Solutions to Chapter 3

1. We start with the definition of the CDF of Y,

$$\begin{aligned} F_Y(y) &\triangleq P[Y \leq y] \\ &= P[aX + b \leq y] \\ &= P[aX \leq y - b] \\ &= P\left[X \geq \frac{y-b}{a}\right], \qquad \text{for } a < 0. \end{aligned}$$

Now for any value x, we have $P[X \geq x] + P[X < x] = 1$ and $P[X < x] = F_X(x) - P[X = x]$. Hence for $a < 0$,

$$\begin{aligned} F_Y(y) &= P\left[X \geq \frac{y-b}{a}\right] \\ &= 1 - P\left[X < \frac{y-b}{a}\right] \\ &= 1 - \left(F_X(\frac{y-b}{a}) - P\left[X = \frac{y-b}{a}\right]\right) \\ &= 1 - F_X(\frac{y-b}{a}) + P\left[X = \frac{y-b}{a}\right]. \end{aligned}$$

2. We are given that X is a Gaussian random variable distributed as $N(0,1)$, i.e. $\mu = 0$ and $\sigma^2 = 1$. A second random variable is the result of the transformation $y = g(x)$, (Fig. 1)

$$g(x) = \begin{cases} x, & x \geq 0, \\ x^2, & x < 0. \end{cases}$$

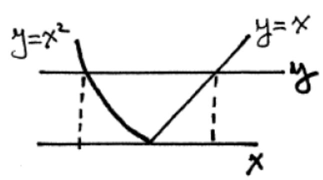


Figure 1:

We are asked to find the pdf of $Y = g(X)$. We proceed using the indirect method of first finding the CDF. From the figure we can see that, for $y > 0$, the event

$$\{Y \leq y\} = \{-\sqrt{y} \leq X \leq y\}.$$

For $y < 0$, the event $\{Y \leq y\} = \phi$, the null event, with probability zero. So the sought after CDF becomes

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[-\sqrt{y} \leq X \leq y] \\ &= F_X(y) - F_X(-\sqrt{y}). \end{aligned}$$

Taking the derivative with respect to y, we get

$$\begin{aligned} f_Y(y) &\triangleq \frac{dF_Y(y)}{dy} \\ &= \frac{dF_X(y)}{dy} - \frac{dF_X(-\sqrt{y})}{dy} \\ &= f_X(y) - f_X(-\sqrt{y})\frac{-d\sqrt{y}}{dy} \\ &= f_X(y) + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}. \end{aligned}$$

Now plugging in the standard Normal (Gaussian) distribution for X, we get

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} + \frac{1}{2\sqrt{2\pi y}}e^{-\frac{1}{2}y}, \qquad y > 0.$$

For $y < 0$, the density $f_Y(y) = 0$. Since X is a continuous random variable, there is no probability mass at $y = 0$, so we can set $f_Y(y) = 0$ there too. The overall pdf for Y then becomes

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} + \frac{1}{2\sqrt{2\pi y}}e^{-\frac{1}{2}y}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

3. (*function of RV*) For a given value of $Y = y$, with $y > 0$, there are two solutions for x, i.e. $x_1(y) = \frac{1}{2}y$ and $x_2(y) = -y$ as shown in the Fig. 2.

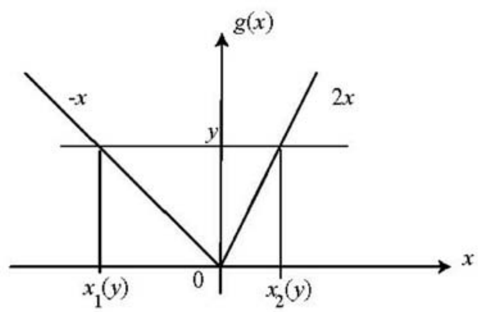


Figure 2:

Thus the pdf of Y is given as

$$\begin{aligned} f_Y(y) &= \left|\frac{dx_1(y)}{dy}\right| f_X(x_1(y)) + \left|\frac{dx_2(y)}{dy}\right| f_X(x_2(y)) \\ &= \left|\frac{1}{2}\right| f_X(y/2) + |-1|\, f_X(-y) \\ &= \frac{1}{2}f_X(y/2) + f_X(-y). \end{aligned}$$

When $Y < 0$, there are no solutions, so the general form of the pdf of Y is

$$f_Y(y) = \left(\frac{1}{2}f_X(y/2) + f_X(-y)\right)u(y),$$

where $u(y)$ is the unit step function. For this problem, we have $X \sim N(0,25)$,thus

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}5} e^{-\frac{1}{2}\frac{x^2}{25}} \\ &= \frac{1}{\sqrt{2\pi}5} e^{-\frac{x^2}{50}}, \end{aligned}$$

so that

$$\begin{aligned} f_Y(y) &= \left(\frac{1}{2} \frac{1}{\sqrt{2\pi}5} e^{-\frac{(y/2)^2}{50}} + \frac{1}{\sqrt{2\pi}5} e^{-\frac{(-y)^2}{50}} \right) u(y) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{10} e^{-\frac{y^2}{200}} + \frac{1}{5} e^{-\frac{y^2}{50}} \right) u(y). \end{aligned}$$

4. We are given that X is a Gaussian random variable distributed as $N(0,2)$, i.e. $\mu = 0$ and $\sigma^2 = 2$. A second random variable is the result of the transformation $y = g(x)$,

$$g(x) = \begin{cases} x, & x \geq 0, \\ 2x^2, & x < 0. \end{cases}$$

We are asked to find the probability density function (pdf) of $Y = g(X)$. We find the CDF first and then differentiate to find the pdf. From the equation for g we can see that, for $y > 0$, the event

$$\{Y \leq y\} = \{-\sqrt{\frac{y}{2}} \leq X \leq y\}.$$

For $y < 0$, the event $\{Y \leq y\} = \phi$, the null event, with probability zero. So the sought after CDF becomes

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[-\sqrt{\frac{y}{2}} \leq X \leq y] \\ &= F_X(y) - F_X(-\sqrt{\frac{y}{2}}). \end{aligned}$$

Taking the derivative with respect to y, we get

$$\begin{aligned} f_Y(y) &\triangleq \frac{dF_Y(y)}{dy} \\ &= \frac{dF_X(y)}{dy} - \frac{dF_X(-\sqrt{\frac{y}{2}})}{dy} \\ &= f_X(y) - f_X(-\sqrt{\frac{y}{2}}) \frac{-d\sqrt{\frac{y}{2}}}{dy} \\ &= f_X(y) + f_X(-\sqrt{\frac{y}{2}}) \frac{1}{4\sqrt{\frac{y}{2}}}. \end{aligned}$$

Now plugging in the standard Normal (Gaussian) distribution for X, we get

$$f_Y(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y^2} + \frac{1}{8\sqrt{\pi \frac{y}{2}}} e^{-\frac{1}{8}y}, \qquad y > 0.$$

For $y < 0$, the density $f_Y(y) = 0$. Since X is a continuous random variable, there is no probability mass at $y = 0$, so we can set $f_Y(y) = 0$ there too. The overall pdf for Y then becomes

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}y^2} + \frac{1}{8\sqrt{\pi\frac{y}{2}}} e^{-\frac{1}{8}y}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

5. In both parts of this problem the random variable X has its probability density given as exponential with parameter $\alpha(> 0)$,i.e. $f_X(x) = \alpha e^{-\alpha x} u(x)$.

   (a) Here the function is given as $y = g(x) = x^3$, as shown in Fig. 3.

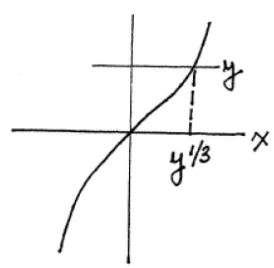


Figure 3:

Using the *indirect method*, we see that the distribution function of Y satisfies $F_Y(y) = F_X(y^{1/3})$ since the event $\{Y \leq y\} = \{X \leq y^{1/3}\}$. Upon differentiation we get

$$\begin{aligned} f_Y(y) &= dF_X(y^{1/3})/dy \\ &= f_X(y^{1/3})\frac{dy^{1/3}}{dy} \\ &= \alpha e^{-\alpha y^{1/3}} u(y^{1/3})\frac{1}{3}y^{-2/3} \\ &= \frac{\alpha}{3y^{2/3}} e^{-\alpha y^{1/3}} u(y). \end{aligned}$$

Alternatively, and since the function g is monotonic, we can easily use the *direct method* to get the pdf of $Y = g(X) = X^3$. We get

$$\begin{aligned} f_Y(y) &= f_X(y^{1/3})|\frac{dx}{dy}| \\ &= f_X(y^{1/3})/\left|\frac{dy}{dx}\right|. \end{aligned}$$

Now $\frac{dy}{dx} = 3x^2 = 3y^{2/3}$, so

$$\begin{aligned} f_Y(y) &= \alpha e^{-\alpha y^{1/3}} u(y^{1/3})/3y^{2/3} \\ &= \frac{\alpha}{3y^{2/3}} e^{-\alpha y^{1/3}} u(y), \quad \text{same as above.} \end{aligned}$$

(b) For this part, only the function g has changed, this time $g(x) = 2x + 3$. We choose to use the indirect method, and find

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P\left[X \le \frac{y-3}{2}\right] \\ &= F_X\left(\frac{y-3}{2}\right). \end{aligned}$$

Taking the derivative with respect to the free variable y, we get

$$f_Y(y) = \frac{1}{2} f_X\left(\frac{y-3}{2}\right),$$

which, for the exponential distribution with parameter α, becomes

$$\begin{aligned} f_Y(y) &= \frac{1}{2}\alpha e^{-\alpha\frac{y-3}{2}} u(\frac{y-3}{2}) \\ &= \frac{\alpha}{2} e^{-\frac{\alpha}{2}(y-3)} u(y-3). \end{aligned}$$

6. By definition $F_Y(y) = P[Y \le y] = P[g(X) \le y]$, so $F_Y(y) = 1$ for all $y \ge +1$ for this transformation g. When $-1 < y < +1$, we can write

$$\begin{aligned} F_Y(y) &= P[g(X) \le y] \\ &= P[X \le y] \\ &= F_X(y), \end{aligned}$$

then calculating the distribution function for the Laplacian density, we get by running integration

$$F_X(x) = \begin{cases} 1 - \frac{1}{2}e^{-x}, & x \ge 0, \\ \frac{1}{2}e^{+x}, & x < 0. \end{cases}$$

Thus, we have

$$\begin{aligned} F_Y(y) &= F_X(y), \qquad -1 < y < +1, \\ &= \begin{cases} 1 - \frac{1}{2}e^{-y}, & 1 > y \ge 0, \\ \frac{1}{2}e^{+y}, & -1 < y < 0. \end{cases} \end{aligned}$$

Clearly, for $y < -1$, we must have $F_Y(y) = 0$. Combining these results, we obtain

$$F_Y(y) = \begin{cases} 1, & y \ge 1, \\ 1 - \frac{1}{2}e^{-y}, & 1 > y \ge 0, \\ \frac{1}{2}e^{+y}, & 0 > y > -1, \\ 0, & -1 \ge y. \end{cases}$$

7. We present two methods to solve this transformation problem:

Method (1): For $y > 0$

$$P[Y \le y] = P[e^X \le y] = P[X \le \ln y] = F_X(\ln y).$$

Hence

$$f_Y(y) = \frac{dF_X(\ln y)}{dy} = \frac{dF_X(\ln y)}{d(lny)}\frac{d(\ln y)}{dy} = \frac{1}{y}f_X(\ln y).$$

For $y \le 0$

$$P[Y \le y] = P[e^X \le y] = P[\phi] = 0.$$

Hence

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\ \sigma y}\exp[-\frac{1}{2}(\frac{\ln y - \mu}{\sigma})^2]u(y).$$

Method (2): A plot of $y = g(x)$ is is $x = \ln y$, for $y > 0$ as given in Fig. 4.

