Chapter 9 solutions

1. (a) $E[X_a(t)] = E[X[n]] = \mu_X$ for $n \leq t < n+1$, for all integers n. So the mean of the output analog process $\mu_{X_a}(t) = \mu_X$, a constant.

   (b) Assume times t_1 and t_2 satisfy $\boxed{\begin{array}{c} m \leq t_1 < m+1, \\ \hline n \leq t_2 < n+1, \end{array}}$ then

$$\begin{aligned} E[X_a(t_1)X_a(t_2)] &= R_X[m-n] \\ &= R_X[\lfloor t_1 \rfloor - \lfloor t_2 \rfloor], \end{aligned}$$

   where $\lfloor \cdot \rfloor$ is the least integer function, i.e. truncates down to the next lower integer.

2. (a)

$$\begin{aligned} \mu_X(t) &\triangleq E[X(t)] \\ &= \sum_{n=-\infty}^{+\infty} E[X[n]] \frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \\ &= \mu_X \sum_{n=-\infty}^{+\infty} 1 \frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \\ &= \mu_X \cdot 1 \quad \text{(since } g(t)=1 \text{ is bandlimited with samples } g(nT)=1) \\ &= \mu_X. \end{aligned}$$

   (b)

$$\begin{aligned} R_{XX}(t_1,t_2) &\triangleq E[X(t_1)X^*(t_2)] \\ &= \sum_{\text{all } m,n} R_{XX}[m-n] \text{sinc}\left(\frac{t_1-mT}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right), \end{aligned}$$

   where the sinc function is defined as $\text{sinc}(\tau) \triangleq \frac{\sin \pi\tau}{\pi\tau}$, so we can write the above as

$$\begin{aligned} R_{XX}(t_1,t_2) &= R_{XX}[0] \cdot \sum_n \text{sinc}\left(\frac{t_1-nT}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right) \\ &\quad + R_{XX}[1] \cdot \sum_n \text{sinc}\left(\frac{t_1-(n+1)T}{T}\right) \text{sinc}\left(\frac{t_2-nT}{T}\right) + \cdots \\ &= \sum_m R_{XX}[m] r_m(t_1,t_2), \end{aligned}$$

   where $r_m(t_1,t_2) \triangleq \sum_n \text{sinc}\left(\frac{t_1-(n+m)T}{T}\right)\text{sinc}\left(\frac{t_2-nT}{T}\right)$, so then we have

$$\begin{aligned} R_{XX}(t_1,t_2) &= \sum_m R_{XX}[m] \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right) \quad \text{(see below)} \\ &= R_{XX}(t_1-t_2) \quad \text{and so } X(t) \text{ is WSS.} \end{aligned}$$

   To see that $r_m(t_1,t_2) = \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right)$, we proceed as follows: Fix m and t_1 and define the function $g(t_2) \triangleq \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right)$. Clearly g is bandlimited with coefficients $g(nT) = \text{sinc}\left(\frac{t_1-(n+m)T}{T}\right)$, so that

$$\begin{aligned}
g(t_2) &= \text{sinc}\left(\frac{(t_1-t_2)-mT}{T}\right) \\
&= \sum_n g(nT)\text{sinc}\left(\frac{t_2-nT}{T}\right) \\
&= \sum_n \text{sinc}\left(\frac{t_1-(n+m)T}{T}\right)\text{sinc}\left(\frac{t_2-nT}{T}\right).
\end{aligned}$$

3. The random sequence $B[n]$ is Bernoulli and its values ± 1 occur with equal probabilities $1/2$. We have $X(t) \triangleq \sqrt{p}\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)$ where $\sqrt{p}$ and f_0 are given real numbers.

   (a)

$$\begin{aligned}
\mu_X(t) &\triangleq E[X(t)] \\
&= E\left[\sqrt{p}\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)\right] \\
&= \sqrt{p}E\left[\sin\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)\right] \\
&= \sqrt{p}\left(\frac{1}{2}\sin\left(2\pi f_0 t + \frac{\pi}{2}\right) + \frac{1}{2}\sin\left(2\pi f_0 t - \frac{\pi}{2}\right)\right) \\
&= \sqrt{p}\left(\frac{1}{2}\cos(2\pi f_0 t) + \frac{1}{2}(-\cos(2\pi f_0 t))\right) \\
&= \sqrt{p}\left(\frac{1}{2}\cos(2\pi f_0 t) - \frac{1}{2}\cos(2\pi f_0 t)\right) \\
&= 0.
\end{aligned}$$

   (b) For this real-valued process, $K_{XX}(t,s) = E[X(t)X(s)]$ since the means are zero. To evaluate $E[X(t)X(s)]$, we consider two cases:

   (i) Case 1: $nT \le t, s < (n+1)T$, i.e. t and s are in the same half-open interval $[nT, (n+1)T)$. Then

$$\begin{aligned}
E[X(t)X(s)] &= \frac{1}{2}\sqrt{p}\sin\left(2\pi f_0 t + \frac{\pi}{2}\right)\sqrt{p}\sin\left(2\pi f_0 s + \frac{\pi}{2}\right) + \frac{1}{2}\sqrt{p}\sin\left(2\pi f_0 t - \frac{\pi}{2}\right)\sqrt{p}\sin\left(2\pi f_0 s - \frac{\pi}{2}\right) \\
&= \frac{1}{2}p\cos(2\pi f_0 t)\cos(2\pi f_0 s) + \frac{1}{2}p\cos(2\pi f_0 t)\cos(2\pi f_0 s) \\
&= p\cos(2\pi f_0 t)\cos(2\pi f_0 s).
\end{aligned}$$

   (ii) Case 2: $nT \le t < (n+1)T, mT \le s < (m+1)T$, with $n \ne m$, i.e. t and s are in different intervals. In this case $X(t)$ and $X(s)$ are independent, so $E[X(t)X(s)] = E[X(t)]E[X(s)]$, but here the means are zero, hence $E[X(t)X(s)] = 0$.

   Combining the two cases we can write

$$\begin{aligned}
K_{XX}(t,s) &= E[X(t)X(s)] \\
&= \begin{cases} p\cos(2\pi f_0 t)\cos(2\pi f_0 s), & nT \le t, s < (n+1)T \text{ for some integer } n. \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

4.

$$Y(t) = \sum_{n=0}^{N-1} A_n X(t - nT).$$

(a)

$$\begin{aligned}
E[Y(t)Y(t+\tau) &= \sum_{n,m} E[A_n A_m] E[X(t)X(t+\tau)] \\
&= \sum_{n=0}^{N-1} \sigma_A^2 R_{XX}(\tau) \qquad \text{(terms for } n \neq m \text{ are zero)} \\
&= N\sigma_A^2 R_{XX}(\tau).
\end{aligned}$$

(b)

$$\begin{aligned}
\Phi_Y(\omega) &\triangleq E[e^{+j\omega Y(t)}] \\
&= E\left[e^{+j\omega \sum_n A_n X(t-nT)}\right] \\
&= E\left[\prod_n \exp\left(j\omega A_n X(t-nT)\right)\right].
\end{aligned}$$

Now the A_n are jointly independent and also independent of the random process $X(t)$. Also $R_{XX}(nT) = 0$, all of the $X(t-nT)$ for fixed t, are also jointly independent, since Gaussian. Therefore, all the terms in the above product are jointly independent, thus

$$\Phi_Y(\omega) = \prod_n E[\exp\left(j\omega A_n X(t-nT)\right)].$$

Next, use conditional expectation to write $E[\exp\left(j\omega A_n X(t-nT)\right)] = E[E[\exp\left(j\omega A_n X(t-nT)\right) | A_n]]$, where $E[\exp\left(j\omega A_n X(t-nT)\right) | A_n = a_n] = \exp(-\frac{1}{2}R_{XX}(0)\omega^2 a_n^2)$. Then we have

$$\begin{aligned}
E[\exp\left(j\omega A_n X(t-nT)\right)] &= E[\exp(-\frac{1}{2}R_{XX}(0)\omega^2 A_n^2)] \\
&= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma_X^2\omega^2 a^2} \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{1}{2}\frac{a^2}{\sigma_A^2}}\, da \qquad \text{(with } \sigma_X^2 = R_{XX}(0)\text{)} \\
&= \frac{1}{\sqrt{2\pi\sigma_A^2}} \int_{-\infty}^{+\infty} e^{-\frac{a^2}{2}\left(\sigma_X^2\omega^2 + \frac{1}{\sigma_A^2}\right)} da \\
&= \left(\sigma_X^2\sigma_A^2\omega^2 + 1\right)^{-\frac{1}{2}}.
\end{aligned}$$

Hence

$$\Phi_Y(\omega) = \left(\sigma_X^2\sigma_A^2\omega^2 + 1\right)^{-\frac{N}{2}}.$$

(c)

$$\begin{aligned}
\Phi_Y\left(\frac{\omega}{\sqrt{N}}\right) &= \frac{1}{\left(\sigma_X^2\sigma_A^2\frac{\omega^2}{N} + 1\right)^{\frac{N}{2}}} \\
&\simeq \left(e^{-\frac{\sigma_X^2\sigma_A^2\omega^2}{N}}\right)^{N/2} \\
&= e^{-\frac{1}{2}\sigma_X^2\sigma_A^2\omega^2}.
\end{aligned}$$

Hence the asymptotic CDF of $\frac{1}{\sqrt{N}}Y(t)$, as $N \longrightarrow \infty$, is Normal (Gaussian): $N(0, \sigma_X^2\sigma_A^2)$.

(d) The number of taps N is Poisson now, so we must re-do (a) and (b). First

$$E[Y(t+\tau)Y(t)|N] = N\sigma_A^2 R_{XX}(\tau),$$

which implies

$$\begin{aligned} E[Y(t+\tau)Y(t)] &= E[N\sigma_A^2 R_{XX}(\tau)] \\ &= E[N]\sigma_A^2 R_{XX}(\tau) \\ &= \lambda\sigma_A^2 R_{XX}(\tau). \end{aligned}$$

Then

$$\begin{aligned} \Phi_Y(\omega) &= E[e^{+j\omega Y(t)}] \\ &= E[E[e^{+j\omega Y(t)}|N]] \\ &= E\left[\left(\frac{1}{\sigma_X^2\sigma_A^2\omega^2+1}\right)^{N/2}\right] \\ &= \sum_{n=0}^{\infty} d^{n/2}e^{-\lambda}\frac{\lambda^n}{n!}, \qquad \text{where} \quad d \triangleq \frac{1}{\sigma_X^2\sigma_A^2\omega^2+1}, \\ &= \left(\sum_{n=0}^{\infty}\frac{(\lambda\sqrt{d})^n}{n!}\right)e^{-\lambda} \\ &= \exp(\lambda\sqrt{d}-\lambda) \\ &= \exp(\lambda(\sqrt{d}-1)). \end{aligned}$$

5. By definition of Poisson process with parameter $\lambda(>0)$, we have

$$\begin{aligned} P_N(n;t) &= P[N(t)=n] \\ &= \frac{\lambda t}{n!}e^{-\lambda t}u[n], \end{aligned}$$

where $u[n]$ is the unit-step function.

(a) Let $t_2 \geq t_1$, then be independent increments property, $N(t_2) - N(t_1)$ and $N(t_1)$ are independent RVs. Also the increment is Poisson distributed with the same parameter λ. Hence

$$\begin{aligned} P_N(n_1, n_2; t_1, t_2) &= P[N(t_1) = n_1, N(t_2) = n_2] \\ &= P[N(t_1) = n_1]P[N(t_2) - N(t_1) = n_2 - n_1] \\ &= \frac{\lambda t_1}{n_1!}e^{-\lambda t_1}u[n_1]\frac{\lambda(t_2-t_1)}{(n_2-n_1)!}e^{-\lambda(t_2-t_1)}u[n_2-n_1] \\ &= \frac{\lambda t_1}{n_1!}e^{-\lambda t_1}\frac{\lambda(t_2-t_1)}{(n_2-n_1)!}e^{-\lambda(t_2-t_1)}u[n_1]u[n_2-n_1] \\ &= \frac{\lambda^2 t_1(t_2-t_1)}{n_1!(n_2-n_1)!}e^{-\lambda t_2}u[n_1]u[n_2-n_1]. \end{aligned}$$

(b) Since the t_i's are increasing, $N(t)$ has independent increments, which can be recursively applied to conclude

$$\begin{aligned}
P_N(n_1, n_2, ..., n_K; t_1, t_2, ..., t_K) &= P[N(t_1) = n_1, N(t_2) = n_2, ..., N(t_K) = n_K] \\
&= P[N(t_1) = n_1]P[N(t_2) - N(t_1) = n_2 - n_1] \cdots \\
&\qquad \cdots P[N(t_K) - N(t_{K-1}) = n_K - n_{K-1}] \\
&= \frac{\lambda^K t_1(t_2 - t_1) \cdots (t_K - t_{K-1})}{n_1!(n_2 - n_1)! \cdots (n_K - n_{K-1})!} e^{-\lambda t_K} u[n_1]u[n_2 - n_1] \cdots u[n_K - n_{K-1}].
\end{aligned}$$

6. (a) Use property (3) for $t_1 = 0$ and $t_2 = t$. Then by the property (1), $N(0) = 0$. So, (3) becomes:

$$P[N(t) = n] = \frac{\left(\int_0^t \lambda(s)ds\right)^n}{n!} e^{-\left(\int_0^t \lambda(s)ds\right)^n} \quad \text{for } n \geq 0.$$

Then since $N(t)$ is Poisson distributed, we recognize the mean as

$$\mu_N(t) = \int_0^t \lambda(s)ds, \qquad t \geq 0.$$

(b) Take $t_2 \geq t_1 \geq 0$, and write $E[N(t_1)N(t_2)] = E[N(t_1)[N(t_1) + (N(t_2) - N(t_1))]]$. Then using the linearity of the expectation operator E and the independent increments property (2), we get

$$\begin{aligned}
R_N(t_1, t_2) &\triangleq E[N(t_1)N(t_2)] \\
&= E[N^2(t_1)] + E[N(t_1)]E[N(t_2) - N(t_1)].
\end{aligned}$$

We then recognize the first term on the rhs as the second moment of the Poisson, therefore

$$E[N^2(t_1)] = \int_0^t \lambda(s)ds + \left(\int_0^t \lambda(s)ds\right)^2.$$

Now $E[N(t_2) - N(t_1)] = \int_{t_1}^{t_2} \lambda(s)ds$, and from part (a) $E[N(t_1)] = \int_0^t \lambda(s)ds$, so, putting these together, we get

$$\begin{aligned}
R_N(t_1, t_2) &= E[N^2(t_1)] + E[N(t_1)]E[N(t_2) - N(t_1)], \qquad t_2 \geq t_1 \geq 0, \\
&= \int_0^{t_1} \lambda(s)ds + \left(\int_0^{t_1} \lambda(s)ds\right)^2 + \left(\int_0^{t_1} \lambda(s)ds\right)\left(\int_{t_1}^{t_2} \lambda(s)ds\right) \\
&= \left(\int_0^{t_1} \lambda(s)ds\right)\left(1 + \int_0^{t_2} \lambda(s)ds\right), \qquad t_2 \geq t_1 \geq 0.
\end{aligned}$$

For the general case, from the symmetry of the correlation function $R_N(t_1, t_2) \triangleq E[N(t_1)N(t_2)]$, we can write

$$R_N(t_1, t_2) = \left(\int_0^{\min(t_1,t_2)} \lambda(s)ds\right)\left(1 + \int_0^{\max(t_1,t_2)} \lambda(s)ds\right), \qquad t_1, t_2 \geq 0.$$

(c) We have to show properties (1), (2), and (3):

(1) $N_u(0) \triangleq N(t(0)) = N(0) = 0.$ ✓

(2) Let $\tau_1 \leq \tau_2 \leq \tau_3 \leq \cdots \leq \tau_k$. Then since $t(\tau)$ is monotone increasing (since it is the integral of a positive $\lambda(s)$), we have $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_k$ where $t_i \triangleq t(\tau_i)$. Thus $N_u(\tau_i) \triangleq N(t(\tau_i)) = N(t_i)$. So, by definition, $N(t_1), N(t_2) - N(t_1), ...,$ $N(t_k) - N(t_{k-1})$ are jointly independent. But $N_u(\tau_i) - N_u(\tau_{i-1}) = N(t_i) - N(t_{i-1})$, so the $N_u(\tau)$ process also has independent increments.

