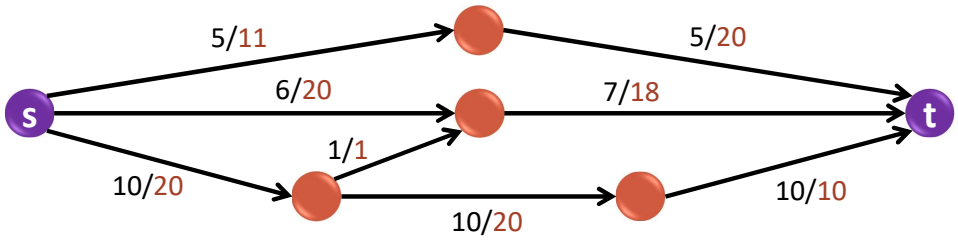
Network Flow

EECS 336: Design and Analysis of Algorithms.
Konstantin Makarychev

Network

- Network:
  - Directed graph $G = (V, E)$.
  - Two designated nodes (terminals): s-source and t-sink.
  - Every edge e has a capacity $c(e)$.

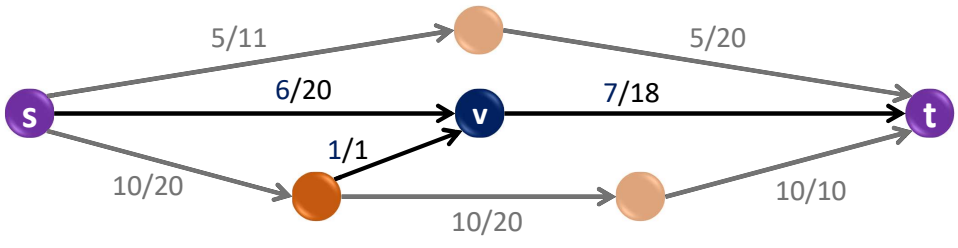


Feasible Flow

Flow $f: E \to \mathbb{R}^+$. We route $f(e)$ units of flow over each edge e.

a. Flow conservation constraints: for all vertices v but s and t,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$



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a. Flow conservation constraints: for all vertices v but s and t,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

b. Capacity constraints: for all edges e:

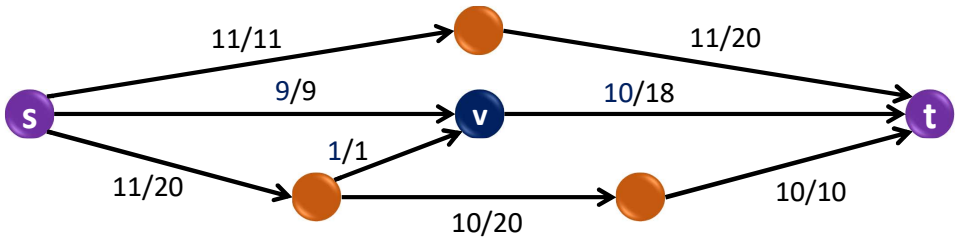
$$f(e) \leq c(e)$$

The value of the flow f equals

$$\sum_{s \to v \in E} f(s \to v) - \sum_{s \to v \in E} f(s \to v)$$

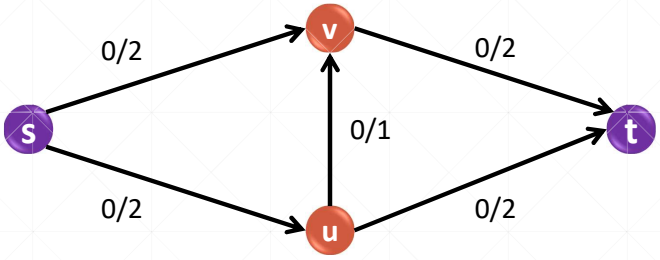
Maximum Flow Problem

Given a network $G = (V, E)$, find the maximum feasible flow from s to t.



Ford–Fulkerson algorithm

Naïve Greedy Algorithm Fails



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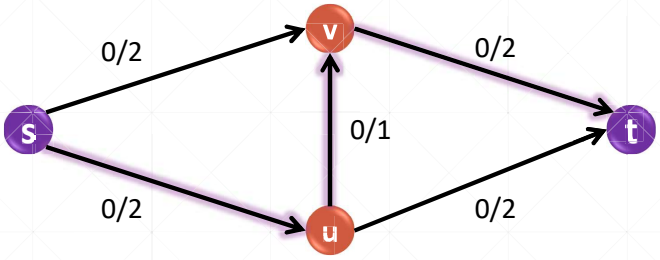

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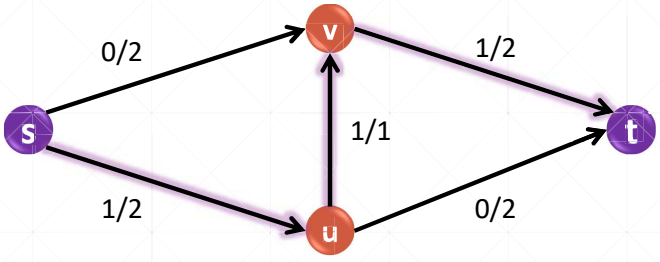

