Solutions to Chapter 4

1. The sample mean for the set of numbers is given by

$$X_s = \frac{1}{12}\sum_{i=1}^{12} X_i \approx 1.07.$$

The standard deviation is given by

$$\sigma_s = \sqrt{\frac{1}{12}\sum_{i=1}^{12}(X_i - X_s)^2} \approx 3.98.$$

2. We are given that X is Bernoulli distributed over $\{0,1\}$ with parameter $p(1 \geq p \geq 0)$ with the PMF

$$P_X(k) = \begin{cases} p, & k=1, \\ 1-p, & k=0 \\ 0, & \text{else.} \end{cases} .$$

So

$$E[X] = \sum_{k=-\infty}^{+\infty} kP_X(k) = 1p + 0(1-p) = p.$$

3. Since the random variable X is the constant value c, then its pdf $f_X(x) = \delta(x-c)$, so

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\ &= \int_{-\infty}^{\infty} x\delta(x-c)dx \\ &= c. \end{aligned}$$

4. Here the random variable X is binomial distributed with parameters n a positive integer and $p(1 \geq p \geq 0)$. We are asked for the expected value $E[X]$. The PMF is given as

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \left(u(k) - u(n+1-k)\right).$$

Thus

$$\begin{aligned} E[X] &= \sum_{k=-\infty}^{+\infty} kP_X(k) \\ &= \sum_{k=-\infty}^{+\infty} k\binom{n}{k} p^k (1-p)^{n-k} \left(u(k) - u(n+1-k)\right) \\ &= \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}. \end{aligned}$$

To evaluate this sum, we attempt to modify it so that it looks like the binomial sum that we know. To this end, we pull out n from $\binom{n}{k}$ and pull out the variable p also, to obtain

$$
\begin{aligned}
E[X] &= \sum_{k=1}^{n} k\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \qquad \text{since the } k=0 \text{ term will be 0,}\\
&= np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}p^{(k-1)}(1-p)^{n-k}\\
&= np\sum_{k=0}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}p^{(k-1)}(1-p)^{(n-1)-(k-1)}\\
&= np\sum_{i=0}^{n-1}\frac{(n-1)!}{i!(n-1-i)!}p^{i}(1-p)^{(n-1)-i} \quad \text{with the substitution } i \triangleq k-1,
\end{aligned}
$$

$$
\begin{aligned}
&= np\sum_{i=0}^{n-1}\binom{n-1}{i}p^{i}(1-p)^{(n-1)-i}\\
&= np \times (p+(1-p))^{n-1},\\
&= np \times 1 = np.
\end{aligned}
$$

5. The pdf of the uniform random variable X is given as

$$f_X(x) = \frac{1}{b-a}[u(x-a)-u(x-b)].$$

The expectation of X, which is nothing but the mean of X, is

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx\\
&= \frac{1}{b-a}\frac{x^2}{2}|_a^b = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2}.
\end{aligned}
$$

6. (i)

$$F_X(x) = \int_{-\infty}^{x} f_X(v)\,dv,$$

so for $x < 0$, we get $F_X(x) = 0$ since $f_X = 0$ there. For $0 \le x < 1$, we then have

$$
\begin{aligned}
F_X(x) &= \int_0^x 2v\,dv\\
&= x^2.
\end{aligned}
$$

For $x > 1$, we get $F_X(x) = F_X(1) = 1$ since $f_X = 0$ for $x > 1$. Combining these results, we have

$$F_X(x) = \begin{cases} 0, & x < 0,\\ x^2, & 0 \le x < 1,\\ 1, & x \ge 1.\end{cases}$$

(ii)

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} x f_X(x)\,dx \\ &= \int_0^1 x2x\,dx \\ &= 2\frac{x^3}{3}\bigg|_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

(iii)

$$\begin{aligned} \sigma_X^2 &= E[X^2] - (E[X])^2 \\ &= \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx - \left(\frac{2}{3}\right)^2 \\ &= \int_0^1 x^2 2x\,dx - \frac{4}{9} \\ &= 2\frac{x^4}{4}\bigg|_0^1 - \frac{4}{9} \\ &= \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}. \end{aligned}$$

7. The hypergeometric PMF gives the probability of x successes in k draws from a population of size $m+n$ where m is the number of success states in the population of size $m+n$. It is given by

$$P_X(x) = \binom{m+n}{k}^{-1} \binom{m}{x} \binom{n}{k-x}$$

and

$$\sum_{x=0}^{k} P_X(x) = \sum_{x=0}^{k} \binom{m+n}{k}^{-1} \binom{m}{x} \binom{n}{k-x} = 1.$$

Now consider

$$\begin{aligned} E[X] &= \sum_{x=0}^{k} x P_X(x) = \sum_{x=0}^{k} x \binom{m+n}{k}^{-1} \binom{m}{x} \binom{n}{k-x} \\ &= m \binom{m+n}{k}^{-1} \sum_{x=1}^{k} \binom{m-1}{x-1} \binom{n}{k-1-(x-1)}. \end{aligned}$$

Make the following substitutions:

$$j \triangleq x-1, m' \triangleq m-1, \text{ and } k' \triangleq k-1,$$

and obtain

$$E[X] = m \binom{m+n}{k}^{-1} \sum_{j=0}^{k'} \binom{m'}{j} \binom{n}{k'-j}.$$

The next-to-last step is to recognize that $\begin{pmatrix} m+n \\ k \end{pmatrix} = \frac{m+n}{k}\begin{pmatrix} m' \\ k' \end{pmatrix}$ so that finally

$$E[X] = m\left(\frac{m+n}{k}\right)^{-1}\begin{pmatrix} m' \\ k' \end{pmatrix}^{-1}\sum_{j=0}^{k'}\begin{pmatrix} m' \\ j \end{pmatrix}\begin{pmatrix} n \\ k'-j \end{pmatrix}.$$

But the term to the right of the product sign is unity; hence $E[X] = mk/(m+n)$.

8. By using Eq. 4.1-9, we get

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{+\infty} g(x)f_X(x)dx \\ &= \frac{1}{b-a}\int_a^b x^2 dx \\ &= \frac{1}{3(b-a)}(b^3-a^3). \end{aligned}$$

To use Eq. 5.4-1, we must compute the density f_Y . We proceed in an indirect fashion by finding the CDF F_Y first.,

We have

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[X^2 \leq y] \\ &= P[-\sqrt{y} \leq X \leq \sqrt{y}] \\ &= F_X(\sqrt{y}) - F_Y(\sqrt{y}). \end{aligned}$$

Taking derivatives, we then get

$$f_Y(y) = \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}}f_X(-\sqrt{y}).$$

But $f_X(x) = 0$ for $x < a$ or $x > b$, hence with $a > 0$, the second term in the above equation is zero, and so

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) \\ &= \frac{1}{2\sqrt{y}}\frac{1}{b-a}[u(y-a^2) - u(y-b^2)], \end{aligned}$$

where u is the unit step function. We then get for the expectation

$$\begin{aligned} E[Y] &= \int_{-\infty}^{+\infty} y f_Y(y)dy \\ &= \frac{1}{2(b-a)}\int_{a^2}^{b^2}\sqrt{y}dy \\ &= \frac{1}{2(b-a)}\left(\frac{2}{3}y^{\frac{3}{2}}\Big|_{a^2}^{b^2}\right) \\ &= \frac{1}{2(b-a)}\frac{2}{3}(b^3-a^3) \\ &= \frac{1}{3(b-a)}(b^3-a^3), \quad \text{the same as above.} \end{aligned}$$

9. Now $E[Y] = E[X^2+1] = E[X^2]+1$. But from Problem 4.6 we have that $E[X^2] = \sigma_X^2 + \mu_X^2 = 1/18 + 4/9 = 9/18$. Hence $E[Y] = E[X^2] + 1 = 27/18 = 1.5$. To compute σ_Y^2 we proceed as follows:

$$\begin{aligned} E[Y^2] &= E[(X^2+1)^2] \\ &= E[X^4 + 2X^2 + 1] \\ &= E[X^4] + 2 \times 9/18 + 1 \\ &= E[X^4] + 2. \end{aligned}$$

But $E[X^4] = \int_0^1 x^4(2x)dx = 1/3$. Hence $\sigma_Y^2 = E[Y^2] - E^2[Y] = 14/6 - 9/4 = 1/12$.

10. We are given $Y = X^2 + b$, therefore

$$\begin{aligned} E[Y] &= \sum_{k=0}^{\infty}(k^2+b)\frac{e^{-\alpha}\alpha^k}{k!} \\ &= e^{-\alpha}\sum_{k=0}^{\infty}k^2\frac{\alpha^k}{k!} + be^{-\alpha}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}. \end{aligned}$$

Looking at sum in the second term, we see that

$$e^{-\alpha}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!} = 1.$$

Next considering the sum in the first term above, we have

$$\begin{aligned} \sum_{k=0}^{\infty}k^2\frac{\alpha^k}{k!} &= \sum_{k=1}^{\infty}k\frac{\alpha^k}{(k-1)!} \\ &= \sum_{k=1}^{\infty}[(k-1)+1]\frac{\alpha^k}{(k-1)!} \\ &= \sum_{k=2}^{\infty}\frac{\alpha^k}{(k-2)!} + \sum_{k=1}^{\infty}\frac{\alpha^k}{(k-1)!} \\ &= \alpha^2\sum_{k=2}^{\infty}\frac{\alpha^{k-2}}{(k-2)!} + \alpha\sum_{k=1}^{\infty}\frac{\alpha^{k-1}}{(k-1)!} \\ &= (\alpha^2+\alpha)e^{\alpha}. \end{aligned}$$

Combining these results, we get

$$\begin{aligned} E[Y] &= e^{-\alpha}[\alpha^2+\alpha]e^{\alpha} + b \\ &= \alpha^2 + \alpha + b. \end{aligned}$$

11. We start from the definition

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\sigma y + \mu) e^{-\frac{1}{2}y^2} dy, \quad \text{with } y \triangleq \frac{x-\mu}{\sigma}, \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y e^{-\frac{1}{2}y^2} dy + \mu \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} dy, \\
&= \frac{\sigma}{\sqrt{2\pi}} \times 0 + \mu \times 1 \\
&= \mu,
\end{aligned}
$$

where the first integral is 0 because it is an integral of an odd function over even limits, and the second integral is recognized as that of the standard Gaussian density over all its domain, and hence is 1.

12. (a) The CDF of the momentum $P = MV$ is given as

$$F_P(p) = \int_0^{\infty} f_V(v) \left(\int_0^{p/v} f_M(m) dm \right) dv.$$

Then we find the pdf $f_P(p)$ by differentiation as

$$
\begin{aligned}
f_P(p) &= dF_P(p)/dp \\
&= \int_0^{\infty} f_V(v) \left(f_M(m)|_{m=p/v} \right) \frac{d(p/v)}{dp} dv \\
&= \int_0^{\infty} f_V(v) f_M(p/v) \frac{1}{v} dv \\
&= \int_0^{\infty} f_M(m) f_V(p/m) \frac{1}{m} dm.
\end{aligned}
$$

(b) By independence $\mu_P = E[P] = E[MV] = E[M]E[V] = \mu_M \mu_V$.

13. We are to prove the general inequality $|E[X]| \leq E|X|$. We start with our definition (for continuous random variables) $E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$. Thus

$$
\begin{aligned}
|E[X]| &= |\int_{-\infty}^{+\infty} x f_X(x) dx| \\
&\leq \int_{-\infty}^{+\infty} |x f_X(x)| dx \\
&= \int_{-\infty}^{+\infty} |x| f_X(x) dx \quad \text{since } f_X \geq 0, \\
&= E[|X|],
\end{aligned}
$$

as was to be shown.

14. If $E[g_i(X)]$ exists, then

$$\begin{aligned} E\left[\sum_{i=1}^{n} g_i(X)\right] &= \int_{-\infty}^{+\infty} \sum_{i=1}^{n} g_i(x) f_X(x) dx \\ &= \sum_{i=1}^{n} \int_{-\infty}^{+\infty} g_i(x) f_X(x) dx \\ &= \sum_{i=1}^{n} E[g_i(X)]. \end{aligned}$$

15. This is a simple statistics calculation of mean and conditional mean.

    (a) The average number of children per household $\overline{X_S}$ is given as

$$\begin{aligned} \overline{X_S} &= \frac{1}{20} \sum X_i = \frac{32}{20} \\ &\doteq 1.6. \end{aligned}$$

    (b) Now, the number of households with children are 13, thus $\overline{X_{S|C}}$ (i.e. given that children are in the household) is

$$\begin{aligned} \overline{X_{S|C}} &= \frac{32}{13} \\ &\doteq 2.46. \end{aligned}$$

16. We know from our prior work that

$$f_X(x|B) = \begin{cases} 0, & x \le a, \\ \frac{f_X(x)}{P[B]}, & a < x \le b, \\ 0, & x > b. \end{cases}$$

Hence

$$E[X|B] = \int_a^b x f(x) dx / P[B].$$

Here

$$\begin{aligned} P[B] &= F_X(b) - F_X(a) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{2} \exp(-\frac{1}{2}x^2) dx \\ &= \operatorname{erf}(2) - \operatorname{erf}(-1) \\ &= \operatorname{erf}(2) + \operatorname{erf}(1) \\ &\simeq .82. \end{aligned}$$

Then

$$E[X|B] = \frac{1}{.82} \frac{1}{\sqrt{2\pi}} \int_{-1}^{2} x e^{-\frac{1}{2}x^2} dx = (e^{-\frac{1}{2}} - e^{-2})/2.06 \simeq .22$$

17. For any $h(x)$, we have $E[h(X)] = \int_{-\infty}^{+\infty} h(x) f_X(x) dx$.

(a) Using Taylor's series, we write

$$\begin{aligned} h(x) &= h(\mu) + h'(\mu)(x-\mu) + h''(\mu)\frac{(x-\mu)^2}{2!} + ... \\ &\simeq h(\mu) + h'(\mu)(x-\mu) + h''(\mu)\frac{(x-\mu)^2}{2!}, \end{aligned}$$

if $|h^{(n)}(\mu)|$ is sufficiently small and $|x-\mu|$ is not too large. So, using this approximation, we have

$$\begin{aligned} E[h(X)] &= \int_{-\infty}^{+\infty} h(x) f_X(x) dx \\ &\simeq E\left[h(\mu) + h'(\mu)(X-\mu) + h''(\mu)\frac{(X-\mu)^2}{2!}\right] \\ &= E[h(\mu)] + E[h'(\mu)(X-\mu)] + E\left[h''(\mu)\frac{(X-\mu)^2}{2!}\right] \\ &= h(\mu) + h'(\mu)E[X-\mu] + \frac{h''(\mu)}{2}E[(X-\mu)^2] \\ &= h(\mu) + h'(\mu)\left(E[X]-\mu\right) + \frac{h''(\mu)}{2}\sigma^2 \\ &= h(\mu) + \frac{h''(\mu)}{2}\sigma^2. \end{aligned}$$

