Quiz 7 - Math 257

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1 Problem 1

(a). To find a basis for the kernel of T, we must find all vectors $\boldsymbol{v}$ such that $A\boldsymbol{v} = 0$. Let's compute this

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{cases} x + y = 0 \\ -x + 2y = 0 \\ y = 0 \end{cases} \implies x + 0 = 0 \implies x = 0$$

Therefore, the kernel has only the trivial solution:

$$\boxed{\ker(T) = \{0\}}$$

(b). To find the range of the linear transformation represented by A, let's first test for linear independence of the column matrices by setting up the equation as follows:

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_2 + 2c_2 = 0 \\ c_2 = 0 \end{cases}$$

The only solution to this is the trivial solution where $c_1 = c_2 = 0$, therefore, the column matrices are linearly independent. Since the column matrices are linearly independent, a basis for the range of T is

$$\boxed{range(T) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}}$$

2 Problem 2

To do this problem, we will have to compute $P^{-1}AP$, since

$$A' = P^{-1}AP$$

First, let's compute A:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = (1, 1+2(0), 1+0+0) = (1,1,1) \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = (0, 0+2(1), 0+1+2(0)) = (0,2,1)$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = (0, 0+2(0), 0+03(1)) = (0,0,3)$$

Therefore,

$$A = \begin{bmatrix}1 & 0 & 0\\1 & 2 & 0\\1 & 1 & 3\end{bmatrix}$$

Now, let's find the transition matrix from B' to B. We will call this P:

$$[BB'] \xrightarrow{RREF} [I_3P] \implies \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & -1 & 0 & 1\\0 & 0 & 1 & 0 & 1 & -1\end{bmatrix} \implies P = \begin{bmatrix}1 & 0 & 0\\-1 & 0 & 1\\0 & 1 & -1\end{bmatrix}$$

Now, let's find P^{-1}:

$$\begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\-1 & 0 & 1 & 0 & 1 & 0\\0 & 1 & -1 & 0 & 0 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 1 & 1 & 1\\0 & 0 & 1 & 1 & 1 & 0\end{bmatrix}$$

Therefore,

$$P^{-1} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & 0\end{bmatrix}$$

Now, all that's left is the computation for $P^{-1}AP$:

$$P^{-1}AP = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\1 & 2 & 0\\1 & 1 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-1 & 0 & 1\\0 & 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-1 & 0 & 2\\0 & 3 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{bmatrix}$$

Therefore,

$$\boxed{A' = \begin{bmatrix}1 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{bmatrix}}$$