Solutions to HW Problems Chapter 6

6.1 The rearrangement leads to $Y_2 = X_1, Y_1 = X_2, Y_4 = X_3, etc.$ so the sequence $X_1, X_2, X_3, ..., X_n$ becomes the sequence $Y_2, Y_1, Y_4, Y_3, ..., Y_n, Y_{n-1}$. Since there is no size ordering or any other functional relationship among the Y's , the elements of the $\{Y_i, i = 1,...,n\}$ are i.i.d. as well.

6.2 $f_{X_1X_2X_3}(x_1,x_2,x_3) = \prod_{i=1}^{3}(2\pi)^{-1/2}\exp\left(-\frac{1}{2}x_i^2\right) = (2\pi)^{-3/2}\exp\left(-\frac{1}{2}\sum_{i=1}^{3}x_i^2\right)$, To get the pdf of $Y \triangleq X_1 + X_2 + X_3$, we need only to remember that the sum of n i.i.d $N(\mu,\sigma^2)$ RVs is distributed as $N(n\mu, n\sigma^2)$. Hence $f_Y(y) = (6\pi)^{-1}\exp\left(-\frac{1}{2}(y/\sqrt{3})^2\right)$.

6.3 Let $X_i = \begin{cases} 1, \text{ if exposed villager dies; } P[X_i = 1] \triangleq p \\ \quad 0, \text{else; } P[X_i = 0] = 1 - p = q \end{cases}$.

Then $\hat{p} = \sum_{i=1}^{n} X_i / n$. Since X_i is Bernoulli, $n\hat{p} = \sum_{i=1}^{n} X_i$ is binomial with $E[\hat{p}] = p,\ Var[\hat{p}] = p(1-p)/n$. Since n>>1, we use the Normal approximation and write

$$P[-a < (\hat{p} - p) < a] = 0.95 = 2erf\left[\frac{a}{\sqrt{p(1-p)/n}}\right]$$

or $F_{SN}(z_{0.975}) = F_{SN}\left(a\sqrt{n}/\sqrt{p(1-p)}\right) = 0.975$. From the Tables of the SN we get

$z_{0.975} = 1.96$ and $a = 1.96\sqrt{p(1-p)/n}$. Finally we solve $(\hat{p} - p)^2 = 3.84p(1-p)/n$ and obtain $p_2 = 0.58, p_1 = 0.46$ and the margin of error is $\pm 6\%$.

6.4 Expand $(p - \hat{p})^2 - (9/n)p(1-p) = 0$ and obtain

$p^2[1 + (9/n)] - p[2\hat{p} + (9/n)] + \hat{p}^2 = 0$,which is of the form $ap^2 + bp + c = 0$; then use the quadratic root formula: $p = -(b/2a) \pm \sqrt{(b/2a)^2 - (c/a)}$ to get the desired result.

6.5 This can be done in Excel as:

1. create a column X that varies from0.05 to 0.95 in steps of 0.05;

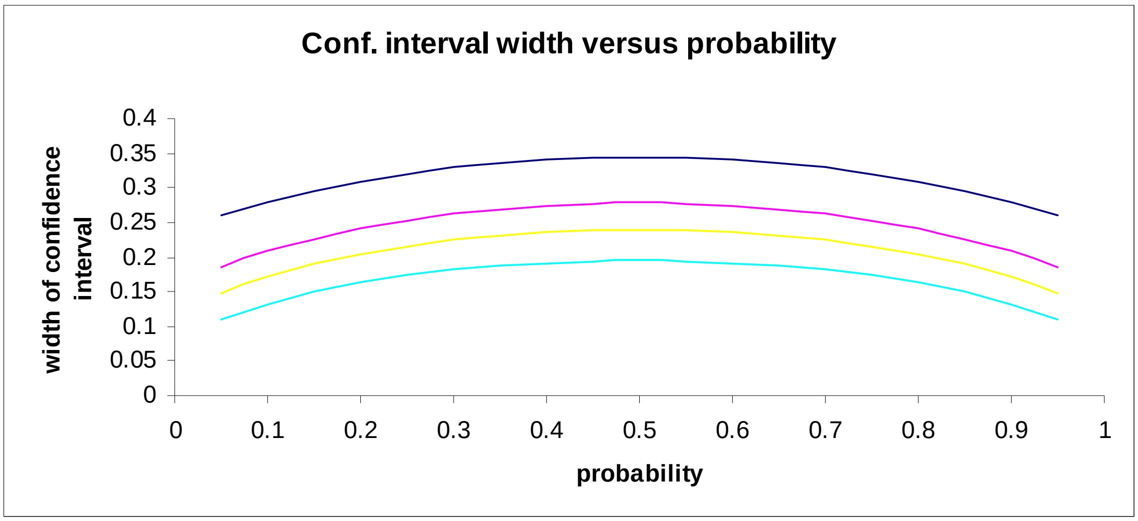
2.Fix the parameter *n*;

3. Create a column as

$$W = 2*SQRT(((2*X+(9/n))/2*(1+(9/n))^2-X^2/(1+(9/n)))$$

4. Create additional columns as you change *n*.

5. Use the Chart Wizard to get the curves below.



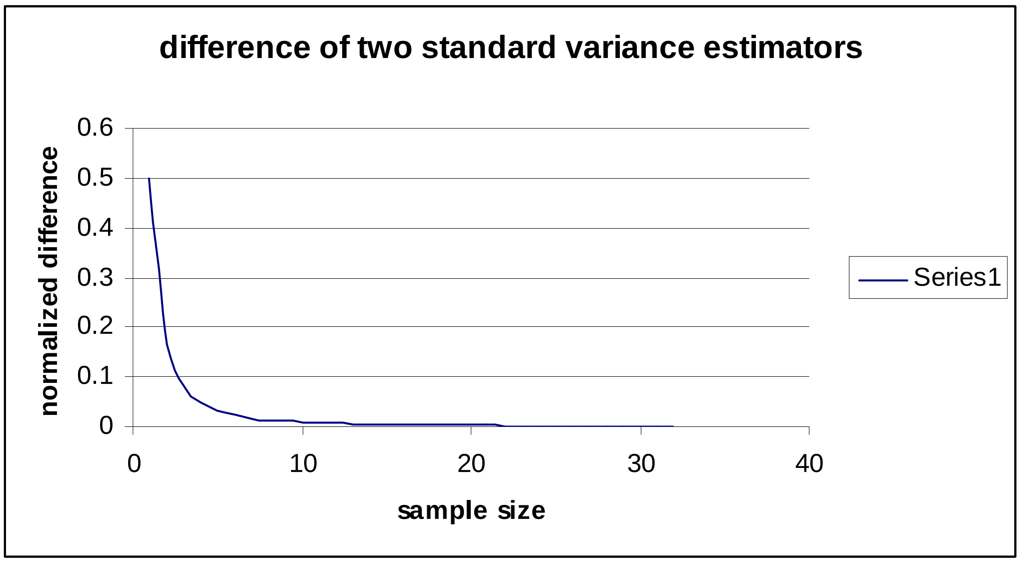
6.6 A reasonable way to do this (here we borrow from the next chapter) is to compute the so-called *distance* from the actual data yield to the *theoretical* data yield if the coin was unbiased. Thus, assume we flip the coin n times where $n >> 1$.If the coin is fair then we expect, on the average $n/2$ heads and $n/2$ tails. Let Y denote the observed number of heads; then it is approximately true (by the Central Limit Theorem)

$$\frac{Y-n/2}{\sqrt{n/2}} : N(0,1) \text{ and } \frac{n/2-Y}{\sqrt{n/2}} : N(0,1)$$

and $V \triangleq \left(\frac{Y-n/2}{\sqrt{n/2}}\right)^2 + \left(\frac{n/2-Y}{\sqrt{n/2}}\right)^2$ is a “distance statistic”, which is χ^2 with one degree of freedom. Now if a realization of this distance statistic is too large we would reject the notion that that the coin is fair. On the other hand if the distance is small we would accept that the coin is fair. To find the 95 percent cutoff point we go to the tables of the Chi-square distribution and look up the CDF with DOF=1 and $0.95 = F_{\chi^2}(x_{0.95};1)$,which yields $x_{0.95} = 3.84$. Thus if $V > 3.84$ we would reject that the coin is fair.

