}^3$.

Determine whether the set S is linearly independent or linearly dependent

$$S = \{7 - 3x + 4x^2, 6 + 2x - x^2, 1 - 8x + 5x^2\}$$

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1(7 - 3x + 4x^2) + c_2(6 + 2x - x^2) + c_3(1 - 8x + 5x^2) = 0$$

$$7c_1 + 6c_2 + c_3 + 2c_2x - 3c_1x - 8c_3x + 4c_1x^2 - c_2x^2 + 5c_3x^2 = 0 + 0x + 0x^2$$

$$7c_1 + 6c_2 + c_3 = 0$$

$$-3c_1 + 2c_2 - 8c_3 = 0$$

$$4c_1 - c_2 + 5c_3 = 0$$

$$\Longrightarrow \begin{bmatrix} 7 & 6 & 1 & 0 \\ -3 & 2 & -8 & 0 \\ 4 & -1 & 5 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow c_1 = c_2 = c_3 = 0$$

Therefore, S is linearly indepdent

4.5 Basis and Dimension

4.5.1 Theory

Definition of a Basis

A set of vectors $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ in a vector space V is a **basis** for V when the conditions below are true.

1. S spans V
2. S is lineraly independent

Standard Bases for Common Vector Spaces

1. $\mathbb{R}^3 : \{(1,0,0),(0,1,0),(0,0,1)\}$
2. $M_{22} : \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
3. $P_5 : \left\{1, x, x^2, x^3, x^4, x^5\right\}$

Bases and Linear Dependence

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every set containing more than n vectors is linearly dependent.

Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors

Definition of the Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then the number n is the **dimension** of V, denoted by $\dim V = n$. When V consists of the zero vector alone, the dimension of V is defined as zero.

Basis Test in an n-dimensional Space

Let V be a vector space of dimension n.

- If $S = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
- If $S = \{v_1, v_2, \ldots, v_n\}$ spans V, then S is a basis for V.

4.5.2 Examples

Explain why S is not a basis for the given vector space $\mathbb{R}^3$:

$$S = \{(0,0,0), (1,0,0), (0,1,0)\}$$

Representing S:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $c_2 = c_3 = 0$, but c_1 can be non-zero. Since this isn't the trivial solution, S is not a linearly independent set. Therefore, S is not a basis.

Explain why S is not basis for the given vector space $\mathbb{R}^3$:

$$S = \{(1,1,1), (0,1,1), (1,0,1), (0,0,0)\}$$

S cannot be a basis for $\mathbb{R}^3$, because it contains the zero vector and too many vectors. Therefore, S is linearly dependent.

4.6 Rank of a Matrix

4.6.1 Theory

Definitions of Row Space and Column Space of a Matrix

Let A be an $m \times n$ matrix

- The **row space** of A is the subspace of R^n spanned by the row vectors of A
- The **column space** of A is the subspace of R^m spanned by the column vectors of A

Row-Equivalent Matrices Have the Same Row Space

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

Basis for the Row Space of a Matrix

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row-space of A

Row and Column Spaces Have Equal Dimensions

The row space and column space of an $m \times n$ matrix A have the same dimension

Definition of the Rank of a Matrix

The dimension of the row (or column) space of a matrix A is the **rank** of A and is denoted by $rank(A)$.

Solutions of a Homogenous System

If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $Ax = 0$ is a subspace of $\mathbb{R}^n$ called the **nullspace** of A and is denoted $N(A)$. So,

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$$

The dimensions of the nullspace of A is the **nullity** of A.

Dimension of the Solution Space

If A is a $m \times n$ matrix of rank r, then the dimension of the solution space of $Ax = 0$ is $n - r$. That is $n = rank(A) + N(A)$

Solutions of a System of Linear Equations

The system $Ax = b$ is consistent if and only if b is in the column space of A.

Summary of Equivalent Conditions for Square Matrices

If A is an $n \times n$ matrix, then the conditions below are equivalent

- A is invertible
- $Ax = b$ has a unique solution for any $n \times 1$ matrix b
- $Ax = 0$ has only the trivial solution
- A is row-equivalent to I_n
- $|A| \neq 0$
- $Rank(A) = n$
- The n row vectors of A are linearly indepdent
- The n column vectors of A are linearly independent

4.7 Coordinates and Change of Basis

4.7.1 Theory

Coordinate Representation Relative to a Basis

Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ be an ordered basis for a vector space V and let x be a vector in V such that

$$\boldsymbol{x} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_n\boldsymbol{v_n}$$

The scalars $c_1, c_2, \ldots, c_n$ are the **coordinates of** x **relative to the basis** B. The **coordinate matrix** (or **coordinate vector**) **of** x **relative to** B is the column matrix in $\mathbb{R}^n$ whose components are the coordinates of x.

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Transition Matrix B to B'

Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ and $B' = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ be two bases for $\mathbb{R}^n$. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $\begin{bmatrix} B' & B \end{bmatrix}$, as shown below.

$$\begin{bmatrix} B' & B \end{bmatrix} = \begin{bmatrix} I_n & P^{-1} \end{bmatrix}$$

The Inverse of a Transition Matrix

If P is a transition matrix from a basis B' to a basis B in $\mathbb{R}^n$, then P in invertible and the transition matrix from B to B' is P^{-1}.