(3) Since $N_u(\tau_2) - N_u(\tau_1) = N(t_2) - N(t_1)$ with mean value

$$\begin{aligned}\int_{t_1}^{t_2} \lambda(s)ds &= \int_0^{t_2} \lambda(s)ds - \int_0^{t_1} \lambda(s)ds \\ &= \tau_2 - \tau_1\end{aligned}$$

which means that $N(\tau)$ has λ parameter equal to 1.

7. The Poisson PMF is given as

$$P_N(n;t) = \frac{\left(\int_0^t \lambda(\nu)d\nu\right)^n}{n!} e^{-\left(\int_0^t \lambda(\nu)d\nu\right)^n} u[n].$$

(a)

$$\begin{aligned}\mu_N(t) &= E[N(t)] \\ &= \int_0^t \lambda(\nu)d\nu, \qquad t \geq 0, \\ &= \int_0^t (1+2\nu)d\nu \\ &= \nu + \nu^2\Big|_0^t \\ &= t + t^2, \qquad t \geq 0.\end{aligned}$$

(b)Let $t_2 \geq t_1 \geq 0$, then

$$N(t_1)N(t_2) = N(t_1)[N(t_1) + (N(t_2) - N(t_1))],$$

so

$$R_N(t_1,t_2) = E[N^2(t_1)] + \mu_N(t_1)\left(\mu_N(t_2) - \mu_N(t_1)\right).$$

Now

$$\begin{aligned}E[N^2(t)] &= \int_0^t \lambda(\nu)d\nu + \left(\int_0^t \lambda(\nu)d\nu\right)^2 \\ &= \left(t+t^2\right) + \left(t+t^2\right)^2.\end{aligned}$$

So for $t_2 \geq t_1$,

$$\begin{aligned}R_N(t_1,t_2) &= \left(t_1+t_1^2\right) + \left(t_1+t_1^2\right)^2 + \left(t_1+t_1^2\right)\left(\left(t_2+t_2^2\right) - \left(t_1+t_1^2\right)\right) \\ &= \left(t_1+t_1^2\right) + \left(t_1+t_1^2\right)\left(t_2+t_2^2\right).\end{aligned}$$

In general, we have

$$R_N(t_1,t_2) = \left[\min(t_1,t_2) + (\min(t_1,t_2))^2\right]\left[1 + \max(t_1,t_2) + (\max(t_1,t_2))^2\right].$$

(c)

$$\begin{aligned} P[N(t) \geq t] &= \sum_{n\geq \lceil t\rceil} P_N(n;t) \\ &= \sum_{n\geq \lceil t\rceil} \frac{(t+t^2)^n}{n!} e^{-(t+t^2)}, \qquad t \geq 0. \end{aligned}$$

(d) Use the CLT with $\mu_N(t) = t + t^2$ and $\sigma_N^2 = t + t^2$ to yield

$$\begin{aligned} P[N(t) &\geq t] \approx \frac{1}{2} + \operatorname{erf}\left(\frac{t^2}{\sqrt{t+t^2}}\right) \\ &\approx \frac{1}{2} + \operatorname{erf}(t). \end{aligned}$$

We remember

$$\begin{aligned} \operatorname{erf}(x) &= \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}v^2} dv \\ &= P[X_{SN} \leq x] \quad \text{for} \quad X_{SN} : N(0,1). \end{aligned}$$

8. (a) We know

$$f_T(t;n) = \underbrace{f_T(t) * f_T(t) * \cdots f_T(t)}_{n \text{ times}},$$

with $f_T(t) = \lambda e^{-\lambda t} u(t)$. Therefore

$$\begin{aligned} f_T(t;2) &= f_T(t) * f_T(t) \\ &= \left(\int_0^t \lambda e^{-\lambda\tau} \lambda e^{-\lambda(t-\tau)} d\tau\right) u(t) \\ &= \lambda^2 e^{-\lambda t} \left(\int_0^t d\tau\right) u(t) \\ &= \lambda^2 t e^{-\lambda t} u(t). \end{aligned}$$

For $n = 3$, we have

$$\begin{aligned}
f_T(t;3) &= f_T(t;2) * f_T(t) \\
&= \lambda^2 te^{-\lambda t}u(t) * \lambda e^{-\lambda t}u(t) \\
&= \lambda^3 \left(\int_0^t \tau e^{-\lambda\tau} e^{-\lambda(t-\tau)} d\tau \right) u(t) \\
&= \lambda^3 e^{-\lambda t} \left(\int_0^t \tau d\tau \right) u(t) \\
&= \lambda^3 \frac{t^2}{2} e^{-\lambda t} u(t).
\end{aligned}$$

Now we can guess the general result

$$f_T(t;n) \stackrel{?}{=} \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t} u(t).$$

To prove this result for general n, we appeal to mathematical induction, and assume the result $f_T(t;n-1) = \lambda^{n-1} \frac{t^{n-2}}{(n-2)!} e^{-\lambda t} u(t)$ and calculte the result at n as follows:

$$\begin{aligned}
f_T(t;n) &= f_T(t;n-1) * f_T(t) \\
&= \left(\lambda^{n-1} \frac{t^{n-2}}{(n-2)!} e^{-\lambda t} u(t) \right) * \lambda e^{-\lambda t} u(t) \\
&= \lambda^n \left(\int_0^t \frac{\tau^{n-2}}{(n-2)!} e^{-\lambda\tau} e^{-\lambda(t-\tau)} d\tau \right) u(t) \\
&= \lambda^n \left(\int_0^t \frac{\tau^{n-2}}{(n-2)!} d\tau \right) e^{-\lambda t} u(t) \\
&= \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t} u(t).
\end{aligned}$$

(b)

$$\begin{aligned}
F_{\tau'[i]}(\tau'|\tau[i] &\geq t) = P[\tau[i] \leq \tau' + t | \tau[i] \geq t] \\
&= \frac{\int_t^{t+\tau'} \lambda e^{-\lambda\tau} d\tau}{\int_t^{\infty} \lambda e^{-\lambda\tau} d\tau} \\
&= \frac{-e^{-\lambda\tau}|_t^{t+\tau'}}{-e^{-\lambda\tau}|_t^{\infty}} \\
&= 1 - e^{-\lambda\tau'},
\end{aligned}$$

thus, by differentiation, $f_{\tau'[i]}(\tau') = |\tau[i] \geq t) = \lambda e^{-\lambda\tau'} u(\tau')$, the same density as τ.

(c)

$$f_{\tau'[i]}(\tau') = \int_{-\infty}^{+\infty} f_{\tau'[i]}(\tau'|T = t) f_T(t) dt,$$

but from part (b), the conditional pdf $f_{\tau'[i]}(\tau') = |\tau[i] \geq t)$ is independent of the variable t, so

$$\begin{aligned}
f_{\tau'[i]}(\tau') &= f_{\tau'[i]}(\tau'|T = t) \int_{-\infty}^{+\infty} f_T(t) dt \\
&= f_{\tau'[i]}(\tau'|T = t) \\
&= f_{\tau[i]}(\tau'), \qquad \text{using the result of (b).}
\end{aligned}$$

9. (a) Use the general conditional expectation property $E[X] = E[E[X|Y]]$ to conclude

$$\begin{aligned}
\mu_N(t) &= E[N(t)] \\
&= E[E[N(t)|S(t)]] \\
&= E[\int_0^t S(\tau)d\tau] \\
&= \int_0^t \mu_S(\tau)d\tau \\
&= \mu_0 t.
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma_N^2(t) &= E[N^2(t)] - (E[N(t)])^2 \\
&= E[E[N^2(t)|S(t)]] - (\mu_0 t)^2 \\
&= E\left[\int_0^t S(\tau)d\tau + \left(\int_0^t S(\tau)d\tau\right)^2\right] - (\mu_0 t)^2 \\
&= \mu_0 t + \int_0^t \int_0^t K_S(\tau_1, \tau_2)d\tau_1 d\tau_2.
\end{aligned}$$

10.

$$\begin{aligned}
P_K(k;t) &= \sum_{n=k}^{\infty} P[K(t) = k, N(t) = n] \\
&= \sum_{n=k}^{\infty} P[K(t) = k|N(t) = n]P[N(t) = n] \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k q^{n-k} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad k \geq 0, t \geq 0, \\
&= \sum_{n'=0}^{\infty} \frac{p^k q^{n'}}{n'!k!} (\lambda t)^{n'+k} e^{-\lambda t} \quad \text{with } n' \triangleq n - k, \\
&= \left(\sum_{n'=0}^{\infty} \frac{(q\lambda t)^{n'}}{n'1}\right) \frac{p^k (\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{q\lambda t} \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \\
&= \frac{(p\lambda t)^k}{k!} e^{-p\lambda t}, \quad k \geq 0, t \geq 0,
\end{aligned}$$

a Poisson RV with parameter $p\lambda t$.

11. (a) Use conditional expectation and write

$$\begin{aligned}
E[N(x)] &= E[E\{N(x)|S(x)]] \\
&= E[\lambda(x)] \\
&= E[S(x) + \lambda_0]T \\
&= (E[S(x)] + \lambda_0)\, T \\
&= \lambda_0 T.
\end{aligned}$$

Then for the variance, we have

$$\begin{aligned} E\left[(N(x)-E[N(x)])^2\right] &= E\left[E[(N(x)-E[N(x)])^2]|S(x)\right] \\ &= \sigma_S^2 T^2 + \lambda_0 T. \end{aligned}$$

(b)

$$\begin{aligned} R_N(x_1,x_2) &\triangleq E[N(x_1)N(x_2)] \\ &= E[E[N(x_1)N(x_2)|S(x_1),S(x_2)]] \\ &= E[E[N(x_1)|S(x_1)]E[N(x_2)|S(x_2)] \quad \text{by conditional independence of } N \text{ given } S, \\ &= E[(S(x_1)] + \lambda_0)\,T\,(S(x_2)] + \lambda_0)\,T] \\ &= E[S(x_1)S(x_2)]T^2 + (\lambda_0 T)^2 \qquad \text{since } S \text{ is zero mean,} \\ &= \left(\sigma_S^2 \exp(-\alpha|x_1-x_2|) + \lambda_0^2\right) T^2. \end{aligned}$$

12. (a) In each 'state,' the time to the next transition is governed by the *interarrival time sequence* $\tau[n]$ for the Poisson process. These are i.i.d. exponentially distributed RVs. Hence, future behavior of $X(t)$ only depends on this current value, the state.

(b) For δ small and positive, we write

$$P[X(t)=+1] = P[X\,(t-\delta)=+1](1-\lambda\delta) + P[X(t-\delta)=-1]\lambda\delta,$$

so in steady state, letting $P_1 \triangleq P_X(1,\infty)$ and $P_{-1} \triangleq P_X(-1,\infty)$, where $P_{-1}=1-P_1$, we have $P_1 = P_1(1-\lambda\delta) + P_{-1}\lambda\delta$ or $P_1\lambda\delta = P_{-1}\lambda\delta$. Hence $P_1 = 1/2$.

(c) Rewriting the first equation in (b) using PMF notation, we have

$$P_X(1;t) - P_X(1;t-\delta) = -\lambda P_X(1;t-\delta)\delta + \lambda P_X(-1;t-\delta)\delta.$$

Dividing this equation by δ, and taking limits as $\delta \to 0$, we get

$$\frac{dP_X(1;t)}{dt} = -\lambda P_X(1;t) + \lambda P_X(-1;t),$$

and similarly we obtain

$$\frac{dP_X(-1;t)}{dt} = -\lambda P_X(-1;t) + \lambda P_X(1;t).$$

13. (a)

$$\begin{aligned} P_N(n;t) &= P[T[n] \le t, T[n+1\} > t], \qquad n \ge 1, \\ &= P[\{T[n] \le t\} - \{T[n+1] \le t\}], \text{ where minus sign indicates } \textit{event subtraction}, \\ &= P[[T[n] \le t] - P[T[n+1] \le t] \\ &= F_T(t;n) - F_T(t;n+1). \end{aligned}$$

(b)

$$P_N(0;t) = \begin{cases} 1-t, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

$$P_N(1;t) = \begin{cases} t - \frac{t^2}{2}, & 0 \le t \le 1, \\ 2 - 2t + \frac{t^2}{2}, & 1 < t \le 2 \\ 0 & t > 2. \end{cases}$$

These results come from

$$F_T(t;1) = \begin{cases} t, & 0 \le t \le 1, \\ 1, & t > 1, \end{cases}$$

and

$$F_T(t;2) = \begin{cases} \frac{t^2}{2}, & 0 \le t \le 1, \\ -1 + 2t - \frac{t^2}{2}, & 1 < t \le 2, \\ 1, & t > 2, \end{cases}$$

which, in turn, come from running integration, i.e. $\int_{-\infty}^{t}$, on the pdf's

$$\begin{aligned} f_T(t;1) &= f_\tau(t) \\ &= u(t) - u(t-1), \end{aligned}$$

and

$$f_T(t;2) = \begin{cases} t, & 0 \le t \le 1, \\ 2 - t, & 1 < t \le 2. \end{cases}$$

For $P_N(2;t)$, we get

$$P_N(2;t) = \begin{cases} \frac{t}{2} - \frac{t^3}{6}, & 0 \le t \le 1, \\ -\frac{3}{2} + \frac{7}{2}t - 2t^2 + \frac{t^3}{3}, & 1 < t \le 2, \\ \frac{9}{2} - \frac{9}{2}t + \frac{3}{2}t^2 - \frac{t^3}{3}, & 2 < t \le 3, \\ 0, & t > 3, \end{cases}$$

which is supported by the pdf

$$f_T(t;3) = \begin{cases} \frac{t^2}{2}, & 0 \le t \le 1, \\ -\frac{3}{2} + 3t - t^2, & 1 \le t \le 2, \\ \frac{9}{2} - 3t + \frac{t^2}{2}, & 2 < t \le 3, \\ 0, & t > 3, \end{cases}$$

and CDF

$$F_T(t;3) = \begin{cases} \frac{1}{6}t^3, & 0 \le t \le 1, \\ \frac{1}{2} - \frac{3}{2}t + \frac{3}{2}t^2 - \frac{t^3}{3}, & 1 \le t \le 2, \\ -\frac{7}{2} + \frac{9}{2}t - \frac{3}{2}t^2 + \frac{t^3}{6}, & 2 < t \le 3, \\ 1, & t > 3. \end{cases}$$

(c) We have

$$T[n] = \sum_{k=1}^{n} \tau[k], \quad \text{and CF} \quad \Phi_\tau(\omega) = \left(\frac{\sin \omega/2}{\omega/2} \right) e^{+j\omega/2} \ ,$$

so

$$\begin{aligned} \Phi_{T[n]}(\omega) &= (\Phi_\tau(\omega))^n \\ &= \left(\frac{\sin \omega/2}{\omega/2} \right)^n e^{+j\omega n/2}. \end{aligned}$$

Note: In the 1st printing, part (c) asks for the CF of the renewal process $N(t)$, but the CF of the interarrival time sequence $T[n]$ is all that is needed in part (d) of this problem, in order to obtain an approximate expression for the PMF $P_N(n;t)$ via the result of part (a).

(d) Now, since $\left(\frac{\sin\omega/2}{\omega/2}\right) \approx 1 - \frac{1}{6}\left(\frac{\omega}{2}\right)^2$ for small ω, which in turn is $\approx \exp(-\frac{1}{6}\left(\frac{\omega}{2}\right)^2)$, via $e^{-x} \approx 1 - x$, we get

$$\begin{aligned} \Phi_{T[n]}(\omega) &= \left(\frac{\sin\omega/2}{\omega/2}\right)^n e^{+j\omega n/2} \\ &\approx \exp(-\frac{n}{12}\omega^2 + j\omega n/2), \end{aligned}$$

which is Normal with mean $n/2$ and variance $n/12$, i.e. $T[n] : N(n/2, n/12)$. Thus

$$F_T(t;n) \approx \begin{cases} \frac{1}{2} - \operatorname{erf}\left(\frac{n/2-t}{\sqrt{n/12}}\right), & t \leq n/2, \\ \frac{1}{2} + \operatorname{erf}\left(\frac{t-n/2}{\sqrt{n/12}}\right), & t > n/2, \end{cases}$$

where the approximation is good near the mean, i.e. within a few std's, and for large n.