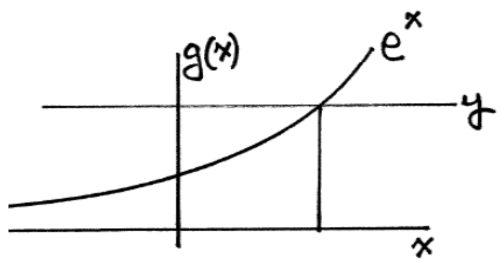


Figure 4:

$$f_Y(y) = \sum_{i=1}^{r}\frac{f_X(x_i)}{|g'(x_i)|} = f_X(\ln y)/e^x|_{x=\ln y}.$$

For $y \le 0$, there is no real solution to $y - g(x) = 0$; hence for $y \le 0$, the pdf of Y equals zero there. Combining solutions for both regions of y, we get

$$f_Y(y) = \frac{f_X(\ln y)}{y}u(y).$$

8. (a) Note that the peak is approximately at $\ln y = \mu$ or $y = \exp\mu$ when $\mu > \sigma$. To investigate the behavior near zero, to aid in a hand plot, we note

$$\begin{aligned}
\frac{1}{y}e^{-(\ln y)^2} &= \frac{1}{y}e^{-(\ln y)(\ln y)} \\
&= \frac{1}{y}\left(e^{-\ln y}\right)^{\ln y} \\
&= \frac{1}{y}\left(\frac{1}{y}\right)^{\ln y} \\
&= y^{-\ln y - 1}.
\end{aligned}$$

Now near zero, the exponent $-\ln y - 1$ takes on large positive values, thus $y^{-\ln y - 1}$ converges to zero as y approaches zero from the right. So the hand plots should just look to first order like a Gaussian density, with a log scale for the horizontal axis.

Precise plotting with $\mu = 2$ and $\sigma = 1$, results in the Fig. 5. The plots are in terms of conventional y versus x axis plots.

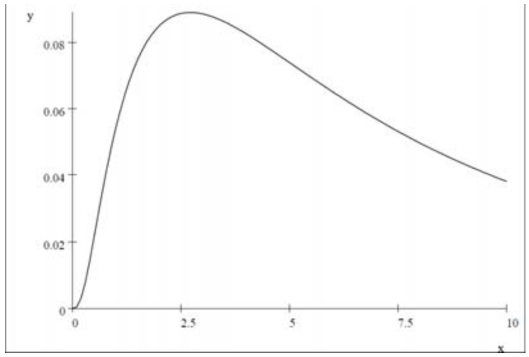


Figure 5:

$\frac{1}{\sqrt{2\pi}x} \exp -\frac{(\ln x-2)^2}{2}$
A second plot is with $\mu = 0$ and $\sigma = 4$ is given in Fig. 6.
$\frac{1}{\sqrt{2\pi}4x} \exp -\frac{(\ln x)^2}{32}$

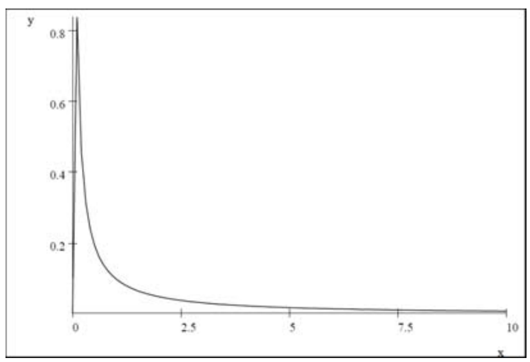


Figure 6:

(b) We first integrate the density $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma y} \exp -\frac{(\ln y-\mu)^2}{2\sigma^2}\, u(y)$. We have

$$F_Y(y) = 0 \text{ for } y < 0.$$

For $y \geq 0$, we write

$$F_Y(y) = \int_0^y \frac{1}{\sqrt{2\pi}\sigma s} \exp -\frac{(\ln s - \mu)^2}{2\sigma^2}\, ds.$$

We then make the substitution $t \triangleq \ln s$ to get (with $ds = e^t dt$)

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}\sigma e^t} \exp -\frac{(t-\mu)^2}{2\sigma^2}\, e^t dt \\ &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}\sigma} \exp -\frac{(t-\mu)^2}{2\sigma^2}\, dt \\ &= \frac{1}{2} + \operatorname{erf}\left(\frac{\ln y - \mu}{\sigma}\right). \end{aligned}$$

The overall solution for the distribution function (CDF) is then

$$F_Y(y) = \left[\frac{1}{2} + \operatorname{erf}\left(\frac{\ln y - \mu}{\sigma}\right)\right] u(y).$$

The alternative solution is easier and starts with the distribution function

$$\begin{aligned}
F_Y(y) &\triangleq P[Y \le y] \\
&= P[e^X \le y] \\
&= P[X \le \ln y], \qquad \text{because } \ln(\cdot) \text{ is a monotonic increasing function,} \\
&= F_X(\ln y) \qquad \text{on } y > 0, \\
&= \frac{1}{2} + \operatorname{erf}\left(\frac{\ln y - \mu}{\sigma}\right), \qquad \text{on } y > 0, \\
&= \left[\frac{1}{2} + \operatorname{erf}\left(\frac{\ln y - \mu}{\sigma}\right)\right] u(y), \quad \text{since } F_Y(y) = 0 \ \text{ for } \quad y \le 0.
\end{aligned}$$

9. *Method* (1):

$$P[Y \le y] = P[\ln X \le y] = P[X \le e^y] = F_X(e^y)$$

$$f_Y(y) = \frac{dF}{d(e^y)} \frac{d(e^y)}{dy} = f_X(e^y)e^y = \frac{1}{3}e^{-\frac{1}{3}e^y}u(e^y)e^y$$

But since $e^y \ge 0$ all real y, $u(e^y) = 1$ everywhere. Hence

$$f_Y(y) = \frac{1}{3}e^{-\frac{1}{3}(\exp(y) - 3y)}$$

Method (2): We plot $y = g(x)$ first, in Fig. 7. The only solution to $y - g(x) = 0$ is $x = \ln y$

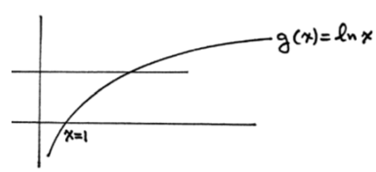


Figure 7:

for $-\infty < y < \infty$. Hence

$$\begin{aligned}
f_Y(y) &= \frac{f_X(e^y)}{|dg/dx|_{x=e^y}} \\
&= f_X(e^y)e^y. \qquad (1)
\end{aligned}$$

10. Here is the plot of $y = g(x)$ in Fig. 8: $y - g(x) = 0$ is $y = \sqrt{x}$, or $x = y^2$. For $y < 0$, no real solutions exist to $y - g(x) = 0$. Hence

$$f_Y(y) = \frac{f_X(x)}{g'(x)}|_{x=y^2}.$$

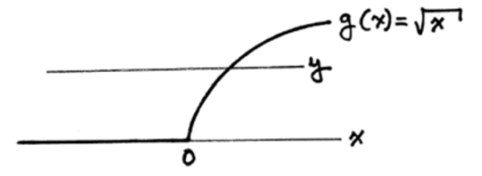


Figure 8:

Now

$$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}|_{x=y^2} = \frac{1}{2y},$$

thus

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}}ye^{-\frac{1}{2}y^4}, & y > 0, \\ 0, & y < 0. \end{cases}$$

But at $y = 0$ there is a probability mass, jump in the distribution function, and impulse in the density function. We have

$$\begin{aligned} P[Y = 0] &= P[X \leq 0] \\ &= F_Y(0) = \frac{1}{2}, \end{aligned}$$

thus

$$F_Y(0) = \frac{1}{2}\int_{-\epsilon}^{\epsilon} \delta(y)dy.$$

So, combining results, we have the following answer valid for all y,

$$f_Y(y) = \frac{1}{2}\delta(y) + \sqrt{\frac{2}{\pi}}ye^{-\frac{1}{2}y^4}u(y).$$

11. (a) From the function g given in Fig. P3.11 in the text, we see that the event $\{Y \leq y\} = \{-\infty < X < +\infty\} = \Omega$[1] for $y \geq 1$. Hence $F_Y(y) = 1$ there. Next consider $0 < y < 1$, again from the figure, we see that in this region of the function g with slope 1, $\{Y \leq y\} = \{X \leq y\} \cup \{X > 2\}$, a disjoint union. So in this region of y values, it must be that $F_Y(y) = F_X(y) + (1 - F_X(2))$. Right at the point $y = 0$, we have $\{Y \leq y\} = \{X \leq 0\} \cup \{X > 2\}$. Now consider the remaining region $y < 0$, there the event $\{Y \leq y\} = \phi$ the null event, since there are no x values that map to $y < 0$ for the given function g shown in Fig. 9. So $F_Y(y) = 0$ there. Since $X : N(0, 1)$, we can write

$$\begin{aligned} F_Y(y) &= \begin{cases} 0, & y < 0, \\ \frac{1}{2} + \operatorname{erf}(y) + \left[1 - \left(\frac{1}{2} + \operatorname{erf}(2)\right)\right], & 0 \leq y < 1, \\ 1, & 1 \leq y, \end{cases} \\ &= \begin{cases} 0, & y < 0, \\ 1 + \operatorname{erf}(y) - \operatorname{erf}(2), & 0 \leq y < 1, \\ 1, & 1 \leq y. \end{cases} \end{aligned}$$

[1] Actually, we only said that random variables take on finite values with probability one. So there could be some events of probability zero that actually attain $X = \pm\infty$. However, since they are probability zero, the distribution function is not affected.

Taking derivatives with respect to the free variable y, we get *within* each interval

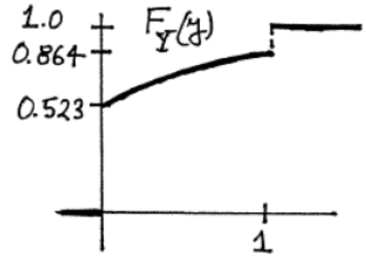


Figure 9:

$$f_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, & 0 < y < 1, \\ 0, & 1 < y. \end{cases}$$

This is not the complete story though as there are jumps in the CDF $F_Y(y)$ at both $y = 0$ and $y = 1$. Thus we must add to this pdf two impulses located at these two jump points. From the CDF of Y given above, we see that

$$\begin{aligned} F_Y(1) - F_Y(1^-) &= 1 - (1 + \text{erf}(1) - \text{erf}(2)) \\ &= \text{erf}(2) - \text{erf}(1) \\ &\doteq 0.137 \end{aligned}$$

so this is the area of the impulse needed at $y = 1$. It thus becomes $0.137\delta(y-1)$. The jump in the CDF $F_Y(y)$ at $y = 0$ is found as

$$\begin{aligned} F_Y(0) - F_Y(0^-) &= (1 + \text{erf}(0) - \text{erf}(2)) - 0 \\ &= 1 - \text{erf}(2) \\ &\doteq 0.523, \end{aligned}$$

so the needed impulse in the density at $y = 0$ becomes $0.523\delta(y)$. Combining these results, we get the pdf $f_Y(y)$ over the full domain of y as

$$f_Y(y) = 0.523\delta(y) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} (u(y) - u(y-1)) + 0.137\delta(y-1).$$

(b) Equation 3.2-23 in the book will only give us the portion of the solution for $0 < y < 1$, where in fact there is one root $x_1(y) = y$ and $n = 1$. Using this method, we thus get $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$ and miss the impulses at $y = 0$ and $y = 1$. The problem here is that our function g has regions of zero slope. Over such regions, there are uncountably infinitely many x values corresponding to the one y value. Thus Equation 3.2-23 does not hold there.