4.7.2 Examples

Given the coordinate matrix of x relative to a constandard basis B for $\mathbb{R}^n$, find the coordinate matrix of x relative to the standard basis.

$$B = \{(2, -1), (0, 1)\}, [x]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}; B' = \{(1, 0), (0, 1)\}$$

Method 1

$$4\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies [x]_{B'} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

Method 2 Transition matrix from B to B' (Find P^{-1})

$$\begin{bmatrix} B' & B \end{bmatrix} \implies \begin{bmatrix} I_n & P^{-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \implies P^{-1} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

Find the coordinate matrix of x in $\mathbb{R}^n$ relative to the standard basis $B^{'}$

$$B' = \{(4,3,3), (-11,0,11), (0,9,2)\}, x = (11,18,-7)$$

$$c_1 \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -11 \\ 0 \\ 11 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \\ -7 \end{bmatrix}$$

$$\begin{cases} 4c_1 - 11c_2 = 11 \\ 3c_1 + 9c_3 = 18 \\ 3c_1 + 11c_2 + 2c_3 = -7 \end{cases} \implies \begin{bmatrix} 4 & -11 & 0 & 11 \\ 3 & 0 & 9 & 18 \\ 3 & 11 & 2 & -7 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\implies \begin{cases} c_1 = 0 \\ c_2 = -1 \\ c_3 = 2 \end{cases} \implies [x]_{B'} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

4.8 Applications of Vector Spaces

4.8.1 Theory

Solutions of a Linear Homogenous Differential Equation

Every n-th order linear homogeneous differential equation

$$y^{(n)} + g_{n-1}(x)y^{n-1} + \ldots + g_1(x)y^{'} + g_0(x)y = 0$$

has n linearly independent solutions. Moreover, if $\{y_1, y_2, \ldots, y_n\}$ is a set of linearly independent solutions, then every solution is of the form

$$y = C_1y_1 + C_2y_2 + \ldots C_ny_n$$

where $C_1, C_2, \ldots, C_n$ are real numbers

Definition of the Wronskian of a Set of Functions

Let $\{y_1, y_2, \ldots, y_n\}$ be a set of functions, each of which has $n-1$ derivatives on an interval I. The determinant

$$W(y_1, y_2, \ldots, y_n) = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1} \end{bmatrix}$$

is the **Wronskian** of the set of functions

Wronskian Test for Linear Independence

Let $\{y_1, y_2, \ldots, y_n\}$ be a set of n solutions of an nth-order linear homogeneous differential equation. This set is linearly independent if and only if the Wronskian is not identically equal to zero.

4.8.2 Examples

Determine which functions are solutions of the linear differential equation

$$x^2y'' - 2y = 0$$

1. $\frac{1}{x^2}$
2. x^2
3. e^{x^2}

1.
$$\frac{1}{x^2} = x^{-2} \implies (x^{-2})' = -2x^{-3} \implies (-2x^{-3})' = \frac{6}{x^4}$$

$$\implies x^2\left(\frac{6}{x^4}\right) - 2\left(\frac{1}{x^2}\right) \implies \frac{6}{x^2} - \frac{2}{x^2} = \frac{4}{x^2} \neq 0$$

2.
$$x^2 \implies (x^2)' = 2x \implies (2x)' = 2$$

$$\implies x^2(2) - 2(x^2) = 2x^2 - 2x^2 = 0$$

3.
$$y = e^{x^2} \implies \left(e^{x^2}\right)' = 2xe^{x^2} \implies \left(2xe^{x^2}\right)' = 2e^{x^2} + 4x^3e^{x^2}$$

$$\implies x^2(2e^{x^2} + 4x^2e^{x^2}) - 2\left(e^{x^2}\right) = 2x^2e^{x^2} + 4x^3e^{x^2} - 2e^{x^2} \implies 2e^{x^2} = 2x^2e^{x^2} + 4x^3e^{x^2} \implies 2e^{x^2} \neq 0$$

Therefore, only 2 satisfies the differential equation

Find the Wronskian of the set of functions

$$\{x, \sin x, \cos x\}$$

$$W = \begin{bmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{bmatrix}$$

$$W = x \begin{bmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{bmatrix} - \sin x \begin{bmatrix} 1 & -\sin x \\ 0 & -\cos x \end{bmatrix} + \cos x \begin{bmatrix} 1 & \cos x \\ 0 & -\sin x \end{bmatrix}$$

$$W = x \left(-\cos^2 x - \sin^2 x\right) - \sin x \left(-\cos x\right) + \cos x \left(-\sin x\right)$$

$$W = -x\cos^2 x - x\sin^2 x + \sin x \cos x - \sin x \cos x \implies W = -x\cos^2 x - x\sin^2 x \implies \boxed{W = -x}$$

Find the Wronskian of the set of functions

$$\{1, e^x, e^{2x}\}$$

$$W = \begin{bmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{bmatrix}$$

$$W = 1 \begin{bmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{bmatrix} - e^x \begin{bmatrix} 0 & 2e^{2x} \\ 0 & 4e^{2x} \end{bmatrix} + e^{2x} \begin{bmatrix} 0 & e^x \\ 0 & e^x \end{bmatrix} = \left(4e^{3x} - 2e^{3x}\right) - e^x(0) + e^{2x}(0)$$

$$\boxed{W = 2e^{3x}}$$

Chapter 5

Unit 5

5.1 Length and Dot Product in $\mathbb{R}^n$

5.1.1 Theory

Definition of the Length of a Vector in $\mathbb{R}^n$

The **length**, or **norm** of a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 \ldots + v_n^2}$$

The length of a vector is also called its **magnitude**. If $\|v\| = 1$, then the vector $\boldsymbol{v}$ is a **unit vector**.

Length of a Scalar Multiple

Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then,

$$\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$$

where $|c|$ is the absolute value of c

Unit Vector in the Direction of $\boldsymbol{v}$

If $\boldsymbol{v}$ is a nonzero vector in $\mathbb{R}^n$, then the vector

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

has length 1 and has the same direction as $\boldsymbol{v}$. This vector $\boldsymbol{u}$ is the **unit vector in the direction of v**.

Definition of Distance Between Two Vectors

The distance between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$$

Definition of Dot Product in $\mathbb{R}^n$

The **dot product** of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is the scalar quantity

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Properties of the Dot Product

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
2. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
3. $c(\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v})$
4. $\boldsymbol{v} \cdot \boldsymbol{v} = \|v\|^2$
5. $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$, and $\boldsymbol{v} \cdot \boldsymbol{v} = 0 \iff \boldsymbol{v} = 0$

Definition of the Angle Between Two Vectors in $\mathbb{R}^n$

The **angle** θ between two nonzero vectors in $\mathbb{R}^n$ can be found using

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}, 0 \leq \theta \leq \pi$$

Definition of Orthogonal Vectors

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are **orthogonal** when

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0$$

The Triangle Inequality

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$, then

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

The Pythagorean Theorem

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are othogonal if and only if

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

5.1.2 Examples

Examples are trivial and not worth including

5.2 Inner Product Spaces

5.2.1 Theory

Definition of Inner Product

Let $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in a vector space V, and let c be any scalar. An **inner product** on V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and satisfies the axioms listed below.

