Question 9

Solve the following questions from the Discrete Math zyBook:

a: Exercise 4.1.3, sections b, c

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$?

b: $f(x) = \frac{1}{x^2-4}$

Not a function, because x = 2 or x = -2, then $\frac{1}{x^2-4} = \frac{1}{2^2-4} = \frac{1}{0}$

c. $f(x) = \sqrt{x^2}$

This is a function. The square root of x^2 will be |x|

The range will be $\mathbb{R}^+$.

b. Exercise 4.1.5, sections b, d, h, i, l

b.

$A = \{2, 3, 4, 5\}$
$f\colon A \to \mathbb{Z}$, such that $f(x) = x^2$
{4, 9, 16,25}

d. $f: \{0, 1\}^5 \rightarrow \mathbb{Z}$

$f(x)$ is the number of 1s that occur in x

{0,1,2,3,4,5}

h. $A = \{1, 2, 3\}$

$f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$

$f(x, y) = (y, x)$

{(1,1),(1,2),(,1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)}

i. $A = \{1, 2, 3\}$

$f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$

$f(x, y) = (x, y + 1)$

{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)}

l. $A = \{1, 2, 3\}$

$f: P(A) \rightarrow P(A)$

For $X \subseteq A,\ f(X) = x - \{1\}$

{∅,{2},{3},{2,3}}

Question 10

I. Solve the following questions from the Discrete Math zyBook:

Exercise 4.2.2, sections c, g, k

c. h: $\mathbb{Z} \to \mathbb{Z}$. h(x)=$x^3$

One-to-one.

Each element in the domain corresponds to a unique element in the codomain.

Not onto:

There is no way for f(x) to equal 2, so the function is not onto.

g. f: $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. f(x,y) = (x+1,2y)

One-to-one.

Each element in the domain corresponds to a unique element in the codomain.

Not onto:

Since y is an integer, 2y is by definition an even integer.

Therefore there is no value of 2y that equals an odd integer.

k. f: $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$. f(x,y) = $2^x + y$

Not one-to-one.

f(3,1) = f(2,5) = 9.

Not onto:

There is no positive integer pair such that f(x,y) = 1.

Exercise 4.2.4, sections b, c, d, g

b. $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$

The output of f is obtained by taking the input string and replacing the first bit with 1.

Not onto:

It is impossible to get an output that starts with 0, such as 010.

Not one-to-one:

f(011) = f(111) = 111

c. . $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$

The output of f is obtained by taking the output strings and reversing the bits.

Both one-to-one and onto

d. $f: \{0, 1\}^3 \rightarrow \{0, 1\}^4$

The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end.

Not onto:

It would be impossible to get a string where the first and last bit are not the same, such as 0001 or 1000.

One-to-one:

Each input has a unique output.

g. $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$B = \{1\}$

$f: P(A) \rightarrow P(A)$

For $X \subseteq P(A)$, $f(X) = A - B$

Not one-to-one:

$f(\{1, 2, 3\}) = f(\{2, 3\} = \{2, 3\}$

Not onto:

There is no way for f(X) to equal {1}

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto.

$$f(x) = \begin{cases} x+6 & \text{If } x>0 \\ -4(x)+5 & \text{if } x \leq 0 \end{cases}$$

b. Onto, but not one-to-one

f(x)=$x^2 + 1$

c. One-to-one and onto

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 2x+1 & \text{If } x \leq 1 \end{cases}$$

d. Neither one-to-one or onto

$f(x) = |x| \star 2$

Question 11

Solve the following questions from the Discrete Math zyBook:

Exercise 4.3.2, sections c, d, g, i

c. $f: \mathbb{R} \to \mathbb{R}$. f(x)=2x+3

$f^{-1}(x) = \frac{x-3}{2}$

d. $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$f: P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

For $X \subseteq A, f(X) = |X|$

This function does not have a well defined inverse.

g. $f: \{0, 1\}^3 \to \{0, 1\}^3$

Output of f is determined by obtaining the input string and reversing the bits.

The output of f^{-1} would be obtained the same way.

i. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. $f(x. y) = (x + 5, y - 2)$

$f^{-1}(x, y) = (x - 5, y + 2)$

Exercise 4.4.8, sections c, d

$f(x) = 2x + 3$

$g(x) = 5x + 7$

$h(x) = x^2 + 1$

c. $f \circ h = f(h(x)) = 2(x^2 + 1) + 3 = 2x^2 + 5$

d. $h \circ f = h(f(x)) = (2x + 3)^2 + 1 = 4x^2 + 12x + 10$

Exercise 4.4.2, sections b-d

$f(x) = x^2$

$g(x) = 2^x$

$h(x) = \lceil \frac{x}{5} \rceil$

b. $f \circ h(52) = f(h(52))$

$h(52) = \lceil \frac{52}{5} \rceil = 11$

$f(11) = 11^2 = 121$

c. $g \circ h \circ f(4) = g(h(f(4)))$

$f(4) = 4^2 = 16$

$h(16) = \lceil \frac{16}{5} \rceil = 4$

$g(4) = 2^4 = 16$

d. $h \circ f = h(f(x)) = \lceil \frac{x^2}{5} \rceil$

Exercise 4.4.6, sections c-e

$f\{0, 1\}^3 \rightarrow \{0, 1\}^3$

Output is taken by taking the first bit and replacing it with 1

$g\{0, 1\}^3 \rightarrow \{0, 1\}^3$

Output is taking the input string and reversing the bits

$h\{0, 1\}^3 \rightarrow \{0, 1\}^3$

Replaces the last bit with a copy of the first bit

c. $h \circ f(010) = h(f(010))$

$$f(010) = 110$$

$$h(110) = 111$$

d. Range of $h \circ f$: $\{111,101\}$

e. Range of $g \circ f$: $\{100, 101, 110, 111\}$

Extra Credit: Exercise 4.4.4, sections c, d

c.

If f is not one-to-one, then $g \circ f$ would not be one to one.

If f is not one-to-one, then there are 2 elements, x_1 and x_2, such that $x_1 \neq x_2$, and $f(x_1) = f(x_2)$

Because of that, $g \circ f(x_1) = g \circ f(x_2)$

d. It is possible for g to not be one-to-one and $g \circ f$ to be one-to-one.

For example:

$X = \{1, 2, 3\}$

$Y = \{a, b, c, d\}$

$Z = \{1, 2, 3\}$

f: $X \rightarrow Y$.

$f(1) = b,\ f\ (2) = c,\ f(3) = a$

g: $Y \rightarrow Z$

$g(a) = 1, g(b) = 2, g(c) = 3,\ g(d) = 3$

$g \circ f(1) = 2,\ g \circ f(2) = 3,\ g \circ f(3) = 1$