Chapter 10 solutions

1. (a) It was shown in the text that the m.s. limit is a linear operator, namely if $X[n] \to X$ and $Y[n] \to Y$ in the m.s.-sense, then

$$\lim_{n\to\infty} (aX[n] + bY[n]) = aX + bY. \quad \text{(m.s.)} \tag{1}$$

We are here interested in extending this relationship to derivatives, by showing:

$$\begin{aligned}
\frac{d}{dt}\left[aX_1(t) + bX_2(t)\right] &= \lim_{\epsilon\to 0}\left[\frac{(aX_1(t+\epsilon) + bX_2(t+\epsilon)) - (aX_1(t) + bX_2(t))}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0}\left[a\left(\frac{X_1(t+\epsilon) - X_1(t)}{\epsilon}\right) + b\left(\frac{X_2(t+\epsilon) - X_2(t)}{\epsilon}\right)\right] \\
&= a\lim_{\epsilon\to 0}\left(\frac{X_1(t+\epsilon) - X_1(t)}{\epsilon}\right) + b\lim_{\epsilon\to 0}\left(\frac{X_2(t+\epsilon) - X_2(t)}{\epsilon}\right), \quad \text{by } (1), \\
&= a\frac{d}{dt}X_1(t) + b\frac{d}{dt}X_2(t).
\end{aligned}$$

(b) By definition, we have

$$E[X_1(t)X_2'(t)] = E\left[X_1(t)\lim_{\epsilon\to 0}\left(\frac{X_2(t+\epsilon) - X_2(t)}{\epsilon}\right)\right],$$

which reduces to

$$\lim_{\epsilon\to 0} E\left[X_1(t)\left(\frac{X_2(t+\epsilon) - X_2(t)}{\epsilon}\right)\right],$$

since we know that $\lim_{n\to\infty} E[X[n]] = E[X]$ if $X[n] \to X$ in the m.s. sense. Now, in turn,

$$\begin{aligned}
\lim_{\epsilon\to 0} E\left[X_1(t)\left(\frac{X_2(t+\epsilon) - X_2(t)}{\epsilon}\right)\right] &= \lim_{\epsilon\to 0}\left(\frac{R_{X_1X_2}(t_1, t_2+\epsilon) - R_{X_1X_2}(t_1, t_2)}{\epsilon}\right) \\
&= \frac{\partial}{\partial t_2} R_{X_1X_2}(t_1, t_2)
\end{aligned}$$

2. (a) We need the existence of $\left.\frac{\partial^2 K_{XX}}{\partial t \partial s}\right|_{t=s}$ for the existence of $X'(t)$. For the given covariance function, we get

$$\begin{aligned}
\frac{\partial K_{XX}(t,s)}{\partial s} &= +\sigma^2 \frac{\partial}{\partial s}\cos\omega_0(t-s) \\
&= \sigma^2\omega_0 \sin\omega_0(t-s).
\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\left(\frac{\partial K_{XX}(t,s)}{\partial s}\right) = \sigma^2\omega_0^2\cos\omega_0(t-s).$$

Thus $\left.\frac{\partial^2 K_{XX}}{\partial t \partial s}\right|_{t=s} = \sigma^2\omega_0^2 < \infty$. Therefore the m.s. derivative exists for all finite t.

(b) We just found that $\frac{\partial^2 K_{XX}}{\partial t \partial s} = \sigma^2\omega_0^2\cos\omega_0(t-s)$ and this is equal to $K_{X'X'}(t,s)$.

(c) $R_{X'X'}(t,s) = K_{X'X'}(t,s) + \mu_{X'}(t)\mu^*_{X'}(s)$. But $\mu_{X'}(t) = \frac{d}{dt}\mu_X(t)$ and $\mu_X(t)$ is constant here. Thus $\mu_{X'}(t) = 0$ and so $R_{X'X'}(t,s) = K_{X'X'}(t,s)$.

3. (a) The condition on $X'(t)$ is the existence of $\left.\frac{d^2R_{XX}(\tau)}{d\tau^2}\right|_{\tau=0}$. The overall condition for $Y = 3X + 2X'$ becomes the existence of $R_{YY}(\tau)|_{\tau=0} = 9R_{XX}(0) - 4R''_{XX}(0)$.

(b)

$$
\begin{aligned}
R_{YY}(\tau) &\triangleq E[Y(t+\tau)Y^*(t)] \\
&= E[(3X(t+\tau) + 2X'(t+\tau))(3X(t) + 2X'(t))^*] \\
&= 9R_{XX}(\tau) + 6R_{X'X}(\tau) + 6R_{XX'}(\tau) + 4R_{X'X'}(\tau) \\
&= 9R_{XX}(\tau) + 6R'_{XX}(\tau) + 6R'^*_{XX}(-\tau) - 4R''_{XX}(\tau) \\
&= 9R_{XX}(\tau) - 4R''_{XX}(\tau) \quad \text{since the given } R_{XX} \text{ is real and even,} \\
&= \sigma^2 e^{-(\tau/T)^2}\left[9 + \frac{8}{T^2}\left(1 - \frac{2\tau^2}{T^2}\right)\right].
\end{aligned}
$$

This since $R'_{XX}(\tau) = -2\frac{\tau}{T^2}\sigma^2 e^{-(\tau/T)^2}$ and $R''_{XX}(\tau) = \left(\frac{2\tau^2}{T^2}\right)^2 \sigma^2 e^{-(\tau/T)^2} - \frac{2}{T^2}\sigma^2 e^{-(\tau/T)^2}$.

4. The random process $X(t)$ is stationary with mean μ_X and covariance function

$$K_{XX}(\tau) = \frac{\sigma_X^2}{1 + \alpha^2\tau^2}.$$

(a) We have to show that $R_{XX}(\tau)$ has derivatives up to order two. Because $X(t)$ is stationary, the mean is constant, so that $\mu'_X(t) = 0$, therefore

$$
\begin{aligned}
\frac{dR_{XX}(\tau)}{d\tau} &= \frac{dK_{XX}(\tau)}{d\tau} \\
&= \frac{-\alpha^2\tau\sigma_X^2}{(1+\alpha^2\tau^2)^2}, \quad \text{which exists for all finite } \tau.
\end{aligned}
$$

Next

$$
\begin{aligned}
\frac{d^2K_{XX}(\tau)}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{-2\alpha^2\tau}{(1+\alpha^2\tau^2)^2}\right)\sigma_X^2 \\
&= \frac{-2\alpha^2(1+\alpha^2\tau^2)^2 + 2\alpha^2\tau \cdot 2 \cdot 2\alpha^2\tau(1+\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^4}\sigma_X^2 \\
&= \frac{-2\alpha^2(1+\alpha^2\tau^2) + 8\alpha^4\tau^2}{(1+\alpha^2\tau^2)^3}\sigma_X^2 \\
&= \frac{-2\alpha^2(1-3\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^3}\sigma_X^2
\end{aligned}
$$

which exists for all τ, and hence, for $\tau = 0$. Therefore the m.s. derivative exists for all finite t.

(b) We have

$$
\begin{aligned}
\mu_{\dot{X}}(t) &\triangleq E[\dot{X}(t)] \\
&= E\left[\lim_{\epsilon\to 0}\frac{X(t+\epsilon)-X(t)}{\epsilon}\right] \\
&= \lim_{\epsilon\to 0} E\left[\frac{X(t+\epsilon)-X(t)}{\epsilon}\right], \quad \text{because of m.s. existence,} \\
&= \lim_{\epsilon\to 0}\frac{E[X(t+\epsilon)]-E[X(t)]}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\mu_X-\mu_X}{\epsilon}, \quad \text{by stationarity,} \\
&= \lim_{\epsilon\to 0}\frac{0}{\epsilon} \\
&= \lim_{\epsilon\to 0} 0 \\
&= 0.
\end{aligned}
$$

For the covariance of $\dot{X}(t)$, we have

$$
\begin{aligned}
K_{\dot{X}\dot{X}}(\tau) &= -\frac{d^2K_{XX}(\tau)}{d\tau^2} \\
&= \frac{2\alpha^2(1-3\alpha^2\tau^2)}{(1+\alpha^2\tau^2)^3}\sigma_X^2, \quad \text{from result in part (a).}
\end{aligned}
$$

5. Let $I_1 \triangleq \int_0^T X_1(t)dt$ and $I_2 \triangleq \int_0^T X_2(t)dt$, and consider

$$
\begin{aligned}
&\left\|\sum_{i=1}^{N}(a_1X_1(t_i)+a_2X_2(t_i))\Delta t_i-(a_1I_1+a_2I_2)\right\| \\
\leq\ & |a_1|\left\|\sum_{i=1}^{N}X_1(t_i)\Delta t_i-I_1\right\|+|a_2|\left\|\sum_{i=1}^{N}X_2(t_i)\Delta t_i-I_2\right\| \\
\leq\ & |a_1|\delta_1+|a_2|\delta_2,
\end{aligned}
$$

where both δ_1 and δ_2, both positive, can be taken arbitrarily small by the m.s. existence of the integrals I_1 and I_2. Therefore, it must be that the m.s. limit of $\sum_{i=1}^{N}(a_1X_1(t_i)+a_2X_2(t_i))\Delta t_i$ exists in the limit of $N \nearrow \infty$, ($\Delta t_i \searrow 0$). Also, we see that this m.s. limit must equal $a_1I_1+a_2I_2$. This m.s. limit is given the *symbol*

$$\int_0^T (a_1X_1(t)+a_2X_2(t))dt$$

6. Let

$$I_{12} \triangleq \int_{t_1}^{t_2} X(t)dt \ , I_{23} \triangleq \int_{t_2}^{t_3} X(t)dt, \text{ and } I_{13} \triangleq \int_{t_1}^{t_3} X(t)dt.$$

For each positive integer N, define the averages over increasingly fine partitions for $t_i < t_j$

$$I_{ij}(N) \triangleq \sum_{k=1}^{N} X\left(t_i + k\Delta t_{ij}\right)\Delta t_{ij}$$

with $\Delta t_{ij} \triangleq (t_j - t_i)/N$ and $i,j = 1,2$ and $2,3$, Then we define

$$I_{13}(N) \triangleq I_{12}(N) + I_{23}(N),$$

and note that it is a valid and increasingly fine partition for the combined interval (t_1, t_3). Then, we know that $I_{ij}(N) \to I_{ij}$ (m.s.) for $i,j = 1,2,3$. So, from the triangle inequality,

$$\|I_{12} + I_{23} - I_{13}(N)\| \leq \|I_{12} - I_{12}(N)\| + \|I_{23} - I_{23}(N)\|.$$

Since the two terms in the rhs go to zero as $N \to \infty$, it must be that $\|I_{12} + I_{23} - I_{13}(N)\| \to 0$, thus

$$\begin{aligned} I_{12} + I_{23} &= \lim_{N\to\infty} I_{13}(N) \quad \text{(m.s.)} \\ &= I_{13}. \end{aligned}$$

7. We have that $I(T) \triangleq \frac{1}{T}\int_0^T X(t)dt, \qquad T > 0.$

(a)

$$\begin{aligned} E[I(T)] &= E\left[\frac{1}{T}\int_0^T X(t)dt\right] \\ &= \frac{1}{T}E\left[\int_0^T X(t)dt\right] \\ &= \frac{1}{T}\int_0^T E[X(t)]dt \\ &= \frac{1}{T}\int_0^T \mu_X dt \\ &= \mu_X. \end{aligned}$$

(b)

$$\begin{aligned} \sigma^2_{I(T)} &= E[(I(T) - E[I(T)])^2] \\ &= E\left[\left(\frac{1}{T}\int_0^T X(t)dt - \mu_X\right)^2\right] \\ & \quad E\left[\left(\frac{1}{T}\int_0^T (X(t) - \mu_X)\, dt\right)^2\right] \\ &= \frac{1}{T^2}\int_0^T\int_0^T K_{XX}(t-s)dtds \\ &= \frac{1}{T^2}\int_{-T}^T (T - |\tau|)K_{XX}(\tau)d\tau \\ &= \frac{1}{T}\int_{-T}^T \left(\frac{T-|\tau|}{T}\right) K_{XX}(\tau)d\tau, \end{aligned}$$

where we have made the substitution $\tau \triangleq t - s, \xi \triangleq t + s$, two lines above, and then integrated out in the variable ξ.

