Chapter 8 solutions

1. We recall that the set $\{A_n\}_{n=1}^N = \{A_1, A_2, ..., A_N\}$, and wish to prove the chain rule

$$P[A_1A_2\cdots A_N] = P[A_1]P[A_2|A_1]\cdots P[A_N|A_1A_2\cdots A_{N-1}]. \tag{1}$$

We can use mathematical induction in this case. By definition of conditional probability, we know

$$P[A_1A_2] = P[A_1]P[A_2|A_1], \tag{2}$$

thus the proposition is true for $N = 2$. Following mathematical induction, we assume that (1) is true for some positive integer $N = K$, for simplicity denoting the joint event $B \triangleq A_1A_2\cdots A_K$, then by using (2), we have

$$\begin{aligned} P[A_1A_2\cdots A_{K+1}] &= P[BA_{K+1}] \\ &= P[B]P[A_{K+1}|B] \\ &= P[A_1A_2\cdots A_K]P[A_{K+1}|A_1A_2\cdots A_K] \\ &= P[A_1]P[A_2|A_1]\cdots P[A_K|A_1A_2\cdots A_{K-1}]P[A_{K+1}|A_1A_2\cdots A_K], \end{aligned}$$

since (1) is assumed true for K. Thus we have shown that (1) being true for $N = K$ implies that it is true also for $N = K + 1$. Thus by the principle of mathematical induction, since we also know (1) is true for $N = 2$, it must be true for all positive integers N.

Expressing this result in terms of joint CDFs $F(x_1, x_2, ..., x_N)$, we have

$$F(x_1, x_2, ..., x_N) = F(x_1)F(x_2|x_1)\cdots F(x_N|x_1, x_2, ..., x_{N-1}).$$

For joint pdf's $f(x_1, x_2, ..., x_N)$, we similarly have

$$f(x_1, x_2, ..., x_N) = f(x_1)f(x_2|x_1)\cdots f(x_N|x_1, x_2, ..., x_{N-1}).$$

2. Given the $N-$dimensional vector $(x_1, x_2, ..., x_N)$ whose components are *pairwise independent*,

$$\text{i.e.}\quad f(x_i, x_j) = f(x_i)f(x_j) \quad \text{for all } i \neq j,$$

we want to show that it is possible that,

$$f(x_1, x_2, ..., x_N) \neq f(x_1)f(x_2)\cdots f(x_N)$$

i.e. *joint independence* does not follow. Consider a case with $N = 3$: $f(x_3, x_2, x_1)$. By the chain rule for pdf's we then have $f(x_3, x_2, x_1) = f(x_3|x_2, x_1)f(x_2|x_1)f(x_1)$ and from pairwise independence we have $f(x_2, x_1) = f(x_2)f(x_1)$, $f(x_3, x_1) = f(x_3)f(x_1)$, and $f(x_3, x_2) = f(x_3)f(x_2)$, substituting in, we conclude

$$f(x_3, x_2, x_1) = f(x_3|x_2, x_1)f(x_2)f(x_1).$$

The question is now whether $f(x_3, x_1) = f(x_3)f(x_1)$, and $f(x_3, x_2) = f(x_3)f(x_2)$ provide enough information to conclude $f(x_3|x_2, x_1) = f(x_3)$. Alas, this is not so. [1]

[1] There is one exception to this and that is the case where the RVs are jointly Gaussian distributed.

Here is a specific counterexample: Let X_1 and X_2 be two independent RVs, each uniformly distributed on the interval $[-\pi, +\pi]$, i.e. $X_i : U[-\pi, +\pi], i = 1, 2$. In terms of pdf's, we have

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{2\pi}, & |x_i| \leq \pi, \\ 0, & \text{else.} \end{cases}$$

Next, define a third RV by $X_3 \triangleq (X_1 + X_2) \bmod \pi$, meaning

$$X_3 = \begin{cases} X_1 + X_2 - 2\pi, & X_1 + X_2 > \pi, \\ X_1 + X_2, & |X_1 + X_2| \leq \pi, \\ X_1 + X_2 + 2\pi, & X_1 + X_2 < -\pi. \end{cases}$$

Upon some reflection, we see

$$f_{X_3|X_1}(x_3|x_1) = \frac{1}{2\pi}, \quad |x_3| \leq \pi,$$

and the same for $f_{X_3|X_2}$, and thus since X_1 and X_2 are independent, we can conclude that X_1, X_2, X_3 are pairwise independent. However, by the definition of X_3, we see that $(X_1 + X_2) \bmod \pi$ determines X_3, specifically

$$f_{X_3|X_1,X_2}(x_3|x_1, x_2) = \delta(x_3 - (x_1 + x_2) \bmod \pi).$$

Thus, joint independence does not prevail.

3. We are given $X_i = X_{i-1} + B_i = \sum_{j=1}^{i} B_j, \ \ 1 \leq i \leq 5.$

(a) Thus

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(b) Writing $\mathbf{B}^T = (B_1, B_2, B_3, B_4, B_5)^T$, we have $\boldsymbol{\mu}_{\mathbf{X}} = E[\mathbf{AB}] = \mathbf{A}\boldsymbol{\mu}_{\mathbf{B}} = \frac{1}{2}\mathbf{A1}$ where $\mathbf{1}$ is a column vector of all 1s.

(c) $\mathbf{K_B} = E[\mathbf{B}_c\mathbf{B}_c^T]$ where $\mathbf{B}_c \triangleq \mathbf{B} - \boldsymbol{\mu}_{\mathbf{B}} = \mathbf{B} - \frac{1}{2}\mathbf{1}$. Now

$$(B_c)_i = \begin{cases} +\frac{1}{2}, & p = \frac{1}{2}, \\ -\frac{1}{2}, & p = \frac{1}{2}, \end{cases}$$

thus $E[(B_c)_i^2] = \frac{1}{4}$ and $E[(B_c)_i\,(B_c)_j] = 0$ for $i \neq j$, thus

$$\mathbf{K_B} = \frac{1}{4}\mathbf{I}.$$

(d)

$$
\begin{aligned}
\mathbf{K_X} &= E[(\mathbf{AB}_c)\,(\mathbf{AB}_c)^T] \\
&= \mathbf{A}E[\mathbf{B}_c\mathbf{B}_c^T]\mathbf{A}^T \\
&= \mathbf{AK_BA}^T \\
&= \frac{1}{4}\mathbf{AA}^T \\
&= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.
\end{aligned}
$$

4. (a) Yes. The θ_k are then N outcomes ς (zeta) in the sample space $\Omega = \{\theta_k\}_{k=0}^{N-1}$. The field $\mathcal{F}$ of events is the collection of all 2^N subsets of Ω. The P measure of the event $E \in \mathcal{F}$ is $P[E] = n/N$ where $n = \#$ of outcomes in E. The random variables $X(n, \varsigma)$ map the sample space Ω into the linear space of real-valued sequences.

(b)

$$
\begin{aligned}
E[X[n]] &= E\left[\cos\left(\frac{2\pi n}{5} + \Theta\right)\right] \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\cos\left(\frac{2\pi n}{5} + \frac{2\pi k}{N}\right) \\
&= \cos\left(\frac{2\pi n}{5}\right)\left(\frac{1}{N}\sum_{k=0}^{N-1}\cos\left(\frac{2\pi k}{N}\right)\right) - \sin\left(\frac{2\pi n}{5}\right)\left(\frac{1}{N}\sum_{k=0}^{N-1}\sin\left(\frac{2\pi k}{N}\right)\right) \\
&= \cos\left(\frac{2\pi n}{5}\right)\cdot 0 - \sin\left(\frac{2\pi n}{5}\right)\cdot 0 \\
&= 0.
\end{aligned}
$$

Here, we first used the trigonometry formula for the cosine of the sum of two angles $\cos(A+B)$ and then made indirect use of the formula for the sum of a geometric series, in this case powers of $e^{\frac{j2\pi}{N}}$, as:

$$
\begin{aligned}
\sum_{k=0}^{N-1} e^{\frac{j2\pi k}{N}} &= \sum_{k=0}^{N-1}\cos\left(\frac{2\pi k}{N}\right) - j\left(\frac{1}{N}\sum_{k=0}^{N-1}\sin\left(\frac{2\pi k}{N}\right)\right) \\
&= \frac{1 - e^{j2\pi}}{1 - e^{\frac{j2\pi}{N}}} = 0 + j0.
\end{aligned}
$$

(c)

$$
\begin{aligned}
E[X[n]X[m]] &= E\left[\cos\left(\frac{2\pi n}{5}+\Theta\right)\cos\left(\frac{2\pi m}{5}+\Theta\right)\right] \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\cos\left(\frac{2\pi n}{5}+\frac{2\pi k}{N}\right)\cos\left(\frac{2\pi m}{5}+\frac{2\pi k}{N}\right) \\
&= \frac{1}{2}\cos\left(\frac{2\pi(n-m)}{5}\right)\cdot\left(\frac{1}{N}\sum_{k=0}^{N-1}\cos(0)\right)+\frac{1}{2}\cos\left(\frac{2\pi(n+m)}{5}\right)\cdot\left(\frac{1}{N}\sum_{k=0}^{N-1}\cos\left(\frac{4\pi k}{N}\right)\right) \\
&= \frac{1}{2}\cos\left(\frac{2\pi(n-m)}{5}\right)\cdot 1+\frac{1}{2}\cos\left(\frac{2\pi(n+m)}{5}\right)\cdot 0 \\
&= \frac{1}{2}\cos\left(\frac{2\pi(n-m)}{5}\right),
\end{aligned}
$$

where we have made use of the trigonometry formula $\cos A\cdot\cos B$ and also the geometric sum

$$
\begin{aligned}
\sum_{k=0}^{N-1} e^{\frac{j4\pi k}{N}} &= \sum_{k=0}^{N-1}\cos\left(\frac{4\pi k}{N}\right)-j\left(\frac{1}{N}\sum_{k=0}^{N-1}\sin\left(\frac{4\pi k}{N}\right)\right) \\
&= \frac{1-e^{j4\pi}}{1-e^{\frac{j4\pi}{N}}}=0+j0, \quad \text{for } N>2.
\end{aligned}
$$

5. We know that an RV X is defined as a mapping from a sample space Ω to the real line R^1, written symbolically as

$$X:\Omega\longrightarrow R^1,$$

with field of events $\mathcal{F}$ and probability measure P assigned to these events. Let us assume in this case that our sample space is a copy the real line itself R^1. We can then write $X: R^1\longrightarrow R^1$ or $X(x)=x$ for all $x\in R^1$. Let the field be the Borel field of subsets of R^1 generated by the intervals $(a,b]$ for all $a<b$ which are half-open half-closed. Then any interval in R^1 can be given (represented) by using countable intersections of these intervals or their complements. For example: $(a,\infty)=(-\infty,a]^c$ and $(a,b)=\lim_{n\longrightarrow\infty}(a,b-\frac{1}{n}]$. We can also assign a probability to these events (in this case intervals) and therefore all together we have created an underlying probability space $(\Omega,\mathcal{F},P)$. To do this we use the CDF of the RV X and write

$$P[(a,b]]=F_X(b)-F_X(a).$$

6. (a) Let events S_1 and S_2 be defined as follows for two times $t_2>t_1>0$:

$$
\begin{aligned}
S_1 &\triangleq \{\text{no photon emitted prior to time } t_1\} \\
S_2 &\triangleq \{\text{ at least one photon emitted prior to time } t_2\}.
\end{aligned}
$$

By definition

$$
\begin{aligned}
P[S_2|S_1] &= \frac{P[S_2S_1]}{P[S_1]} \text{ and} \\
P[S_1] &= 1-\int_0^{t_1}\lambda e^{-\lambda t}dt=e^{-\lambda t_1}.
\end{aligned}
$$

Thus

$$P[S_2S_1] = \int_{t_1}^{t_2} \lambda e^{-\lambda t} dt = e^{-\lambda t_1} - e^{-\lambda t_2}$$

and so

$$\begin{aligned} P[S_2|S_1] &= \frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda t_1}} \\ &= 1 - e^{-\lambda(t_2 - t_1)}. \end{aligned}$$

(b) Let us define four events as follows:

$$\begin{aligned} A &\triangleq \{\text{at least one photon emitted prior to time } t_2 \text{ from 3 independent sources}\}, \\ S_1 &\triangleq \{\text{no photon emitted from source 1 prior to time } t_2\}, \\ S_2 &\triangleq \{\text{no photon emitted from source 2 prior to time } t_2\}, \quad \text{and} \\ S_3 &\triangleq \{\text{no photon emitted from source 3 prior to time } t_2\}. \end{aligned}$$