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
3. $c\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle c\boldsymbol{u}, \boldsymbol{v} \rangle$
4. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = 0$

Properties of Inner Products

Let $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in an inner product space V, and let c be any real number.

1. $\langle 0, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, 0 \rangle = 0$
2. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
3. $\langle \boldsymbol{u}, c\boldsymbol{v} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

Definitions of Length Distance and Angle

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product space V

1. The **length** (or **norm**) of $\boldsymbol{u}$ is $\|\boldsymbol{u}\| = \sqrt{(\boldsymbol{u}, \boldsymbol{u})}$
2. The **distance** between $\boldsymbol{u}$ and $\boldsymbol{v}$ is $(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$
3. The **angle** between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be found using

$$\cos\theta = \frac{(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}, 0 \leq \theta \leq \pi$$

4. $\boldsymbol{u}$ and $\boldsymbol{v}$ are **orthogonal** when $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$

Generalized Theorems from the Dot Product

1. Cauchy-Schwarz Inequality: $\|(\boldsymbol{u}, \boldsymbol{v})\| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$
2. Triangle Inequality: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$
3. Pythagorean Theorem: $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal iff

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

Definition of Orthogonal Projection

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in an inner product space V, such that $\boldsymbol{v} \neq 0$. Then the **orthogonal projection** of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is

$$proj_{\boldsymbol{v}}\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}\boldsymbol{v}$$

5.2.2 Examples

Show that the function defines an inner product on $\mathbb{R}^3$, where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$.

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$$

Let $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3 = 2v_1u_1 + 3v_2u_2 + v_3u_3 = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u}, \boldsymbol{v}+\boldsymbol{w} \rangle = 2u_1(v_1+w_1)+3u_2(v_2+w_2)+u_3(v_3+w_3) = 2u_1v_1+2u_1w_1+3u_2v_2+3u_2w_2+u_3v_3+u_3w_3 = 2u_1v_1 + 3u_2v_2 + u_3v_3 + 2u_1w_1 + 3u_2w_2 + u_3w_3 = \langle \boldsymbol{u} + \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
3. $c\langle \boldsymbol{u}, \boldsymbol{v} \rangle = c(2u_1v_1 + 3u_2v_2 + u_3v_3) = 2cu_1v_1 + 3cu_2v_2 + cu_3v_3 = 2(cu_1)v_1 + 3(cu_2)v_2 + (cu_3)v_3 = \langle c\boldsymbol{u}, \boldsymbol{v} \rangle$
4. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 2u_1u_1 + 3u_2u_2 + u_3u_3 \geq 0$ with equality $\iff$ $\boldsymbol{u} = 0$

Show that the function does not define an inner product on $\mathbb{R}^3$, where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_2^2$$

Let's just show that axiom 3 fails. Let $\boldsymbol{u} = (1, 2, 3)$, $\boldsymbol{v} = (1, 0, 0)$, and $c = 4$

$$c\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 4(1^21^2 + 2^20^2 + 3^20^2) = 4(1) = 4$$

$$\langle c\boldsymbol{u}, \boldsymbol{v} \rangle = \langle (4, 8, 12), (1, 0, 0) \rangle = 4^21^2 + 8(0) + 12(0) = 16 \neq 4$$

5.3 Orthogonal Bases: Gram-Schmidt Process

5.3.1 Theory

Defintions of Orthogonal and Orthogonal Sets

A set S of vectors in an inner product space V is **orthogonal** when every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector, then S is **orthonormal**.
For $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ this definition has the form below.

- **Orthogonal:** $\langle \boldsymbol{v_i}, \boldsymbol{v_j} \rangle = 0, i \neq j$
- **Orthonormal:** $\langle \boldsymbol{v_i}, \boldsymbol{v_j} \rangle = 0, i \neq j \quad \|v_i\| = 1, i = 1, 2, \ldots, n$

If S is a basis, then it is an **orthogonal basis** or an **orthonormal basis**, respectively.

Orthogonal Sets are Linearly Independet

If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

Corollary

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Coordinates Relative to an Orthonormal Basis

If $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ is an orthonormal basis for an inner product space V, then the coordinate system of a vector $\boldsymbol{w}$ relative to B is

$$\boldsymbol{w} = \langle \boldsymbol{w_1}, \boldsymbol{v_1} \rangle \boldsymbol{v_1} + \langle \boldsymbol{w_2}, \boldsymbol{v_2} \rangle \boldsymbol{v_2} + \ldots + \langle \boldsymbol{w_n}, \boldsymbol{v_n} \rangle \boldsymbol{v_n}$$

Gram-Schmidt Orthonormalization Process

1. Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ be a basis for an inner product space V
2. Let $B' = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_n}\}$ where

$$\boldsymbol{w_1} = \boldsymbol{v_1}$$

$$\boldsymbol{w_2} = \boldsymbol{v_2} - \frac{\langle \boldsymbol{v_2}, \boldsymbol{w_1} \rangle}{\langle \boldsymbol{w_1}, \boldsymbol{w_1} \rangle}\boldsymbol{w_1}$$

$$\boldsymbol{w_3} = \boldsymbol{v_3} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_1} \rangle}{\langle \boldsymbol{v_1}, \boldsymbol{v_1} \rangle}\boldsymbol{w_1} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_2} \rangle}{\langle \boldsymbol{v_2}, \boldsymbol{v_2} \rangle}\boldsymbol{w_2}$$

$$\vdots$$

$$\boldsymbol{w_n} = \boldsymbol{v_n} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_1} \rangle}{\langle \boldsymbol{v_1}, \boldsymbol{v_1} \rangle}\boldsymbol{w_1} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_2} \rangle}{\langle \boldsymbol{v_2}, \boldsymbol{v_2} \rangle}\boldsymbol{w_2} - \dots - \frac{\langle \boldsymbol{v_n}, \boldsymbol{w_{n-1}} \rangle}{\langle \boldsymbol{w_{n-1}}, \boldsymbol{w_{n-1}} \rangle}\boldsymbol{w_{n-1}}$$

   Then B' is an orthogonal basis for V

3. Let $\boldsymbol{u_i} = \frac{\boldsymbol{w_i}}{\|\boldsymbol{w_i}\|}$. Then $B'' = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}\}$ is an orthonormal basis for V.