8. Given that $X(t)$ is an WSS and Gaussian random process, we have to show that $\dot{Y}(t) = 2X(t)\dot{X}(t)$ in m.s. sense when $Y(t) \triangleq X^2(t)$. We assume that the m.s. derivative $\dot{X}(t)$ exists and that $X(t)$ is second order, i.e. $E[|X^2(t)|] < \infty$. We proceed as follows

$$\begin{aligned}
\dot{Y}(t) &= \lim_{\epsilon \to 0} \left(\frac{Y(t+\epsilon) - Y(t)}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left(\frac{X^2(t+\epsilon) - X^2(t)}{\epsilon} \right) \\
&= \lim_{\epsilon \to 0} \left\{ [X(t+\epsilon) + X(t)] \left(\frac{X^2(t+\epsilon) - X^2(t)}{\epsilon} \right) \right\} \\
&\stackrel{?}{=} \lim_{\epsilon \to 0} [X(t+\epsilon) + X(t)] \cdot \lim_{\epsilon \to 0} \left(\frac{X^2(t+\epsilon) - X^2(t)}{\epsilon} \right) \\
&= 2X(t)\dot{X}(t) \quad ?
\end{aligned}$$

To show these last two lines are true in this case, we consider the general case where $X^{(n)}(t) \to X(t)$ and $Y^{(n)}(t) \to Y(t)$ (m.s.). Under what conditions is it then true that $X^{(n)}(\mathrm{t})\mathrm{Y}^{(n)}(\mathrm{t}) \to X(t)Y(t)$. We will show that the answer is yes in the Gaussian case. Using the norm notation of the Hilbert space of RVs, we can write for any t

$$\begin{aligned}
\sqrt{E[|X^{(n)}(t)Y^{(n)}(t) - X(t)Y(t)|^2]} &= \left\| X^{(n)}Y^{(n)} - XY \right\| \\
&= \left\| X^{(n)}Y^{(n)} - (X^{(n)}Y - X^{(n)}Y) - XY \right\| \\
&\leq \left\| X^{(n)}Y^{(n)} - X^{(n)}Y \right\| + \left\| X^{(n)}Y - XY \right\|, \text{ by triangle inequality,} \\
&= \left\| X^{(n)}(Y^{(n)} - Y) \right\| + \left\| (X^{(n)} - X)Y \right\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\left\| X^{(n)}(Y^{(n)} - Y) \right\|^2 &= E[|X^{(n)}(t)(Y^{(n)}(t) - Y(t))|^2] \\
&= E[|X^{(n)}(t)|^2 |Y^{(n)}(t) - Y(t)|^2] \\
&\leq \sqrt{E[|X^{(n)}(t)|^4} \cdot \sqrt{E[|Y^{(n)}(t) - Y(t)|^4]}, \quad \text{by Schwarz inequality,}
\end{aligned}$$

and similarly for the second term above, $\left\| (X^{(n)} - X)Y \right\|^2 \leq \sqrt{E[|X^{(n)}(t) - X(t)|^4} \cdot \sqrt{E[|Y(t)|^4]}$. In general, we would not know whether these 4th order moments converge or not, but in this Gaussian case, we can use the stated Gaussian 4th order moment property to write

$$E[|X^{(n)}(t) - X(t)|^4] = 3\left(E[|X^{(n)}(t) - X(t)|^2] \right)^2 \to 0 \text{ as } n \to \infty,$$

and similarly $\sqrt{E[|Y^{(n)}(t) - Y(t)|^4} \to 0$ as $n \to \infty$. Then since $E[|Y(t)|^4] < \infty$ and $E[|X^{(n)}(t)|^4] < \infty$, by the same 4th order moment property, we get the general conclusion that $X^{(n)}(\mathrm{t})\mathrm{Y}^{(n)}(\mathrm{t}) \to X(t)Y(t)$ (m.s.) for the Gaussian case.

As for the relevant correlation function relation,

$$\begin{aligned}
R_{YY}(\tau) &= E[X^2(t+\tau)]E[X^2(t)] + 2\left(E[X(t+\tau)X(t)] \right)^2, \\
&\qquad \text{by the same 4}^{\text{th}}\text{order moment property,} \\
&= R_{XX}^2(0) + 2R_{XX}^2(\tau).
\end{aligned}$$

Then

$$\begin{aligned}
R_{\dot{Y}\dot{Y}}(\tau) &= -\frac{d^2}{d\tau^2}R_{YY}(\tau) \\
&= -4\left(R_{XX}(\tau)\frac{d^2R_{XX}(\tau)}{d\tau^2} + \left(\frac{dR_{XX}(\tau)}{d\tau}\right)^2\right).
\end{aligned}$$

9. Let

$$\begin{aligned}
Y_n'(t) &\triangleq n[Y(t+\frac{1}{n}) - Y(t)] \\
&= n\left[X(t+\frac{1}{n})\cos 2\pi f_0(t+\frac{1}{n}) - X(t)\cos 2\pi f_0 t\right],
\end{aligned}$$

and then use the substitution $\cos 2\pi f_0 t\ \cos 2\pi\frac{f_0}{n} - \sin 2\pi\frac{f_0}{n}\ \sin 2\pi f_0 t$ for the first cosine term, to get

$$\begin{aligned}
Y_n'(t) - Y'(t) &= [nX(t+\frac{1}{n})\cos 2\pi f_0 t\ \cos 2\pi\frac{f_0}{n} - X(t+\frac{1}{n})\sin 2\pi\frac{f_0}{n}\ \sin 2\pi f_0 t \\
&\quad -X(t)\cos 2\pi f_0 t] + 2\pi f_0 \sin 2\pi f_0 t\ X(t) - \cos 2\pi f_0 t\ X'(t) \\
&= \cos 2\pi f_0 t\left[n\left(X(t+\frac{1}{n})\cos 2\pi\frac{f_0}{n} - X(t)\right) - X'(t)\right] \\
&\quad +\sin 2\pi f_0 t\left[2\pi f_0\ X(t) - n\sin 2\pi\frac{f_0}{n}\ X(t+\frac{1}{n})\right].
\end{aligned}$$

So, in the Hilbert space of RVs, for each t,

$$\begin{aligned}
\left\|Y_n'(t) - Y'(t)\right\| &\leq |\cos 2\pi f_0 t|\ \left\|n\left(X(t+\frac{1}{n})\cos 2\pi\frac{f_0}{n} - X(t)\right) - X'(t)\right\| \\
&\quad +|\sin 2\pi f_0 t|\ \left\|2\pi f_0\ X(t) - n\sin 2\pi\frac{f_0}{n}\ X(t+\frac{1}{n})\right\|,
\end{aligned}$$

by use of the triangle inequality and scalar multiplication property $\|aX\| = |a|\ \|X\|$ for X an RV.

It then remains to show that each of the two norms in the above equation, call them (A) and (B), respectively, tends to zero as $n \to \infty$. Since $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for θ small, e.g. $2\pi\frac{f_0}{n}$ for n large, we expect that both these norms should tend to zero, since $X'(t)$ is the m.s. derivative of $X(t)$, and hence, $X(t)$ is m.s. continuous. The remainder of this solution is thus devoted to showing this. First we consider (A) and subtract/add $X(t+\frac{1}{n})$ inside the round brackets to obtain

$$\begin{aligned}
\left\|n\left(X(t+\frac{1}{n})\cos 2\pi\frac{f_0}{n} - X(t)\right) - X'(t)\right\| &= \left\|n\left(X(t+\frac{1}{n})\cos 2\pi\frac{f_0}{n} - X(t+\frac{1}{n}) + X(t+\frac{1}{n}) - X(t)\right) - X'(t)\right\| \\
&\leq \left\|n\left(\cos 2\pi\frac{f_0}{n} - 1\right)\right\| + \left\|n\left(X(t+\frac{1}{n}) - X(t)\right) - X'(t)\right\| \\
&\leq |n(\cos 2\pi\frac{f_0}{n} - 1)|\ \left\|X(t+\frac{1}{n})\right\| + \left\|n\left(X(t+\frac{1}{n}) - X(t)\right) - X'(t)\right\|.
\end{aligned}$$

Now, the first term tends to zero, as $n \to \infty$ since $n(\cos 2\pi\frac{f_0}{n} - 1) = O(\frac{1}{n})$ and the second term is the defining approximant for the m.s. derivative $X'(t)$. Note: We take $\|X(t)\| < \infty$ since $X(t)$ is said to be a second order random process, i.e. $E[|X(t)|^2] < \infty$.

Turning to the second norm (B), we proceed as above, but now subtract/add $2\pi f_0 X(t+\frac{1}{n})$. Then, using the triangle inequality again, we obtain

$$\left\|2\pi f_0\ X(t) - n\sin 2\pi\frac{f_0}{n}\ X(t+\frac{1}{n})\right\| \le |2\pi f_0|\ \left\|X(t) - X(t+\frac{1}{n})\right\| + |2\pi f_0 - n\sin 2\pi\frac{f_0}{n}|\ \left\|X(t+\frac{1}{n})\right\|.$$

Here, the first term on the right-hand side goes to zero as $n \to \infty$ by m.s. continuity, and the second term does via $|2\pi f_0 - n\sin 2\pi\frac{f_0}{n}| \to 0$ using $\sin\theta \approx \theta$ with $\theta = 2\pi\frac{f_0}{n}$ for n large. Thus we have $\|Y_n'(t) - Y'(t)\| \to 0$ for the specified formula for $Y'(t)$ as required to be shown.

10. (a) By definition (10.2-1), since $U(t)$ has independent increments, if we let $t_2 > t_1$,

$$\begin{aligned} E[U(t_1)(U(t_2) - U(t_1))] &= E[U(t_1)]E[U(t_2) - U(t_1)] \\ &= 0, \quad \text{since } U(t) \text{ is zero mean.} \end{aligned}$$

Thus

$$E[U(t_1)U(t_2)] = E[U^2(t_1)].$$

If $t_1 > t_2$, then by the symmetry of the situation, we would get $E[U(t_1)U(t_2)] = E[U^2(t_2)]$. So, in general we have

$$\begin{aligned} E[U(t_1)U(t_2)] &= E[U^2(\min(t_1,t_2))] \\ &= f(\min(t_1,t_2)). \end{aligned}$$

(b) Since $U(t)$ has independent increments, if we have $t_1 < t_2 < t_3 < \cdots < t_n$ and then set

$$\delta \triangleq \min_{1\le i\le n-1}(t_{i+1} - t_i),$$

then the increments

$$U(t_1+\delta) - U(t_1), U(t_2+\delta) - U(t_2), \cdots, U(t_n+\delta) - U(t_n),$$

will be jointly independent for all smaller δ. Then, by dividing by δ, and taking the m.s. limit, we get that the derivatives

$$U'(t_1), U'(t_2), \cdots, U'(t_n),$$

are jointly independent for all $t_1 < t_2 < t_3 < \cdots < t_n$ and all positive integers n. Thus be definition $U'(t)$ is an independent process.