Then $P[A] = 1 - P[S_1S_2S_3]$, and because the three sources are independent $P[S_1S_2S_3] = P[S_1]P[S_2]P[S_3]$. Furthermore $P[S_i] = 1 - \int_0^{t_2} \lambda e^{-\lambda t} dt = e^{-\lambda t_2}$. Thus

$$\begin{aligned} P[A] &= 1 - P[S_1]P[S_2]P[S_3] \\ &= 1 - e^{-3\lambda t_2}. \end{aligned}$$

7. (a) We use the general result $E[X] = E[[X|Y]]$. In this instance, it becomes

$$E[e^{j\omega X}] = E[E[e^{j\omega X}|M]].$$

Now $E[e^{j\omega X}|M = m] = \exp(j\omega m - \frac{1}{2}\sigma^2\omega^2)$. Therefore the characteristic function for X can be written as

$$\begin{aligned} \Phi_X(\omega) &= E[e^{j\omega X}] \\ &= E[\exp(j\omega M - \frac{1}{2}\sigma^2\omega^2)] \\ &= e^{-\frac{1}{2}\sigma^2\omega^2} E[\exp j\omega M] \\ &= e^{-\frac{1}{2}\sigma^2\omega^2} \Phi_M(\omega). \end{aligned}$$

Now $\Phi_M(\omega) = E[e^{j\omega M}] = \int_0^\infty e^{j\omega m} \lambda e^{-\lambda m} dm = \lambda/(\lambda - j\omega)$. Thus

$$\Phi_X(\omega) = \frac{\lambda e^{-\frac{1}{2}\sigma^2\omega^2}}{\lambda - j\omega}.$$

(b) For the mean we write

$$\begin{aligned} E[X] &= E[E[X|M]] \\ &= E[M], \end{aligned}$$

i.e. the mean of X is the mean of M, and for the variance

$$\begin{aligned} \sigma_X^2 &= E[X^2] - \mu_X^2 \\ &= E[\Sigma^2 + M^2] - \mu_X^2 \\ &= \mu_{\Sigma^2} + E[M^2] - \mu_X^2 \\ &= \mu_{\Sigma^2} + \sigma_M^2. \end{aligned}$$

8. (a) First we define the dummy RV $S \triangleq X+Y$, then we have $X=SR$ and $Y=S-SR$. Then $f_{R,S}(r,s)=f_{X,Y}(x,y)|J|$ for $0 \le r \le 1, 0 \le s < \infty$, and the Jacobian J is given as

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{bmatrix} = \det \begin{bmatrix} s & r \\ -s & 1-r \end{bmatrix} = s.$$

Since X and Y are independent

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y) \\ &= \lambda^2 e^{-\lambda(x+y)} \quad \text{(by the problem statement)} \\ &= \lambda^2 e^{-\lambda s}. \end{aligned}$$

So $f_{R,S}(r,s)=\lambda^2 s e^{-\lambda s}$. Then integrating out the variable s, we get

$$\begin{aligned} f_R(r) &= \int_0^\infty \lambda^2 s e^{-\lambda s} ds \\ &= 1. \end{aligned}$$

Remembering that the range of r is $[0,1]$, we have finally

$$f_R(r) = \begin{cases} 1, & 0 \le r \le 1, \\ 0, & \text{else.} \end{cases}$$

(b) Since $A \triangleq \{X < 1/Y\}$, we can write

$$\begin{aligned} P[X \le x | A, Y=y] &= \frac{P[X \le \min(x,1/y), Y=y]}{P[A, Y=y]} \quad \text{(since the RVs are continuous)} \\ &= \frac{P[X \le \min(x,1/y)]P[Y=y]}{P[Y=y]P[A|Y=y]} \\ &= \frac{P[X \le \min(x,1/y)]}{P[A|Y=y]} \quad \text{(cancelling like terms }^2\text{)} \\ &= \frac{P[X \le \min(x,1/y)]}{P[X \le 1/y]} \quad \text{(since } X \text{ is a continuous RV)} \\ &= \frac{1-e^{-\lambda \min(x,1/y)}}{1-e^{-\lambda/y}}. \end{aligned}$$

Thus the conditional pdf is the derivative of this CDF with respect to x. Taking this derivative, we obtain

$$\begin{aligned} f_X(x|A, Y=y) &= \frac{d}{dx}\left(\frac{1-e^{-\lambda \min(x,1/y)}}{1-e^{-\lambda/y}}\right) \\ &= \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda/y}} \quad \text{for } \; 0 < x \le 1/y. \end{aligned}$$

Elsewhere $f_X(x|A.Y=y)=0$, therefore the total solution is given as

$$f_X(x|A, Y=y) = \begin{cases} \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda/y}}, & 0 < x \le 1/y \\ 0, & \text{else.} \end{cases}$$

(c) Let $\widehat{X}$ denote the minimum mean-square error (MSE) estimate of X. Then

$$\begin{aligned}
\widehat{X} &= E[X|A, Y=y] \\
&= \int_0^\infty x f_X(x|A, Y=y)dx \\
&= \int_0^{1/y} x \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda/y}} dx \\
&= \frac{1}{\lambda} - \frac{e^{-\lambda/y}}{y\left(1-e^{-\lambda/y}\right)}, \qquad \text{for } \; y>0.
\end{aligned}$$

9. For two RVs of finite variance A and B, the Schwarz inequality can be written as $|E[AB^*]| \leq \sqrt{E[|A|^2]E[|B|^2]}$. So, let $A \triangleq X[n+m]$ and $B \triangleq X[n]$ to obtain $R_X[m] = E[X[n+m]X^*[n]] = E[AB^*]$, and also $E[|A|^2] = E[|B|^2] = R_X[0]$ by the WSS property of $X[n]$. Thus by Schwarz,

$$\begin{aligned}
|R_X[m]| &\leq \sqrt{R_X[0]R_X[0]} \\
&= R_X[0],
\end{aligned}$$

where we have used the fact that $R_X[0]$ is real-valued and non-negative.

10. We are given

$$X_i = \frac{2}{5}(X_{i-1} + X_{i+1}) + W_i \quad \text{for } 2 \leq i \leq 9,$$

plus $X_1 = \frac{1}{2}X_2 + \frac{5}{4}W_1$ and $X_{10} = \frac{1}{2}X_9 + \frac{5}{4}W_{10}$. Thus

$$\begin{aligned}
X_2 &= \frac{2}{5}\left(\frac{1}{2}X_2 + \frac{5}{4}W_1 + X_3\right) + W_2 \\
&= \frac{1}{5}X_2 + \frac{1}{2}W_1 + \frac{2}{5}X_3 + W_2.
\end{aligned}$$

Solving for X_2, we obtain

$$X_2 = \frac{5}{4}W_2 + \frac{5}{8}W_1 + \frac{1}{2}X_3. \tag{3}$$

Next, we take the X_3 equation, and use this result to eliminate X_2 obtaining

$$X_3 = \frac{5}{4}W_3 + \frac{5}{8}W_2 + \frac{5}{16}W_1 + \frac{1}{2}X_4. \tag{4}$$

We continue in this manner till $i=9$, and there obtain

$$X_9 = \frac{5}{4}W_9 + \frac{5}{8}W_8 + \frac{5}{16}W_7 + \cdots + \frac{5}{1024}W_1 + \frac{1}{2}X_{10}.$$

We now solve this equation together with $X_{10} = \frac{1}{2}X_9 + \frac{5}{4}W_{10}$ to finally obtain for X_9

$$X_9 = \frac{5}{6}W_{10} + \frac{5}{3}W_9 + \frac{5}{6}W_8 + \cdots + \frac{5}{768}W_1.$$

At this point, we can work backwards, eliminating the X_{i+1} term in such equations as (3-4). At this point, we may see the general form for this answer is

$$X_i = \frac{5}{3}\sum_{k=1}^{10} \rho^{|i-k|} W_k\,, \qquad 1 \leq i \leq 10, \tag{5}$$

which can then be used for answering (a-c). Alternatively, and in terms of vectors and matrices, upon setting $\mathbf{W} = \mathbf{TX}$, we can see from the hint given in this problem, that $\mathbf{T} = \beta^2\mathbf{A}^{-1}$ with $\beta = 3/5$, $\alpha = 2/5$, and $\rho = 1/2$. Then using the hint, we have

$$\begin{aligned}
\mathbf{X} &= \mathbf{T}^{-1}\mathbf{W} \\
&= \left(\beta^2\mathbf{A}^{-1}\right)^{-1}\mathbf{W} \\
&= \frac{1}{\beta^2}\mathbf{AW} \\
&= \frac{1+\rho^2}{1-\rho^2}\mathbf{AW} \\
&= \frac{5}{3}\mathbf{AW},
\end{aligned}$$

which is the matrix version of (5). Using the result (5), we now answer the questions.

(a) $E[X_i] = \frac{5}{3}\sum_{k=1}^{10}\rho^{|i-k|}E[W_k] = \sum 0 = 0$, since the W_k are zero mean. Thus the mean $\boldsymbol{\mu}_X = \mathbf{0}$.

(b) Since the mean is zero, the covariance and correlation matrices agree, and so

$$\begin{aligned}
(\mathbf{K_X})_{i,j} &= E[X_iX_j] \\
&= \left(\frac{5}{3}\right)^2 E\left[\sum_{k=1}^{10}\rho^{|i-k|}W_k\sum_{l=1}^{10}\rho^{|j-l|}W_l\right] \\
&= \left(\frac{5}{3}\right)^2 \sum_{k=1}^{10}\sum_{l=1}^{10}\rho^{|i-k|}\rho^{|j-l|}E[W_kW_l] \\
&= \left(\frac{5}{3}\right)^2 \sigma^2\sum_{k=1}^{10}\rho^{|i-k|}\rho^{|j-l|}\sigma^2\delta_{k-l}, \qquad \text{where } \delta \text{ is the Kronecker delta,} \\
&= \left(\frac{5}{3}\right)^2 \sigma^2\sum_{k=1}^{10}\sum_{l=1}^{10}\rho^{|i-k|}\rho^{|k-j|},
\end{aligned}$$

which can be given in matrix form as

$$\mathbf{K_X} = \left(\frac{5}{3}\right)^2\sigma^2\mathbf{AA}^T.$$

(c) Since the W_i are i.i.d. Laplacian with variance σ^2, we can write

$$\begin{aligned}
f_{\mathbf{W}}(\mathbf{w}) &= \prod_{i=1}^{10} f_{W_i}(w_i) \\
&= \left(\frac{1}{\sqrt{2}\sigma}\right)^{10}\exp\left(-\frac{\sqrt{2}}{\sigma}\sum_{i=1}^{10}|w_i|\right) \\
&= \left(\frac{1}{\sqrt{2}\sigma}\right)^{10}\exp\left(-\frac{\sqrt{2}}{\sigma}\mathbf{1}^T|\mathbf{w}|\right), \qquad \text{with } \mathbf{1} \text{ a vector of 1's.}
\end{aligned}$$

Now $\mathbf{W} = \mathbf{TX}$ and $\mathbf{X} = \mathbf{T}^{-1}\mathbf{W}$, so with $J = \det(\mathbf{T}^{-1}) = 1/\det\mathbf{T}$, we get

$$f_{\mathbf{X}}(\mathbf{x}) = \left(\frac{1}{\sqrt{2}\sigma|J|}\right)^{10}\exp\left(-\frac{\sqrt{2}}{\sigma}\mathbf{1}^T|\mathbf{Tx}|\right).$$

Here $\mathbf{T}$ is given as

$$\begin{aligned}\mathbf{T} &= \frac{3}{5}\mathbf{A}^{-1}\\ &= \begin{bmatrix} 6/5 & -2/5 & 0 & \dots & 0 \\ -2/5 & 1 & -2/5 & 0 & \dots \\ 0 & -2/5 & 1 & \dots & 0 \\ \dots & 0 & \dots & \dots & -2/5 \\ 0 & \dots & 0 & -2/5 & 6/5 \end{bmatrix}.\end{aligned}$$

11. To prove the Corollary to Theorem 8.1-1, note that the sequence of events is *decreasing* here, i.e. $B_1 \supset B_2 \supset B_3 \cdots$, so equivalently the sequence of complementary sets is increasing, i.e. $B_1^c \subset B_2^c \subset B_3^c \cdots$. So, if we apply Theorem 8.1-1 to the sequence of increasing events B_n^c, upon defining $B_\infty^c \triangleq \bigcup_{n=1}^{\infty} B_n^c$, we get that

$$\lim_{n\to\infty} P[B_n^c] = P[B_\infty^c] \ .$$

So

$$\begin{aligned}\lim_{n\to\infty} P[B_n] &= \lim_{n\to\infty} (1 - P[B_n^c]) \\ &= 1 - \lim_{n\to\infty} P[B_n^c] \\ &= 1 - P[B_\infty^c] \\ &= P[B_\infty].\end{aligned}$$

with $B_\infty \triangleq \bigcap_{n=1}^{\infty} B_n$ for this decreasing sequence of events. Note that by the definitions of infinite unions and intersections, $\bigcap_{n=1}^{\infty} B_n$ means the set of outcomes that are in *every* B_n and $\bigcup_{n=1}^{\infty} B_n^c$ means the set of outcomes that are in *any* of the B_n^c. Thus $\bigcup_{n=1}^{\infty} B_n^c = \left(\bigcap_{n=1}^{\infty} B_n\right)^c$, and the two expressions above for B_∞ are the same.