6.7 Let $\hat{\sigma}_1^2 \triangleq (n-1)^{-1}\sum_{i=1}^{n}\left(X_i - \hat{\mu}_X\right)^2$ and $\hat{\sigma}_2^2 \triangleq n^{-1}\sum_{i=1}^{n}\left(X_i - \hat{\mu}_X\right)^2$. We plot

$(\hat{\sigma}_1^2 - \hat{\sigma}_2^2) / \sum_{i=1}^{n}\left(X_i - \hat{\mu}_X\right)^2 \triangleq$ normalized difference=$1/[(n-1)n]$ using, say, Excel to obtain the figure shown below.



We see that for n>20, the difference becomes extremely small.

6.8 Since $X : N(1,1)$ i.e. $\sigma_X = 1$, we can rewrite $P[|\hat{\mu}_X(n) - \mu_X| \le 0.1]$ as

$$P[\frac{-0.1}{1/\sqrt{n}} \le \frac{\hat{\mu}_X(n) - \mu_X}{1/\sqrt{n}} \le \frac{0.1}{1/\sqrt{n}}] = P[-0.1\sqrt{n} \le Y \le 0.1\sqrt{n}]$$

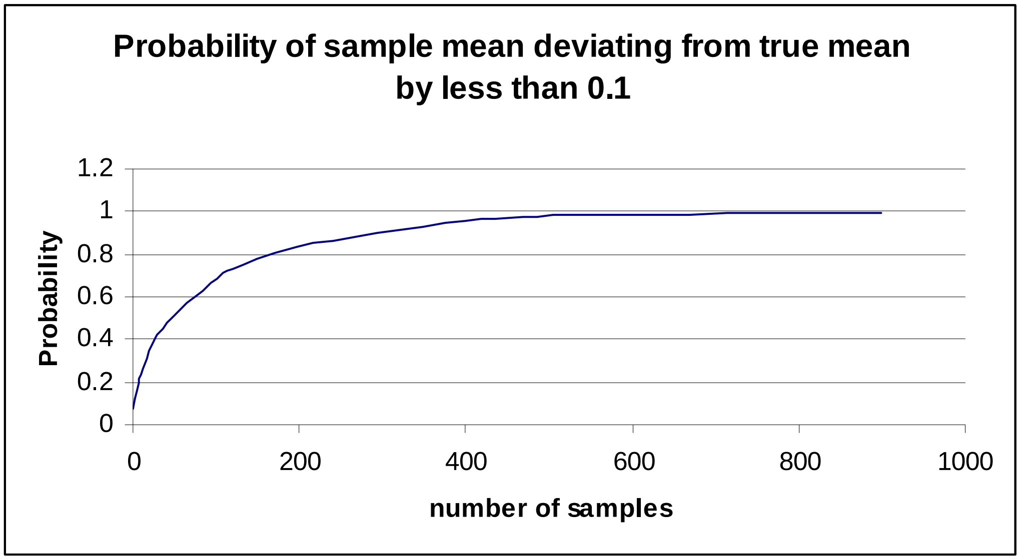
$= 2 \times erf(0.1\sqrt{n})$, where Y is the standard Normal RV.

From Table 2.4-1 we get

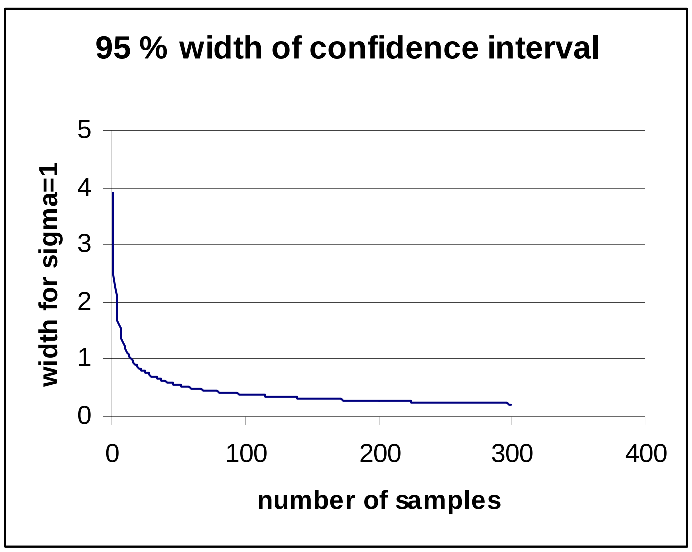
1	0.07966
2.25	0.118
4	0.15852
6.25	0.19742
9	0.23582
25	0.38292
36	0.4515
64	0.57628
100	0.68268

121	0.72866
196	0.83848
400	0.95448
552	0.9812
900	0.99728

Where the first column is n and the second column is P. When plotted we get



6.9 Since the variance is known we can use the formula in Equation 6.3-13:
$W_\delta = 2 \times z_{(1+\delta)/2} \times \sigma_X / \sqrt{n}$, which yields, for $\delta = 0.95$ and $\sigma_X = 1 : W_\delta = 3.92 / \sqrt{n}$.



6.10 To simplify things let's temporarily denote $K \triangleq (\alpha!\beta^{\alpha+1})^{-1}$. Then the MGF is given by $M(t) = K\int_0^\infty x^\alpha e^{-x/\beta} e^{tx} dx = K\int_0^\infty x^\alpha e^{-x(1/\beta - t)} dx$. Now let $u \triangleq x(\beta^{-1} - t)$ so that $x = u/(\beta^{-1} - t)$ and $dx = du/(\beta^{-1} - t)$. This transformation yields $M(t) = K \times (1/\beta - t)^{-(\alpha+1)} \int_0^\infty u^\alpha e^{-u} du = K \times (1/\beta - t)^{-(\alpha+1)} \times \alpha!$

Finally, substituting for K enables us to write $M(t) = (\alpha!)^{-1}\alpha!/[\beta^{(\alpha+1)} \times (\beta^{-1} - t)^{\alpha+1}] = \dfrac{1}{(1-\beta t)^{\alpha+1}}$, which is the desired result.

6.11 There are at least two ways to do this problem: Let $X_1, ..., X_n$ denote the n i.i.d observations on X. (1) Start with $X_1 : N(\mu, \sigma)$ and $Y_1 \triangleq \dfrac{X_1 - \mu}{\sigma}$. Then $Y_1 : N(0,1)$ and let $W_1 \triangleq Y_1^2$. Then $P[W_1 \leq w] = F_{W_1}(w) = F_Y(\sqrt{w}) - F_Y(-\sqrt{w})$ or, equivalently, by differentiation, $f_{W_1}(w) = \dfrac{1}{\sqrt{2\pi w}} e^{-w/2} u(w)$. Next, we consider $W_2 \triangleq Y_1^2 + Y_2^2$ and since Y_1^2 and Y_2^2 are i.i.d

$$f_{W_2}(w) = \frac{1}{\sqrt{2\pi w}} e^{-w/2} u(w) * \frac{1}{\sqrt{2\pi w}} e^{-w/2} u(w)$$
$$= \frac{1}{2} e^{-w/2} u(w) \text{ (see Example 4.11).}$$

Proceeding in this way, by repeated convolutions, we arrive at the result:

$$f_{W_n}(w) \triangleq f_W(w; n) = ([n/2) - 1]! 2^{n/2})^{-1} w^{(n/2)-1} \exp(-(1/2)w) \times u(w)$$

Another way to obtain the result is to compute the moment generating function of $W_n \triangleq \sum_{i=1}^n Y_i^2$. This is easily done as

$$M_n(t) = (2\pi)^{-n/2} \int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \exp\left(t\sum_{i=1}^n y_i^2\right) \exp\left(-(1/2)\sum_{i=1}^n y_i^2\right) \prod_{i=1}^n dy_i$$
$$= \left[\int_{-\infty}^\infty \frac{1}{2\pi} \exp\left(-(1/2) y_i^2 (1-2t)\right) dy_i\right]^n = \left(\frac{1}{1-2t}\right)^{n/2}.$$

The MGF of $W_1 \triangleq Y_1^2$ is $M_1(t) = \dfrac{1}{\sqrt{(1-2t)}}\left(\dfrac{1}{(2\pi)^{1/2}}\int_{-\infty}^\infty e^{-\frac{1}{2}y^2} dy\right) = \dfrac{1}{\sqrt{(1-2t)}}$. Hence the MGF of $W_n \triangleq Y_1^2 + Y_2^2 + \cdots Y_n^2$ is $\dfrac{1}{(1-2t)^{1/2}} \times \cdots \times \dfrac{1}{(1-2t)^{1/2}}$ (n times)$=\dfrac{1}{(1-2t)^{n/2}}$.

