Frequency and Phase Response

A Two-Zero Filter

Filter 'equation'

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2]$$

z-transform:

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z)$$

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = a_0 + a_1 z^{-1} + a_2 z^{-2}$$

To calculate frequency and phase response, evaluate:

$$H(z)\Big|_{z=e^{j\hat{\omega}}}$$

If $z^{-1} = -\frac{a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$ yields a pair of complex conjugate roots

$$\begin{aligned} a_0 + a_1 z^{-1} + a_2 z^{-2} &= (1 - Re^{j\theta} z^{-1})(1 - Re^{-j\theta} z^{-1}) = 0 \\ &= 1 - R(e^{j\theta} + e^{-j\theta}) z^{-1} + R^2 z^{-2} \\ &= 1 - 2R\cos\theta z^{-1} + R^2 z^{-2} \end{aligned}$$

then $a_2 = R^2 a_0 \rightarrow R = \sqrt{a_2/a_0}$

and $a_1 = -2a_0 R \cos\theta \rightarrow \theta = \cos^{-1} \frac{a_1}{-2a_0 R}$

Frequency and Phase Response

A Two-Zero Filter

To calculate frequency and phase response, evaluate:

$$H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$\begin{aligned}
H(e^{j\hat{\omega}}) &= a_0 + a_1 e^{-j\hat{\omega}} + a_2 e^{-2j\hat{\omega}} \\
&= a_0 + a_1(\cos\hat{\omega} - j\sin\hat{\omega}) + a_2(\cos 2\hat{\omega} - j\sin 2\hat{\omega}) \\
&= (a_0 + a_1\cos\hat{\omega} + a_2\cos 2\hat{\omega}) + j(-a_1\sin\hat{\omega} - a_2\sin 2\hat{\omega}) \\
&= x + iy
\end{aligned}$$

where

$$x = a_0 + a_1\cos\hat{\omega} + a_2\cos 2\hat{\omega}$$

and

$$y = -a_1\sin\hat{\omega} - a_2\sin 2\hat{\omega}$$

Then frequency response:

$$|H(e^{j\hat{\omega}})| = \sqrt{x^2 + y^2}$$

and phase response:

$$\angle H(e^{j\hat{\omega}}) = \tan^{-1}\frac{y}{x} \qquad = \tan^{-1}\frac{-(a_1\sin\hat{\omega} + a_2\sin 2\hat{\omega})}{a_0 + a_1\cos\hat{\omega} + a_2\cos 2\hat{\omega}}$$

M-Zero Filter

Filter 'equation'

$$y[n] = a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-M]$$

z-transform:

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + \cdots + a_M z^{-M} X(z)$$

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}$$

To calculate frequency and phase response, evaluate:

$$H(z)\Big|_{z=e^{j\hat{\omega}}}$$

We can generalize the 2-Zero case to get

$$\begin{aligned} H(e^{j\hat{\omega}}) &= a_0 + a_1 e^{-j\hat{\omega}} + \cdots + a_M e^{-jN\hat{\omega}} \\ &= a_0 + a_1(\cos\hat{\omega} - j\sin\hat{\omega}) + \cdots + a_M(\cos M\hat{\omega} - j\sin M\hat{\omega}) \\ &= (a_0 + a_1\cos\hat{\omega} + \cdots + a_M\cos M\hat{\omega}) + j(-a_1\sin\hat{\omega} - \cdots) \\ &= x + jy, \quad x = \sum_{k=0}^{M} a_k \cos k\hat{\omega} \text{ and } y = -\sum_{k=1}^{M} a_k \sin k\hat{\omega} \end{aligned}$$

Gain and Phase:

$$|H(e^{j\hat{\omega}})| = \sqrt{x^2 + y^2}; \Theta = \tan^{-1}\frac{y}{x}$$

M-Zero, N-Pole Filter

This is the most general case of filter we can discuss.

Filter 'equation'

$$y[n] = a_0 x[n] + \cdots + a_M x[n-M] - b_1 y[n-1] - \cdots - b_1 y[n-N]$$

Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + \cdots + b_N z^{-N}}$$

Proceeding as we did in the N-Zero case

$$\begin{aligned} H(e^{j\hat{\omega}}) &= \frac{a_0 + a_1 e^{-j\hat{\omega}} + \cdots + a_M e^{-jM\hat{\omega}}}{1 + b_1 e^{-j\hat{\omega}} + \cdots + b_N e^{-jN\hat{\omega}}} \\ &= \frac{x_z + j y_z}{x_p + j y_p} \end{aligned}$$

$$\text{where } x_z = \sum_{k=0}^{M} a_k \cos k\hat{\omega}, \; y_z = -\sum_{k=1}^{M} a_k \sin k\hat{\omega}$$

$$x_p = \sum_{i=0}^{N} a_i \cos k\hat{\omega} \text{ and } y_p = -\sum_{i=1}^{N} a_i \sin k\hat{\omega}$$

M-Zero, N-Pole Filter (contd.)

Since the magnitude of a ratio of two complex numbers is the ratio of the magnitudes and the argument of the ratio is the difference of the arguments

$$|H(e^{j\hat{\omega}})| = \frac{\sqrt{x_z^2 + y_z^2}}{\sqrt{x_p^2 + y_p^2}}$$

$$\Theta = \tan^{-1}\frac{y_z}{x_z} - \tan^{-1}\frac{y_p}{x_p}$$

So, if a given filter equation cannot be simplified nicely along the lines of our very first, 1-zero example, by knowing the coefficients a_i and b_i we can calculate the gain and phase response at an angle $\hat{\omega}$.

The algorithm in the perl code `filt-resp.pl` simply considers k different angles in the region $0 \ldots \pi$ and evaluates $|H(e^{j\hat{\omega}})|$ and Θ above at each value.

Sound Filters

- To avoid distortion, audio filters should normally have linear phase characteristics
- Normally start with a specification of frequency behaviour of a filter and the problem becomes one of determining the system equation a_i and b_i that'll deliver the required performance within a specified tolerance limit
- We saw earlier that an arbitrary system can be "factored" into several simpler ones that are cascaded together; this occurs quite frequently in practise