12. We have that $S_k \triangleq X_1 + X_2 + \ldots + X_k$ where the X_i are i.i.d. as $N(0,1)$. We start by writing

$$\begin{aligned}f_{S_n,S_m}(s_n, s_m) &= f_{S_m,S_n-S_m}(s_m, s_n - s_m), \qquad n > m \\ &= f_{S_m}(s_m)\, f_{S_n-S_m}(s_n - s_m),\end{aligned}$$

since for $n > m$, $S_n - S_m$ and S_n must be independent. Since they are both sums of i.i.d. standard Gaussians, they are also Gaussian, with means $E[S_m] = E[S_n - S_m] = 0$ and variances $\mathrm{Var}[S_m] = m$ and $\mathrm{Var}[S_n - S_m] = n - m (> 0)$. Hence

$$f_{S_n,S_m}(s_n, s_m) = \frac{1}{2\pi\sqrt{m(n-m)}} \exp - \left(\frac{s_m^2}{2m} + \frac{(s_n - s_m)^2}{2(n-m)}\right), \quad n > m \geq 1.$$

13. No, they need not be continuous from the left. For example, consider a discrete random variable X with the following PMF

$$P_X(x) = \begin{cases} \frac{1}{6}, & 1,2,3,4,5,6, \\ 0, & \text{else.} \end{cases}.$$

Let $F_X(x)$ be the corresponding CDF. Then, for example, $F_X(5) = \frac{5}{6}$, but for any $0 < \epsilon < 1$, we have $F_X(5-\epsilon) = \frac{4}{6}$. Therefore, $\lim_{\epsilon \longrightarrow 0} F_X(5-\epsilon) \neq F_X(5)$. Thus CDFs need not be continuous from the left.

14. (a) Denoting the outcomes as ζ_i, we have

$$\begin{aligned} \mu_X[n] &\triangleq E[X[n]] \\ &= \sum_{\zeta_i} P[\{\zeta_i\}] X[n, \zeta_i] \\ &= \frac{1}{3}\left(3\delta[n] + u[n-1] + \cos\frac{\pi n}{2}\right). \end{aligned}$$

(b)

$$\begin{aligned} R_X[m,n] &\triangleq E[X[m]X^*[n]] \\ &= \sum_{\zeta_i} P[\{\zeta_i\}] X[m, \zeta_i] X^*[n, \zeta_i] \\ &= \frac{1}{3}\left(9\delta[m]\delta[n] + u[m-1]u[n-1] + \cos\frac{\pi m}{2}\cos\frac{\pi n}{2}\right). \end{aligned}$$

(c) We can summarize the RVs $X[0]$ and $X[1]$ with the following table.

ζ_i	p	$X[0]$	$X[1]$
a	$\frac{1}{3}$	3	0
b	$\frac{1}{3}$	1	1
c	$\frac{1}{3}$	0	0

Thus $P[X[0] = 3, X[1] = 0] = P[\{a\}] = \frac{1}{3}$. The respective marginal probabilities are found as $P[X[0] = 3] = \frac{1}{3}$ and $P[X[1] = 0] = \frac{2}{3}$. Multiplying, we find

$$\begin{aligned} P[X[0] &= 3, X[1] = 0] = \frac{1}{3} \\ &\neq \frac{1}{3}\frac{2}{3} \\ &= P[X[0] = 3]P[X[1] = 0], \end{aligned}$$

therefore the RVs $X[0]$ and $X[1]$ are not independent.

15. (a) The random variables $X[n]$ and $X[n-1]$ are jointly Gaussian distributed with zero means and covariance matrix

$$\mathbf{K} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \quad \text{with} \quad |\rho| < 1.$$

The determinant of this matrix is $\det \mathbf{K} = \sigma^4(1-\rho^2)$, and the inverse matrix is found as

$$\mathbf{K}^{-1} = \frac{1}{\sigma^4(1-\rho^2)} \begin{bmatrix} \sigma^2 & -\rho\sigma^2 \\ -\rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

We can then write their joint pdf as

$$f_X(x_n, x_{n-1}) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}\left(x_n^2 - 2\rho x_n x_{n-1} + x_{n-1}^2\right)\right).$$

Also the marginal pdf for $X[n-1]$ is given directly as

$$f_X(x_{n-1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_{n-1}^2}{2\sigma^2}\right).$$

We can the write the conditional density

$$\begin{aligned}
f_X(x_n|x_{n-1}) &= \frac{f_X(x_n, x_{n-1})}{f_X(x_{n-1})} \\
&= \frac{1}{\sqrt{2\pi}\sigma\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma^2(1-\rho^2)}\left(x_n - \rho x_{n-1}\right)^2\right),
\end{aligned}$$

after recognizing the perfect square $x_n^2 - 2\rho x_n x_{n-1} + \rho^2 x_{n-1}^2 \equiv (x_n - \rho x_{n-1})^2$. Recognizing that this conditional density is $N(\rho x_{n-1}, \sigma^2(1-\rho^2))$, we can immediately write its conditional mean

$$E[X[n]|X[n-1]] = \rho X[n-1].$$

(b) This predictor minimizes the mean square error over all functions $gX[n-1))$, i.e. it minimizes $E[(X[n] - g(X[n-1])^2]$ over all functions g. c.f. Example 4.3-4.

16. (a)

$$\begin{aligned}
\mu_Y[n] &\triangleq E[Y[n]] \\
&= E\left[\sum_k h[k]X[n-k]\right] \\
&= \sum_k h[k]\mu_X[n-k] \\
&= \frac{1}{2}\mu_X[n] + \frac{1}{2}\mu_X[n-1].
\end{aligned}$$

(b)

$$\begin{aligned}
R_{YY}[n_1, n_2] &\triangleq E[Y[n_1]Y^*[n_2]] \\
&= \sum_{k,l} h[k]h^*[l]R_{XX}[n_1-k, n_2-l] \\
&= \frac{1}{4}\left(R_{XX}[n_1, n_2] + R_{XX}[n_1-1, n_2] + R_{XX}[n_1, n_2-1] + R_{XX}[n_1-1, n_2-1]\right).
\end{aligned}$$

(c)

$$\begin{aligned}
K_{YY}[n_1, n_2] &\triangleq E[(Y[n_1] - \mu_Y[n_1])(Y^*[n_2] - \mu_Y^*[n_2])] \\
&= R_{YY}[n_1, n_2] - \mu_Y[n_1]\mu_Y^*[n_2] \\
&= \frac{1}{4}\left(R_{XX}[n_1, n_2] + R_{XX}[n_1-1, n_2] + R_{XX}[n_1, n_2-1] + R_{XX}[n_1-1, n_2-1]\right) \\
&\quad - \left(\frac{1}{2}\mu_X[n_1] + \frac{1}{2}\mu_X[n_1-1]\right)\left(\frac{1}{2}\mu_X^*[n_2] + \frac{1}{2}\mu_X^*[n_2-1]\right) \\
&= \frac{1}{4}\left(K_{XX}[n_1, n_2] + K_{XX}[n_1-1, n_2] + K_{XX}[n_1, n_2-1] + K_{XX}[n_1-1, n_2-1]\right).
\end{aligned}$$

(d) Set $\mathbf{Y} \triangleq \begin{bmatrix} Y[n_1] \\ Y[n_2] \end{bmatrix}$, then $\mathbf{Y}$ is distributed as $N(\boldsymbol{\mu}_Y, \mathbf{K}_{YY})$, where

$$\boldsymbol{\mu}_Y = \begin{bmatrix} \mu_Y[n_1] \\ \mu_Y[n_2] \end{bmatrix} \quad \text{and} \quad \mathbf{K}_{YY} = \begin{bmatrix} K_{YY}[n_1, n_1] & K_{YY}[n_1, n_2] \\ K_{YY}[n_2, n_1] & K_{YY}[n_2, n_2] \end{bmatrix}.$$

The pdf of vector $\mathbf{Y}$ is given as

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)|\det \mathbf{K}_{YY}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}_Y)^T \mathbf{K}_{YY}^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y)\right).$$

17. We need the joint pdf $f_T(t_2, t_1; 10, 5)$. Now

$$\begin{aligned} T[10] &= \sum_{k=1}^{10} \tau[k] \\ &= T[5] + \sum_{k=6}^{10} \tau[k]. \end{aligned}$$

Calling $X \triangleq \sum_{k=6}^{10} \tau[k]$, we see that by definition, X and $T[5]$ are independent. This since $T[5]$ is a sum of earlier $\tau[k]'s$ not included in the sum that is X. Thus

$$\begin{aligned} f_T(t_2, t_1; 10, 5) &= f_T(t_2|t_1; 10, 5) f_T(t_1; 5) \\ &= f_T(t_2 - t_1; (10 - 5)) f_T(t_1; 5) \\ &= f_T(t_2 - t_1; 5) f_T(t_1; 5) \\ &= \frac{(\lambda (t_2 - t_1))^4}{4!} \lambda e^{-\lambda(t_2 - t_1)} \frac{(\lambda t_1)^4}{4!} \lambda e^{-\lambda t_1}, \quad t_2 \geq t_1 \geq 0. \end{aligned}$$

18. (a)

$$T[n] = \sum_{k=1}^{n} \tau[k], \qquad n \geq 1.$$

$$\begin{aligned} \Phi_T(\omega; n) &\triangleq E[e^{j\omega T[n]}] \\ &= \prod_{k=1}^{n} E\left[e^{+j\omega\tau[k]}\right] \\ &= (\Phi_\tau(\omega))^n. \end{aligned}$$

In turn

$$\begin{aligned} \Phi_\tau(\omega) &= E[e^{j\omega\tau}] \\ &= \int_0^\infty \lambda e^{-\lambda\tau} e^{+j\omega\tau} d\tau \\ &= \lambda \left(\left. \frac{e^{(-\lambda + j\omega)\tau}}{-\lambda + j\omega} \right|_0^\infty \right) \\ &= \frac{\lambda}{\lambda - j\omega}, \quad \text{since} \quad \lambda > 0. \end{aligned}$$

Thus

$$\Phi_T(\omega; n) = \left(\frac{\lambda}{\lambda - j\omega}\right)^n.$$

(b)

$$\begin{aligned}
\mu_T[n] &= m_1 \\
&= \frac{1}{j}\Phi_T^{(1)}(0; n) \\
&= \frac{1}{j}\lambda^n \left(\left.\frac{(\lambda - j\omega)^n \cdot 0 + nj\,(\lambda - j\omega)^{n-1}}{(\lambda - j\omega)^{2n}}\right|_{\omega=0} \right) \\
&= \frac{1}{j}\lambda^n \frac{nj}{\lambda^{n+1}} \\
&= \frac{n}{\lambda}, \qquad n \geq 1.
\end{aligned}$$

This answer is correct because each of the n interarrival times $\tau[k]$ has average value $1/\lambda$.