(c) We have already shown:

$$\begin{aligned} &\text{i)} \quad E[U'(t)] = 0, \text{ a constant,} \\ &\text{ii)}\ R_{U'U'}(t_1,t_2) = 0 = \frac{\partial^2 f(\min(t_1,t_2))}{\partial t_1 \partial t_2}, \quad \text{for } t_1 \ne t_2\ . \end{aligned}$$

Now

$$\frac{\partial f(\min(t_1,t_2))}{\partial t_1} = \begin{cases} f'(t_1), & t_1 < t_2\ , \\ 0, & t_1 > t_2, \end{cases}$$

so

$$\begin{aligned} \frac{\partial}{\partial t_2}\left(\frac{\partial f(\min(t_1,t_2))}{\partial t_1}\right) &= \frac{\partial}{\partial t_2}\left(f'(t_1)u(t_2 - t_1)\right) \\ &= f'(t_1)\delta(t_2 - t_1). \end{aligned}$$

Hence, in order for U' to be WSS, we need either $f'(t)$ to be constant, thus $f(t) = at + b$, i.e. a linear function of time.

11. (a)

$$\int_0^t \int_0^t a(t,\tau_1)R_{XX}(\tau_1,\tau_2)a^*(t,\tau_2)d\tau_1 d\tau_2 \quad < \quad \infty,$$

i.e. exists and is finite.

(b)

$$\mu_Y(t) = \int_0^t a(t,\tau)\mu_X(\tau)d\tau$$

(c)

$$K_{YY}(t,s) = \int_0^t \int_0^s a(t,\tau_1)K_{XX}(\tau_1,\tau_2)a^*(s,\tau_2)d\tau_1 d\tau_2$$

12. (a) Let

$$I = \int_{-\infty}^t e^{-(t-s)}X(s)ds,$$

with existence in the m.s. sense if

$$\int_{-\infty}^t \int_{-\infty}^t e^{-(t-s_1)}e^{-(t-s_2)}R_{XX}(s_1,s_2)ds_1 ds_2 < \infty.$$

(b) Given that $Y(t)$ exists, we have

$$\begin{aligned} R_{YY}(t_1,t_2) &\triangleq E[Y(t_1)Y^*(t_2)] \\ &= \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-(t_1-s_1)}e^{-(t_2-s_2)}R_{XX}(s_1,s_2)ds_1 ds_2. \end{aligned}$$

(c) The m.s. existence of the derivative $dY(t)/dt$ depends on whether

$$\frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2} \text{ exists at } t_1 = t_2 = t.$$

We compute this derivative in two steps, as follows:

<u>step1:</u> Take first partial with respect to (wrt) t_1.

$$\begin{aligned} \frac{\partial R_{XX}(t_1,t_2)}{\partial t_1} &= \frac{\partial}{\partial t_1}\left(\int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-(t_1-s_1)}e^{-(t_2-s_2)}R_{XX}(s_1,s_2)ds_1 ds_2\right) \\ &= e^{-(t_1-t_1)}\int_{-\infty}^{t_2} e^{-(t_2-s_2)}R_{XX}(t_1,s_2)ds_2 \\ &\quad - \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-(t_1-s_1)}e^{-(t_2-s_2)}R_{XX}(s_1,s_2)ds_1 ds_2. \end{aligned}$$

<u>step2:</u> Take partial of the result in step 2, this time wrt t_2.

$$\begin{aligned} \frac{\partial}{\partial t_2}\left(\frac{R_{XX}(t_1,t_2)}{\partial t_1}\right) &= R_{XX}(t_1,t_2) - \int_{-\infty}^{t_2} e^{-(t_2-s_2)}R_{XX}(t_1,s_2)ds_2 \\ &\quad - \int_{-\infty}^{t_1} e^{-(t_1-s_1)}R_{XX}(s_1,t_2)ds_1 + R_{YY}(t_1,t_2). \end{aligned}$$

So,

$$\left.\frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=t} = R_{XX}(t,t) - \int_{-\infty}^{t} e^{-(t-s)} R_{XX}(t,s)ds - \int_{-\infty}^{t} e^{-(t-s)} R_{XX}(s,t)ds + R_{YY}(t,t) < \infty,$$

where

$$R_{YY}(t,t) = \int_{-\infty}^{t}\int_{-\infty}^{t} e^{-(t-s_1)}e^{-(t-s_2)} R_{XX}(s_1,s_2)ds_1 ds_2,$$

as in part (a).

13. (a) We have that $X[n] \to Y$ in the m.s. sense, and hence in probability and distribution. Each $X[n]$ is Gaussian distributed with mean μ_n and variance σ_n^2. By m.s. convergence, $\mu_n \to \mu = E[Y]$ and $\sigma_n^2 \to \sigma^2 =Var[Y]$, thus the distribution of $X[n]$ must be converging to $N(\mu,\sigma^2)$. Since each one is $N(\mu_n,\sigma_n^2)$ and since the Gaussian pdf/CDF is continuous in its parameters (for $\sigma^2 > 0$), we get that Y must be distributed as $N(\mu,\sigma^2)$, i.e. be Gaussian too. Note that here, to avoid degeneracy, we need $\sigma^2 > 0$.

    (b) Here, we have the vector case $\mathbf{X}[n] \to \mathbf{Y}$ in the m.s. sense. Since m.s. convergence for random vectors implies term-wise m.s. convergence, i.e.

    $$E[\|\mathbf{X}[n]-\mathbf{Y}\|^2] = \sum_{i=1}^{K} E[(X_i[n]-Y_i)^2], \qquad \text{(Please don't confuse this vector norm with the Hilbert space for RVs norm)}$$

    we have that

    $$\boldsymbol{\mu}_n \triangleq E[\mathbf{X}[n]] \to E[\mathbf{Y}] \triangleq \boldsymbol{\mu}$$

    and

    $$\mathbf{K}_n \triangleq E[(\mathbf{X}[n]-\boldsymbol{\mu}_n)(\mathbf{X}[n]-\boldsymbol{\mu}_n)^T] \to E[(\mathbf{Y}-\boldsymbol{\mu})(\mathbf{Y}-\boldsymbol{\mu})^T] \triangleq \mathbf{K}.$$

    Then, since the $\mathbf{X}[n]$ are Gaussian distributed as $N(\boldsymbol{\mu}_n,\mathbf{K}_n)$, we have that its m.s. limit random vector $\mathbf{Y}$ is also Gaussian distributed as $N(\boldsymbol{\mu},\mathbf{K})$.

    (c) By definition, a random process is Gaussian if all its finite order distributions are Gaussian. But these are all vectors, thus the result of part (b) applies. So, if $X_n(t) \to Y(t)$ in the m.s. sense, then for any K times t_i, the corresponding K dimensional vectors converge by

    $$E[\|\mathbf{X}[n]-\mathbf{Y}\|^2\,] = \sum_{i=1}^{K} E[|X_n(t_i)-Y(t_i)|^2],$$

    so the result of part (b) implies that $Y(t)$ is Gaussian distributed at these K samples. Since this result holds for arbitrary positive integers K and for arbitrary times $t_1 < t_2 < \cdots < t_K$, it follows that $Y(t)$ is a Gaussian random process.

14. First note that for each $n > 0$, we need to consider n-vectors at arbitrary times $t_1 < t_2 < \cdots < t_n$ and show that these vectors are *jointly* Gaussian. This is all that is required by the Gaussian random process definition. But the m.s. derivative is an m.s. limit, and we know that convergence in mean square implies convergence in distribution by the Chebyshev inequality. To get the vector version of this, proceed as in problem 10.14 which introduces m.s.

convergence and Chebyshev inequality for random vectors. The result then follows because each finite difference approximation to the derivative vector is just a linear transformation on jointly Gaussian data, and hence is Gaussian distributed. Since every term in the vector sequence is Gaussian distributed, the limit distribution must be Gaussian distributed also. The only possible problem occurs if $\mathbf{K}_X[n] \to \mathbf{K}_X$ which was singular. Then, one could write the joint characteristic function, but the joint density function would not exist as it requires the inverse of $\mathbf{K}_X$.

15. (a) Denote the psd of $X(t)$ as $S_X(\omega)$ and note that for the given conditions $S_X(\omega) = 0$ for $|\omega| > \omega_c < \omega_1$. Let the ideal lowpass filter be given as

$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_1, \\ 0, & \text{else.} \end{cases}$$

Then $S_X(\omega) = H(\omega)S_X(\omega)$ for all ω. Now

$$\begin{aligned} E[|X(t) - Y(t)|^2] &= E[(X(t) - Y(t))\,(X^*(t) - Y^*(t))] \\ &= E[|X(t)|^2] - E[X(t)Y^*(t)] - E[X^*(t)Y(t)] + E[|Y(t)|^2]. \end{aligned}$$

Here $Y(t)$ is given by the m.s. integral

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau,$$

so we can insert this expression into the last three expectation terms above, the first of which is

$$\begin{aligned} E[X(t)Y^*(t)] &= E\left[X(t)\int_{-\infty}^{+\infty} h^*(t-\tau)X^*(\tau)d\tau\right] \\ &= \int_{-\infty}^{+\infty} h^*(t-\tau)E[X(t)X^*(\tau)]d\tau, \quad \text{by Theorem10.1-5 (2),} \\ &= \int_{-\infty}^{+\infty} h^*(t-\tau)R_X(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} h^*(\tau)R_X(\tau)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H^*(-\omega)S_X(\omega)d\omega \quad \text{by Parseval's theorem,} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)d\omega \quad \text{because } S_X(\omega) = H^*(\omega)S_X(\omega), \\ &= R_X(0). \end{aligned}$$

Similarly for the other two terms, again invoking Theorem 10.1-5 (2) and (4), to permit interchange of the expectation operator and the m.s. integral. Here the existence condition is that $X(t)$ and $Y(t)$ have finite m.s. values, i.e. $R_X(0) < \infty$. Thus finally we have $E[|X(t) - Y(t)|^2] = R_X(0) - 2R_X(0) + R_X(0) = 0$, and so we can write $X(t) = Y(t)$ (m.s.). Note it is not valid to simply assert that $Y(t) = X(t)$ since $h(t)$ acts like $\delta(t)$ for the bandlimited $X(t)$, effectively treating the operation as an ordinary integral. The m.s. integral is more complicated and needs the second-order treatment given above.

(b) By Chebyshev's inequality, m.s. convergence implies convergence in probability. But in fact here $X(t) = Y(t)$ with probability one. To see this, let $\epsilon > 0$, then

$$\begin{aligned} P[|X(t) - Y(t)| > \epsilon] &\leq \frac{E[|X(t) - Y(t)|^2]}{\epsilon^2} \\ &= 0. \end{aligned}$$

Then since ϵ is arbitrary, it must be that $P[|X(t) - Y(t)| \neq 0] = 0$.

16. (a) The *zero-state solution* is

$$\int_{t_0}^{t} e^{-3(t-\tau)} u(\tau) d\tau,$$

where the input $u(\tau)$ is assumed to start at $\tau = t_0$. (This is just $h(t)*u(t)$, where $h(t) = e^{-3t}\times$ (unit step function). The *zero-input solution* is:

$$x_0 e^{-3(t-t_0)}.$$

The *total solution* $x(t)$ is then the sum of these two components:

$$x(t) = x_0 e^{-3(t-t_0)} + \int_{t_0}^{t} e^{-3(t-\tau)} u(\tau) d\tau, \qquad t \geq t_0.$$

(b) Now, for second-order processes, we can also write, for RV X_0 and input random process $U(t)$,

$$X(t) = X_0 e^{-3(t-t_0)} + \int_{t_0}^{t} e^{-3(t-\tau)} U(\tau) d\tau, \qquad t \geq t_0.$$

The first term will be well defined in the m.s. sense if X_0 is a second-order RV, i.e. $E[|X_0|^2] < \infty$. The second term is an m.s. integral, and is well defined if

$$\int_{t_0}^{t} \int_{t_0}^{t} e^{-3(t-\tau_1)} e^{-3(t-\tau_2)} R_{UU}(\tau_1, \tau_2) d\tau_1 d\tau_2 < \infty.$$

Clearly, this will be the case for $R_{UU}(t_1, t_2) = \frac{1}{4} \exp -2|t_1 - t_2|$, which is the correlation function corresponding to the psd $S_{UU}(\omega) = \frac{1}{\omega^2+4}$.