12. The given pdf of X is uniform over $[0, 2]$. Thus $f_X(x) = \frac{1}{2}(u(x) - u(x-2))$. The function g (Fig. 10) is given as

$$g(x) = \begin{cases} 0, & x < 0, \\ 2x, & 0 \le x < \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \le x < 1, \\ 0, & 1 \le x, \end{cases}$$

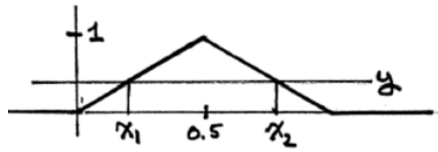


Figure 10:

For $y \geq 1$, for the given function g, the event $\{Y \leq y\} = \{-\infty < X < +\infty\}$, so for all $y \geq 1$, the CDF $F_Y(y) = 1$. When $0 < y \leq 1$, there are two solutions to the equation $y = g(x)$, and they can be given as $x_1(y) = y/2$ and $x_2(y) = 1 - y/2$. So, using Equation 3.2-23, we get

$$f_Y(y) = \frac{1}{2} f_X(y/2) + \frac{1}{2} f_X(1 - y/2), \qquad 0 < y \leq 1.$$

At $y = 0$, we see the probability mass $P[Y = 0] = P[X \leq 0] + P[X \geq 1]$, so we must add an impulse of this area to the density $f_Y(y)$ at $y = 0$. The overall answer for the general pdf f_Y then becomes,

$$f_Y(y) = \left(\frac{1}{2} f_X(y/2) + \frac{1}{2} f_X(1 - y/2)\right)(u(y) - u(y-1)) + (P[X \leq 0] + P[X \geq 1])\,\delta(y).$$

For the given uniform density of random variable X, i.e. $X : U[0, 2]$, we have $P[X \leq 0] = 0$ and $P[X \geq 1] = 1/2$. Also

$$f_X(y/2) = \frac{1}{2}(u(y/2) - u(y/2 - 2)) = \frac{1}{2}(u(y) - u(y-4))$$

and

$$f_X(1 - y/2) = \frac{1}{2}(u(1 - y/2) - u(1 - y/2 - 2)) = \frac{1}{2}(u(2 - y) - u(-y - 2))$$

so that

$$f_Y(y) = \left(\frac{1}{4}(u(y) - u(y-4)) + \frac{1}{4}(u(2-y) - u(-y-2))\right) \times (u(y) - u(y-1)) + \frac{1}{2}\delta(y).$$

If we define a function

$$\mathrm{rect}(x) \triangleq \begin{cases} 1, & -\frac{1}{2} < x < +\frac{1}{2}, \\ 0, & \text{else}, \end{cases}$$

then we can write $f_X(x) = \frac{1}{2}\mathrm{rect}(\frac{x-1}{2})$ and so $f_X(y/2) = \frac{1}{2}\mathrm{rect}\left(\frac{y/2-1}{2}\right) = \frac{1}{2}\mathrm{rect}(\frac{y-2}{4})$ and $f_X(1 - y/2) = \frac{1}{2}\mathrm{rect}\left(\frac{1-y/2-1}{2}\right) = \frac{1}{2}\mathrm{rect}(\frac{-y}{4}) = \frac{1}{2}\mathrm{rect}(\frac{y}{4})$. The overall answer for the density f_Y can then be written as

$$\begin{aligned} f_Y(y) &= \left(\frac{1}{4}\mathrm{rect}(\frac{y-2}{4}) + \frac{1}{4}\mathrm{rect}\left(\frac{y}{4}\right)\right)\mathrm{rect}(y - \frac{1}{2}) + \frac{1}{2}\delta(y), \\ &= \frac{1}{2}\mathrm{rect}(y - \frac{1}{2}) + \frac{1}{2}\delta(y). \end{aligned}$$

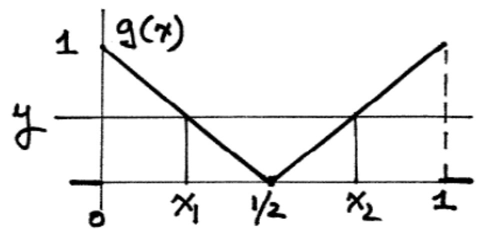


Figure 11:

13. First we plot $y = g(x)$ in Fig. 11. For $0 < x < \frac{1}{2}$: $g(x) = 1 - 2x$. For $\frac{1}{2} < x < 1$: $g(x) = 2x - 1$; elsewhere $g(x) = 0$. For $y > 1$, no real roots to $y - g(x) = 0$, then pdf $= 0$ there. For $y = 1$, $P[Y = 1] = P[X = 0] + P[X = 1] = 0$ since X is a continuous random variable. For $y < 0$, there are no real roots to $y - g(x) = 0$, then pdf=0. For $0 < y < 1$, there are two roots: $x_1 = \frac{1}{2} - \frac{y}{2}$; $x_2 = \frac{1}{2} + \frac{y}{2}$. So

$$f_Y(y) = \sum_{i=1}^{N} \frac{f_X(x_i)}{|dg/dx|_{x_i}} = \frac{1}{2} f_X(\frac{1}{2} - \frac{y}{2}) + \frac{1}{2} f_X(\frac{1}{2} + \frac{y}{2}).$$

For $y = 0$, since $X : U(0, 2)$

$$P[Y = 0] = P[X < 0] + P[X > 1] = \frac{1}{2}.$$

Therefore at $y = 0$, $f_Y(y) = \frac{1}{2}\delta(y)$. To construct $f_Y(y)$:

$$f_X(x) = \frac{1}{2}\text{rect}\left(\frac{x-1}{2}\right).$$

Therefore for $0 < y < 1$:

$$f_Y(y) = \frac{1}{4}\text{rect}\left(\frac{-y-1}{4}\right) + \frac{1}{4}\text{rect}\left(\frac{y-1}{4}\right).$$

And $\text{rect}(x) = \text{rect}(-x)$ in general. Therefore

$$f_Y(y) = \frac{1}{4}\text{rect}\left(\frac{y+1}{4}\right) + \frac{1}{4}\text{rect}\left(\frac{y-1}{4}\right).$$

for $0 < y < 1$, with sketch (Fig. 12) valid in $(0, 1)$, For all y, we thus have,

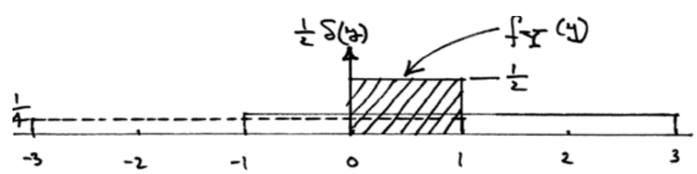


Figure 12:

$$f_Y(y) = \begin{cases} \frac{1}{2}\text{rect}(y-\frac{1}{2}), & 0 < y < 1, \\ \frac{1}{2}\delta(y), & y = 0, \\ 0, & y < 0 \quad \text{or} \quad y > 1. \end{cases}$$

14. (a) See the sketch in Fig. 13.

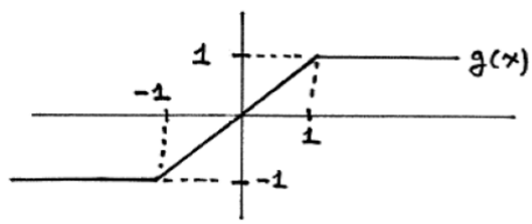


Figure 13:

(b) $F_Y(y)$ can be computed indirectly without computing $f_Y(y)$, but it is easier to do part (c) first and then come back to (b)

(c) For $y < -1$: no real solutions to $g(x) - y = 0$. For $y = -1$: $P[Y = -1] = P[X \leq -1] = P[X > 1] = \frac{1}{2} - \text{erf}(1) \doteq 0.159$, so $f_Y(y) = 0.159\delta(y+1)$. For $-1 < y < 1$: $g(x) = x$, $y - x = 0$, or $x = y$. Therefore $f_Y(y) = f_X(y)$. For $y = 1$: $P[Y = 1] = P[X > 1] \simeq 0.159$, so $f_Y(y) = 0.159\delta(y-1)$. For $y > 1$: no real solutions to $g(x) - y = 0$ exist. Putting all these pieces together we get:

$$f_Y(y) = 0.159\delta(y+1) + \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\text{rect}(\frac{y}{2}) + 0.159\delta(y-1),$$

which is sketched in Fig. 14. (b) Now that we have $f_Y(y)$, we obtain $F_Y(y)$ from

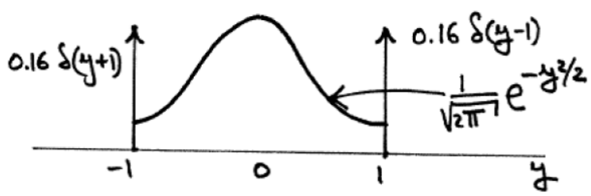


Figure 14:

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(u)du \\ &= 0.159u(y+1) + 0.159u(y-1) \\ &\quad + \begin{cases} \text{erf}(1) - \text{erf}(-y), & -1 < y < 0, \\ \text{erf}(1) + \text{erf}(y), & 0 < y < 1. \end{cases} \end{aligned}$$

Note that this last part, involving the erf function is to be added to the step functions on the line above. Another way of writing the solution comes from noticing that for

$-1 < y < +1$, the distribution function of Y must agree with that of X. Thus in this region, we have the total answer $F_Y(y) = F_X(y) = 0.5+\text{erf}(y)$. Then we can write the solution as

$$F_Y(y) = \begin{cases} 1, & y \geq 1, \\ 0.5 + \text{erf}(y), & -1 \leq y < +1, \\ 0, & y < -1, \end{cases}$$

which is the same as that a few lines above. A sketch of the distribution function is provided in Fig. 15.

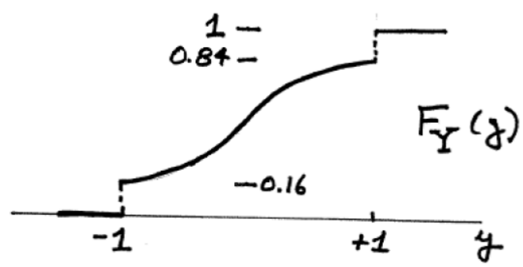


Figure 15:

15.

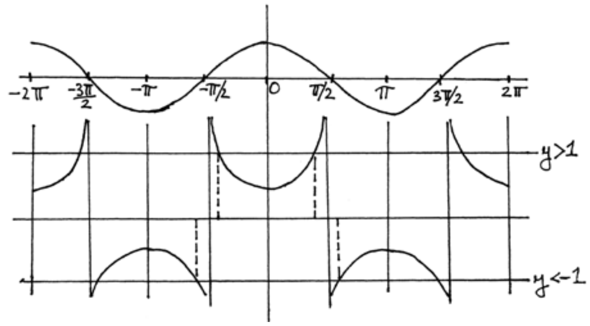
$$\begin{aligned} F_Y(y) &\triangleq P[Y \leq y] \\ &= P[\frac{a}{X} \leq y] \\ &= P[X \geq \frac{a}{y}] \\ &= 1 - P[X < \frac{a}{y}] \\ &= 1 - P[X \leq \frac{a}{y}], \text{ since } X \text{ is continuous,} \\ &= 1 - F_X(\frac{a}{y}). \end{aligned}$$

So, upon differentiation

$$\begin{aligned} f_Y(y) &= \frac{a}{y^2} f_X(\frac{a}{y}) \\ &= \frac{a}{y^2} \frac{\alpha/\pi}{\alpha^2 + (\frac{a}{y})^2} \\ &= \frac{a}{\alpha\pi} \frac{1}{y^2 + (a/\alpha)^2}. \end{aligned}$$

16.

$$\begin{aligned} Y &= \sec X \\ &= \frac{1}{\cos X} \end{aligned}$$



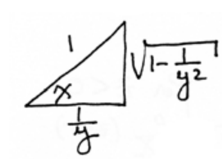
For $y > 1$, there are two roots in the interval $(-\pi, +\pi)$

$$\begin{aligned} y &= g(x) \\ &= \frac{1}{\cos x} \\ \Longrightarrow & \quad x = \cos^{-1}(1/y). \end{aligned}$$

The two roots are

$$x_1 = +\cos^{-1}(1/y) \quad \text{and} \quad x_2 = -\cos^{-1}(1/y).$$

Also $g(x) = (\cos x)^{-1} \quad \Longrightarrow \quad g'(x) = \frac{\sin x}{\cos^2 x}$. Can then use the right triangle



to show that $|g'(x_i(y))| = y\sqrt{y^2-1}$. Hence

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{2} \frac{f_X(x_i(y))}{|g'(x_i(y))|} \\ &= \frac{1}{\pi|y|} \frac{1}{y^2-1}. \end{aligned}$$

For $y < -1$, there are again two roots and we actually obtain the same result as above.

For $-1 < y < +1$, there are no solutions to $y - g(x) = 0$, thus $f_Y(y) = 0$ there.

Overall we have the solution

$$f_Y(y) = \begin{cases} \frac{1}{\pi|y|}\frac{1}{y^2-1}, & |y| > 1, \\ 0, & \text{else.} \end{cases}$$

17. Since the random variables X and Y are independent, we can find the density of Z via convolution of the two uniform densities f_X and f_Y, thus

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{+\infty} f_X(z-x) f_Y(x) dx. \end{aligned}$$

Here, by the problem statement $X : U(-1, +1)$ and hence

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < +1, \\ 0, & \text{else.} \end{cases}$$

Similarly $Y : U(-2, +2)$ and so

$$f_Y(y) = \begin{cases} \frac{1}{4}, & -2 < y < +2, \\ 0, & \text{else.} \end{cases}$$

Computing the convolution graphically, we see that the resulting function f_Z will be constant when the short pulse f_X is completely contained inside the longer pulse f_Y, and that this will occur for $-1 \leq z \leq +1$, for which the area is easily computed as $\frac{1}{2} \times \frac{1}{4} \times 2 = \frac{1}{4}$. From graphical considerations, we can also easily see that the output function f_Z must be zero when the two pulses do not overlap, and that this will occur for all $|z| > 3$. For $3 \geq |z| > 1$, we then just connect these result together via straight lines, to obtain

$$f_Z(z) = \begin{cases} 0, & z < -3, \\ \frac{1}{8}(z+3), & -3 \leq z < -1 \\ \frac{1}{4}, & -1 \leq z \leq +1, \\ \frac{1}{8}(-z+3), & +1 < z \leq +3 \\ 0, & z > +3 \end{cases},$$

which is graphed as Fig. 16.