19. No. Since $\tau[n]$ and $\tau[n-1]$ can have any joint pdf. which has not been specified. If $\tau[n]$ has independent increments and if the increments $\tau[n] - \tau[n-1]$ are identically distributed, then because of $T[n] = \sum_{k=1}^{n} \tau[k]$, it follows that

$$\Phi_T(\omega) = \Phi_\tau^n(\omega), \text{ which implies } \quad \Phi_\tau(\omega) = \sqrt[n]{\Phi_T(\omega)},$$

where we take the positive real nth root since it must hold that $\Phi_\tau(0) = 1$. By this result, we have that $\tau[n]$ is exponentially distributed with parameter λ. In this case, the joint pdf would be

$$\begin{aligned}
f_T(t_n, t_{n-1}; n, n-1) &= f_T(t_{n-1}; n-1) f_\tau(t_n - t_{n-1}; n) \\
&= \frac{(\lambda t_{n-1})^{n-2}}{(n-2)!}\lambda e^{-\lambda t_{n-1}} \cdot \lambda e^{-\lambda(t_n - t_{n-1})} u(t_n - t_{n-1}) u(t_{n-1}).
\end{aligned}$$

Actually, the following weaker assumption will do here: that $T[n-1]$ and $T[n] - T[n-1]$ are independent for all $n \geq 1$.

20. We have $X[n, \boldsymbol{\zeta}] = \sum_{i=-\infty}^{i=+\infty} A(\zeta_i) h[n-i]$, with $h[n] = \begin{cases} 1/4 & n = -1 \\ 1/2 & n = 0 \\ 1/4 & n = +1 \\ 0 & \text{else,} \end{cases}$, thus

(a) $E[X[n]] = \sum_{i=-1}^{i=+1} E[A_i] h[n-i] = \lambda(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}) = \lambda = \mu$. Here, for the vector outcome $\boldsymbol{\zeta}$, with ζ_i as it's $\mathrm{i^{th}}$ component, we call the random variable $A(\zeta_i) \triangleq A_i$.

(b) $\sigma_X^2(n) = E[(X[n]-\lambda)^2] = E[X^2[n]] - \lambda^2$, with

$$
\begin{aligned}
E[X^2[n]] &= \sum_{i=n-1}^{i=n+1}\sum_{j=n-1}^{j=n+1} E[A_iA_j]h[n-i]h[n-j] \\
&= \sum_{i=n-1}^{i=n+1} E[A_i^2]h^2[n-i] + \sum_{i\neq j} E[A_i]E[A_j]h[n-i]h[n-j] \\
&= (\lambda^2+\lambda)\left(\frac{1}{16}+\frac{1}{4}+\frac{1}{16}\right) + \lambda^2\left(\frac{1}{4}\frac{1}{2}+\frac{1}{4}\frac{1}{4}+\frac{1}{2}\frac{1}{4}+\frac{1}{4}\frac{1}{2}+\frac{1}{4}\frac{1}{4}\right) \\
&= \frac{3}{8}(\lambda^2+\lambda) + \frac{5}{8}\lambda^2 = \frac{3}{8}\lambda + \lambda^2.
\end{aligned}
$$

Thus $\sigma_X^2[n] = \frac{3}{8}\lambda + \lambda^2 - \lambda^2 = \frac{3}{8}\lambda = \sigma^2$ and $X[n] : N(\lambda, 3\lambda/8)$.

(c) Since the $X's$ will be correlated, we need to specify the correlation coefficient ρ to complete the expression for this joint pdf.

$$
\begin{aligned}
E[X[n]X[n+1]] &= E\left[\sum_{i=n-1}^{i=n+1} A_ih[n-i]\sum_{j=n}^{j=n+2} A_jh[n+1-j]\right] \\
&= \sum_{i=n-1}^{i=n+1}\sum_{j=n}^{j=n+2} E[A_iA_j]h[n-i]h[n+1-j] \\
= (\lambda^2+\lambda)\left(\frac{1}{8}+\frac{1}{8}\right) &+ \lambda^2\left(\frac{1}{4}\frac{1}{4}+\frac{1}{4}\frac{1}{2}+\frac{1}{4}\frac{1}{4}+\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{4}+\frac{1}{4}\frac{1}{4}+\frac{1}{4}\frac{1}{4}\right) \\
&= \frac{1}{4}\lambda^2 + \frac{1}{4}\lambda + \frac{3}{4}\lambda^2 \\
&= \frac{1}{4}\lambda + \lambda^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\rho &= \frac{\operatorname{cov}[X[n]X[n+1]]}{\sigma_X[n]\sigma_X[n+1]} \\
&= \frac{E[X[n]X[n+1]] - \lambda^2}{\sigma^2} \\
&= \frac{2}{3}.
\end{aligned}
$$

So, the joint pdf becomes

$$
\begin{aligned}
f_{X[n],X[n+1]}(x_1,x_2) &= \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu)^2}{\sigma^2} - 2\rho\frac{(x_1-\mu)(x_2-\mu)}{\sigma^2}\frac{(x_2-\mu)^2}{\sigma^2}\right]\right\} \\
&= \frac{1}{2\pi\frac{3}{8}\lambda\sqrt{1-\left(\frac{2}{3}\right)^2}}\exp\left\{-\frac{1}{2(1-\left(\frac{2}{3}\right)^2)}\left[\frac{(x_1-\lambda)^2}{\frac{3}{8}\lambda} - 2\frac{2}{3}\frac{(x_1-\lambda)(x_2-\lambda)}{\frac{3}{8}\lambda}\frac{(x_2-\lambda)^2}{\frac{3}{8}\lambda}\right]\right\} \\
&= \frac{1}{\frac{\sqrt{5}}{4}\pi\lambda}\exp\left\{-\frac{24}{10}\left[\frac{(x_1-\lambda)^2}{\lambda} - \frac{4}{3}\frac{(x_1-\lambda)(x_2-\lambda)}{\lambda}\frac{(x_2-\lambda)^2}{\lambda}\right]\right\}.
\end{aligned}
$$

(Alternate simple solution to parts a and b using CFs)

$$\begin{aligned}\Phi_X(\omega) &= E[e^{j\omega X}] \\ &= E\left[e^{+j\omega(\frac{1}{4}A_{-1}+\frac{1}{2}A_0+\frac{1}{4}A_1)}\right] \\ &= \Phi_A^2(\frac{\omega}{4})\Phi_A(\frac{\omega}{2}).\end{aligned}$$

Now $\Phi_A(\omega) = E[e^{+j\omega A}] = \exp(j\omega\lambda - \frac{1}{2}\omega^2\lambda)$ since $A : N(\lambda, \lambda)$. Thus

$$\Phi_A(\frac{\omega}{4}) = \exp(j\omega\frac{\lambda}{4} - \frac{1}{2}\omega^2\frac{\lambda}{16}) \text{ and } \Phi_A(\frac{\omega}{2}) = \exp(j\omega\frac{\lambda}{2} - \frac{1}{2}\omega^2\frac{\lambda}{4}),$$

so

$$\Phi_X(\omega) = \exp\left(j\omega\lambda - \frac{1}{2}\omega^2\frac{3}{8}\lambda\right) \quad \text{and so} \quad X[n] : N(\lambda, 3\lambda/8).$$

21. First we compute

$$\begin{aligned}f_X(x_1) &= \int_{-\infty}^{+\infty} f_X(x_1|x_0)\delta(x_0)dx_0 \\ &= \alpha e^{-\alpha x_1}u(x_1).\end{aligned}$$

Then

$$\begin{aligned}f_X(x_2) &= \int_{-\infty}^{+\infty} f_X(x_2|x_1)f_X(x_1)dx_1 \\ &= \int_{-\infty}^{+\infty} \alpha e^{-\alpha(x_2-x_1)}u(x_2-x_1)\alpha e^{-\alpha x_1}u(x_1)dx_1 \\ &= \alpha^2 e^{-\alpha x_2}\int_{-\infty}^{+\infty} u(x_2-x_1)u(x_1)dx_1 \\ &= \alpha^2 e^{-\alpha x_2}u(x_2)\left(\int_0^{x_2} dx_1\right) \\ &= \alpha^2 x_2 e^{-\alpha x_2}u(x_2).\end{aligned}$$

(b) Doing this again for $n = 3$, we would get $f_X(x_3) = \frac{\alpha^3 x_3^2}{2!}e^{-\alpha x_3}u(x_3)$, thus we guess the Erlang pdf

$$f_X(x_n) = \frac{\alpha^n x_n^{n-1}}{(n-1)!}e^{-\alpha x_n}u(x_n).$$

Then, using mathematical induction, we must calculate

$$
\begin{aligned}
f_X(x_n) &= \int_{-\infty}^{+\infty} f_X(x_n|x_{n-1}) f_X(x_{n-1}) dx_{n-1} \\
&= \int_{-\infty}^{+\infty} \alpha e^{-\alpha(x_n - x_{n-1})} u(x_n - x_{n-1}) f_X(x_{n-1}) dx_{n-1} \\
&= \alpha e^{-\alpha x_n} \left(\int_0^{x_n} e^{\alpha x_{n-1}} f_X(x_{n-1}) dx_{n-1} \right) u(x_n) \\
&= \alpha e^{-\alpha x_n} \left(\int_0^{x_n} e^{\alpha x_{n-1}} \frac{\alpha^{n-1} x_{n-1}^{n-2}}{(n-2)!} e^{-\alpha x_{n-1}} dx_{n-1} \right) u(x_n) \\
&= \frac{\alpha^n}{(n-2)!} e^{-\alpha x_n} \left(\int_0^{x_n} x_{n-1}^{n-2} dx_{n-1} \right) u(x_n) \\
&= \frac{\alpha^n}{(n-2)!} e^{-\alpha x_n} \frac{x_n^{n-1}}{n-1} u(x_n) \\
&= \frac{\alpha^n x_n^{n-1}}{(n-1)!} e^{-\alpha x_n} u(x_n), \qquad \text{as was to be shown.}
\end{aligned}
$$

22. Let the system be represented by operator L as $y[n] = L\{x[n]\}$. From the definition$h[n] = L\{\delta[n]\}$ with $\delta[n]$ being the discrete time impulse function $\delta[n] \triangleq \begin{cases} 1, & n = 0, \\ 0, & \text{else.} \end{cases}$ Next, using the shifting representation, we write the input sequence as $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$. Then we can compute

$$
\begin{aligned}
y[n] &= L\{x[n]\} \\
&= L\left\{ \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \right\} \\
&= \sum_{k=-\infty}^{+\infty} x[k] L\{\delta[n-k]\}, \qquad \text{by linearity for a continuous operator } L, \\
&= \sum_{k=-\infty}^{+\infty} x[k] h[n-k]. \\
&= x[n] * h[n].
\end{aligned}
$$

Therefore $Y[n] = X[n] * h[n]$ too. Note that in order to interchange the operator L and the infinite summation operator $\sum_{k=-\infty}^{+\infty}$, we generally need that $h[n]$ be absolutely summable, i.e. $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$, a stable system. Stable operators L are *continuous* in the sense that a small change in the input sequence x results in a bounded change in the output sequence y.

(b) $A(\omega) \triangleq \sum_{n=-\infty}^{+\infty} a[n]e^{-j\omega n}$ and so,

$$\begin{aligned}
a[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)e^{+j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{+\infty} a[m]e^{-j\omega m}\right)e^{+j\omega n}d\omega \\
&= \sum_{n=-\infty}^{+\infty} a[m]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{+j\omega(n-m)}d\omega\right), \quad \text{by interchanging the infinite sum and the integral,} \\
&= \sum_{n=-\infty}^{+\infty} a[m]\delta[n-m] \\
&= a[n],
\end{aligned}$$

where the interchange of the infinite sum and the integral is permitted if the sequence a is absolutely summable, i.e. $\sum_{n=-\infty}^{+\infty} |a[n]| < \infty$.

(c) We have $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$, thus

$$\begin{aligned}
Y(\omega) &= \sum_{n=-\infty}^{+\infty}\left(\sum_{k=-\infty}^{+\infty} x[k]h[n-k]\right)e^{-j\omega n} \\
&= \sum_{k=-\infty}^{+\infty} x[k]\left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega n}\right), \quad \text{by interchanging the infinite sums,} \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k}\left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega n}e^{+j\omega k}\right) \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k}\left(\sum_{n=-\infty}^{+\infty} h[n-k]e^{-j\omega(n-k)}\right) \\
&= \sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k}\left(H(\omega)\right) \\
&= H(\omega)\sum_{k=-\infty}^{+\infty} x[k]e^{-j\omega k} \\
&= H(\omega)X(\omega).
\end{aligned}$$

Note that the interchange of the infinite sums in the steps above can be justified if the infinite sum $\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ converges uniformly. This occurs when $\sum_{k=-\infty}^{+\infty} |x[k]|\cdot|h[n-k]| < \infty$.