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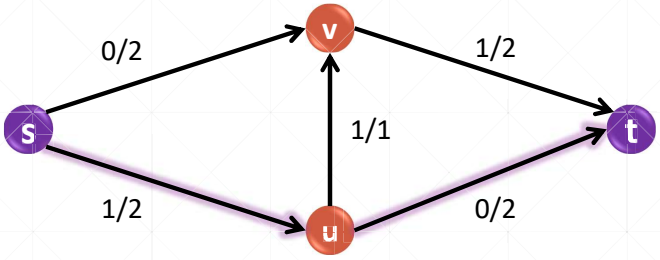

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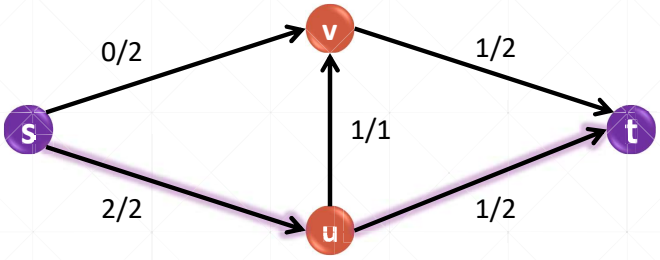

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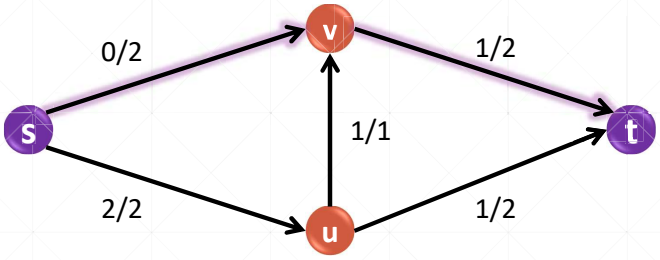

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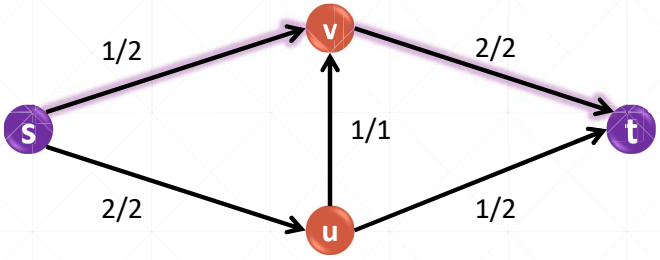

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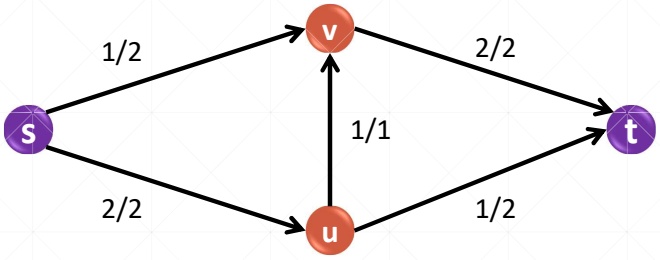
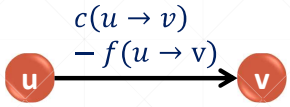
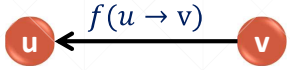


Residual network G_f

- Residual network is a graph on the same set of vertices as the original network G.
- For every original edge $u \to v$, G_f has edge $u \to v$ if $f(u \to v) < c(u \to v)$.
  - $c_f(u \to v) = c(u \to v) - f(u \to v)$



- For every original edge $u \to v$, G_f has edge $v \to u$ if $f(u \to v) > 0$.
  - $c_f(v \to u) = f(u \to v)$



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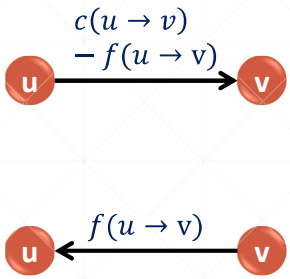
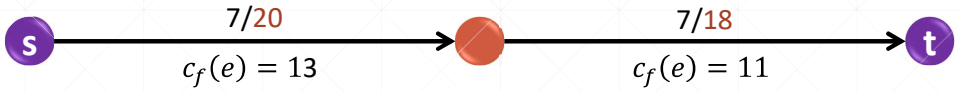


Bottleneck Edge

- Bottleneck edge in P is the edge with the min residual capacity.
- After augmenting the flow along P, the bottleneck edge disappears.



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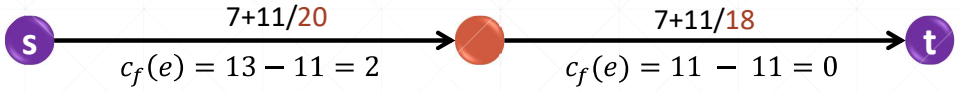

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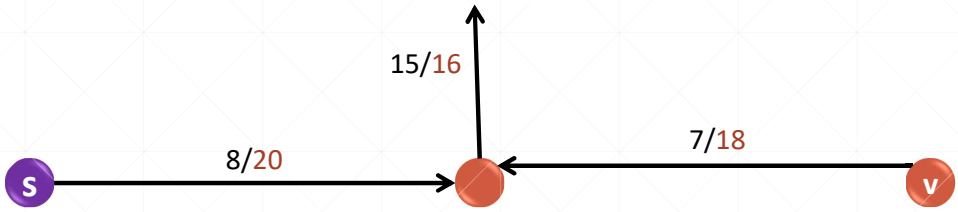