6.12 The solution to this problem is given in Appendix F. However you might want to try a different approach as follows.

Assume that we have n i.i.d $X_i : N(\mu,\sigma^2), i=1,...,n$ RVs. For simplicity let

$Y_i \triangleq \dfrac{X_i-\mu}{\sigma}, i=1,...,n$. The joint pdf of the $Y_i,\ i=1,...,n$ is $f_{\mathbf{Y}}(\mathbf{y})=(2\pi)^{-n/2}e^{-\frac{1}{2}\sum_{i=1}^{n}y_i^2}$,where $\mathbf{Y}\triangleq\{Y_i, i=1,...,n\}$ and $\mathbf{y}\triangleq\{y_i, i=1,...,n\}$.

Now let $m\triangleq\dfrac{1}{n}\sum_{i=1}^{n}y_i$ and $s\triangleq\dfrac{1}{n}\sum_{i=1}^{n}(y_i-\mu_s)^2$. We observe that $\sum_{i=1}^{n}y_i^2=ns+nm^2$, so that, symbolically at least, we can write

$f_{\mathbf{Y}}(\mathbf{y})=(2\pi)^{-n/2}e^{-\frac{1}{2(1/n)}\left[m^2+s^2\right]}u(s)$. This form suggests that we define two new RVs $M\triangleq\dfrac{1}{n}\sum_{i=1}^{n}Y_i,$ and $S\triangleq\dfrac{1}{n}\sum_{i=1}^{n}(Y_i-M)^2$, which we recognize as the sample mean $\hat{\mu}$ and sample variance $\hat{\sigma}^2$ respectively. Try to compute $f_{MS}(m,s)$ and show that it factors as $f_{MS}(m,s)=f_M(m)f_S(s)$ thus proving the independence of $\hat{\mu}$ and $\hat{\sigma}^2$.

6.13 (a) For simplicity we leave out the subscript n on W_n so that it becomes W. Clearly if X and W are independent so are $Y\triangleq(X-\mu)/\sigma$ and W Now $Y:N(0,1)$ and $W:\chi_n^2$ so that $f_Y(y)=(2\pi)^{-1/2}e^{-\frac{1}{2}y^2}$ and $f_W(w;n)=\dfrac{1}{[(n/2)-1]!2^{n/2}}w^{(n/2)-1}e^{-(1/2)w}u(w)$. Now take the product to obtain $f_{YW}(y,w;n)=\dfrac{1}{(2\pi)^{1/2}}e^{-\frac{1}{2}y^2}\dfrac{1}{[(n/2)-1]!2^{n/2}}w^{(n/2)-1}e^{-(1/2)w}u(w)$.

(b) We create two new RVs as $T\triangleq\dfrac{Y}{\sqrt{W/n}}$ and $S\triangleq W$ (an auxiliary RV). Now consider the functional transformations:

$t=y/(\sqrt{w/n})$ and $s=w$. The Jacobian is

$$\begin{vmatrix}\dfrac{\sqrt{n}}{\sqrt{s}} & \dfrac{-t}{s}\\ 0 & 1\end{vmatrix}=\frac{\sqrt{n}}{\sqrt{s}}.$$

Now replacing $t(\sqrt{w/n}) = y$ and $w = s$ in

$f_{YW}(y,w;n) = \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}y^2} \frac{1}{[(n/2)-1]!2^{n/2}} w^{(n/2)-1} e^{-(1/2)w} u(w)$ and dividing by $\frac{\sqrt{n}}{\sqrt{s}}$ yields

$f(s,t;n) = \frac{1}{(2\pi n)^{1/2}} e^{-\frac{1}{2}s(\frac{t^2}{n}+1)} \frac{1}{[(n/2)-1]!2^{n/2}} s^{(n-1)/2} u(s)$. Finally integrating out with respect to s and recalling that $[(n-1)/2]! \triangleq \int_0^\infty s^{(n-1)/2} e^{-s} ds$ yields the desired result. We use the symbol $[(n-1)/2]! \triangleq \int_0^\infty s^{(n-1)/2} e^{-s} ds$ whether or nor $[(n-1)/2]$ is an integer.

6.14 The RV $F \triangleq \frac{W_m/m}{V_n/n}$ has the F-distribution with m, n degrees of freedom. From the solution of Problem 6.13 we see that $T \triangleq \frac{Y}{\sqrt{W_n/n}}$ has the T-distribution with n degrees of freedom. Now $T^2 \triangleq \frac{Y^2}{W_n/n}$ and $Y : N(0,1)$ so $Y^2 \triangleq W_1$ is χ_1^2. Thus $T^2 \triangleq \frac{W_1/1}{W_n/n} = F_{1,n}$.

6.15 For this problem we used the program Excel.

The first matrix of 20 rows and 6 columns are explained below. Note that in order to save space we show only 20 results instead of the 50 the problem called for.

first column: each cell represent the sample mean of 50 Normal random numbers $\{x_i : 1 = 1, ..., 50\}$ called by the function NORMSINV(RAND());

second column: the sample standard deviation computed from

$$\sigma_s = \left(\frac{1}{49} \sum_{i=1}^{50} \left(x_i - \left[\frac{1}{50} \sum_{i=1}^{50} x_j \right] \right)^2 \right)^{1/2} ;$$

third column: for $\delta = 0.95$, the number $t_{(1+\delta)/2}$: $F_T(t_{(1+\delta)/2}) = (1+\delta)/2$;

fourth column: the lower limit: $\mu_s - (\sigma_s \times t_{(1+\delta)/2} / \sqrt{50})$;

fifth column: the upper limit $\mu_s + (\sigma_s \times t_{(1+\delta)/2} / \sqrt{50})$;

sixth column: interval width $2(\sigma_s \times t_{(1+\delta)/2} / \sqrt{50})$

0.152	1.01	2.01	-0.13514	0.439143	0.574286

sample mean of 50 samples	sample sigma	t-number for 95%CI	lower limit	upper limit	interval width
0.167	0.984	2.01	-0.11275	0.446751	0.559502
-0.139	0.974	2.01	-0.41591	0.137908	0.553816
0.186	0.871	2.01	-0.06163	0.433625	0.49525
0.023	0.96	2.01	-0.24993	0.295928	0.545856
0.089	0.95	2.01	-0.18108	0.359085	0.54017
-0.027	0.96	2.01	-0.29993	0.245928	0.545856
0.031	1.032	2.01	-0.2624	0.324397	0.586795
-0.133	1.047	2.01	-0.43066	0.164662	0.595324
0.114	1.147	2.01	-0.21209	0.440092	0.652184
-0.038	1.155	2.01	-0.36637	0.290366	0.656733
0.21	1.094	2.01	-0.10102	0.521024	0.622048
-0.07	1.056	2.01	-0.37022	0.230221	0.600441
0.201	0.902	2.01	-0.05544	0.457438	0.512877
0.17	1.065	2.01	-0.13278	0.472779	0.605559
0.193	1	2.01	-0.0913	0.4773	0.5686
0.27	0.874	2.01	0.021522	0.518478	0.496956
0.016	1.137	2.01	-0.30725	0.339249	0.646498
-0.132	1.15	2.01	-0.45894	0.194945	0.65389
0.22	0.97	2.01	-0.05577	0.495771	0.551542

The second matrix shown below repeats the experiment but assumes that the sigma σ_X of the distribution is known to be unity. Again, to save space , we show only 20 outcomes rather than the 50 the problem called for. We n ote that in only one case, marked in bold, did the interval fail to cover the true mean $\mu_X = 0$.