14. The standard Wiener process $W(t)$ is distributed as $N(0,t)$ and defined on $[0,\infty)$. Using the independent increments property as in problem 9.5, we have for any $t_2 > t_1$,

$$\begin{aligned} f_W(a_1,a_2;t_1,t_2) &= f_W(a_1;t_1) f_W(a_2-a_1;t_2-t_1) \\ &= \frac{1}{\sqrt{2\pi t_1}} \exp\left(-\frac{1}{2}\frac{a_1^2}{t_1}\right) \frac{1}{\sqrt{2\pi\,(t_2-t_1)}} \exp\left(-\frac{1}{2}\frac{(a_2-a_1)^2}{t_2-t_1}\right). \end{aligned}$$

15. Since W_1 and W_2 are independent Wiener processes, there difference is Gaussian distributed. Since the mean of $W_2(t)$ is zero, $-W_2(t)$ still has the correlation function $\alpha_2 \min(t_1,t_2)$. Thus

(a)

$$R_X(t_1,t_2) = (\alpha_1+\alpha_2)\min(t_1,t_2)$$

and

(b)

$$X(t) : N(0, (\alpha_1+\alpha_2)t).$$

16. (a) $Y(t) = X'(t)$, so $\mu_Y(t) = \frac{d}{dt}\mu_X(t) = \frac{d}{dt}(4) = 0$.

(b) Since $\mu_X(t) = 4$,

$$\begin{aligned} R_{YY}(t_1,t_2) &= \frac{\partial^2}{\partial t_1 \partial_{t_2}}[5\min{}^2(t_1,t_2)+4] \\ &= \frac{\partial}{\partial t_1}\left[\frac{\partial}{\partial_{t_2}}\left(5\min{}^2(t_1,t_2)\right)\right] \\ &= \frac{\partial}{\partial t_1}\left[10\min(t_1,t_2)u(t_1-t_2)\right] \\ &= 10\min(t_1,t_2)\delta(t_1-t_2), \end{aligned}$$

where we have written 0 for $u(t_2-t_0)u(t_1-t_2)$, since it is only non-zero at the one point $t_1=t_2$, and there only takes on the finite value 1. So

$$\begin{aligned} K_{YY}(t_1,t_2) &= R_{YY}(t_1,t_2) \\ &= 10\min(t_1,t_2)\delta(t_1-t_2) \\ &= 10t_1\delta(t_1-t_2) = 10t_2\delta(t_1-t_2). \end{aligned}$$

(c) (i) $\mu_Y(t)=0$.

(ii) $R_{YY}(t+\tau,t)=(t+\tau)\delta(\tau)=t\delta(\tau)$, not WSS.

(d) One way to get this covariance is to multiply together two jointly independent Wiener processes $W_1(t)$ and $W_2(t)$, and then scale the result by $\sqrt{5}$. Call the resulting process

$$X_c(t) \triangleq \sqrt{5}W_1(t)W_2(t).$$

Since the W_i have covariance function $\min(t_1,t_2)$, we get

$$\begin{aligned} K_{X_cX_c}(t_1,t_2) &= E[\sqrt{5}W_1(t_1)W_2(t_1)\sqrt{5}W_1(t_2)W_2(t_2)] \\ &= 5E[W_1(t_1)W_1(t_2)W_2(t_1)W_2(t_2)] \\ &= 5E[W_1(t_1)W_1(t_2)]E[W_2(t_1)W_2(t_2)] \\ &= 5\min(t_1,t_2)\min(t_1,t_2) \\ &= 5\min{}^2(t_1,t_2). \end{aligned}$$

17. (a) $X(t)=W^2(t)$ and $f_W(w;t)=\frac{1}{\sqrt{2\pi t}}\exp(-\frac{w^2}{2t})$, $t>0$. So the transformation is $x=w^2$ with two roots at $w=+\sqrt{x}$ and $-\sqrt{x}$ for positive x. Thus

$$f_X(x;t)=f_W(\sqrt{x};t)\frac{1}{|J|}+f_W(-\sqrt{x};t)\frac{1}{|J|}, \quad x>0,$$

where the Jacobian $|J|=|dx/dw|=|2w|=2\sqrt{x}$ for positive x. Combining, we have

$$\begin{aligned} f_X(x;t) &= f_W(\sqrt{x};t)\frac{1}{|J|}+f_W(-\sqrt{x};t)\frac{1}{|J|} \\ &= \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x}{2t})\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{2\pi t}}\exp(-\frac{x}{2t})\frac{1}{2\sqrt{x}} \\ &= \frac{1}{\sqrt{2\pi xt}}\exp(-\frac{x}{2t}) \quad \text{for } x,t>0. \end{aligned}$$

For $x<0$, clearly $f_X(x;t)=0$. Thus the complete answer is

$$f_X(x;t)=\frac{1}{\sqrt{2\pi xt}}\exp(-\frac{x}{2t})u(x) \quad \text{for } t>0.$$

(b) We first find the joint pdf

$$\begin{aligned} f_X(x_1,x_2) &= f_W(\sqrt{x_1},\sqrt{x_2})\frac{1}{|J|}+f_W(-\sqrt{x_1},\sqrt{x_2})\frac{1}{|J|} \\ &\quad +f_W(\sqrt{x_1},-\sqrt{x_2})\frac{1}{|J|}+f_W(-\sqrt{x_1},-\sqrt{x_2})\frac{1}{|J|}. \end{aligned}$$

Since $\frac{1}{|J|} = \left|\det \begin{bmatrix} 2w_1 & 0 \\ 0 & 2w_2 \end{bmatrix}\right| = 4\sqrt{x_1x_2}$, and using the independent increments property of the Wiener process, we have

$$\begin{aligned} f_X(x_1,x_2) &= f_{W_1}(\sqrt{x_1})f_{W_2-W_1}(\sqrt{x_2}-\sqrt{x_1})\frac{1}{4\sqrt{x_1x_2}} + f_{W_1}(-\sqrt{x_1})f_{W_2-W_1}(\sqrt{x_2}+\sqrt{x_1})\frac{1}{4\sqrt{x_1x_2}} \\ &\quad + f_{W_1}(\sqrt{x_1})f_{W_2-W_1}(-\sqrt{x_2}-\sqrt{x_1})\frac{1}{4\sqrt{x_1x_2}} + f_{W_1}(-\sqrt{x_1})f_{W_2-W_1}(-\sqrt{x_2}-\sqrt{x_1})\frac{1}{4\sqrt{x_1x_2}} \\ &= \frac{1}{2\pi\sqrt{x_1t_1x_2(t_2-t_1)}}\left[\frac{1}{2}\exp\left(-\frac{x_1}{2t_1}-\frac{(\sqrt{x_2}-\sqrt{x_1})^2}{2(t_2-t_1)}\right)+\frac{1}{2}\exp\left(-\frac{x_1}{2t_1}-\frac{(\sqrt{x_2}+\sqrt{x_1})^2}{2(t_2-t_1)}\right)\right], \end{aligned}$$

when $t_2 > t_1$, and $x_1, x_2 > 0$. Then

$$\begin{aligned} f_X(x_2|x_1) &= \frac{f_X(x_1,x_2)}{f_X(x_1)} \\ &= \frac{1}{\sqrt{2\pi x_2(t_2-t_1)}}\left[\frac{1}{2}\exp\left(-\frac{(\sqrt{x_2}-\sqrt{x_1})^2}{2(t_2-t_1)}\right)+\frac{1}{2}\exp\left(-\frac{(\sqrt{x_2}+\sqrt{x_1})^2}{2(t_2-t_1)}\right)\right], \end{aligned}$$

when $t_2 > t_1$, and $x_1, x_2 > 0$.

(c) Yes, it is Markov. Consider $f_X(x_3,x_2,x_1;t_3,t_2,t_1) = f_X(x_3,x_2,x_1)$. Calculating, we find

$$\begin{aligned} f_X(x_3,x_2,x_1) &= \frac{1}{8\sqrt{x_3x_2x_1}}f_W(\sqrt{x_1})\left[f_{\Delta W}(\sqrt{x_2}-\sqrt{x_1}) + f_{\Delta W}(\sqrt{x_2}+\sqrt{x_1})\right] \\ &\qquad \times \left[f_{\Delta W}(\sqrt{x_3}-\sqrt{x_2}) + f_{\Delta W}(\sqrt{x_3}+\sqrt{x_2})\right]. \end{aligned}$$

Then, from part (b), we have

$$f_X(x_2,x_1) = \frac{1}{4\sqrt{x_2x_1}}f_W(\sqrt{x_1})\left[f_{\Delta W}(\sqrt{x_2}-\sqrt{x_1}) + f_{\Delta W}(\sqrt{x_2}+\sqrt{x_1})\right],$$

so

$$\begin{aligned} f_X(x_3|x_2,x_1) &= \frac{f_X(x_3,x_2,x_1)}{f_X(x_2,x_1)} \\ &= \frac{1}{2\sqrt{x_3}}\left[f_{\Delta W}(\sqrt{x_3}-\sqrt{x_2}) + f_{\Delta W}(\sqrt{x_3}+\sqrt{x_2})\right] \\ &= f_X(x_3|x_2), \end{aligned}$$

which satisfies the Markov requirement.

(d) From part (a) and part (b), we know

$$f_X(x_2,x_1;t_2,t_1) \neq f_X(x_1;t_1)f_X(x_2-x_1;t_2,t_1),$$

so $X(t)$ does not have the property of independent increments.

18. $X(t)$ is a Markov random process on $[0,\infty)$ with $f_X(x;0) = \delta(x-1)$ and

$$f_X(x_2|x_1;t_2,t_1) = \frac{1}{\sqrt{2\pi\,(t_2-t_1)}}\exp\left(-\frac{(x_2-x_1)^2}{2\,(t_2-t_1)}\right), \qquad t_2 > t_1.$$

(a) We know

$$\begin{aligned} f_X(x_2;t_2) &= \int_{-\infty}^{+\infty} f_X(x_2|x_1;t_2,0) f_X(x_1;0)dx_1 \\ &= \frac{1}{\sqrt{2\pi t_2}} \exp\left(-\frac{(x_2-1)^2}{2t_2}\right), \end{aligned}$$

so $f_X(x;t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-1)^2}{2t}\right)$ for all $t > 0$.

(b) Here we are given $X(0) : N(0,1)$, thus, following the method in (a), we have

$$\begin{aligned} f_X(x_2;t_2) &= \int_{-\infty}^{+\infty} f_X(x_2|x_1;t_2,0) f_X(x_1;0)dx_1 \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t_2}} \exp\left(-\frac{(x_2-x_1)^2}{2t_2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right) dx_1. \end{aligned}$$

By completing the square (see math. Appendix A), we have

$$\begin{aligned} f_X(x_2;t_2) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{(2\pi)^2 t_2}} \exp\left(-\frac{x_2^2}{2(t_2+1)} - \frac{(t_2+1)\left(x_1 - \frac{x_2}{t_2+1}\right)^2}{2}\right) dx_1 \\ &= \frac{1}{\sqrt{2\pi(t_2+1)}} \exp\left(-\frac{x_2^2}{2(t_2+1)}\right), \end{aligned}$$

which is the same as $f_X(x;t) = \frac{1}{\sqrt{2\pi(t+1)}} \exp\left(-\frac{x^2}{2(t+1)}\right)$, i.e. $X(t) : N(0, t+1)$, for all $t > 0$.

19. (a)

$$\begin{aligned} P[\textit{remain} \text{ in state 2 until time } t|X(0) &= 2] = P[\text{no transition to 3}]P[\text{no transition to 1}] \\ &= e^{-\lambda_2 t} e^{-\mu_2 t} \\ &= \exp -(\lambda_2 + \mu_2)t. \end{aligned}$$

(b)

$$\begin{bmatrix} P_1(t+\Delta t) \\ P_2(t+\Delta t) \\ P_3(t+\Delta t) \end{bmatrix} \approx \begin{bmatrix} 1-\lambda_1 \Delta t & \mu_2 \Delta t & 0 \\ \lambda_1 \Delta t & 1-(\lambda_2+\mu_2)\Delta t & \mu_3 \Delta t \\ 0 & \lambda_2 \Delta t & 1-\mu_3 \Delta t \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix}.$$

Then taking the limit as   goes to zero, we get the vector equations

$$\frac{d\mathbf{P}}{dt} = \begin{bmatrix} -\lambda_1 & \mu_2 & 0 \\ \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 \\ 0 & \lambda_2 & -\mu_3 \end{bmatrix} \mathbf{P}(t),$$

where $\mathbf{P}(t) \triangleq (P_1(t), P_2(t), P_3(t))^T$.

(c) In the steady state $d\mathbf{P}/dt = 0$ and from the above vector equation we obtain

$$\begin{aligned} -\lambda_1 P_1 + \mu_2 P_2 &= 0 \quad \text{and} \\ \lambda_2 P_2 - \mu_3 P_3 &= 0. \end{aligned}$$

Solving these two, we obtain

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 \qquad \text{and} \qquad P_3 = \frac{\lambda_2}{\mu_3} P_2.$$

Now, together with $P_1 + P_2 + P_3 = 1$, which always holds, we get

$$\begin{aligned} P_1 + P_2 + P_3 &= P_1 + \frac{\lambda_1}{\mu_2} P_1 + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} P_1 \\ &= P_1 \left(1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}\right) \\ &= 1, \end{aligned}$$

so,

$$\begin{aligned} P_1 &= \frac{1}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\ P_2 &= \frac{\frac{\lambda_1}{\mu_2}}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\ P_3 &= \frac{\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}. \end{aligned}$$

20. (a)

$$P[X(t) =' good'] = P[X(t-\delta) =' \text{good}'](1 - \lambda_1\delta) + P[X(t-\delta) =' \text{bad}']\lambda_2\delta.$$

For steady state, set $P_G \triangleq P[X(\infty) = '\text{good}']$ and $P_B \triangleq P[X(\infty) = '\text{bad}']$, to get

$$P_G = P_G(1 - \lambda_1\delta) + P_B\lambda_2\delta.$$

Together with $P_B = 1 - P_G$, we can the solve to find:

$$\begin{aligned} P_G &= \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{9}{10}, \\ P_B &= \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{1}{10}. \end{aligned}$$

(b) The average probability of error $= P_G \cdot 0 + P_B\epsilon = 0 + 10^{-1}10^{-2} = 10^{-3}$.

21. (a)

$$f_X(x(t)) = \frac{1}{\sqrt{2\pi\alpha t}} \exp\left(-\frac{x(t)^2}{2\alpha t}\right), \qquad -\infty < x(t) < +\infty.$$

(b)

$$f_X(x(t)|x(s)) = \frac{1}{\sqrt{2\pi\alpha(t-s)}} \exp\left(-\frac{(x(t)-x(s))^2}{2\alpha(t-s)}\right), \qquad t > s.$$

(c)

$$\frac{1}{\sqrt{2\pi 2\alpha\Delta}} \exp\left(-\frac{(x_3 - x_1)^2}{2\alpha 2\Delta}\right) = \frac{1}{2\pi\alpha\Delta} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x_3 - x_2)^2 + (x_2 - x_1)^2}{2\alpha\Delta}\right) dx_2.$$

We then complete the square as

$$(x_3 - x_2)^2 + (x_2 - x_1)^2 = 2\left[\left(x_2 - \frac{x_1 + x_3}{2}\right)^2 + \left(\frac{x_1 - x_3}{2}\right)^2\right],$$

so

$$\begin{aligned}
\frac{1}{\sqrt{2\pi 2\alpha\Delta}} \exp\left(-\frac{(x_3 - x_1)^2}{2\alpha 2\Delta}\right) &= \frac{\sqrt{2\pi\alpha\Delta/2}}{2\pi\alpha\Delta} \exp\left(-\frac{2(x_1 - x_3)^2/4}{2\alpha\Delta}\right) \\
&= \frac{1}{\sqrt{2\pi 2\alpha\Delta}} \exp\left(-\frac{(x_1 - x_3)^2}{4\alpha\Delta}\right).
\end{aligned}$$

22. The covariance function is just a function of the difference of the two times t and s. Also the mean $\mu_{X'} = 0$ is constant. This satisfies the definition of wide sense stationarity. For (strict sense) stationarity, one would need to know that all higher order moment functions were also shift invariant. So, we cannot conclude stationarity here.