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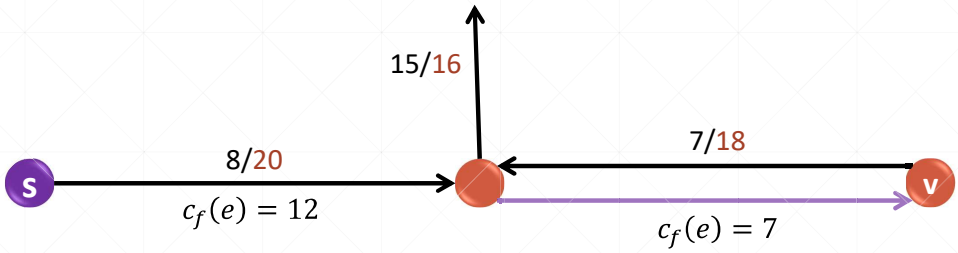

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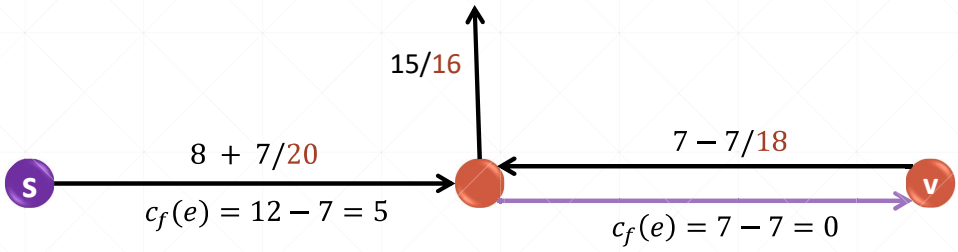

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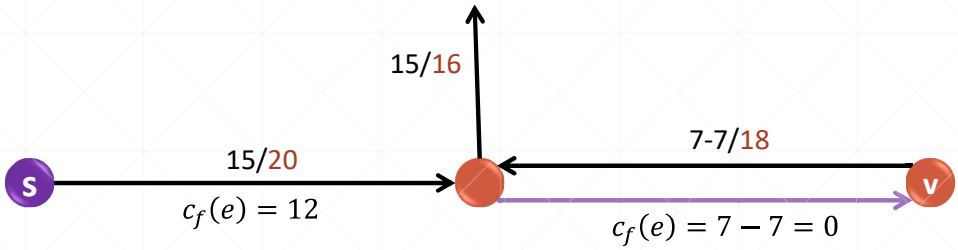


Ford–Fulkerson algorithm

Input: network $G = (V, E)$ with capacities c, source node s, sink node t

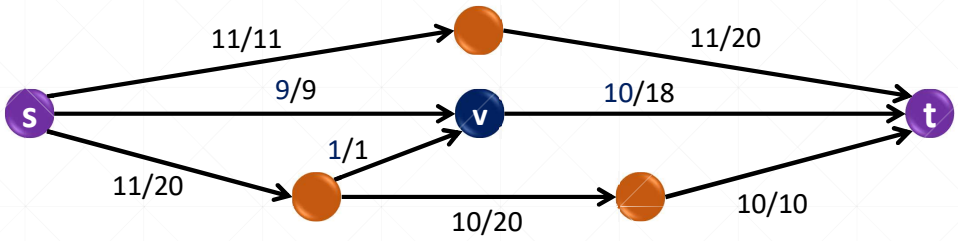
- Initialize flow: Let $f(u, v) = 0$ for all $(u, v) \in E$
- **Repeat:**
  - Compute residual flow G_f.
  - Find* a path P from s to t in G_f.
  - Augment the flow along P by the bottleneck capacity.
  - If there is no path from s to t in G_f return f.

Ford–Fulkerson algorithm

- **Claim:** If all capacities are integral, then Ford-Fulkerson algorithm eventually terminates and outputs a flow in G.
- Is this flow maximum possible?

Maximum Flow Problem

Given a network $G = (V, E)$, find the maximum feasible flow from s to t.



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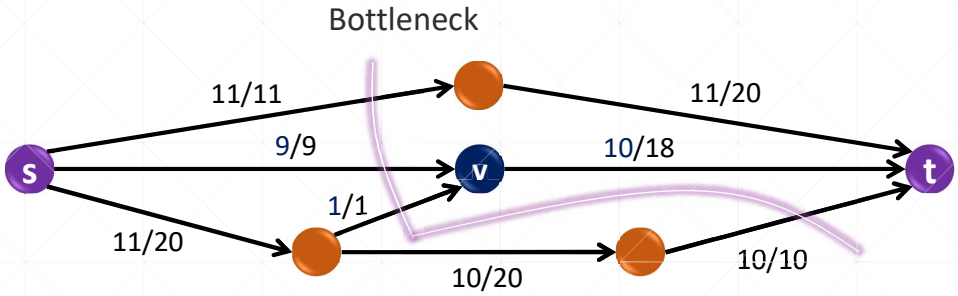
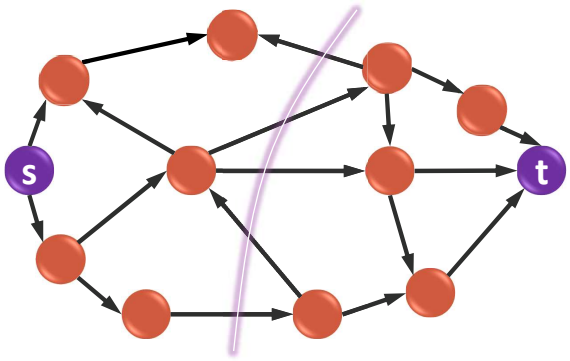


st-Cuts

- Cut (A, B) is partitioning of the set of all vertices into two sets A and B s.t. $s \in A$ and $t \in B$.
- Capacity of the cut:

$$cut(A, B) = \sum_{e \in A \to B} c(e)$$



Min Cut = Max Flow = Ford-Fulkerson

- We prove that

$$\textbf{Min Cut} \geq \textbf{Max Flow} \geq \textbf{Ford} - \textbf{Fulkerson} \geq \textbf{Min Cut}$$

