## Dynamic Programming Algorithms

# Typographic Alignment

CS 336: Design and Analysis of Algorithms
Konstantin Makarychev

*Eye of the Hurricane: An Autobiography*

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It also has a very interesting property as an adjective, and that it's impossible to use the
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# Samples

**Plain**

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**Latex**

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**Word**

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# Typography

| Good | Bad |
|---|---|
| Good fonts. | Bad fonts. |
| Proportional or variable width fonts . | Monospaced or fixed-width fonts. |
| … | … |
| Good text alignment or justification. | |

# Full Justification

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# How to Break Lines in Paragraph

**Given:** Paragraph of text.

**Goal:** Break lines in the optimal way.

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**Goal:** Break lines in the optimal way.

a. Each line should fit on the page.

b. All lines should have about the same width.

## A Simple Model of Text Justification

**Assumption:** Fixed-width font. Width of the page is $m$ letters.

**Given:** Paragraph of text. The width of word $i$ is $w_i$. #words: $n$.

**Goal:** Partition paragraph in lines:

The width of line $j$ is

$$\text{width}(p_j, p_{j+1}) = \sum_{i=p_j}^{p_{j+1}-1} w_i + \underbrace{(p_{j+1} - p_j - 1)}_{\text{white spaces}}$$

| |
|---|
| $w_1, w_2, \dots, w_{p_1-1}$ |
| $w_{p_1}, w_{p_1+1}, \dots, w_{p_2-1}$ |
| $w_{p_2}, w_{p_2+1}, \dots, w_{p_3-1}$ |
| … |
| $w_{p_k}, w_{p_k+1}, \dots, w_n$ |

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| |
|---|
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| $w_{p_1}, w_{p_1+1}, \dots, w_{p_2-1}$ |
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| … |
| $w_{p_k}, w_{p_k+1}, \dots, w_n$ |

$$\text{badness}(p_j, p_j + 1) = \begin{cases} \infty, & \text{if width}(p_j, p_{j+1}) > m \\ \left(m - \text{width}(p_j, p_{j+1})\right)^3, & \text{otherwise} \end{cases}$$

**Given:** Paragraph of text. The width of word $i$ is $w_i$. #words: $n$.

**Goal:** Partition paragraph in lines, so as to

$$\text{minimize} \sum_j \text{badness}(p_j, p_j + 1)$$

| |
|---|
| $w_1, w_2, \dots, w_{p_1-1}$ |
| $w_{p_1}, w_{p_1+1}, \dots, w_{p_2-1}$ |
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## Why cube $\left(m - \mathrm{width}(p_j, p_{j+1})\right)^3$?

$$\mathrm{badness}(p_j, p_j + 1) = \begin{cases} \infty, & \text{if } \mathrm{width}(p_j, p_{j+1} > m) \\ \left(m - \mathrm{width}(p_j, p_{j+1})\right)^3, & \text{otherwise} \end{cases}$$

## How to design a DP Algorithm?

1. Need to find one or several subproblems.
2. Express the cost of each subproblem in terms of smaller subproblems.

## Subproblem

- **Suffix starting at word #$\boldsymbol{i}$**: $w_i, \dots, w_n$.
- **Subproblem:** Break suffix $w_i, \dots, w_n$ into lines.
- **Recurrence relation?**

| $w_i, w_{i+1}, \dots, w_{p_1 - 1}$ |
|---|
| $w_{p_1}, w_1, \dots,$<br>$\dots,$<br>$\dots, w_n$ |

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- Need to find the best place to break the first line.

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$$\text{cost}(i) = \min_{p_1 > i}\big(\text{badness}\,(i, p_1) + \text{cost}(p_1)\big)$$

## Recursive Algorithm

**function** TypographicAlignment ($i$)

$$\text{result} = \min_{p_1 > i}\left(\text{badness}\ (i, p_1) + \text{TypographicAlignment}(p_1)\right)$$

return result;

Running time?

## Top-Down DP Algorithm

**function** TypographicAlignment ($i$)

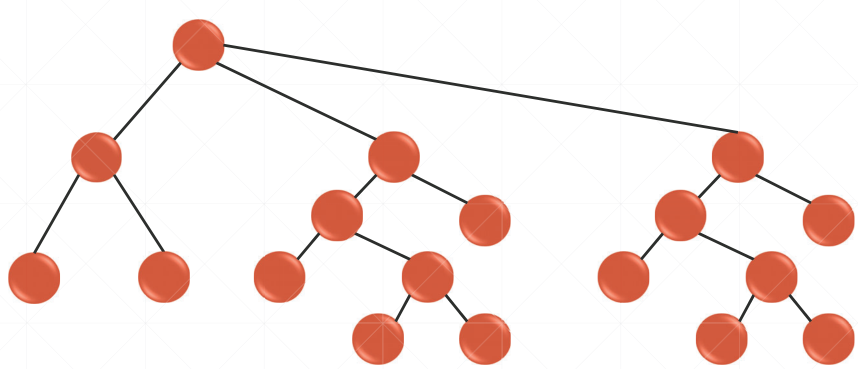
Lookup the result for $i$ in the table/cache.

$$\text{result} = \min_{p_1 > i}\left(\text{badness}\ (i, p_1) + \text{TypographicAlignment}(p_1)\right)$$

Store result in the table/cache.
return result;

Running time?