(b) Let $g(x) = h^2(x)$. Then using the above approximate Taylor series on function g gives:

$$\begin{aligned} g(x) &\simeq g(\mu) + g'(\mu)(x-\mu) + g''(\mu)\frac{(x-\mu)^2}{2!} \\ &= h^2(\mu) + 2h(\mu)h'(\mu)(x-\mu) + 2[h(\mu)h''(\mu) + (h'(\mu))^2]\frac{(x-\mu)^2}{2!}. \end{aligned}$$

Thus, applying the result of part a), now to the function g, we have

$$\begin{aligned} E[h^2(X)] &= E[g(X)] \\ &= g(\mu) + \frac{g''(\mu)}{2}\sigma^2 \\ &= h^2(\mu) + \frac{2[h(\mu)h''(\mu) + (h'(\mu))^2]}{2}\sigma^2 \\ &= h^2(\mu) + [h(\mu)h''(\mu) + (h'(\mu))^2]\sigma^2. \end{aligned}$$

18. (a) The joint pdf of X and Y is

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) f_{Y|X}(y|x) \\ &= \begin{cases} 3x^2 \cdot 2y/x^2 = 6y, & 0 < y < x < 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

(b) The conditional mean of Y given $X = x$ is

$$\begin{aligned} E[Y|x] &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy \\ &= \begin{cases} \int_0^x y\left(2y/x^2\right) dy = \frac{2}{3}x, & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

(c) The marginal pdf of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \begin{cases} \int_y^1 6y dx = 6y(1-y), & 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Then

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 6y^2(1-y)dy = \frac{1}{2}.$$

19. Let the number of units manufactured at the various sites be denoted n_A, n_B, and n_C, with total number of units simply n. Then from the problem statement we know that

$$n_A = 3n_B \quad \text{and} \quad n_B = 2n_C,$$

and of course $n = n_A + n_B + n_C$. Then from classical probabilities, we get the probability of a unit selected 'at random' as

$$P[A] = \frac{n_A}{n} = \frac{6}{9}, P[B] = \frac{n_B}{n} = \frac{2}{9}, \quad \text{and} \quad P[C] = \frac{n_C}{n} = \frac{1}{9},$$

where we define event $A \triangleq \{\text{unit comes from plant } A\}$, and so forth for events B and C. Now we can use the concept of conditional expectation to write

$$\begin{aligned} E[X] &= E[X|A]P[A] + E[X|B]P[B] + E[X|C]P[C] \\ &= \frac{1}{5}\int_0^{\infty} xe^{-x/5}dx\frac{6}{9} + \frac{1}{6.5}\int_0^{\infty} xe^{-x/6.5}dx\frac{2}{9} + \frac{1}{10}\int_0^{\infty} xe^{-x/10}dx\frac{1}{9} \\ &= 5\frac{6}{9} + 6.5\frac{2}{9} + 10\frac{1}{9} \approx 5.89 \text{ years.} \end{aligned}$$

20. We are asked to compute the expected value $E[Y]$ of the received signal. Now

$$\begin{aligned} E[Y] &= E[E[Y|\Theta]] \\ &= \int_{-\infty}^{\infty} E[Y|\theta]f(\theta)d\theta. \end{aligned}$$

Now

$$E[Y|\Theta = \theta] = \theta \quad \text{by inspection of the given conditional Normal density.}$$

Thus

$$E[Y] = \frac{1}{2\pi}\int_0^{2\pi} \theta d\theta = \pi.$$

21. This problem computes the variances of a) Bernoulli, b) binomial, c) Poisson, d) Gaussian, and e) Rayleigh random variables.

(a) $E[X^2] = 1^2 p + 0^2(1-p) = p$, $E[X] = p$, so $\sigma^2 = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$.

(b) For the binomial random variable X, we have $E[X] = np$ from Problem 4.4. Then, letting $q = 1 - p$, we have

$$\begin{aligned}
E[X^2] &= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \\
&= \sum_{k=1}^{n} ((k-1)+1) \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k} \\
&= n(n-1)p^2 + np.
\end{aligned}$$

Then $\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2 = np(1-p) = npq$.

(c) From Example 4.1-6, we have $E[X] = a$ for the Poisson random variable with parameter $a(> 0)$. Remember the Poisson PMF is given as $P_X(k) = \frac{a^k}{k!} e^{-a} u(k)$. To compute the second moment, we proceed

$$\begin{aligned}
E[X^2] &= \sum_{k=-\infty}^{+\infty} k^2 P_X(k) \\
&= \sum_{k=1}^{\infty} k^2 \frac{a^k}{k!} e^{-a} \quad \text{since the } k = 0 \text{ term will be zero,} \\
&= a \left(\sum_{k=1}^{\infty} (k-1+1) \frac{a^{k-1}}{(k-1)!} \right) e^{-a} \\
&= ae^{-a} \left(\sum_{k=1}^{\infty} (k-1) \frac{a^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} 1 \frac{a^{k-1}}{(k-1)!} \right) \\
&= ae^{-a} \left(\sum_{k'=0}^{\infty} k' \frac{a^{k'}}{k'!} + \sum_{k'=0}^{\infty} 1 \frac{a^{k'}}{k'!} \right) \quad \text{with the substitution } k' = k - 1, \\
&= ae^{-a}(ae^{+a} + e^{+a}) \\
&= a^2 + a.
\end{aligned}$$

Then $\sigma^2 = E[X^2] - (E[X])^2 = a^2 + a - a^2 = a(> 0)$.

(d) We must compute the variance of the Gaussian random variable with parameters μ and σ^2, i.e. $X : N(\mu, \sigma^2)$. For the mean $E[X] = \mu$ see Example 4.1-2 in textbook. For the variance, consider the random variable $X - \mu$ with variance $E[(X-\mu)^2]$.

$$\begin{aligned}
E[(X-\mu)^2] &= \int_{-\infty}^{+\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{-\frac{z^2}{2}} dz \quad \text{with substitution } z \triangleq (x-\mu)/\sigma.
\end{aligned}$$

Next we integrate by parts with $u = z$ and $dv = ze^{-\frac{z^2}{2}}dz$, yielding $du = dz$ and $v = -e^{-\frac{z^2}{2}}$, so that, the above integral becomes

$$\begin{aligned}\int_{-\infty}^{+\infty} z^2 e^{-\frac{z^2}{2}}dz &= \left(-ze^{-\frac{z^2}{2}}\right)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}dz \\ &= -0 + 0 + \sqrt{2\pi},\end{aligned}$$

where the last term is due to the fact that the standard normal density $N(0,1)$ integrates to 1. Thus we have $E[(X-\mu)^2] = \frac{\sigma^2}{\sqrt{2\pi}}\sqrt{2\pi} = \sigma^2$, and thus the parameter σ^2 in the Gaussian density is shown to be the variance of the random variable $X - \mu$, which is the same as the variance of the random variable X.

(e) We are concerned here with the Rayleigh random variable with parameters μ and σ^2. The mean is calculated as follows.

$$\begin{aligned}E[X] &= \int_0^\infty x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx \\ &= \int_0^\infty \left(\frac{x}{\sigma}\right)^2 e^{-\frac{x^2}{2\sigma^2}}dx \\ &= \sigma\int_0^\infty y^2 e^{-\frac{y^2}{2}}dy \quad \text{with the transformation } y \triangleq \frac{x}{\sigma}, \\ &= \sigma\sqrt{\frac{\pi}{2}} \quad \text{from the similar } z \text{ integral done above.}\end{aligned}$$

To calculate the variance, we again rely on the indirect method of calculating $E[X^2]$ first.

We have

$$\begin{aligned}E[X^2] &= \int_0^\infty x^2\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx \\ &= \int_0^\infty \frac{x^3}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx,\end{aligned}$$

which can be calculated directly. An easier way though is to observe from Example 3.3-11 that the Raleigh random variable results from the square root of the sum of the squares of two independent Gaussians, each distributed as $N(0,\sigma^2)$. Calling this Rayleigh random variable Z, then $Z^2 = X^2 + Y^2$, and so $E[Z^2] = E[X^2 + Y^2] = E[X^2] + E[Y^2] = 2\sigma^2$. Hence the variance of the Rayleigh random variable Z with parameter σ is given via $\sigma^2 = E[Z^2] - \mu^2$, as $2\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2$.

22. (a) We use conditional expectation to write

$$\begin{aligned}E[T] &= E[E[T|\text{type}]] \\ &= 0.3E[T_1] + 0.7E[T_2] \\ &= 0.3\mu_1 + 0.7\mu_2.\end{aligned}$$

(b) We again use conditional expectation to write

$$\begin{aligned}E[T^2] &= E[E[T^2|\text{type}]] \\ &= 0.3E[T_1^2] + 0.7E[T_2^2].\end{aligned}$$

Now for the exponential random variable X with mean μ_X, we remember[1] that $E[X^2] = 2\mu_X^2$. Thus we have $E[T_i^2] = 2\mu_i^2$ for $i = 1, 2$ and so then

$$\begin{aligned} E[T^2] &= 0.3E[T_1^2] + 0.7E[T_2^2] \\ &= 0.6\mu_1^2 + 1.4\mu_2^2. \end{aligned}$$

(c) We start from the general expression $\sigma^2 = E[X^2] - E^2[X]$, so that here

$$\begin{aligned} \sigma_T^2 &= E[T^2] - E^2[T] \\ &= 0.6\mu_1^2 + 1.4\mu_2^2 - (0.3\mu_1 + 0.7\mu_2)^2 \\ &= 0.51\mu_1^2 - 0.42\mu_1\mu_2 + 0.91\mu_2^2. \end{aligned}$$

Thus finally $\sigma_T = \sqrt{0.51\mu_1^2 - 0.42\mu_1\mu_2 + 0.91\mu_2^2}$.

23. Taking the indirect approach, we start with calculation of the CDF $F_Z(z)$ for $z \geq 0$ as

$$\begin{aligned} F_Z(z) &= \iint_{\sqrt{x^2+y^2} \leq z} e^{-\frac{1}{2}(x^2+y^2)} dxdy \\ &= \int_0^z \int_0^{2\pi} e^{-\frac{1}{2}r^2} r dr d\theta, \\ \text{(with the transformation} &: \quad x = r\cos\theta, y = r\sin\theta, \quad \text{and} \quad dxdy = rdrd\theta), \\ &= 1 - e^{-\frac{1}{2}z^2} \quad \text{for} \quad z \geq 0. \end{aligned}$$

Since $F_Z(z) = 0$ for $z < 0$, we have the total solution $F_Z(z) = (1 - e^{-\frac{1}{2}z^2})u(z)$. Then the pdf is given as

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} \\ &= \frac{d[(1 - e^{-\frac{1}{2}z^2})u(z)]}{dz} \\ &= \frac{d(1 - e^{-\frac{1}{2}z^2})}{dz} u(z) + 0\delta(z) \\ &= ze^{-\frac{1}{2}z^2} u(z). \end{aligned}$$

We then calculate the mean as

$$\begin{aligned} E[Z] &= \int_{-\infty}^{+\infty} z f_Z(z) dz \\ &= \int_0^{+\infty} z^2 e^{-\frac{1}{2}z^2} dz \\ &= \sqrt{\frac{\pi}{2}}, \end{aligned}$$

[1] Alternatively, it is found using integration by parts twice.

by manipulation of the integral for the variance of a standard Normal. Then, the mean square value is

$$\begin{aligned} E[Z^2] &= \int_{-\infty}^{+\infty} z^2 f_Z(z)dz \\ &= \int_0^{+\infty} z^3 e^{-\frac{1}{2}z^2} dz \\ &= 2, \end{aligned}$$

by using the substitution $v \triangleq \frac{1}{2}z^2$. Finally, the variance can then be found as

$$\begin{aligned} \mathrm{Var}[Z] &= E[Z^2] - E^2[Z] \\ &= 2 - (\sqrt{\frac{\pi}{2}})^2 \\ &= 2 - \frac{\pi}{2} \\ &\simeq 0.43. \end{aligned}$$

24. (a) Although it may not look like it, this is a standard transformation of RVs problem. First we partition the x_1, x_2, x_3space into six disjoint regions $\Re_1, \Re_2, \Re_3, \Re_4, \Re_5$, and $\Re_6$,where

$$\begin{aligned} \Re_1 &= \{x_1 < x_2 < x_3\}, \Re_2 = \{x_1 < x_3 < x_2\}, \\ \Re_3 &= \{x_2 < x_1 < x_3\}, \Re_4 = \{x_2 < x_3 < x_1\}, \\ \Re_5 &= \{x_3 < x_1 < x_2\}, \text{ and } \Re_6 = \{x_3 < x_2 < x_1\}. \end{aligned}$$