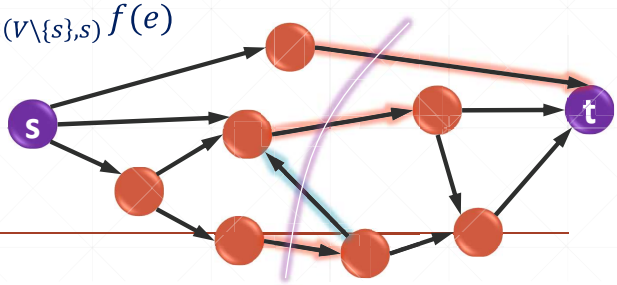
Flow-value Lemma

Network $G = (V, E)$. Feasible flow F from s to t. st-Cut $(S, \bar{S})$.

$$\text{val}(f) = \sum_{e \in (S,\bar{S})} f(e) - \sum_{e \in (\bar{S},S)} f(e)$$

where

- $\text{val}(f) = \sum_{e \in (s,V\setminus\{s\})} f(e) - \sum_{e \in (V\setminus\{s\},s)} f(e)$
- $f(e) = 0$, if $u \notin E$



Flow-value Lemma

Network $G = (V, E)$. Feasible flow F from s to t. st-Cut $(S, \bar{S})$.

$$\text{val}(f) = \sum_{e \in (S,\bar{S})} f(e) - \sum_{e \in (\bar{S},S)} f(e) = (*)$$

Proof:

$$val(f) = \sum_{u \in S} \left[\sum_{v:(u,v) \in E} f(e) - \sum_{v:(v,u) \in E} f(e) \right] = (*)$$

Details are on the board.

Min Cut = Max Flow = Ford-Fulkerson

We prove that

a. **Min Cut ≥ Max Flow:** By the flow-value lemma, the value of the flow equals the flow across the min cut, which is upper bounded by **Min Cut**.

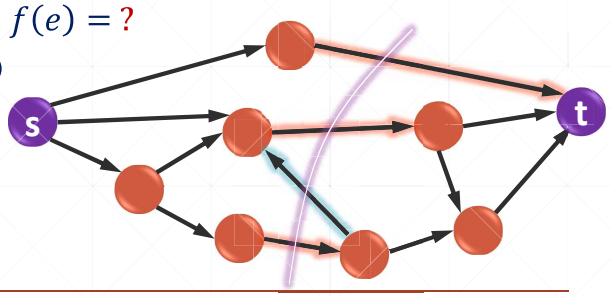
b. **Max Flow ≥ Ford − Fulkerson:** Ford-Fulkerson gives a feasible solution to the Max Flow Problem.

c. **Ford − Fulkerson ≥ Min Cut** (next slide)

Ford–Fulkerson ≥ Min Cut

Let f be the flow returned by Ford-Fulkerson and S be the set of nodes reachable from s in G_f. By the flow-value lemma:

$$\text{val}(f) = \sum_{e \in (S,\bar{S})} f(e) - \sum_{e \in (\bar{S},S)} f(e) = ?$$

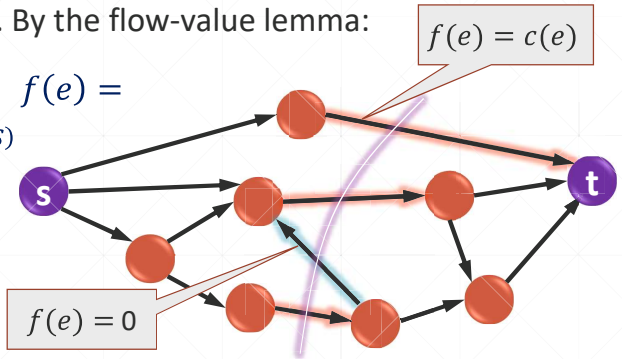


Ford–Fulkerson ≥ Min Cut

Let f be the flow returned by Ford-Fulkerson and S be the set of nodes reachable from s in G_f. By the flow-value lemma:

$$\text{val}(f) = \sum_{e \in (S,\bar{S})} f(e) - \sum_{e \in (\bar{S},S)} f(e) =$$

$$= \sum_{e \in (S,\bar{S})} c(e) \geq \textbf{Min Cut}$$



- Min cut – the minimum st-cut.
- Max flow – max flow from s to t.

Max-flow min-cut theorem:

The value of max flow = The capacity of min cut

Integral flow theorem:

If each edge in a flow network has integral capacity, then there exists an integral maximum flow.

Edmonds-Karp Algorithm

- Start with a feasible flow $f = 0$.
- while (f is not max flow)
  - Construct residual network G_f wrt flow f.
  - Find the *shortest path* P from s to t in G_f.
  - Push/augment flow along path P.
- return f

Lemma 1

- Throughout the algorithm, the length of a shortest augmenting path never decreases.