23. (a) Using Z transforms

$$Y(z) + \alpha z^{-1}Y(z) = X(z) \quad \text{and} \quad X(z) = \frac{1}{1-\beta z^{-1}},$$

so

$$Y(z) = \frac{1}{(1+\alpha z^{-1})(1-\beta z^{-1})} = \frac{z^2}{(z+\alpha)(z-\beta)}.$$

Now via the residue method, evaluating the complex integral

$$y[n] = \frac{1}{2\pi j} \oint Y(z) z^{n-1} dz, \qquad \text{see Appendix A.3 (discrete time),}$$

$$\begin{aligned} y[n] &= \frac{(-\alpha)^{n+1}}{-\alpha-\beta} + \frac{\beta^{n+1}}{\alpha+\beta} \\ &= \frac{1}{\alpha+\beta}\left(\beta^{n+1} - (-\alpha)^{n+1}\right) \quad n \geq 0. \end{aligned}$$

The answer is zero for $n < 0$, so the full answer is

$$y[n] = \begin{cases} \frac{1}{\alpha+\beta}\left(\beta^{n+1} - (-\alpha)^{n+1}\right), & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Alternatively, we can use the partial fraction method, and first express

$$\begin{aligned} Y(z) &= \frac{A}{1+\alpha z^{-1}} + \frac{B}{1-\beta z^{-1}} \\ &= \frac{\alpha/(\alpha+\beta)}{1+\alpha z^{-1}} + \frac{\beta/(\alpha+\beta)}{1-\beta z^{-1}}, \end{aligned}$$

and upon inverse Z-transform obtaining the same answer.

24. (a) We are given that ρ is a real constant, but in general α could be complex.

$$\begin{aligned} K_{YY}[m] &\triangleq E[Y[n+m]Y^*[n]] \\ &= E\left[(X[n+m] - \alpha X[n+m-1])(X[n] - \alpha X[n-1])^*\right] \\ &= K_{XX}[m] - \alpha K_{XX}[m-1] - \alpha^* K_{XX}[m+1] + |\alpha|^2 K_{XX}[m] \\ &= (1+|\alpha|^2) K_{XX}[m] - \alpha K_{XX}[m-1] - \alpha^* K_{XX}[m+1] \\ &= \sigma^2 \left[(1+|\alpha|^2)\rho^{|m|} - \alpha\rho^{|m-1|} - \alpha^*\rho^{|m+1|}\right]. \end{aligned}$$

(b) To get white noise, we try α real and take $m \geq 1$, then we set

$$\begin{aligned} 0 &= (1+\alpha^2)\rho^m - \alpha\rho^{m-1} - \alpha\rho^{m+1} \\ &= \rho^m(1+\alpha^2 - \alpha/\rho - \alpha\rho) \\ &\Longrightarrow \alpha = \rho. \end{aligned}$$

This also works, i.e. gives zero for $K_{YY}[m]$ for $m < 0$, thus $\alpha = \rho$ is a solution. The value $\alpha = \rho^{-1}$ also works to produce white noise at the system output.

(c) For $m = 0$, we then get the variance of the white noise sequence Y

$$\begin{aligned} \sigma_Y^2 &= \sigma^2(1+\alpha^2 - \alpha\rho - \alpha\rho) \\ &= \sigma^2(1-\rho^2), \quad \text{with the choice } \alpha = \rho. \end{aligned}$$

Alternatively, with the choice $\alpha = \rho^{-1}$, we get $\sigma_Y^2 = (\rho^{-2} - 1)$.

25. (a) From the given LCCDE, we can see that $X[n-1]$ is a linear combination of only the $W[n-i]'$ for $i > 0$. Thus, since the $W[n]$ are all jointly independent, it follows that $X[n-1]$ and $W[n]$ are independent.

(b)

$$\begin{aligned}
\Phi_{X[n]}(\omega) &\triangleq E[e^{+j\omega X[n]}] \\
&= E[e^{+j\omega(\rho X[n-1]+W[n])}] \\
&= E[e^{+j\omega\rho X[n-1]}]E[e^{+j\omega W[n]}], \quad \text{by independence,} \\
&= \Phi_{X[n-1]}(\rho\omega)\Phi_W(\omega) \\
&= \Phi_X(\rho\omega)\Phi_W(\omega), \qquad \text{since } \; X[n] \text{ is stationary.}
\end{aligned}$$

Note that we must have $|\rho| < 1$ for stationarity.

(c) Since $W[n]$ is Gaussian with zero mean, we have

$$\Phi_W(\omega) = e^{-\frac{1}{2}\sigma_W^2\omega^2}.$$

Now from the answer to (b), we have upon iteration,

$$\begin{aligned}
\Phi_X(\omega) &= \Phi_X(\rho\omega)\Phi_W(\omega) \\
&= \Phi_X(\rho^2\omega)\Phi_W(\rho\omega)\Phi_W(\omega) \\
&= \Phi_X(\rho^3\omega)\Phi_W(\rho^2\omega)\Phi_W(\rho\omega)\Phi_W(\omega) \\
&\cdots \\
&= \Phi_X(\rho^k\omega)\Phi_W(\rho^{k-1}\omega)\Phi_W(\rho^{k-2}\omega)\cdots\Phi_W(\omega).
\end{aligned}$$

Now in the limit, as $k \to \infty$, the condition $|\rho| < 1$ forces $\rho^k\omega$ to zero for any finite ω, therefore assuming continuity in the CF, we get

$$\begin{aligned}
\Phi_X(\omega) &= \prod_{k=0}^{\infty}\Phi_W(\rho^k\omega) \\
&= \exp\left(-\sum_{k=0}^{\infty}\frac{1}{2}\sigma_W^2\rho^{2k}\omega^2\right) \\
&= \exp\left(-\frac{1}{2}\sigma_W^2\omega^2\sum_{k=0}^{\infty}\rho^{2k}\right) \\
&= \exp\left(-\frac{1}{2}\sigma_W^2\omega^2\frac{1}{1-\rho^2}\right) \\
&= \exp\left(-\frac{1}{2}\frac{\sigma_W^2}{1-\rho^2}\omega^2\right).
\end{aligned}$$

(d) We recognize the CF of Gaussian noise in the result of (c), therefore it must be that $\sigma_X^2 = \frac{\sigma_W^2}{1-\rho^2}$.

26. (a) From the problem, $Y[n] = h[n] * [W[n] + X[n]]$, so

$$\begin{aligned}
\mu_Y[n] &= h[n] * (\mu_W[n] + 3) \\
&= \sum_{k=0}^{\infty} \rho^k (\mu_W[n-k] + 3) \\
&= \sum_{k=0}^{\infty} \rho^k (2+3) \\
&= 5 \sum_{k=0}^{\infty} \rho^k \\
&= \frac{5}{1-\rho.}
\end{aligned}$$

(b) The second moment of the real-valued random sequence Y is given as:

$$\begin{aligned}
E[Y^2[n]] &= E\left[\left(\sum_{k=0}^{\infty} h[k](W[n-k]+3)\right)^2\right] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]E[(W[n-k]+3)\,(W[n-l]+3)] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](\sigma_W^2 \delta[l-k] + 4 + 9 + 6 + 6) \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](\sigma_W^2 \delta[l-k] + 25) \\
&= \sum_{k=0}^{\infty} h^2[k]\sigma_W^2 + \sum_{(k,l)\geq 0}^{\infty} h[k]h[l](25) \\
&= \left(\sum_{k=0}^{\infty} h^2[k]\right)\sigma_W^2 + \left(\sum_{k=0}^{\infty} h[k]\right)^2 25 \\
&= \left(\sum_{k=0}^{\infty} \rho^{2k}\right)\sigma_W^2 + \left(\sum_{k=0}^{\infty} \rho^k\right)^2 25 \\
&= \frac{\sigma_W^2}{1-\rho^2} + \frac{25}{(1-\rho)^2}.
\end{aligned}$$

(c) For the covariance function of Y, we have

$$\begin{aligned}
K_{YY}[m,n] &= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]K_{WW}[m-k,n-l] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[m-k-(n-l)] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[(m-n)-(k-l)] \\
&= \sum_{(k,l)\geq 0}^{\infty} h[k]h[l]\sigma_W^2\delta[(m-n)-(k-l)] \\
&= \sum_{k=0}^{\infty} h[k]h[k-(m-n)]\sigma_W^2 \\
&= g(m-n),
\end{aligned}$$

where $g(m) = K_{YY}[m]$, the WSS covariance function. Continuing on,

$$\begin{aligned}
K_{YY}[m] &= \sum_{k=0}^{\infty} h[k]h[k-m]\sigma_W^2 \\
&= \sum_{k=\max(0,m)}^{\infty} \rho^k\rho^{k-m}\sigma_W^2 \\
&= \left(\sum_{k=\max(0,m)}^{\infty} \rho^{2k}\right)\rho^{-m}\sigma_W^2 \\
&= \frac{\rho^{2\max(0,m)}}{1-\rho^2}\rho^{-m}\sigma_W^2 \\
&= \rho^{|m|}\frac{\sigma_W^2}{1-\rho^2}.
\end{aligned}$$

Thus $K_{YY}[m,n] = K_{YY}[m-n] = \rho^{|m-n|}\frac{\sigma_W^2}{1-\rho^2}$.

27. We will show $E[X[n]X[n+1]] \neq 0$. Assume that α is real. Then

$$X[n] = \sum_{m=1}^{n} \alpha^{n-m}W[m] \quad \text{and} \quad X[n+1] = \sum_{l=1}^{n+1} \alpha^{n+1-l}W[l].$$

So

$$\begin{aligned}
E[X[n]X[n+1]] &= \sum_{m=1}^{n}\sum_{l=1}^{n+1} \alpha^{n-m}\alpha^{n+1-l}E[W[m]W[l]] \\
&= \alpha^{2n+1}\sum_{l=1}^{n}\alpha^{-2l} \\
&= \alpha^{2n-1}\sum_{i=0}^{n-1}\alpha^{-2i} \\
&= \alpha\left(\frac{1-\alpha^{2n}}{1-\alpha^2}\right) \\
&= \alpha\cdot \mathrm{Var}[X] \\
&\neq 0.
\end{aligned}$$

28. We start with

$$X[0]=0 \quad \text{and} \quad X[n]=\rho X[n-1]+W[n] \ \text{ for } \ n\geq 1.$$

(a) The solution to this recursion equation is

$$X[n]=\sum_{i=1}^{n}\rho^{n-i}W[i], n\geq 1,$$

so taking expectation of both sides we get

$$\begin{aligned}
E[X[n]] &= E\left[\sum_{i=1}^{n}\rho^{n-i}W[i]\right] \\
&= \sum_{i=1}^{n}\rho^{n-i}E[W[i]] \\
&= \sum_{i=1}^{n}\rho^{n-i}\cdot 0 \\
&= 0.
\end{aligned}$$

(b) For the covariance, we have by definition

$$\begin{aligned}
K_X[m,n] &\triangleq E[(X[m]-\mu)(X[n]-\mu)] \\
&= E[X[m]X[n]] \quad \text{since } \mu=0 \text{ here,} \\
&= E\left[\left(\sum_{i=1}^{m}\rho^{m-i}W[i]\right)\left(\sum_{j=1}^{n}\rho^{n-j}W[j]\right)\right] \\
&= \sum_{i,j}^{m,n}\rho^{m-i}\rho^{n-j}E[W[i]W[j]],
\end{aligned}$$

where we have written the second sum with dummy index 'j'. Next, let $m\leq n$, and note that $E[W[i]Wj]]=0$ for $i\neq j$ since the input $W[n]$ is an independent random sequence, i.e.

independent of itself at other times, and also has zero mean. Thus, we get

$$\begin{aligned} K_X[m,n] &= \sum_{i=1}^{m} \rho^{m-i}\rho^{n-i}E[W^2[i]] \\ &= \left(\sum_{i=1}^{m} \rho^{m+n-2i}\right)\sigma_W^2, \quad \text{for} \quad m \le n. \end{aligned}$$

To compute the sum, we proceed as follows:

$$\begin{aligned} \sum_{i=1}^{m} \rho^{m+n-2i} &= \rho^{n-m}\sum_{i=1}^{m} \left(\rho^2\right)^{m-i} \\ &= \rho^{n-m}\left(\frac{1-\rho^{2m}}{1-\rho^2}\right). \end{aligned}$$

So

$$\begin{aligned} K_X[m,n] &= \rho^{n-m}\left(\frac{1-\rho^{2m}}{1-\rho^2}\right)\sigma_W^2 \\ &= \frac{\sigma_W^2}{1-\rho^2}[\rho^{n-m}-\rho^{n+m}], \quad \text{for} \quad m \le n. \end{aligned}$$

The answer for $n \le m$, is found by symmetry, then the overall answer for all m, n can be written as

$$\begin{aligned} K_X[m,n] &= \begin{cases} \frac{\sigma_W^2}{1-\rho^2}[\rho^{n-m}-\rho^{n+m}], & m \le n, \\ \frac{\sigma_W^2}{1-\rho^2}[\rho^{m-n}-\rho^{m+n}], & n \le m, \end{cases} \\ &= \frac{\sigma_W^2}{1-\rho^2}[\rho^{|m-n|}-\rho^{m+n}], \quad 1 \le m, n < \infty. \end{aligned}$$

(c) Let $|\rho| < 1$, then as $m, n \to \infty$, we see that the above

$$K_X[m,n] \to \frac{\sigma_W^2}{1-\rho^2}\rho^{|m-n|},$$

and K_X becomes asymptotically just a function of $m-n$, i.e. $K_X[m-n]$ a one parameter covariance function.