(c) Here are two ways to solve for $R_{UU}(t_1, t_2)$:

(1) Guess the answer is some constant $k \exp -2|\tau|$ and take Fourier transform to get

$$k \left(\int_{0}^{\infty} e^{(-2-j\omega)\tau} d\tau + \int_{-\infty}^{0} e^{(+2-j\omega)\tau} d\tau \right) = \frac{1}{\omega^2 + 4}.$$

Then solve for $k = \frac{1}{4}$ to fit.

(2) Use Laplace transforms, and write $S_{UU}(s) = \frac{1}{(s+2)(-s+2)}$, $-2 < \mathrm{Re}(s) < +2$, by substituting $s/j = \omega$. Then do the partial fraction expansion to get

$$\begin{aligned} S_{UU}(s) &= \frac{1}{(s+2)(-s+2)} \\ &= \frac{A}{s+2} + \frac{B}{-s+2}, \end{aligned}$$

with appropriate regions of convergence for each of these terms. We find

$$A = \left.\frac{(s+2)}{(s+2)(-s+2)}\right|_{s=-2} = \frac{1}{4} \quad \text{and} \quad B = \left.\frac{(-s+2)}{(s+2)(-s+2)}\right|_{s=+2} = \frac{1}{4}.$$

Now, for stability, the region of convergence (ROC) of the $\frac{\frac{1}{4}}{s+2}$ term is $\mathrm{Re}(s) > -2$, while the ROC of the $\frac{\frac{1}{4}}{-s+2}$ term is $\mathrm{Re}(s) < +2$. Note that the overlap of these two ROCs is $-2 < \mathrm{Re}(s) < +2$, as required above. From these two terms, we get a causal component $\frac{1}{4}e^{-2\tau}u(\tau)$, where, in this part of the problem, we use u to denote the unit step function, and an anticausal component $\frac{1}{4}e^{+2\tau}u(-\tau)$.

17. (a) We have the following ordinary differential equation for the mean function,

$$\begin{aligned} \dot{\mu}_Y(t) + 2\mu_Y(t) &= \mu_X(t), \quad t > 0, \\ &= 5\cos 2t, \end{aligned}$$

with $\mu_Y(0) = E[Y(0)] = E[0] = 0$. Using elementary methods, we have the solution to this deterministic linear differential equation as

$$\begin{aligned} \mu_Y(t) &= Ae^{-2t} + \mathrm{Re}\left[\frac{5e^{+j2t}}{j2+2}\right] \\ &= 0 \quad \text{at} \quad t = 0. \end{aligned}$$

Now $\frac{5e^{+j2t}}{j2+2} = \frac{5}{2\sqrt{2}}e^{+j(2t-\frac{\pi}{4})}$, so we can write the solution for $t > 0$ simply as

$$\begin{aligned} \mu_Y(t) &= \frac{5}{2\sqrt{2}}\left[\cos(2t - \frac{\pi}{4}) - \frac{1}{\sqrt{2}}e^{-2t}\right] \\ &= \frac{5}{4}\left[\cos 2t + \sin 2t - e^{-2t}\right]. \qquad \text{(alternative form)} \end{aligned}$$

(b) The covariance function is just the correlation function of the centered process $Y_c(t) \triangleq Y(t) - \mu_Y(t)$, which by the linearity of the equation is just the m.s. solution to

$$\frac{dY_c(t)}{dt} + 2Y_c(t) = W(t), \quad t > 0,$$

with initial condition $Y_c(0) = Y(0) - \mu_Y(0) = 0 - 0 = 0$. After solving this first-order differential equation, we can get the covariance function as follows:

$$K_{YY}(t_1, t_2) = \begin{cases} \frac{\sigma^2}{4}e^{-2t_2}(e^{+2t_1} - e^{-2t_1}), & t_1 \leq t_2, \\ \frac{\sigma^2}{4}(1 - e^{-4t_2})e^{-2(t_1 - t_2}, & t_1 > t_2. \end{cases}$$

(c) The variance function $\sigma_Y^2(t) = K_{YY}(t,t) = \frac{\sigma^2}{4}(1 - e^{-4t})$, $t > 0$. Using the fact that $Y(t)$ must be Gaussian (since it is the m.s. limit of Gaussian), we have

$$\begin{aligned} P[|Y(t) - \mu_Y(t)| \leq 0.1] &= P\left[\left|\frac{Y(t) - \mu_Y(t)}{\sigma_Y(t)}\right| \leq \frac{0.1}{\sigma_Y(t)}\right] \\ &= 2\,\mathrm{erf}\left(\frac{0.1}{\sigma_Y(t)}\right) \\ &\geq 0.99, \\ \text{or} \quad & \mathrm{erf}\left(\frac{0.1}{\sigma_Y(t)}\right) \geq 0.495. \end{aligned}$$

Then, from Table, we get approximately

$$\frac{0.1}{\sigma_Y(t)} \geq 2.555, \quad \text{so} \quad \sigma_Y(t) \leq \frac{0.1}{2.555}.$$

Then since $\max \sigma_Y^2(t) = \sqrt{\frac{\sigma^2}{4}} = \frac{\sigma}{2}$ for $t > 0$, we need

$$\sigma \leq \frac{0.2}{2.555} = 0.078.$$

18. Given

$$\frac{\partial R_{XY}(t_1,t_2)}{\partial t_2} + \alpha R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2) \quad \text{and} \tag{18.1}$$

$$\frac{\partial R_{YY}(t_1,t_2)}{\partial t_1} + \alpha R_{YY}(t_1,t_2) = R_{XY}(t_1,t_2), \tag{18.2}$$

we want to prove that

$$E\left[\left|\dot{Y}(t) + \alpha Y(t) - X(t)\right|^2\right] = 0.$$

Thus we can compute

$$E\left[\left(\dot{Y}(t_1) + \alpha Y(t_1) - X(t_1)\right)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right],$$

where we assume the differential equation parameter α is real. Then, we must show that this quantity is zero for $t_1 = t_2 = t$. Expanding, we get

$$E\left[\left(\dot{Y}(t_1) + \alpha Y(t_1) - X(t_1)\right)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right] = E\left[\dot{Y}(t_1)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right]$$

$$+E\left[\alpha Y(t_1)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right] - E\left[X(t_1)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right]$$

$$= \left(\frac{\partial^2 R_{YY}(t_1,t_2)}{\partial t_1 \partial t_2} + \alpha \frac{\partial R_{YY}(t_1,t_2)}{\partial t_1} - \frac{\partial R_{YX}(t_1,t_2)}{\partial t_1}\right) + \alpha\left(\frac{\partial R_{YY}(t_1,t_2)}{\partial t_2} + \alpha R_{YY}(t_1,t_2) - R_{YX}(t_1,t_2)\right)$$

$$-\left(\frac{\partial R_{XY}(t_1,t_2)}{\partial t_2} + \alpha R_{XY}(t_1,t_2) - R_{XX}(t_1,t_2)\right), \tag{18.3}$$

where we have ample use of the linearity of the m.s. derivative and the expectation rules of Theorem 10.1-5. We notice that the last term is zero by (18.1), thus the rhs reduces to

$$\left(\frac{\partial^2 R_{YY}(t_1,t_2)}{\partial t_1 \partial t_2} + \alpha \frac{\partial R_{YY}(t_1,t_2)}{\partial t_1} - \frac{\partial R_{YX}(t_1,t_2)}{\partial t_1}\right) + \alpha\left(\frac{\partial R_{YY}(t_1,t_2)}{\partial t_2} + \alpha R_{YY}(t_1,t_2) - R_{YX}(t_1,t_2)\right).$$

Here the last term is zero by (18.2). To see this, we must first take the conjugate of (18.2) obtaining

$$\begin{aligned}\frac{\partial R_{YY}(t_2,t_1)}{\partial t_1} + \alpha R_{YY}(t_2,t_1) &= R_{XY}^*(t_1,t_2) \\ &= R_{YX}(t_2,t_1)\end{aligned}$$

and then since t_1 and t_2 are arbitrary, we can exchange them to obtain

$$\frac{\partial R_{YY}(t_1,t_2)}{\partial t_2} + \alpha R_{YY}(t_1,t_2) - R_{YX}(t_1,t_2). \tag{18.4}$$

So, now we have the rhs of (18.3) reduced to,

$$\frac{\partial^2 R_{YY}(t_1,t_2)}{\partial t_1 \partial t_2} + \alpha \frac{\partial R_{YY}(t_1,t_2)}{\partial t_1} - \frac{\partial R_{YX}(t_1,t_2)}{\partial t_1} = \frac{\partial}{\partial t_1}\left(\frac{\partial R_{YY}(t_1,t_2)}{\partial t_2} + \alpha R_{YY}(t_1,t_2) - R_{YX}(t_1,t_2)\right).$$

Now this last term is zero because it is the derivative of (18.4). We thus conclude that

$$E\left[\left(\dot{Y}(t_1) + \alpha Y(t_1) - X(t_1)\right)\left(\dot{Y}^*(t_2) + \alpha Y^*(t_2) - X^*(t_2)\right)\right] = 0,$$

and so setting $t_1 = t_2 = t$, is is specially true that

$$E\left[\left|\dot{Y}(t) + \alpha Y(t) - X(t)\right|^2\right] = 0.$$

19.

$$\begin{aligned} S_{XX}(\omega) &= \frac{1}{(\omega^2+1)^2} \\ &= \frac{1}{(j\omega+1)(j\omega+1)(-j\omega+1)(-j\omega+1)} \\ &= H(\omega)H(-\omega) \\ &= H(\omega)H^*(\omega). \end{aligned}$$

So, to be stable and causal, we choose the two factors $(j\omega+1)(j\omega+1)$ to compose $H(\omega)$, and thus obtain

$$H(\omega) = \frac{1}{(j\omega+1)(j\omega+1)},$$

or

$$\begin{aligned} H(s) &= \frac{1}{(s+1)^2} \\ &= \frac{1}{s^2+2s+1}. \end{aligned}$$

The differential equation is then

$$\frac{d^2X(t)}{dt^2} + 2\frac{dX(t)}{dt} + X(t) = W(t).$$

20. We are given the generalized m.s. differential equation

$$\frac{dX}{dt} + 3X(t) = \frac{dW}{dt} + 2W(t), \quad t \geq t_0,$$

where $X(t_0) = X_0 \perp W(t), \mu_{X_0} = \mu_W(t) = 0$, and psd $S_{WW}(\omega) = 1$ for all ω, i.e. $-\infty < \omega < +\infty$.