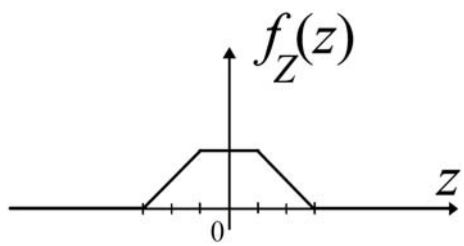


Figure 16:

18. We wish to calculate the pdf of $Z \triangleq X - Y$, given that X and Y are independent and exponentially distributed as $f_X(x) = f_Y(x) = \alpha \exp(-\alpha x) u(x)$. Our approach is to first find the density of $V \triangleq -Y$ and then make use of the convolution of densities property to find the density of $Z = X + V$. Now $P[V \leq v] = P[Y \geq -v] = 1 - F_Y(-v)$, where we have made use of the fact that Y is a continuous random variable. Therefore

$$\begin{aligned} f_V(v) &\triangleq \frac{dF_V(v)}{dv} \\ &= \frac{d(1 - F_Y(-v))}{dv} \\ &= f_Y(-v). \end{aligned}$$

Then, since X and V are independent,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_X(x) f_V(z-x)dx \\
&= \int_{-\infty}^{+\infty} f_X(x) f_Y(-(z-x))dx \\
&= \int_{-\infty}^{+\infty} f_X(x) f_Y(x-z)dx \\
&= \int_{-\infty}^{+\infty} \alpha e^{-\alpha x} u(x) \alpha e^{-\alpha(x-z)} u(x-z)dx \\
&= \int_{\max(0,z)}^{+\infty} \alpha e^{-\alpha x} \alpha e^{-\alpha(x-z)} dx \\
&= \int_{\max(0,z)}^{+\infty} \alpha e^{-\alpha x} \alpha e^{-\alpha(x-z)} dx \\
&= \alpha^2 e^{+\alpha z} \int_{\max(0,z)}^{+\infty} e^{-2\alpha x} dx \\
&= \alpha^2 e^{+\alpha z} \begin{cases} \frac{1}{2\alpha}, & z < 0, \\ \frac{1}{2\alpha} e^{-2\alpha z}, & z \geq 0, \end{cases} \\
&= \frac{\alpha}{2} \begin{cases} e^{+\alpha z}, & z < 0, \\ e^{-\alpha z}, & z \geq 0, \end{cases} \\
&= \frac{\alpha}{2} \exp(-\alpha |z|), \qquad -\infty < z < +\infty.
\end{aligned}$$

19. (a) The joint pdf $f_{V,W}(v,w)$ is given as

$$f_{V,W}(v,w) = f_{X,Y}\left(\frac{1}{3}(v+w), \frac{1}{3}(2v-w)\right) \left|\frac{\partial(\phi,\psi)}{\partial(v,w)}\right|$$

with the transformation and inverse given as

$$\begin{array}{ll} v = x+y & x = \phi(v,w) = \frac{1}{3}(v+w) \\ w = 2x - y & y = \psi(v,w) = \frac{1}{3}(2v-w)\,. \end{array}$$

We proceed to evaluate the absolute value of the Jacobian as

$$|J| = \left|\det\begin{bmatrix} \frac{\partial\phi}{\partial v} & \frac{\partial\phi}{\partial w} \\ \frac{\partial\psi}{\partial v} & \frac{\partial\psi}{\partial w} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\right| = \left|-\frac{1}{9} - \frac{2}{9}\right| = \frac{1}{3}.$$

Thus we finally obtain

$$f_{V,W}(v,w) = \frac{1}{3} f_{X,Y}\left(\frac{1}{3}(v+w), \frac{1}{3}(2v-w)\right).$$

(b) We start by noting that Z is just the V of part (a). Hence $f_Z(z)$ is just the marginal density for $f_{Z,W}(z,w)$ where

$$f_{Z,W}(z,w) = \frac{1}{3} f_{X,Y}\left(\frac{1}{3}(z+w), \frac{1}{3}(2z-w)\right).$$

Combining we have

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{3} f_{X,Y}\left(\frac{1}{3}(z+w), \frac{1}{3}(2z-w)\right) dw.$$

Upon the substitution $x = \frac{1}{3}(z+w)$ inside the integral, we get

$$dx = \frac{1}{3}dw \quad \text{and} \quad \frac{1}{3}(2z-w) = \frac{1}{3}(2z-(3x-z)) = z-x,$$

so that the integral expression then becomes

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\, dx.$$

To get the desired result, we need that X and Y be independent RVs, because then the joint density $f_{X,Y}(x, z-x)$ will factor into $f_X(x)f_Y(z-x)$, which is the desired integrand.

20. In this problem

$$g(x) = \begin{cases} x, & |x| \leq 1, \\ 0, & \text{else,} \end{cases}$$

so the range of y is $0 \leq y \leq 1$. For $0 < y \leq 1$, i.e. when the first inequality is strict, there is only one root root to $y - g(x) = y - x = 0$: at $x_1 = y$. Hence

$$f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dg}{dx}\right|} = \frac{f_X(x_1)}{1} \quad \text{for } 0 < y \leq 1.$$

Since $f_X(x) = e^{-x}u(x)$, it follows that

$$\begin{aligned} f_Y(y) &= f_X(x_1) \\ &= e^{-y}u(y) \\ &= e^{-y} \quad \text{there, i.e. for } 0 < y \leq 1. \end{aligned}$$

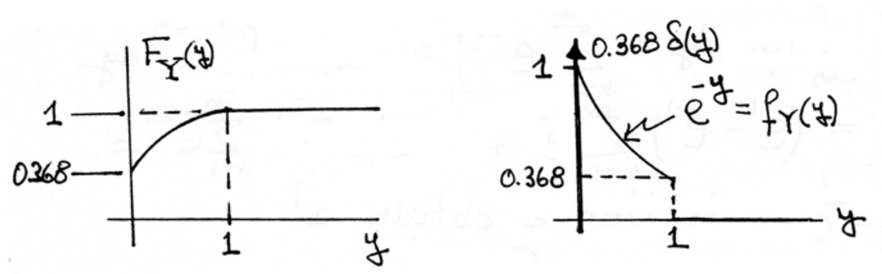
However, according to the mapping $g(x)$, the event $\{Y=0\} = \{X \leq 0\} \cup \{X > 1\}$, so

$$\begin{aligned} P[Y=0] &= P[X \leq 0] + P[X > 1] \\ &= 0 + \int_1^{\infty} e^{-x} dx \\ &= e^{-1}. \end{aligned}$$

Thus the complete answer for the pdf of Y is:

$$\begin{aligned} f_Y(y) &= e^{-1}\delta(y) + e^{-y}[u(y) - u(y-1)] \\ &\simeq 0.368\delta(y) + e^{-y}[u(y) - u(y-1)]. \end{aligned}$$

Labeled sketches of the CDF and pdf follow:



21. The only real roots occur when $-1 < y < +1$, and, as can be seen from the diagram, these occur at $x_n = y + 2n$ for $n = ..., -2, -1, 0, 1, 2, ...$ Since $f_X(x) = e^{-x}u(x)$ and $|g'(x))|_{x=x_n} = 1$, we obtain

$$f_Y(y) = \sum_{n=-\infty}^{+\infty} e^{-(y+2n)}\text{rect}(y/2)u(y+2n),$$

where $\text{rect}(x) \triangleq u(x+\frac{1}{2}) - u(x-\frac{1}{2}))$. We note that the product $\text{rect}(y/2)u(y+2n) = \text{rect}(y/2)$ for $n \geq 1$. For $n < 0$, this product is zero. At $n = 0$, only half of $\text{rect}(y/2)$ is overlapped by $u(y)$. Hence, for $n = 0$, the product is $\text{rect}(y - \frac{1}{2})$. Thus,

$$f_Y(y) = \sum_{n=1}^{\infty} e^{-(y+2n)}\text{rect}(y/2) + e^{-y}\text{rect}(y - \frac{1}{2}).$$

To show that the pdf f_Y is a legitimate pdf, we note first that it is non-negative, and next that

$$\begin{aligned}
\int_{-\infty}^{+\infty} f_Y(y)dy &= \sum_{n=1}^{\infty} e^{-2n} \int_{-1}^{+1} e^{-y}dy + \int_0^{+1} e^{-y}dy \\
&= \left(\sum_{n=1}^{\infty} e^{-2n}\right)(e - e^{-1}) + (1 - e^{-1}) \\
&= \left(\frac{1}{1-e^{-2}} - 1\right)(e - e^{-1}) + (1 - e^{-1}) \\
&= \left(\frac{e}{e-e^{-1}} - 1\right)(e - e^{-1}) + (1 - e^{-1}) \\
&= e - (e - e^{-1}) + (1 - e^{-1}) \\
&= 1 \quad \text{as required for a legitimate pdf.}
\end{aligned}$$

22. Let X_n and X_{n+1} be two numbers generated in sequence by the random number generator at the nth construct. Form $Y_n = X_n + X_{n+1}$. If the $\{X_i\}$ are generated independently, the pdf of Y_n is $f_{Y_n}(y) = f_{X_n}(y) * f_{X_{n+1}}(y)$ and looks like the sketch in Fig. 17. So the form is right but the mean is wrong! Therefore generate $Z_n = Y_n - 1$. The pdf of Z_n looks like the sketch in Fig. 18. Equivalently one could subtract 0.5 from X_n and X_{n+1} before adding. In either case one can get $f_{Z_n}(z) = \text{tri}(z/2)$. Here the *triangle function* tri is defined as

$$\text{tri}(x) \triangleq \begin{cases} 1 - 2|x|, & -\frac{1}{2} < x < +\frac{1}{2}, \\ 0, & \text{else.} \end{cases}$$

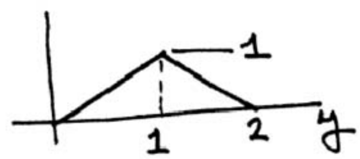


Figure 17:

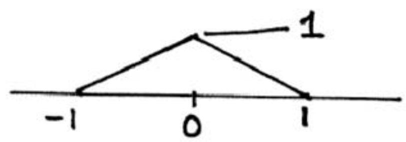
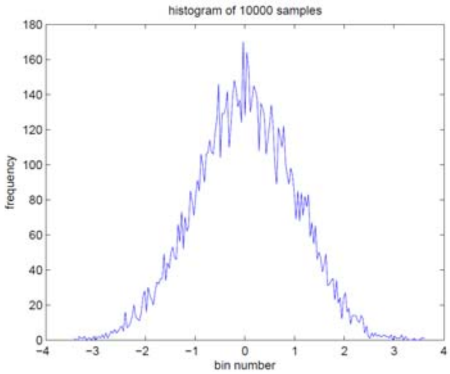


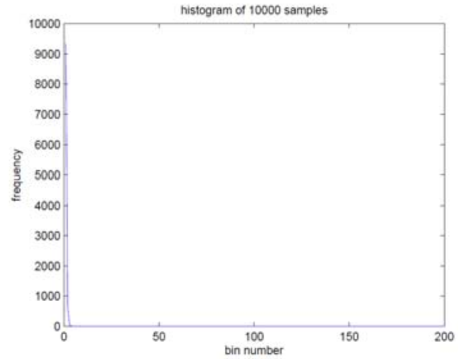
Figure 18:

23. We present a MATLAB function `histonorm` as follows:

```
function [alpha] = histonorm( samples, sumrange, bins );
sum = zeros( 1, samples );
for i = 1 : bins
x(i) = i;
end
for i = 1 : samples
for j = 1 : sumrange
sum(i) = sum(i) + rand;
end
sum(i) = sum(i) - 6;
end
[z, x] = hist( sum, bins );
plot( x, z);
title( ['histogram of ', num2str(samples), ' samples'] );
xlabel( 'bin number' )
ylabel( 'frequency' )
```

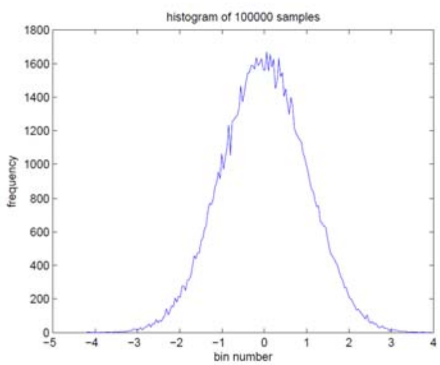
The multiple figures in Fig. **??** show the output of `histonorm` for 10,000 with "smart binning" on the left and "dumb binning" on the right.