0.028102	1	1.959963	-0.24912	0.305324	0.554445
-0.01337	1	1.959963	-0.29059	0.263852	0.554445
0.246472	1	1.959963	-0.03075	0.523694	0.554445
-0.26262	1	1.959963	-0.53984	0.014602	0.554445
-0.34849	**1**	**1.959963**	**-0.62571**	**-0.07127**	**0.554445**
0.164584	1	1.959963	-0.11264	0.441806	0.554445
0.137643	1	1.959963	-0.13958	0.414865	0.554445
0.130803	1	1.959963	-0.14642	0.408025	0.554445
-0.21185	1	1.959963	-0.48907	0.065372	0.554445
0.037	1	1.959963	-0.24022	0.314222	0.554445
-0.02536	1	1.959963	-0.30258	0.251862	0.554445
-0.04149	1	1.959963	-0.31871	0.235732	0.554445
0.021513	1	1.959963	-0.25571	0.298735	0.554445
0.05607	1	1.959963	-0.22115	0.333292	0.554445
0.214857	1	1.959963	-0.06237	0.492079	0.554445
-0.20623	1	1.959963	-0.48345	0.070992	0.554445
0.064295	1	1.959963	-0.21293	0.341517	0.554445
0.119289	1	1.959963	-0.15793	0.396511	0.554445
0.00526	1	1.959963	-0.27196	0.282482	0.554445

-0.16481	1	1.959963	-0.44203	0.112412	0.554445
sample mean	**given sigma**	**Normal number for 95% CI**	**lower limit**	**upper limit**	**interval width**

Warning: for sensitive and/or important simulations double check that the random number generator indeed gives data that are Normal in distribution.

6.16 We need to show that $E\left[(n-1)^{-1}\sum_{i=1}^{n}\left(X_i-\hat{\mu}_X\right)^2\right]=\sigma_X^2$. This is easily done by expanding the expression in the square bracket as:

$$E\left[(n-1)^{-1}\left(\sum_{i=1}^{n}X_i^2-2n\hat{\mu}_X^2+n\hat{\mu}_X^2\right)\right]$$
$$=E\left[(n-1)^{-1}\left(\sum_{i=1}^{n}X_i^2-n\hat{\mu}_X^2\right)\right]$$
$$=(n-1)^{-1}\left(E[\sum_{i=1}^{n}X_i^2]-nE[\hat{\mu}_X^2]\right)$$
$$=(n-1)^{-1}\left[\left(n\sigma_X^2+n\mu_X^2\right)-nE[\hat{\mu}_X^2]\right]$$

Now

$$E[\hat{\mu}_X^2]=E[\frac{1}{n^2}\left(\sum_{i=1}^{n}X_i^2+\sum_{i=1}^{n}\sum_{j\neq i}^{n}X_iX_j\right)]$$
$$=\frac{1}{n^2}\left(n\mu_X^2+n\sigma_X^2+n(n-1)\mu_X^2\right)$$

so that

$$(n-1)^{-1}\left[\left(n\sigma_X^2+n\mu_X^2\right)-nE[\hat{\mu}_X^2]\right]$$
$$=\frac{1}{n-1}\left(n\sigma_X^2+n\mu_X^2-\sigma_X^2-\mu_X^2-n\mu_X^2+\mu_X^2\right)$$
$$=\sigma_X^2$$

6.17 We need to show that $Var[\hat{\hat{\sigma}}_X^2]\triangleq Var\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\mu}_X\right)^2\right]\xrightarrow{n\to\infty}0$, where $\hat{\hat{\sigma}}_X^2$ is the biased estimator given by Equation 6.3-4 and the double hat $\hat{\hat{}}$ is used to differentiate the biased estimator from the unbiased estimator in Equation 6.3-3 i.e., $\hat{\sigma}_X^2\triangleq\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\hat{\mu}_X\right)^2$. However, we have already shown in the text (Section 6.4) that $\hat{\sigma}_X^2$ is consistent for σ_X^2 and $\hat{\sigma}_X^2=k_n\hat{\hat{\sigma}}_X^2$ where $k_n\triangleq\frac{n}{n-1}$. Hence

$k_n\hat{\hat{\sigma}}_X^2$ is consistent for σ_X^2 and since $k_n \triangleq \frac{n}{n-1} \xrightarrow{n\to\infty} 1$, we deduce that $\hat{\hat{\sigma}}_X^2$ is consistent for σ_X^2.

6.18

6.19 From the data we compute that $\hat{\mu}_X = 0.014$ and $\hat{\sigma}_X^2 = 3.89$. We would not be surprised if the data were from $N(0,4)$. In addition, we compute $\sum_{i=1}^{15}\left(X_i - \hat{\mu}_X\right)^2 = 54.45$. Using that the confidence interval is given by

$$\left\{\frac{1}{x_{(1+\delta)/2}}\sum_{i=1}^{15}\left(X_i - \hat{\mu}_X\right)^2, \frac{1}{x_{(1-\delta)/2}}\sum_{i=1}^{15}\left(X_i - \hat{\mu}_X\right)^2\right\}$$

where

$$F_{\chi^2}(x_{(1+\delta)/2};14) = \frac{1+\delta}{2} \text{ and } F_{\chi^2}(x_{(1-\delta)/2};14) = \frac{1-\delta}{2}$$

and $\delta = 0.95$, we compute that that a 95% interval is {2.09, 9.67}. The width of the interval is 7.58. To reduce the width, more data samples would be needed.

6.20 Start with $P[-a \le \frac{\hat{p}-p}{\sqrt{pq/n}} \le a] = \delta$ and let $X \triangleq \frac{\hat{p}-p}{\sqrt{pq/n}}$. Then

$P[-a \le X \le a] = 2erf(a) = F_X(a) - F_X(-a) = \delta.$ We recognize that X is $N(0,1)$ under the assumption that the CLT applies. Now a careful examination of the Normal curve and its symmetry reveals that

$F_X(a) + F_X(-a) = 1$ or $F_X(-a) = 1 - F_X(a).$ Thus $2F_X(a) - 1 = \delta$ or $F_X(a) = (1+\delta)/2.$

Solving for a yields $a = x_{(1+\delta)/2}$, the $x_{(1+\delta)/2}$ percentile of X.

6.21 We do this problem using the Excel program. In the first column (A) we list 20 RNs drawn from a $N(0,2)$ population. In B we enter $\max(A_{2n+1}, A_{2n+2})$ for $n = 0,1,...,9.$ In (C) a single estimate of $\hat{\sigma}_X$. The bold number 1.512 in the next-to-last row is the average estimate of $\hat{\sigma}_X$ using Equation 6.4-13. The estimated variance is in the last row at 2.287. The next-to-last entry in column D is the estimated standard deviation using the square root of Equation 6.4-2; it yields a value of 1.505. The associated variance is 2.266. The estimates yielded by the two methods are quite similar.