29. (a)

$$\begin{aligned} \mu_X[n] &= E[X[n]] \\ &= E[A\cos\omega n + B\sin\omega n] \\ &= E[A]\cos\omega n + E[B]\sin\omega n \\ &= 0 + 0 = 0, \text{ a constant.} \end{aligned}$$

As for the correlation function

$$\begin{aligned} R_X[m] &= E[X[n+m]X[n]] \\ &= E[(A\cos\omega(n+m) + B\sin\omega(n+m))(A\cos\omega n + B\sin\omega n)] \\ &= \sigma^2(\cos\omega(n+m)\cos\omega n + \sin\omega(n+m)\sin\omega n) \\ &= \sigma^2(\cos\omega(n+m) - \omega n) \\ &= \sigma^2\cos\omega m. \end{aligned}$$

Thus by the definition, the random sequence $X[n]$ is WSS.

(b) Consider the third moment function

$$\begin{aligned}
E[X^3[n]] &= E[(A\cos\omega n + B\sin\omega n)^3] \\
&= E[A^3]\cos^3\omega n + 3E[A^2B]\cos^2\omega n\sin\omega n \\
&\quad +3E[AB^2]\cos\omega n\sin^2\omega n + E[B^3]\sin^3\omega n \\
&= E[A^3]\cos^3\omega n + 3E[A^2]E[B]\cos^2\omega n\sin\omega n \\
&\quad +3E[A]E[B^2]\cos\omega n\sin^2\omega n + E[B^3]\sin^3\omega n \\
&= E[A^3]\cos^3\omega n + E[B^3]\sin^3\omega n \\
&= m_3(\cos^3\omega n + \sin^3\omega n),
\end{aligned}$$

since $E[B] = E[A] = 0$ and $E[A^3] = m_3$. Now, since m_3 is a non-zero constant and the function $\cos^3\omega n + \sin^3\omega n$ can be easily seen to be non-constant, we have that the third moment function is time varying, i.e. not consistent with stationarity (strict sense).

30. (a) $K_{XX}[0]$ must be non-negative and here $K_{XX}[0] = p^2 - \mu^2$, so $p^2 \geq \mu^2$. Set $\sigma^2 \triangleq p^2 - \mu^2$ (≥ 0).

(b) The $N \times N$ covariance matrix is then

$$\mathbf{K_{XX}} = \begin{bmatrix} \sigma^2 & & -\mu^2 \\ & \ddots & \\ -\mu^2 & & \sigma^2 \end{bmatrix},$$

so for the all 1s vector $\mathbf{a}$, we get $\mathbf{aKa}^\dagger = N\sigma^2 - N(N-1)\mu^2 \geq 0$, so it must be that

$$\begin{aligned}
\mu^2 &\leq \frac{N\sigma^2}{N(N-1)} \\
&= \frac{\sigma^2}{N-1}.
\end{aligned}$$

(c) By taking a sequence of ever increasing N values, we conclude that μ must be zero, for any finite p^2 and hence σ^2.

31. We know that for LSI filtering of a WSS random sequence in problem 8.16,

$$\begin{aligned}
S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\
&= \left|\frac{1}{2}(1+e^{-j\omega})\right|^2 S_{XX}(\omega) \\
&= \frac{1}{4}(1+e^{-j\omega})(1+e^{+j\omega})S_{XX}(\omega) \\
&= \frac{1}{4}(2+2\cos\omega)S_{XX}(\omega) \\
&= \frac{1}{2}(1+\cos\omega)S_{XX}(\omega).
\end{aligned}$$

Now, the rest of the problem was meant to be taken generally and not with reference to the special filtering problem given in 8.16. We thus consider a general WSS random sequence $X[n]$ with correlation function $R_{XX}[m]$ and psd $S_{XX}(\omega)$.

(a) We start with the definition

$$\begin{aligned}
S_{XX}(\omega) &\triangleq \sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} R_{XX}[-m]e^{+j\omega m}, \quad \text{by replacing } m \leftarrow -m.
\end{aligned}$$

Now we know that always $R_{XX}[m] = R^*_{XX}[-m]$ from the basic definition for any WSS random sequence, so, continuing on

$$\begin{aligned}
S_{XX}(\omega) &= \sum_{m=-\infty}^{+\infty} R_{XX}[-m]e^{+j\omega m} \\
&= \left(\sum_{m=-\infty}^{+\infty} R^*_{XX}[-m]e^{-j\omega m} \right)^* \\
&= \left(\sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{-j\omega m} \right)^* \\
&= S^*_{XX}(\omega).
\end{aligned}$$

(b) If the random sequence $X[n]$ is real valued, then $R_{XX}[m] = R_{XX}[-m]$, and by substitution we get

$$\begin{aligned}
S_{XX}(\omega) &\triangleq \sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} R_{XX}[-m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{+j\omega m} \\
&= S_{XX}(-\omega),
\end{aligned}$$

i.e. the psd of a real WSS random sequence is an even function.

(c) Use the same argument as in the text on p. 489.

32. We are given $R_{XX}[m] = 10e^{-\lambda_1|m|} + 5e^{-\lambda_2|m|}$ with $\lambda_1 > 0$ and $\lambda_2 > 0$. We assume $\lambda_1 \neq \lambda_2$

and offer the general solution.

$$
\begin{aligned}
S_{XX}(\omega) &\triangleq \sum_{m=-\infty}^{+\infty} R_{XX}[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} 10e^{-\lambda_1|m|}e^{-j\omega m} + \sum_{m=-\infty}^{+\infty} 5e^{-\lambda_2|m|}e^{-j\omega m} \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-\lambda_1 m}e^{-j\omega m} + \sum_{m=-\infty}^{-1} e^{+\lambda_1 m}e^{-j\omega m}\right) \\
&\quad +5\left(\sum_{m=0}^{+\infty} e^{-\lambda_2 m}e^{-j\omega m} + \sum_{m=-\infty}^{-1} e^{+\lambda_2 m}e^{-j\omega m}\right) \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-(\lambda_1+j\omega)m} + \sum_{m=-\infty}^{0} e^{+(\lambda_1-j\omega)m} - 1\right) \\
&\quad +5\left(\sum_{m=0}^{+\infty} e^{-(\lambda_2+j\omega)m} + \sum_{m=-\infty}^{0} e^{+(\lambda_2-j\omega)m} - 1\right) \\
&= 10\left(\sum_{m=0}^{+\infty} e^{-(\lambda_1+j\omega)m} + \sum_{m'=0}^{+\infty} e^{-(\lambda_1-j\omega)m'} - 1\right) \\
&+ 5\left(\sum_{m=0}^{+\infty} e^{-(\lambda_2+j\omega)m} + \sum_{m'=0}^{+\infty} e^{-(\lambda_2-j\omega)m'} - 1\right), \qquad \text{with sub } m' \triangleq -m, \\
&= 10\left(\frac{1}{1-e^{-(\lambda_1+j\omega)}} + \frac{1}{1-e^{-(\lambda_1-j\omega)}} - 1\right) + 5\left(\frac{1}{1-e^{-(\lambda_2+j\omega)}} + \frac{1}{1-e^{-(\lambda_2-j\omega)}} - 1\right) \\
&= 10\left(\frac{1-e^{-2\lambda_1}}{1-2\cos\omega\; e^{-\lambda_1}+e^{-2\lambda_1}}\right) + 5\left(\frac{1-e^{-2\lambda_2}}{1-2\cos\omega\; e^{-\lambda_2}+e^{-2\lambda_2}}\right).
\end{aligned}
$$

33.

$$
\begin{aligned}
S_{XX}(\omega) &= \frac{1}{((1+\alpha^2)-2\alpha\cos\omega)^2} \\
&= G^2(\omega) \quad \text{with} \quad G(\omega) \triangleq 1/(1+\alpha^2) - 2\alpha\cos\omega.
\end{aligned}
$$

Equivalently in Z transforms, with the substitution $z = e^{j\omega}$, we would have

$$
\begin{aligned}
G(z) &= \frac{1}{(1+\alpha^2)-2\alpha(z+z^{-1})} \\
&= \frac{1}{(1-\alpha z)(1-\alpha z^{-1})} \\
&= \frac{(-\frac{1}{\alpha})z^{-1}}{(1-\frac{1}{\alpha}z^{-1})(1-\alpha z^{-1})} \\
&= \frac{1}{1-\alpha^2}\left(\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\frac{1}{\alpha}z^{-1}}\right).
\end{aligned}
$$

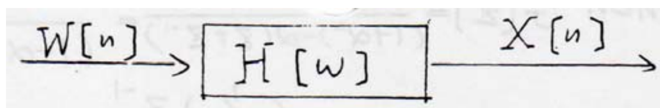
Then, upon inverse Z transform, we obtain

$$\begin{aligned} g[n] &= \frac{1}{1-\alpha^2}(\alpha^n u[n] - \alpha^{-1}u[-n-1]) \\ &= \frac{\alpha^{|n|}}{1-\alpha^2}. \end{aligned}$$

Then

$$\begin{aligned} R_{XX}[m] &= \text{IFT}\{S_{XX}(\omega)\} \\ &= g[m] * g[m] \\ &= \frac{2\alpha^{|m|+1}}{(1-\alpha^2)^3} + \frac{(|m|+1)\,\alpha^{|m|}}{(1-\alpha^2)^2}. \end{aligned}$$

34. α



We know that

$$\begin{aligned} \mu_W[n] &= 0, K_{WW}[m] = \delta[m], \text{and} \\ X[n] &= \sum_{k=-\infty}^{+\infty} h[k]W[n-k] = \sum_{k=-\infty}^{+\infty} h[n-k]W[k]. \end{aligned}$$

So

$$\begin{aligned} K_{XW}[n] &\triangleq E[X[m+n]W^*[m]] \\ &= E\left[W^*[m]\sum_{k=-\infty}^{+\infty} h[k]W[n+m-k]\right] \\ &= \sum_{k=-\infty}^{+\infty} h[k]E[W^*[m]W[n+m-k]], \quad \text{if the sum converges,} \\ &= \sum_{k=-\infty}^{+\infty} h[k]K_{WW}[n-k] \\ &= \sum_{k=-\infty}^{+\infty} h[k]\delta[n-k] \\ &= h[n]. \end{aligned}$$

By Fourier transformation, also we have that the cross-power spectral density

$$S_{XW}(\omega) = H(\omega).$$

35. (a) $S_{YY}(\omega) = |H(\omega)|^2(S_{XX}(\omega) + S_{VV}(\omega))$.

(b) $Y[n] = \sum_{k=-\infty}^{+\infty} h[k](X[n-k]+V[n-k])$. Now $E[X[n+m]Y^*[n]] = \sum_{k=-\infty}^{+\infty} h^*[k]E[X[n+m]X^*[n-k]]$, so

$$\begin{aligned}
R_{XY}[m] &= \sum_{k=-\infty}^{+\infty} h^*[k]R_{XX}[m+k] \\
&= \sum_{k'=-\infty}^{+\infty} h^*[-k']R_{XX}[m-k'], \quad \text{with} \quad k' \triangleq -k, \\
&= R_{XX}[m] * h^*[-m].
\end{aligned}$$

Hence $S_{XY}(\omega) = H^*(\omega)S_{XX}(\omega)$.