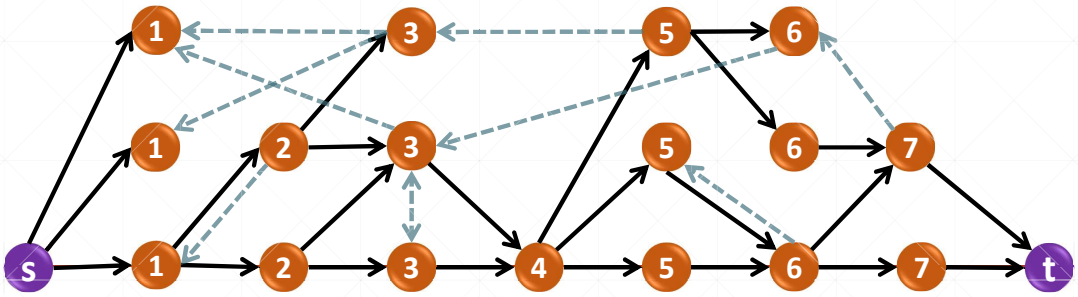
Lemma 2

- After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

Lemma 1 + Lemma 2 imply the Edmonds-Karp algorithm terminates in mn steps.

Level Graph

- Let $d(u)$ be the shortest path distance from s to u. Let $L = d(t)$.
- Place vertex u in row $d(u)$.
- Forward edges go from i to $j \geq i$. Backward edges go from j to $i < j$.

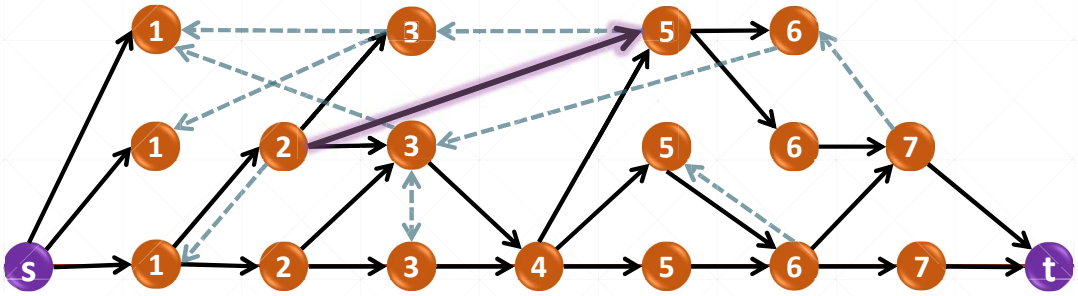


Level Graph

Claim A: If $u \rightarrow v$ is a forward edge, then

level of v = level of u + 1

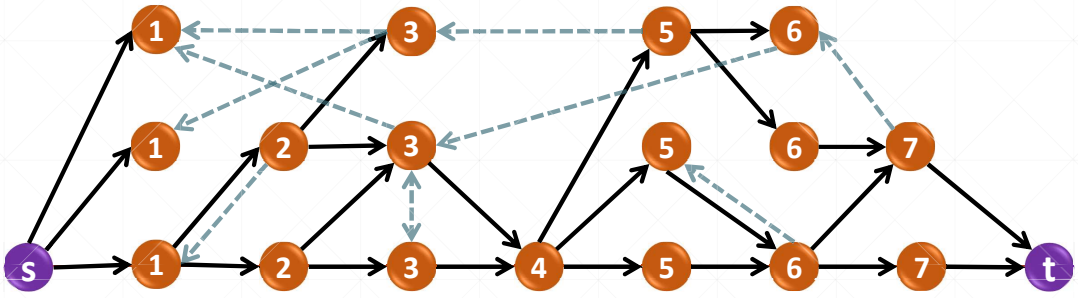
If graph satisfies (A) then: Every path $s \rightarrow t$ of length L consists only of forward edges, where L is the level of t. Distance from s to t is $\geq L$.



Level Graph

Claim B: If we add arbitrary ***backward*** edges and remove arbitrary edges, then the distance from s to t will still be $\geq L$.

Proof: After the update, the graph still satisfies **(A)**
(note: we don't change the positions of vertices).



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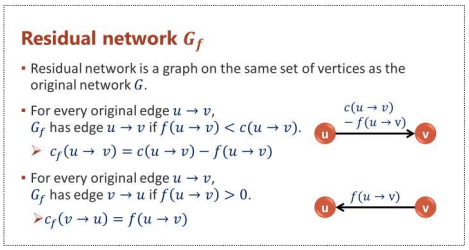

Proof:

- f and f'— flow before and after a shortest-path augmentation along path P.
- G_f and $G_{f'}$ — corresponding residual networks.

Consider the level graph for G_f. We update capacities in f along the path P. Hence, only edges from $\text{reverse}(P)$ may get added to $G_{f'}$. Hence, we don't add forward edges. Lemma 1 follows from Claim B.

Lemma 2

- After at most m shortest-path augmentations, the length of the shortest augmenting path strictly increases.

Proof: Consider the phase of ALG when $d(s,t) = L$.

- Fix the level graph at the first step when $d(s,t) = L$.
- At every step, in this phase, all shortest paths $s \to t$ consist only of forward edges.
- Thus, no forward edges are added, but at least one forward edge – the bottleneck edge – is deleted.
- After m steps all forward edges are deleted or $d(s,t)$ increases.