A	B	C	D
-0.0258	0.5796	2.166806	0.380812
0.5796			1.494506
-0.0132	-0.0132	1.115972	0.396434
-1.592			0.900791
-1.211	-1.211	-1.00695	0.322738
-1.257			0.377119
-0.6477	0.7753	2.513675	2.3E-05
0.7753			2.011291
1.25	1.25	3.355059	3.58307
-2.462			3.309125
-1.284	0.5348	2.0874	0.411009
0.5348			1.386977
-3.521	-2.248	-2.84498	8.28346
-2.248			2.576346
3.237	3.237	6.876923	15.05362
-2.102			2.128973
-0.4927	-0.4927	0.266205	0.02256
-1			0.12752
-1.07	-0.308	0.593577	0.182414
-0.308			0.112158
-0.64289		**1.512369**	**1.505**
		2.287259	**2.266366**

6.22 We write. Then

$$V_1 = (1/3\sigma_X)\times(2X_1 - X_2 - X_3)$$
$$V_2 = (1/3\sigma_X)\times(2X_2 - X_1 - X_3)$$
$$V_1 = (1/3\sigma_X)\times(2X_3 - X_2 - X_1)$$

and

$V_1^2+V_2^2+V_3^2=(1/3\sigma_X^2)\times 2\left(X_1^2+X_2^2+X_3^2-X_1X_2+X_1X_3+X_2X_3\right)$

$=(1/3\sigma_X^2)\left(R_1^2+R_2^2+R_3^2\right)$ where $R_1\triangleq X_1-X_2, R_2\triangleq X_1-X_3, R_3\triangleq X_2-X_3$.

Note that $R_3=R_2-R_1$. Hence

$V_1^2+V_2^2+V_3^2=(2/3\sigma_X^2)\times\left(R_1^2+R_1^2-2R_1R_2\right)$ (so only two RV's are involved!) To get rid of the cross-terms we introduce S_1 and S_2 as

$S_1\triangleq aR_1+bR_2,\ S_2\triangleq cR_1+dR_2$ such that

$S_1^2+S_2^2=V_1^2+V_2^2+V_3^2=(2/3\sigma_X^2)\times\left(R_1^2+R_1^2-2R_1R_2\right)$. To compute the coefficients *a, b, c, d* use the procedure in Example 5.5-2.

6.23 The table below summarizes the experiment. In column A we show 20 RNs drawn from $N(0,2)$. Call these $x_i, i=1,...,20$; the bold number in position A21 is the numerical mean $\overline{x}=\frac{1}{20}\sum_{i=1}^{20}x_i$. The numbers in column B are $(x_i-\overline{x})^2, i=1,...,20$ and the bold number in position B21 is $\sum_{i=1}^{20}(x_i-\overline{x})^2$ .Next we find numbers *a,b* such that $F_{\chi^2}(b;19)=0.975$ and $F_{\chi^2}(a;19)=0.025$. This yields *b*=32.9 and *a*=8.91. The last two bold numbers in column A are the lower and upper limits on the 95 percent interval on σ_X^2 . Thus the 95 percent interval is (1.309, 4.833). Another run is shown in columns C and D. Column C is the analogue of A and column D is the analogue of B. There we found the 95 percent confidence interval to be (1.174, 4.338).

A	B	C	D
-0.0258	0.3808	-0.877	0.737228
0.5796	1.494482	0.03777	0.003153
-0.01327	0.396421	0.356	0.14016
-1.592	0.90081	1.198	1.47958
-1.211	0.322749	0.277	0.087249
-1.257	0.377131	2.016	4.138702
-0.6477	2.31E-05	-1.266	1.556556
0.7753	2.011263	1.43	2.097805
1.25	3.583033	-2.99	8.830525
-2.462	3.309161	1.27	1.659923
-1.284	0.411022	-2.17	4.629469
0.5348	1.386954	-0.208	0.035956
-3.521	8.283517	1.66	2.816959
-2.248	2.576378	-0.833	0.663606
3.237	15.05355	-0.4275	0.167379
-2.102	2.129002	-0.2136	0.038111

-0.4927	0.022557	0.5647	0.339982
-1	0.127528	-2.243	4.948934
-1.07	0.182423	2.05	4.278196
-0.308	0.112151	0.001	0.000376
-0.64289	**43.06095**	**-0.01838**	**38.64985**
1.308843	lower limit	**1.174767**	
4.832879	upper limit	**4.337806**	

6.24 We have two i.i.d. RVs X_1 and X_2 and create two new RVs as

$V \triangleq X + (Y/\sqrt{2}\ )$ and $W \triangleq X - (Y/\sqrt{2}\ )$.

Then $E[VW] = E[\left(X + (Y/\sqrt{2}\ )\right)\left(X - (Y/\sqrt{2}\ )\right)] = 1 - 1/2 = 1/2$.

We compute $E[V^2] = E[W^2] = 3/2$. Hence $\rho_{VW} = \dfrac{E[VW]}{\sqrt{\sigma_V^2}\sqrt{\sigma_W^2}} = \dfrac{1/2}{\sqrt{3/2}\sqrt{3//2}} = 1/3$.

To check this out numerically you might call the NORMSINV(RAND()) in Excel to generate two columns of 20 numbers each. Call these columns X_i and Y_i. Next create two new columns of 20 numbers each. Call these columns V_i and W_i and compute $V_i = X_i + Y_i/1.41$ and $W_i = X_i - Y_i/1.41$. Next create column $(V_i - V_s) \times (W_i - W_s)$ where $V_s \triangleq (1/20)\sum_{i=1}^{20} V_i$ and $W_s \triangleq (1/20)\sum_{i=1}^{20} W_i$. Finally compute $\hat{\rho}_{vw} = \sum_{i=1}^{20}(V_i - V_s)\times(W_i - W_s) / \left(\sqrt{\sum_{i=1}^{20}(V_i - V_s)^2}\sqrt{\sum_{i=1}^{20}(W_i - W_s)^2}\right)$. Do not be alarmed if your result differs from 0.333. The random number generator may not yield properly decorrelated numbers.

6.25 The covariance function is defined as

$c_{11} \triangleq E\left[\left(X - \mu_X\right)(Y - \mu_Y)\right] = E(XY) - \mu_X \mu_Y$.

We wish to show that

$\hat{c}_{11} \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_X(n))\times(Y_i - \hat{\mu}_Y(n))$ where $\{(X_iY_i), i=1,...,n\}$ are paired observations, is an unbiased, consistent estimator for c_{11}.

(a) Unbiasedness: Start with

$$\hat{c}_{11} \triangleq \frac{1}{n-1}\sum_{i=1}^{n}\left(X_iY_i + \hat{\mu}_X(n)\hat{\mu}_Y(n) - X_i\hat{\mu}_Y(n) - Y_i\hat{\mu}_X(n)\right)$$ and observe that

$$E[\hat{c}_{11}] \triangleq \frac{1}{n-1}\sum_{i=1}^{n}\left(E[X_iY_i] + E[\hat{\mu}_X(n)\hat{\mu}_Y(n)] - E[X_i\hat{\mu}_Y(n)] - [Y_i\hat{\mu}_X(n)]\right)$$

because of the linearity of the expectation operator.

Now use:

$$E[\hat{c}_{11}] \triangleq \frac{1}{n-1}\sum_{i=1}^{n}\left(E[X_iY_i] + E[\hat{\mu}_X(n)\hat{\mu}_Y(n)] - E[X_i\hat{\mu}_Y(n)] - [Y_i\hat{\mu}_X(n)]\right)$$

where

$$E[X_iY_i] = c_{11} + \mu_X\mu_Y$$

$$E[X_i\hat{\mu}_Y(n)] = n^{-1}E[\sum_{i=j}^{n}Y_jX_i] = n^{-1}\left(c_{11} + \mu_X\mu_Y\right) + n^{-1}(n-1)\mu_X\mu_Y$$

$$E[Y_i\hat{\mu}_X(n)] = n^{-1}E[\sum_{i=j}^{n}X_jY_i] = n^{-1}\left(c_{11} + \mu_X\mu_Y\right) + n^{-1}(n-1)\mu_X\mu_Y$$

$$E[\hat{\mu}_X(n)\hat{\mu}_Y(n)] = n^{-2}E[\sum_{l=1}^{n}X_l\sum_{k=1}^{n}Y_k] = n^{-2}\sum_{l=1}^{n}\sum_{k=1}^{n}E[X_lY_k]$$

$$= n^{-2}\left(n(c_{11} + \mu_X\mu_Y)\right) + n^{-2}\left(n(n-1)\mu_X\mu_Y\right)$$

Now put all these terms back into the expression for $\hat{c}_{11}$ and find that $E[\hat{c}_{11}] = c_{11}$.