In $\Re_1$, write $y_1 \triangleq g_1(x_1, x_2, x_3) = x_1, y_2 \triangleq h_1(x_1, x_2, x_3) = x_2, y_3 \triangleq q_1(x_1, x_2, x_3) = x_3$.
In $\Re_2$, write $y_1 \triangleq g_2(x_1, x_2, x_3) = x_1, y_2 \triangleq h_2(x_1, x_2, x_3) = x_3, y_3 \triangleq q_2(x_1, x_2, x_3) = x_2$.
In $\Re_3$, write $y_1 \triangleq g_3(x_1, x_2, x_3) = x_2, y_2 \triangleq h_3(x_1, x_2, x_3) = x_1, y_3 \triangleq q_3(x_1, x_2, x_3) = x_3$.
In $\Re_4$, write $y_1 \triangleq g_4(x_1, x_2, x_3) = x_2, y_2 \triangleq h_4(x_1, x_2, x_3) = x_3, y_3 \triangleq q_4(x_1, x_2, x_3) = x_1$.
In $\Re_5$, write $y_1 \triangleq g_5(x_1, x_2, x_3) = x_3, y_2 \triangleq h_5(x_1, x_2, x_3) = x_1, y_3 \triangleq q_5(x_1, x_2, x_3) = x_2$.
In $\Re_6$, write $y_1 \triangleq g_6(x_1, x_2, x_3) = x_3, y_2 \triangleq h_6(x_1, x_2, x_3) = x_2, y_3 \triangleq q_6(x_1, x_2, x_3) = x_1$.
In each region we have a different transformation. The magnitude of the Jacobian for each transformation is unity. For example in $\Re_1 = \{x_1 < x_2 < x_3\}$:

$$|J_1| = \mathrm{mag} \begin{vmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \partial g_1/\partial x_3 \\ \partial h_1/\partial x_1 & \partial h_1/\partial x_2 & \partial h_1/\partial x_3 \\ \partial q_1/\partial x_1 & \partial q_1/\partial x_2 & \partial q_1/\partial x_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Computing the Jacobian for the other transformations, we get

$$|J_i| = mag \begin{vmatrix} \partial g_i/\partial x_1 & \partial g_i/\partial x_2 & \partial g_i/\partial x_3 \\ \partial h_i/\partial x_1 & \partial h_i/\partial x_2 & \partial h_i/\partial x_3 \\ \partial q_i/\partial x_1 & \partial q_i/\partial x_2 & \partial q_i/\partial x_2 \end{vmatrix} = 1, \quad i = 1, ..., 6.$$

Finally, again considering region $\Re_1$, we find that the only solution to $y_1 - g_1(x_1, x_2, x_3) = 0$ is $x_1^{(1)} = y_1$;$y_2 - h_1(x_1, x_2, x_3) = 0$ is $x_2^{(1)} = y_2$;$y_3 - q_1(x_1, x_2, x_3) = 0$ is $x_3^{(1)} = y_3$, where the superscript has been added to remind us that the solution applies to region

1. For example in $\Re_2$ we would have: $x_1^{(2)} = y_1; x_2^{(2)} = y_3; x_3^{(2)} = y_2$; To get the final solution we use

$$\begin{aligned} f_{Y_1Y_2Y_3}(y_1, y_2, y_3) &= \sum_{i=1}^{6} f_{X_1X_2X_3}(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})/|J_i| \\ &= \begin{cases} 6(2\pi)^{-3/2}\exp[-(y_1^2 + y_2^2 + y_3^2)/2], & y_1 < y_2 < y_3, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

(b) To get, say, $E[Y_1]$, we need $f_{Y_1}(y_1)$ to compute $E[Y_1] = \int_{-\infty}^{\infty} y_1 f_{Y_1}(y_1)dy_1$. But $f_{Y_1}(y_1)$ is computed as

$$\begin{aligned} f_{Y_1}(y_1) &= 6\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2}\frac{1}{\sqrt{2\pi}}\int_{y_3=y_1}^{\infty} e^{-\frac{1}{2}y_3^2}\left(\frac{1}{\sqrt{2\pi}}\int_{y_1}^{y_3} e^{-\frac{1}{2}y_2^2}dy_2\right)dy_3, \\ &= 6\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2}\frac{1}{\sqrt{2\pi}}\int_{y_3=y_1}^{\infty} e^{-\frac{1}{2}y_3^2}\left(F_{SN}(y_3) - F_{SN}(y_1)\right)dy_3. \end{aligned}$$

To get $f_{Y_2}(y_2)$ to compute $E[Y_2] = \int_{-\infty}^{\infty} y_2 f_{Y_2}(y_2)dy_2$ we proceed as follows:

$$\begin{aligned} f_{Y_2}(y_2) &= 6\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_2^2}\frac{1}{\sqrt{2\pi}}\int_{y_2}^{\infty} e^{-\frac{1}{2}y_3^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y_2} e^{-\frac{1}{2}y_1^2}dy_1\right)dy_3 \\ &= 6\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_2^2}F_{SN}(y_2)\left(1 - F_{SN}(y_2)\right). \end{aligned}$$

Hence $E[Y_2] = \int_{-\infty}^{\infty} y_2 f_{Y_2}(y_2)dy_2 = 0$, since the integrand is odd and the interval of integration is even.

25. The support of interest is shown below:

$$\begin{aligned} E[Y] &= 2\int_{y=0}^{1}\int_{x=0}^{y} y\,dxdy = 2/3, \\ E[Y^2] &= 2\int_{y=0}^{1}\int_{x=0}^{y} y^2dxdy = 1/2. \end{aligned}$$

Hence $\sigma_Y^2 = E[Y^2] - E^2[Y] = 1/2 - 4/9 = 1/18.$

26. We first compute the marginal density $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f_{XY}(x, y)dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}}\exp\left(-\frac{x^2 + y^2 - 2\rho xy}{2\sigma^2(1-\rho^2)}\right)dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right) \end{aligned}$$

Y is Gaussian random variable with distribution $N(0,\sigma^2)$, so $E[Y]=0$.

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{f_{XY}(x,y)}{\int_{-\infty}^{+\infty} f_{XY}(x,y)dy} \\
&= \frac{\frac{1}{2\pi\sigma^2\sqrt{1-\rho}^{1/2}}\exp\left(-\frac{x^2+y^2-2\rho xy}{2\sigma^2(1-\rho^2)}\right)}{\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)} \\
&= \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\exp\left(-\frac{(y-\rho x)^2}{2\sigma^2(1-\rho^2)}\right)
\end{aligned}$$

$$\begin{aligned}
E[Y|X=x] &= \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy \\
&= \int_{-\infty}^{+\infty} \frac{y}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}}\exp\left(-\frac{(y-\rho x)^2}{2\sigma^2(1-\rho^2)}\right)dy \\
&= \rho x
\end{aligned}$$

This result shows that although $E[Y]=0$, the expected value of Y given X is ρX. The best predictor Y_ρ of Y is $Y_\rho = \rho X$. Of course this result only makes sense if we can observe X

27. (a) $\mu_Z = E[Z] = E[\frac{1}{2}(X+Y)] = \frac{1}{2}E[X] + \frac{1}{2}E[Y] = 0$. The variance is given by

$$\begin{aligned}
\sigma_Z^2 &= E[Z^2]-\mu_Z^2 \\
&= E\left[\left(\frac{X+Y}{2}\right)^2\right]-0 \\
&= \frac{1}{4}E[X^2+2XY+Y^2] \\
&= \frac{1}{4}(\sigma^2+2E[XY]+\sigma^2) \\
&= \frac{1}{4}(2\sigma^2+0) \\
&\quad \text{(since } X,\ Y \text{ are independent,} E[XY]=E[X]E[Y]) \\
&= \frac{\sigma^2}{2}.
\end{aligned}$$

(b) Even if X and Y are dependent, the mean of Z would remain the same. Hence, $E[Z]=0$. The variance of Z is given by

$$\begin{aligned}
\sigma_Z^2 &= E[\frac{1}{4}(X+Y)^2]-0 \\
&= \frac{1}{4}E[X^2+2XY+Y^2] \\
&= \frac{1}{4}(\sigma^2+2E[XY]+\sigma^2) \\
&= \frac{1}{4}(2\sigma^2+2\rho\sigma^2) \\
&\quad \text{(because } E[(X-0)(Y-0)]=\rho\sigma_X\sigma_Y=\rho\sigma^2) \\
&= \frac{1}{2}\sigma^2(1+\rho).
\end{aligned}$$

(c) For $\rho = -1$, 0, and 1, the values of σ_Z^2 are 0, $\frac{\sigma^2}{2}$, and σ^2, respectively. The variance of the sample average does not reduce when the random variables are perfectly correlated. If X and Y are uncorrelated, $\rho = 0$, and since they are Gaussian, they are independent. In that case, the variance of the sample average goes down with the number of samples. For values of ρ other than 1, the variance of the sample average is less than the variance of each sample.

28. Write $f_{XY}(x,y) = f_{Y|X}(y|x)f_X(x)$. The from Example 2.6-14, we have that in the jointly Normal case,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma^2}}e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)\sigma^2}} \quad \text{and} \quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}.$$

Recall that

$$\lim_{a\to\infty} ae^{-\pi a^2x^2} = \delta(x).$$

Then setting $a \triangleq 1/\sqrt{2\pi(1-\rho^2)\sigma^2}$, then $\pi a^2 = 1/(2(1-\rho^2)\sigma^2)$, and

$$f_{Y|X}(y|x) = ae^{-\pi a^2(y-\rho x)^2},$$

thus

$$\begin{aligned}\lim_{\rho\to 1} f_{Y|X}(y|x) &= \lim_{a\to\infty} ae^{-\pi a^2(y-\rho x)^2} \\ &= \delta(y-x).\end{aligned}$$

Hence, and also in the limit as $\rho \to 1$, we get

$$\lim_{\rho\to 1} f_{XY}(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\delta(y-x).$$

29. (a) Calculating, we obtain $P[X=-1] = P[\{\zeta=-1\}] = \frac{1}{5}$, $P[Y=1] = P[\{\zeta=-1 \text{ or } +1\}] = P[\{\zeta=-1\}] + P[\{\zeta=+1\}] = \frac{2}{5}$, and $P[X=-1, Y=1] = P[\{\zeta=-1\}] = \frac{1}{5}$. Then we have

$$\begin{aligned}P[X=-1]P[Y=1] &= \frac{1}{5}\cdot\frac{2}{5} \\ &\neq \frac{1}{5} \\ &= P[X=-1, Y=1].\end{aligned}$$

Hence, by definition, X and Y are not independent RVs.

(b)

$$\begin{aligned}E[X] &= \sum xP[X=x] \\ &= \sum_{i=1}^{5} \zeta_i P[\{\zeta_i\}] \\ &= \frac{1}{5}(-1 - \frac{1}{2} + 0 + \frac{1}{2} + 1) \\ &= 0.\end{aligned}$$

$$\begin{aligned} E[Y] &= \sum yP[Y=y] \\ &= \sum_{i=1}^{5} \zeta_i^2 P[\{\zeta_i\}] \\ &= \frac{1}{5}(1+\frac{1}{4}+0+\frac{1}{4}+1) \\ &= \frac{1}{5}\frac{5}{2}=\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} E[XY] &= \sum xyP[X=x,Y=y] \\ &= \sum_{i=1}^{5} \zeta_i^3 P[\{\zeta_i\}] \\ &= \frac{1}{5}(-1-\frac{1}{8}+0+\frac{1}{8}+1) \\ &= 0. \end{aligned}$$

Thus $E[XY]=0=0\cdot\frac{1}{2}=E[X]E[Y]$, and so, X and Y are uncorrelated RVs.

30. The conditional mean is always the mean of the conditional density. Since this conditional density is $N(\alpha x,\sigma^2)$, it follows that the conditional mean is αx, i.e. $E[Y|X=x]=\alpha x$, then by definition of the conditional mean as a random variable, we have

$$E[Y|X]=\alpha X.$$

31. We want to maximize

$$H(X) \triangleq -\int_{-\infty}^{+\infty} f(x)\ln f(x)dx, \qquad \text{(A)}$$

subject to the constraints:

$$\begin{aligned} \int_{-\infty}^{+\infty} xf(x)dx &= \mu, \\ \int_{-\infty}^{+\infty} x^2 f(x)dx &= \mu^2+\sigma^2, \\ \text{and} \int_{-\infty}^{+\infty} f(x)dx &= 1. \end{aligned}$$

Now, via the substitution into (A) with $y=x-\mu$, it is easily shown that $H(X)$ is invariant with respect to the mean value μ, so that we can simplify to minimization of:

$$-H(X)=\int_{-\infty}^{+\infty} f(x)\ln f(x)dx,$$

subject to the constraints:

$$\begin{aligned} \int_{-\infty}^{+\infty} x^2 f(x)dx &= \sigma^2, \\ \text{and} \int_{-\infty}^{+\infty} f(x)dx &= 1, \end{aligned}$$

where we now assume $\mu = 0$. To solve this problem, we introduce two Lagrange multipliers, and write the augmented functional

$$g(f) = \int_{-\infty}^{+\infty} f(x)\ln f(x)dx + \lambda_1\left(\int_{-\infty}^{+\infty} f(x)dx - 1\right) + \lambda_2\left(\int_{-\infty}^{+\infty} x^2 f(x)dx - \sigma^2\right).$$

Then we form the functional derivatives

$$\begin{aligned} \frac{\partial g(f)}{\partial f} &= \ln f(x) + 1 + \lambda_1 + \lambda_2 x^2 \\ &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} f(x) &= e^{-1-\lambda_1} e^{-\lambda_2 x^2} \\ &= c e^{-\lambda_2 x^2}, \end{aligned}$$

for some normalization constant c. Thus we must have a Normal random variable for maximum entropy. Bringing back the mean parameter μ, the only choice is $X : N(\mu, \sigma^2)$ which satisfies all the given constraints.

32. Write for positive integer n,

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx.$$

If n is odd, since the integrand is then an odd function and the integration region is even, the integral is zero. Next let

$$\alpha \triangleq \sigma^{-2} \quad \text{or} \quad \sigma = \alpha^{-\frac{1}{2}},$$

and assume that m is any positive integer, then

$$\int_{-\infty}^{+\infty} x^m e^{-\frac{1}{2}\alpha x^2} dx = \sqrt{2\pi}\alpha^{-\frac{1}{2}}.$$

Now, differentiating m times with respect to α, we get

$$\int_{-\infty}^{+\infty} \underbrace{x^2 x^2 \cdots x^2}_{m \text{ times}} (-\frac{1}{2})^m e^{-\frac{1}{2}\alpha x^2} dx = \sqrt{2\pi}\left(\underbrace{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots}_{m \text{ times}}\right)\alpha^{-\frac{(2m+1)}{2}},$$

or

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^{2m} e^{-\frac{x^2}{2\sigma^2}} dx = 1\cdot 3\cdot 5\cdots(2m-1)\sigma^{2m}.$$

Then, for n even, we can substitute $n = 2m$ to obtain

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx = 1\cdot 3\cdot 5\cdots(n-1)\sigma^n.$$

33.

$$
\begin{aligned}
& E\left[\frac{c_{11}}{c_{20}}(X-\mu_X)-(Y-\mu_Y)\right]^2 \\
= & \frac{c_{11}^2}{c_{20}^2}c_{20}+c_{02}-2\frac{c_{11}}{c_{20}}c_{11} \\
= & c_{02}-\frac{c_{11}^2}{c_{20}}
\end{aligned}
$$

But here $c_{11}^2 = c_{02}c_{20}$, so $c_{02} - \frac{c_{11}^2}{c_{20}} = c_{02} - \frac{c_{02}c_{20}}{c_{20}} = 0$. Now define a random variable

$$Z \triangleq \frac{c_{11}}{c_{20}}(X-\mu_X)-(Y-\mu_Y).$$

Then we have shown that $E[Z^2]=0$, so we can conclude that Z itself is zero by the Chebyshev inequality, i.e. $Z=0$. Then, rearranging the right-hand side of the above equation, we obtain

$$
\begin{aligned}
Y &= \frac{c_{11}}{c_{20}}X+(\mu_Y-\frac{c_{11}}{c_{20}}\mu_X) \\
&= \alpha X+\beta, \text{ with} \\
\alpha &= \frac{c_{11}}{c_{20}}, \text{ and } \beta=\mu_Y-\frac{c_{11}}{c_{20}}\mu_X.
\end{aligned}
$$

34. From the text: $\alpha_o = \rho\frac{\sigma_Y}{\sigma_X}$ and $\beta_o = \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X$. Then

$$
\begin{aligned}
\epsilon_{\min}^2 &= E[(Y-\alpha_o X-\beta_o)^2] \\
&= E[((Y-\mu_Y)-\alpha_o(X-\mu_X))^2] \\
&= \sigma_Y^2-2\rho\frac{\sigma_Y}{\sigma_X}\rho\sigma_X\sigma_Y+\left(\rho\frac{\sigma_Y}{\sigma_X}\right)^2\sigma_X^2 \\
&= \sigma_Y^2(1-\rho^2).
\end{aligned}
$$

When $\rho=1$, then Y is a linear function of X and there is no error in prediction.

