Question 7

a. Solve Exercise 8.2.2, section b from the Discrete Math zyBook.

$f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$

First, we need to prove that $f = O(n^3)$

Proof:

Select c = 8 and $n_0 = 1$. We will show that for any $n \geq 1$, $f(n) \leq 8 * n^3$.

For any $n \geq 1,\ n^3 \geq n^2\ and\ n^2 \geq n$.

Therefore, $n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$.

Thus, $f \leq 8n^3$, which means $f = O(n^3)$.

Then, we need to prove that $f = \Omega(n^3)$.

Proof:

Select c = 1 and $n_0 = 1$. We will show that for any $n \geq 1$, $1 * n^3 \leq f(n)$.

For any $n \geq 1,\ n^3 \geq n^2\ and\ n^2 \geq n$.

Therefore, $n^3 \leq n^3 + n^2$, which also means that $n^3 \leq n^3 + 3n^2 + 4$.

Thus, $n^3 \leq f$, which means $f = \Omega(n^3)$.

Since we have proven that $f = O(n^3)$ and $f = \Omega(n^3)$, $f = \Theta(n^3)$. ▮

b. Use the definition of Θ to show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

$f(n) = \sqrt{7n^2 + 2n - 8}.$

First we need to prove that $f = O(n)$.

Proof:

Select c = 3 and $n_0 = 1$. We will show that for any n ≥ 1, f(n) $\leq 3n$.

$\sqrt{7n^2 + 2n - 8} \leq \sqrt{7n^2 + 2n}$. Since $n \geq 1$ $n^2 \geq n$.

Therefore, $\sqrt{7n^2 + 2n} \leq \sqrt{7n^2 + 2n^2}$

$= \sqrt{9n^2}$

$= 3n.$

Then we need to prove that $f = \Omega(n)$

Proof:

Select c = 1 and $n_0 = 1$. We will show that for any $n \geq 1$, c * $n \leq f(n)$.

Since $n \geq 1$, $\sqrt{7n^2 + 2n - 8n} \geq \sqrt{n^2} = n$.

Therefore, 1 * n $\leq \sqrt{7n^2 + 2n - 8n}$.

$f = \Omega(n)$.

Since we have shown that $f = O(n)$ and $f = \Omega(n)$, $f = \Theta(n)$. ▮

c. Solve Exercise 8.3.5, sections a-e from the Discrete Math zyBook

a.

There are a few inputs for this algorithm. The first is a, a sequence of numbers, and the second input is n, the length of a, and the third is p, a number.

There is one pointer, i, that is at the start of a, equal to 1 Another pointer, j, is at the end of a, equal to the length of a.

There is a while loop that runs while i is less than j. While the first loop is running, there are two internal while loops. The first loop runs while i is less than j and if the value of a at position i is less than p. If those conditions are true, the value of i is incremented by 1.

The second loop checks if i is less than j and if the value of a at position j is greater than or equal to p. If those conditions are true, j is decremented by 1.

At the end of each iteration, if i is less than j, a at position of i is swapped with a at position of j.

The output of the algorithm will be the sequence of numbers sorted so that all the values that are less than p are at the beginning of the list, and the values greater than p are after.

b.

The total number of times "i := i + 1" or "j := j -1" will execute is equivalent to the length of the sequence n - 1.

c.

The number of times the swap operation is completed depends on the actual values of the numbers in the sequence. Specifically the amount of numbers less than p.

Maximizing the number of times the swap operation occurs would require all of the numbers greater than *p* being in the left half of the sequence and all the numbers less than *p* being in the right half of the sequence. This would run at most $\frac{n}{2}$ times.

Minimizing the number of times the swap operation occurs would require there being no numbers less than p in the sequence, which would mean the operation occurs 0 times.

d.

This function is Ω(n), because the two while loops incrementing i and decrementing j will run a total of n -1 times. The swap operation can only occur when i is less than j, and thus it will occur less times than the two while loops.

e.

This function is O(n), because in the worst case scenario it will run n - 1 times.