23. (a)

$$\begin{aligned}
\mu_X(t) &= \mu_A \cos 2\pi f t + \mu_B \sin 2\pi f t \\
&= 0 \cdot \cos 2\pi f t + 0 \cdot \sin 2\pi f t \\
&= 0.
\end{aligned}$$

$$K_{XX}(t, s) = E[X(t)X(s)]$$

$$\begin{aligned}
&= E[(A \cos 2\pi f t + B \sin 2\pi f t)(A \cos 2\pi f s + B \sin 2\pi f t s)] \\
&= E[A^2] \cos 2\pi f t \cos 2\pi f s + E[B^2] \sin 2\pi f t \sin 2\pi f t s \\
&= \sigma^2 (\cos 2\pi f t \cos 2\pi f s + \sin 2\pi f t \sin 2\pi f t s) \\
&= \sigma^2 \cos 2\pi f (t - s).
\end{aligned}$$

(b) It can be shown that $E[X^3(t)]$ is not constant, as follows.

$$\begin{aligned}
E[X^3(t)] &= E[(A \cos 2\pi f t + B \sin 2\pi f t)^3] \\
&= E[A^3 \cos^3 2\pi f t + 3A^2 B \cos^2 2\pi f t \sin 2\pi f t \\
&\quad + 3AB^2 \cos 2\pi f t \sin^2 2\pi f t + B^3 \sin^3 2\pi f t] \\
&= \mu(\cos^3 2\pi f t + \sin^3 2\pi f t) \\
&\neq \text{constant}.
\end{aligned}$$

The second to last line is because $E[A^2 B] = E[A^2]E[B] = 0$ and $E[AB^2] = E[A]E[B^2] = 0$, since $E[A] = E[B] = 0$. We also write $\mu = E[A^3] = E[B^3]$. The last line is because $\mu \neq 0$ and $\cos^3 2\pi f t + \sin^3 2\pi f t$ is not a constant with respect to time.

24. We are asked to prove that

$$K_{YY}(t_1, t_2) = L_{t_1} L_{t_2} \{K_{XX}(t_1, t_2)\}.$$

We start with the definition

$$
\begin{aligned}
K_{YY}(t_1,t_2) &\triangleq E\left[(Y(t_1)-\mu_Y(t_1))(Y^*(t_2)-\mu_Y^*(t_2))\right] \\
&= R_{YY}(t_1,t_2)-\mu_Y(t_1)\mu_Y^*(t_2) \\
&= L_{t_1}L_{t_2}\{R_{XX}(t_1,t_2)\}-L_{t_1}L_{t_2}\{\mu_X(t_1)\mu_X(t_2)\} \\
&= L_{t_1}L_{t_2}\{K_{XX}(t_1,t_2)\},
\end{aligned}
$$

as was to be shown.

25. (a)

$$
\dot{\mu}_Y(t)+a\mu_Y(t)=\mu_X \qquad \text{and} \qquad h(t)=e^{-at}u(t), \quad \text{so}
$$

$$
\begin{aligned}
\mu_Y(t) &= \mu_X\int_0^\infty e^{-at}dt \\
&= \mu_X/a.
\end{aligned}
$$

(b) From (9.5-5a)

$$
\begin{aligned}
R_{XY}(\tau) &= h^*(-\tau)*R_{XX}(\tau) \\
&= \int_{-\infty}^{\infty} h^*(-t)R_{XX}(\tau-t)dt \\
&= \int_{-\infty}^{\infty} h^*(t')R_{XX}(\tau+t')dt', \qquad \text{with } t'\triangleq -t, \\
&= \int_0^\infty e^{-at'}[\delta(\tau+t')+\mu_X^2]dt' \\
&= \begin{cases} e^{a\tau}+\mu_X^2/a, & \tau\le 0, \\ 0+\mu_X^2/a, & \tau>0, \end{cases} \\
&= e^{a\tau}u(-\tau)+\mu_X^2/a \qquad \text{for all } \tau.
\end{aligned}
$$

Then, from (9.5-5b),

$$
\begin{aligned}
R_{YY}(\tau) &= \int_{-\infty}^{\infty} h(\tau_1)R_{XY}(\tau-\tau_1)d\tau_1 \\
&= \int_0^\infty e^{-a\tau_1}[e^{a(\tau-\tau_1)}+\mu_X^2/a]d\tau_1, \qquad \text{for} \quad \tau<0, \\
&= e^{a\tau}\int_0^\infty e^{-2a\tau_1}d\tau_1+\mu_X^2/a, \\
&= e^{a\tau}/2a+(\mu_X/a)^2, \ \tau<0.
\end{aligned}
$$

For $\tau>0$,

$$
\begin{aligned}
R_{YY}(\tau) &= e^{a\tau}\int_\tau^\infty e^{-2a\tau_1}d\tau_1+(\mu_X/a)^2, \\
&= e^{-a\tau}/2a+(\mu_X/a)^2, \tau<0.
\end{aligned}
$$

Thus for all τ, we have

$$
R_{YY}(\tau)=e^{-a|\tau|}/2a+(\mu_X/a)^2.
$$

(c) Since $\mu_Y=\mu_X/a$, then for the covariance, we have

$$
K_{YY}(\tau)=e^{-a|\tau|}/2a \quad \text{and} \quad \sigma_Y^2=1/2a.
$$

26. No. For example for $t_1 = 1.0$ and $t_2 = 1.9$,

$$E[X(t_1)X(t_2)] = R_{XX}[0].$$

For $t_1 = 1.2$ and $t_2 = 1.2 + 0.9 = 2.1$,

$$E[X(t_1)X(t_2)] = R_{XX}[1] \neq R_{XX}[0].$$

Hence, $X(t)$ is not WSS.

27.

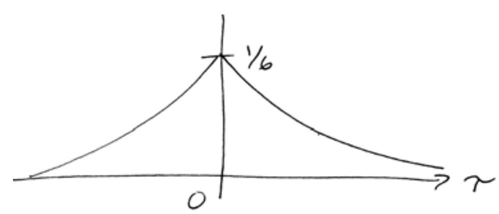
$$\begin{aligned} S_{UU}(\omega) &= \frac{1}{\omega^2+9} \\ &= \frac{1}{(j\omega+3)(-j\omega+3)} \\ &= \frac{a}{j\omega+3} + \frac{b}{-j\omega+3}, \qquad \text{using partial fraction expansion,} \\ &= \frac{1/6}{j\omega+3} + \frac{1/6}{-j\omega+3}. \end{aligned}$$

So

$$S_{UU}(s) = \frac{1/6}{s+3} + \frac{1/6}{-s+3}, \qquad |\operatorname{Re}(s)| < 3.$$

Taking the inverse Laplace transform

$$\begin{aligned} R_{UU}(\tau) &= \frac{1}{6}e^{-3\tau}u(\tau) + \frac{1}{6}e^{+3\tau}u(-\tau) \\ &= \frac{1}{6}e^{-3|\tau|}, \qquad -\infty < \tau < +\infty. \end{aligned}$$



(also see Example A.3-1 in Appendix A.)

28. Using white noise for system indentification.

$$\begin{aligned} K_{WW}(\tau) &= \delta(\tau) \\ K_{XW}(\tau) &= K_{WW}(\tau) * h(\tau) \\ &= \delta(\tau) * h(\tau) \\ &= h(\tau). \end{aligned}$$

Therefore

$$\begin{aligned} H(\omega) &= FT\{h(\tau)\} \\ &= S_{XW}(\omega). \end{aligned}$$

29. (a)

$$\begin{aligned} \mu_Y(t) &= E[X(t) + 0.3X'(t)] \\ &= \mu_X(t) + 0.3\mu'_X(t) \\ &= 5t + 0.3 \times 5 \\ &= 5t + 1.5. \end{aligned}$$

(b)

$$K_{YY}(t_1, t_2) = E[\left(X_c(t_1) + 0.3X'_c(t_1)\right)\left(X_c(t_2) + 0.3X'_c(t_2)\right)],$$

where $X_c(t) \triangleq X(t) - \mu_X(t)$, i.e. the *centered version* of X. Then we can write

$$K_{YY}(t_1, t_2) = K_{XX}(t_1, t_2) + 0.3\frac{\partial K_{XX}(t_1, t_2)}{\partial t_1} + 0.3\frac{\partial K_{XX}(t_1, t_2)}{\partial t_2} + 0.09\frac{\partial^2 K_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}.$$

Now, from the given K_{XX},

$$\frac{\partial K_{XX}(t_1, t_2)}{\partial t_1} = \frac{-\sigma^2 2\alpha(t_1 - t_2)}{(1 + \alpha(t_1 - t_2)^2)^2},$$

and

$$\begin{aligned} \frac{\partial K_{XX}(t_1, t_2)}{\partial t_2} &= \frac{-\sigma^2 2\alpha(t_2 - t_1)}{(1 + \alpha(t_2 - t_1)^2)^2} \\ &= -\frac{\partial K_{XX}(t_1, t_2)}{\partial t_1}. \end{aligned}$$

Thus the two cross-terms cancel, and inserting

$$\frac{\partial^2 K_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{-6\alpha^2\sigma^2 2\alpha(t_1 - t_2)^2 + 2\alpha\sigma^2}{(1 + \alpha(t_1 - t_2)^2)^3},$$

we have finally

$$K_{YY}(t_1, t_2) = \frac{\sigma^2}{1 + \alpha\tau^2}\left(1 + \frac{-0.54\alpha^2\tau^2 + 0.18\alpha}{(1 + \alpha\tau^2)^2}\right), \quad \text{with } \tau \triangleq t_1 - t_2.$$

Equivalently

$$K_{YY}(\tau) = \frac{\sigma^2}{1 + \alpha\tau^2}\left(1 + \frac{-0.54\alpha^2\tau^2 + 0.18\alpha}{(1 + \alpha\tau^2)^2}\right).$$

(c) No, not WSS. The covariance function is shift-invariant, but not the mean function.

30. (a) We have $H(\omega) = \frac{1}{j\omega+1}$ and $S_{WW}(\omega) = 1$, so

$$\begin{aligned} S_{XX}(\omega) &= |H(\omega)|^2 S_{WW}(\omega) \\ &= H(\omega)H^*(\omega)S_{WW}(\omega) \\ &= \frac{1}{j\omega+1}\frac{1}{-j\omega+1}\,1 \\ &= \frac{1}{\omega^2+1}. \end{aligned}$$

(b) Using the residue method, we have

$$R_{XX}(\tau) = \begin{cases} \text{Res}\left[\frac{e^{s\tau}}{(s+1)(-s+1)}; s=-1\right], & \tau \geq 0, \\ \text{Res}\left[\frac{e^{s\tau}}{(s+1)(-s+1)}; s=+1\right], & \tau \leq 0. \end{cases}$$

Now,

$$\begin{aligned} \text{Res}\left[\frac{e^{s\tau}}{(s+1)(-s+1)}; s=-1\right] &= \left\{\frac{e^{s\tau}(s+1)}{(s+1)(-s+1)}\right|_{s=-1} \\ &= \frac{1}{2}e^{-\tau}, \tau \geq 0, \end{aligned}$$

and

$$\begin{aligned} \text{Res}\left[\frac{e^{s\tau}}{(s+1)(-s+1)}; s=+1\right] &= \left\{\frac{e^{s\tau}(-s+1)}{(s+1)(-s+1)}\right|_{s=+1} \\ &= \frac{1}{2}e^{+\tau}, \tau \leq 0. \end{aligned}$$

Overall, we then have

$$R_{XX}(\tau) = \frac{1}{2}e^{-|\tau|}, -\infty < \tau < +\infty.$$

(c) Consider $X'(t) + X(t) = W(t)$ for $t > 0$, subject to initial condition $X(0)$ with $X(0) \perp W(t)$. Then, we can write

$$\begin{aligned} X(t) &= X(0)e^{-t} + \int_0^t e^{-(t-\tau)}W(\tau)d\tau, \\ &= X(0)e^{-t} + e^{-t}\int_0^t e^{\tau}W(\tau)d\tau,\ t \geq 0. \end{aligned}$$

Thus

$$\begin{aligned}
R_{XX}(t,s) &= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \int_0^t \int_0^s e^{\tau_1+\tau_2} E[W(\tau_1)W(\tau_2)] d\tau_1 d\tau_2 \\
&= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \int_0^t \int_0^s e^{\tau_1+\tau_2} \delta(\tau_1 - \tau_2) d\tau_1 d\tau_2 \\
&= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \int_0^s e^{\tau_2} \left(\int_0^t e^{\tau_1} \delta(\tau_1 - \tau_2) d\tau_1 \right) d\tau_2 \\
&= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \int_0^s e^{\tau_2} \left\{ \begin{array}{ll} e^{\tau_2}, & \tau_2 < t \\ 0, & \tau_2 > t \end{array} \right\} d\tau_2 \\
&= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \int_0^{\min(t,s)} e^{2\tau_2} d\tau_2 \\
&= E[X^2(0)]e^{-(t+s)} + e^{-(t+s)} \frac{1}{2} [e^{2\min(t,s)} - 1] \\
&= E[X^2(0)]e^{-(t+s)} + \frac{1}{2} [e^{-(t-s)} - e^{-(t+s)})].
\end{aligned}$$

Choosing initial average power $E[X^2(0)] = 1/2$, we get

$$R_{XX}(t,s) = \frac{1}{2} e^{-(t-s)} \quad \text{for all } t \geq 0, s \geq 0.$$

With the initial mean $\mu_{X(0)} = E[X(0)] = 0$ also, we get finally $\mu_X(t) = E[X(t)] = 0$. Thus the desired initial conditions are

$$\mu_{X(0)} = 0 \quad \text{and} \quad \sigma^2_{X(0)} = \frac{1}{2}.$$

31. Let

$$\begin{aligned}
r(\tau) &\triangleq \int_{-\infty}^{+\infty} h^*(-u) h(\tau - u) du \\
&= h^*(-\tau) * h(\tau).
\end{aligned}$$

We need to show that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j^* r(\tau_i - \tau_j) \geq 0, \quad \text{for arbitrary } a_i, \tau_i.$$

Consider

$$\begin{aligned}
r(\tau_i - \tau_j) &= \int_{-\infty}^{+\infty} h^*(-u) h(\tau_i - \tau_j - u) du \\
&= \int_{-\infty}^{+\infty} h(\tau_i - \tau_j - u) h^*(-u) du
\end{aligned}$$

Making the substitution $u' \triangleq -u - \tau_j$, we then get

$$r(\tau_i - \tau_j) = \int_{-\infty}^{+\infty} h(\tau_i + u') h^*(\tau_j + u') du'.$$

So now,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j^* r(\tau_i - \tau_j) &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j^* \int_{-\infty}^{+\infty} h(\tau_i + u')h^*(\tau_j + u')du' \\
&= \int_{-\infty}^{+\infty} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j^* h(\tau_i + u')h^*(\tau_j + u') \right) du' \\
&= \int_{-\infty}^{+\infty} \left| \sum_{i=1}^{N} a_i h(\tau_i + u') \right|^2 du' \\
&\geq 0, \quad \text{since integral of non-negative function.}
\end{aligned}$$

32. (a) The input process $X(t)$ has constant mean 128. So $\mu_Y(t) = \mu_X H(0) = 128 \times 1 = 128$.