36. For this system,

$$h[n] = \frac{1}{5}\left(\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2]\right)$$

and

$$\begin{aligned}
H(\omega) &= \frac{1}{5}(1 + 2\cos\omega + 2\cos 2\omega) \\
&= \frac{1}{5}\frac{\sin\frac{5}{2}\omega}{\sin\frac{1}{2}\omega}.
\end{aligned}$$

Then

(a)

$$\begin{aligned}
S_{YY}(\omega) &= |H(\omega)|^2 S_{XX}(\omega) \\
&= \frac{1}{25}(1 + 2\cos\omega + 2\cos 2\omega)^2 \cdot 2 \\
&= \frac{2}{25}\left(\frac{\sin\frac{5}{2}\omega}{\sin\frac{1}{2}\omega}\right)^2.
\end{aligned}$$

(b)

$$\begin{aligned}
R_{YY}[m] &= h[m] * h[-m] * [\delta[m] \\
&= \frac{2}{25}\text{triag}[m].
\end{aligned}$$

Here, the triangular finite-support sequence triag[$\cdot$] is specified as follows:

n	0	±1	±2	±3	±4	else
triag[n]	5	4	3	2	1	0

37. (a)

$$\begin{aligned} S_{WW}(\omega) &= |G(\omega)|^2 S_{YY}(\omega) \\ &= \frac{1}{S_{YY}(\omega)} S_{YY}(\omega) \\ &= 1. \end{aligned}$$

For cross-power spectral density, we go back to the time domain first, $R_{XW}[m] = g[m] * R_{XX}[m]$, so

$$\begin{aligned} S_{XW}(\omega) &= G(\omega) S_{XX}(\omega) \\ &= \frac{1}{\sqrt{S_{YY}(\omega)}} S_{XX}(\omega). \end{aligned}$$

(b) We have the FIR estimator

$$\widehat{X}[n] = \sum_{k=0}^{N-1} h[k] W[n-k],$$

with orthogonality condition

$$(\widehat{X}[n] - X[n]) \perp W[m], \text{ for } m = n, n-1, ..., n-(N-1),$$

which is to say $E[(\widehat{X}[n] - X[n]) W^*[m]] = 0$, or equivalently $E[\widehat{X}[n] W^*[m]] = E[X[n] W^*[m]]$, for $m = n, n-1, ..., n-(N-1)$. Next, we plug in the assumed FIR form for our estimate $\widehat{X}$ to get

$$\sum_{k=0}^{N-1} h[k] \underbrace{E[W[n-k] W^*[m]]}_{R_{WW}[(n-m)-k]} = R_{XW}[n-m], \quad m = n, n-1, ..., n-(N-1).$$

Since $R_{WW}[n] = \delta[n]$ here, we get $h[n-m] = R_{XW}[n-m]$, which is equivalent to:

$$h[n] = R_{XW}[n], \quad n = 0, ..., N-1.$$

(c) In the limit as $N \nearrow \infty$, the above FIR convolution becomes unconstrained by finite order:

$$h[n] = R_{XW}[n], \quad -\infty < n < +\infty,$$

so that

$$\begin{aligned} H(\omega) &= S_{XW}(\omega) \\ &= \frac{S_{XX}(\omega)}{\sqrt{S_{YY}(\omega)}}. \end{aligned}$$

38. We have

$$Y[n] = \sum_{k_1} h[k_1] X[n-k_1],$$

so:

(a)

$$\begin{aligned}
R_Y[m_1, m_2] &\triangleq E[Y[n+m_1]Y[n+m_2]Y^*[n]] \\
&= \sum_{k_1,k_2,k_3} h[k_1]h[k_2]h^*[k_3]E[X[n+m_1-k_1]X[n+m_2-k_2]X^*[n-k_3]] \\
&= \sum_{k_1,k_2,k_3} h[k_1]h[k_2]h^*[k_3]R_X[m_1-k_1+k_3, m_2-k_2+k_3],
\end{aligned}$$

by choosing $n = k_3$, and then realizing that $R_X[m_1', m_2'] = E[X[m_1']X[m_2']X^*[0]]$.

(b)

$$\begin{aligned}
S_Y(\omega_1, \omega_2) &\triangleq \sum_{m_1,m_2} R_Y[m_1, m_2]e^{-j(\omega_1 m_1 + \omega_2 m_2)} \\
&= \sum_{m_1,m_2} \left(\sum_{k_1,k_2,k_3} h[k_1]h[k_2]h^*[k_3]R_X[m_1-k_1+k_3, m_2-k_2+k_3] \right) e^{-j(\omega_1 m_1 + \omega_2 m_2)}.
\end{aligned}$$

We re-write the argument of the complex exponential $-j(\omega_1 m_1 + \omega_2 m_2)$ as follows:

$$-j\omega_1 k_1 \; -j\omega_2 k_2 \; -j(m_1 - k_1 + k_3) \; -j(\omega_1 + \omega_2)k_3 \; -j\left[\omega_1(m_1 - k_1 + k_3) + \omega_2(m_2 - k_2 + k_3)\right],$$

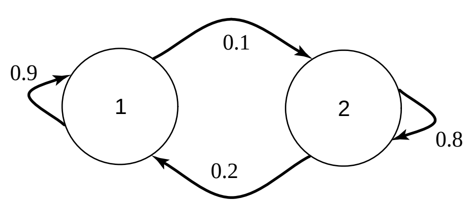
and then factor this complex exponential, to obtain

$$\begin{aligned}
S_Y(\omega_1, \omega_2) &= \sum_{k_1} h[k_1]e^{-j\omega_1 k_1} \left(\sum_{k_2} h[k_2]e^{-j\omega_2 k_2} \left(\sum_{k_3} h^*[k_3]e^{+j(\omega_1+\omega_2)k_3} \left(\sum_{m_1,m_2} R_X[m_1-k_1+k_3, m \right.\right.\right. \\
&= \left(\sum_{k_1} h[k_1]e^{-j\omega_1 k_1} \right) \left(\sum_{k_2} h[k_2]e^{-j\omega_2 k_2} \right) \left(\sum_{k_3} h^*[k_3]e^{+j(\omega_1+\omega_2)k_3} \right) S_X(\omega_1, \omega_2) \\
&= H(\omega_1)H(\omega_2)H^*(\omega_1 + \omega_2)S_X(\omega_1, \omega_2).
\end{aligned}$$

39. (a) With the two states $X = 1, 2$, we have state probability vector $\mathbf{p}$ at time n (≥ 0) given as

$$\begin{aligned}
\mathbf{p}[n] &= (P[X[n] = 1, P[X[n] = 2]) \\
&= (P[X[n-1] = 1]p_{11} + P[X[n-1] = 2]p_{21}, P[X[n-1] = 1]p_{12} + P[X[n-1] = 2]p_{22}) \\
&= \mathbf{p}[n-1]\mathbf{P}, \quad \text{with } \mathbf{P} \text{ the state-transition matrix,} \\
&= \mathbf{p}[n-2]\mathbf{P}^2, \\
&\;\;\vdots \\
&= \mathbf{p}[0]\mathbf{P}^n.
\end{aligned}$$

(b)



(c) Let p be the probability of the event $\{$*first* transition to state 2 occurring at time $n\}$. Then, given $X[0]=1$, we have

$$\begin{aligned} p &= p_{11}^{n-1} p_{12} \\ &= (0.9)^{n-1}(0.1) \\ &= (0.1)\,(0.9)^{n-1}. \end{aligned}$$

40. (a)

$$H(\omega) = \frac{1}{1 - re^{-j\omega}} \qquad \text{and} \qquad h[n] = r^n u[n],$$

so

$$\begin{aligned} S_{XX}(\omega) &= |H(\omega)|^2 S_{ZZ}(\omega) \\ &= \frac{1}{|1 - re^{-j\omega}|^2}\sigma_Z^2 \\ &= \frac{\sigma_Z^2}{1 + r^2 - 2r\cos\omega}. \end{aligned}$$

(b) We know $R_{XX}[m] = (h[m] * h^*[-m]) * \sigma_Z^2 \delta[m]$. Here, we have

$$\begin{aligned} h[n] * h^*[-n] &= \sum_{k=-\infty}^{+\infty} h[k] h^*[-(n-k)] \\ &= \sum_{k=-\infty}^{+\infty} r^k u[k] r^{-(n-k)} u[k-n] \\ &= r^{-n} \sum_{k=0}^{+\infty} r^{2k} u[k-n] \\ &= \begin{cases} \frac{r^n}{1-r^2}, & n \geq 0, \\ \frac{r^{-n}}{1-r^2}, & n \leq 0, \end{cases} \\ &= \frac{r^{|n|}}{1-r^2}, \qquad \text{for all } n. \end{aligned}$$

Thus

$$\begin{aligned} R_{XX}[m] &= (h[m] * h^*[-m]) * \sigma_Z^2 \delta[m] \\ &= \frac{r^{|m|}}{1-r^2} * \sigma_Z^2 \delta[m] \\ &= \left(\frac{r^{|m|}}{1-r^2} * \delta[m] \right) \sigma_Z^2 \\ &= \frac{r^{|m|}}{1-r^2} \sigma_Z^2. \end{aligned}$$

41. Using the short notation, we rewrite the two-step pdf $f_X(x|y; n, n-2)$ as $f_X(x_n|x_{n-2})$ as

$$f_X(x_n|x_{n-2}) = f_X(x_n|x_{n-2}; n, n-2).$$

Then

(a)

$$\begin{aligned} f_X(x_n|x_{n-2}) &= \int_{-\infty}^{+\infty} f_X(x_n, x_{n-1}|x_{n-2})dx_{n-1} \\ &= \int_{-\infty}^{+\infty} f_X(x_n|x_{n-1}, x_{n-2})f_X(x_{n-1}|x_{n-2})dx_{n-1}, \quad \text{by condl. prob. def.,} \\ &= \int_{-\infty}^{+\infty} f_X(x_n|x_{n-1})f_X(x_{n-1}|x_{n-2})dx_{n-1}, \quad \text{by Markov property.} \end{aligned}$$

(b)

$$\begin{aligned} f_X(x_n|x_{n-N}) &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_X(x_n, x_{n-1}, ..., x_{n-N+1}|x_{n-N})d\ x_{n-1}dx_{n-2}\cdots dx_{n-N+1} \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_X(x_n|x_{n-1})f_X(x_{n-1}|x_{n-2})\cdots f_X(x_{n-N+1}|x_{n-N})d\ x_{n-1}dx_{n-2}\cdots dx_{n-N+1} \\ &\qquad \text{by application of the Markov property } N-1 \text{ times.} \end{aligned}$$

42. (a)

$$\begin{aligned} \mu_X[n] &\triangleq E[X[n]] \\ &= E\left[\sum_{k=1}^{n} W[k]\right] \\ &= \sum_{k=1}^{n} E[W[k]] \\ &= \sum_{k=1}^{n} \frac{1}{2}s_1 + \frac{1}{2}(-s_2) \\ &= n\frac{1}{2}(s_1 - s_2). \end{aligned}$$

(b) Since $X[n]$ is real-valued, we can dispense with the conjugate and write

$$\begin{aligned} R_X[n_1, n_2] &\triangleq E[X[n_1]X[n_2]] \\ &= \sum_{k_1,k_2=1}^{n_1,n_2} E[W[k_1]W[k_2]] \\ &= \sum_{k=1}^{n_2} E[W^2[k]] + \sum_{k_1 \neq k_2} E[W[k_1]]E[W[k_2]], \qquad , n_2 \leq n_1, \\ &= n_2\frac{1}{2}(s_1^2 + s_2^2) + (n_1n_2 - n_2)(\frac{1}{2}(s_1 - s_2))^2 \\ &= \frac{n_2}{4}(s_1 + s_2)^2 + \frac{n_1n_2}{4}(s_1 - s_2)^2. \end{aligned}$$

In general, we can then write

$$R_X[n_1, n_2] = \frac{\min(n_1, n_2)}{4}(s_1 + s_2)^2 + \frac{n_1n_2}{4}(s_1 - s_2)^2.$$

43. We will use the simplified notation here.

$$f_X(x_n|x_{n+1},...,x_{100}) = \frac{f_X(x_n,x_{n+1},...,x_{100})}{f_X(x_{n+1},...,x_{100})}. \tag{6}$$

Now, by the chain rule of probability theory, the numerator of Equation 6 is equal to

$$\begin{aligned} & f_X(x_n,x_{n+1})f_X(x_{n+2}|x_{n+1},x_n)f_X(x_{n+3}|x_{n+2},x_{n+1},x_n)...f_X(x_{100}|x_{99},x_{98},...,x_n) \\ = \quad & f_X(x_n,x_{n+1})\prod_{k=2}^{100-n} f_X(x_{n+k}|x_{n+k-1}), \quad \text{by the Markov property.} \end{aligned}$$