Also, $span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}\} = span\{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ for $k = 1, 2, \dots, n$.

5.3.2 Examples

(a) Determine whether set of vectors is orthogonal. (b) If the set is orthogonal, then determine whether it is orthormal. (c) Determine whether the set is a basis for $\mathbb{R}^n$.

$$\{(4, -1, 1), (-1, 0, 4), (-4, -17, -1)\}$$

Starting with part a, we will find the inner product of all $\binom{n}{2}$ possible vectors where n is the number of vectors in the set.

$$\langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle = -4 + 4 = 0$$

$$\langle \boldsymbol{v_1}, \boldsymbol{v_3} \rangle = -16 + 17 - 1 = 0$$

$$\langle \boldsymbol{v_2}, \boldsymbol{v_3} \rangle = 4 + 0 - 4 = 0$$

Since all $\binom{3}{2}$ inner products equal 0, the vectors are orthogonal. Now, for part b, the vectors all must be unit vectors for the set of vectors to be orthonormal. Let's check $\boldsymbol{v_1}$

$$\|\boldsymbol{v_1}\| = \sqrt{\langle \boldsymbol{v_1}, \boldsymbol{v_1} \rangle} = \sqrt{16 + 1 + 1} = 3\sqrt{2} \neq 1$$

Therefore, the set is not orthonormal. Finally, for part c, we must check for linear independence.

$$c_1 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ -17 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -1 & -4 & 0 \\ -1 & 0 & -17 & 0 \\ 1 & 4 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, $c_1 = c_2 = c_3 = 0$ and the set is linearly independent. Since the set is linearly independent and has 3 vectors, the set is a basis for $\mathbb{R}^3$.

(a) Show that the set of vectors in $\mathbb{R}^n$ is orthogonal, and (b) normalize the set to produce an orthonormal set.

$$\{(\sqrt{3}, \sqrt{3}, \sqrt{3}), (-\sqrt{2}, 0, \sqrt{2})\}$$

Starting with part a, as in the previous problem, we need to show that all combinations of 2 vectors in the set have an inner product of zero. There are only two vectors, so this simplifies the process

$$\langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle = -\sqrt{6} + \sqrt{6} = 0$$

Now, for part b, we must ensure that all of the vectors in the set are unit vectors. To do this, we will divide the vectors by their magnitude.

$$\|\boldsymbol{v_1}\| = \sqrt{\langle \boldsymbol{v_1}, \boldsymbol{v_1} \rangle} = \sqrt{3+3+3} = 3$$

$$\frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|} = \frac{(\sqrt{3}, \sqrt{3}, \sqrt{3})}{3} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)$$

$$\|\boldsymbol{v_2}\| = \sqrt{\langle \boldsymbol{v_2}, \boldsymbol{v_2} \rangle} = \sqrt{2+2} = 2$$

$$\frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|} = \frac{(-\sqrt{2}, 0, \sqrt{2})}{2} = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)$$

Therefore, the orthogonal set is

$$\left\{ \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right), \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right) \right\}$$

Find the coordinate matrix of $\boldsymbol{w}$ relative to the orthonormal basis B in $\mathbb{R}^n$

$$\boldsymbol{w} = (2, -2, 1), B = \left\{ \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right), (0, 1, 0), \left(\frac{-3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right) \right\}$$

$$\langle \boldsymbol{w}, \boldsymbol{v_1} \rangle = (2, -2, 1) \cdot \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right)$$

$$= \frac{2\sqrt{10}}{10} + 0 + \frac{3\sqrt{10}}{10} = \frac{5\sqrt{10}}{10} = \boxed{\frac{\sqrt{10}}{2}}$$

$$\langle \boldsymbol{w}, \boldsymbol{v_2} \rangle = (2, -2, 1) \cdot (0, 1, 0) = \boxed{-2}$$

$$\langle \boldsymbol{w}, \boldsymbol{v_3} \rangle = (2, -2, 1) \cdot \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right)$$

$$= -\frac{3\sqrt{10}}{5} + \frac{\sqrt{10}}{10} = \boxed{-\frac{\sqrt{10}}{2}}$$

$$\implies [\boldsymbol{w}]_B = \begin{bmatrix} \frac{\sqrt{10}}{2} \\ -2 \\ -\frac{\sqrt{10}}{2} \end{bmatrix}$$

Apply the Gram-Schmidt orthonormalizatoin process to transform the given basis for $\mathbb{R}^n$ into an orthonormal basis. Use the vectors in the order in which they are given

$$B = \{(4, -3, 0), (1, 2, 0), (0, 0, 4)\}$$

$$\boldsymbol{w_1} = \boldsymbol{v_1} = (4, -3, 0)$$

$$\boldsymbol{w_2} = \boldsymbol{v_2} - \frac{\langle \boldsymbol{v_2}, \boldsymbol{w_1} \rangle}{\langle \boldsymbol{w_1}, \boldsymbol{w_1} \rangle} \boldsymbol{w_1}$$

$$= (1, 2, 0) - \frac{1(4) + 2(-3) + 0(0)}{16 + 9}(4, -3, 0) = (1, 2, 0) + \frac{2}{25}(4, -3, 0) = (1, 2, 0) + \left(\frac{8}{25}, -\frac{6}{25}, 0 \right)$$