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Lemma 2

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Lemma 1 + Lemma 2 imply the Edmonds-Karp algorithm terminates in mn steps.

Applications of Max Flow and Min Cut

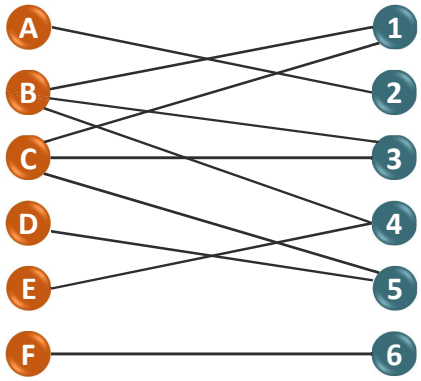
Routing

- Routing vehicles on the road.
- Routing electricity.
- Routing internet traffic.

Bipartite Matching

Bipartite Matching

- Alice, Bob, Carol, Dan, Eve, and Frank want to organize a party. They would like to prepare dishes 1,…,6. Each of them knows how to cook some subset of these dishes. What dish should each person prepare?
- Need to find a matching.



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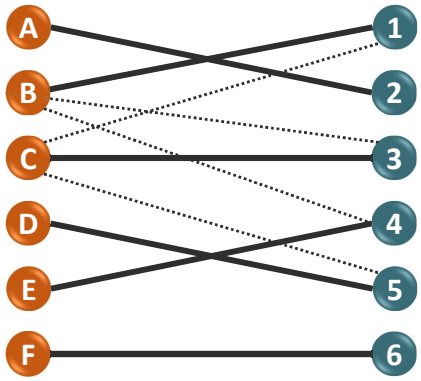
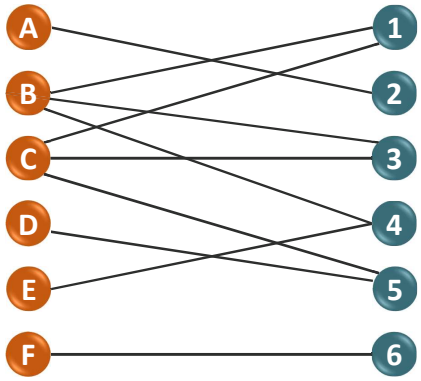


Bipartite Matching

Given: Bipartite graph

$$G = (L, R, E)$$

Goal: Find a perfect matching i.e., a subset of edges $M \subset E$ s.t. each vertex is incident to exactly one edge in M.



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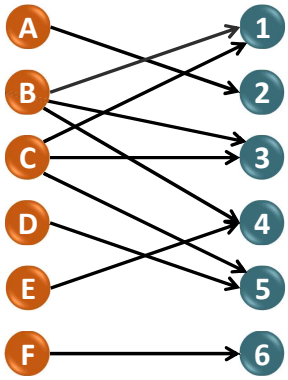


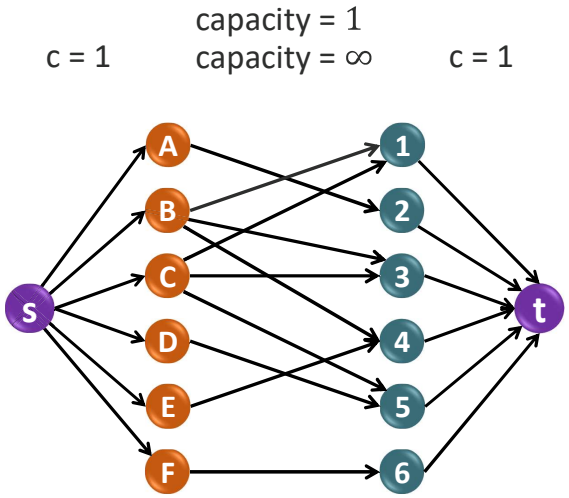
Bipartite Matching

Given: Bipartite graph

$$G = (L, R, E)$$

Goal: Find a perfect matching i.e., a subset of edges $M \subset E$ s.t. each vertex is incident to exactly one edge in M.

Consider a flow network with edges of capacity 1 and ∞.



Bipartite Matching

Given: Bipartite graph G.

Goal: Find a perfect matching.

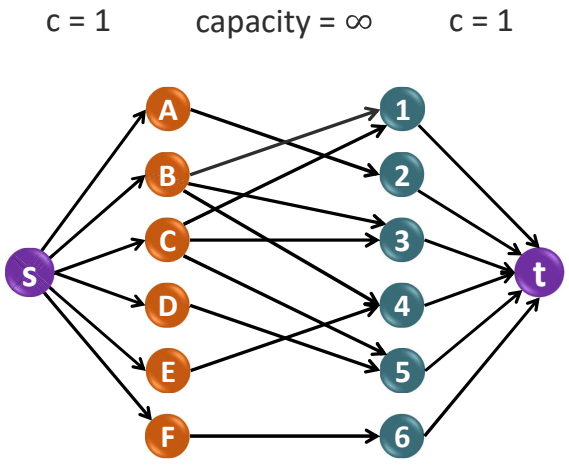
n = #vertices on each side

Thm:

There is perfect matching in G

⇕

the max flow equals n

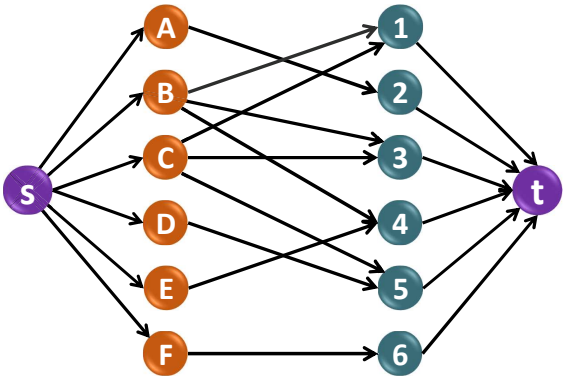


Bipartite Matching

Proof:

⇒ If there is a perfect matching in G, then we can route the flow via this matching.

c = 1 capacity = ∞ c = 1



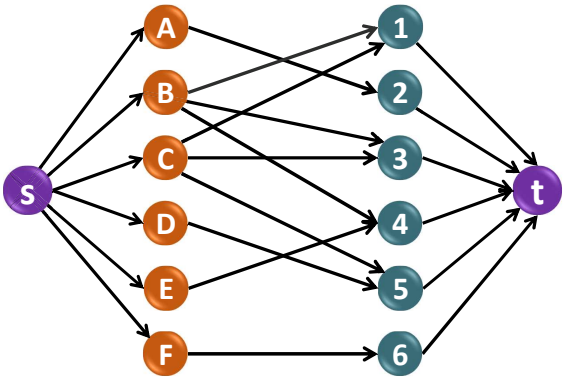
Bipartite Matching

Proof:

⇒ If there is a perfect matching in G, then we can route the flow via this matching.

⇐ Since all capacities are integral, there exists an integral max flow ($f(e) \in \mathbb{Z}$). Drop all edges with $f(e) = 0$.

c = 1 capacity = ∞ c = 1



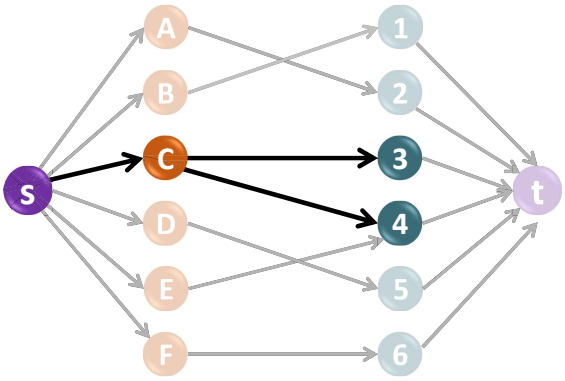
Bipartite Matching

Proof:

⇐ Since all capacities are integral, there exists an integral max flow ($f(e) \in \mathbb{Z}$). Drop all edges with $f(e) = 0$.

The flow to each vertex on the left ≤ 1. It is 1 because max flow $= n$. Hence, flow out of each vertex must be also 1.

c = 1 capacity = ∞ c = 1

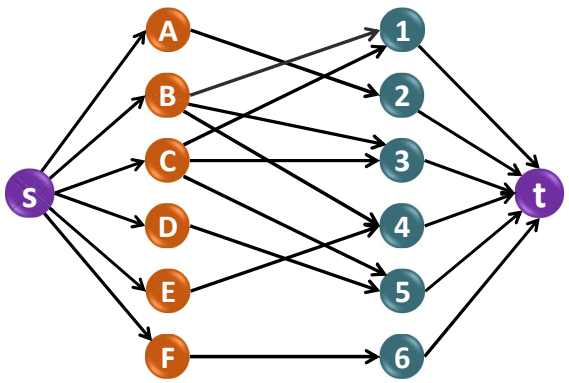


Algorithm

- Solve the Max Flow problem.
- Return all edges e between L and R with $f(e) = 1$.

When does the algorithm succeed?

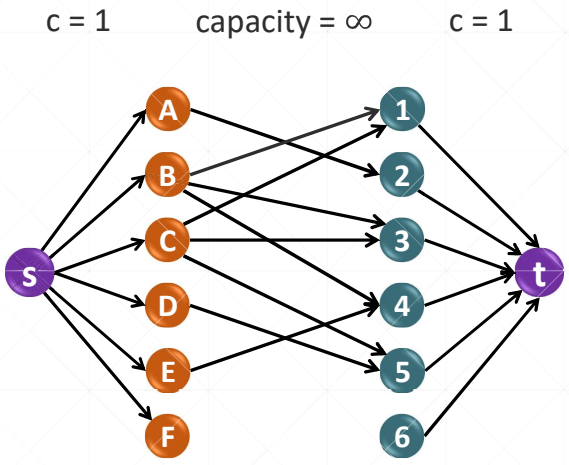
Min cut = Max flow $= n$.



Maximum Matching

Algorithm for Maximum Matching

- Solve the Max Flow problem.
- Return all edges e between L and R with $f(e) = 1$.