(b) For the covariance function,

$$K_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2 S_{X_cX_c}(\omega) e^{+j\omega\tau} d\omega, \tag{p32eq1}$$

where $X_c \triangleq X - \mu_X$ is the centered version of X. Also

$$\begin{aligned}
|H(\omega)|^2 &= H(\omega)H^*(\omega) \\
&= \frac{1}{1+j\omega}\frac{1}{1-j\omega} \\
&= \frac{1}{1+\omega^2}.
\end{aligned}$$

The PSD $S_{X_cX_c}$ is determined as the FT of K_{XX} :

$$\begin{aligned}
S_{X_cX_c}(\omega) &= \int_{-\infty}^{+\infty} 1000 e^{-10|\tau|} e^{-j\omega\tau} d\tau \\
&= 1000 \left(\int_{-\infty}^{0} e^{+(10-j\omega)\tau} d\tau + \int_{0}^{+\infty} e^{-(10+j\omega)\tau} d\tau \right) \\
&= 1000 \left(\frac{1}{10-j\omega} + \frac{1}{10+j\omega} \right) \\
&= \frac{20,000}{100+\omega^2}.
\end{aligned}$$

Now, we can plug into (p32eq1) to obtain

$$\begin{aligned}
K_{YY}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+\omega^2}\frac{20,000}{100+\omega^2} e^{+j\omega\tau} d\omega \\
&= \frac{20,000}{2\pi(99)}\int_{-\infty}^{+\infty} \left(\frac{1}{1+\omega^2} - \frac{1}{100+\omega^2} \right) e^{+j\omega\tau} d\omega \\
&= \frac{20,000}{99}\left(\frac{1}{2}e^{-|\tau|} - \frac{1}{20}e^{-10|\tau|} \right) \\
&\doteq 101.01 e^{-|\tau|} - 10.10 e^{-10|\tau|}.
\end{aligned}$$

33. (a) In general, we have $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$. Here $S_X(\omega) = 2$ and

$$\begin{aligned} H(\omega) &= \frac{1}{4}\int_{-2}^{+2} e^{-j\omega t} dt \\ &= \frac{e^{-jw2} - e^{+j\omega 2}}{-j2(2\omega)} \\ &= \frac{\sin 2\omega}{2\omega}. \end{aligned}$$

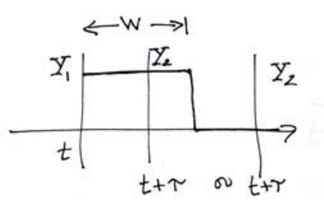
Thus

$$\begin{aligned} S_Y(\omega) &= |H(\omega)|^2 S_X(\omega) \\ &= \left(\frac{\sin 2\omega}{2\omega}\right)^2 2. \end{aligned}$$

34. We first express

$$\begin{aligned} R_{YY}(\tau) &\triangleq E[Y(t+\tau)Y(t)] \\ &= E_W[E[Y(t+\tau)Y(t)|W]], \end{aligned}$$

where the outer expectation $E_W[\cdot]$ is on the random variable W, which indicates the width of a pulse. Next, let $W = w$ and evaluate $E[Y(t+\tau)Y(t)|W = w]$. Since this quantity is not a function of t, by the WSS condition on Y, we can take t at the start of a pulse, as shown below.



Now for $\tau < w$, we have $Y_1 = Y_2$, so $Y_1Y_2 = Y_1^2$, while for $\tau > w$, we have Y_1Y_2 with $Y_1 \perp Y_2$, thus

$$\begin{aligned} E[Y(t+\tau)Y(t)|W = w] &= E[Y_1Y_2] \\ &= \begin{cases} (E[Y])^2, & \tau > w, \\ E[Y^2], & \tau < w. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} E[Y(t+\tau)Y(t)] &= E_W[E[Y(t+\tau)Y(t)|W]] \\ &= (E[Y])^2 P[W < \tau] + E[Y^2]P[W > \tau]. \end{aligned}$$

Calculating, we get

$$\begin{aligned} P[W > \tau] &= \int_{\tau}^{\infty} \lambda e^{-\lambda w} dw \\ &= e^{-\lambda\tau}, \end{aligned}$$

and $E[Y]=0$ and $E[Y^2]=\sigma_X^2$ since, by the problem statement, the independent pulse amplitude X has zero mean and variance σ_X^2. Combining results, we get

$$\begin{aligned} R_{YY}(\tau) &= (E[Y])^2\, P[W<\tau]+E[Y^2]P[W>\tau] \\ &= 0+E[Y^2]P[W>\tau] \\ &= \sigma_X^{22} e^{-\lambda\tau}, \qquad \tau>0. \end{aligned}$$

Since $R_{YY}(\tau)$ must be even in the delay variable τ, we finally have

$$R_{YY}(\tau)=\sigma_X^{22}\exp(-\lambda|\tau|), \qquad \text{for all } -\infty<\tau+\infty.$$

The psd is then given as

$$S_{YY}(\omega)=\frac{2\lambda\sigma_X^2}{\omega^2+\lambda^2}.$$

For $\tau=0$, we get $R_{YY}(0)=\sigma_X^2$, the mean-square value of X, as it should be. For $\tau=\infty$, we get $R_{YY}(\infty)=0$, the mean value of the process squared. Widely separated elements become uncorrelated.

35. (a)

$$S_{XX}(\omega)=2\pi\mu_X^2\delta(\omega)+\frac{10}{(\omega-10)^2+4}+\frac{10}{(\omega+10)^2+4},$$

where we have used the known Fourier transform pair $\exp(-\alpha\,|\tau|)\Longleftrightarrow 2\alpha/(\omega^2+\alpha^2)$.

(b)

$$\frac{1}{2\pi}\int_{\omega_1}^{\omega_2}(\cdot)d\omega+\frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1}(\cdot)d\omega.$$

(c)

$$S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega),$$

with

$$\begin{aligned} |H(\omega)|^2 &= \frac{j\omega+4}{(j\omega+6)(j\omega+5)}+\frac{-j\omega+4}{(-j\omega+6)(-j\omega+5)} \\ &= \frac{\omega^2+16}{(\omega^2+36)(\omega^2+25)}. \end{aligned}$$

36. (a) Given $K_{XX}(\tau)=\frac{1}{\tau_0}e^{-|\tau|/\tau_0}$, since X is zero mean, $K_{XX}=R_{XX}$, hence

$$S_{XX}(\omega)=\frac{1}{1+(\omega\tau_0)^2}.$$

Therefore

$$\begin{aligned} S_{YY}(\omega) &= S_{XX}(\omega)|G(\omega)|^2 \\ &= \begin{cases} \frac{1}{1+(\omega\tau_0)^2}, & |\omega|\le\omega_0, \\ 0, & |\omega|>\omega_0. \end{cases} \end{aligned}$$

(b) $S_{WW}(\omega)=FT\{K_{WW}(\tau)\}=FT\{\delta(\tau)\}=1$, where $K(\tau)=R(\tau)$ for a zero-mean process. Thus,

$$S_{VV}(\omega)=\begin{cases} 1, & |\omega|\le\omega_0, \\ 0, & |\omega|>\omega_0. \end{cases}$$

(c) If $|\omega_0 \tau_0| \ll 1$, then $1 + (\omega\tau_0)^2 \approx 1$ for $|\omega| \le \omega_0$, therefore

$$\begin{aligned} S_{YY}(\omega) &\approx \begin{cases} 1, & |\omega| \le \omega_0, \\ 0, & |\omega| > \omega_0. \end{cases} \\ &= S_{VV}(\omega). \end{aligned}$$

For the correlation function error

$$\begin{aligned} R_{VV}(0) - R_{YY}(0) &= \frac{1}{2\pi} \int_{-\omega_0}^{+\omega_0} \left(1 - \frac{1}{1 + (\omega\tau_0)^2} \right) d\omega \\ &= \frac{2}{2\pi} \int_0^{+\omega_0} \frac{(\omega\tau_0)^2}{1 + (\omega\tau_0)^2} d\omega \\ &< \frac{2}{2\pi} \int_0^{+\omega_0} \frac{(\omega_0\tau_0)^2}{1 + (\omega_0\tau_0)^2} d\omega, \quad \text{since the integrand is increasing function,} \\ &= \frac{2}{2\pi} \frac{(\omega_0\tau_0)^2}{1 + (\omega_0\tau_0)^2} \int_0^{+\omega_0} d\omega \\ &= \frac{\omega_0 (\omega_0\tau_0)^2}{\pi \left(1 + (\omega_0\tau_0)^2\right)}. \end{aligned}$$

37. The processes $X(t)$ and $N(t)$ are WSS and mutually uncorrelated with zero means and psd's S_{XX} and S_{NN}, respectively.

(a) Let $Z(t) = X(t) + N(t)$, then

$$\begin{aligned} R_{ZZ}(\tau) &= E[(X(t+\tau) + N(t+\tau))(X^*(t) + N^*(t))] \\ &= R_{XX}(\tau) + R_{NN}(\tau). \end{aligned}$$

Therefore,

$$\begin{aligned} S_{YY}(\omega) &= |H(\omega)|^2 S_{ZZ}(\omega) \\ &= |H(\omega)|^2 (S_{XX}(\omega) + S_{NN}(\omega)). \end{aligned}$$

(b) For the cross-correlation,

$$\begin{aligned} R_{XY}(\tau) &= E[X(t+\tau)(h^*(t) * (X^*(t) + N^*(t))] \\ &= \int_{-\infty}^{+\infty} h^*(s) E[X(t+\tau) X^*(t-s) ds + \int_{-\infty}^{+\infty} h^*(s) E[X(t+\tau) N^*(t-s)] ds \\ &= \int_{-\infty}^{+\infty} h^*(s) R_{XX}(\tau + s) ds + \int_{-\infty}^{+\infty} h^*(s)\, 0\, ds \\ &= \int_{-\infty}^{+\infty} h^*(-s') R_{XX}(\tau - s') ds' \qquad \text{with} \quad s' \triangleq -s, \\ &= h^*(-\tau) * R_{XX}(\tau). \end{aligned}$$

Therefore $S_{XY}(\omega) = H^*(\omega) S_{XX}(\omega)$. Following the same path, we can show that cross-power spectral density $S_{YX}(\omega) = H(\omega) S_{XX}(\omega)$.

(c) For the error $\xi(t) \triangleq Y(t) - X(t)$, we get

$$\begin{aligned} R_{\xi\xi}(\tau) &= E[(Y(t+\tau) - X(t+\tau))(Y^*(t) - X^*(t))] \\ &= R_{YY}(\tau) - R_{XY}(\tau) - R_{YX}(\tau) + R_{XX}(\tau). \end{aligned}$$

Proceeding to the psd's and using results from parts (a) and (b),

$$\begin{aligned} S_{\xi\xi}(\omega) &= S_{YY}(\omega) - S_{XY}(\omega) - S_{YX}(\omega) + S_{XX}(\omega) \\ &= |H(\omega)|^2(S_{XX}(\omega) + S_{NN}(\omega)) - H^*(\omega)S_{XX}(\omega) - H^*(\omega)S_{XX}(\omega) + S_{XX}(\omega) \\ &= |H(\omega) - 1|^2 S_{XX}(\omega) + |H(\omega)|^2 S_{NN}(\omega). \end{aligned}$$

(d) Given $h(t) = a\delta(t)$, then $H(\omega) = a$. Letting a be real, we then have

$$S_{\xi\xi}(\omega) = (a^2 - 2a + 1)S_{XX}(\omega) + a^2 S_{NN}(\omega),$$

or equivalently

$$R_{\xi\xi}(\tau) = (a^2 - 2a + 1)R_{XX}(\tau) + a^2 R_{NN}(\tau).$$

To find the minimum of $R_{\xi\xi}(0)$ with respect to the variable a, we differentiate and set the derivative to zero, obtaining

$$\begin{aligned} \frac{dR_{\xi\xi}(0)}{da} &= (2a-2)R_{XX}(0) + 2aR_{NN}(0) \\ &= 0. \end{aligned}$$

Thus

$$a_{\min} = \frac{R_{XX}(0)}{R_{XX}(0) + R_{NN}(0)}.$$

To see that this is a true minimum, and not merely a stationary point, we calculate the second derivative and find

$$\begin{aligned} \frac{d^2 R_{\xi\xi}(0)}{da^2} &= 2R_{XX}(0) + 2R_{NN}(0) \\ &\geq 0 \quad \text{always for valid correlation functions, and} \\ &> 0 \quad \text{if } X(t) \text{ has positive average power.} \end{aligned}$$

38. (a)

$$\begin{aligned} \mu_X(t) &= E[N\cos(2\pi f_0 t + \Theta)] \\ &= E[N]E[\cos(2\pi f_0 t + \Theta)] \\ &= E[N]\int_{-\pi}^{+\pi} \cos(2\pi f_0 t + \theta)\frac{d\theta}{2\pi} \\ &= 0, \end{aligned}$$

since the above integral will have value 0 independent of $2\pi f_0 t$.

(b)

$$\begin{aligned} K_X(t,s) &= R_X(t,s) \quad \text{since} \quad \mu_X(t) = 0, \ \text{for all } t, \\ &= E[N^2\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0 s + \Theta)] \\ &= E[N^2]E[\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0 s + \Theta)] \\ &= E[N^2]\int_{-\pi}^{+\pi} \cos(2\pi f_0(t-s) + \theta)\cos\theta \frac{d\theta}{2\pi}. \end{aligned}$$

Now, we remember the trig identities

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B, \quad \text{and}\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B.\end{aligned}$$

From these two, we get the further identity

$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)].$$

Using this identity on the above integrand, we get

$$\cos(2\pi f_0(t-s)+\theta)\cos\theta = \frac{1}{2}[\cos(2\pi f_0(t-s)+2\theta)+\cos(2\pi f_0(t-s))].$$

But $\int_{-\pi}^{+\pi}\cos(2\pi f_0(t-s)+2\theta)\frac{d\theta}{2\pi} = \int_{-\pi}^{+\pi}\cos 2\theta\frac{d\theta}{2\pi} = 0$ and $\int_{-\pi}^{+\pi} c\frac{d\theta}{2\pi} = c$ for any constant c. Thus

$$\begin{aligned}K_X(t,s) &= E[N^2]\int_{-\pi}^{+\pi}\cos(2\pi f_0(t-s)+\theta)\cos\theta\frac{d\theta}{2\pi}\\ &= \frac{1}{2}E[N^2]\int_{-\pi}^{+\pi}[\cos(2\pi f_0(t-s)+2\theta)+\cos(2\pi f_0(t-s))]\frac{d\theta}{2\pi}\\ &= \frac{1}{2}E[N^2][0+\cos(2\pi f_0(t-s))]\\ &= \frac{1}{2}E[N^2]\cos(2\pi f_0(t-s)).\end{aligned}$$

Since the Poisson RV has mean square value $E[N^2]=\lambda+\lambda^2$, we finally get

$$K_X(t,s) = \frac{1}{2}\left(\lambda+\lambda^2\right)\cos(2\pi f_0(t-s)).$$

(c) By the definition of WSS, we can conclude that $X(t)$ is WSS by the answers in parts (a) and (b) since the mean function is constant and the covariance function is shift-invariant.

(d) To show (strict sense) stationarity, we must show that for any positive integer m, the mth order distribution function $F_X(x_1,...,x_m;t_1,...,t_m)$ is invariant to the joint shift $t_i \longrightarrow t_i+t$ for all t. Now

$$\begin{aligned}F_X(x_1,...,x_m;t_1,...,t_m) &\triangleq P\left[\bigcap_{i=1}^{m}\{X(t_i)\le x_i\}\right]\\ &= \sum_{n=1}^{\infty}P_N(n)P\left[\bigcap_{i=1}^{m}\{\cos(2\pi f_0 t_i+\Theta)\le\frac{x_i}{n}\}\right]+P_N(0).\end{aligned}$$

So, to prove stationarity, we must then prove invariance wrt t of the probability

$$P\left[\bigcap_{i=1}^{m}\{\cos(2\pi f_0 t_i+2\pi f_0 t+\Theta)\le x_i'\}\right], \quad \text{with} \quad x_i' = x_i/n.$$

Now, since the RV Θ is uniformly distributed, i.e. $U[-\pi,+\pi]$, after defining a new RV as

$$\Theta' \triangleq \Theta + 2\pi f_0 t,$$

we can conclude that Θ' is $U[2\pi f_0 t-\pi, 2\pi f_0 t+\pi]$. Then, since all the functions $\cos(2\pi f_0 t_i + 2\pi f_0 t)$ are periodic with period 2π, we can just as well integrate them over $[-\pi, +\pi]$. Thus we have

$$P\left[\bigcap_{i=1}^{m}\{\cos(2\pi f_0 t_i + 2\pi f_0 t + \Theta) \leq x_i'\}\right] = P\left[\bigcap_{i=1}^{m}\{\cos(2\pi f_0 t_i + \Theta') \leq x_i'\}\right],$$

which is independent of t. Hence this $X(t)$ is stationary.