$$= \left(\frac{33}{25}, \frac{44}{25}, 0 \right)$$

$$\boldsymbol{w_3} = \boldsymbol{v_3} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_1} \rangle}{\langle \boldsymbol{w_1}, \boldsymbol{w_1} \rangle} \boldsymbol{w_1} - \frac{\langle \boldsymbol{v_3}, \boldsymbol{w_2} \rangle}{\langle \boldsymbol{w_2}, \boldsymbol{w_2} \rangle} = (0, 0, 4) - 0 \left(\frac{33}{25}, \frac{44}{25}, 0 \right) - 0(4, -3, 0)$$

$$= (0, 0, 4)$$

5.4 Mathematical Models and Least Squares Analysis

5.4.1 Theory

Least Squares Problem

Given an $m \times n$ matrix A and a vector $\boldsymbol{b}$ in $\mathbb{R}^m$, the **least squares problem** is to find $\boldsymbol{x}$ in $\mathbb{R}^n$ such that $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ is minimized.

Remark The term **least squares** come from the fact that minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|$ is equivalent to minimizing $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$, which is a sum of squares.

Definition of Orthogonal Subspaces

The subspaces S_1 and S_2 of $\mathbb{R}^n$ are **orthogonal** when $\boldsymbol{v_1} \cdot \boldsymbol{v_2} = 0 \; \forall \; \boldsymbol{v_1} \in S_1, \forall \; \boldsymbol{v_2} \in S_2$

Orthogonal Complement

If S is a subspace of $\mathbb{R}^n$, then the **orthogonal complement** of S is the set $S^\perp = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{u} = 0 \; \forall \; \boldsymbol{v} \in S\}$

Definition of Direct Sum

Let S_1 and S_2 be two subspaces of $\mathbb{R}^n$. If each vector $\boldsymbol{x} \in \mathbb{R}^n$ can be uniquely written as a sum of a vector $\boldsymbol{s_1}$ from S_1 and a vector $\boldsymbol{s_2}$ from S_2, $\boldsymbol{x} = \boldsymbol{s_1} + \boldsymbol{s_2}$, then $\mathbb{R}^n$ is the **direct sum** of S_1 and S_2 and you can write $\mathbb{R}^n = S_1 \oplus S_2$

Properties of Orthogonal Subspaces

Let S be a subspace of $\mathbb{R}^n$. Then the properties listed below are true:

1. $\dim S + \dim S^\perp = n$
2. $\mathbb{R}^n = S \oplus S^\perp$
3. $\left(S^\perp\right)^\perp = S$

Projection onto a Subspace

] If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_t}\}$ is an orthonormal basis for the subspace S of $\mathbb{R}^n$, and $\boldsymbol{v} \in \mathbb{R}^n$, then

$$proj_S \boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{u_1})\boldsymbol{u_1} + (\boldsymbol{v} \cdot \boldsymbol{u_2})\boldsymbol{u_2} + \ldots + (\boldsymbol{v} \cdot \boldsymbol{u_t})\boldsymbol{u_t}$$

Funbdamental Subspaces of a Matrix

Recall that if A is an $m \times n$ matrix, then the column space of A is a subspace of $\mathbb{R}^m$ consisting of all vectors of the form $A\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^n$. The four **fundamental subspaces** of the matrix A are listed below:

- $N(A)$ = nullspace of A
- $N(A^T)$ = nullspace of A^T
- $R(A)$ = column space of A
- $R(A^T)$ = column space of A^T

5.4.2 Examples

Determine whether the subspaces are orthogonal

$$S_1 = span\left\{\begin{bmatrix}3\\2\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \quad S_2 = span\left\{\begin{bmatrix}2\\-3\\0\end{bmatrix}\right\}$$

$$\boldsymbol{v_1} \cdot \boldsymbol{u_1} = (3, 2, -2) \cdot (2, -3, 0) = 6 - 6 = 0$$

$$\boldsymbol{v_2} \cdot \boldsymbol{u_1} = (0, 1, 0) \cdot (2, -3, 0) = 0 - 3 + 0 = -3 \neq 0$$

Therefore, S_1 and S_2 are not orthogonal

Find (a) the orthogonal complement and (b) the direct sum

$$S_1 = span\left\{\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}$$

a.

$$(0, 1, -1, 1) \cdot (x_1, x_2, x_3, x_4) = 0$$

$$x_2 - x_3 + x_4 = 0$$

$$\begin{cases} x_1 = t \\ x_2 = x_3 - x_4 \\ x_3 \\ x_4 \end{cases} \implies \begin{bmatrix}t\\r-w\\r\\w\end{bmatrix} = \begin{bmatrix}t\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\r\\r\\0\end{bmatrix} + \begin{bmatrix}0\\-w\\0\\w\end{bmatrix}$$

Thus

$$S^{\perp} = \left\{ t\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + r\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + w\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} : t, r, w \in \mathbb{R} \right\} = span\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$$

b.

$$S \oplus S^{\perp} = \mathbb{R}^4$$

Find the projection of the vector $\boldsymbol{v}$ onto the subspace S

$$S = span\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \quad \boldsymbol{v} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

First, we will start with the Gram-Schmidt process:

$$\boldsymbol{w_1} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\boldsymbol{w_2} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\1\\\frac{1}{2}\end{bmatrix}$$

$$\implies \left\{\frac{(1,0,1)}{\sqrt{2}}, \frac{(-\frac{1}{2},1,\frac{1}{2})}{\sqrt{\frac{3}{2}}}\right\} = \left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}\right\}$$

This is the orthonormal basis for S. Now, we can get the projection of $\boldsymbol{v}$ onto S

$$proj_s\boldsymbol{v} = \left(\begin{bmatrix}2\\3\\4\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + \left(\begin{bmatrix}2\\3\\4\end{bmatrix}\cdot\begin{bmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}\right)\begin{bmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}$$

$$= \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right)\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + \left(-\frac{2}{\sqrt{6}} + \frac{6}{\sqrt{6}} + \frac{4}{\sqrt{6}}\right)\begin{bmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix} = \frac{6}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + \frac{8}{\sqrt{6}}\begin{bmatrix}-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}$$

$$= \begin{bmatrix}3\\0\\3\end{bmatrix} + \begin{bmatrix}= \frac{8}{6}\\\frac{16}{6}\\\frac{8}{6}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{5}{3}\\\frac{8}{3}\\\frac{13}{3}\end{bmatrix}}$$

