Luke Gearty
Net ID: lg3878
Homework 2

Question 5

a) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.12.2, sections b, e:

1.12.2.b

$p \rightarrow (q \land r)$
$\neg q$
―――――
$\therefore \neg p$

1. $\neg q$	hypothesis
2. $\neg q \lor \neg r$	Addition, 1
3. $\neg(q \land r)$	DeMorgan's Law, 2
4. $p \rightarrow (q \land r)$	hypothesis
5. $\neg p$	Modus Tollens, 3 and 4

1.12.2.e

$p \lor q$
$\neg p \lor r$
$\neg q$
―――――
$\therefore r$

1. $p \lor q$	hypothesis
2. $\neg p \lor r$	hypothesis
3. $q \lor r$	Resolution, 1 and 2
4. $\neg q$	hypothesis
5. r	Disjunctive Syllogism, 3 and 4

2. Exercise 1.12.3, section c

1.12.3.c

Prove that Disjunctive syllogism is valid using the laws of propositional logic and any of the other rules of inference besides disjunctive syllogism

$p \lor q$
$\neg p$
___
$\therefore q$

1. $\neg p$	hypothesis
2. $p \lor q$	hypothesis
3. $\neg p \rightarrow q$	Conditional identities, 2
4. q	Modus Ponens, 1 and 3

3. Exercise 1.12.5, sections c, d

c: I will buy a new car
h: I will buy a new house
j: I will get a job

1.12.5.c

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
___
I will not buy a new car.

Argument form:

$(c \land h) \rightarrow j$
$\neg j$
___
$\therefore \neg c$

This argument is invalid if $c \equiv T,\ h \equiv F\ and\ j \equiv F$

If c is false, h is false and j is false, then all hypotheses are True, but the conclusion, $\neg c$, will evaluate to false.

1.12.5.d

I will buy a new car and a new house only if I get a job.
I am not going to get a job.

I will buy a new house.

I will not buy a new car.

Form of the argument:

$(c \land h) \to j$

$\neg j$

h

$\therefore \neg c$

This is a valid argument.

1. $\neg j$	hypothesis
2. $(c \land h) \to j$	hypothesis
3. $\neg(c \land h)$	Modus Tollens, 1 and 2
4. $\neg c \lor \neg h$	DeMorgan's Law, 3
5. $\neg h \lor \neg c$	Commutative Law, 4
6. h	hypothesis
7. $\neg c$	Disjunctive Syllogism, 5 and 6

b) Solve the following questions from the Discrete Math zyBook:

1. Exercise 1.13.3, section b

   1.13.3.b

   Show that the given argument is invalid by giving values for the predicates P and Q over the domain {a, b}

   $\exists x(P(x) \lor Q(x))$

   $\exists x \neg Q(x)$

   $\therefore \exists x P(x)$

	P	Q
a	F	T
b	F	F

The first hypothesis is true, because $P(a) \lor Q(a)$ is true, and the second hypothesis is true, because $\neg Q(b)$ is true. But there is no value of P where it is true, so the conclusion is false.

2. Exercise 1.13.5, sections d, e

   Domain: Students in the class

   A(x): x got an A.

   M(x): x missed class.

   D(x): x got a detention

   1.13.5.d

   Every student who missed class got a detention.

   Penelope is a student in the class.

   Penelope did not miss class.

   Penelope did not get a detention.

   Form of the argument:

   $\forall x(M(x) \to D(x))$

   Penelope is a student in the class.

   $\neg M(Penelope)$

   $\therefore \neg D(Penelope)$

   This argument is invalid if M(Penelope)$\equiv F$, D(Penelope)$\equiv T$. This makes all the hypotheses true, but the conclusion is false.

   1.13.5.e

3. Every student who missed class or got a detention did not get an A.

   Penelope is a student in the class.

   Penelope got an A.

   Penelope did not get a detention.

   Form of the argument:

   $\forall x((M(x) \lor D(x) \to \neg A(x))$

   Penelope is a student in the class.

   $A(Penelope)$

   $\therefore \neg D(Penelope)$

   This is a valid argument.

1. Penelope is a student in the class	hypothesis
2. $\forall x((M(x) \lor D(x) \rightarrow \neg A(x))$	hypothesis
3. $A(Penelope)$	hypothesis
4. $(M(Penelope) \lor D(Penelope)) \rightarrow \neg A(Penelope)$	Universal Instantiation, 1 and 2
5. $\neg(M(Penelope) \lor D(Penelope)$	Modus Tollens, 3 and 4
6. $\neg M(Penelope) \land \neg D(Penelope)$	DeMorgan's Law, 5
7. $\neg D(Penelope)$	Simplification, 6

Question 6

Solve Exercise 2.4.1, section d; Exercise 2.4.3, section b, from the Discrete Math zyBook:

2.4.1.D:

Theorem: The product of two odd integers is an odd integer.

Proof:

Let x and y be odd integers. We will prove that $x \cdot y$ is an odd integer.

By definition of an odd integer, $x = 2m + 1$, for some integer m. Also by definition of an odd integer, $y = 2n + 1$, for some integer n.

Plugging in $2m + 1$ and $2n + 1$ into $x \cdot y$: $x \cdot y = (2m + 1)(2n + 1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

Since x and y are integers, 2mn + m+ n are also integers.