35. We compute

$$
\begin{aligned}
m_r &= \int_{-\infty}^{\infty}x^r f_X(x)dx=\int_0^2 x^r(1-\frac{x}{2})dx \\
&= \int_0^2 x^r dx-\int_0^2(x^{r+1}/2)dx \\
&= \left.\frac{x^{r+1}}{r+1}\right|_0^2-\left.\frac{x^{r+2}}{2(r+2)}\right|_0^2.
\end{aligned}
$$

Thus $m_r = \frac{2^{r+1}}{(r+1)(r+2)}$. We find that $m_0 = 1, m_1 = 2/3, m_2 = 2/3, \ldots.$

36. Computing the mean:

$$\begin{aligned} E\left[\hat{\mu}_N\right] &= E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] \\ &= \frac{1}{N}\sum_{i=1}^{N} E[X_i] \\ &= \frac{1}{N} N\mu \\ &= \mu \end{aligned}$$

Computing the variance:

$$\begin{aligned} E\left[\hat{\mu}_N^2\right] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)^2\right] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[X_i X_j] \\ &= \frac{1}{N^2}\left(\sum_{i=1}^{N} E\left[X_i^2\right] + \sum_{1\le i,j\le N, i\neq j} E[X_i X_j]\right) \\ &= \frac{1}{N^2}\left(N\left(\mu^2+\sigma^2\right) + N(N-1)\mu^2\right) \\ &= \frac{1}{N}\sigma^2 + \mu^2 \end{aligned}$$

then

$$\begin{aligned} \mathrm{Var}\left[\hat{\mu}_N\right] &= E\left[\hat{\mu}_N^2\right] - \left(E\left[\hat{\mu}_N\right]\right)^2 \\ &= \frac{1}{N}\sigma^2 + \mu^2 - \mu^2 \\ &= \frac{1}{N}\sigma^2 \end{aligned}$$

37. By Chebyshev inequality:

$$\begin{aligned} P\left[|\hat{\mu}_N - \mu| > 0.1\sigma\right] &\le \frac{\mathrm{Var}\left[\hat{\mu}_N\right]}{(0.1\sigma)^2} \\ &= \frac{\frac{1}{N}\sigma^2}{0.01\sigma^2} \\ &= \frac{100}{N} \end{aligned}$$

To ensure $P\left[|\hat{\mu}_N - \mu| > 0.1\sigma\right] \le 0.01$, one can set $\frac{100}{N} \le 0.01 \Leftrightarrow N \ge 10,000$.

38. a) Moment-generating function:

$$\begin{aligned} M_X(t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{tx} dx \\ &= \frac{1}{t}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) \\ &= \frac{\sinh(t/2)}{t/2} \end{aligned}$$

b)

$$\begin{aligned} \frac{d}{dt}M_X(t) &= \frac{d}{dt}\left(\frac{1}{t}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)\right) \\ &= \frac{1}{t}\frac{d}{dt}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) + \left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)\frac{d}{dt}\left(\frac{1}{t}\right) \\ &= \frac{1}{2t}\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\right) - \frac{1}{t^2}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right). \end{aligned}$$

We note that $M_X'(0)$ is undefined. However, we can evaluate its limiting form as t approaches zero and use that to determine the first moment. To do this, we first substitute Taylor series expansions for the exponential terms using the well-known series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4), \qquad \text{which converges for all finite } x.$$

It turns out that we will have to keep terms up to $(t/2)^3$ here because of the $(1/t^2)$ term in $M_X'(t)$. Thus we get the approximation

$$\begin{aligned} e^{\frac{1}{2}t} &\approx 1 + \frac{1}{2}t + \frac{\left(\frac{1}{2}t\right)^2}{2!} + \frac{\left(\frac{1}{2}t\right)^3}{3!} \\ &= 1 + \frac{1}{2}t + \frac{1}{8}t^2 + \frac{1}{48}t^3. \end{aligned}$$

with corresponding approximation for $e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}$ and $e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}$ given by

$$\begin{aligned} e^{\frac{1}{2}t} + e^{-\frac{1}{2}t} &\approx 2 + \frac{1}{4}t^2 \quad \text{and} \\ e^{\frac{1}{2}t} - e^{-\frac{1}{2}t} &\approx t + \frac{1}{24}t^3. \end{aligned}$$

Finally, inserting these approximations into our expression for $M_X'(t)$, we get the following

$$\begin{aligned} M_X'(t) &= \frac{1}{2t}\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\right) - \frac{1}{t^2}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) \\ &\approx \frac{1}{2t}\left(2 + \frac{1}{4}t^2\right) - \frac{1}{t^2}\left(t + \frac{1}{24}t^3\right) \\ &= t^{-1} + \frac{1}{8}t - t^{-1} - \frac{1}{24}t \\ &= \frac{1}{12}t, \qquad \text{good near } t = 0. \end{aligned}$$

Then

$$\begin{aligned} E[X] &= \left.\frac{d}{dt}M_X(t)\right|_{t=0} \\ &= \left.\frac{1}{12}t\right|_{t=0} \\ &= 0, \end{aligned}$$

the correct value for the mean of a uniform random variable centered on 0.

39. (a)

$$\begin{aligned} M(t) &= \sum_{k=0}^{\infty} e^{tk}\frac{a^k}{k!}e^{-a} \\ &= \left(\sum_{k=0}^{\infty}\frac{\left(ae^t\right)^k}{k!}\right)e^{-a} \\ &= e^{ae^t}e^{-a} \\ &= e^{a(e^t-1)} \\ &= \exp(a(e^t-1)). \end{aligned}$$

(b)

$$\begin{aligned} \frac{dM(t)}{dt} &= \frac{de^{ae^t}}{d(ae^t)}\frac{d(ae^t)}{dt}e^{-a} \\ &= e^{ae^t}ae^te^{-a} \\ &= \left(e^{ae^t}e^{-a}\right)ae^t \\ &= M(t)ae^t. \end{aligned}$$

So, evaluating this derivative at $t = 0$, we obtain

$$\begin{aligned} \left.\frac{dM(t)}{dt}\right|_{t=0} &= M(t)ae^t\big|_{t=0} \\ &= M(0)ae^0 \\ &= 1a\cdot 1 \\ &= a = m_1, \text{ the mean.} \end{aligned}$$

40. There is the relevant series expansion (in *Discrete Distributions* by L.L. Johnson and S. Kotz, Wiley and Sons, 1969) whose coefficients are closely related to the negative binomial distribution. The function and its expansion are:

$$\begin{aligned} \left(Q-Pe^t\right)^{-N} &= \sum_{k=0}^{\infty}\binom{N+k-1}{N-1}\left(\frac{Pe^t}{Q}\right)^k\left(1-\frac{P}{Q}\right)^N, \quad \text{for } t \text{ small}, \\ &= \sum_{k=0}^{\infty}e^{kt}\binom{N+k-1}{N-1}\left(\frac{P}{Q}\right)^k\left(1-\frac{P}{Q}\right)^N \\ &= \sum_{k=0}^{\infty}e^{kt}P_X(k). \end{aligned}$$

Therefore

$$M_X(t) = \left(Q - Pe^t\right)^{-N}.$$

41. By definition:

$$M(t) \triangleq \left[\alpha!\beta^{\alpha+1}\right]^{-1} \int_0^\infty x^\alpha e^{-x(\frac{1}{\beta}-t)} dx.$$

From any table of *definite exponential integrals* we find that for α an integer:

$$\int_0^\infty x^\alpha e^{-x(\frac{1}{\beta}-t)} dx = \frac{\alpha!}{(1/\beta - t)^{\alpha+1}}.$$

Hence

$$\begin{aligned} M(t) &\triangleq \left[\alpha!\beta^{\alpha+1}\right]^{-1} \int_0^\infty x^\alpha e^{-x/\beta} e^{tx} dx \\ &= \left[\alpha!\beta^{\alpha+1}\right]^{-1} \frac{\alpha!}{(1/\beta - t)^{\alpha+1}} \\ &= \frac{1}{(1-\beta t)^{\alpha+1}}. \end{aligned}$$

42. We can do this simply using the MGF of the gamma distribution. From Problem 4.41 we have $M(t) = (1-\beta t)^{-(\alpha+1)}$. Hence $M'(t) = (1-\beta t)^{-(\alpha+2)}(-)(\alpha+1)(-)\beta$ and

$$\begin{aligned} M'(0) &= (\alpha+1)\beta \\ &= E[X]. \end{aligned}$$

To get the variance we compute $M''(t)|_{t=0} = E[X^2]$. We find that

$$M''(t) = -(\alpha+1)(-\beta)(-1)(\alpha+2)(-\beta)(1-\beta t)^{-(\alpha+3)}$$

so that $M''(0) = (\alpha+1)(\alpha+2)\beta^2$. Hence

$$\begin{aligned} \sigma_X^2 &= M''(0) - [M'(0)]^2 \\ &= (\alpha+1)\beta^2. \end{aligned}$$

43. The Chernoff bound on $P[X \geq a]$, where X is an exponential random variable, is given by $P[X \geq a] \leq e^{-at} M_X(t)$, where $M_X(t) \triangleq E[e^{tX}]$, the moment generating function of X is given as

$$\begin{aligned} M_X(t) &= \int_{-\infty}^\infty e^{tx} f_X(x)\, dx \\ &= \lambda \int_0^\infty e^{tx} e^{-\lambda x}\, dx \\ &= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx \text{ (will converge if } Re(t) < \lambda \text{)} \\ &= \frac{\lambda}{\lambda - t}. \end{aligned}$$

Therefore

$$P[X \geq a] \leq e^{-at} \frac{\lambda}{\lambda - t}.$$

Denoting the right-hand side as $h(t)$, we seek its minimum by setting the derivative of h to zero. Now

$$\begin{aligned}\frac{dh}{dt} &= \frac{\lambda}{\lambda - t}(-a)e^{-at} + e^{-at}\lambda(-)\left(\frac{1}{\lambda - t}\right)^2(-) \\ &= 0,\end{aligned}$$

implies $\lambda - \frac{1}{a}$ (after a little algebra). We must note however that t must be non-negative, so that this value only holds for $a\lambda \geq 1$. Hence, for such a, we have

$$P[X \geq a] \leq e^{-a\left(\lambda - \frac{1}{a}\right)} \frac{\lambda}{\lambda - (\lambda - \frac{1}{a})} = a\lambda e^{(-a\lambda+1)}.$$

So, the Chernoff bound is

$$\begin{aligned}P[X \geq a] &\leq a\lambda e^{-a\lambda+1} \\ &= (a\lambda e)e^{-a\lambda}, \quad \text{for} \ \ a \geq 1/\lambda.\end{aligned}$$

44. We combine both parts (a) and (b) by doing the case for general positive integer N. By the Chernoff bound

$$P[X \geq k] \leq e^{-tk}[Q - Pe^t]^{-N}.$$

Next, let $h(t) \triangleq g_1(t)[g_2(t)]^{-N}$ with $g_1(t) \triangleq e^{-tk}$ and $g_2(t) \triangleq Q - Pe^t$. Then

$$\frac{dh(t)}{dt} = g_1(-N)[g_2(t)]^{-N-1}\frac{dg_2(t)}{dt} + [g_2(t)]^{-N}\frac{dg_1(t)}{dt}.$$

Here,

$$\frac{dg_2(t)}{dt} = -Pe^t \quad \text{and} \quad \frac{dg_2(t)}{dt} = -ke^{tk}.$$

Setting the derivative $\frac{dh(t)}{dt} = 0$, then gives

$$e^{-tk}(-N)[Q - Pe^t]^{-N-1} - Pe^t + [Q - Pe^t]^{-N}(-k)e^{-tk} = 0,$$

which implies

$$e^t = \frac{kQ}{NP + kP} \quad \text{or} \quad t = \ln\left[\frac{k(Q/P)}{N+k}\right].$$

Then, recalling that $e^{\ln x} = x$, we get

$$\begin{aligned}P[X \geq k] &\leq e^{-k\ln\left[\frac{k(Q/P)}{N+k}\right]}\left[Q - P\frac{k(Q/P)}{N+k}\right]^{-N} \\ &= \left[\frac{k(Q/P)}{N+k}\right]^{-k}\left[\frac{QN}{N+k}\right]^{-N}.\end{aligned}$$

In the special case of part (a), where $N = 1$, we have

$$\begin{aligned}P[X \geq k] &\leq \left[\frac{k(Q/P)}{1+k}\right]^{-k}\left[\frac{Q}{1+k}\right]^{-1} \\ &= \frac{(k+1)}{Q}\frac{(k+1)^k}{[k(Q/P)]^k} \\ &= \frac{(k+1)^{k+1}}{Q\,[k(Q/P)]^k}.\end{aligned}$$

If also, we have $k = 1$, then

$$\begin{aligned} P[X &\geq 1] \leq \frac{(2)^2}{Q\left[2(Q/P)\right]^2} \\ &= \frac{P^2}{Q^3} = \frac{P^2}{(1+P)^3}. \end{aligned}$$

45. The characteristic function of the Cauchy random variable X, with density function $f_X(x) = \frac{\alpha}{\pi(\alpha^2+x^2)}$, is given by

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} \frac{\alpha}{\pi(\alpha^2+x^2)}\, dx = S(-\omega),$$

where $S(\omega) \triangleq FT\left[\frac{\alpha}{\pi(\alpha^2+x^2)}\right] = \int_{-\infty}^{\infty} e^{-j\omega x} \frac{\alpha}{\pi(\alpha^2+x^2)}\, dx$.

