Quiz 3 - Math 257

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1 Problem 1

If I want to find the inverse of matrix B, I must first set up an additional matrix A, to do so. A will be a 3×6 matrix with matrix B on the left and the identity matrix on the right:

$$A = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix}$$

I will then do the following elementary row operations to get an identity matrix on the left side rather than the right:

$$2R_1 - R_2 \implies R_2$$

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 + R_3 \implies R_3$$

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$R_1 + R_2 \implies R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$2R_2 - R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 3 & -2 & -1 \end{bmatrix}$$

$$R_2 + R_3 \implies R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & -1 & 3 & -2 & -1 \end{bmatrix}$$

$$R_1 - R_3 \implies R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & -1 & 3 & -2 & -1 \end{bmatrix}$$

$$-R_3 \implies R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

Therefore, with my identity matrix on the left

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

2 Problem 2

To write matrix A as a product of elementary matrices, I must use elementary row operations to get matrix A in to the identity matrix:

$$A = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$4R_1 - 3R_2 \implies R_2$$

$$\begin{bmatrix} -3 & 2 \\ 0 & 5 \end{bmatrix}$$

$$5R_1 - 2R_2 \implies R_1$$

$$\begin{bmatrix} -15 & 0 \\ 0 & 5 \end{bmatrix}$$

$$-\frac{1}{15}R_1 \implies R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\frac{1}{5}R_2 \implies R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, I can represent each elementary row operation as an elementary matrix:

$$4R_1 - 3R_2 \implies R_2$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$$

$$5R_1 - 2R_2 \implies R_1$$

$$E_2 = \begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix}$$

$$-\frac{1}{15}R_1 \implies R_1$$

$$E_3 = \begin{bmatrix} -\frac{1}{15} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{5}R_2 \implies R_2$$

$$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

Now that I have that, I must think about the order in which I originally got the identity matrix. Multiplying the identity matrix by E_1 first wouldn't make much sense since the last step I did before getting the identity matrix was E_4. With this in mind, I can set up the following equation:

$$E_4E_3E_2E_1A = I$$

This makes sense considering I first multiplied A by E_1 and then by E_2, etc. Using the identity that $AA^{-1} = I$, I can start multiplying both sides by the inverses of $E_1 \dots E_4$:

$$\left(E_4E_4^{-1}\right) E_3E_2E_1A = E_4^{-1}$$

$$\left(E_3E_3^{-1}\right) E_2E_1A = E_3^{-1}E_4^{-1}$$

$$\left(E_2E_2^{-1}\right) E_1A = E_2^{-1}E_3^{-1}E_4^{-1}$$

$$\left(E_1E_1^{-1}\right) A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

To find the inverse of each of the respective matrices, I will use the trick for a 2×2 matrix:

$$E_4^{-1} = 5\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$E_3^{-1} = -15\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{15} \end{bmatrix} = \begin{bmatrix} -15 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -15 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$