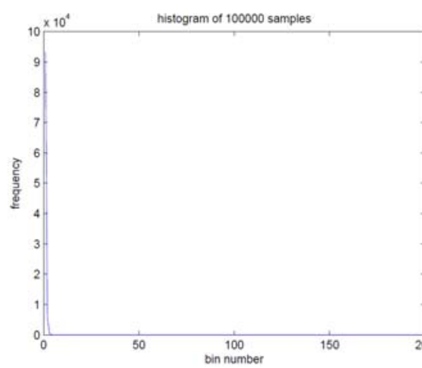


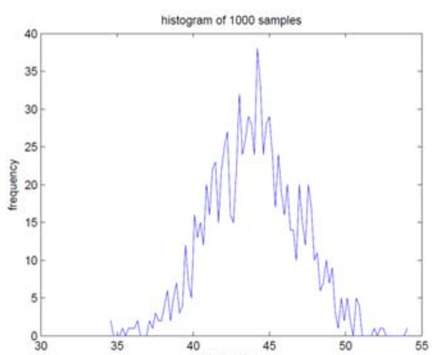


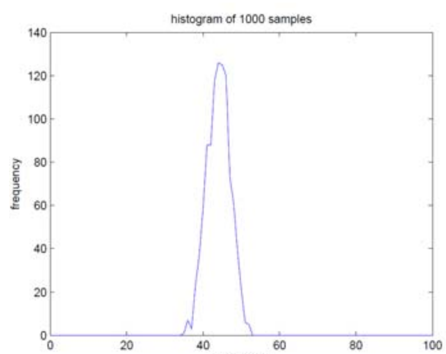
Note that the 'dumb binning' result on the right requires a slight modification of the function as indicated in the comments inside it. The 'smart binning' result which is the default only plots the domain where the bins are not empty.

While only 10,000 runs were requested in the problem statement, we next show in Fig. **??** the result of a larger number of trials 100,000 (Fig. **??** and **??**), and also a smaller number 1000 trials (Fig. **??** and **??**). We can see that the histogram begins to look strongly Gaussian at the high value of 100,000 trials. A better idea of its Gaussian-ness could be obtained by viewing a log plot, which is not done here however.









24. (a) We have to calculate the running integral of the density $f_Y(y) = \frac{c}{2}\exp(-c|y|)$. Now

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(u)du \\ &= \int_{-\infty}^{y} \frac{c}{2}\exp(-c|u|)du. \end{aligned}$$

Because of the absolute value sign, it is easier to consider the two cases $y \leq 0$ and $y \geq 0$ separately. First we evaluate for $y \leq 0$, where $f_Y(y) = \frac{c}{2}\exp(+cy)$. We find

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{c}{2}\exp(cu)du \\ &= \frac{c}{2}\left(\frac{1}{c}\exp(cu)|_{-\infty}^{y}\right) \\ &= \frac{1}{2}\exp\left(cy\right), \qquad \text{for } y \leq 0. \end{aligned}$$

Now we consider the case $y \geq 0$, where $f_Y(y) = \frac{c}{2}\exp(-cy)$. We note that by symmetry $\int_{-\infty}^{0} \frac{c}{2}\exp(cu)du = 1/2$, so we can write

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{c}{2}\exp(-c|u|)du \\ &= \frac{1}{2} + \int_{0}^{y} \frac{c}{2}\exp(-cu)du \\ &= \frac{1}{2} + \frac{c}{2}\left(\frac{1}{c}\exp(-cu)|_{0}^{y}\right) \\ &= \frac{1}{2} + \frac{1}{2}\left(1 - \exp\left(-cy\right)\right) \\ &= 1 - \frac{1}{2}\exp\left(-cy\right), \qquad \text{for } \; y \geq 0. \end{aligned}$$

Now, as a check, we note that both results agree at their common point $y = 0$ as they should. Overall, we can write the Laplacian distribution function as

$$F_Y(y) = \begin{cases} \frac{1}{2}\exp\left(cy\right), & y < 0, \\ 1 - \frac{1}{2}\exp\left(-cy\right), & y \geq 0. \end{cases}$$

(b) Probably the first thing to do here is to note that since $X : U[0,1]$, we have that $F_X(x) = x\{u(x) - u(x-1)\}$, i.e. just a straight line segment with slope 1 on $[0,1]$. Since the distribution function F_Y is monotone increasing, we have

$$F_Z(z) = P[X \leq g^{-1}(z)]$$

Next, we note that since $g = F_Y^{-1}$, so $g^{-1} = F_Y$ and hence

$$\begin{aligned} F_Z(z) &= P[X \leq g^{-1}(z)] \\ &= F_X(F_Y(z)) \\ &= F_Y(z)\left(u(F_Y(z)) - u(F_Y(z) - 1)\right) \\ &= F_Y(z)\left(1 - 0\right) \\ &= F_Y(z). \end{aligned}$$

Note: Although it was not asked for in this problem, to actually use this method on a computer, we would need to calculate $g(x)$. It is given on $(0,1)$ as

$$g(x) = \begin{cases} \frac{1}{c}\ln 2x, & 0 < x < \frac{1}{2}, \\ \frac{1}{c}\ln\frac{1}{2(1-x)}, & \frac{1}{2} \leq x < 1. \end{cases}$$

(c) This method strictly speaking will not work with either jumps or flat regions in the desired distribution function F_Y. In a flat region of F_Y, the corresponding g would be a vertical line, not be a valid function! At a jump of F_Y, $g = F_Y^{-1}$ will not be defined for some of the input values. This won't work either. On the other hand, there are simple modifications of this method that can get around these problems and make the basic method useful in both cases. One simply has to remove the flat regions from F_Y before finding the inverse function. At the jumps, where the inverse function would have a gap, just fill it in with a horizontal line. With these changes the basic method extends to both mixed and discrete distribution functions.

25. First we solve $f_Z(z)$ is as follows. Let $Z = X - Y$, and $f_X(x) = f_Y(x) = \alpha e^{-\alpha x}u(x)$. Let $V \triangleq -Y$, then $P[V \leq v] = P[Y \geq -v] = 1 - F_Y(-v)$, and so $f_V(v) = \frac{dF_V}{dv} = f_Y(-v)$. Therefore

$$f_V(v) = \alpha e^{\alpha v}u(-v).$$

Now we can write $Z = X + V$, so that we have the density of Z given as the convolution

$$f_Z(z) = \int_{-\infty}^{\infty} f_V(v)f_X(z-v)dv.$$

Evaluating for $z < 0$ (Fig. 19), we get

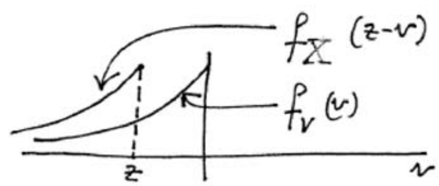


Figure 19:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_V(v)f_X(z-v)dv. \\ &= \int_{-\infty}^{z} e^{2\alpha v}dv \cdot e^{-\alpha z}\alpha^2 \\ &= \frac{\alpha}{2}e^{\alpha z}. \end{aligned}$$

For $z \geq 0$, we get

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{0} e^{2\alpha v}dv \cdot e^{-\alpha z}\alpha^2 \\ &= \frac{\alpha}{2}e^{-\alpha z}. \end{aligned}$$

Combining these two results, we get the formula valid for all z,

$$f_Z(z) = \frac{\alpha}{2}e^{-\alpha|z|},$$

as found in problem 3.18. Now let

$$W \triangleq |Z| = \begin{cases} Z, & Z \geq 0; \\ -Z, & Z < 0. \end{cases}$$

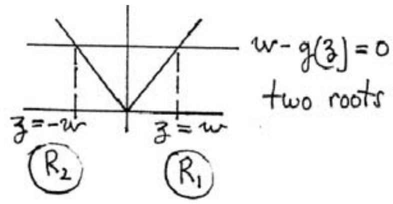


Figure 20:

Consider $w = g(z) = |z|$ (Fig. 20): At root R_1: $g'(z)|_w = 1$; At root R_2: $g'(z)|_{-w} = -1$. Then

$$f_W(w) = \begin{cases} f_Z(w)/1 + f_Z(-w)/1, & \text{for } w \geq 0; \\ 0, & \text{for } w < 0. \end{cases}$$

Hence if $f_Z(w) = f_Z(-w)$ as it is for here, then finally

$$\begin{aligned} f_w(W) &= 2f_Z(w)u(w) \\ &= \alpha e^{-\alpha w} u(w). \end{aligned}$$

26. We are told here that X and Y are continuous random variables and are independent. We are asked for the distribution function and probability density function of $Z \triangleq \min(X, Y)$. For the distribution function, we write

$$\begin{aligned} F_Z(z) &\triangleq P[Z \leq z] \\ &= 1 - P[Z > z] \\ &= 1 - P[X > z]P[Y > z] \\ &= 1 - (1 - P[X \leq z])\,(1 - P[Y \leq z]) \\ &= 1 - (1 - F_X(z)\,(1 - F_Y(z)) \\ &= F_X(z) + F_Y(z) - F_X(z)F_Y(z). \end{aligned}$$

For the pdf, we can write

$$\begin{aligned} f_Z(z) &\triangleq \frac{dF_Z(z)}{dz} \\ &= \frac{d\,(F_X(z) + F_Y(z) - F_X(z)F_Y(z))}{dz} \\ &= f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z). \end{aligned}$$

For X and Y uniformly distributed as $U[0,1]$, we then get,

$$\begin{aligned} F_Z(z) &= z + z - z^2 \quad \text{for } z \in [0,1], \text{ or generally,} \\ &= \begin{cases} 2z - z^2, & 0 \leq z \leq 1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

The pdf becomes,

$$\begin{aligned} f_Z(z) &= 1 + 1 - 2z \quad \text{for } z \in [0,1], \text{ or generally,} \\ &= \begin{cases} 2 - 2z, & 0 \leq z \leq 1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Here are the plots:

$f_Z(z) = 2 - 2z$ (Fig. 21):

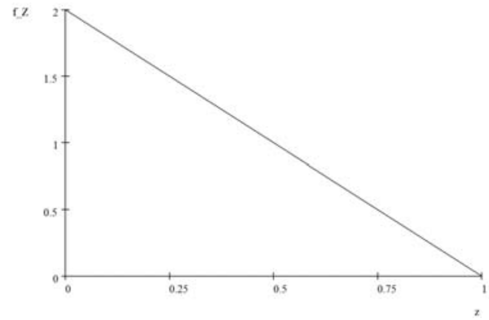


Figure 21:

$F_Z(z) = 2z - z^2$ (Fig. 22):

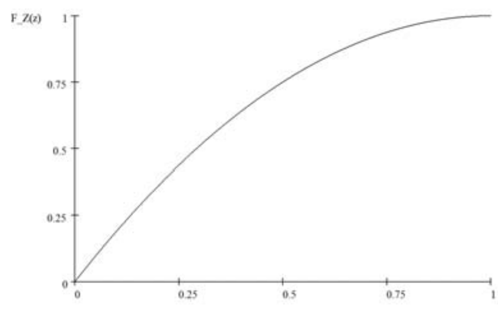


Figure 22:

Next we repeat the plots for the case where X and Y are independent and exponentially distributed with parameter $\alpha (> 0)$, i.e. $f_X(x) = f_Y(x) = \alpha \exp(-\alpha x) u(x)$. Plugging into the equations, we get $F_Z(z) = (1 - \exp(-2\alpha z))u(z)$ and $f_Z(z) = 2\alpha \exp(-2\alpha z)u(z)$. We notice that, unlike the case of the two uniform random variables, the minimum of two independent exponential random variables remains exponential, but with twice the parameter value. The plots then become, for $\alpha = 2$,

$f_Z(z) = 4 \exp(-4z)$ (Fig. 23):

$F_Z(z) = 1 - \exp(-4z)$ (Fig. 24):

27.

$$
\begin{aligned}
F_Z(z) &= F_{X_1,X_2}(z,z) \\
&= F_{X_1}(z) F_{X_2}(z) \\
&= [1 - \exp(-z/\mu)]\,[1 - \exp(-z/\mu)]\;\, u^2(z) \\
&= [1 - \exp(-z/\mu)]^2\;\, u(z) \\
& \quad\; (1 - e^{-z/\mu})^2 u(z).
\end{aligned}
$$

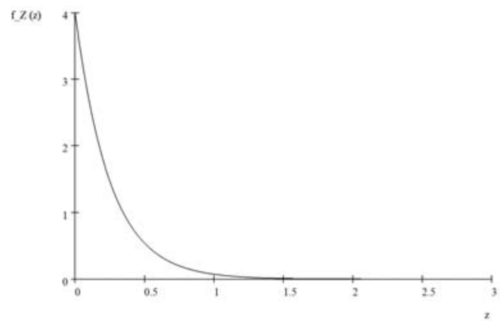


Figure 23:

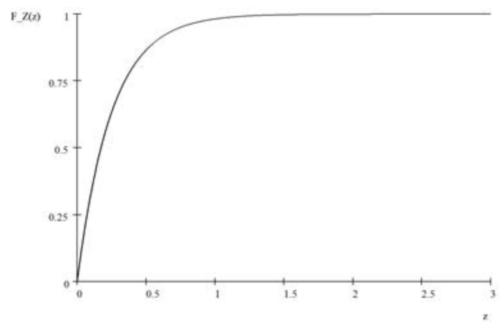


Figure 24:

$$\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz}\left([1-\exp(-z/\mu)]^2 \; u(z) \right) \\
&= \frac{d}{dz}\, [1-\exp(-z/\mu)]^2 \; u(z) \\
&= \frac{2}{\mu} \exp(-z/\mu) \, [1-\exp(-z/\mu)] \; u(z) \\
&= \frac{2}{\mu} e^{-z/\mu} (1-e^{-z/\mu}) u(z)
\end{aligned}$$

This problem has an interpretation for the time to failure of two independent machines. If we regard the two random variables X_1 and X_2 as waiting times (to failure), then Z would be the time till both machines fail. This has clear application in high-reliability computing.