39. (a) Since $X(t)$ is an independent increments process, we have

$$\begin{aligned}
\Phi_{X(t)}(\omega) &\triangleq E[e^{+j\omega X(t)}] \\
&= E[e^{+j\omega(X(t)-X(0))}e^{+j\omega X(0)}] \\
&= E[e^{+j\omega(X(t)-X(0)}]E[e^{+j\omega X(0)}] \\
&= \Phi_{X(t)-X(0)}(\omega)\Phi_{X_0}(\omega).
\end{aligned}$$

(b) By definition

$$\begin{aligned}
\Phi_{X(t_2),X(t_1)}(\omega_2,\omega_1) &\triangleq E[e^{+j\omega_2 X(t_2)+j\omega_1 X(t_1)}] \\
&= E[e^{+j\omega_1 X(t_1)+j\omega_2 X(t_1)}e^{+j\omega_2(X(t_2)-X(t_1))}] \\
&= E[e^{+j\omega_2(X(t_2)-X(t_1))}]E[e^{+j\omega_1 X(t_1)+j\omega_2 X(t_1)}] \\
&= E[e^{+j\omega_2(X(t_2)-X(t_1))}]E[e^{+j(\omega_1+\omega_2)(X(t_1)-X(0)+X(0)}] \\
&= E[e^{+j\omega_2(X(t_2)-X(t_1))}]E[e^{+j(\omega_1+\omega_2)(X(t_1)-X(0)}]E[e^{+j(\omega_1+\omega_2)X(0)}] \\
&= \Phi_{X(t_2)-X(t_1)}(\omega_2)\Phi_{X(t_1)-X(0)}(\omega_1+\omega_2)\Phi_{X_0}(\omega_1+\omega_2).
\end{aligned}$$

(c) Since the Markov provess given in problem 9.18(b) happens to be an independent increments process, we can apply the result of part (a) of this problem as

$$\begin{aligned}
\Phi_{X_0}(\omega) &= \exp(-\frac{1}{2}\omega^2), \\
\Phi_{X(t)-X(0)}(\omega) &= \exp(-\frac{1}{2}t\omega^2), \quad \text{and} \\
\Phi_{X(t)}(\omega) &= \Phi_{X(t)-X(0)}(\omega)\Phi_{X_0}(\omega) \\
&= \exp(-\frac{1}{2}t\omega^2 - \frac{1}{2}\omega^2) = \\
&= \exp\left(-\frac{1}{2}(t+1)\omega^2\right).
\end{aligned}$$

Thus, for the pdf, we get

$$f_X(x;t) = \frac{1}{\sqrt{2\pi(t+1)}}\exp\left(-\frac{1}{2}\frac{x^2}{t+1}\right).$$

40. (a) See definition 9.2-1.

(b) See definition 9.2-4.

(c)

$$\begin{aligned}
f(x_{t_n}|x_{t_{n-1}}, x_{t_{n-2}}, ..., x_{t_1}) &= f(x_{t_n} - x_{t_{n-1}}|x_{t_{n-1}}, x_{t_{n-2}}, ..., x_{t_1}), \quad \text{since } x_{t_{n-1}} \text{ is conditionally known,} \\
&= f(x_{t_n} - x_{t_{n-1}}), \quad \text{by independent increments property,} \\
&= f(x_{t_n}|x_{t_{n-1}}), \quad \text{by Markov property.}
\end{aligned}$$

41. $Y(t) = X(t) - X(T)$ for $t \geq T$. So $\mu_Y(t) = \mu_X(t) - \mu_X(T) = \mu_0 - \mu_0 = 0$, for $t \geq T$. Next consider the covariance function of $Y(t)$.

$$\begin{aligned} K_{YY}(t_1, t_2) &= E[Y(t_1)Y(t_2)] \\ &= E[(X(t_1) - X(T))\,(X(t_2) - X(T))]. \end{aligned}$$

Consider the case $t_2 > t_1$, then we can write the second factor $(X(t_2) - X(T))$ as

$$(X(t_2) - X(t_1)) + (X(t_1) - X(T))\,,$$

with both terms independent because of the independent increments property. So for the above expectation, we have

$$\begin{aligned} E[(X(t_1) - X(T))\,(X(t_2) - X(T))] &= E[(X(t_1) - X(T))\,(X(t_2) - X(t_1))] \\ &\quad + E[(X(t_1) - X(T))\,(X(t_1) - X(T))] \\ &= E[X(t_1) - X(T)]E[X(t_2) - X(t_1)] + E[(X(t_1) - X(T))^2] \\ &= 0 \times 0 + E[(X(t_1) - X(T))^2] \\ &= E[(X(t_1) - X(T))^2], \quad \text{for all } T \leq t_1 < t_2 \end{aligned}$$

So, we now have to determine $E[(X(t_1) - X(T))^2]$. Exploiting independent increments again, we can write

$$X(t_1) = (X(t_1) - X(T)) + X(T),$$

where the two terms are independent, since $t_1 \geq T$. Thus variances add, and we have

$$\sigma_X^2(t_1) = E[(X(t_1) - X(T))^2] + \sigma_X^2(T),$$

or $E[(X(t_1) - X(T))^2] = \sigma_X^2(t_1) - \sigma_X^2(T)$. So for $t_2 > t_1 (\geq T)$, we have $K_Y(t_1, t_2) = \sigma_X^2(t_1) - \sigma_X^2(T)$. By the symmetry of covariance function, it must be that $K_Y(t_1, t_2) = \sigma_X^2(t_2) - \sigma_X^2(T)$ when $t_2 < t_1$. Note that at $t_1 = t_2$ both answers would be valid. Thus the full answer is

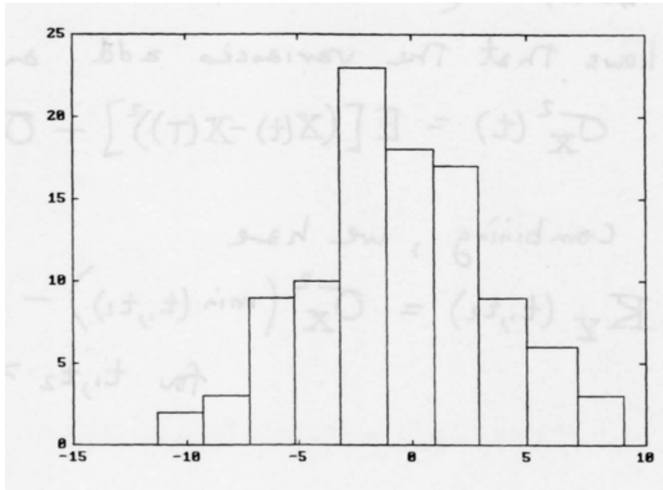
$$K_{YY}(t_1, t_2) = \sigma_X^2 \left(\min(t_1, t_2) - \sigma_X^2(T)\right), \quad \text{for all } t_1, t_2 \geq T.$$

42. We first set $s = \sqrt{\alpha T} = \sqrt{2} \times 0.1$, then the required MATLAB program uses 100 calls `wiener.m` as follows:

```
clear
for i=1:100
wiener;
y(i)=x(1000);
end
hist(y)
mean(y)
std(y)
```

Note though that you have to remove the line with the command `clear` from `wiener.m` for this to work. Also the array size for `x` can be reduced from 4000 to 1000 for this problem. Finally check that your version of wiener.m is using `alpha=2`.

A representative output plot then is:



and mean(y)=-0.5204, std(y)=3.9923, and var=15.9385. The theoretical values are $E[X(t)] = 0$ and $\text{Var}[X(t)] = \alpha t = 2t$. Now since $N = 1000$, we get $t = NT = 10$, so that $\text{Var}[X(t)] = 2t = 20$. Thus the measured variance at 15.9 was somewhat low, but the measured mean at -0.5 is relatively close to zero.

43. We are given the third-order random (stochastic) differential equation

$$\frac{d^3Y(t)}{d^3t} + a_2\frac{d^2Y(t)}{d^2t} + a_1\frac{dY(t)}{dt} + a_0Y(t) = X(t).$$

(a) Putting this into vector form, we have with $\mathbf{Y}(t) \triangleq (Y(t), \dot{Y}(t), \ddot{Y}(t))^T$,

$$\dot{\mathbf{Y}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{Y}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} X(t), \qquad ((1))$$

then

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) We have

$$\dot{\mathbf{Y}}^\dagger(t) = \mathbf{Y}^\dagger(t)\mathbf{A}^\dagger + X(t)\mathbf{B}^\dagger. \qquad ((2))$$

Turning to correlations, then

$$\begin{aligned} \mathbf{R}_{X\dot{\mathbf{Y}}}(\tau) &= E[X(t+\tau)\dot{\mathbf{Y}}^\dagger(t)] \\ &= E[X(t+\tau)\mathbf{Y}^\dagger(t)\mathbf{A}^\dagger] + E[X(t+\tau)X(t)\mathbf{B}^\dagger] \\ &= \mathbf{R}_{X\mathbf{Y}}(\tau)\mathbf{A}^\dagger + R_{XX}(\tau)\mathbf{B}^\dagger \\ &= -\frac{d\mathbf{R}_{X\mathbf{Y}}(\tau)}{d\tau}. \end{aligned}$$

(c) Using above equation (1)

$$\begin{aligned} E[\dot{\mathbf{Y}}(t+\tau)\mathbf{Y}^{\dagger}(t)] &= E[\mathbf{A}\mathbf{Y}(t+\tau)\mathbf{Y}^{\dagger}(t)] + E[\mathbf{B}X(t+\tau)\mathbf{Y}^{\dagger}(t)] \\ &= \mathbf{A}\mathbf{R}_{\mathbf{YY}}(\tau) + \mathbf{B}\mathbf{R}_{X\mathbf{Y}}(\tau) \\ &= \frac{d\mathbf{R}_{\mathbf{YY}}(\tau)}{d\tau}. \end{aligned}$$

(d) Taking the Fourier transforms of the results in part (b), we get

$$-j\omega\mathbf{S}_{X\mathbf{Y}}(\omega) = \mathbf{S}_{X\mathbf{Y}}(\omega)\mathbf{A}^{\dagger} + S_{XX}(\omega)\mathbf{B}^{\dagger}.$$

Solving for $\mathbf{S}_{X\mathbf{Y}}$, we then obtain

$$\mathbf{S}_{X\mathbf{Y}}(\omega) = S_{XX}(\omega)\mathbf{B}^{\dagger}(-j\omega\mathbf{I} - \mathbf{A}^{\dagger})^{-1}.$$

From part (c), taking Fourier transforms, we obtain $(j\omega\mathbf{I} - \mathbf{A})\mathbf{S}_{\mathbf{YY}}(\omega) = \mathbf{B}\mathbf{S}_{X\mathbf{Y}}(\omega)$,with solution

$$\mathbf{S}_{\mathbf{YY}}(\omega) = (j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{S}_{X\mathbf{Y}}(\omega).$$

Combining these two results, we finally get the matrix psd

$$\mathbf{S}_{\mathbf{YY}}(\omega) = (j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}S_{XX}(\omega)\mathbf{B}^{\dagger}(-j\omega\mathbf{I} - \mathbf{A}^{\dagger})^{-1}.$$

44. (a) We start with a definition:

$$\begin{aligned} \mathbf{R}_{\mathbf{YX}}(\tau) &\triangleq E[\mathbf{Y}(t+\tau)\mathbf{X}^{\dagger}(t)] \\ &= E[\mathbf{Y}(t)\mathbf{X}^{\dagger}(t-\tau)] \\ &= E[(\mathbf{h}(t) * \mathbf{X}(t))\mathbf{X}^{\dagger}(t-\tau)] \\ &= \int_{-\infty}^{+\infty} \mathbf{h}(\alpha)E[\mathbf{X}(t-\alpha))\mathbf{X}^{\dagger}(t-\tau)]d\alpha \\ &= \int_{-\infty}^{+\infty} \mathbf{h}(\alpha)\mathbf{R}_{\mathbf{XX}}(\tau-\alpha)d\alpha \\ &= \mathbf{h}(\tau) * \mathbf{R}_{\mathbf{XX}}(\tau). \end{aligned}$$

Here, the superscript daggar indicates Hermitian transpose. We also have

$$\begin{aligned} \mathbf{R}_{\mathbf{YY}}(\tau) &\triangleq E[\mathbf{Y}(t+\tau)\mathbf{Y}^{\dagger}(t)] \\ &= E[\mathbf{Y}(t+\tau)\int_{-\infty}^{+\infty} \mathbf{X}^{\dagger}(t-\alpha)\mathbf{h}^{\dagger}(\alpha)d\alpha] \\ &= \int_{-\infty}^{+\infty} E[\mathbf{Y}(t+\tau)\mathbf{X}^{\dagger}(t-\alpha)]\mathbf{h}^{\dagger}(\alpha)d\alpha \\ &= \int_{-\infty}^{+\infty} \mathbf{R}_{\mathbf{YX}}(\tau+\alpha)\mathbf{h}^{\dagger}(\alpha)d\alpha \\ &= \mathbf{R}_{\mathbf{YX}}(\tau) * \mathbf{h}^{\dagger}(-\tau). \end{aligned}$$

Then, combining these two results we have

$$\begin{aligned} \mathbf{R}_{\mathbf{YY}}(\tau) &= \mathbf{R}_{\mathbf{YX}}(\tau) * \mathbf{h}^{\dagger}(-\tau) \\ &= \mathbf{h}(\tau) * \mathbf{R}_{\mathbf{XX}}(\tau) * \mathbf{h}^{\dagger}(-\tau). \end{aligned}$$

(b) Define the centered random vectors as

$$\mathbf{X}_c(t) \triangleq \mathbf{X}(t) - \boldsymbol{\mu}_{\mathbf{X}}(t) \quad \text{and} \quad \mathbf{Y}_c(t) \triangleq \mathbf{Y}(t) - \boldsymbol{\mu}_{\mathbf{Y}}(t).$$

Then, by the equation $\mathbf{Y}(t) = \mathbf{h}(t) * \mathbf{X}(t)$, we have

$$\boldsymbol{\mu}_{\mathbf{Y}}(t) = \mathbf{h}(t) * \boldsymbol{\mu}_{\mathbf{X}}(t).$$

Thus

$$\begin{aligned} \mathbf{Y}_c(t) &= \mathbf{h}(t) * \mathbf{X}(t) - \boldsymbol{\mu}_{\mathbf{Y}}(t) \\ &= \mathbf{h}(t) * \mathbf{X}(t) - \mathbf{h}(t) * \boldsymbol{\mu}_{\mathbf{X}}(t) \\ &= \mathbf{h}(t) * (\mathbf{X}(t) - \boldsymbol{\mu}_{\mathbf{Y}}(t)) \\ &= \mathbf{h}(t) * \mathbf{X}_c(t), \end{aligned}$$

by linearity. So, upon application of the result of part (a) to this equation, we immediately have

$$\mathbf{R}_{\mathbf{Y}_c\mathbf{Y}_c}(\tau) = \mathbf{h}(\tau) * \mathbf{R}_{\mathbf{X}_c\mathbf{X}_c}(\tau) * \mathbf{h}^{\dagger}(-\tau).$$

But $\mathbf{R}_{\mathbf{X}_c\mathbf{X}_c}(\tau) = \mathbf{K}_{\mathbf{XX}}(\tau)$ and $\mathbf{R}_{\mathbf{Y}_c\mathbf{Y}_c}(\tau) = \mathbf{K}_{\mathbf{YY}}(\tau)$, so finally

$$\mathbf{K}_{\mathbf{YY}}(\tau) = \mathbf{h}(\tau) * \mathbf{K}_{\mathbf{XX}}(\tau) * \mathbf{h}^{\dagger}(-\tau).$$

45. (a) $Y(t) = \mathbf{A}^T\mathbf{X}(t)$, so

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t+\tau)Y^*(t)] \\ &= E[\mathbf{A}^T\mathbf{X}(t+\tau)\mathbf{X}^{\dagger}(t)\mathbf{A}^*] \\ &= \mathbf{A}^T\mathbf{R}_{\mathbf{XX}}(\tau)\mathbf{A}^*. \end{aligned}$$

Thus

$$\begin{aligned} S_{YY}(\omega) &\triangleq FT\{R_{YY}(\tau)\} \\ &= FT\{\mathbf{A}^T\mathbf{R}_{\mathbf{XX}}(\tau)\mathbf{A}^*\} \\ &= \sum_{i,k} a_i a_k^* FT\{(\mathbf{R}_{\mathbf{XX}}(\tau))_{i,k} \\ &= \mathbf{A}^T\mathbf{S}_{\mathbf{XX}}(\omega)\mathbf{A}^*. \end{aligned}$$

(b) Since $S_{YY}(\omega) \geq 0$ for all ω and for all vectors $\mathbf{A}$, then part (a) shows that $\mathbf{S}_{\mathbf{XX}}(\omega)$ must be a positive semidefinite matrix for each frequency ω.