Now consider the function $y(x) = e^{-\alpha|x|}$. The Fourier transform of this function is given by $Y(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}$. Therefore, $FT\left[\frac{1}{2\pi} y(x)\right] = \frac{1}{2\pi} Y(\omega) = \frac{\alpha}{\pi(\alpha^2+\omega^2)}$.

From the duality of the Fourier transform, we know that if $FT\{y(x)\} = Y(\omega)$, then $FT[Y(x)] = 2\pi y(-\omega)$. Therefore, $S(\omega) = FT\left[\frac{\alpha}{\pi(\alpha^2+x^2)}\right] = 2\pi \frac{1}{2\pi} y(-\omega) = y(-\omega) = e^{-\alpha|\omega|}$.

Therefore,

$$\Phi_X(\omega) = S(-\omega) = e^{-\alpha|\omega|} \quad \text{for all} \quad \omega.$$

46. Since $f_X(x) = \frac{1}{\pi(1+(x-a)^2)}$ is a pdf for any value of a including $a = 0$, *we* immediately deduce that $\int_{-\infty}^{\infty} \frac{1}{\pi(1+(x-a)^2)} dx = 1$. Then

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} \frac{x}{\pi(1+(x-a)^2)} dx \\ &= \int_{-\infty}^{\infty} \frac{u+a}{\pi(1+u^2)} du \\ &= \int_{-\infty}^{\infty} \frac{u}{\pi(1+u^2)} du + \int_{-\infty}^{\infty} \frac{a}{\pi(1+u^2)} du \\ &= 0 + a = a. \end{aligned}$$

We took advantage that the first integral after the third equal sign is 0 because the integrand is odd and the interval about zero is even. To calculate the variance we would have to consider an integral of the form $E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\pi(1+(x-a)^2)} dx$, which is readily seen not to converge in any sense. See what happens to the integrand when x approaches infinity.

47. The solution proceeds as

$$\begin{aligned} \Phi_X(\omega) &\triangleq E[e^{+j\omega X}] \\ &= \int_0^{\infty} \frac{1}{\mu} e^{(j\omega - 1/\mu)x} dx \\ &= \frac{1}{\mu} \left. \frac{e^{(j\omega-1/\mu)x}}{j\omega - \frac{1}{\mu}} \right|_0^{\infty} \\ &= \frac{1}{\mu} \frac{-1}{j\omega - \frac{1}{\mu}} \\ &= \frac{1}{1 - j\omega\mu}, \qquad -\infty < \omega < +\infty. \end{aligned}$$

48. Calculating the CF when mean parameter $\alpha = \mu = 0$, when X is Cauchy distributed, we get $\Phi_X(\omega) \triangleq E[e^{+j\omega X}] = e^{-|\omega|}$ by a known Fourier transform relation. It is easily checked by finding the pdf that corresponds to this CF as follows:

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_X(\omega)e^{-j\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|\omega|}e^{-j\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{+\omega}e^{-j\omega x}d\omega + \frac{1}{2\pi}\int_{0}^{+\infty} e^{-\omega}e^{-j\omega x}d\omega \\ &= \frac{1}{\pi\left(1+x^2\right)}. \end{aligned}$$

So for the given case, with mean μ, we have $X_i = X + \mu$, and so its CF is given as

$$\begin{aligned} \Phi_{X_i}(\omega) &\triangleq E[e^{+j\omega X_i}] \\ &= E[e^{+j\omega(X_i+\mu)}] \\ &= e^{j\omega\mu}e^{-|\omega|}. \end{aligned}$$

Now turning to the sum Y, we have $Y = \frac{1}{N}\sum_{i=1}^{N} X_i$. Then, since the X_i are i.i.d., the CF of Y is given as

$$\begin{aligned} \Phi_Y(\omega) &= E[e^{+j\omega Y}] \\ &= \prod_{i=1}^{N}\int_{-\infty}^{+\infty} e^{+j\frac{\omega}{N}x_i}\frac{1}{\pi[1+(x_i-\mu)^2}dx_i \\ &= \prod_{i=1}^{N} e^{+j\frac{\omega}{N}\mu}e^{-\frac{1}{N}|\omega|}, \quad \text{with the substitution:} \quad z_i = x_i - \mu, \\ &= e^{+j\omega\mu-|\omega|}, \end{aligned}$$

which is equal to the $\Phi_{X_i}(\omega)$, which are all equal since the X_i are i.i.d. So, since $\Phi_Y(\omega) = \Phi_{X_i}(\omega)$, it must be that $f_Y(y) = f_{X_i}(y)$, as was to be shown. We conclude that the average of i.i.d. Cauchy RVs is also Cauchy, and with the same mean μ.

49. If X is uniform in (a, b), then $E[X] = \frac{a+b}{2}$ (see problem 4.5).

$$E[Z] = E[X+Y] = E[X] + E[Y] = \frac{a-a}{2} + \frac{na+(n-2)a}{2} = (n-1)a.$$

Since the convolution of two *rect* functions is a triangle function of twice the width with the center at the mean, the density function of Z, obtained by convolving the uniform density functions[2] of X and Y, is obtained as seen in Fig. 1.

Equivalently, $\Phi_X(\omega) = \frac{sin(\omega a)}{\omega a}$; $\Phi_Y(\omega) = \frac{sin(\omega a)}{\omega a}e^{-j\omega a(n-1)}$.

$$f_Z(z) = FT^{-1}\left[\left(\frac{sin(\omega a)}{\omega a}\right)^2 e^{-j\omega a(n-1)}\right],$$

[2] Note that the uniform density function is merely a shifted and scaled rect. function.

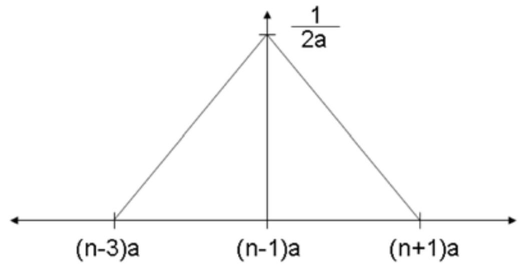


Figure 1:

which is the triangle function, merely shifted by $(n-1)a$, and results in a shift in the density function.

50.

$$\begin{aligned} E[X_n X_{n-k}] &= E[(Z_n - aZ_{n-1})(Z_{n-k} - aZ_{n-k-1})] \\ &= E[Z_n Z_{n-k}] - aE[Z_n Z_{n-k-1}] - aE[Z_{n-1} Z_{n-k}] + a^2 E[Z_{n-1} Z_{n-k+1}]. \end{aligned}$$

Hence

$$\begin{aligned} R_n(k) &= R_Z(k) - aR_Z(k+1) - aR_Z(k-1) + a^2 R_Z(k) \\ &= (1+a^2) R_Z(k) - aR_Z(k+1) - aR_Z(k-1). \end{aligned}$$

Now $R_Z(k) = \sigma^2 \delta(k)$ as given, so

$$R_n(k) = \begin{cases} \left(1+a^2\right)\sigma^2, & k = 0, \\ -a\sigma^2, & k = \pm 1, \\ 0, & \text{else.} \end{cases}$$

51. We here assume some kind of statistical steady state, taking the form

$$E[X_n^2] = K, \quad \text{a constant.}$$

So then,

$$\begin{aligned} R_n(0) &= E[X_n^2] = K, \\ R_n(1) &= E[X_n X_{n-1}] = E[(bX_{n-1} + Z_n) X_{n-1}] = bK, \\ R_n(2) = E[X_n X_{n-2}] &= E[(bX_{n-1} + Z_n) X_{n-2}] \\ &= E[(b(bX_{n-2} + Z_{n-1}) + Z_n) X_{n-2}] = b^2 K, \\ &\vdots \\ R_n(k) = E[X_n X_{n-k}] &= b^k K, \quad \text{for } k \geq 0. \end{aligned}$$

By symmetry and since $E[X_n^2] = K$, a constant, we must have the value of $R_n(k)$ for $k < 0$, which gives then general solution

$$R_n(k) = b^{|k|} K \qquad , -\infty < k < +\infty.$$

Note, for this to make sense, we need to have $|b| < 1$. It turns out that this is also a necessary condition for our assumption $E[X_n^2] = K$, a constant, to be true.

52. With $Z = aX + bY$ and $W = cX + dY$ we can write:

$$\begin{aligned}\Phi_{ZW}(\omega_1,\omega_2) &= E\left[e^{j(\omega_1 Z+\omega_2 W)}\right] \\ &= E\left[e^{j(\omega_1(aX+bY)+\omega_2(cX+dY))}\right] \\ &= E\left[e^{j((a\omega_1+c\omega_2)Z+(b\omega_1+d\omega_2)W)}\right] \\ &= \Phi_{XY}(a\omega_1+c\omega_2, b\omega_1+d\omega_2)\end{aligned}$$

53. We note that X and Y are independent since

$$\begin{aligned}f_{XY}(x,y) &= 16e^{-4(x+y)}u(x)u(y) \\ &= 4e^{-4x}u(x)4e^{-4y}u(y).\end{aligned}$$

Hence the joint MGF is

$$M_{XY}(t_1,t_2) = M_X(t_1)M_Y(t_2)$$

where

$$\begin{aligned}M_X(t_1) &= 4\int_0^\infty e^{-4x}e^{t_1 x}dx \\ &= 4\int_0^\infty e^{-(4-t_1)xt_1 x}dx \\ &= \frac{1}{1-t_1/4}\end{aligned}$$

and

$$\begin{aligned}M_Y(t_2) &= 4\int_0^\infty e^{-4y}e^{t_2 y}dy \\ &= 4\int_0^\infty e^{-(4-t_2)y}dy \\ &= \frac{1}{1-t_2/4}.\end{aligned}$$

Hence

$$\begin{aligned}M_{XY}(t_1,t_2) &= (1-t_1/4)^{-1}(1-t_2/4)^{-1} \\ &= \frac{16}{(4-t_1)(4-t_2)}.\end{aligned}$$

To get the CF, just replace t_i by $j\omega_i$ on the right-hand side. We get

$$\begin{aligned}\Phi_{XY}(\omega_1,\omega_2) &= (1-j\omega_1/4)^{-1}(1-j\omega_2/4)^{-1} \\ &= \frac{16}{(4-j\omega_1)(4-j\omega_2)}.\end{aligned}$$

54. Two random variables X: Poisson(a) and Y: Poisson (b) are given, where $a = 2$, and $b = 3$, $Z = X + Y$, and the characteristic function of Z is given by

$$\begin{aligned}\Phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] \\ &= E[e^{j\omega X}e^{j\omega Y}] \\ &= E[e^{j\omega X}]E[e^{j\omega Y}] \text{ (because } X \text{ and } Y \text{ are independent)} \\ &= \Phi_X(\omega)\Phi_Y(\omega).\end{aligned}$$

The characteristic function of X is given by

$$\begin{aligned}\Phi_X(\omega) &= E[e^{j\omega X}] \\ &= \sum_{k=0}^{\infty} e^{j\omega k} P_X(k) \\ &= \sum_{k=0}^{\infty} e^{j\omega k} \frac{e^{-a}a^k}{k!} \\ &= e^{-a} \sum_{k=0}^{\infty} \frac{(ae^{j\omega})^k}{k!} \\ &= e^{-a}e^{ae^{j\omega}} \\ &= \exp(-a + ae^{j\omega}) \\ &= \exp(-a(1 - e^{j\omega})).\end{aligned}$$

Similarly, we obtain $\Phi_Y(\omega) = \exp(-b(1 - e^{j\omega}))$. Therefore,

$$\begin{aligned}\Phi_Z(\omega) &= \exp(-a(1 - e^{j\omega}))\exp(-b(1 - e^{j\omega})) \\ &= \exp(-(a + b)(1 - e^{j\omega})).\end{aligned}$$

This implies that Z is Poisson$(a + b)$ and has a density function $f_Z(z) = \frac{e^{-5}5^n}{n!}$, for $n = 0, 1, 2, \ldots$, for $a = 2$ and $b = 3$.

55. We use the central limit theorem (CLT)

Let $X_i, i = 1, \ldots, 2000$ denote the state of the ith toaster

$$X_i = \begin{cases} 1 & \text{if toaster dented with probability } p \\ 0 & \text{if toaster OK with probability } q = 1 - p \end{cases}$$

Let $W \triangleq \sum_{i=1}^{n} X_i$, then

$$E[W] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = np$$

$$
\begin{aligned}
\mathrm{Var}[W] &= E[W^2] - (E[W])^2 \\
&= E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\
&= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} X_i\, X_j\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\
&= \sum_{i=1}^{n} E\left[X_i^2\right] + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E\left[X_i X_j\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\
&= np + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E[X_i]E[X_j] - n^2p^2 X_i \text{ and } X_j \text{ are independent} \\
&= np + n(n-1)p^2 - n^2p^2 \\
&= npq
\end{aligned}
$$

Therefore $E[W] = np = 2000 \times 0.05 = 100$, $\mathrm{Var}[W] = npq = 2000 \times 0.05 \times 0.95 = 95$.

$$
\begin{aligned}
P[110 \leq W \leq 2000] &= \frac{1}{\sqrt{2\pi \times 95}} \int_{110}^{2000} e^{-\frac{1}{2}\left[\frac{x-100}{\sqrt{95}}\right]^2} dx \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{1.03}^{+\infty} e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{2} - \mathrm{erf}(1.03) \\
&= 0.15
\end{aligned}
$$

56. Now

$$
\begin{aligned}
P[L \leq x] &= F_L(x) \\
&= 1 - e^{-0.002x} \\
&= 1 - e^{-x/\mu.}
\end{aligned}
$$

Hence, $\mu = 1/0.002 = 500$.