Find bases for the four fundamental subspaces of the matrix A

$$\begin{bmatrix}1&0&0\\0&1&1\\1&1&1\\1&2&2\end{bmatrix}$$

Let's start with $N(A)$ (or the nullspace):

$$\begin{bmatrix}1&0&0&0\\0&1&1&0\\1&1&1&0\\1&2&2&0\end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \implies \begin{cases}x_1 = 0\\x_2 + x_3 = 0\end{cases} \implies \begin{cases}x_1 = 0\\x_2 = -t\\x_3 = t\end{cases}$$

Therefore, $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is the basis for the nullspace of A. From the reduced row-echelon form of the matrix above, we can see that the leading ones are in columns one and two. Therefore the basis for $R(A)$ is ,

$$R(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Now, for $N(A^T)$:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \xRightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -x_3 - 2x_4 \end{cases} \implies \begin{cases} x_1 = -t - r \\ x_2 = -t - 2r \\ x_3 = t \\ x_4 = r \end{cases}$$

$$\begin{bmatrix} -t-r \\ -t-2r \\ t \\ r \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -r \\ -2r \\ 0 \\ r \end{bmatrix} \implies N(A^T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

From the work above, we can see that the leading ones are in columns one and two. Therefore, for $R(A^T)$, we have:

$$R(A^T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

5.5 Applications of Inner Product Spaces

5.5.1 Theory

Definition of the Cross Product of Two Vectors

Let $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ be vectors in $\mathbb{R}^3$. The **cross product** of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

$$\boldsymbol{u} \times \boldsymbol{v} = (u_2v_3 - u_3v_2)\boldsymbol{i} - (u_1v_3 - u_3v_1)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}$$

A convenient way to remember the formula for the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is to use the determinant form:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Algebraic Properties of the Cross Product

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\mathbb{R}^3$ and c is a scalar, then the properties listed below are true

1. $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$
2. $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{w})$
3. $c(\boldsymbol{u} \times \boldsymbol{v}) = c\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times c\boldsymbol{v}$
4. $\boldsymbol{u} \times 0 = 0 \times \boldsymbol{u} = 0$
5. $\boldsymbol{u} \times \boldsymbol{u} = 0$
6. $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{w}) \cdot \boldsymbol{w}$

Geometric Properties of the Cross Product

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in $\mathbb{R}^3$, then the properties listed below are true.

1. $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$
2. The angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ is found using $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$
3. $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel iff $\boldsymbol{u} \times \boldsymbol{v} = 0$
4. The parallelogram having $\boldsymbol{u}$ and $\boldsymbol{v}$ as adjacent sides has an area of $\|\boldsymbol{u} \times \boldsymbol{v}\|$

Fourier Approximation

On the interval $(0, 2\pi]$, the least squares approximation of a continuous function f with respect to the vector space spanned by

$$\{1, \cos x, \ldots, \cos nx, \sin x, \ldots, \sin nx\}$$

is

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \ldots + a_n \cos nx + b_1 \sin x + \ldots + b_n \sin nx$$

where the **Fourier coefficients** $a_0, a_1, \ldots a_n, b_1, \ldots, b_n$ are

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx dx \quad j = 1, 2, \ldots n$$

$$b_j = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin jx dx \quad j = 1, 2, \ldots, n$$

5.5.2 Examples

Find $\boldsymbol{u} \times \boldsymbol{v}$, and show that it it orhtogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{u} = (0, 1, -2) \quad \boldsymbol{v} = (1, -1, 0)$$

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k} = (-2, -2, -1)$$

Now, to show that both $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to $\boldsymbol{u} \times \boldsymbol{v}$, we have

$$(0, 1, -2) \cdot (-2, -2, -1) = -2 + 2 = 0$$

$$(1, -1, 0) \cdot (-2, -2, -1) = -2 + 2 = 0$$

Find the area of the parrallelogram that has the vectors as adjacent sides.

$$\boldsymbol{u} = (3, 2, -1) \quad \boldsymbol{v} = (1, 2, 3)$$

The area of the parallelogram will be $\|\boldsymbol{u} \times \boldsymbol{v}\|$. Let's first compute $\boldsymbol{u} \times \boldsymbol{v}$

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 8\boldsymbol{i} - 10\boldsymbol{j} + 4\boldsymbol{k}$$

Now, let's compute $\|\boldsymbol{u} \times \boldsymbol{v}\|$

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{64 + 100 + 16} = \sqrt{180} = 6\sqrt{5}$$

Find the Fourier approximation with the specified order of the function on the interval $[0, 2\pi]$

Third Order:

$$f(x) = \pi - x$$

$$g(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} (\pi - x) dx = -\frac{1}{\pi} \frac{(\pi - x)^2}{2} \bigg|_0^{2\pi} = -\frac{1}{2\pi} (\pi - x)^2 \bigg|_0^{2\pi}$$

$$= -\frac{1}{2\pi} \left((\pi - 2\pi)^2 - (\pi - 0)^2 \right) = -\frac{1}{2\pi} (\pi^2 - \pi^2) = 0$$

$$a_j = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\cos jx dx = \left(\frac{1}{\pi j}(\pi - x)\sin jx - \frac{1}{\pi j^2}\cos jx\right)\Big|_0^{2\pi}$$

$$= \left(\frac{1}{\pi j}(\pi - 2\pi)\cdot 0 - \frac{1}{\pi j^2}(1)\right) - \left(\frac{1}{\pi j}(\pi - 0)(0) - \frac{1}{\pi j^2}(1)\right) = -\frac{1}{\pi j^2} + \frac{1}{\pi j^2} = 0$$

$$b_j = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin jx dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)\sin jx dx = \left(-\frac{1}{\pi j}(\pi - x)\cos jx - \frac{1}{\pi j^2}\sin jx\right)\Big|_0^{2\pi}$$

$$= \left(-\frac{1}{\pi j}(\pi - 2\pi)(1) - 0\right) - \left(-\frac{1}{\pi j}(\pi)(1) - 0\right) = -\frac{1}{\pi j}(-\pi) + \frac{1}{j} = \frac{1}{j} + \frac{1}{j} = \frac{2}{j}$$

$$\therefore g(x) = \frac{0}{2} + 0 + 0 + 0 + 2\sin x + 1\cdot\sin 2x + \frac{2}{3}\sin 3x = 2\sin x + \sin 2x + \frac{2}{3}\sin 3x$$