(a) So, for $t > t_0$, we get

$$X(t) = Ae^{-3(t-t_0)} + \int_{t_0}^{t} h(t-\nu)W(\nu)d\nu,$$

with $h(t) = \delta(t) - e^{-3t}u(t)$. This becomes

$$X(t) = (X_0 - W(t_0)e^{-3(t-t_0)} + W(t) - \int_{t_0}^{t} e^{-3(t-\nu)}W(\nu)d\nu. \qquad \text{((A))}$$

(b)

$$\begin{aligned} S_{XX}(\omega) &= |H(j\omega)|^2 S_{WW}(\omega) \\ &= \left|\frac{j\omega+2}{j\omega+3}\right|^2 \cdot 1 \\ &= \frac{\omega^2+4}{\omega^2+9}. \end{aligned}$$

(c)

$$\begin{aligned} S_{XX}(s) &= \frac{(s+2)(-s+2)}{(s+3)(-s+3)} \cdot 1 \\ &= \left(1 - \frac{1}{s+3}\right)\left(1 - \frac{1}{-s+3}\right) \\ &= 1 - \frac{1}{s+3} - \frac{1}{-s+3} + \frac{1}{(s+3)(-s+3)}. \end{aligned}$$

Which implies the autocorrelation function result

$$\begin{aligned} R_{XX}(\tau) &= \begin{cases} \delta(\tau) - e^{-3\tau} + \frac{1}{3+3}e^{-3\tau}, & \tau \geq 0, \\ \delta(\tau) - e^{+3\tau} + \frac{1}{3+3}e^{+3\tau}, & \tau \leq 0, \end{cases} \\ &= \delta(\tau) - \frac{5}{6}e^{-3|\tau|}, \quad -\infty < \tau < +\infty. \end{aligned}$$

(d) By equation (A) above, even if $W(t)$ is Gaussian, still $X(t)$ is not Markov. This because, as we can see from (A), for $t > t_0$, $X(t)$ has direct dependence on more than just $X(t_0)$.

21. We have

$$\widehat{X}(t) = \int_{-\infty}^{+\infty} h(t-\tau)(X(\tau)+N(\tau))d\tau \qquad \text{(if it exists)},$$

so the constraint $E[|\widehat{X}(t)|^2] < \infty$ becomes

(a)

$$E[|\widehat{X}(t)|^2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t-\tau_1)h^*(t-\tau_2)[R_{XX}(\tau_1-\tau_2) + R_{NN}(\tau_1-\tau_2)]d\tau_1 d\tau_2 < \infty,$$

where we have used the orthogonality of the X and N processes.

(b) In the Fourier transform domain we have

$$\begin{aligned}
E[|\widehat{X}(t)|^2] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\widehat{X}\widehat{X}}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 \left(S_{XX}(\omega)+S_{NN}(\omega)\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{S_{XX}(\omega)}{S_{XX}(\omega)+S_{NN}(\omega)}\right)^2 \left(S_{XX}(\omega)+S_{NN}(\omega)\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{S_{XX}^2(\omega)}{S_{XX}(\omega)+S_{NN}(\omega)} d\omega < \infty.
\end{aligned}$$

(c) Since the WSS process $X(t)$ is second-order, i.e. $E[|X(t)|^2] < \infty$, we know $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega < \infty$. Thus, we can show the condition in part (b) as follows

$$\begin{aligned}
E[|\widehat{X}(t)|^2] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)\left(\frac{S_{XX}(\omega)}{S_{XX}(\omega)+S_{NN}(\omega)}\right) d\omega \\
&\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega, \quad \text{since } 0 \le \frac{S_{XX}(\omega)}{S_{XX}(\omega)+S_{NN}(\omega)} \le 1, \\
&< \infty, \quad \text{since } X \text{ is second-order.}
\end{aligned}$$

22. We have two hypotheses

$$\left.\begin{array}{ll} H_0: & R(t) = W(t) \\ H_1: & R(t) = A + W(t) \end{array}\right\} 0 \le t \le T$$

$$\Lambda = \int_0^T R(t)dt$$

(a) (i) Under hypothesis H_0 :

$$\begin{aligned}
E[\Lambda|H_0] &= E\left[\int_0^T R(t)dt|H_0\right] \\
&= E\left[\int_0^T W(t)dt\right] \\
&= E[0] = 0.
\end{aligned}$$

(ii) Under hypothesis H_1 :

$$\begin{aligned}
E[\Lambda|H_1] &= E\left[\int_0^T R(t)dt|H_1\right] \\
&= E\left[\int_0^T (A + W(t))\, dt\right] \\
& \quad E\left[\int_0^T Adt\right] \\
&= E[AT] = AT.
\end{aligned}$$

(b) For the variances, under H_0 :

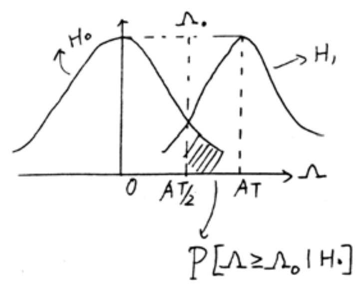
$$\begin{aligned}
\sigma_\Lambda^2 &= E[(\Lambda-\mu_\Lambda)^2|H_0] \\
&= E\left[\left(\int_0^T W(t)dt - \int_0^T \mu_W(t)dt\right)^2 |H_0\right] \\
&= E\left[\left(\int_0^T W(t)dt - 0\right)^2\right] \\
&= E\left[\int_0^T\int_0^T W(t_1)W(t_2)dt_1dt_2\right] \\
&= \int_0^T\int_0^T \sigma^2\delta(t_1-t_2)dt_1dt_2 \\
&= \sigma^2 T.
\end{aligned}$$

Under hypothesis H_1, the variance is the same since there is only a shift in the DC value. Thus under each hypothesis $\sigma_\Lambda^2 = \sigma^2 T$.

(c)

$$P[\Lambda \geq \Lambda_0|H_0] = \frac{1}{\sqrt{2\pi\sigma_\Lambda^2}}\int_{\Lambda_0}^\infty e^{-\frac{\alpha^2}{2\sigma_\Lambda^2}}d\alpha,$$

where $\Lambda_0 \triangleq AT/2$. Let $\frac{\alpha}{\sigma_\Lambda} = \eta$, then $d\alpha = \sigma_\Lambda d\eta$. Also $\alpha = \frac{AT}{2}$, which implies $\eta = \frac{AT}{2\sigma_\Lambda}$.



Then

$$\begin{aligned}
P[\Lambda \geq \Lambda_0|H_0] &= \frac{\sigma_\Lambda}{\sqrt{2\pi}\sigma_\Lambda}\int_{\frac{AT}{2\sigma_\Lambda}}^\infty e^{-\frac{\eta^2}{2}}d\eta \\
&= \frac{1}{2} - \operatorname{erf}\left(\frac{AT}{2\sigma_\Lambda}\right) \\
&= \frac{1}{2} - \operatorname{erf}\left(\frac{AT}{2\sigma\sqrt{T}}\right) \\
&= \frac{1}{2} - \operatorname{erf}\left(\frac{A\sqrt{T}}{2\sigma}\right).
\end{aligned}$$

23. (a) Let

$$I_T \triangleq \frac{1}{2T}\int_{-T}^{+T} X(t)dt.$$

Then, if $E[I_T] = E[X(t)] = \mu_X$, a constant, and if

$$\sigma_{I_T}^2 \triangleq E[(I_T - \mu_X)^2] \underset{T\to\infty}{\longrightarrow} 0,$$

then $X(t)$ is ergodic in the mean.

(b) Using the sufficient condition, i.e.

$$\begin{aligned}\int_{-\infty}^{+\infty} K_{XX}(\tau)d\tau &< \infty, \text{ or} \\ \int_{-\infty}^{+\infty} R_{XX}(\tau)d\tau &< \infty,\end{aligned}$$

in the case of zero mean, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \sigma^2 e^{-\alpha|\tau|}\cos(2\pi f\tau)d\tau &= 2\int_{0}^{+\infty} \sigma^2 e^{-\alpha\tau}\left(\frac{e^{j2\pi f\tau}+e^{-j2\pi f\tau}}{2}\right)d\tau \\ &= \sigma^2\left[\frac{e^{(-\alpha+j2\pi f)\tau}}{-\alpha+j2\pi f} - \frac{e^{-(\alpha+j2\pi f)\tau}}{\alpha+j2\pi f}\right]\Bigg|_0^\infty \\ &= -\sigma^2\left[\frac{1}{-\alpha+j2\pi f} - \frac{1}{\alpha+j2\pi f}\right] \\ &= \frac{2\alpha\sigma^2}{\alpha^2+(2\pi f)^2} < \infty.\end{aligned}$$

Therefore, the given random process $X(t)$ is ergodic in the mean.

24. Since $X(t)$ is ergodic in the mean,

$$\frac{1}{2T}\int_{-2T}^{+2T}\left(1-\frac{|\tau|}{2T}\right)K_{XX}(\tau)d\tau \underset{T\to\infty}{\longrightarrow} 0.$$

Now, as shown in Theorem 10.4-4, this can only happen together with $K_{XX}(\tau) \underset{|\tau|\to\infty}{\longrightarrow}$ a constant. If this constant is zero, i.e.

$$\lim_{|\tau|\to\infty} K_{XX}(\tau) = 0.$$

But, always we have

$$R_{XX}(\tau) = K_{XX}(\tau) + |\mu_X(\tau)|^2,$$

so

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = |\mu_X(\tau)|^2$$

under the stated conditions.

25. (a) Let $X_c = X - \mu_X$, then

$$\begin{aligned}E[|\widehat{M}-\mu_X|^2] &= \frac{1}{N^2}E\left[\left|\sum_{n=1}^{N} X_c[n]\right|^2\right] \\ &= \frac{1}{N^2}\sum_{n,m=1}^{N} K_{XX}[n-m] \underset{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

(b)

$$\begin{aligned} E[|\widehat{M}-\mu_X|^2] &= \frac{1}{N^2}[NK_{XX}[0]+(N-1)(K_{XX}[+1]+K_{XX}[-1])+ \\ &\quad +(N-2)(K_{XX}[+2]+K_{XX}[-2])+\cdots] \\ &= \frac{1}{N}\sum_{m=-N}^{+N}\left(1-\frac{|m|}{N}\right)K_{XX}[m] \underset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

(c) We calculate

$$\begin{aligned} \sum_{m=-\infty}^{+\infty}|K_{XX}[m]| &\leq 5\sum_{m=-\infty}^{+\infty}0.9^{|m|}+15\sum_{m=-\infty}^{+\infty}0.8^{|m|} \\ &< \infty. \end{aligned}$$

So,

$$\frac{1}{N}\sum_{m=1}^{N}|K_{XX}[m]| \underset{N\to\infty}{\longrightarrow} 0,$$

which implies that $X[n]$ is ergodic in the mean.

26. Define the one-parameter covariance function $K_{XX}(\tau) \triangleq K_{XX}(s+\tau, s) = \sigma^2\cos\omega_0\tau$, which is seen to be periodic in τ and independent of s. Then since the mean μ_X is constant, we have a WSS periodic random process. For these processes, the Fourier series expansion coefficients are orthogonal, i.e. the Fourier series basis set is also the Karhunen-Loeve basis set. The period of $K_{XX}(\tau)$ is $T \triangleq \frac{2\pi}{\omega_0}$. Hence, any interval of the time axis of width T will do for the expansion.

27. By the equation (10.5-3),

$$\int_{-T/2}^{T/2}K_{XX}(t_1,t_2)\phi_1(t_2)\,dt_2 = \lambda_1\phi_1(t_1),$$

we have

$$\begin{aligned} \int_{-T/2}^{T/2}\phi_1(t)\phi_2^*(t)dt &= \frac{1}{\lambda_1}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}K_{XX}(t,t_2)\phi_1(t_2)\phi_2^*(t)dtdt_2 \\ &= \frac{1}{\lambda_1}\int_{-T/2}^{T/2}\phi_1(t_2)\left(\int_{-T/2}^{T/2}K_{XX}^*(t_2,t)\phi_2^*(t)dt\right)dt_2 \\ &= \frac{1}{\lambda_1}\int_{-T/2}^{T/2}\phi_1(t_2)\lambda_2^*\phi_2^*(t_2)dt_2 \\ &= \frac{\lambda_2^*}{\lambda_1}\int_{-T/2}^{T/2}\phi_1(t)\phi_2^*(t)dt \\ &= \frac{\lambda_2}{\lambda_1}\int_{-T/2}^{T/2}\phi_1(t)\phi_2^*(t)dt, \text{ since the } \lambda_i \text{ are real.} \end{aligned}$$

Because $\lambda_1 \neq \lambda_2$, it must be that the indicated integral is zero.