(a)

$$
E[L] = 500, \text{ and } E\left[\sum L_i\right] = 400 \cdot 500 = 200K\text{bytes}.
$$

(b)

$$
\sigma_X = 1/0.002 = 500 \quad \text{and } n = 400,
$$

so

$$
\begin{aligned}
P\left[\sum L_i > 420\right] &= 1 - \Phi\left(\frac{520 - 500}{(500/20)}\right) \\
&= 1 - \Phi(0.80) \\
&= 0.21.
\end{aligned}
$$

57. We have 100 independent and i.i.d. random variables, say X_i, with means μ and variances σ^2. We form their sample mean

$$\widehat{\mu}_{100} = \frac{1}{100}\sum_{i=1}^{100} X_i.$$

Since the X_i are independent, we have the mean and variance of the sample mean given by

$$\mu_{\widehat{\mu}_{100}} = \mu \quad \text{and} \quad \sigma^2_{\widehat{\mu}_{100}} = \frac{1}{100}\sigma^2.$$

We can now write Chebyshev's inequality for this sample mean random variable $\widehat{\mu}_{100}$ as

$$P[|\widehat{\mu}_{100} - \mu| > \delta] \leq \frac{\sigma^2_{\widehat{\mu}_{100}}}{\delta^2}.$$

Setting $\delta = \sigma/5$, we obtain

$$\begin{aligned} P[|\widehat{\mu}_{100} - \mu| > \sigma/5] &\leq \frac{\sigma^2_{\widehat{\mu}_{100}}}{(\sigma/5)^2} \\ &= \frac{1}{100}\sigma^2\frac{25}{\sigma^2} \\ &= \frac{1}{4}. \end{aligned}$$

58. We use the central limit theorem (CLT)

Let $X_i, i = 1, \ldots, 2000$ denote the state of the ith panel

$$X_i = \begin{cases} 1 & \text{if panel is bad, with probability } p \\ 0 & \text{if panel is good, with probability } q = 1 - p \end{cases}$$

Let $W \triangleq \sum_{i=1}^{n} X_i$, then

$$E[W] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = np$$

$$\begin{aligned} \mathrm{Var}[W] &= E[W^2] - (E[W])^2 \\ &= E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\ &= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} X_i\, X_j\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\ &= \sum_{i=1}^{n} E\left[X_i^2\right] + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} E\left[X_i X_j\right] - \left(\sum_{i=1}^{n} E[X_i]\right)^2 \\ &= np + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} E[X_i]E[X_j] - n^2p^2 X_i \text{ and } X_j \text{ are independent} \\ &= np + n(n-1)p^2 - n^2p^2 \\ &= npq \end{aligned}$$

Therefore $E[W] = np = 2000 \times 0.03 = 60$, $\mathrm{Var}[W] = npq = 2000 \times 0.03 \times 0.97 = 58.2$.

$$
\begin{aligned}
P[71 \le W \le 2000] &= \frac{1}{\sqrt{2\pi \times 58.2}} \int_{71}^{2000} e^{-\frac{1}{2}\left[\frac{x-60}{\sqrt{58.2}}\right]^2} dx \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{1.44}^{+\infty} e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{2} - \mathrm{erf}(1.44) \\
&= 0.074
\end{aligned}
$$

59. (a)

$$
\begin{aligned}
E[Z_n] &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E[W_i] \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{E[X_i] - p}{\sqrt{pq}} \\
&= 0 \qquad \text{since } E[X_i] = p
\end{aligned}
$$

(b) The Poisson approximation is obtained by setting $a \triangleq np$ and using $p(k_1) = e^{-a}\frac{a^{k_1}}{k_1!}$
The CLT uses the following. Let X be the number of successes. Then with $W \triangleq \frac{X-np}{\sqrt{npq}}$, $E[W] = 0, \mathrm{Var}[W] = 1$, and

$$b[k_1; n, p] \approx P[c_L \le W \le c_U]$$

where

$$
\begin{aligned}
c_U &\triangleq (k_1 + \frac{1}{2} - np)/\sqrt{npq} \\
c_L &\triangleq (k_1 - \frac{1}{2} - np)/\sqrt{npq}
\end{aligned}
$$

and $W : N(0,1)$. Note that X is the sum of a large number (n) of Bernoulli RVs, which is the basic for using the CLT.

The 3 mini-programs are summarized into one MATLAB function called "CalcBinProb(p, k1, n)". The test cases for $n = 2000, p = 0.05, k_1 = \{110, 120, 150, 170\}$ are included in MATLAB script "problem_4_59.m". The results in table 1 were obtained from the test:

Table 1: $P[k$ successes in n tries$]$

	$k_1 = 110$	$k_1 = 120$	$k_1 = 150$	$k_1 = 170$
EXACT	0.0235	4.7×10^{-4}	3.5×10^{-7}	1.45×10^{-11}
POISSON	0.0234	5.8×10^{-4}	6.5×10^{-7}	5.1×10^{-11}
CLT	$\underbrace{0.0242}_{1\sigma \text{ error}}$	$\underbrace{3.6 \times 10^{-4}}_{3\sigma \text{ error}}$	$\underbrace{8.0 \times 10^{-8}}_{5\sigma \text{ error}}$	$\underbrace{2.6 \times 10^{-13}}_{10\sigma \text{ error}}$

60. These are repeated Bernoulli trials resulting in the binomial distribution with $n = 1000$ and $p = 0.001$. Let X_i be the individual random variables, taking on value 1 for an erroneous line and 0 for an error-free line. Then we can write the sum or total of the errors as

$$Z = \sum_{i=1}^{n} X_i.$$

Then Z is Binomial with $\mu_Z = np = 1$ and $\sigma_Z^2 = npq = 0.999$. We can use the Poisson approximation to the Binomial with $a = np = 1$ here. Then

$$\begin{aligned} P[2 &\leq Z \leq 1000] = 1 - P_Z(0) - P_Z(1) \\ &\approx 1 - e^{-a} - ae^{-a} \\ &= 1 - 2e^{-1} \\ &\doteq 0.264. \end{aligned}$$

The CLT approximation gives a Normal distribution with mean $\mu = 1$ and $\sigma = \sqrt{0.999} = 0.9995$. However, it is not as accurate here since the mean μ_Z is only approximately one standard deviation away from 0, the minimum value of a Bernoulli random variable. Calculating the CLT approximate answer, we find

$$\begin{aligned} P[2 \leq Z \leq 1000] &\approx \frac{1}{\sqrt{2\pi \times 0.999}} \int_2^{1000} e^{-\frac{1}{2}\left[\frac{z-1}{\sqrt{0.999}}\right]^2} dz \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{1.0005}^{+\infty} e^{-\frac{1}{2}x^2} dx \\ &= 0.5 - \operatorname{erf}(1.0005) \\ &\doteq 0.5 - 0.341 \\ &= 0.159, \quad \text{not very accurate here.} \end{aligned}$$

61. The pdf of each random variable is given as shown in Fig. **??**.

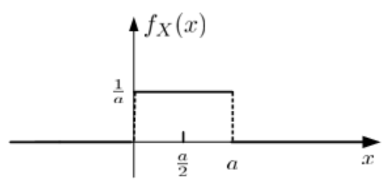


Figure 2:

(a). $E[Z] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = nE[X_1]$ since the X_i are i.i.d.

$$E[X_1] = \frac{1}{a}\int_0^a x dx = \frac{a}{2}$$

Therefore $E[Z] = \frac{na}{2}$.

(b) Define $X_i' \triangleq X_i - \frac{a}{2}$, then the X_i' are i.i.d. and $\text{Var}[X_i'] = \text{Var}[X_i]$ since shift does not affect variance.

$$\text{Var}[X_i'] = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 dx = \left.\frac{x^3}{3a}\right|_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{a^2}{12}$$

Since X_i' are i.i.d.,

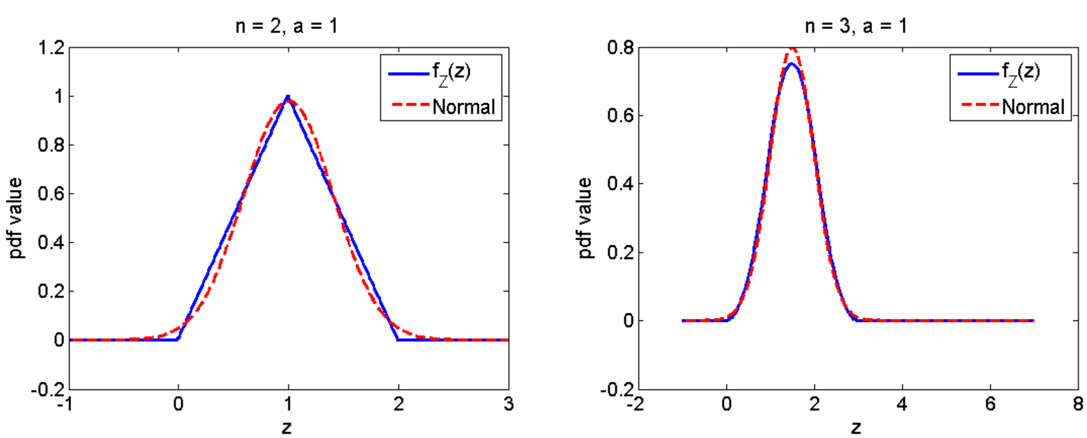
$$\begin{aligned}\text{Var}[\sum_{i=1}^{n} X_i'] &= \sum_{i=1}^{n} \text{Var}(X_i) \\ &= \frac{na^2}{12}.\end{aligned}$$

(c)(d). Characteristic function of X_i is

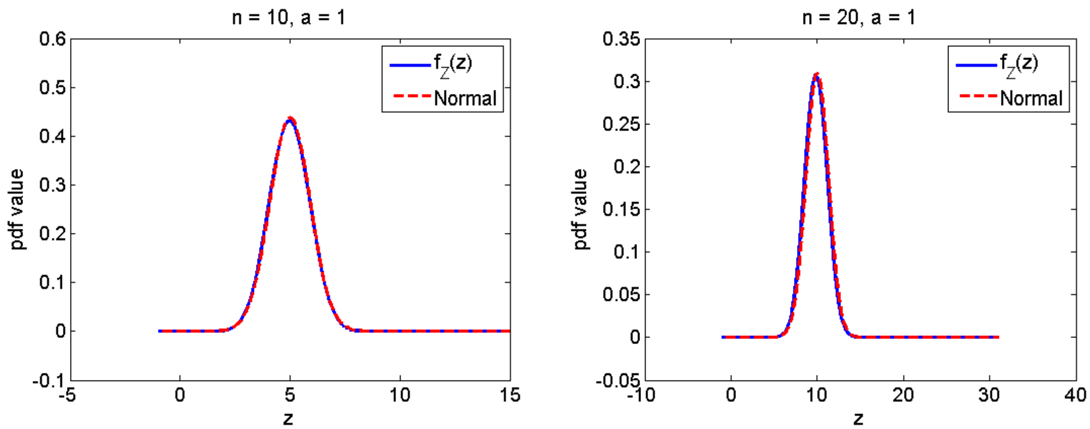
$$\begin{aligned}\Phi_{X_i}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{a} e^{j\omega x} \text{rect}\left(\frac{x - a/2}{a}\right) dx \\ &= e^{\frac{j\omega a}{2}} \frac{\sin(\omega a/2)}{\omega a/2} \\ \Phi_{Z_n}(\omega) &= [\Phi_{X_i}(\omega)]^n \\ &= e^{\frac{jn\omega a}{2}} \left[\frac{\sin(\omega a/2)}{\omega a/2}\right]^n\end{aligned}$$

To get $f_{Z_n}(z)$, we can apply FFT to $\Phi_{Z_n}(\omega)$.

The MATLAB function to compute f_{Z_n} and the interval probability $P[\mu_n - k\sigma_n \le Z_n \le \mu_n + k\sigma_n]$ is called `CltVSNorm(n, a, L, Ks)`. You can run MATLAB script `problem_4_59.m` to call this function for different n. The four figures below (together called Figure 2) compare $f_{Z_n}(z)$ with Gaussian pdf's $N(\frac{na}{2}, \frac{na^2}{12})$ for $n = 2, 3, 10, 20$.



Plots of $f_Z(z)$ and its Normal approximation



(e). The probability $P(\mu - k\sigma \le Z \le \mu + k\sigma)$ with different n and k are shown in table 2.

Table 2: Interval probabilities $P(\mu - k\sigma \le Z \le \mu + k\sigma)$ in problem 4.39

	$k = 0.1$	$k = 0.5$	$k = 1$	$k = 2$	$k = 3$
$n = 2$	0.0766	0.3650	0.6475	0.9677	1.0000
$n = 3$	0.0702	0.3646	0.6667	0.9583	1.0000
$n = 10$	0.0806	0.3754	0.6783	0.9559	0.9982
$n = 20$	0.0809	0.3803	0.6770	0.9536	0.9976
Normal approximation	0.0797	0.3829	0.6827	0.9545	0.9973

62.

$$\begin{aligned} E[X] &= \int_0^{\infty} xf(x)dx + \int_{-\infty}^{0} xf(x)dx \\ &= \int_0^{\infty} xf(x)dx - \int_0^{\infty} xf(-x)dx \\ &= \int_0^{\infty} xf(x)dx - \int_0^{\infty} xf(x)dx, \quad \text{since} \quad f(x) = f(-x), \\ &= 0. \end{aligned}$$

The converse, i.e.$\{E[X] = 0\} \Longrightarrow \{\, f(x) = f(-x)\}$, is not true. Just consider a nonsymmetrical pdf that happens to have zero mean, such as

$$f(x) = \begin{cases} \frac{2}{9}, & 0 \le x < 3, \\ \frac{1}{18}, & -6 \le x < 0, \\ 0, & \text{else.} \end{cases}$$

As you can verify, for this nonsymmetrical density has zero mean.