Since $x \cdot y = 2r + 1$, where r = 2mn + m + n, $x \cdot y$ is an odd integer. ∎

2.4.3.B:

Theorem: If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Proof:

Let x be a real number and $x \leq 3$. We will show that $12 - 7x + x^2 \geq 0$.

If we subtract x from both sides of the inequality $x \leq 3$, we get $0 \leq 3 - x$.

$4 - x$ is one more than $3 - x$. Since $3 - x$ is greater than or equal to 0, $4 - x \geq 0$.

Since both equations are at least equal to 0, their product should be at least equal to 0, if not greater.

$(4 - x)(3 - x) \geq 0$.

$= 12 - 4x - 3x + x^2 \geq 0$

$= 12 - 7x + x^2 \geq 0$ ∎

Question 7

Solve Exercise 2.5.1, section d; Exercise 2.5.4, sections a, b; Exercise 2.5.5, section c, from the Discrete Math zyBook:

2.5.1.d:

For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof:

By contraposition, for every integer n, if n is even, then $n^2 - 2n + 7$ is odd.

Let n be an even integer. By definition $n = 2k$, where k is some integer.

Plugging into $n^2 - 2n + 7$:

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

Since k is an integer, $2k^2 - 2k + 3$ is also an integer.

Since $n^2 - 2n + 7 = 2r + 1$, where $r = 2k^2 - 2k + 3$, $n^2 - 2n + 7$ is odd. ∎

2.5.4.a:

For every pair of real numbers x and y, if $x^3 + xy^2 \le x^2y + y^3$, then $x \le y$

Proof:

By contraposition, assume x and y are real numbers and $x > y$. We shall prove $x^3 + xy^2 > x^2y + y^3$.

Since $x > y$, one of the numbers, x or y, is not 0.

If we subtract $x^2y + y^3$, from both sides of the inequality, we get: $x^3 + xy^2 > x^2y + y^3$

$$= x^3 + xy^2 - x^2y + y^3 > 0$$

If we factor out x in $x^3 + xy^2$, we get $x(x^2 + y^2)$. Similarly, if we factor out y in $x^2y + y^3$, we get $y(x^2 + y^2)$.

$$\begin{aligned} &x^3 + xy^2 - x^2y + y^3 > 0 \\ &= x(x^2 + y^2) - y(x^2 + y^2) > 0 \\ &= (x - y)(x^2 + y^2) > 0 \end{aligned}$$

Since either x or y is not 0, then $x^2 + y^2 > 0$. Therefore, $x^3 + xy^2 - x^2y + y^3 > 0$, and thus $x^3 + xy^2 > x^2y + y^3$ ∎

2.5.4.b:
For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Proof:
By contraposition,assume x and y are real numbers and $x \leq 10$ and $y \leq 10$. We shall prove that $x + y \leq 20$.

Adding the inequalities $x \leq 10$ and $y \leq 10$ together gives us $x + y \leq 20$. ∎

2.5.5.c
For every non-zero real number, if x is irrational, then 1/x is also irrational.

Proof:
By contraposition, assume 1/x is rational. We shall prove x is rational.

A number, r, is rational if there are integers a and b, such that $b \neq 0$ and $r = a/b$.

Since 1/x is rational, $1/x = a/b$. Flipping both sides, we get $x/1 = b/a$, or $x = b/a$.

Since b and a are both integers, and $a = 1$ and $a \neq 0$, x is rational. ∎

Question 8

Solve Exercise 2.6.6, sections c, d, from the Discrete Math zyBook:

2.6.6.c:

The average of three real numbers is greater than or equal to at least one of the numbers.

Proof:

Suppose there are 3 real numbers, x, y, and z, and the average, a, is less than all three of the numbers.

And since a is the average of x, y, and z, $a = (x + y + z)/3$.

Since a is less than all three, $a < x, a < y, a < z$.

Adding the inequalities, we get $x + y + z > 3a$.

Plugging in to $x + y + z > 3a$: $x + y + z > 3a$

$=x + y + z > 3((x + y + z)/3)$

$=x + y + z > x + y + z$

$x + y + z$ cannot be less than $x + y + z$, thus the assumption of the average of 3 real numbers being less than all three is false. ∎

2.6.6.d:

There is no smallest integer.

Proof:

Assume there is a smallest integer, x.

If x is an integer, then $x - 1$ is also an integer.

Since x is the smallest integer, $x < x - 1$.

This contradicts the assumption that x is the smallest integer, because x cannot be less than $x - 1$. ∎

Question 9

Solve Exercise 2.7.2, section b, from the Discrete Math zyBook:

2.7.2.b:

If integers x and y have the same parity, then $x + y$ is even.

Proof:

Case 1: x and y are even.

Since x is an even integer, then $x = 2m$, where m is some integer. If y is an even integer, then $y = 2k$, where k is some integer.

Plugging in to $x + y$:
$$x + y = 2m + 2k$$
$$= 2(m + k)$$

Since m and k are integers, $m + k$ is an integer.

Since $x + y = 2r$, where $r = m + k$, $x + y$ is even.

Case 2: x and y are odd.

Since x is an odd integer, then $x = 2n + 1$, where n is some integer. If y is an odd integer, then $y = 2a + 1$, where a is some integer.

Plugging in to $x + y$:
$$x + y = (2n + 1) + (2a + 1)$$
$$= 2n + 2a + 2$$
$$= 2(n + a + 1)$$

Since n and a are integers, $n + a + 1$ is also an integer.

Since $x + y = 2b$, where $b = n + a + 1$, $x + y$ is even. ∎