46. (a)

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) + S_{UU}(\omega) \quad \text{since} \quad X \perp U.$$

(b)

$$S_{X\widehat{X}}(\omega) = H^*(\omega)G^*(\omega)S_{XX}(\omega) \quad \text{since} \quad X \perp U.$$

(c)

$$\begin{aligned} S_{\varepsilon\varepsilon}(\omega) &= FT\{R_{\varepsilon\varepsilon}(\tau)\} \\ &= FT\{E[(\widehat{X}(t+\tau) - X(t+\tau))(\widehat{X}(t) - X(t))^*]\} \\ &= FT\{R_{\widehat{X}\widehat{X}}(\tau) - R_{X\widehat{X}}(\tau) - R_{\widehat{X}X}(\tau) + R_{XX}(\tau)\} \\ &= S_{\widehat{X}\widehat{X}}(\omega) - S_{X\widehat{X}}(\omega) - S_{\widehat{X}X}(\omega) + S_{XX}(\omega). \end{aligned}$$

We still need to find $S_{\widehat{X}\widehat{X}}$ and proceed as follows

$$S_{\widehat{X}\widehat{X}}(\omega) = |H(\omega)G(\omega)|^2 S_{XX}(\omega) + |G(\omega)|^2 S_{UU}(\omega) \quad \text{since} \quad X \perp U.$$

Then combining results, we have

$$\begin{aligned}
S_{\varepsilon\varepsilon}(\omega) &= S_{\widehat{X}\widehat{X}}(\omega) - S_{X\widehat{X}}(\omega) - S_{\widehat{X}X}(\omega) + S_{XX}(\omega) \\
&= |HG|^2 S_{XX} + |G|^2 S_{UU} - H^* G^* S_{XX} - HGS_{XX} + S_{XX} \\
&= (|H|^2|G|^2 - 2\,\mathrm{Re}(HG) + 1) S_{XX} + |G|^2 S_{UU} \\
&= |HG - 1|^2 S_{XX} + |G|^2 S_{UU}.
\end{aligned}$$

(d) For those ω such that $S_{XX}(\omega) >> S_{UU}(\omega)$, we then have

$$S_{\varepsilon\varepsilon}(\omega) \approx |H(\omega)G(\omega) - 1|^2 S_{XX}(\omega),$$

which is minimum (actually 0) at $G = H^{-1}$ assuming the inverse exists. Similarly, for $S_{XX}(\omega) << S_{UU}(\omega)$, we have

$$S_{\varepsilon\varepsilon}(\omega) \approx |G(\omega)|^2 S_{UU}(\omega),$$

which is minimized at $G = 0$.

47. (a) Now $S_Z(\omega) = |H(\omega)|^2 S_Y(\omega)$, so

$$S_Y(\omega) = \frac{S_Z(\omega)}{|H(\omega)|^2}.$$

Also

$$\begin{aligned}
H(\omega) &= \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \\
&= \int_0^{+\infty} e^{-t} e^{-j\omega t} dt \\
&= \frac{1}{j\omega + 1,}
\end{aligned}$$

hence

$$|H(\omega)|^2 = \frac{1}{\omega^2 + 1},$$

and so we have

$$\begin{aligned}
S_Y(\omega) &= \frac{S_Z(\omega)}{|H(\omega)|^2} \\
&= (\omega^2 + 1) S_Z(\omega) \\
&= (\omega^2 + 1)\left(\frac{2\beta}{(\omega^2 + \beta^2)(\omega^2 + 1)} + \pi\delta(\omega)\right) \\
&= \frac{2\beta}{\omega^2 + \beta^2} + \pi\delta(\omega).
\end{aligned}$$

Then upon inverse Fourier Transform, we have

$$R_Y(\tau) = e^{-\beta|\tau|} + \frac{1}{2}, \qquad -\infty < \tau < +\infty.$$

(b) For this part, we assume the input random process $X(t)$ is zero-mean.

$$\begin{aligned} E[Y(t)Y(t+\tau)] &= E[X^2(t)X^2(t+\tau)] \\ &= E[X(t)X(t)X(t+\tau)X(t+\tau)] \\ &= R_X(0)R_X(0) + R_X(\tau)R_X(\tau) + R_X(\tau)R_X(\tau), \end{aligned}$$

by 4^{th}-order moment property for zero-mean Gaussian RVs (see problem 4.66), and so

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau).$$

Combining with the result of part (a), we get

$$e^{-\beta|\tau|} + \frac{1}{2} = R_X^2(0) + 2R_X^2(\tau).$$

Plugging in $\tau = 0$, we find that $R_X^2(0) = \frac{1}{2}$, and hence $R_X(0) = \frac{1}{\sqrt{2}}$ since we must take the non-negative root. But then

$$e^{-\beta|\tau|} + \frac{1}{2} = \frac{1}{2} + 2R_X^2(\tau),$$

with non-negative definite solution

$$R_X(\tau) = \frac{1}{\sqrt{2}} \exp -\frac{\beta}{2}|\tau|.$$

48.

(a)

$$\begin{aligned} P_1(t+dt) &= P_1(t)(1-1dt) + P_2(t)(2dt) \\ P_2(t+dt) &= P_1(t)(1dt) + P_2(t)(1-2dt), \end{aligned}$$

or, forming derivative terms on the left-hand side,

$$\begin{aligned} \frac{P_1(t+dt)-P_1(t)}{dt} &= -P_1(t) + 2P_2(t) \\ \frac{P_2(t+dt)-P_2(t)}{dt} &= P_1(t) - 2P_2(t), \end{aligned}$$

(b) For steady state, $dP_i/dt = 0$, which then implies $P_1 = 2P_2$, but always $P_1 + P_2 = 1$, so we have asympotically, the steady-state probabilities:

$$P_1 = \frac{2}{3} \quad \text{and} \quad P_2 = \frac{1}{3}.$$

49. Since X and Y are jointly WSS,

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{+j\omega\tau}d\omega,$$

then directly from part (a), we get

$$\left|\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{+j\omega\tau}d\omega\right| \leq \sqrt{\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega \int_{-\infty}^{+\infty} S_{YY}(\omega)d\omega}.$$

Then, putting $X(t)$ and $Y(t)$ through narrow-band filters with gain 1, bandwidth ϵ (a very small positive number), and centered at frequency ω_0, call the respective outputs $\widetilde{X}(t)$ and $\widetilde{Y}(t)$ Then, written for processes $\widetilde{X}(t)$ and $\widetilde{Y}(t)$, this above inequality becomes

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} S_{\widetilde{X}\widetilde{Y}}(\omega)e^{+j\omega\tau}d\omega\right| &\approx S_{XY}(\omega_0)|e^{+j\omega_0\tau}|\epsilon \\ &= S_{XY}(\omega_0)\epsilon \\ &\leq \sqrt{S_{XX}(\omega_0)\epsilon\, S_{YY}(\omega_0)\epsilon} \\ &= \sqrt{S_{XX}(\omega_0)S_{YY}(\omega_0)}\epsilon.\end{aligned}$$

Then, cancelling out ϵ and letting $\epsilon \searrow 0$, we get the exact inequality

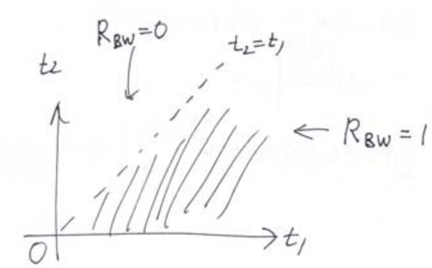
$$S_{XY}(\omega_0) \leq \sqrt{S_{XX}(\omega_0)S_{YY}(\omega_0)}.$$

Finally since ω_0 is an arbitrary frequency, this inequality must be true for all $-\infty < \omega < +\infty$, as was to be shown.

50. (a)

$$\begin{aligned}R_{BW}(t_1,t_2) &= E\left[\int_0^{t_1} W(\tau_1)d\tau_1 W(t_2)\right] \\ &= \int_0^{t_1} R_{WW}(\tau_1,t_2)d\tau_1 \\ &= \int_0^{t_1} \delta(\tau_1 - t_2)d\tau_1 \\ &= u(t_1 - t_2).\end{aligned}$$

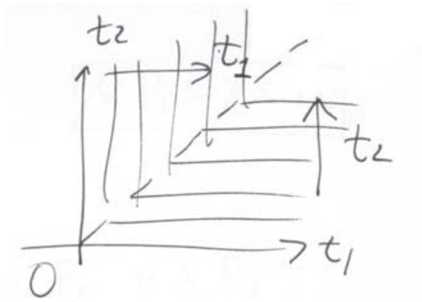
Here is a sketch.



(b) Let the two times $t_1, t_2 \geq 0$, then

$$\begin{aligned}R_{BB}(t_1,t_2) &= E\left[\int_0^{t_1}\int_0^{t_2} W(\tau_1)W(\tau_2)d\tau_1 d\tau_2\right] \\ &= \int_0^{t_1}\int_0^{t_2} R_{WW}(\tau_1,\tau_2)d\tau_1 d\tau_2 \\ &= \int_0^{t_1}\int_0^{t_2} \delta(\tau_1-\tau_2)d\tau_1 d\tau_2 \\ &= \int_0^{t_1}\left(\int_0^{t_2} \delta(\tau_1-\tau_2)d\tau_2\right)d\tau_1 \\ &= \int_0^{t_1} u(t_2-\tau_1)d\tau_1 \\ &= \min(t_1,t_2), t_1,t_2 \geq 0.\end{aligned}$$

Here is a sketch.



51. (a)

$$\begin{aligned} p_0(t+\delta t) &= (1-2\mu\,\delta t)p_0(t) + \lambda\,\delta t\, p_1(t) + 0 \\ p_1(t+\delta t) &= 2\mu p_0(t) + (1-(\mu+\lambda)\delta t)p_1(t) + 2\lambda\,\delta t\, p_2(t) \\ p_2(t+\delta t) &= 0 + \mu\,\delta t\, p_1(t) + (1-2\lambda\delta t)p_2(t), \end{aligned}$$

which yeilds $d\mathbf{p}(t)/dt = \mathbf{A}\mathbf{p}(t)$ with

$$\mathbf{A} \triangleq \begin{bmatrix} -2\mu & +\lambda & 0 \\ +2\mu & -(\mu+\lambda) & +2\lambda \\ 0 & +\mu & -2\lambda \end{bmatrix}.$$

(b) In steady-state, the state probabilities $p_i(t)$ are constant, so the flow or rate of increase over δt must balance out between adjacent states $(i-1)$ and i. If all the net flows are zero, then $p_i(t)$ cannot change. Looking at the state-transition diagram, we see the flow from state 0 to state 1 is $2\mu p_0(t)$, while the flow from state 1 to state 0 is $\lambda p_1(t)$, so to balance, we must have $2\mu p_0 = \lambda p_1$ in steady state. Between states 1 and 2, we similarly balance the flow by setting $\mu p_1 = 2\lambda p_2$. To see whether this solution solves $\mathbf{A}\mathbf{p} = \mathbf{0}$, we write its resulting three equations

$$\begin{aligned} -2\mu p_0 + \lambda p_1 \qquad &= 0, \\ +2\mu p_0 - (\mu+\lambda)p_1 + 2\lambda p_2 &= 0, \\ +\mu p_1 - 2\lambda p_2 &= 0. \end{aligned}$$

We can see that the two flow solutions are equivalent to the first and third equation arising from $\mathbf{A}\mathbf{p} = \mathbf{0}$. Since the middle equation is obtained by summing the first and last one, and then multiplying by -1, we conclude that $\mathbf{A}\mathbf{p} = \mathbf{0}$ is satisfied.

(c) From our two flow equations

$$p_1 = \frac{2\mu}{\lambda}p_0 \quad \text{and} \quad p_2 = \frac{\mu}{2\lambda}p_1 = \left(\frac{\mu}{\lambda}\right)^2 p_0.$$

But also,

$$\begin{aligned} 1 &= p_0 + p_1 + p_2 \\ &= p_0\left(1 + \frac{2\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right). \end{aligned}$$

Thus, we have the steady-state solution

$$\begin{aligned}
p_0 &= \frac{1}{1+\frac{2\mu}{\lambda}+\left(\frac{\mu}{\lambda}\right)^2} = \frac{\lambda^2}{\lambda^2+2\mu\lambda+\mu^2}, \\
p_1 &= \frac{2\mu\lambda}{\lambda^2+2\mu\lambda+\mu^2}, \\
p_2 &= \frac{\mu^2}{\lambda^2+2\mu\lambda+\mu^2}.
\end{aligned}$$

For $\lambda = 0.001$ and $\mu = 0.1$, we get $\lambda^2+2\mu\lambda+\mu^2 \approx 10^{-2}$, and then

$$\begin{aligned}
p_0 &\approx \frac{10^{-6}}{10^{-2}} = 10^{-4}, \\
p_1 &\approx \frac{2\times 10^{-4}}{10^{-2}} = 2\times 10^{-2}, \quad \text{and} \\
p_2 &\approx 1-2\times 10^{-2} = 0.98.
\end{aligned}$$

52. (a) We have $Y_1 = h * X_1 + g * X_2$, thus

$$\begin{aligned}
S_{Y_1X_1}(\omega) &= FT\{R_{Y_1X_1}(\tau)\} \\
&= FT\{E[(h*X_1)(t+\tau)X_1^*(t)] + E[(g*X_2)(t+\tau)X_1^*(t)]\} \\
&= FT\{h(\tau)*R_{X_1X_1}(\tau) + g(\tau)*R_{X_2X_1}(\tau)\} \\
&= FT\{h(\tau)*R_{X_1X_1}(\tau)\}, \quad \text{since } X_1 \perp X_2, \\
&= H(\omega)S_{X_1X_1}(\omega).
\end{aligned}$$

(b)

$$\begin{aligned}
S_{Y_2X_2}(\omega) &= FT\{E[Y_2(t+\tau)X_2^*(t)]\} \\
&= FT\{b(\tau)*R_{X_2X_2}(\tau)\}, \quad \text{since } \ U \perp X_2, \\
&= B(\omega)S_{X_2X_2}(\omega).
\end{aligned}$$

(c)

$$\begin{aligned}
E[Y_1(t+\tau)Y_2^*(t)] &= E[(h*X_1+g*X_2)(t+\tau)(U^*+b^* * X_2^*)(t)] \\
&= E[(g*X_2)(t+\tau)(b^* * X_2^*)(t)], \quad \text{since } \ X_1 \perp X_2, X_1 \perp U, \text{ and } X_2 \perp U, \\
&= g(\tau)*b^*(-\tau)*R_{X_2X_2}(\tau).
\end{aligned}$$

Thus

$$\begin{aligned}
S_{Y_1Y_2}(\omega) &= FT\{g(\tau)*b^*(-\tau)*R_{X_2X_2}(\tau)\} \\
&= G(\omega)\,B^*(\omega)S_{X_2X_2}(\omega).
\end{aligned}$$

53. (a)

$$\begin{aligned} E[Y(t_1)Y^*(t_2)] &= E\left[\sum_{n_1,n_2}^{N-1,N-1} A_{n_1}X(t_1-n_1T)A^*_{n_2}X^*(t_2-n_2T)\right] \\ &= \sum_{n_1,n_2}^{N-1,N-1} E[A_{n_1}A^*_{n_2}]E[X(t_1-n_1T)X^*(t_2-n_2T)] \\ &= \sum_{n_1,n_2}^{N-1,N-1} R_{AA}(n_1,n_2)R_{XX}(t_1-t_2-(n_1-n_2)T) \\ &= R_{YY}(t_1,t_2). \end{aligned}$$

(b) From part (a), we can see

$$\begin{aligned} R_{YY}(t+\tau,t) &= \sum_{n_1,n_2}^{N-1,N-1} R_{AA}(n_1,n_2)R_{XX}(\tau-(n_1-n_2)T) \\ &= R_{YY}(\tau), \quad \text{independent of } t. \end{aligned}$$

Also

$$\mu_Y(t) = \left(\sum_n \mu_A[n]\right)\mu_X = \text{ constant wrt } t.$$

So, we don't need A_n to be a WSS random sequence, in order to make $Y(t)$ be a WSS random process.