Hall's Theorem

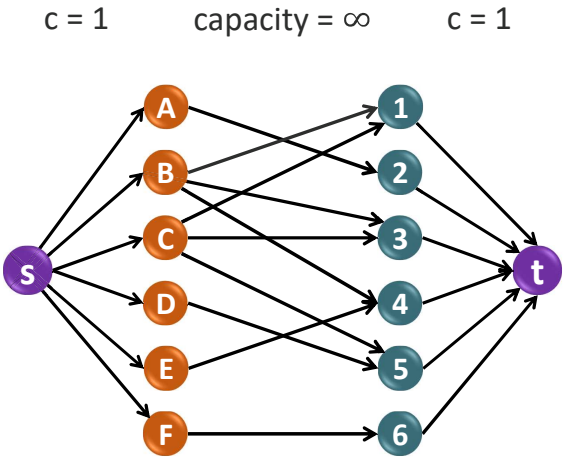
Hall's Theorem

There is a perfect matching in G

$\Updownarrow$

for every $S \subset L$, we have
$|N(S)| \geq |S|$

Here, $N(S)$ is the set of neighbors of S.



Hall's Theorem

There is a perfect matching in G

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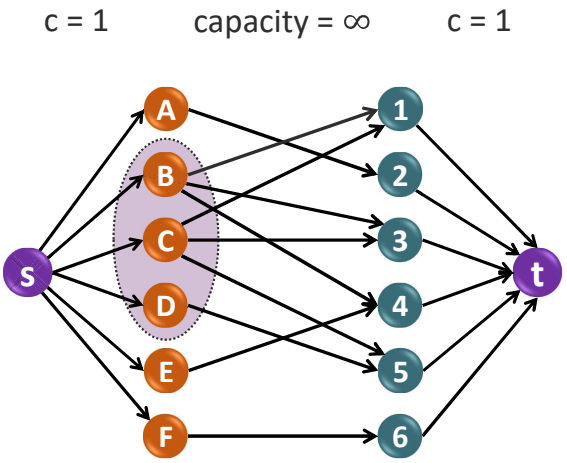
for every $S \subset L$, we have
$|N(S)| \geq |S|$

Here, $N(S)$ is the set of neighbors of S.

$\Rightarrow$ If there is a perfect matching in G, then "the image of S" contains at least $|S|$ elements.



Hall's Theorem

There is a perfect matching in G

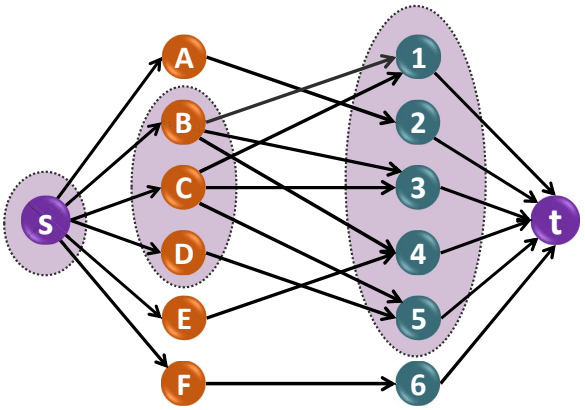
$\Updownarrow$

for every $S \subset L$, we have $|N(S)| \geq |S|$

$\Leftarrow$ Consider min cut (A, B). Its capacity $< \infty$. Thus, edges between L and R cannot cross it. Let $S = A \cap L$; $T = A \cap R$.

Then, $N(S) \subset T$. We get

$$cut(A, B) = n - |S| + |T|$$
$$\geq n - |S| + |N(S)| \geq n$$

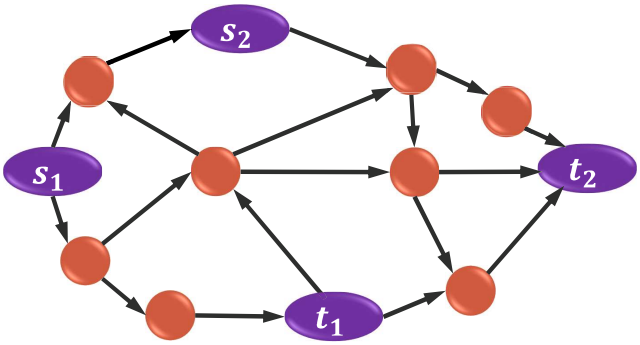


Multiple Sources & Sinks

Multiple Sources & Sinks

Given: Flow network w/ many sources: $s_1, \ldots, s_{k'}$ and many sinks: $t_1, \ldots, t_{k''}$.

Goal: Maximize the total amount of flow from all sources to all sinks.



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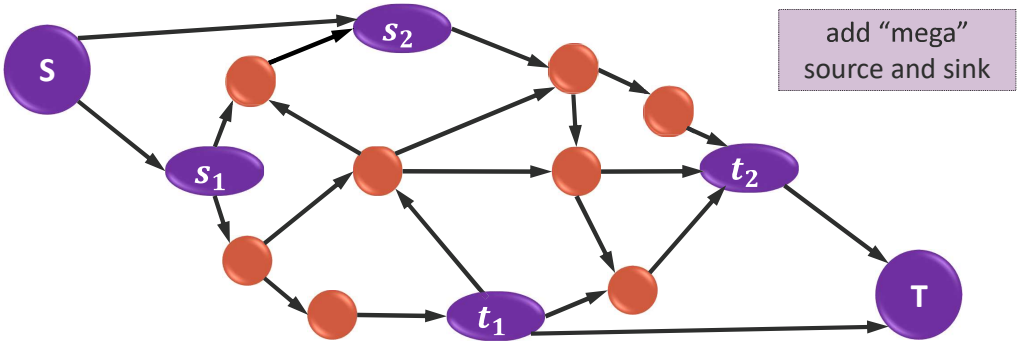

Questions