Chapter 6

Unit 6

6.1 Introduction to Linear Transformations

6.1.1 Theory

Images and Preimages of Functions

In this chapter, we discuss functions that **map** a vector space V into a vector space W. This type of function is denoted by

$$T : V \to W$$

The standard terminology is used for such functions. For instance, V is the **domain** of T, and W is the **codomain** of T. If $\boldsymbol{v}$ is in V and $\boldsymbol{w}$ is in W such that $T(\boldsymbol{v}) = \boldsymbol{w}$, then $\boldsymbol{w}$ is the **image** of $\boldsymbol{v}$ under T. The set of all images of vectors in V is the **range** of T, and the set of all $\boldsymbol{v}$ in V such that $T(\boldsymbol{v}) = \boldsymbol{w}$ is the **preimage** of $\boldsymbol{w}$.

Definition of a Linear Transformation

Let V and W be vector spaces. The function

$$T : V \to W$$

is a **linear transformation** of V into W when the two properties below are true for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in V and for any scalar c.

1. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
2. $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$

Properties of Linear Transformations

Let T be a linear transformation from V into W, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are in V. Then the properties listed below are true.

1. $T(0) = 0$
2. $T(-\boldsymbol{v}) = -T(\boldsymbol{v})$
3. $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$
4. If $\boldsymbol{v} = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ then $T(\boldsymbol{v}) = T(c_1v_1 + c_2v_2 \ldots c_nv_n) = c_1T(v_1) + c_2T(v_2) + \ldots + c_nT(v_n)$

Linear Transformation Given by a Matrix

Let A be an $m \times n$ matrix. The function T is defined by

$$T(\boldsymbol{v}) = A\boldsymbol{v}$$

is a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent vectors in $\mathbb{R}^n$ and $m \times 1$ matrices represent the vectors in $\mathbb{R}^m$.

Some Other Examples of Linear Transformations

$$T : \mathbb{R}^2 \to \mathbb{R}^2, A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This rotates a vector counterclockwise about the origin through the angle θ.

$$T : \mathbb{R}^3 \to \mathbb{R}^3, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

T maps every vector in $\mathbb{R}^3$ to its orthogonal projection on the xy-plane

6.1.2 Examples

Determine whether the function is a linear transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x + 1, y + 1, z + 1)$$

$$T(1, 1, 1) + T(1, 1, 1) =\neq T(2, 2, 2)$$

$$(2, 2, 2) + (2, 2, 2) = (4, 4, 4) \neq T(2, 2, 2) = (3, 3, 3)$$

Therefore, T is not a linear transformation

Determine whether the function is a linear transformation

$$T : M_{2,2} \to R, T(A) = a + b + c + d, where A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T(A_1) + T(A_2) = T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= a_1 + b_1 + c_1 + d_1 + a_2 + b_2 + c_2 + d_2 = (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) + (d_1 + d_2) = T(A_1 + A_2)$$

Therefore, T preserves addition.

$$T(kA) = ka + kb + kc + kd == k(a + b + c + d) = kT(A)$$

Therefore, $T(A)$ preserves scalar multiplication.

Use the function to find (a) the image of $\boldsymbol{v}$ and (b) the preimage of $\boldsymbol{w}$

$$T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2), \boldsymbol{v} = (2, 1, 4), \boldsymbol{w} = (-1, 2)$$

$$a. \qquad T(\boldsymbol{v}) = (2(2) + 1, , 1) = (5, 1)$$

$$b. \qquad T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2) = (-1, 2)$$

$$\begin{cases} 2v_1 + v_2 = -1 \\ v_1 - v_2 = 2 \end{cases} \implies \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_2} \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix} \implies \begin{cases} v_1 = \frac{1}{3} \\ 2\left(\frac{1}{3}\right) + v_2 = -1 \implies v_2 = -\frac{5}{3} \\ v_3 = t \end{cases}$$

Therefore, the preimage of $\boldsymbol{w}$ is

$$\left\{ \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

Use the function to find (a) the image of $\boldsymbol{v}$ and (b) the preimage of $\boldsymbol{w}$

$$T(v_1, v_2) = \left(\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2, v_1 - v_2, v_2\right) \quad \boldsymbol{v} = (2, 4) \quad \boldsymbol{w} = (\sqrt{3}, 2, 0)$$

$$a. \qquad T(2, 4) = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}(4), 2 - 4, 4\right) = (\sqrt{3} - 2, -2, 4)$$

$$b. \qquad \left(\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_1, v_1 - v_2, v_2\right) = (\sqrt{3}, 2, 0)$$

$$\begin{cases} \frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2 = \sqrt{3} \\ v_1 - v_2 = 2 \\ v_2 = 0 \implies v_1 = 2 \end{cases}$$

Therefore, the preimage of $\boldsymbol{w}$ is (2,0).

Define the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(v) = A\boldsymbol{v}$. Find the dimensions of $\mathbb{R}^n$ and $\mathbb{R}^m$.

6.2 The Kernel and Range of a Linear Transformation

6.2.1 Theory

Definition of Kernel of a Linear Transformation

Let $T : V \to W$ be a linear transformation. Then the set of all vectors $\boldsymbol{v}$ in V that satisfy $T(\boldsymbol{v}) = \mathbf{0}$ is the textbfkernel of T and is denoted by $\ker(T)$.