28. Using Cauchy convergence, we consider $E[|\widehat{S}_N(t) - \widehat{S}_M(t)|^2]$, where

$$\begin{aligned}\widehat{S}_N(t) &\triangleq \sum_{n=1}^{N} \frac{\sigma^2_{S_n}}{\sigma^2_{S_n} + \sigma^2_{N_n}} X_n \phi_n(t) \\ &= \sum_{n=1}^{N} \beta_n X_n \phi_n(t), \quad \text{with } \beta_n \triangleq \frac{\sigma^2_{S_n}}{\sigma^2_{S_n} + \sigma^2_{N_n}} \ .\end{aligned}$$

We get three terms, the general form of which is the middle term

$$\begin{aligned}E[\widehat{S}_N(t)\widehat{S}^*_M(t)] &= \sum_{n,m=1}^{N,M} \beta_n \beta_m E[X_n X^*_m] \phi_n(t) \phi^*_m(t) \\ &= \sum_{n=1}^{\min(N,M)} \beta^2_n \lambda_n \left|\phi_n(t)\right|^2, \ \text{ by orthogonality of the } X_n.\end{aligned}$$

Now, we know from Mercer's Theorem that

$$R(t,t) = \sum_{n=1}^{\infty} \lambda_n \left|\phi_n(t)\right|^2 < \infty,$$

for the second-order process $X(t)$. So, since the β_n are non-negative and less than one, we must have

$$\begin{aligned}\sum_{n=1}^{\infty} \beta^2_n \lambda_n \left|\phi_n(t)\right|^2 &\le \sum_{n=1}^{\infty} \lambda_n \left|\phi_n(t)\right|^2 \\ &< \infty,\end{aligned}$$

so it must exist and be finite, call it A. Thus

$$\begin{aligned}E[|\widehat{S}_N(t) - \widehat{S}_M(t)|^2] &= E[|\widehat{S}_N(t)|^2] - 2E[\widehat{S}_N(t)\widehat{S}^*_M(t)] + E[|\widehat{S}_M(t)|^2] \\ &\to A - 2A + A, \\ &= 0 \quad \text{as } N, M \to \infty.\end{aligned}$$

29. a) Since

$$\begin{aligned}\int_{-T/2}^{+T/2} \sigma^2_W \delta(t-s) \phi_n(s) ds &= \sigma^2_W \phi_n(t) \\ &= \lambda_n \phi_n(t),\end{aligned}$$

we have $\lambda_n = \sigma^2_W$ for all $n (\ge 1)$. Therefore, any set of orthonormal functions $\{\phi_n(t)\}$ will satisfy the K-L integral equation for the white noise process $W(t)$.

b) Since $R_{XX}(t,s) = R_{SS}(t,s) + \sigma^2_W \delta(t-s)$

$$\begin{aligned}\lambda^{(X)}_n \phi^{(X)}_n(t) &= \int_{-T/2}^{+T/2} R_{XX}(t,s) \phi^{(X)}_n(s) ds \\ &= \int_{-T/2}^{+T/2} \left(R_{SS}(t,s) + \sigma^2_W \delta(t-s)\right) \phi^{(X)}_n(s) ds \\ &= \int_{-T/2}^{+T/2} R_{SS}(t,s) \phi^{(X)}_n(s) ds + \int_{-T/2}^{+T/2} \sigma^2_W \delta(t-s) \phi^{(X)}_n(s) ds \\ &= \int_{-T/2}^{+T/2} R_{SS}(t,s) \phi^{(X)}_n(s) ds + \sigma^2_W \phi^{(X)}_n(t),\end{aligned}$$

so we have

$$\int_{-T/2}^{+T/2} R_{SS}(t,s)\phi_n^{(X)}(s)ds = (\lambda_n^{(X)} - \sigma_W^2)\phi_n^{(X)}(t),$$

which means that both $S(t)$ and $X(t)$ have the same K-L orthonormal basis functions, i.e. $\phi_n^{(X)}(t) = \phi_n^{(S)}(t)$ and the eigenvalues of $S(t)$ are $\lambda_n^{(S)} = \lambda_n^{(X)} - \sigma_W^2$, or equivalently $\lambda_n^{(X)} = \lambda_n^{(S)} + \sigma_W^2$. We can then express the K-L expansion of $X(t)$ in terms of the basis functions $\phi_n^{(S)}(t)$ notationally as

$$X(t) = S(t) + W(t) \sim \{\lambda_n^{(S)} + \sigma_W^2, \phi_n^{(S)}(t)\}.$$

c) Let the linear estimate be

$$\widehat{E}[S_n|X_n] = \alpha X_n,$$

then by the orthogonality principle, we must have

$$\left(S_n - \widehat{E}[S_n|X_n]\right) \perp X_n,$$

which is the same as

$$\begin{aligned}
0 &= E\left[\left(S_n - \widehat{E}[S_n|X_n]\right) X_n^*\right] \\
&= E[(S_n - \widehat{S}_n)X_n^*] \\
&= E[(S_n - \alpha X_n)X_n^*],
\end{aligned}$$

so

$$\begin{aligned}
\alpha &= \frac{E[S_n X_n^*]}{E[|X_n|]} \\
&= \frac{E[S_n(S_n+W_n)^*]}{E[|S_n+W_n|^2]} \\
&= \frac{\sigma_{S_n}^2}{\sigma_{S_n}^2 + \sigma_W^2},
\end{aligned}$$

using the fact that $S_n \perp W_n$.

d) For all $t, \tau \in [0,T]$ we can write

$$\begin{aligned}
E\left[\left(S(t) - \widehat{S}(t)\right) X^*(\tau)\right] &= \sum_{n,m} E[(S_n - \widehat{S}_n)X_m^*]\phi_n(t)\phi_m^*(\tau) \\
&= \sum_{n} E[(S_n - \widehat{S}_n)X_n^*]\phi_n(t)\phi_n^*(\tau),
\end{aligned}$$

by orthogonality of the coefficients for distinct m and n and the orthogonality of S_n and X_m. But $E[(S_n - \widehat{S}_n)X_n^*] = 0$ by result of part c), thus

$$E\left[\left(S(t) - \widehat{S}(t)\right) X^*(\tau)\right] = 0,$$

and the result follows that for all $t, \tau \in [0,T]$, we can write

$$\begin{aligned}
\widehat{S}(t) &= \sum_{n=1}^{\infty} \widehat{S}_n \phi_n(t) \\
&= \sum_{n=1}^{\infty} \frac{\sigma_{S_n}^2}{\sigma_{S_n}^2 + \sigma_W^2} X_n \phi_n(t).
\end{aligned}$$

30. (a) Since the noise process is Gaussian, the K-L expansion ensures independence of the transformed coefficients. The other $R_k's$ are thus independent of R_{k_o}, which is the only one containing the message. Thus

$$P[R_{k_o} \leq r | \{ \text{ all other } R_k's \}] = P[R_{k_o} \leq r].$$

(b) Since λ_k is the noise mean-square level on basis function (channel) k, we want the smallest λ_k for the signaling channel. So we want $k_o = \infty$. Of course, practical conditions would intercede in reality, forcing a lower finite choice.

31. Assuming that N is even, let's define

$$X_k \triangleq \sum_{n=-N/2}^{+N/2} X[n]\phi_k^*[n],$$

where the $\phi_k[n]$ constitute orthonormal sequences. Then

$$\begin{aligned} E[X_k X_l^*] &= \sum_{n=-N/2}^{+N/2} \sum_{m=-N/2}^{+N/2} E[X[n]X^*[m]]\phi_k^*[n]\phi_l[m] \\ &= \sum_{n,m} R[n,m]\phi_k^*[n]\phi_l[m]. \end{aligned}$$

But, from Mercer's Theorem

$$R[n,m] = \sum_{k=1}^{N+1} \lambda_k \phi_k[n]\phi_k^*[m].$$

Therefore, upon changing the dummy variable of this summation from k to i, we get

$$\begin{aligned} E[X_k X_l^*] &= \sum_{n,m} \left(\sum_{i=1}^{N+1} \lambda_i \phi_i[n]\phi_i^*[m] \right) \phi_k^*[n]\phi_l[m] \\ &= \sum_{i=1}^{N+1} \lambda_i \left(\sum_n \phi_i[n]\phi_k^*[n] \right) \left(\sum_m \phi_l[m]\phi_i^*[m] \right) \\ &= \sum_{i=1}^{N+1} \lambda_i \delta[i-k]\delta[l-i] \\ &= \lambda_k \delta[k-l], \end{aligned}$$

that is, the X_k are orthogonal random variables.

$$\begin{aligned} E\left[\left| X[n] - \sum_{k=1}^{N+1} X_k \phi_k[n] \right|^2 \right] &= R[n,n] - E\left[\sum_{k=1}^{N+1} X[n] X_k^* \phi_k^*[n] \right] \\ &\quad - E\left[\sum_{k=1}^{N+1} X^*[n] X_k \phi_k[n] \right] + E\left[\sum_{k,l} X_k X_l^* \phi_k^*[n]\phi_l[n] \right]. \end{aligned}$$

But,

$$\begin{aligned}
E\left[\sum_{k=1}^{N+1} X[n]X_k^*\phi_k^*[n]\right] &= E\left[\sum_{k,m} X[n]X^*[m]\phi_k[m]\phi_k^*[n]\right] \\
&= \sum_{k,m} E[X[n]X^*[m]]\phi_k[m]\phi_k^*[n] \\
&= \sum_k \left(\sum_m R[n,m]\phi_k[m]\right)\phi_k^*[n] \\
&= \sum_{k=1}^{N+1} \lambda_k\phi_k[n]\phi_k^*[n] \\
&= R[n,n].
\end{aligned}$$

In the same way, one can show that

$$\begin{aligned}
E\left[\sum_{k=1}^{N+1} X^*[n]X_k\phi_k[n]\right] &= \sum_{k=1}^{N+1} \lambda_k\phi_k[n]\phi_k^*[n] \\
&= R[n,n].
\end{aligned}$$

Therefore, plugging in these results, we have that

$$\begin{aligned}
E\left[\left|X[n] - \sum_{k=1}^{N+1} X_k\phi_k[n]\right|^2\right] &= R[n,n] - 2R[n,n] + R[n,n] \\
&= 0.
\end{aligned}$$

So we can say

$$\begin{aligned}
X[n] &= \sum_{k=1}^{N+1} X_k\phi_k[n] \qquad \text{(m.s.)}, \\
\text{with } \; X_k &\triangleq \sum_{n=-N/2}^{+N/2} X[n]\phi_k^*[n].
\end{aligned}$$

32. $K_{XX}(t,s) = 3 + 2\cos\frac{2\pi t}{T}\cos\frac{2\pi s}{T} + \cos\frac{4\pi t}{T}\cos\frac{4\pi s}{T}$ on the interval $[0,T]$.

