Question 5

a. Use mathematical induction to prove that for any positive integer n,

   3 divides $n^3 + 2n$ (leaving no remainder).

   Let $P(n)$ be the proposition that 3 divides $n^3 + 2n$ evenly when n is a positive integer.

   Proof: by Induction on n

   I. Base Case: $n = 1$

   Plugging $n = 1$ into the equation:

   $1^3 + 2(1) = 1 + 2 = 3$

   3 divides 3 evenly

   $P(n)$ is true when $n = 1$

   II. Induction Step:

   Assume that $P(k)$ is true, that is $k^3 + 2k$ is evenly divisible by 3, where k is some positive integer. We will prove that $P(k + 1)$ is also true, that is

   $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

   $(k + 1)^3 + 2(k + 1)$

   = $k^3 + 3k^2 + 3k + 1 + 2k + 2$

   =\` $(k^3 + 2k) + 3k^2 + 1 + 2$

   = $(k^3 + 2k) + 3(k^2 + k + 1)$

   Because k is an integer, $k^2 + k + 1$ is also an integer.

   By the inductive hypothesis, the first part of the equation, $k^3 + 2k$ is evenly divisible by 3.

   The second part of the equation, $3(k^2 + k + 1)$, is divisible by 3, because it is 3 times an integer.

   ∴ If $P(k)$ is true, $P(k + 1)$ is also true. ∎

b. Use strong induction to prove that any positive integer n ($n \geq 2$) can be written as a product of primes.

Let $P(n)$ be the proposition that, for any positive integer n ($n \geq 2$), can be written as a product of primes.

Proof: by strong induction on n

I. Base Case: $n = 2$

P(2) is true because 2 is a prime number and it can be written as a product of one prime.

II. Induction Step:

Assume $k \geq 2$, and any integer j ,where $2 \leq j \leq k$, can be written as a product of primes. We will prove $P(k + 1)$ is also true, that is $k + 1$ can also be written as a product of primes.

Consider 2 cases:

Case 1: $k + 1$ is prime. Then $P(k + 1)$ is true, because $k + 1$ can be written as a product of one prime.

Case 2: $k + 1$ is a composite number. If $k + 1$ is composite, then there are 2 numbers, x and y, such that $2 \leq x \leq y < k + 1$, and $xy = k + 1$.

By the inductive hypothesis, x and y can be written as the product of primes.

Since x and y can be written as the product of primes, and $xy = k + 1$, $k + 1$ can be written as the product of primes. ∎

Question 6

Solve the following questions from the Discrete Math zyBook:

a. Exercise 7.4.1, sections a-g

Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

a. $P(3)$

Left hand side of the equation:

$$1^2 + 2^2 + 3^2 = 14$$

Right hand side of the equation:

$$\frac{3(3+1)(2(3)+1)}{6} = \frac{3*4*7}{6} = \frac{84}{6} = 14$$

b. $P(k)$

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

c. $P(k + 1)$

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

d. In an inductive proof, for $P(n)$, $P(1)$ must be proven in the base case.

e. In an inductive proof, for the induction step, $P(k + 1)$ must be proven.

f. In my previous answers, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, would be the inductive hypothesis.

g. Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $P(n)$ be the proposition that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

I. Base Case: P(1)

Left hand side of the equation:

$1^2 = 1$

Right hand side of the equation:

$\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$

1 = 1

$P(n)$ is true when $n = 1$.

II. Induction Step:

Assume $P(k)$ is true, that is $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.

We will prove that $P(k + 1)$ is also true, that is

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2$ By separating out the last term

$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$ By the inductive hypothesis

$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$

$= \frac{(k+1)(2k^2+k+6k+6)}{6}$

$$=\frac{(k+1)(2k^2+7k+6)}{6}$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ If $P(k)$ is true, $P(k+1)$ is also true. ▮

b. Exercise 7.4.3, section c

Hint: you may want to use the following fact: $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$

c. Prove that for $n \geq 1$, $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Let $P(n)$ be the proposition that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$, for $n \geq 1$.

I. Base Case: P(1)

Left hand side of the equation: $\frac{1}{1^2} = \frac{1}{1} = 1$

Right hand side of the equation: $2 - \frac{1}{1} = 2 - 1 = 1$

$1 \leq 1$

$P(n)$ is true when $n = 1$.

II. Induction Step:

Assume $P(k)$ is true, that is for $k \geq 1$, $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$.

We will prove that $P(k+1)$ is also true, that is

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$ By separating out the last term

$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ By the inductive hypothesis.

$\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$, therefore

$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$

$= 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)}$

$= 2 - \frac{1}{k+1}$

$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$

$\therefore$ If $P(k)$ is true, $P(k + 1)$ is also true. ∎

c. Exercise 7.5.1, section a

a. Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Let $P(n)$ be the proposition that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

I. Base Case: P(1)

Plugging 1 into the equation:

$3^{2(1)} - 1 = 3^2 - 1 = 8$

$8 \div 4 = 2$

$P(n)$ is true when $n = 1$.

II. Induction Step:

Assume $P(k)$ is true, that is for some positive integer k, $3^{2k} - 1$ is evenly divided by 4.

We will prove $P(k + 1)$ is also true, that is $3^{2(k+1)} - 1$ is also evenly divided by 4.

$3^{2(k+1)} - 1$

$=3^{2k+2} - 1$

$=3^2 * 3^{2k} - 1$

$=9(3^{2k} - 1)$

$=(8 * 3^{2k}) + (3^{2k} - 1)$

By the inductive hypothesis, $(3^{2k} - 1)$ is divisible by 4. Because 8 is divisible by 4, $(8 * 3^{2k})$ is also divisible by 4.

∴ If $P(k)$ is true, $P(k + 1)$ is also true. ▮