63. Since the X_i are i.i.d., we write

$$\begin{aligned} E[Z_n] &= E\left[\sum_{i=n}^{n} X_i^2\right] \\ &= \sum_{i=n}^{n} E[X_i^2]. \end{aligned}$$

but, if the $X_i : N(0,1)$, then $E[X_i^2] = 1$, so $E[Z_n] = \sum_{i=n}^{n} 1 = n$.

$$\begin{aligned} \text{Var}[Z_n] &= E[(Z_n - n)^2] \\ &= E[Z_n^2] - n^2. \end{aligned}$$

Now for $E[Z_n^2]$, we get

$$\begin{aligned} E[Z_n^2] &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i^2 X_j^2\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} E[X_i^2 X_j^2] \\ &= \sum_{i=n}^{n} E[X_i^4] + \sum_{i\neq j}^{n} E[X_i^2 X_j^2] \\ &= \sum_{i=n}^{n} E[X_i^4] + \sum_{i\neq j}^{n} E[X_i^2]E[X_j^2] \\ &= \sum_{i=n}^{n} E[X_i^4] + n(n-1). \end{aligned}$$

One way to evaluate the fourth-order Gaussian moment $E[X_i^4]$ is to use the result of problem 4.66, which specialized to the case $X_i : N(0,1)$ yields $E[X_i^4] = 3\left(E[X_i^2]\right)^2 = 3$. So, then

$$\begin{aligned} E[Z_n^2] &= \sum_{i=n}^{n} E[X_i^4] + n(n-1) \\ &= 3n + n(n-1), \end{aligned}$$

and, from the above result for the variance,

$$\begin{aligned} \text{Var}[Z_n] &= E[Z_n^2] - n^2 \\ &= 3n + n(n-1) - n^2 \\ &= 2n. \end{aligned}$$

64. For n odd, the solution is easy. For n even, a little work is required. For odd n, we can write $n = 2k+1$ for $k = 0, 1, 2, \cdots$, and find

$$\begin{aligned} E[(X-\mu)^n] &= E[(X-\mu)^{2k+1}] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^{2k+1} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\,dx \\ &= \frac{\sigma^{2k+1}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^{2k+1} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\,dx \\ &= \frac{\sigma^{2k+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^{2k+1} e^{-\frac{1}{2}y^2}\,dy, \quad \text{with } y \triangleq \left(\frac{x-\mu}{\sigma}\right), \\ &= 0, \end{aligned}$$

because $y^{2k+1}e^{-\frac{1}{2}y^2}$ is an odd function and the integral has even limits.

Now consider the even n case, and write $n = 2k, k = 0, 1, 2, \cdots$. After the transformation $Y \triangleq X - \mu$ with necessarily $f_Y(y) = f_X(y + \mu)$, we then have the characteristic function

$$\begin{aligned}
\Phi_Y(\omega) &= \int_{-\infty}^{\infty} f_Y(y) e^{+j\omega y} dy \\
&= \int_{-\infty}^{\infty} f_Y(y) \left(\sum_{n=0}^{\infty} \frac{(j\omega y)^n}{n!} \right) dy \\
&= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \left(\int_{-\infty}^{\infty} y^n f_Y(y) dy \right) \\
&= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} E[Y^n].
\end{aligned}$$

As given $X : N(\mu, \sigma^2)$ so that $Y : N(0, \sigma^2)$ and hence

$$\begin{aligned}
\Phi_Y(\omega) &= \exp(-\frac{1}{2}\omega^2\sigma^2) \\
&= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\omega^2\sigma^2 \right)^n / n!
\end{aligned}$$

Then, equating terms in these two series for $n = 2k$, we get

$$\begin{aligned}
\frac{(j\omega)^{2k}}{(2k)!} E[Y^{2k}] &= (-1)^{2k} \frac{\omega^{2k}}{2^k k!} \sigma^{2k} \\
&= \frac{\omega^{2k}}{2^k k!} \sigma^{2k}.
\end{aligned}$$

Hence, finally we have

$$E[Y^{2k}] = \frac{(2k)!}{2^k k!} \sigma^{2k}.$$

65. The MATLAB program for this problem is included in file `problem_4_65.m`. For the $\chi^2 : Z = \sum_{i=1}^{n} X_i^2$, where $X_i : N(0,1)$ are i.i.d., then

$$\begin{aligned}
\Phi_Z(\omega) &= E[e^{j\omega Z}] \\
&= E[e^{j\omega \sum_{i=1}^{n} X_i^2}] \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \prod_{i=1}^{n} \left(e^{j\omega x_i^2} e^{-\frac{1}{2}x_i^2} \right) dx_1 \ldots dx_n \\
&= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2j\omega)} dx \right]^n \\
&= \left[\frac{1}{\sqrt{1-2j\omega}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\beta^2} d\beta}_{=1} \right]^n \qquad \beta = x\sqrt{1-2j\omega} \\
&= (1-2j\omega)^{-\frac{n}{2}}
\end{aligned}$$

(a). The graphs of the Chi-square distribution for $n = 30, 40, 50$ are shown in figure 3.

(b). Table 3 shows the probability of $P[\mu - \sigma \le Z_n \le \mu + \sigma]$ with $n = 30, 40, 50$ and Gaussian approximation.

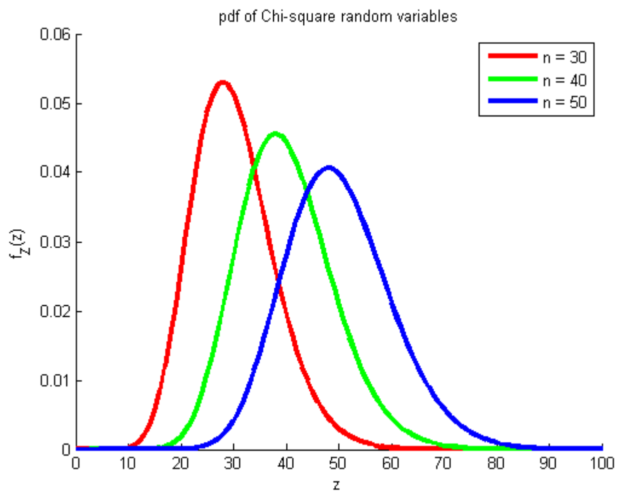


Figure 3: Graphs of Chi-square distribution

Table 3: Interval probabilities $P(\mu - \sigma \le Z_n \le \mu + \sigma)$ in problem 4.43

$n = 30$	$n = 40$	$n = 50$	$N(n, 2n)$
0.6726	0.6620	0.6610	0.6827

66. This problem is somewhat advanced for this chapter as it requires knowledge of the multidimensional Gaussian law and its characteristic function that is presented later in Chapter 5 on Random Vectors (see almost identical problem 5.34). Nevertheless, the solution is presented here. First we form the correlation matrix $\mathbf{R}$ of the random vector $\mathbf{X} = (X_1, X_2, X_3, X_4)$. Then since $\mathbf{X}$ is zero mean, upon setting $\boldsymbol{\omega} \triangleq (\omega_1, \omega_2, \omega_3, \omega_4)^T$, we have

$$\Phi_{\mathbf{X}}(\boldsymbol{\omega}) = \exp -\frac{1}{2}\boldsymbol{\omega}^T \mathbf{K} \boldsymbol{\omega} \ .$$

Now, by definition

$$\Phi_{\mathbf{X}}(\boldsymbol{\omega}) \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 x_4)} f_{\mathbf{X}}(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4,$$

so that

$$E[X_1 X_2 X_3 X_4] = \left. \frac{\partial^4 \Phi_{\mathbf{X}}(\boldsymbol{\omega})}{\partial\omega_1 \partial\omega_2 \partial\omega_3 \partial\omega_4} \right|_{\boldsymbol{\omega}=\mathbf{0}} \ .$$

In calculating this 4-th order partial derivative, it is helpful to write out

$$\boldsymbol{\omega}^T \mathbf{K} \boldsymbol{\omega} = \sum_{i=1}^{4}\sum_{l=1}^{4} K_{il}\omega_i\omega_l,$$

then

$$\begin{aligned}
\frac{\partial \Phi_{\mathbf{X}}}{\partial \omega_1} &= -\Phi_{\mathbf{X}} \sum_{l=1}^{4} K_{1l}\omega_l \quad \text{at} \quad \omega_1 = 0, \\
\frac{\partial^2 \Phi_{\mathbf{X}}}{\partial \omega_1 \partial \omega_2} &= -\Phi_{\mathbf{X}} \left[K_{12} - \sum_{l=1}^{4}\sum_{k=1}^{4} K_{1l}K_{2k}\omega_l\omega_k \right] \quad \text{at} \quad \omega_1 = \omega_2 = 0, \\
\frac{\partial^3 \Phi_{\mathbf{X}}(\boldsymbol{\omega})}{\partial \omega_1 \partial \omega_2 \partial \omega_3} &= \Phi_{\mathbf{X}} \left[2K_{13}K_{23} + K_{23}\sum_{l=1}^{4} K_{1l}\omega_l + K_{13}\sum_{l=1}^{4} K_{2l}\omega_l \right] \\
&+ \Phi_{\mathbf{X}} \left[\left(K_{12} - \sum_{l=1}^{4}\sum_{k=1}^{4} K_{1l}K_{2k}\omega_l\omega_k \right) \sum_{l=1}^{4} K_{3l}\omega_l \right] \quad \text{at} \quad \omega_1 = \omega_2 = \omega_3 = 0.
\end{aligned}$$

Finally

$$\begin{aligned}
\left. \frac{\partial^4 \Phi_{\mathbf{X}}(\boldsymbol{\omega})}{\partial \omega_1 \partial \omega_2 \partial \omega_3 \partial \omega_4} \right|_{\boldsymbol{\omega}=\mathbf{0}} &= K_{12}K_{34} + K_{13}K_{24} + K_{23}K_{14} \\
&= E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] \\
&\qquad + E[X_2X_3]E[X_1X_4],
\end{aligned}$$

since the X_i are zero mean.

67. For the $\chi^2 : Z = \sum_{i=1}^{n} X_i^2$, where $X_i : N(0,1)$ are i.i.d. Then

$$\begin{aligned}
\Phi_Z(\omega) &= E[e^{j\omega Z}] \\
&= E[e^{j\omega \sum_{i=1}^{n} X_i^2}] \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \prod_{i=1}^{n} \left(e^{j\omega x_i^2} e^{-\frac{1}{2}x_i^2} \right) dx_1 \ldots dx_n \\
&= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2(1-2j\omega)} dx \right]^n \\
&= \left[\frac{1}{\sqrt{1-2j\omega}} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\beta^2} d\beta}_{=1} \right]^n \qquad (\beta = x\sqrt{1-2j\omega}) \\
&= (1-2j\omega)^{-\frac{n}{2}}
\end{aligned}$$

68. Let $E[X_i] = \mu$, $\mathrm{Var}[X_i] = \sigma^2$. We wish to estimate μ with the *sample mean*

$$\widehat{\mu} \triangleq \frac{1}{N} \sum_{i=1}^{N} X_i.$$

The mean of $\widehat{\mu}$:

$$E[\widehat{\mu}] = E\left[\frac{1}{N} \sum_{i=1}^{N} X_i \right] = \frac{1}{N} \sum_{i=1}^{N} E[X_i] = \frac{1}{N} \sum_{i=1}^{N} \mu = \mu$$

The variance of $\widehat{\mu}$:

$$
\begin{aligned}
\operatorname{Var}[\widehat{\mu}] &= E\left[\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right)^2\right] \\
&= \frac{1}{N^2} E\left[\sum_{i=1}^{N}(X_i - \mu)\right]^2 \\
&= \frac{1}{N^2} E\left[\sum_{i=1}^{N}\sum_{j=1}^{N}(X_i - \mu)(X_j - \mu)\right] \\
&= \frac{1}{N^2} \sum_{i=1}^{N}\sum_{j=1}^{N} E\left[(X_i - \mu)(X_j - \mu)\right] \\
&= \frac{1}{N^2} \left[\sum_{i=1}^{N} E\left[(X_i - \mu)^2\right] + \sum_{i=1}^{N}\sum_{j\neq i} E\left[(X_i - \mu)(X_j - \mu)\right]\right]
\end{aligned}
$$

Since X_i's are independent, so they are uncorrelated, which means $E\left[(X_i - \mu)(X_j - \mu)\right] = 0$ for all $i \neq j$. Besides, we have $E\left[(X_i - \mu)^2\right] = \operatorname{Var}[X_i] = \sigma^2$. Thus,

$$
\operatorname{Var}[\widehat{\mu}] = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2 = \frac{\sigma^2}{N}.
$$

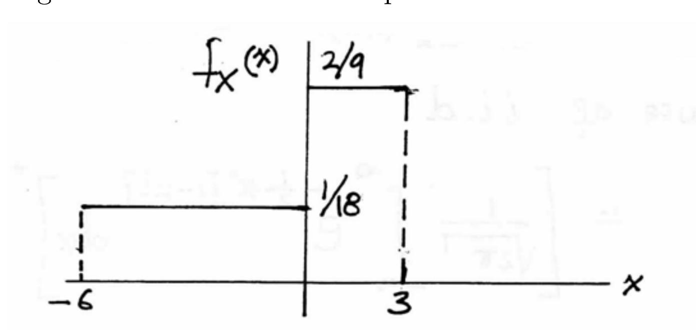
The second moment is then

$$
\begin{aligned}
E[\widehat{\mu}^2] &= \operatorname{Var}[\widehat{\mu}] + (E[\widehat{\mu}])^2 \\
&= \frac{\sigma^2}{N} + \mu^2.
\end{aligned}
$$

69. We need to find a nonsymmetrical function $f \geq 0$ such that

$$
\int_{-\infty}^{+\infty} f(x)dx = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} xf(x)dx = 0.
$$

Now, the following function satisfies both requirements:



It can be written analytically as

$$f(x) = \begin{cases} \frac{2}{9}, & 0 < x \leq 3, \\ \frac{1}{18}, & < -6x \leq 0, \\ 0, & \text{else.} \end{cases}$$

Clearly $f(x) \neq f(-x)$ and the function f is non-negative. Computing its integral, we get

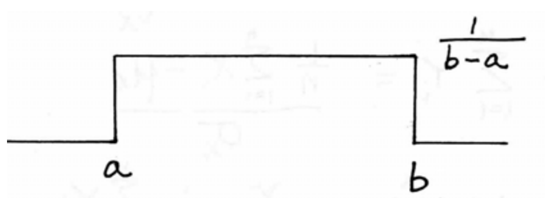
$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)dx &= \frac{1}{18} \times 6 + 3 \times \frac{2}{9} \\ &= 1, \end{aligned}$$

with mean value

$$\begin{aligned} \int_{-\infty}^{+\infty} xf(x)dx &= \frac{1}{18}\int_{-6}^{0} xdx + \frac{2}{9}\int_{0}^{3} xdx \\ &= -1+1 \\ &= 0. \end{aligned}$$

Thus, a mean of zero does not imply that the pdf is symmetric. We note however, that concerning the inverse, it is true that a symmetric pdf, i.e. symmetry about 0, does imply that the mean is zero.