   (a) By definition, positive semidefinite is:

$$\sum_k\sum_l a_k a_l^* K_{XX}(t_k,t_l) \geq 0, \qquad \text{for all } a_k, t_k, \text{ and for all positive } n.$$

   Plugging in, answer substituting for t and s, we obtain

$$\begin{aligned}
&\sum_k\sum_l a_k a_l^*\left(3 + 2\cos\frac{2\pi t_k}{T}\cos\frac{2\pi t_l}{T} + \cos\frac{4\pi t_k}{T}\cos\frac{4\pi t_l}{T}\right) \\
&= 3\left|\sum_k a_k\right|^2 + 2\left|\sum_k a_k\cos\frac{2\pi t_k}{T}\right|^2 + \left|\sum_k a_k\cos\frac{4\pi t_k}{T}\right|^2 \\
&\geq 0, \quad \text{since sum of three non-negative quantities.}
\end{aligned}$$

(b) Mercer's Theorem:

$$R_{XX}(t,s) = \sum_{n=0}^{\infty} \lambda_n \phi_n(t)\phi_n^*(s).$$

Hence, $\phi_0(t) = k, \phi_1(t) = k_1 \cos\frac{2\pi t}{T}, \phi_2(t) = k_2 \cos\frac{4\pi t}{T}$. Normalizing these orthogonal functions by $\int_0^T |\phi_n(t)|^2 dt = 1$, we get

$$k = \sqrt{\frac{1}{T}}, k_1 = \sqrt{\frac{2}{T}} = k_2.$$

Then, we can compute the lambdas as: $\lambda_0 = 3T, \lambda_1 = T$, and $\lambda_2 = T/2$. These are the variances of the three RV coefficients $X_{0,}X_1$, and X_2, the only non-zero terms in the K-L expansion for the given covariance function:

$$X(t) = \sum_{n=0}^{2} X_n \phi_n(t),$$

where the coefficient means $\mu_{X_n} = 0$, and $E[X_k X_l^*] = 0$ for $k \neq l$.

(c) For non-zero mean function:

$$\begin{aligned}
\lambda_n \phi_n(t) &= \int_0^T R_{XX}(t,s)\phi_n(s)ds \\
&= \int_0^T \left(K_{XX}(t,s) + \mu_X(t)\mu_X^*(s)\right)\phi_n(s)ds \\
&= \int_0^T K_{XX}(t,s)\phi_n(s)ds + \left(\int_0^T \mu_X^*(s)\phi_n(s)ds\right)\mu_X(t)
\end{aligned}$$

This effectively expands the mean function in the basis function set $\{\phi_n(t)\}$ also.

(d) By Definition 9.6-1, in the zero-mean case, wide-sense periodic means

$$K_{XX}(t,s) = K_{XX}(t+T,s) = K_{XX}(t,s+T).$$

Clearly, this is true for the $K_{XX}(t,s)$ given in the present problem. However, the periodicity is trivial here, since the whole observation interval $[0,T]$ is the 'period.'

33. We must calculate

$$E\left[\int_0^T X_1(t)X_2^*(t)dt\right],$$

where $X_1(t) = \sum_{n=1}^{\infty} X_{n,1}\phi_n(t)$ and $X_2(t) = \sum_{n=1}^{\infty} X_{n,2}\phi_n(t)$, so

$$\begin{aligned}
E\left[\int_0^T X_1(t)X_2^*(t)dt\right] &= E\left[\int_0^T \left(\sum_{n=1}^{\infty} X_{n,1}\phi_n(t)\right)\left(\sum_{m=1}^{\infty} X_{m,2}\phi_m(t)\right)^* dt\right] \\
&= \int_0^T \sum_{n,m=1}^{\infty} E[X_{n,1}X_{m,2}^*]\phi_n(t)\phi_m^*(t)dt \\
&= \sum_{n,m=1}^{\infty} E[X_{n,1}X_{m,2}^*]\int_0^T \phi_n(t)\phi_m^*(t)dt \\
&= \sum_{n,m=1}^{\infty} E[X_{n,1}X_{m,2}^*]\delta_{n,m}, \quad \text{by orthonormality of basis,} \\
&= \sum_{n,m=1}^{\infty} E[X_{n,1}X_{n,2}^*]
\end{aligned}$$

34. By the equation:

$$R_{ZZ}(\tau) = R_{UU}(\tau) + R_{\breve{U}\breve{U}}(\tau) - jR_{U\breve{U}}(\tau) + jR_{\breve{U}U}(\tau),$$

we have

$$\begin{aligned}
S_{ZZ}(\omega) &= S_{UU}(\omega) + |H(\omega)|^2 S_{UU}(\omega) - jH^*(\omega)S_{UU}(\omega) + jH(\omega)S_{UU}(\omega) \\
&= S_{UU}(\omega) + S_{UU}(\omega) + \operatorname{sgn}(\omega)S_{UU}(\omega) + \operatorname{sgn}(\omega)S_{UU}(\omega) \\
&= 4S_{UU}(\omega)u(\omega),
\end{aligned}$$

where $u(\cdot)$ is the unit step function, here the unit step in frequency.

35. If we repeat a segment of $X(t)$ to construct the periodic process, we may lose the stationarity, so that the periodic process may not have an orthogonal Fourier series expansion. Alternatively, if we repeat the correlation function periodically with period T, we get a WSS periodic process but it is not the same process we get by repeating a segment of $X(t)$. So this won't work either. The difficulty arises because a segment of length T from a WSS process determines $R(\tau)$ for $-T \le \tau \le T$, which implies a period of $2T$ not T.

36. Let $X(t)$ be a WSS periodic random process, i.e. $R_{XX}(\tau + T) = R_{XX}(\tau)$ with period $T = 1/f_0$. We then take

$$\phi_n(t) = \frac{1}{\sqrt{T}}e^{+j2\pi f_0 t},$$

so that

$$\begin{aligned}
\lambda_n\phi_n(t) &= \int_0^T R_{XX}(s)\phi_n(t-s)ds \\
&= \int_0^T \frac{1}{\sqrt{T}}e^{+j2\pi f_0(t-s)}R_{XX}(s)ds \\
&= \frac{1}{\sqrt{T}}e^{+j2\pi f_0 t}\int_0^T R_{XX}(s)e^{-j2\pi f_0 s}ds.
\end{aligned}$$

We thus see

$$\lambda_n = \int_0^T R_{XX}(s)e^{-j2\pi f_0 s}ds,$$

and therefore $\phi_n(t) = \frac{1}{\sqrt{T}}e^{+j2\pi f_0 t}$ are the Karhunen-Loeve basis functions for a WSS periodic random process with period T.

37.

$$\begin{aligned} U(t) &\triangleq \operatorname{Re}\{Z(t)e^{-j\omega_0 t}\} \\ &= X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t, \end{aligned}$$

since $Z \triangleq X + jY$. Then

$$\begin{aligned} E[U(t+\tau)U(t)] &= E\{[X(t+\tau)\cos\omega_0(t+\tau) + Y(t+\tau)\sin\omega_0(t+\tau)]\,[X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t]\} \\ &= R_{XX}(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t + R_{YY}(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t \\ &\quad + R_{XY}(\tau)\cos\omega_0(t+\tau)\sin\omega_0 t + R_{YX}(\tau)\sin\omega_0(t+\tau)\cos\omega_0 t. \end{aligned}$$

Now, as in the text, let $R_{XX}(\tau) = R_{YY}(\tau)$ and $R_{YX}(\tau) = -R_{XY}(\tau)$, and use common trig identities to obtain

$$\begin{aligned} E[U(t+\tau)U(t)] &= R_{XX}(\tau)\cos\omega_0\tau + R_{YX}(\tau)\sin\omega_0\tau \\ &= R_{UU}(\tau). \end{aligned}$$

So $U(t)$ is then WSS if we can show its mean is constant, but

$$\begin{aligned} \mu_U(t) &= E[U(t)] \\ &= \mu_X\cos\omega_0 t + \mu_Y\sin\omega_0 t, \end{aligned}$$

hence we also need $\mu_X = \mu_X = 0$ to make $\mu_U(t) = \mu_U$, a constant, and then we have $\mu_U = 0$. Then the bandpass random process $U(t)$ will be WSS.

38.

$$\begin{aligned} E[|X(t+T) - X(t)|^2] &= E[|X(t+T)|^2 + |X(t)|^2 - X(t+T)X^*(t) - X(t)X^*(t+T)] \\ &= R_{XX}(t+T, t+T) + R_{XX}(t,t) - R_{XX}(t+T,t) - R_{XX}(t,t+T) \\ &= 2R_{XX}(t,t) - R_{XX}(t,t) - R_{XX}(t,t) \\ &= 0, \end{aligned}$$

since by the definition in Chapter 9, Section 9.6, for a wide-sense periodic random process, we have

$$\begin{aligned} \mu_X(t) &= \mu_X(t+T) \quad \text{and} && (38.1) \\ K_{XX}(t_1,t_2) &= K_{XX}(t_1+T,t_2) = K_{XX}(t_1,t_2+T), && (2) \end{aligned}$$

from which it immediately follows that $K_{XX}(t_1,t_2) = K_{XX}(t_1+T,t_2+T)$ also. Here we have used the correlation function version of (38.1) which follows immediately after substitution. Thus the wide-sense periodic property implies mean-square periodicity.

Secondly, we are asked to show using Chebyshev's inequality that for such processes, we actually have $X(t) = X(t+T)$ (pr. 1). Now, by Chebyshev

$$\begin{aligned} P[|X(t+T) - X(t)| > \epsilon] &\leq \frac{E[|X(t+T) - X(t)|^2]}{\epsilon^2} \\ &= \frac{0}{\epsilon^2} \\ &= 0 \quad \text{for all } \epsilon > 0. \end{aligned}$$

Then, define the events

$$A_n \triangleq \left\{ |X(t+T) - X(t)| > \frac{1}{n} \right\}, \qquad n \geq 1.$$

Now, consider the countable union of the A_n, it must contain the event $\{|X(t+T) - X(t)| \neq 0\}$, i.e.

$$\cup_{n=1}^{\infty} A_n \supseteq \{|X(t+T) - X(t)| \neq 0\},$$

thus by the continuity of probability measure P, we have

$$\begin{aligned} P[\{|X(t+T) - X(t)| \neq 0\}] &\leq P[\cup_{n=1}^{\infty} A_n] \\ &\leq \sum_{n=1}^{\infty} P[A_n] \\ &= \sum_{n=1}^{\infty} 0 \\ &= 0. \end{aligned}$$

So, we can say $X(t) = X(t+T)$ (pr. 1).

39. Because of zero mean, we can work with either covariance or correlation function equally well here. We choose to use the correlation function notation. The results given are, for $X(t)$ and $Y(t)$ processes,

$$R_{XX}(t_1, t_2) = R_{YY}(t_1, t_2) \quad \text{and} \quad R_{XY}(t_1, t_2) = -R_{YX}(t_1, t_2)$$

So

$$\begin{aligned} E[S(t_1)S(t_2)] &= E[(X(t_1)\cos\omega_0 t_1 + Y(t_1)\sin\omega_0 t_1)(X(t_2)\cos\omega_0 t_2 + Y(t_2)\sin\omega_0 t_2)] \\ &= R_{XX}(t_1, t_2)[\cos\omega_0 t_1 \cos\omega_0 t_2 + \sin\omega_0 t_1 \sin\omega_0 t_2] \\ &\qquad + R_{XY}(t_1, t_2)[\cos\omega_0 t_1 \sin\omega_0 t_2 - \sin\omega_0 t_1 \cos\omega_0 t_2] \\ &= R_{XX}(t_1 - t_2)\cos\omega_0(t_1 - t_2) - R_{XY}(t_1 - t_2)\sin\omega_0(t_1 - t_2) \\ &= \text{ a function only of the time difference } t_1 - t_2. \end{aligned}$$

For the mean, we have

$$\begin{aligned} E[S(t)] &= E[X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t] \\ &= E[X(t)]\cos\omega_0 t + E[Y(t)]\sin\omega_0 t \\ &= 0\cos\omega_0 t + 0\sin\omega_0 t \\ &= 0, \text{ a constant.} \end{aligned}$$