(b) Consistency:

Here we use a “little trick” that mathematicians would not approve of. To prove consistency we have to let *n*, the number of samples get very large. This being the case we replace

$\hat{\mu}_X(n)$ by μ_X and $\hat{\mu}_Y(n)$ by μ_Y since for large *n*, $\hat{\mu}_X(n) \rightarrow \mu_X$ and $\hat{\mu}_Y(n) \rightarrow \mu_Y$.

Also $n-1 \rightarrow n$. Consider then the estimator

$\hat{c}_{11} \triangleq \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)\times(Y_i - \mu_Y)$ instead of $\hat{c}_{11} \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_X(n))\times(Y_i - \hat{\mu}_Y(n))$.

This will save a lot of algebra and yield the same result. Thus

$$E[\hat{c}_{11}^2] = n^{-2}E\left[\sum_{i=1}^{n}\ \sum_{j=1}^{n}(X_i - \mu_X)(Y_i - \mu_Y)\times(X_j - \mu_X)(Y_j - \mu_Y)\right]$$

$$E[\hat{c}_{11}^2] = n^{-2} E\left[\sum_{j=1}^{n} (X_j - \mu_X)^2 (Y_j - \mu_Y)^2\right]$$
$$+ n^{-2} E\left[\sum_{i=1}^{n} \sum_{j \neq i}^{n} (X_i - \mu_X)(Y_i - \mu_Y) \times (X_j - \mu_X)(Y_j - \mu_Y)\right]$$
$$= n^{-2}\left[nc_{22} + n(n-1)c_{11}^2\right] = \frac{c_{22}}{n} + c_{11}^2 - \frac{c_{11}^2}{n}$$

Finally the variance is

$$\sigma_{\hat{c}_{11}}^2 = E[\hat{c}_{11}^2] - c_{11}^2 = \frac{c_{22} - c_{11}^2}{n} \to 0, \text{ as } n \to \infty.$$

Thus the estimator is consistent as long as c_{22} is finite!

6.26 The geometric distribution is often written as

$P_K(k) = pq^k, k = 0,1,...$ To compute the mean we use

$$\mu = E[K] = \sum_{k=0}^{\infty} kpq^k$$
$$= pq \sum_{k=0}^{\infty} kq^{k-1}$$
$$= pq \frac{d}{dq}\left(\sum_{k=0}^{\infty} q^k\right)$$
$$= pq \frac{d}{dq}\left(\frac{1}{1-q}\right)$$
$$= \frac{q}{1-q}$$

where we used that 1=p+q and $\sum_{k=0}^{\infty} q^k = (1-q)^{-1}$ for $0 < q < 1$. With the help of a little algebra, we can easily derive that $q = \frac{\mu}{1+\mu}$ and $p = \frac{1}{1+\mu}$ so that $P_K(k) = \frac{1}{1+\mu}\left(\frac{\mu}{1+\mu}\right)^k$.

To get the variance we first compute

$$E[K^2] = \sum_{k=0}^{\infty} k^2 pq^k$$
$$= \sum_{k=0}^{\infty} \left(k(k-1) + k\right) pq^k$$
$$= pq^2 \sum_{k=0}^{\infty} \left(k(k-1)q^{k-2}\right) + pq \sum_{k=0}^{\infty} kq^{k-1}$$
$$= pq^2 \frac{d^2}{dq^2}\left(\frac{1}{1-q}\right) + pq \frac{d}{dq}\left(\frac{1}{1-q}\right)$$
$$= 2\mu^2 + \mu$$

Hence $Var(K) = E[K^2] - \mu^2 = \mu + \mu^2$.

If n>>1we can use $\hat{\mu} \triangleq \frac{1}{n}\sum_{i=1}^{\infty} K_i$ to get a 95 percent confidence interval on μ. Here the K_i are n i.i.d. observations on K. Using the Central Limit Theorem we approximate $\hat{\mu} : N(\mu, \frac{\mu+\mu^2}{n})$. For a 95 percent interval we seek numbers $-a, a$ such that $P[-a < \frac{\hat{\mu}-\mu}{\sqrt{\left(\mu+\mu^2\right)/n}} < a] = 0.95$ or, equivalently, that $F_{\hat{\mu}}(a) = 0.975$. Thus $a = 1.96$ and we solve $\frac{(\hat{\mu}-\mu)^2}{\left(\mu+\mu^2\right)/n} = 3.84$. This leads to the quadratic equation $\mu^2(1-\frac{3.84}{n}) - \mu(2\hat{\mu}+\frac{3.84}{n}) + \hat{\mu}^2 = 0$. Define $\alpha \triangleq 1-\frac{3.84}{n},\ \beta \triangleq \frac{3.84}{n}+2\hat{\mu},\ \gamma \triangleq \hat{\mu}^2$; then the roots of the quadratic furnish the upper and lower bounds on the 95 confidence interval.

6.27 The event $\{\zeta : -a \le \frac{\hat{p}-p}{\sqrt{pq/n}} \le a\}$ is clearly identical with the event$\{\zeta : -a\sqrt{pq/n} \le \hat{p}-p \le a\sqrt{pq/n}\}$, which for a>0 is clearly identical with the event
$\{\zeta : (\hat{p}-p)^2 \le a^2 pq/n\}$. Hence $P[-a \le \frac{\hat{p}-p}{\sqrt{pq/n}} \le a] = \delta = P[(\hat{p}-p)^2 \le a^2 pq/n]$.

6.28 The likelihood function for n i.i.d Normal RVs is

$$L(\mu_X) = (2\pi\sigma^2)^{-n/2}\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_X)^2\right).$$

The log-likelihood function $L'(\mu_X) \triangleq \ln L(\mu)$ yields

$L'(\mu_X) = (-n/2)\ln(2\pi{\sigma_X}^2) - \frac{1}{2{\sigma_X}^2}\sum_{i=1}^{n}(X_i-\mu_X)^2$, which upon differentiating with respect to μ_X, yields the sample mean (mean estimating function):

$\hat{\mu}_X(n) = (1/n)\sum_{i=1}^{n} X_i$. *Squared error consistency* requires

that $\lim_{n\to\infty} E[(\hat{\mu}_X(n) - \mu_X)^2] = 0$. Expanding and taking expectations before taking limits

yields $E[(\hat{\mu}_X(n) - \mu_X)^2] = \mu_X^2 - (1/n)\mu_X^2 - 2\mu_X^2 + \mu_X^2 + (1/n)\mu_X^2 + (1/n)\sigma_X^2 = (1/n)\sigma_X^2$.

Clearly for a finite variance $\lim_{n\to\infty} E[(\hat{\mu}_X(n) - \mu_X)^2] = \lim_{n\to\infty} E[\sigma_X^2 / n] = 0$.

6.29 The likelihood function for n i.i.d. exponential RVs is

$L(\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i} u(\lambda),\ X_i > 0, i = 1,...,n$. The log-likelihood function yields

$L'(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i = n \ln \lambda - \lambda n \hat{\mu}$ and its derivative with respect to λ, set to zero,

yields $n/\lambda - n\hat{\mu} = 0$ or $\hat{\lambda}_{ML} = 1/\hat{\mu}$. We note that this value is an extreme point that must

maximizes the likelihood function since

$L(0) = L(\infty) = 0$ and $L(\lambda) > 0$ everywhere else.