Question 8:

Solve the following questions from the Discrete Math zyBook:

A. Exercise 5.1.2 section b, c

b. $40^7 + 40^8 + 40^9$

c. $14 * (40^6 + 40^7 + 40^8)$

B. 5.3.2 section a

a. $3 * 2^9 = 1536$

C. 5.3.3 section b, c

b. $10 * 26^4 * 9 * 8$

c. $10 * 26 * 25 * 24 * 23 * 9 * 8$

D. Exercise 5.2.3, sections a, b

a.

$f\colon B^9 \to E_{10}$. The output of f is obtained by counting the numbers of 1s in the input

string. If there is an even number of 1s, then a 0 is appended. If there is an odd number of ones, then a 1 is appended.

For example, $f(000110000) = 0001100000$.

This is a one-to-one function, because each input will have a unique output.

If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

This is an onto function, because all elements in the domain will be accounted for.

Because it is both one-to-one and onto, this is a bijection.

b. Since there is a bijection, $|E_{10}| = |B^9| = 2^9$

Question 9:

Solve the following questions from the Discrete Math zyBook:

A. Exercise 5.4.2, sections a, b

a. $2 * 10^4 = 20000$

b. $2 * 10 * 9 * 8 * 7 = 10080$

B. Exercise 5.5.3, sections a-g

a. $2^{10} = 1024$

b. $1 * 1 * 1 * 2^7 = 128$

c. $1 * 1 * 1 * 2^7 = 128$

$1 * 1 * 2^8 = 256$

$2^7 + 2^8 = 384$

d. $2 * 2 * 2^6 * 1 * 1 = 256$

e. $C(10, 6)= \frac{10!}{(10-6)!6!} = \frac{10!}{4!6!} = \frac{10 * 9 * 8 * 7}{4!} = 210$

f. $C(9,6) = \frac{9!}{(9-6)!6!} = \frac{9!}{3!6!} = \frac{9 * 8 * 7}{3!} = 84$

g. First half: C(5,1) = 5

Second half: C(5,3)= 10

C(5,1) * C(5,3) = 50

C. Exercise 5.5.5, section a

a. $C(30, 10) = \frac{30!}{(30-10)!10!}$

$C(35,10) = \frac{35!}{(35-10)!10!}$

C(30,10) * C(35,10)

D. Exercise 5.5.8, sections c-f

c. C(26, 5) = 65780

d. 13 possible ranks for the first four cards, and then 52-4 possible ranks for the last one, equals to 13*48 = 624.

e. 13 possible ranks for the three cards, 12 possible ranks for the pair.
4 cards to choose the three cards, 4 cards to choose the two cards.

13 * C(4,3) * 12 * C(4, 2) = 3744.

f. C(13,5) to choose five cards of different ranks.
4^5 ways to choose the suit.

$C(13, 5) \ * \ 4^5 \ = \ 1317888$

E. Exercise 5.6.6, sections a, b

a. $C(44,5)$ to choose 5 democrats.

$C(56,5)$ to choose 5 republicans.

$C(44,5) * C(56,5)$

b. $P(44,2)$ to choose the speaker and the vice speaker from the democrats.

$P(44,2) = 1892$

$P(56, 2)$ to choose the speaker and vice speaker from the republicans.

$P(56,2) = 3080)$

$P(44,2) * P(56,2)$

Question 10:

Solve the following questions from the Discrete Math zyBook:

A. Exercise 5.7.2, sections a, b

a. C(52,5) - C(39, 5)

b. C(52,5) - $C(13, 5) \ * \ 4^5$

B. Exercise 5.8.4, sections a, b

a. 5^{20}

b. C(20,4) * C(16,4) * C(12,4) * C(8,4) * C(4,4) = $\frac{20!}{4!4!4!4!4!}$

Question 11:

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a. 4

   0 one-to-one functions

b. 5

   $$\frac{5!}{(5-5)!} = \frac{5*4*3*2*1}{1} = 120$$

c. 5

   $$\frac{6!}{(6-5)!} = \frac{6*5*4*3*2*1}{1!} = 720$$

d. 7

   $$\frac{7!}{(7-5)!} = \frac{7*6*5*4*3*2*1}{2!} = 2520$$