Therefore the random process $U(t)$ is WSS and its one-parameter correlation function is given by

$$R_{SS}(\tau) = R_{XX}(\tau)\cos\omega_0\tau - R_{XY}(\tau)\sin\omega_0\tau.$$

40. Let a WSS random sequence $X[n]$ be m.s. periodic with integer period T. We can then write

$$X[n] = \sum_{k=0}^{T-1} A_k e^{+j\omega_0 nk}, \qquad \text{(m.s.)}$$

where $\omega_0 \triangleq 2\pi/T$ with random Fourier coefficients

$$A_k \triangleq \frac{1}{T}\sum_{n=0}^{T-1} X[n]e^{-j\omega_0 nk}. \tag{40.1}$$

We prove below that the means and correlations of the A_k are given as:

$$E[A_k] = \mu_X\delta[k] \quad \text{and} \quad E[A_k A_l^*] = \alpha_k\delta[k-l],$$

with mean square

$$\alpha_k \triangleq \frac{1}{T}\sum_{m=0}^{T-1} R_X[m]e^{-j\omega_0 mk}. \tag{40.2}$$

Thus the periodic random sequence $X[n]$ can be expanded in a Fourier series whose coefficients are orthogonal. Also we have

$$R_X[m] = \sum_{k=0}^{T-1} \alpha_k e^{+j\omega_0 mk}, \qquad -\infty < m < +\infty.$$

Here is the **proof:** First we show that the A_k are orthogonal. From the definition in (40.1), we can easily see

$$\begin{aligned}
E[A_k] &= \frac{1}{T}\sum_{n=0}^{T-1} E[X[n]]e^{-j\omega_0 nk} \\
&= \frac{1}{T}\sum_{n=0}^{T-1} \mu_X e^{-j\omega_0 nk} \\
&= \mu_X \frac{1}{T}\sum_{n=0}^{T-1} e^{-j\omega_0 nk} \\
&= \mu_X\delta[k],
\end{aligned}$$

since the set of functions $\{e^{-j\omega_0 nk}\}_{k=0}^{T-1}$ is orthogonal in n over $[0, T-1]$ since $\omega_0 = 2\pi/T$. Then

$$\begin{aligned}
E[A_k^* X[n]] &= \frac{1}{T}\sum_{m=0}^{T-1} E[X^*[m]X[n]]e^{+j\omega_0 mk} \\
&= \frac{1}{T}\sum_{m=0}^{T-1} R_X[n-m]e^{+j\omega_0 mk} \\
&= \left(\frac{1}{T}\sum_{m'=0}^{T-1} R_X[m']e^{+j\omega_0 m'k}\right) e^{+j\omega_0 nk}, \quad \text{with} \quad m' \triangleq n-m, \\
&= \alpha_k e^{+j\omega_0 nk}.
\end{aligned}$$

Then we consider

$$\begin{aligned}
E[A_k A_l^*] &= \frac{1}{T}\sum_{n=0}^{T-1} E[A_l^* X[n]] e^{-j\omega_0 nk} \\
&= \alpha_l \frac{1}{T}\sum_{n=0}^{T-1} e^{+j\omega_0 nl} e^{-j\omega_0 nk} \\
&= \alpha_l \delta[k-l].
\end{aligned}$$

It remains to show that $E\left[\left|X[n]-\sum_{k=0}^{T-1} A_k e^{+j\omega_0 nk}\right|^2\right]=0$. Expanding the left-hand side, we have

$$\begin{aligned}
& E[|X[n]|^2] - \sum_k E[A_k^* X[n]] e^{-j\omega_0 nk} - \sum_k E[A_k X^*[n]] e^{+j\omega_0 nk} \\
& \quad + \sum_{k,l} E[A_k A_l^*] e^{+j\omega_0 (k-l)n} \\
&= R_X[0] - \sum_k \alpha_k - \sum_k \alpha_k^* + \sum_k \alpha_k \\
&= R_X[0] - \sum_k \alpha_k, \qquad \text{since the } \alpha_k \text{ are real valued } (\alpha_k = E[|A_k|^2]).
\end{aligned}$$

Now $R_X[m]=\sum_{k=0}^{T-1}\alpha_k e^{+j\omega_0 mk}$ by the orthogonality of $\{e^{-j\omega_0 nk}\}_{k=0}^{T-1}$ applied to (40.2). So $R_X[0]=\sum_k \alpha_k$, and so $R_X[0]-\sum_k \alpha_k$, and thus $E\left[\left|X[n]-\sum_{k=0}^{T-1} A_k e^{+j\omega_0 nk}\right|^2\right]=0$, completing the proof.

41.

42. Consider the random sequence $X[n]$ defined by

$$X[n] = Y$$

where Y is distributed as $P_Y[-1]=P_Y[0]=P_Y[1]=1/3$. Thus the random sequence consists of only three sample functions $X_1[n]=0$, $X_2[n]=1, X_3[n]=-1$. So, consider three sub-ensembles, each with just one of these sample functions. These sub-ensembles are clearly stationary, therefore $X[n]$ is not ergodic because it contains stationary sub-ensembles. For example $E[X[n]]=0$, which is not equal to. $E[[X_2[n]]=1$. Thus in this case the ensemble (statistical) average is not equal to the time average of every sample function.

43. (a) $y_1 = h * x_1 + g * x_2$, so

$$\begin{aligned}
S_{Y_1X_1}(\omega) &= FT\{R_{Y_1X_1}(\tau)\} \\
&= FT\{E[(h*X_1)(t+\tau)X_1^*(t)] + E[(g*X_2)(t+\tau)X_1^*(t)]\} \\
&= FT\{h(\tau)*R_{X_1X_1}(\tau) + g(\tau)*R_{X_2X_1}(\tau)\}, \\
&= FT\{h(\tau)*R_{X_1X_1}(\tau)\}, \quad \text{since } R_{X_2X_1}(\tau)=0 \text{ since } X_2 \perp X_1, \\
&= H(\omega)\, S_{X_1X_1}(\omega).
\end{aligned}$$

(b)

$$\begin{aligned} S_{Y_2X_2}(\omega) &= FT\{E[Y_2(t+\tau)X_2^*(t)]\} \\ &= FT\{b(\tau) * R_{X_2X_2}(\tau)\}, \quad \text{since } U \perp X_2 \\ &= B(\omega)S_{X_2X_2}(\omega). \end{aligned}$$

(c)

$$\begin{aligned} E[Y_1(t+\tau)Y_2^*(t)] &= E[(h * X_1 + g * X_2)(t+\tau)(U^* + b^* * X_2^*)(t)] \\ &= E[(g * X_2)(t+\tau)(b^* * X_2^*)(t)], \quad \text{since } U \perp X_1 \text{ and } X_2, \\ &= g(\tau) * b^*(-\tau) * R_{X_2X_2}(\tau). \end{aligned}$$

Thus $S_{Y_1Y_2}(\omega) = G(\omega)B^*(\omega)S_{X_2X_2}(\omega)$.

44. (a) Since the noise process is Gaussian, the K-L expansion ensures independence of the transformed coefficients. The other $R_k's$ are thus independent of R_{k_o}, which is the only one containing the message. Thus

$$P[R_{k_o} \leq r | \{ \text{ all other } R_k's\}] = P[R_{k_o} \leq r].$$

(b) Since λ_k is the noise mean-square level on basis function (channel) k, we want the smallest λ_k for the signaling channel. So we want $k_o = \infty$. Of course, practical conditions would intercede in reality, forcing a lower finite choice.

45. (a) We need the existence of $\left.\frac{\partial^2 K_{XX}}{\partial t \partial s}\right|_{t=s}$ for the existence of $X'(t)$. Calculating, we find

$$\begin{aligned} \frac{\partial K_{XX}(t,s)}{\partial s} &= \sigma^2 \frac{\partial}{\partial s} \cos\omega_0(t-s) \\ &= \sigma^2 \omega_0 \sin\omega_0(t-s). \end{aligned}$$

Then

$$\begin{aligned} \left.\frac{\partial}{\partial t}\left(\frac{\partial K_{XX}(t,s)}{\partial s}\right)\right|_{t=s} &= \left.\sigma^2\omega_0^2 \cos\omega_0(t-s)\right|_{t=s} \\ &= \sigma^2\omega_0^2 < \infty. \end{aligned}$$

Since this value is finite, the m.s. derivative exists for all t.

(b) From part a), we have

$$\begin{aligned} \frac{\partial^2 K_{XX}(t,s)}{\partial t \partial s} &= \sigma^2\omega_0^2 \cos\omega_0(t-s) \\ &= K_{X'X'}(t,s). \end{aligned}$$

46. (a)

$$\begin{aligned} E[|X(t+\epsilon) - X(t)|^2] &= R_{XX}(t+\epsilon, t+\epsilon) - R_{XX}(t+\epsilon, t) \\ &\quad - R_{XX}(t, t+\epsilon) + R_{XX}(t,t) \\ &\longrightarrow 0 \quad \text{as} \quad \epsilon \searrow 0, \end{aligned}$$

for $R_{XX}(t,s)$ continuous in t and s.

(b) The integral $I = \int_a^b X(t)dt$ exists in the m.s. sense, for $-\infty < a \leq b < +\infty$, if the two-dimensional deterministic integral

$$\int_a^b \int_a^b R_{XX}(t,s)dtds \text{ exists.}$$

But since $R_{XX}(t,s)$ is here assumed to be continuous, by the quoted fact, this ordinary 2-D Riemann integral will exist for all finite values of $a \leq b$.

47. (a) We use the Cauchy criteria on

$$I_N(t) \triangleq \sum_{i=1}^{N} a(t,\tau_i)X(\tau_i)\Delta\tau_i,$$

and investigate the limit of

$$E[|I_M(t) - I_N(t)|^2]$$

as the positive integers $M, N \nearrow \infty$. The general term to consider is

$$E[I_M I_N^*] = \sum_{i,j} a(t,\tau_i)a^*(t,\tau_j)R_{XX}(\tau_i,\tau_j)\Delta\tau_i\Delta\tau_{j.}$$

Now as $M, N \nearrow \infty$, this general term tends to the ordinary 2-D integral, if it exists,

$$\int_0^t \int_0^t a(t,\tau_1)a^*(t,\tau_2)R_{XX}(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

(b)

$$R_{YY}(t,s) = \int_0^t \int_0^s a(t,\tau_1)a^*(s,\tau_2)R_{XX}(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

Thus

$$\begin{aligned}\frac{\partial R_{YY}}{\partial t} &= \int_0^s a(t,t)a^*(s,\tau_2)R_{XX}(t,\tau_2)d\tau_2 \\ &\quad + \int_0^t \int_0^s \frac{\partial a(t,\tau_1)}{\partial t} a^*(s,\tau_2)R_{XX}(\tau_1,\tau_2)d\tau_1 d\tau_2.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial^2 R_{YY}}{\partial t \partial s} &= a(t,t)a^*(s,s)R_{XX}(t,s) + \int_0^s a(t,t)\frac{\partial a^*(s,\tau_2)}{\partial s} R_{XX}(t,\tau_2)d\tau_2 \\ &\quad + \int_0^t \frac{\partial a(t,\tau_1)}{\partial t} a^*(s,s)R_{XX}(\tau_1,s)d\tau_1 + \int_0^t \int_0^s \frac{\partial a(t,\tau_1)}{\partial t}\frac{\partial a^*(s,\tau_2)}{\partial s} R_{XX}(\tau_1,\tau_2)d\tau_1 d\tau_2.\end{aligned}$$

48.

$$\begin{aligned}S_{XX}(\omega) &= \frac{\omega^2+1}{(\omega^2+4)(\omega^2+9)} \\ &= \frac{(j\omega+1)(-j\omega+1)}{(j\omega+2)(-j\omega+2)(j\omega+3)(-j\omega+3)} \\ &= H(\omega)H^*(\omega).\end{aligned}$$

So

$$\begin{aligned}
\frac{Y(s)}{X(s)} &= H(s) \\
&= \frac{s+1}{(s+2)(s+3)} \\
&= \frac{s+1}{s^2+5s+6}.
\end{aligned}$$

This is equivalent to the differential equation

$$Y^{(2)}(t) + 5Y^{(1)}(t) + 6Y(t) = W^{(1)}(t) + W(t) \qquad \text{(m.s.)}$$