28. The event $\{Z \leq z\}$ is the same as

$$\begin{aligned}
& \{\max(X_1, X_2, ..., X_n) \leq z\} \\
= \; & \{X_1 \leq z, X_2 \leq z, ..., X_n \leq z\}
\end{aligned}$$

Since $\{X_i\}$ are independent, then

$$F_Z(z) = F_{X_1}(z)F_{X_2}(z)\cdots F_{X_n}(z).$$

Since $\{X_i\}$ are i.i.d.,

$$F_{X_1}(z) = F_{X_2}(z) = \cdots = F_{X_n}(z),$$

so that

$$\begin{aligned} F_Z(z) &= (F_{X_1}(z))^n \\ &= F_{X_1}^n(z). \end{aligned}$$

29. The event $\{Z > z\} = \{\min(X_1, X_2, ..., X_n) > z\}$ is identical with

$$\{Z > z\} = \{X_1 > z, X_2 > z, ..., X_n > z\}.$$

Thus

$$\begin{aligned} P[Z > z] &= P[\{X_1 > z, X_2 > z, ..., X_n > z\}] \\ &= P[X_1 > z]P[X_2 > z]\cdots P[X_n > z] \\ &= (1 - F_{X_1}(z))(1 - F_{X_2}(z))\cdots(1 - F_{X_n}(z)). \end{aligned}$$

Because the $\{X_i\}$ are i.i.d., this becomes

$$P[Z > z] = (1 - F_{X_1}(z))^n,$$

and so for the complementary event $\{Z \leq z\}$, we have

$$F_Z(z) = 1 - (1 - F_{X_1}(z))^n.$$

30. $Z_n \triangleq \max(X_1, X_2, ..., X_n)$ and the X_is are independent RVs. Then

$$\begin{aligned} F_{Zn}(z) &= P[Z_n \leq z] \\ &= P[X_1 \leq z]P[X_2 \leq z]\cdots P[X_n \leq z] \\ &= (F_X(z))^n \\ &= (1 - e^{-z})^n u(z). \end{aligned}$$

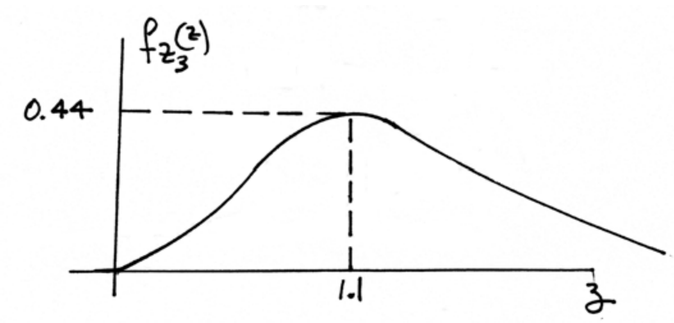
Hence

$$\begin{aligned} f_{Z_n}(z) &= \frac{dF_{Zn}(z)}{dz} \\ &= n((1 - e^{-z})^{n-1}e^{-z}u(z). \end{aligned}$$

The peak of this curve will occur at

$$\begin{aligned} 0 &= f'_{Z_n}(z) = \\ &= n(n-1)(1 - e^{-z})^{n-2}e^{-2z} - n(1 - e^{-z})^{n-1}e^{-z} \\ &= (n-1)e^{-z} - (1 - e^{-z}), \end{aligned}$$

which happens at $ne^{-z_o} = 1$ or $z_o = \ln(n)$. For $n = 3$, $z_o \simeq 1.1$ and $f_{Z_3}(z_o) \simeq 0.444$. See sketch below for $n = 3$.



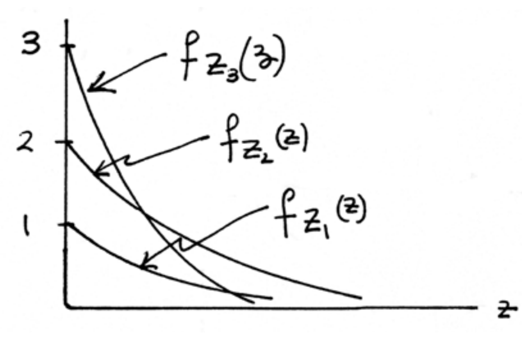
As for the minimum, we have then $Z_n \triangleq \max(X_1, X_2, ..., X_n)$, where again the X_is are independent RVs. Then

$$\begin{aligned}
F_{Z_n}(z) &= P[Z_n \le z] \\
&= 1 - P[Z_n > z] \\
&= 1 - P[X_1 > z]P[X_2 > z] \cdots P[X_n > z] \\
&= 1 - (1 - F_{X_1}(z))(1 - F_{X_2}(z)) \cdots (1 - F_{X_n}(z)) \\
&= 1 - [1 - (1 - e^{-z})u(z)]^n \\
&= \begin{cases} 0, & z < 0, \\ (1 - e^{-nz}) & z \ge 0 \end{cases} \\
&= (1 - e^{-nz})u(z).
\end{aligned}$$

So the pdf is given as

$$\begin{aligned}
f_{Z_n}(z) &= F'_{Z_n}(z) \\
&= ne^{nz}u(z).
\end{aligned}$$

Here is a sketch for $n = 1, 2, 3$.

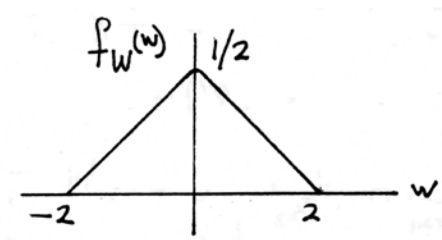


31. We are given $W = X + Y$, where X, Y are independent and identically distributed (i.i.d.)

with $X : U(-1,+1) : Y$. Then

$$\begin{aligned} f_W(w) &= f_X(w) * f_Y(w) \\ &= f_X(w) * f_X(w) \\ &= \mathrm{rect}(\frac{w}{2}) * \mathrm{rect}(\frac{w}{2}) \\ &= \left(\frac{1}{2} - \frac{|w|}{4}\right) \mathrm{rect}(\frac{w}{4}), \end{aligned}$$

with sketch:



Next, consider

$$g(w) = \begin{cases} \sqrt{w}, & w \geq 0, \\ 0, & w < 0, \end{cases}$$

with sketch:

Now, the real roots of $z - g(w) = 0$ are just one $w = z^2, z > 0$, and no real roots when $z < 0$. Hence

$$f_Z(z) = \frac{f_W(z^2)}{|g'(w)|_{w=z^2}},$$

where $g'(w) = \frac{1}{2\sqrt{w}}$ and $|g'(z^2)| = \frac{1}{2z}$. Thus, we have

$$f_Z(z) = \begin{cases} z - \frac{z^3}{2}, & 0 < z \leq \sqrt{2}, \\ 0, & \text{else.} \end{cases}$$

Finally, for $z = 0$, we have the probability mass

$$\begin{aligned} P[Z = 0] &= P[W \leq 0] \\ &= \int_{-2}^{0} f_W(w) dw \\ &= \int_{-2}^{0} \left(\frac{1}{2} - \frac{|w|}{4}\right) dw \\ &= \frac{1}{2}. \end{aligned}$$

We can write this as a delta function in the pdf f_Z as

$$f_Z(z) = \frac{1}{2}\delta(z) + \left(z - \frac{z^3}{2}\right)[u(z) - u(z - \sqrt{2})].$$

32. We have from the problem statement that

$$\begin{aligned} Z &= \alpha X_1 + \alpha X_2 \\ &= X_1' + X_2', \quad \text{with } X_i' \triangleq \alpha X_i \ \text{ for } i = 1, 2. \end{aligned}$$

The pdf's of the primed variables become

$$f_{X_i'}(x') = \frac{1}{\alpha} f_{X_i}(x'/\alpha),$$

and since the X_i, and hence the X_i', are independent, we will have

$$\begin{aligned} f_Z(z) &= f_{X_1'}(z) * f_{X_2'}(z) \\ &= \frac{1}{\alpha^2} f_{X_1}(z/\alpha) * f_{X_2}(z/\alpha). \end{aligned}$$

Now we are given that the X_i are distributed as $U[0, b]$, thus $f_{X_i}(x) = \frac{1}{b}[u(x) - u(x - b)]$, thus

$$\begin{aligned} f_{X_1'}(z) &= \frac{1}{\alpha b}[u(\frac{x}{\alpha}) - u(\frac{x}{\alpha} - b)] \\ &= \frac{1}{\alpha b}[u(x) - u(x - \alpha b)], \quad \text{since } \alpha > 0, \end{aligned}$$

and so

$$f_Z(z) = \frac{1}{\alpha b}[u(z) - u(z - \alpha b)] * \frac{1}{\alpha b}[u(z) - u(z - \alpha b)].$$

Performing this convolution, we get a triangle with support $[0, 2\alpha b]$ on the z axis and height $1/\alpha b$.We can then write down the pdf of Z as

$$f_Z(z) = \begin{cases} \frac{1}{(\alpha\beta)^2} z, & 0 < z \le \alpha b, \\ \frac{1}{\alpha b}\left(2 - \frac{1}{\alpha\beta} z\right), & \alpha b < z \le 2\alpha b, \\ 0, & \text{else.} \end{cases}$$

In this problem, we are given $\alpha = 1/10$ and $b = 100$, so that $\alpha b = 10$, and

$$\begin{aligned} f_Z(z) &= \frac{1}{10}[u(z) - u(z - 10)] * \frac{1}{10}[u(z) - u(z - 10)] \\ &= \begin{cases} \frac{1}{100} z, & 0 < z \le 10, \\ \frac{1}{10}\left(2 - \frac{1}{10} z\right), & 10 < z \le 20, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

We can now calculate the probability that the plane will fly at least 5 hours (area of shaded region in Fig. 25), i.e.

$$\begin{aligned} P[Z &\ge 5] = 1 - F_Z(5), \quad \text{since } Z \text{ is a continuous random variable,} \\ &= 1 - \int_0^5 \frac{1}{100} z dx \\ &= 1 - \frac{1}{100}(\frac{1}{2} z^2|_0^5) \\ &= 1 - \frac{1}{8} = 0.875. \end{aligned}$$

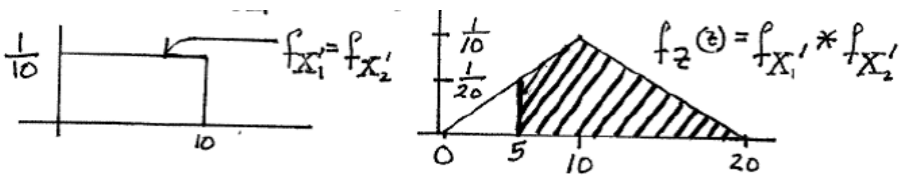


Figure 25:

33. Let $Z = X + Y$, then

$$\begin{aligned} P_Z(n) &= P[Z = n] \\ &= \sum_{k=0}^{\infty} P_X(k) P_Y(n-k) \\ &= e^{-2} e^{-3} \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{(n-k)!} 2^k 3^{n-k} \\ &= \frac{e^{-5}}{n!} \sum_{k=0}^{n} \binom{n}{k} 2^k 3^{n-k} \\ &= \frac{e^{-5}}{n!} 5^n, n \geq 0. \end{aligned}$$

Hence, Z is Poisson distributed with parameter $\lambda_Z = \lambda_X + \lambda_Y$. So

$$\begin{aligned} P[Z \leq 5] &= \sum_{k=0}^{5} P_Z(k) \\ &= e^{-5} \left(1 + 5 + \frac{5^2}{2} + \frac{5^3}{6} + \frac{5^4}{24} + \frac{5^5}{120} \right) \\ &\simeq 0.616 \end{aligned}$$

34. This is an invertible transformation, with inverse

$$x = u + v \qquad \text{and} \qquad y = u - v.$$

The joint density $f_{U,V}$ is then

$$f_{U,V}(u, v) = f_{X,Y}(u + v, u - v) \; |\widetilde{J}|,$$

where the Jacobian of the inverse transformation $\widetilde{J}$ is determined as

$$\begin{aligned} \widetilde{J} &= \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= -2. \end{aligned}$$

Thus we have

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(u+v,u-v)\,|\widetilde{J}| \\ &= 2f_{X,Y}(u+v,u-v) \\ &= 2f_X(u+v)f_Y(u-v), \\ &= \begin{cases} 2, & 0<u+v<1 \text{ and } 0<u-v<1, \\ 0, & \text{else,} \end{cases} \end{aligned}$$

since X and Y are independent and uniformly distributed $U(0,1)$. We can also write the answer in terms of the *unit pulse function* $\text{rect}(\cdot)$ defined as

$$\text{rect}(x) \triangleq \begin{cases} 1, & -0.5<x<+0.5, \\ 0, & \text{else.} \end{cases}$$

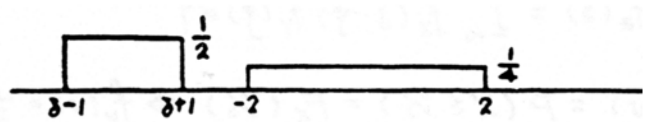
The result can then be written as

$$f_{U,V}(u,v) = 2\,\text{rect}(u+v-0.5)\,\text{rect}(u-v-0.5).$$

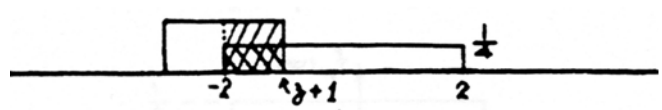
35. (a) $Z = X+Y$ where X and Y are independent, and $X : U(-1,+1)$, i.e. $f_X(x) = \frac{1}{2}\text{rect}(\frac{x}{2})$ and $Y : U(-2,+2)$, i.e. $f_X(x) = \frac{1}{4}\text{rect}(\frac{x}{4})$. Then

$$\begin{aligned} f_Z(z) &= f_X(z) * f_Y(z) \\ &= \int_{-\infty}^{\infty} f_X(z-v)f_Y(v)dv \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\text{rect}(\frac{z-v}{2})\frac{1}{4}\text{rect}(\frac{v}{4})dv \\ &= \frac{1}{8}\int_{-2}^{+2} \text{rect}(\frac{z-v}{2})dv. \end{aligned}$$

To perform this convolution, we start in *region 1*, were $z<-3$ and there is no overlap between the two function supports. Here is the sketch.



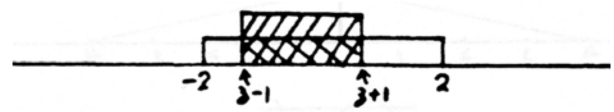
Clearly there is no overlap here in region 1, and so the output $f_Z(z) = 0$ here.
region 2: partial overlap $-3 \leq z < -1$. Here is the sketch:



Using the cross-hatched area as a guide, we can evaluate the integral as

$$\begin{aligned} f_Z(z) &= \frac{1}{8}\int_{-2}^{z+1} dv \\ &= \frac{1}{8}(z+3), \quad -3 \leq z < -1. \end{aligned}$$

Next comes region 3, complete overlap, sketched as



region 3: $-1 \le z < +1$. From the sketch, we can see that the integral must start at $z-1$ and go to $z+1$ in this region. Thus

$$\begin{aligned} f_Z(z) &= \frac{1}{8}\int_{z-1}^{z+1} dv \\ &= \frac{1}{4}. \end{aligned}$$

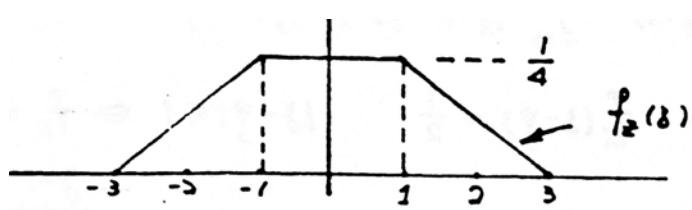
Next comes region 4, partial overlap on the right, where $z-1 < 2$ and $z+1 \ge 2$, which together imply $1 \le z < 3$.
region 4: $1 \le z < 3$. Here

$$\begin{aligned} f_Z(z) &= \frac{1}{8}\int_{z-1}^{2} dv \\ &= \frac{1}{8}(3-z), \quad 1 \le z < 3. \end{aligned}$$

Finally comes region 5, no overlap on right.
region 5: $z \ge 3$ $\qquad f_Z(z) = 0$.
Our complete answer can be written as

$$f_Z(z) = \begin{cases} 0, & z < -3 \\ \frac{1}{8}(z+3), & -3 \le z < -1, \\ \frac{1}{4} & -1 \le z < +1, \\ \frac{1}{8}(3-z), & 1 \le z < 3, \\ 0, & z \le 3, \end{cases}$$

with sketch:

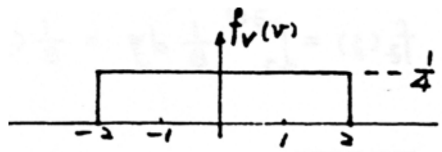


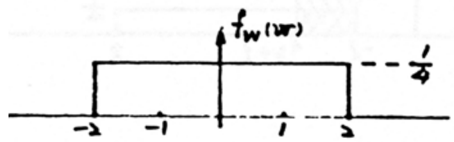
As a check, we note that the total area is $2(\frac{1}{2}(2)\frac{1}{4}) + 2(\frac{1}{4}) = \frac{1}{2} + \frac{1}{2} = 1$.

(b) Here $Z = 2X - Y$, where again X and Y are independent. We let $V = 2X$ and $W = -Y$ to make use of convolution of densities. Now $F_V(v) = P[V \le v] = P[X \le \frac{v}{2}] = F_X(\frac{v}{2}) \Longrightarrow f_V(v) = \frac{1}{2}f_X(\frac{v}{2})$ and $F_W(w) = P[W \le w] = P[Y > -w] = 1 - F_Y(-w) \Longrightarrow f_W(w) = f_Y(-w)$. Hence

$$f_V(v) = \frac{1}{4}\text{rect}(\frac{v}{4}) \quad \text{and} \quad f_W(w) = \frac{1}{4}\text{rect}(\frac{w}{4}),$$

with sketches





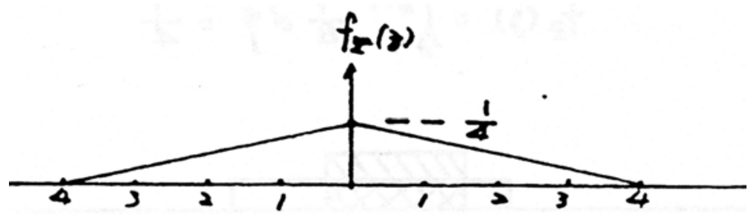
so

$$\begin{aligned} f_Z(z) &= f_V(z) * f_W(z) \\ &= \frac{1}{8}\text{rect}(\frac{z}{4}) * \text{rect}(\frac{z}{4}) \\ &= \frac{1}{4}\text{triag}(\frac{z}{4}), \end{aligned}$$

where function 'triag' is given as

$$\text{triag}(x) \triangleq \begin{matrix} (1-|x|), & |x| \leq 1, \\ 0, & |x| > 1. \end{matrix}$$

A sketch of this f_Z is given as



36. We look at the transformation problem for two independent Normal random variables X and $Y : N(0, \sigma^2)$, transformed to $Z \triangleq X^2 + Y^2$ and $W \triangleq X$. We thus have

$$z = g(x, y) = x^2 + y^2 \quad \text{and} \quad w = h(x, y) = x.$$

This is a non-invertible transformation with two real roots, for $|w| < \sqrt{z}, z > 0$,

$$\begin{aligned} R_1 &: \quad x = w, y = +\sqrt{z - w^2}, \text{ and} \\ R_2 &: \quad x = w, y = -\sqrt{z - w^2}. \end{aligned}$$

Now at both roots the magnitude of the Jacobian is the same,

$$\begin{aligned} |J_1| &= |J_2| = 2\sqrt{z - w^2}, \text{ where} \\ J_{1,2} &= \det \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = 2y = \pm 2\sqrt{z - w^2}. \end{aligned}$$

Hence

$$\begin{aligned} f_{Z,W}(z,w) &= \frac{1}{2\sqrt{z-w^2}}\left(f_{X,Y}(w,\sqrt{z-w^2}) + f_{X,Y}(w,-\sqrt{z-w^2})\right) \\ &= \begin{cases} \frac{1}{2\pi\sigma^2}\frac{1}{\sqrt{z-w^2}}\exp(-z/2\sigma^2), & |w|<\sqrt{z}, z>0 \\ 0, & \text{else.} \end{cases} \end{aligned}$$

We can find the marginal density f_Z either by integrating out the unwanted variable in this joint density, or by using the result of Example 3.3-10 that Z will be Exponential distributed (equivalently Chi-square with 2 degrees of freedom). Either way the answer is

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{+\infty} f_{Z,W}(z,w)dw \\ &= \frac{1}{2\pi\sigma^2}\int_{-\sqrt{z}}^{+\sqrt{z}} \frac{1}{\sqrt{z-w^2}}\exp(-z/2\sigma^2)dw \\ &= \frac{1}{2\sigma^2}\exp(-z/2\sigma^2)\, u(z). \end{aligned}$$

37. We want to find the pdf of the two variables

$$Z = aX + bY$$

$$W = cX + dY$$

The joint pdf of X and Y are given as

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}e^{-Q(x,y)},$$

where $Q(x,y) = \frac{1}{2\sigma^2(1-\rho^2)}[x^2 - 2\rho xy + y^2]$. Consider the inverse transformation, i.e.,

$$X = \hat{a}Z + \hat{b}W$$

$$Y = \hat{c}Z + \hat{d}W$$

Note that from the above transformation, the solution of the two equations are given as $x = \hat{a}z+\hat{b}w$ and $y = \hat{c}z+\hat{d}w$. The term in the exponent of the pdf will be $\frac{1}{2\sigma^2(1-\rho^2)}[x^2-2\rho xy+y^2]$, and this is given as

$$\frac{1}{2\sigma^2(1-\rho^2)}[x^2-2\rho xy+y^2] = \frac{1}{2\sigma^2(1-\rho^2)}\left[(\hat{a}z+\hat{b}w)^2 - 2\rho(\hat{a}z+\hat{b}w)(\hat{c}z+\hat{d}w) + (\hat{c}z+\hat{d}w)^2\right].$$

In this exponent, if the cross terms (terms that contain zw) vanish, then we would be able to split the pdf in to the product of the two marginal pdf's. In other words, if the coefficients of the terms zw is zero, then we would be able to write $f_{Z,W} = f_Z f_W$. Therefore, we need

$$(2\hat{a}\hat{b} + 2\hat{c}\hat{d} - 2\rho\hat{a}\hat{d} - 2\rho\hat{b}\hat{c}) = 0$$