6.30 Consider n i.i.d binomial RVs with $P_{K_i}(k; m, p) \triangleq \binom{m}{k} p^k (1-p)^{m-k}, i = 1,...,n$,

The likelihood function is $L(p) = \binom{m}{K_1}\binom{m}{K_2}\cdots\binom{m}{K_n} p^{\sum_{i=1}^{n} K_i} (1-p)^{\sum_{i=1}^{n}(m-K_i)}$. Then the log-likelihood function is

$L'(p) = \sum_{i=1}^{n} \binom{m}{K_i} + (\sum_{i=1}^{n} K_i) \ln p + (\sum_{i=1}^{n} (m-K_i)) \ln(1-p)$. Setting the derivative with

respect to p to zero yields

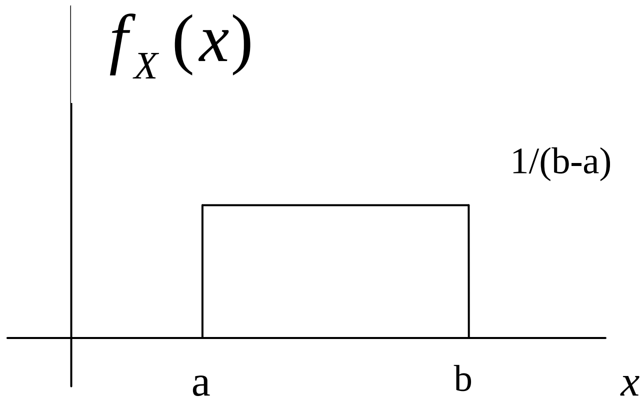
$(\sum_{i=1}^{n} K_i)/p \ln p + (\sum_{i=1}^{n} (m-K_i))/(1-p) = 0$ from which we obtain the ML estimator for

p as

$\hat{p} = \sum_{i=1}^{n} K_i / mn$

6.31 This example illustrates that the ML estimator cannot always be found by

differentiation. The pdf of $f_X(x)$ has the shape



The likelihood function for n i.i.d. RVs $X_1, X_2, ..., X_n$ s is $L(a,b) = (b-a)^{-n}$. To maximize $L(a,b)$ we need to make b as small as possible and a as large as possible subject to $b>a$ and the data constraints. Let $X_m \triangleq \min\{X_1, ...X_n\}$. Let $X_M \triangleq \min\{X_1, ...X_n\}$. Clearly $a \le X_m$ and $b \ge X_M$ otherwise realizations would not come from this pdf. The smallest allowable value of b is $b = X_M$; the largest allowable value of a is $a = X_m$. So the MLE of b is $\hat{b}_{ML} = X_M$ and the MLE of a is $\hat{a}_{ML} = X_m$.

6.32 (i) We consider the vector linear model as $\mathbf{Y} = \mathbf{Ia} + b\mathbf{x} + \mathbf{V}$, where $\mathbf{V}$ is a vector of n $N(0,\sigma^2)$ RVs $V_i, i = 1,...,n$. In terms of scalar equations, we write the linear model as:

$Y_i = a + bx_i + V_i$, $i = 1,...,n$. We note that

$E[Y_i] = a + bx_i + E[V_i] = a + bx_i$, since $E[V_i] = 0$. Also

$\text{Var}[Y_i] = \text{Var}[a + bx_i + V_i] = \text{Var}[V_i] = \sigma^2$. Finally Y_i, exept for a shift, is linearly related to V_i so that $Y_i : N(a + bx_i, \sigma^2)$.

(ii) The likelihood function is

$L(a,b) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \left[\frac{Y_i - (a + bx_i)}{\sigma} \right]^2 \right)$. So clearly, it is maximized when

$l \triangleq \sum_{i=1}^{n} (Y_i - (a + bx)^2$ is minimized. Note that σ^2 is merely a constant here that plays no role in selecting the optimum values of a,b.

(iii) First we compute

$$\frac{dl}{da} = \frac{d}{da}\left(\sum_{i=1}^{n}\left(Y_i - (a+bx_i)\right)^2\right)$$
$$= \sum_{i=1}^{n}\frac{d}{da}\left(Y_i - (a+bx_i)\right)^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}Y_i - \frac{na}{n} - b\frac{1}{n}\sum_{i=1}^{n}x_i = 0$$

With $\hat{\mu}_Y \triangleq \frac{1}{n}\sum_{i=1}^{n}Y_i$ and $\overline{x} \triangleq \frac{1}{n}\sum_{i=1}^{n}x_i$ we get $\hat{a}_{ML} = \hat{\mu}_Y - \hat{b}_{ML}\overline{x}$ .Next we have to compute $\hat{b}_{ML}$ as

$$\frac{dl}{db} = \frac{d}{db}\left(\sum_{i=1}^{n}\left(Y_i - (a_{ML}+bx_i)\right)^2\right)$$
$$= \sum_{i=1}^{n}\frac{d}{db}\left(Y_i - (a_{ML}+bx_i)\right)^2$$
$$= \sum_{i=1}^{n}\left(Y_ix_i - (\hat{\mu}_Y - b\overline{x})x_i - bx_i^2\right)$$
$$= 0$$

Solving for $\hat{b}_{ML}$, we find that

$\hat{b}_{ML} = \dfrac{\sum_{i=1}^{n}Y_i(x_i-\overline{x})}{\sum_{i=1}^{n}(x_i-\overline{x})^2}$, where we used the identity $\sum_{i=1}^{n}(x_i^2 - \overline{x}^2) = \sum_{i=1}^{n}(x_i-\overline{x})^2$

6.33 We are given that $\mu = 220$ and σ=20 and the weight RV is Normal as $N(220,400)$. To compute the 95th percentile we solve

$$0.95 = \int_{-\infty}^{x_{0.95}}(800\pi)^{-1/2}\exp[-0.5\left(\frac{x-220}{20}\right)^2]dx$$

which yields $x_{0.95} = z_{0.95}20 + 220 = 1.645\times 20 + 220 = 253$ lbs . Here $z_{0.95}$ is the 95th percentile of the standard Normal RV.

6.34 We write the geometric distribution as in Table 2.5-1

$F_K(k) = p\dfrac{1-q^{k+1}}{1-q} = 1-q^{k+1}$ since $p = 1-q$ .Assume that q>0.5; then we solve 0.5

$1-q^{k_{0.5}+1} \approx 0.5$ or $q^{k_{0.5}+1} \approx 0.5$ or $k_{0.5} \approx \left[\frac{\ln 0.5}{\ln q} - 1\right]$,where [ ] is the largest integer not exceeding the contents of the brackets.

6.35 Let W_n denote the χ_n^2 RV with n degrees of freedom. Then $W_n \triangleq \sum_{i=1}^{n} X_i^2$, where $X_i : N(0,1)$. Clearly $E[W_n] = E\left(\sum_{i=1}^{n} X_i^2\right) = \sum_{i=1}^{n} E[X_i^2] = n$,

since $E[X_i^2] = Var[X_i] = 1$. Hence the mean of

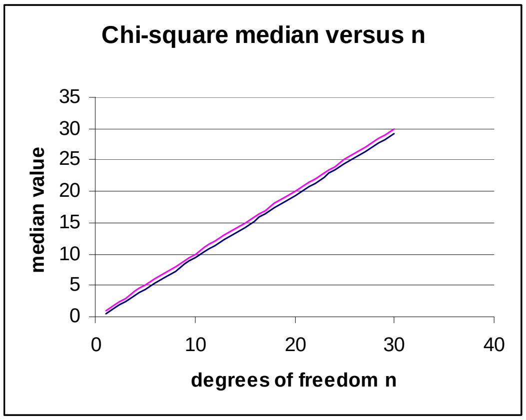
the χ_n^2 RV with n degrees of freedom is n . To compute the median is a little more difficult. First we compute $Var[W_n]$ as

$$
\begin{aligned}
Var[W_n] &= E[(W_n - E[W_n])^2] = E[W_n^2] - n^2 \\
&= \sum_{i=1}^{n} E[X_i^4] + \sum_{i=1}^{n} [X_i^2] \sum_{j\neq i}^{n} E[X_j^2] - n^2 \\
&= 3n + (n-1)n - n^2 = 2n.
\end{aligned}
$$

There are several ways to compute $\sum_{i=1}^{n} E[X_i^4] = 3n$. The formula

$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\pi / a}$ is useful with n=2 and a=1/2.