Note: The kernel is a subspace of the domain V.

Corollary

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then the kernel of T is equal to the solution space of $A\boldsymbol{x} = \mathbf{0}$.

The Range of T is a subspace of W

The range of a linear transformation $T : V \to W$ is a subspace of W

Corollary

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then the column space of A is equal to the range of T.

Definition of Rank and Nullity of a Linear Transformation

Let $T : V \to W$ be a linear transformation. The dimension of the kernel of T is called the **nullity** of T and is denoted by $nullity(T)$ The dimension of the range of T is called the **rank** of T and is denoted by $rank(T)$.

Remark If T is given by a matrix A, then the rank of T is equal to the rank of A, and the nullity of T is equal to the nullity of A.

Sum of Rank and Nullity

Let $T : V \to W$ be a linear transformation from an $n-$dimensional vector space V into a vector space W. Then the sum of the dimensions of the range and kernel is equal to the dimension of the domain. That is,

$$rank(T) + nullity(T) = n$$

or

$$\dim(range) + \dim(kernel) = \dim(domain)$$

Definition of One-to-One and Onto

A function $T : V \to W$ is **one-to-one** when the preimage of every $\boldsymbol{w}$ in the range consists of a single vector. A function $T : V \to W$ is **onto** when every element in W has a preimage in V.

One-to-One Transformations

Let $T : V \to W$ be a linear transformation. Then T is one-to-one iff $\ker(T) = \{0\}$.

Onto Linear Transformations

Let $T : V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto iff the rank of T is equal to the dimension of W.

One-to-One and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation with vector spaces V and W, both of dimension n. Then T is one-to-one iff it is onto

Definition of Isomorphism

A linear transformation $T : V \to W$ that is one-to-one and onto is called an **isomorphism**. Moreover, if V and W are vector-spaces such that there exists an isomorphism from V to W, then V and W are **isomorphic** to each other.

Isomorphic Spaces and DImensions

Two finite-dimensional vector spaces V and W are isomorphic iff they are of the same dimension.

Isomorphic Vector Spaces

The vector spaces below are isomorphic to each other.

1. $\mathbb{R}^4 =$ 4-space
2. $M_{41} =$ space of all 4×1 matrices
3. $M_{22} =$ space of all 2×2 matrices
4. $P_3 =$ space of all polynomials of degree 3 or less
5. $V = \{(x_1, x_2, x_3, x_4, 0) : x_i \in \mathbb{R}\}$ (subspace of $\mathbb{R}^5$)

6.2.2 Examples

Find the kernel of the linear transformation

$$T : P_2 \to P_1, T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$$

$$T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x = \mathbf{0}$$

$$\begin{cases} a_1 = 0 \\ 2a_2 = 0 \end{cases} \implies a_1 = a_2 = 0$$

This implies that a_0 can be any $\mathbb{R}$. Therefore,

$$\ker(T) = \{a_0 : a_0 \in \mathbb{R}\}$$

Find the kernel of the linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (x + 2y, y - x)$$

$$T(x, y) = (x + 2y, y - x) = (0, 0)$$

$$\begin{cases} x + 2y = 0 \\ -x + y = 0 \end{cases} \implies \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$$3y = 0 \implies y = 0 \implies x = 0$$

Therefore,

$$\ker(T) = \{(0, 0)\}$$

Define the linear transformation T by $T(x) = Ax$. Find (a) the kernel of T and (b) the range of T.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

a.

$$T(v) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} v_1 + 2v_2 = 0 \\ 3v_1 + 4v_2 = 0 \end{cases} \implies \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \implies v_1 = v_2 = 0 \implies \ker(T) = \{(0,0)\}$$

b.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, a basis for the range is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and the $range(T) = span \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Define the linear transformation T by $T(x) = Ax$. Find (a) the kernel of T and (b) the range of T.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix}$$

a.

$$T(v) = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 3 & 1 & 2 & -1 & 0 \\ -4 & -3 & -1 & -3 & 0 \\ -1 & -2 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} v_1 = -v_3 \\ v_2 = v_3 \\ v_4 = 0 \end{cases} \implies \left\{ \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} \right\}$$

Therefore,

$$\ker(T) = \left\{ \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} \right\}$$

and a basis for $\ker(T)$ is

$$\ker(T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b.

$$A^T = \begin{bmatrix} 1 & 3 & -4 & -1 \\ 2 & 1 & -3 & -2 \\ -1 & 2 & -1 & 1 \\ 4 & -1 & -3 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$Range(T) = span\{(1,0,-1,0),(0,1,-1,0),(0,0,0,1)\}$$

and the basis for the range of T is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Define the linear transformation T by $T(x) = Ax$. Find (a) the $\ker(T)$, (b) $nullity(T)$, (c) range of T, and (d) $rank(T)$.

$$\begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

a.

$$T(x) = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -3 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies x_1 = x_2 = 0 \implies \ker(T) = \{(0,0)\}$$

b.

$$nullity(T) = \dim(\ker(T)) = 0$$

c.

$$A^T = \begin{bmatrix} 5 & 1 & 1 \\ -3 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \end{bmatrix}$$

A basis for the range of T is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{4} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right\}$$

$$range(T) = \left\{ t \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \right\} = \{(4t, 4r, t-r)\}$$

d.

$$Rank(T) = 2$$

6.3 Matrices for Linear Transformations

6.3.1 Theory

Standard Matrix for a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that, for the standard basis vectors $\boldsymbol{e_i}$ of $\mathbb{R}^n$,

$$T(\boldsymbol{e_1}) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(\boldsymbol{e_2}) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad T(\boldsymbol{e_n}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Composition of Linear Transformations

Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The **composition** $T : \mathbb{R}^n \to \mathbb{R}^p$, defined by $T(\boldsymbol{v}) = T_2(T_1(\boldsymbol{v}))$, is a linear transformationl. Moreover, the standard matrix A for T is the matrix product

$$A = A_2 A_1$$