. If we chose $\hat{a} = \hat{c}, \hat{b} = -\hat{d}$, the coefficient of zw will be zero. Then

$$x = \hat{a}z + \hat{b}w$$

$$y = \hat{a}z - \hat{b}w$$

will give us $\frac{1}{2\sigma^2(1-\rho^2)}[x^2 + y^2 - 2\rho xy] = \frac{1}{2\sigma^2(1-\rho^2)}[2(1-\rho)\hat{a}^2 z^2 + 2(1+\rho)\hat{b}^2 w^2]$. Therefore, $Q(x, y) = \frac{1}{2}\left\{\left[\frac{z\sqrt{1-\rho}}{\sigma/\sqrt{2}\hat{a}}\right]^2 + \left[\frac{w\sqrt{1+\rho}}{\sigma/\sqrt{2}\hat{b}}\right]^2\right\}$. The magnitude of the Jacobian is given as

$$|J| = \text{mag}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \frac{1}{2|\hat{a}\hat{b}|},$$

where $g(x, y) = \frac{x+y}{2\hat{a}}$, $h(x, y) = \frac{x-y}{2\hat{b}}$. With $\sigma_1 \triangleq \sigma\sqrt{1-\rho}/\sqrt{2}\hat{a}$, $\sigma_2 \triangleq \sigma\sqrt{1+\rho}/\sqrt{2}\hat{b}$. Hence,

$$\begin{aligned} f_{ZW}(z, w) &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{z^2}{\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\frac{w^2}{\sigma_2^2}} \\ &= f_Z(z) f_W(w). \end{aligned}$$

38. Let $g(x, y) \triangleq \frac{x^2+y^2}{2}$ and $h(x, y) \triangleq \frac{x^2-y^2}{2}$. The real roots of $g(x, y) = v$, $h(x, y) = w$ occur for $v \geq 0, |v| \geq |w|$ and are four in number.

$$\begin{aligned} x_1 &= +\sqrt{v+w}, \quad y_1 = +\sqrt{v-w} \\ x_2 &= -\sqrt{v+w}, \quad y_2 = +\sqrt{v-w} \\ x_3 &= -\sqrt{v+w}, \quad y_3 = -\sqrt{v-w} \\ x_4 &= +\sqrt{v+w}, \quad y_4 = -\sqrt{v-w}. \end{aligned}$$

We note that w can be negative, but never greater in magnitude than v. The magnitude of J is

$$\text{abs}\begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = 2|xy| = 2\sqrt{v^2 - w^2},$$

and this is the same for all the four roots. Now observe: $x^2 + y^2 = 2v$ for all the roots, and

$$x_1 y_1 = \sqrt{v^2 - w^2} = x_3 y_3$$

$$x_2 y_2 = -\sqrt{v^2 - w^2} = x_4 y_4$$

Hence

$$\begin{aligned} f_{VW}(v, w) &= \frac{1}{4\pi\sqrt{v^2 - w^2}\sqrt{1-\rho^2}} 2\left\{ e^{-\left[\frac{2v - 2\sqrt{v^2-w^2}\rho}{2(1-\rho^2)}\right]} + e^{-\left[\frac{2v + 2\sqrt{v^2-w^2}\rho}{2(1-\rho^2)}\right]} \right\} \\ &= \frac{1}{\pi\sqrt{(1-\rho^2)(v^2 - w^2)}} e^{-\frac{v}{1-\rho^2}} \cosh\left(\frac{\rho\sqrt{v^2 - w^2}}{1-\rho^2}\right), \quad \text{for } v \geq 0, |v| \geq |w|, \end{aligned}$$

where the hyperbolic cosine function $\cosh(x) \triangleq \frac{1}{2}(e^x + e^{-x})$. For $v < 0$ or $|v| < |w|$, there are no real roots of the transformation equations, so that we have $f_{VW}(v, w) = 0$ there.

39. From Eq. 3.4-4 in Example 3.4-1, we have the linear transformation

$$v = x + y, w = x - y,$$

with the one solution, i.e. $n = 1$:

$$x = \phi(v,w) = \frac{v+w}{2}, y = \psi(v,w) = \frac{v-w}{2}.$$

Proceeding with the direct approach, the matrix in the Jacobian $\widetilde{J}$ becomes:

$$\begin{bmatrix} \frac{\partial \phi}{\partial v} & \frac{\partial \phi}{\partial w} \\ \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

with determinant or Jacobian given as

$$\widetilde{J} = \left| \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right| = -\frac{1}{2}.$$

So, the magnitude of the Jacobian is $\frac{1}{2}$. Then by Eq. 3.4-11, we can write

$$\begin{aligned} f_{VW}(v,w) &= |\widetilde{J}| f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right) \\ &= \frac{1}{2} f_{XY}\left(\frac{v+w}{2}, \frac{v-w}{2}\right), \end{aligned}$$

which agrees with Eq. 3.4-4 that was derived in the Example 3.4-1 using the indirect method.

40. Here the transformation is give as

$$\begin{aligned} z &= g(x,y) \triangleq x\cos\theta + y\sin\theta, \qquad (1) \\ w &= h(x,y) \triangleq x\sin\theta - y\cos\theta. \qquad (2) \end{aligned}$$

The random variables X and Y are independent Normal distributed as $N(0,1)$. To find the inverse, we can multiply (1) by $\cos\theta$ and then multiply (2) by $\sin\theta$ and add the results to get

$$x = z\cos\theta + w\sin\theta,$$

since $\cos^2\theta + \sin^2\theta = 1$. Similarly, we multiply (1) by $\sin\theta$ and multiply (2) by $\cos\theta$ and subtract to get

$$y = z\sin\theta - w\cos\theta.$$

Thus there is a unique inverse and one root at $(x,y) = (z\cos\theta + w\sin\theta, z\sin\theta - w\cos\theta)$. Calculating the Jacobian, we get

$$J = \det\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = -1, \quad \text{so that } |J| = 1.$$

The transformed pdf then becomes,

$$\begin{aligned} f_{Z,W}(z,w) &= f_{X,Y}(z\cos\theta + w\sin\theta, z\sin\theta - w\cos\theta)\cdot 1 \\ &= \frac{1}{2\pi}\exp -\frac{1}{2}\left((z\cos\theta + w\sin\theta)^2 + (z\sin\theta - w\cos\theta)^2\right) \\ &= \frac{1}{2\pi}\exp -\frac{1}{2}\left(z^2 + w^2\right), \qquad -\infty < z, w < +\infty. \end{aligned}$$

This independent Normal joint density is thus unchanged by this transformation! Note that this transformation is closely related to a rotation of Cartesian coordinates in the $x-y$ plane. In fact, if w were replaced in the transformation by $-w$, the transformation would be a coordinate rotation by angle $+\theta$ in the $x-y$ plane.

41. We look at the transformation problem for two independent Normal random variables X and $Y\ : N(0,\sigma^2)$, transformed to $Z \triangleq X^2+Y^2$ and $W \triangleq 2Y$. We thus have

$$z = g(x,y) = x^2+y^2 \quad \text{and} \quad w = h(x,y) = 2y.$$

This is a non-invertible transformation with two real roots, for $|w| < \sqrt{z}, z > 0$,

$$\begin{aligned} r_1 &: \quad x = +\sqrt{z-\left(\frac{w}{2}\right)^2}, y = \frac{w}{2}, \text{ and} \\ r_2 &: \quad x = -\sqrt{z-\left(\frac{w}{2}\right)^2}, y = \frac{w}{2}. \end{aligned}$$

Now at both roots the magnitude of the Jacobian is the same,

$$\begin{aligned} |J_1| &= |J_2| = 4\sqrt{z-\left(\frac{w}{2}\right)^2}, \text{ where} \\ J_{1,2} &= \det\begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = 4x = \pm 4\sqrt{z-\left(\frac{w}{2}\right)^2}. \end{aligned}$$

Hence

$$\begin{aligned} f_{Z,W}(z,w) &= \frac{1}{4\sqrt{z-\left(\frac{w}{2}\right)^2}}\left(f_{X,Y}(\sqrt{z-\left(\frac{w}{2}\right)^2},\frac{w}{2}) + f_{X,Y}(-\sqrt{z-\left(\frac{w}{2}\right)^2},\frac{w}{2})\right) \\ &= \begin{cases} \frac{1}{4\pi\sigma^2}\frac{1}{\sqrt{z-\left(\frac{w}{2}\right)^2}}\exp(-z/2\sigma^2), & |\frac{w}{2}| < \sqrt{z}, z>0 \\ 0, & \text{else.} \end{cases} \end{aligned}$$

42. First we determine the value of A

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy \\ &= A\int_0^1\int_{-1}^{+1}(x^2+y^2)dxdy \\ &= A\int_0^1\left(\int_{-1}^{+1}(x^2+y^2)dy\right)dx \\ &\Rightarrow A = \frac{3}{4} \end{aligned}$$

Hence $f_{XY}(x,y) = \frac{3}{4}(x^2+y^2)$ with support on $0 \le x \le 1, -1 \le y \le +1$.

(a) i) For $x<0, y<-1$,

$$F_{XY}(x,y) = 0,$$

i) For $x > 1, y > 1,$

$$F_{XY}(x,y) = 1.$$

iii) $\quad 0 \le x \le 1, y > 1,$

$$\begin{aligned} F_{XY}(x,y) &= \frac{3}{4}\int_0^x \left(\int_{-1}^{+1}(u^2+v^2)dv\right)du \\ &= \frac{3}{4}\left[\frac{x^3}{3}y + \frac{y^3}{3}x\Big|_{-1}^{+1}\right] \\ &= \frac{3}{4}\left[\frac{x^3}{3} + \frac{x}{3} - \left(-\frac{x^3}{3} - \frac{x}{3}\right)\right] \\ &= \frac{3}{4}\left(\frac{2}{3}x^3 + \frac{2}{3}x\right) \\ &= \frac{1}{2}(x^3+x). \end{aligned}$$

iv) For $0 \le x \le 1, -1 < y < 1,$

$$\begin{aligned} F_{XY}(x,y) &= \frac{3}{4}\int_0^x \left(\int_{-1}^{y}(u^2+v^2)dv\right)du \\ &= \frac{1}{4}\left[x^3(y+1) + x(y^3+1)\right] \end{aligned}$$

v) For $x > 1, -1 < y < 1,$

$$\begin{aligned} F_{XY}(x,y) &= \frac{3}{4}\int_0^1 \left(\int_{-1}^{y}(u^2+v^2)dv\right)du \\ &= \frac{1}{4}\left[(y+1) + (y^3+1)\right]. \end{aligned}$$

43. (a) The range of Y, R_Y is a subset of the real line R^1 if Y is a real-valued random variable.

(b) A reasonable probability space for X is $(\Omega, \mathcal{F}, P)$ where Ω is the sample space of the underlying experiment, $\mathcal{F}$ is the $\sigma-$field of events defined on Ω, and P is the set function that assigns to every set (event) $E \in \mathcal{F}$, the number $P[E]$. A reasonable probability space for Y is $(R_X, \mathcal{B}, P_X)$ where R_X, the range of X, is the induced sample space under the mapping X, $\mathcal{B}$ is the Borel field of events over R^1, and P_X is a set function that assigns to every set $B \in \mathcal{B}$, the number $P_X[B]$.

(c) The event $\{\varsigma | Y(\varsigma) \le y\}$ under the mapping Y is the set $(-\infty, y] \in R^1$.

(d) In the I/O viewpoint, the inverse image is computed as follows:

$$\begin{aligned} \{Y \le y\} &= \{2X + 3 \le y\} \\ &= \left\{X \le \frac{y-3}{2}\right\} \\ &= \left(-\infty, \frac{y-3}{2}\right]. \end{aligned}$$

44. Define the event $A = \{\max(T_1, T_2) \leq t\}$. Then

$$F_Y(y,t) = P[Y \leq y|A]P[A] + P[Y \leq y|A^c]P[A^c].$$

Now

$$\begin{aligned} P[A] &= P[T_1 \leq t]P[T_2 \leq t], \quad \text{by independence,} \\ &= (1 - e^{-\lambda t})u(t)(1 - e^{-\lambda t})u(t) \\ &= (1 - e^{-\lambda t})^2 u(t), \end{aligned}$$

and so $P[A^c] = 1 - (1 - e^{-\lambda t})^2 u(t)$. Observe that $Y = 0$ at t given event A, while $Y = X$ at t given event A^c. So

$$P[Y \leq y|A] = u(y) \quad \text{and} \quad P[Y \leq y|A^c] = F_X(y).$$

Hence,

$$\begin{aligned} F_Y(y,t) &= P[Y \leq y|A]P[A] + P[Y \leq y|A^c]P[A^c] \\ &= u(y)(1 - e^{-\lambda t})^2 u(t) + F_X(y)(1 - (1 - e^{-\lambda t})^2 u(t)) \\ &= (1 - e^{-\lambda t})^2 u(y)u(t) + F_X(y)(1 - (1 - e^{-\lambda t})^2 u(t)). \end{aligned}$$

For any fixed $t > 0$ and $y = \infty$, we see $F_Y(\infty, t) = (1 - e^{-\lambda t})^2 + 1(1 - (1 - e^{-\lambda t})^2) = 1$ and for $t = \infty$, and any fixed y, we have $F_Y(y, \infty) = u(y)$. Does this make sense? Think about it!