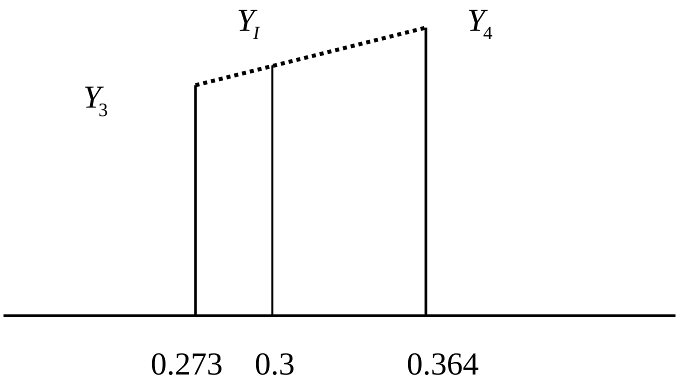
When n is small, the median is easily obtained by table look-up. When n>10 we use the Normal approximation to the χ_n^2 . Then $W_n \sim N(n, 2n)$ and the median is obtained by solving $0.5 = \int_{-\infty}^{x_{0.5}} (4\pi n)^{-1/2} \exp(-0.5\left(\frac{x-n}{\sqrt{2n}}\right)^2)dx$. After a conversion to the standard normal we obtain the median as $w_{0.5} \approx n + z_{0.5}\sqrt{2n}$ where $z_{0.5} = 0$, i.e., the 50 percentile point for the standard Normal. A more general formula for the 100u percentile point is $w_u \approx n + z_u\sqrt{2n}$. The approximation (upper curve) is compared with the true median (lower curve) in the figure below.



6.36 Refer to Example 6.8-4. There we showed that

Y_1 estimates $x_{0.0909}$ i.e. the 9th percentile
Y_2 estimates $x_{0.182}$ i.e. the 18th percentile
Y_3 estimates $x_{0.273}$ i.e. the 27th percentile .
Y_4 estimates $x_{0.364}$ i.e. the 36th percentile

Recall that $x_p : F_{SN}(x_p) = p$ is the $100 \times p$ percentile .Now we wish to use the ordered observations $\{Y_i, i = 1,...,n\}$ to estimate $x_{0.3}$. See the diagram where Y_I is the interpolated value.



Clearly

$$\frac{Y_4 - Y_3}{0.364 - 0.273} = \frac{Y_4 - Y_I}{0.364 - 0.3} \text{ or}$$
$$Y_I = Y_4 + \frac{(0.364 - 0.3) \times (Y_3 - Y_4)}{0.091}$$

6.37 We use the formula $P[Y_1 < x_{0.5} < Y_n] = (1/2)^n \sum_{i=1}^{n-1} \binom{n}{i}$ and find the smallest value of n such that $P[Y_1 < x_{0.5} < Y_n] \geq 0.99$. The table below shows $P[Y_1 < x_{0.5} < Y_n]$ for various values of n:

n	2	3	4	5	6	7	8	9
$P[Y_1 < x_{0.5} < Y_n]$	0.5	0.75	0.88	0.94	0.97	0.98	0.99	0.996

Hence a sample of size will cover the median with probability of 0.99.

6.38 As shown in the text, if $X_1, X_2, \cdots, X_{n-1}, X_n$ are n i.i.d. observations on X with pdf $f(x)$ then their ordered samples $Y_1 < Y_2 < \cdots < Y_n$ have joint pdf

$$f_{\mathbf{Y}}(\mathbf{y}) = n!\, f(y_1) \cdots f(y_n), \text{ where } y_1 < y_2 < \cdots < y_n.$$

Hence $f_{Y_1 Y_n}(y_1, y_n) = n!\, f(y_1)\, f(y_n) \int_{y_1}^{y_{n-1}} \cdots \int_{y_1}^{y_3} f(y_2) \cdots f(y_{n-1}) dy_2 \cdots dy_{n-1}$ i.e. we integrate out with respect to all the y_i except for y_1 and y_n. Now consider the innermost integral (leaving out for simplicity the factor $n!\, f(y_1)\, f(y_n)$) that is the integration with respect y_2: $\int_{y_1}^{y_3} f(y_2) dy_2 = \int_{y_1}^{y_3} dF = F(y_3) - F(y_1)$. Next consider the integration with respect to y_3:

$\int_{y_1}^{y_4} f(y_3)\big(F(y_3) - F(y_1)\big) dy_3 = \int_{y_1}^{y_4} \big(F(y_3) - F(y_1)\big) dF(y_3)$. If we let $\alpha_i \triangleq F(y_i) - F(y_1)$ then $d\alpha_3 = dF(y_3)$ and

$$\int_{y_1}^{y_4}\left(F(y_3)-F(y_1)\right)dF(y_3)=\int_{\alpha_1}^{\alpha_4}\alpha_3 d\alpha_3=\frac{\alpha_3^2}{2}\bigg|_{\alpha_1}^{\alpha_4}=\frac{\alpha_4^2}{2}-\frac{\alpha_1^2}{2}=\frac{\alpha_4^2}{2}=\frac{\left(F(y_4)-F(y_1)\right)^2}{2}.$$

Consider next the integration with respect y_4: this yields

$$\int_{y_1}^{y_5}\frac{\left(F(y_4)-F(y_1)\right)^2}{2}f(y_4)dy_4=\int_{y_1}^{y_5}\frac{\left(F(y_4)-F(y_1)\right)^2}{2}dF(y_4)$$

$$\int_{\alpha_1}^{\alpha_5}\frac{\alpha_4^2}{2}d\alpha_4=\frac{\alpha_4^3}{3\cdot 2}\bigg|_{\alpha_1}^{\alpha_5}=\frac{\alpha_5^3}{3\cdot 2}=\frac{\left(F(y_5)-F(y_1)\right)^3}{3\cdot 2}.$$

It should be clear that after the k^{th} integration we are left with $\frac{\left(F(y_{k+2})-F(y_1)\right)^k}{k!}$. After $n-2$ integrations we have $\frac{\left(F(y_n)-F(y_1)\right)^{n-2}}{(n-2)!}$. Thus the final result (let's not forget the factor $n!\,f(y_1)f(y_n)$) is

$f_{Y_1Y_n}(y_1,y_n)=n(n-1)\left(F(y_n)-F(y_1)\right)^{n-2}f(y_1)f(y_n)$.

6.39 From the previous problem we determined that the joint pdf of

$Y_1\triangleq\min(X_1,X_2,...X_n)$ and $Y_n\triangleq\max(X_1,X_2,...X_n)$ is given by

$f_{Y_1Y_n}(y_1,y_n)=n(n-1)\left(F(y_n)-F(y_1)\right)^{n-2}f(y_n)f(y_1)$,

where $f(y)$ and $F(y)$ are the pdf and CDF of X respectively and the

$X_i, i=1,...,n$ are i.i.d observations on X . To determine the pdf of the *range* $R\triangleq Y_n-Y_1$ we introduce an auxiliary variable $W\triangleq Y_1$. Then the only solution to the functional equations

$r\triangleq y_n-y_1,\ w\triangleq y_1$ is $y_n=r+w,\ y_1=w$.

The Jacobian magnitude of the transformation is

$$\text{mag}\begin{vmatrix}\partial r/\partial y_1 & \partial r/\partial y_n\\ \partial w/\partial y_1 & \partial w/\partial y_n\end{vmatrix}=1$$

hence

$f_{RW}(r,w)=n(n-1)\left(F(r+w)-F(w)\right)^{n-2}f(r+w)f(w)$, where

$0<r<\infty$ and $-\infty<w<\infty$. To get $f_R(r)$ from $f_{RW}(r,w)$ we integrate out with respect to *w*. This yields

$f(r+w)f(w)=1$.

For example, let $f(x)=1,\ 0<x<1,$ and 0 else. Then in the region $0<w<1-r$,

$f(r+w)f(w)=1$; elsewhere the product is zero. In this region

$F(r+w)-F(w)=(r+w)-w=r$. Hence $f_{RW}(r,w)=n(n-1)r^{n-2}f(r+w)f(w)$ so that

integrating out with respect to w yields

$$f_R(r)=\int_0^{1-r} n(n-1)r^{n-2}dw = n(n-1)(1-r)r^{n-2},\ 0<r<1.$$