a) Derive expression for cond. prob pixel (i,j) black given MB

$$E(y,x) = -\eta \sum_{i,j} y_{i,j} x_{i,j} - \beta \sum_{((i,j),(i',j')) \in E} y_{i,j} y_{i',j'}$$

$$P(y,x) = \frac{1}{Z} \exp(-E(y,x)) \quad , \text{ Let } y_{-\ell} = y_{-y_{i,j}}$$

$$p(y_{i,j} = 1 \mid y_m(i,j)) = p(y_{i,j} = 1 \mid y_{-\ell}, X)$$

$$= \frac{p(y_{i,j} = 1, y_{-\ell}, X)}{p(y_{-\ell}, X)}$$

$$= \frac{\exp(-E(y_{i,j} = 1, y_{-\ell}, X))}{\exp(-E(y_{i,j} = 1, y_{-\ell}, X)) + \exp(-E(y_{i,j} = -1, y_{-\ell}, X))}$$

$$= \frac{\exp\left(\eta\left(\sum_{i,j} y_{i,j} x_{i,j} + x_\ell\right) + \beta\left(\sum_{((i,j),(i',j')) \in E} y_{i,j} y_{i',j'} + \sum_{y_m} y_{i',j'}\right)\right)}{// + \exp\left(\eta\left(\sum_{i,j} y_{i,j} x_{i,j} - x_\ell\right) + \beta\left(\sum_{\in E} y_{i,j} y_{i',j'} - \sum_{y_m} y_{i',j'}\right)\right)}$$

$$\eta \sum_{i,j} y_{i,j} x_{i,j} + \eta x_\ell + \beta \sum_{\in E} y_{i,j} y_{i',j'} + \beta y_{i',j'}$$

$$\rightarrow \exp\left(\eta x_\ell + \beta \sum_{y_m} y_{i',j'}\right) \cdot \exp\left(\eta \sum_{i,j} y_{i,j} x_{i,j} + \beta_{\in E} y_{i,j} y_{i',j'}\right)$$

Let $a = \eta x_\ell + \beta \sum_{y_m} y_{i',j'}$, $b = \eta \sum_{i,j} y_{i,j} x_{i,j} + \beta_{\in E} y_{i,j} y_{i',j'}$

Then $p(y_{i,j} = 1 \mid y_m(i,j)) = \dfrac{e^a \cdot e^b}{e^a \cdot e^b + e^{-a} \cdot e^b} = \dfrac{e^a \cdot e^b}{e^b \cdot (e^a + e^{-a})}$

$$= \frac{e^a}{e^a + e^{-a}} \cdot \frac{e^{-a}}{e^{-a}} = \frac{1}{1 + e^{-2a}} = \sigma(2a)$$

$$= \sigma\left(2\left(\eta x_\ell + \beta \sum_{y_m} y_{i',j'}\right)\right)$$