70. The RV X is uniformly distributed on (a, b) where $0 < a < b < \infty$.



Starting from the definition, we find

$$\begin{aligned} \mu_X &\triangleq E[X] \\ &= \frac{1}{b-a}\int_{a}^{b} xdx \\ &= \frac{1}{b-a}\frac{1}{2}(b^2-a^2) \\ &= \frac{1}{2}(a+b), \end{aligned}$$

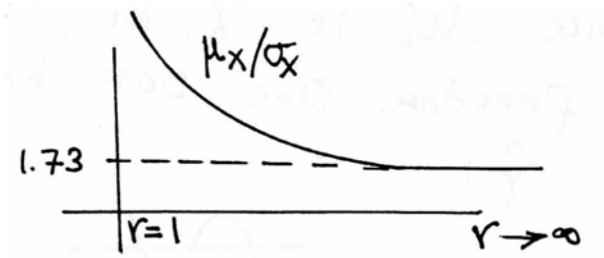
and

$$\begin{aligned} \text{Var}[X] &\triangleq E[(X-\mu_X)^2] \\ &= \frac{1}{b-a}\int_{a}^{b}\left(x-\frac{1}{2}(a+b)\right)^2 dx \\ &= \frac{1}{12}(b-a)^2. \end{aligned}$$

Hence, $\sigma_X \doteq 0.289(b-a)$ and

$$\begin{aligned}\frac{\mu_X}{\sigma_X} &\doteq \frac{b+a}{0.577(b-a)}\\ &\doteq 1.73\frac{b+a}{b-a}\\ &= 1.73\frac{r+1}{r-1} \quad \text{with} \quad r \triangleq \frac{b}{a}.\end{aligned}$$

So, $\lim_{f\longrightarrow\infty} \frac{\mu_X}{\sigma_X} \doteq 1.73.$



71. First, we define $Y_i \triangleq (X_i - \mu_X)/\sigma_X$ which is distributed as $N(0,1)$. Then

$$\begin{aligned}\widehat{Y} &= \frac{1}{n}\sum_{i=1}^{n} Y_i\\ &= \frac{\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu_X}{\sigma_X}.\end{aligned}$$

Hence

$$Y_i - \widehat{Y} = \frac{X_i - \frac{1}{n}\sum_{i=1}^{n} X_i}{\sigma_X},$$

and

$$W_n = \sum_{i=1}^{n}(Y_i - \widehat{Y})^2.$$

Next, compute the moment generating function (MGF)

$$\begin{aligned}M_{W_n}(t) &\triangleq E[e^{tW_n}]\\ &= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{t\sum(y_i-\widehat{y})^2} e^{-\frac{1}{2}y_i^2} dy_1\cdots dy_n\\ &= \left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}}.\end{aligned}$$

Compare with problem 8.67 where we found the MGF of $\sum_{i=1}^{n}\left(\frac{X_i-\mu_X}{\sigma_X}\right)^2$ was $\left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$. Hence the random variable W_n is seen to be χ^2 distributed with $n-1$ degrees of freedom (DOF). So, one DOF has been lost in $\widehat{Y}$!

72. (a) Since $Y = X + N$, the conditional PMF of Y given $X = x$,i.e. given that X is known, is

$$P_{Y|X}(y|x) = P_N(y - x).$$

This is true because given the condition $X = x$, then the equation for Y is $Y = x + N$, a simple translation by the scalar x, with the one solution $N = Y - x$. A more complicated way to see this is to start with the random variable pair (X, N) and consider the transformation to the pair (X, Y). We have $x = g(x) = x$ and $y = h(x, n) = x + n$ with inverse transformation $x = \phi(x) = x$ and $n = \psi(x, y) = y - x$. So

$$\begin{aligned} P_{X,Y}(x, y) &= P_{X,N}(x, y - x) \\ &= P_X(x)P_N(y - x), \end{aligned}$$

since X and N are independent, thus

$$\begin{aligned} P_{Y|X}(y|x) &= \frac{P_{X,Y}(x, y)}{P_X(x)} \\ &= \frac{P_X(x)P_N(y - x)}{P_X(x)} \\ &= P_N(y - x). \end{aligned}$$

Now N is Poisson with $\mu = 5$, then

$$P_N(n) = \frac{5^n}{n!}e^{-5}u(n),$$

and thus finally

$$\begin{aligned} P_{Y|X}(y|x) &= P_N(y - x) \\ &= \frac{5^{y-x}}{(y - x)!}e^{-5}u(y - x), \end{aligned}$$

with support on $y \geq x$ as it should be.

(b) Since the conditional PMF $P_{Y|X}(y|x)$ found in part a is just the Poisson PMF $P_N(n)$ shifted right by x, then the conditional mean $E[Y|X = x]$ must be $\mu + x = x + 5$. Alternatively, we can directly use the linearity of conditional expectation, as follows

$$\begin{aligned} E[Y|X = x] &= E[X + N|X = x] \\ &= E[X|X = x] + E[N|X = x] \\ &= x + E[N] \\ &= x + 5, \end{aligned}$$

where the second to last line follows since X and N are independent.

73. Let $Y \triangleq |X|$ for X real-valued. Then, assuming no jumps in the distribution function F_X,

$$\begin{aligned} F_Y(y) &\triangleq P[Y \leq y] \\ &= P[|X| \leq y] \\ &= P[-y \leq X \leq y] \\ &= F_X(y) - F_X(-y). \end{aligned}$$

Then

$$\begin{aligned} f_Y(y) &\triangleq \frac{dF_Y(y)}{dy} \\ &= \frac{d}{dy}\left[F_X(y) - F_X(-y)\right] \\ &= f_X(y) + f_X(-y) \\ &= 2f_X(y) \qquad \text{since here } f_X(x) = f_X(-x). \end{aligned}$$

Also $E[X] = 0$ so that $E[X^2] = E[|X|^2] = \sigma_X^2 = E[Y^2] = \sigma_Y^2 + \mu_Y^2$. Thus $\sigma_X^2 \geq \mu_Y^2$ or equivalently

$$\sigma_X \geq E[|X|],$$

with equality if and only if $\sigma_Y = 0$. Now

$$\begin{aligned} E[|X|] &= E[Y] \\ &= \int_0^\infty y f_Y(y)dy \\ &= \int_0^{\sigma_X} y f_Y(y)dy + \int_{\sigma_X}^\infty y f_Y(y)dy \\ &\geq \int_{\sigma_X}^\infty y f_Y(y)dy \\ &\geq \sigma_X \int_{\sigma_X}^\infty f_Y(y)dy \\ &= \sigma_X P[|X| \geq \sigma_X]. \end{aligned}$$

Putting these two results together, we get

$$\sigma_X \geq E[|X|] \geq \sigma_X P[|X| \geq \sigma_X].$$

74. (a) First, we consider the special case $E[X] = \mu_X = 0$.

$$\begin{aligned} E[X^2] &= \sigma_X^2 \\ E[XY] &= \mathrm{Cov}[X,Y] + \mu_X\mu_Y = \rho\sigma_X\sigma_Y \\ \mathrm{Cov}[\varepsilon, X] &= E[\varepsilon X] - E[\varepsilon]E[X] \\ &= E[(\alpha X + \beta - Y)X] - 0 \\ &= \alpha E[X^2] + \beta E[X] - E[XY] \\ &= \alpha\sigma_X^2 - \rho\sigma_X\sigma_Y. \end{aligned}$$

When $\mu_X \neq 0$, define $X' = X - \mu_x, \varepsilon' = \alpha X' + \beta - Y$, we can easily get $\mu_{X'} = 0, \sigma_{X'} = \sigma_X, \rho_{X'Y} = \rho_{XY} = \rho, X' - \mu_{X'} = X - \mu_X, \varepsilon' - E[\varepsilon'] = \varepsilon - E[\varepsilon]$.

$$\begin{aligned} \mathrm{Cov}[\varepsilon, X] &= E[(\varepsilon - E[\varepsilon])(X - \mu_X)] \\ &= E[(\varepsilon' - E[\varepsilon'])(X' - E[X'])] \\ &= \mathrm{Cov}[\varepsilon', X'] \\ &= \alpha\sigma_{X'}^2 - \rho_{X'Y}\sigma_{X'}\sigma_Y \\ &= \alpha\sigma_X^2 - \rho\sigma_X\sigma_Y. \end{aligned}$$

When $\mu_X \neq 0$, we can also start from $\text{Cov}[\varepsilon, X] = E[\varepsilon X] - E[\varepsilon]E[X]$ to get the answer.

$$\begin{aligned}
\text{Cov}[\varepsilon, X] &= E[\varepsilon X] - E[\varepsilon]E[X] \\
&= E[(\alpha X + \beta - Y)X] - \mu_X(\alpha\mu_X + \beta - \mu_Y) \\
&= \alpha E[X^2] + \beta\mu_X - E[XY] - \mu_X E[\varepsilon]
\end{aligned}$$

Since $E[X^2] = \sigma_X^2 + \mu_X^2, E[XY] = \rho\sigma_X\sigma_Y + \mu_X\mu_Y$, and $E[\varepsilon] = E[\alpha X + \beta - Y] = \alpha\mu_X + \beta - \mu_Y$,

(b) Set α and β to their optimal values. Then evaluate $\text{Cov}[\varepsilon, X]$ again. To minimize the mean-square error $E[\varepsilon^2]$, we solve α and β that satisfy $\frac{\partial E[\varepsilon^2]}{\partial\alpha} = 0$ and $\frac{\partial E[\varepsilon^2]}{\partial\beta} = 0$.

$$\begin{aligned}
\frac{\partial E[\varepsilon^2]}{\partial\beta} &= E\left[\frac{\partial\varepsilon^2}{\partial\beta}\right] = E\left[2\varepsilon\frac{\partial\varepsilon}{\partial\beta}\right] \\
&= 2E[\varepsilon] = 2(\alpha\mu_X + \beta - \mu_Y) = 0 \\
\Rightarrow \beta &= \mu_Y - \alpha\mu_X
\end{aligned}$$

Hence, $\varepsilon = \alpha X + \mu_Y - \alpha\mu_X - Y = \alpha(X - \mu_X) - (Y - \mu_Y)$.

$$\begin{aligned}
\frac{\partial E[\varepsilon^2]}{\partial\alpha} &= E\left[\frac{\partial\varepsilon^2}{\partial\alpha}\right] = E\left[2\varepsilon\frac{\partial\varepsilon}{\partial\alpha}\right] \\
&= 2E[\varepsilon(X - \mu_X)] = 0 \\
E[\varepsilon(X - \mu_X)] &= E\left\{[\alpha(X - \mu_X) - (Y - \mu_Y)](X - \mu_X)\right\} \\
&= \alpha E[(X - \mu_X)^2] - E[(X - \mu_X)(Y - \mu_Y)] \\
&= \alpha\sigma_X^2 - \rho\sigma_X\sigma_Y = 0 \\
\Rightarrow \alpha &= \frac{\rho\sigma_Y}{\sigma_X} \\
\beta &= \mu_Y - \frac{\rho\sigma_Y\mu_X}{\sigma_X}
\end{aligned}$$

With the optimal α and β are employed, then we obtain $\text{Cov}[\varepsilon, X] = \frac{\rho\sigma_Y}{\sigma_X}\sigma_X^2 - \rho\sigma_X\sigma_Y = 0$. That is to say, the estimate error and the data used to make the estimate (here Y) are uncorrelated.

75. The orthogonality condition gives

$$E[\epsilon X_1] = 0 \text{ and } E[\epsilon X_2] = 0.$$

Since $\epsilon = Y - \hat{Y} = Y - (\alpha_1 X_1 + \alpha_2 X_2)$, we get

$$\begin{aligned}
E[\epsilon X_1] &= E[\{Y - (\alpha_1 X_1 + \alpha_2 X_2)\}X_1] \\
&= E[YX_1] - \alpha_1 E[X_1^2] - \alpha_2 E[X_1 X_2] \\
&= \rho_1\sigma_1\sigma_Y - \alpha_1\sigma_1^2 - \alpha_2\rho_{12}\sigma_1\sigma_2 \\
&= 0 \qquad\qquad \text{(given).}
\end{aligned}$$

Similarly, equating $E[\epsilon X_2] = 0$, we get $\rho_2\sigma_2\sigma_Y - \alpha_1\rho_{12}\sigma_1\sigma_2 - \alpha_2\sigma_2^2 = 0$. The two linear equations in α_1 and α_2 are given by

$$\begin{aligned}\sigma_1^2\alpha_1 + \rho_{12}\sigma_1\sigma_2\alpha_2 &= \rho_1\sigma_1\sigma_Y,\\ \rho_{12}\sigma_1\sigma_2\alpha_1 + \sigma_2^2\alpha_2 &= \rho_2\sigma_2\sigma_Y.\end{aligned}$$

In matrix notation, this can be represented as

$$\begin{pmatrix}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}\rho_1\sigma_1\sigma_Y\\ \rho_2\sigma_2\sigma_Y\end{pmatrix}.$$

If X_1 and X_2 are not perfectly correlated, i.e., $\rho_{12} \neq 1$, then the covariance matrix of X_1 and X_2 will be invertible. The solution to above equation is given by

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} &= \begin{pmatrix}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}\rho_1\sigma_1\sigma_Y\\ \rho_2\sigma_2\sigma_Y\end{pmatrix}\\ &= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}\begin{pmatrix}\sigma_2^2 & -\rho_{12}\sigma_1\sigma_2\\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}\begin{pmatrix}\rho_1\sigma_1\sigma_Y\\ \rho_2\sigma_2\sigma_Y\end{pmatrix}\\ &= \frac{1}{(1-\rho_{12}^2)}\begin{pmatrix}\frac{1}{\sigma_1^2} & \frac{-\rho_{12}}{\sigma_1\sigma_2}\\ \frac{-\rho_{12}}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{pmatrix}\begin{pmatrix}\rho_1\sigma_1\sigma_Y\\ \rho_2\sigma_2\sigma_Y\end{pmatrix}\\ &= \frac{1}{(1-\rho_{12}^2)}\begin{pmatrix}\frac{\rho_1\sigma_Y}{\sigma_1} - \rho_{12}\frac{\rho_2\sigma_Y}{\sigma_1}\\ \frac{\rho_2\sigma_Y}{\sigma_2} - \rho_{12}\frac{\rho_1\sigma_Y}{\sigma_2}\end{pmatrix}.\end{aligned}$$