## Call Graph



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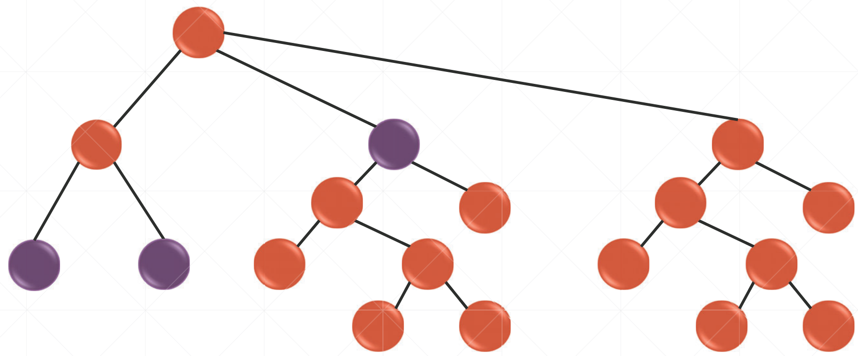

## Top-Down DP Algorithm

**function** TypographicAlignment ($i$)

Lookup the result for $i$ in the table/cache. $//O(1)$ instructions

▶ $\text{result} = \min\limits_{p_1 > i}\left(\text{badness}\ (i, p_1) + \text{TypographicAlignment}(p_1)\right)$

Store result in the table/cache. $//O(1)$ instructions

return result; $//O(1)$ instructions

Running time? $O(1)$ if $i$ is in the table/cache.

## Top-Down DP Algorithm

**function** TypographicAlignment ($i$)

Lookup the result for $i$ in the table/cache. $//O(1)$ instructions

▶ $\text{result} = \min\limits_{p_1 > i}\left(\text{badness}\ (i, p_1) + \text{TypographicAlignment}(p_1)\right)$

Store result in the table/cache. $//O(1)$ instructions

return result; $//O(1)$ instructions

Running time? $O(1)$ for all instructions, but "▶"

## Computing min over a range

$$\min_{p_1 > i}\left(\underbrace{\text{badness}\,(i, p_1) + \text{TypographicAlignment}(p_1)}_{a_{p_1}}\right)$$

for $p_1$ from $i+1$ to $n$

▶ Compute $a_{p_1}$.

min_value = min (min_value, $a_{p_1}$) $//O(1)$ instructions

Running time? $O(1)$ for all instructions excluding "▶"

"▶" = cost of computing badness$(i, p_1)$ and recursive calls.

## Total Running Time

- For each $i$,
  - TypographicAlignment($i$) is called once with $i$ not in the table;
- Running time of each call is

$$O(1) + O(n) + n \times run.time\ (badness)$$

## Total Running Time

- The total running time equals

$$n \times O\big(O(1) + O(n) + n \times run.time\ (badness)\big)$$

If we recompute badness each time, we get a bound of $O(n^3)$.

## Total Running Time

- The total running time equals

$$n \times O\big(O(1) + O(n) + n \times run.time\ (badness)\big)$$

If we recompute badness each time, we get a bound of $O(n^3)$.

We can speed up this algorithm, by precomputing badness.

## Bottom-up Approach

- We fill in table with TypographicAlignment($i$) values from $i = n$ to $i = 0$.

for $i$ from $n$ to $0$

  table[i] $= \min_{p_1 > i}$(badness $(i, p_1)$+table[$p_1$])

return table[0]

## Bottom-up Approach

- We fill in table with TypographicAlignment($i$) values from $i = n$ to $i = 0$.

```
for i from n to 0
  min_value = ∞ ;
  for p_1 from i + 1 to n
     new_option = badness (i, p_1)+table[p_1]
     min_value = min (min_value, new_option)

  table[i] = min_value;

return table[0]
```

## Bottom-up Approach

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```
for i from n to 0
  min_value = ∞ ; //O(1) instructions
  for p_1 from i + 1 to n
    new_option = badness (i, p_1)+table[p_1] //O(1) instructions
    min_value = min (min_value, new_option) //O(1) instr.

  table[i] = min_value; //O(1) instr.

return table[0] //O(1) instr
```

## Backtracking

- We now know the cost or badness of the optimal alignment.
- …but how do we get the actual solutions.
  a. In every table cell, store not only the optimal value but the optimal solution.
  b. In every table cell, store not only the optimal value but the pointer to the optimal $p_1$
  c. In general, store the pointer to the table cells used to compute the optimal solution for the subproblem.

# Questions about text alignmnet?