(c) No. If A_n and $X(t)$ are correlated, we only have $E[A_nX(t-nT)] = E[A_n]E[X(t-nT)]$, not the four term product given in part (a). This is a type of 4th order moment.

54.

(a) Equating probability flows across a cut between the two states, we get $P_G\lambda_{GB}\delta t = P_B\lambda_{BG}\delta t$, so

$$\begin{aligned} P_B &= \frac{\lambda_{GB}}{\lambda_{BG}}P_G \\ &= \frac{\lambda_{GB}}{\lambda_{BG}}(1-P_B), \end{aligned}$$

which solves to

$$P_B = \frac{\lambda_{GB}}{\lambda_{GB}+\lambda_{BG}} \qquad \text{and then,} \quad P_G = \frac{\lambda_{BG}}{\lambda_{GB}+\lambda_{BG}}.$$

(b) In the *bad* state, the transition probability time τ is an exponential RV with pdf

$$f_\tau(t) = \lambda_{BG}e^{-\lambda_{BG}t}u(t).$$

The average value is then $\mu_\tau = 1/\lambda_{BG}$, the average error-burst length.

55. Here $\lambda = 3$.

(a) For $n = 2$ and $t = 4$, we have

$$\begin{aligned}
P_N(2;4) &\triangleq P[N(4)=2] \\
&= \frac{(3\cdot 4)^2}{2!}e^{-3\cdot 4} \\
&= \frac{144}{2}e^{-12} \\
&= 72e^{-12} \\
&\doteq 4.42\times 10^{-4}.
\end{aligned}$$

(b)

$$\begin{aligned}
P_N(1,2;1,2) &\triangleq P[N(1)=1, N(2)=2] \\
&= P[N(1)=1]P[N(1)=1] \\
&= P_N(1;1)P_N(1;1) \\
&= P_N^2(1;1) \\
&= \left(\frac{3^1}{1!}e^{-3\cdot 1}\right)^2 \\
&= 9e^{-6} \\
&\doteq 0.0223.
\end{aligned}$$

56. (a) $S_{YY}\left(\omega\right) = |H(\omega)|^2(S_{XX}(\omega) + S_{VV}(\omega))$.

(b) We see from the figure that $Y(t) = \int h(s)[X(t-s) + V(t-s)]ds$, so

$$\begin{aligned}
R_{XY}(\tau) &= E[X(t+\tau)Y^*(t)] \\
&= \int_{-\infty}^{+\infty} h^*(s)E[X(t+\tau)(X^*(t-s) + V^*(t-s))]ds \\
&= \int_{-\infty}^{+\infty} h^*(s)E[X(t+\tau)X^*(t-s)]ds \qquad \text{since } X \perp V, \\
R_{XY}(\tau) &= \int_{-\infty}^{+\infty} h^*(s)R_{XX}(\tau+s)d\tau \\
&= \int_{-\infty}^{+\infty} h^*(-s')R_{XX}(\tau-s')ds' \qquad \text{with} \quad s' \triangleq -s \\
&= h^*(-\tau) * R_{XX}(\tau),
\end{aligned}$$

so upon Fourier transformation $S_{YY}(\omega) = H^*(\omega)S_{XX}(\omega)$.

57.

$$\begin{aligned}
E[|\widetilde{X}(t)|^2] &= E[|X(t)+U(t)|^2] \\
&= E[|X(t)|^2] + E[|U(t)|^2] \\
&= P + \epsilon \\
&= E[\widetilde{Y}(t)|^2].
\end{aligned}$$

$$\begin{aligned}
E[\widetilde{X}(t_1)\widetilde{Y}^*(t_2)] &= E[X(t_1)(Y^*(t_2)+V^*(t_2))] + E[U(t_1)(Y^*(t_2)+V^*(t_2))] \\
&= E[X(t_1)Y^*(t_2)] \\
&= \rho_{XY}(t_1,t_2)P.
\end{aligned}$$

So $\rho_{\widetilde{X}\widetilde{Y}}(t_1,t_2)=\rho_{XY}(t_1,t_2)\frac{P}{P+\epsilon}$.

58.

(a)

$$\begin{aligned} &S_{YY}\left(\omega\right)=|H(\omega)|^2(S_{XX}(\omega)+S_{VV}(\omega)) \\ =\ & 100w(\omega)\left(\frac{1}{\omega^2+5}+\frac{2\omega^2+8}{(\omega^2+3)\,(\omega^2+5)}\right), \qquad |\omega|\le\pi, \\ =\ & 100\left(\frac{3\omega^2+11}{(\omega^2+3)\,(\omega^2+5)}\right)w(\omega), \end{aligned}$$

where $w(\omega)\triangleq u(\omega+\frac{\pi}{2})-u(\omega-\frac{\pi}{2})$.

(b) Call $W[n]\triangleq X[n]+V[n]$, then

$$\begin{aligned} R_{WW}[m] &= E((X[n+m]+V[n+m])(X^*[n]+V^*[n])] \\ &= R_{XX}[m]+R_{XV}[m]+R_{VX}[m]+R_{VV}[m]. \end{aligned}$$

So, upon Fourier transformation,

$$\begin{aligned} S_{WW}(\omega) &= S_{XX}(\omega)+S_{XV}(\omega)+S_{VX}(\omega)+S_{VV}(\omega) \\ &= S_{XX}(\omega)+S_{XV}(\omega)+S_{XV}(\omega)+S_{VV}(\omega), \end{aligned}$$

since S_{VX} is real valued. Proceeding, we get

$$\begin{aligned} S_{WW}(\omega) &= \frac{3}{\omega^2+5}+\frac{2\omega^2+8}{(\omega^2+3)\,(\omega^2+5)} \\ &= \frac{5\omega^2+17}{(\omega^2+3)\,(\omega^2+5)}. \end{aligned}$$

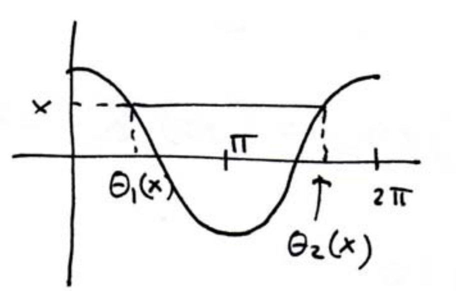
So,

$$\begin{aligned} &S_{YY}\left(\omega\right)=|H(\omega)|^2S_{WW}(\omega) \\ =\ & 100\frac{5\omega^2+17}{(\omega^2+3)\,(\omega^2+5)}w(\omega), \qquad |\omega|\le\pi. \end{aligned}$$

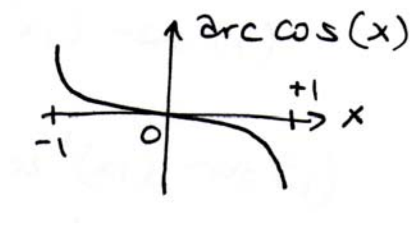
59. The easy way is to first define $\Theta_t\triangleq\omega_0t+\Theta$, and then note that since $\Theta:U[0,2\pi]$, so must Θ_t also be uniformly distributed on $[0,2\pi]$. Then for fixed t, the transformation $X(t)=\cos\Theta_t=x$ has two roots θ_1 and θ_2, given as

$$\theta_1(x)=\cos^{-1}x \quad\text{and}\quad \theta_2(x)=2\pi-\cos^{-1}x\ ,$$

as indicated in the figure below.



Here $\cos^{-1}(\cdot)$ is the principal value arc cosine as indicated in the figure below.



We then have

$$\begin{aligned}\frac{d\theta_1}{dx} &= \frac{d\cos^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \text{and} \\ \frac{d\theta_2}{dx} &= -\frac{d\cos^{-1}x}{dx} = \frac{-1}{\sqrt{1-x^2}},\end{aligned}$$

so

$$\begin{aligned}f_X(x;t) &= f_\Theta(\theta_1(x))\left|\frac{d\theta_1}{dx}\right| + f_\Theta(\theta_2(x))\left|\frac{d\theta_2}{dx}\right| \\ &= \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-x^2}}, \qquad |x|<1, \\ &= \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}(u(x+1)-u(x-1)),\end{aligned}$$

where $u(\cdot)$ denotes the continuous parameter unit step function. Since the pdf $f_X(x;t)$ is independent of t, the process is first-order stationary by definition. The direct method is to work directly with Θ rather than defining Θ_t first. In that case, we would get

$$\theta_1(x) = \cos^{-1}x - \omega_0 t \quad \text{and} \quad \theta_2(x) = 2\pi - \cos^{-1}x - \omega_0 t\ ,$$

with Θ uniformly distributed on $[0, 2\pi]$. The problem then proceeds as above, with the derivatives unchanged. The resulting answer is the same $f_X(x;t)$ that is independent of time t.

To find the conditional density $f_X(x_2|x_1;t_2,t_1)$ for $t_2 > t_1$, we proceed as follows. Given $X(t_1) = x_1 = \cos(\omega_0 t_1 + \theta)$, then the two corresponding roots are $\theta_1 = \cos^{-1}x_1 - \omega_0 t_1$ and $\theta_2 = 2\pi - \cos^{-1}x_1 - \omega_0 t_1$ equally likely in Θ. So $X(t_2) = \cos(\omega_0 t_2 + \cos^{-1}x_1 - \omega_0 t_1)$ or $X(t_2) = \cos(\omega_0 t_2 + 2\pi - \cos^{-1}x_1 - \omega_0 t_1)$, equally likely with probability 1/2. This simplifies to

$$X(t_2) = \cos(\omega_0(t_2-t_1) \pm \cos^{-1}x_1), \quad \text{equally likely.}$$

Thus we end up with conditional probability

$$f_X(x_2|x_1;t_2,t_1) = \frac{1}{2}\delta(x_2 - \cos(\omega_0(t_2-t_1)+\cos^{-1}x_1)) + \frac{1}{2}\delta(x_2-\cos(\omega_0(t_2-t_1)-\cos^{-1}x_1)).$$

Using the trigonometric identities $\cos(A\pm B) = \cos A\cos B \mp \sin A \sin B$, we can also write

$$\cos(\omega_0(t_2-t_1)\pm\cos^{-1}x_1) = x_1\cos\omega_0(t_2-t_1) \mp \sqrt{1-x^2}\sin\omega_0(t_2-t_1),$$

with a consequent change in the expression for $f_X(x_2|x_1;t_2,t_1)$ above.

60.

$$\begin{aligned} U(t) &\triangleq \mathrm{Re}[Z(t)e^{-j\omega_0 t}] \\ &= X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t, \end{aligned}$$

since $Z = X + jY$. Then

$$\begin{aligned} E[U(t+\tau)U(t)] &= \\ & E[(X(t+\tau)\cos\omega_0(t+\tau) + Y(t+\tau)\sin\omega_0(t+\tau))(X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t)] \\ &= R_{XX}(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t + R_{YY}(\tau)\sin\omega_0(t+\tau)\sin\omega_0 t \\ &= R_{XX}(\tau)[\cos\omega_0(t+\tau)\cos\omega_0 t + \sin\omega_0(t+\tau)\sin\omega_0 t], \qquad \text{if} \quad R_{XX}(\tau) = R_{YY}(\tau), \\ &= R_{XX}(\tau)\cos\omega_0\tau, \end{aligned}$$

after making use of the trig identity for $\cos(A - B)$. Since the mean function $\mu_U(t) = 0$, we can then say that the random process $U(t)$ is WSS. The general condition is then $R_{XX}(\tau) = R_{YY}(\tau)$.

61. By equating probability flows, we get the equalities

$$\begin{aligned} \lambda_1 P_1 &= \mu_2 P_2, \\ \lambda_2 P_2 &= \mu_3 P_3, \quad \text{and} \\ \lambda_3 P_3 &= \mu_4 P_4. \end{aligned}$$

From the first equation, $P_2 = \frac{\lambda_1}{\mu_2} P_1$, and then

$$\begin{aligned} P_3 &= \frac{\lambda_2}{\mu_3} P_2 \\ &= \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} P_1, \end{aligned}$$

and

$$\begin{aligned} P_4 &= \frac{\lambda_3}{\mu_4} P_3 \\ &= \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} P_1. \end{aligned}$$

Using the fact that these four probabilities must also sum to one, i.e. $\sum_i P_i = 1$, we finally get

$$\begin{aligned} P_1 &= \frac{1}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} + \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\ P_2 &= \frac{\frac{\lambda_1}{\mu_2}}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} + \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \\ P_3 &= \frac{\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} + \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}, \quad \text{and} \\ P_4 &= \frac{\frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}{1 + \frac{\lambda_1}{\mu_2} + \frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2} + \frac{\lambda_3}{\mu_4}\frac{\lambda_2}{\mu_3}\frac{\lambda_1}{\mu_2}}. \end{aligned}$$

62. (a) The probability of leaving state 2 for the first time at time t is zero, since the waiting time is an exponential RV, a continuous RV.

(b)

$$\begin{aligned}
P_1(t+\delta t) &= (1-\lambda_1\,\delta t)P_1(t) + \mu_2\,\delta t\,P_2(t) + 0P_3(t) \\
P_2(t+\delta t) &= +\lambda_1\,\delta t\,P_1(t) - (\lambda_2+\mu_2)\delta t\,P_2(t) + \mu_3\,\delta t\,P_3(t) \\
P_3(t+\delta t) &= 0P_1(t) + \lambda_2\,\delta t\,P_2(t) + -\mu_3\,\delta t\,P_3(t),
\end{aligned}$$

or

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \underbrace{\begin{bmatrix} -\lambda_1 & +\mu_2 & 0 \\ +\lambda_1 & -(\lambda_2+\mu_2) & +\mu_3 \\ 0 & +\lambda_2 & -\mu_3 \end{bmatrix}}_{\triangleq \mathbf{A}} \mathbf{P}(t) \\
&= \mathbf{A}\mathbf{P}(t).
\end{aligned}$$

(c) We substitute $\exp(\mathbf{A}t)\cdot\mathbf{P}(0)$ into this equation, and then take the term-by-term derivative of the matrix-exponential series, to obtain

$$\begin{aligned}
\dot{\mathbf{P}}(t) &= \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{1}{k!}(\mathbf{A}t)^k\right)\mathbf{P}(0) \\
&= \left(\sum_{k=1}^{\infty}\frac{1}{k!}k\mathbf{A}^k t^{k-1}\right)\mathbf{P}(0) \\
&= \left(\mathbf{A}\sum_{k'=0}^{\infty}\frac{1}{k'!}(\mathbf{A}t)^{k'}\right)\mathbf{P}(0), \quad \text{with} \quad k' \triangleq k-1, \\
&= \mathbf{A}\exp(\mathbf{A}t)\cdot\mathbf{P}(0) \\
&= \mathbf{A}\mathbf{P}(t),
\end{aligned}$$

as was to be shown.