Doing a similar chain rule expansion on the denominator of Equation 6, we get

$$\begin{aligned} & f_X(x_{n+1})f_X(x_{n+2}|x_{n+1})f_X(x_{n+3}|x_{n+2},x_{n+1})...f_X(x_{100}|x_{99},x_{98},...,x_{n+1}) \\ = \quad & f_X(x_{n+1})\prod_{k=2}^{100-n} f_X(x_{n+k}|x_{n+k-1}), \quad \text{again by the Markov property.} \end{aligned}$$

Thus, Equation 6 becomes

$$\begin{aligned} f_X(x_n|x_{n+1},...,x_{100}) &= \frac{f_X(x_n,x_{n+1},...,x_{100})}{f_X(x_{n+1},...,x_{100})} \\ &= \frac{f_X(x_n,x_{n+1})\prod_{k=2}^{100-n} f_X(x_{n+k}|x_{n+k-1})}{f_X(x_{n+1})\prod_{k=2}^{100-n} f_X(x_{n+k}|x_{n+k-1})} \\ &= \frac{f_X(x_n,x_{n+1})}{f_X(x_{n+1})} \\ &= f_X(x_n|x_{n+1}), \quad \text{as was to be shown.} \end{aligned}$$

44. For the three RVs $X[n], X[n-1]$, and $X[n-2]$, we can write from basic probability theory,

$$P[X[n]=x[n]|X[n-2]=x[n-2]] = \sum_{k=-\infty}^{+\infty} P[X[n]=x[n], X[n-1]=x_k|X[n-2]=x[n-2]],$$

where the x_k are the countable set of values that may be taken on by RV $X[n-1]$. Then, by the Markov property, we can rewrite the right-hand side of this equation as

$$= \sum_{k=-\infty}^{+\infty} P[X[n]=x[n]|X[n-1]=x_k]\, P[X[n-1]=x_k|X[n-2]=x[n-2]].$$

By the same line of reasoning, we can write, for $n>2$,

$$\begin{aligned} & P[X[n]=x[n]|X[n-2]=x[n-2], X[n-3]=x[n-3],...] \\ = \quad & \sum_{k=-\infty}^{+\infty} P[X[n]=x[n]|X[n-1]=x_k]\, P[X[n-1]=x_k|X[n-2]=x[n-2], X[n-3]=x[n-3],...] \\ = \quad & \sum_{k=-\infty}^{+\infty} P[X[n]=x[n]|X[n-1]=x_k]\, P[X[n-1]=x_k|X[n-2]=x[n-2]], \\ & \qquad \text{again by the Markov property,} \\ = \quad & P[X[n]=x[n]|X[n-2]=x[n-2]], \quad \text{by first result above.} \end{aligned}$$

45. (a) Plugging in $2n$ for n in the given recursive equation, we get

$$\begin{aligned} X[2n] &= \alpha X[2n-1] + \beta W[2n] \\ &= \alpha(\alpha X[2n-2] + \beta W[2n-1]) + \beta W[2n] \\ &= \alpha^2 X[2n-2] + (\beta W[2n] + \alpha\beta W[2n-1]). \end{aligned}$$

From the definition of Y, we then have

$$\begin{aligned} Y[n] &\triangleq X[2n] \\ &= \alpha^2 Y[n-1] + W'[n], \end{aligned}$$

where $W'[n] \triangleq \beta W[2n] + \alpha\beta W[2n-1]$. Now, since $W'[n]$ involves distinct $W[k]$ in non-overlapping sets, $W'[n]$ is an independent random sequence itself. Thus $Y[n]$ is a Markov random sequence.

(b) As to variance,

$$\sigma_Y^2[n] = \alpha^4 \sigma_Y^2[n-1] + \sigma_{W'}^2, \qquad n \geq 1,$$

subject to $\sigma_Y^2[0] = 0$. Now $\sigma_{W'}^2 = \beta^2\sigma_W^2 + \alpha^2\beta^2\sigma_W^2 = \beta^2(1+\alpha^2)\sigma_W^2$, so we have

$$\begin{aligned} \sigma_Y^2[\infty] &= \frac{1}{1-\alpha^4}\beta^2(1+\alpha^2)\sigma_W^2 \\ &= \frac{\beta^2}{1-\alpha^2}\sigma_W^2. \end{aligned}$$

The homogeneous or transient response is $C\alpha^{4n}$, for some constant C, so the total response is given as

$$\sigma_Y^2[n] = \frac{\beta^2}{1-\alpha^2}\sigma_W^2 + C\alpha^{4n}$$

We determine the constant at $n = 1$ as

$$\begin{aligned} \sigma_Y^2[n] &= \beta^2(1+\alpha^2)\sigma_W^2 \\ &= \frac{\beta^2}{1-\alpha^2}\sigma_W^2 + C\alpha^4, \end{aligned}$$

with solution

$$\begin{aligned} C &= \frac{\beta^2\sigma_W^2\left(1+\alpha^2 - \frac{1}{1-\alpha^2}\right)}{\alpha^4} \\ &= -\frac{\beta^2\sigma_W^2}{1-\alpha^2}. \end{aligned}$$

Then we can get the total solution

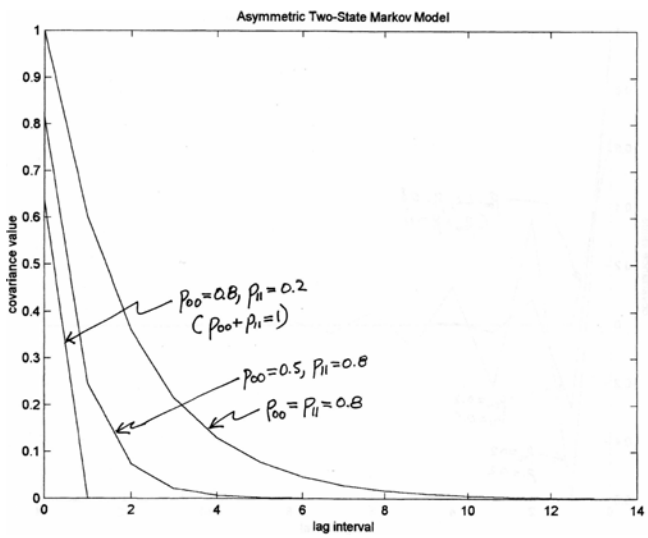
$$\sigma_Y^2[n] = \frac{\beta^2}{1-\alpha^2}\sigma_W^2(1-\alpha^{4n})u[n],$$

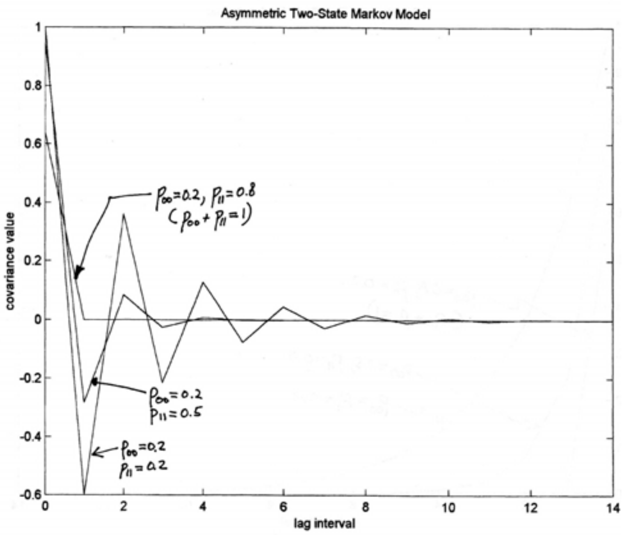
where u is the discrete-time unit step function.

46. Here is the MATLAB function 'triplemarkov':

```
function[cov1,cov2,cov3]=triplemarkov(p001,p002,p003,p111,p112,p113,N,b)
r1=0*ones(1,N);
s1=0*ones(1,N);
r1(1)=1;
s1(1)=0;
for i=2:N
r1(i)=p111*r1(i-1)+{1-p001)*s1(i-1)
s1(i)=(1-p111)*r1(i-1)+p001*s1{i-1)
end
plinf1=(1-p001)/(2-p001-p111);
x=linspace(0,N-1,N);
rt1=r1*plinfl;
cov1=rt1-plinf1^2;
covn1=cov1*(b^2);
end of first run
r2-0*ones(1,N);
s2=0*ones(1,N);
r2(1)=1;
s2(1)=0;
for i=2:N
r2(i)=p112*r2(i-1)+(1-p002)*s2(i-1)
s2(i)=(1-p112)*r2(i-1)+p002*s2(i-1)
end
plinf2=(1-p002)/(2-p002-p112);
rt2=r2*plinf2;
cov2=rt2-plinf2^2;
covn2=cov2*(b^2);
end of run2
r3=0*ones(1,N);
s3=0*ones(1,N);
r3(1)=1;
s3(1)=0;
for i=2:N
r3(i)=p113*r3(i-1)+(1-p003)*s3(i-1)
s3(i)=(1-p113)*r3(i-1)+p003*s3(i-1)
end.
plinf3=(1-p003)/(2-p003-p113);
x=linspace(0,N-1,N);
rt3=r3*plinf3;
cov3=rt3-plinf3^2;
covn3=cov3*(b^2);
plot(x,covnl,x,covn2,x,covn3)
xlabel('lag interval')
ylabel('covariance value')
title('Asymmetric Two-State Markov Model'
```

Here are the plots from the two requested runs





47. (a) Since $\{0\} = [0,1] - (0,1]$, then

$$\begin{aligned} P[\{0\}] &= P[[0,1]] - P[(0,1]] \\ &= P[\Omega] - P[(0,1]] \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

(b) i) Since $X[n,\zeta] = e^{-n\zeta}$, we have

$$\begin{aligned} E[|X[n,\zeta] - 0|^2] &= E[e^{-2n\zeta}] \\ &= \int_0^1 2e^{-2n\zeta} d\zeta \\ &= \frac{e^{-2n} - 1}{-2n} \\ &\longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \end{aligned}$$

Therefore $X[n,\zeta]$ converges (to 0) in the mean-square sense. Also

$$\lim_{n \longrightarrow \infty} X[n,\zeta] = \begin{cases} 1, & \zeta = 0, \\ 0, & 0 < \zeta \leq 1, \end{cases}$$

and therefore converges almost everywhere (a.e.) (also called almost surely (a.s.)).

ii) We ask ourselves the question $X[n,\zeta] = \sin\left(\zeta + \frac{1}{n}\right) \longrightarrow_{?} \sin\zeta$, as follows

$$
\begin{aligned}
E\left[\left|\sin\left(\zeta + \frac{1}{n}\right) - \sin\zeta\right|^2\right] &= E\left[\left|\sin\zeta\cos\frac{1}{n} + \cos\zeta\sin\frac{1}{n} - \sin\zeta\right|^2\right] \\
&= E\left[\left|\sin\zeta(\cos\frac{1}{n} - 1) + \cos\zeta\sin\frac{1}{n}\right|^2\right] \\
&= E[\sin^2\zeta(\cos^2\frac{1}{n} + 1 - 2\cos\frac{1}{n}) + \cos^2\zeta\sin^2\frac{1}{n} \\
&\qquad +2\cos\zeta\sin\frac{1}{n}\sin\zeta(\cos\frac{1}{n} - 1)] \\
&= E[\sin^2\zeta\cos^2\frac{1}{n} + \sin^2\zeta - 2\sin^2\zeta\cos\frac{1}{n} + \cos^2\zeta\sin^2\frac{1}{n} \\
&\quad +2\cos\zeta\sin\zeta\cos\frac{1}{n}\sin\frac{1}{n} - 2\cos\zeta\sin\zeta\sin\frac{1}{n}].
\end{aligned}
$$

Now, when the expectation is taken (the integral is calculated), we can see that as $n \longrightarrow \infty$, the expected value goes to zero. Specifically, we see that

$$\lim_{n\longrightarrow\infty} \sin\left(\zeta + \frac{1}{n}\right) = \sin\zeta \quad \text{for all } \zeta \in \Omega,$$

hence the convergence is everywhere too.

iii) We consider $X[n,\zeta] = \cos^n\zeta$ for $\zeta \in [0,1]$. Since $\cos 0 = 1$ and $|\cos\zeta| < 1$ for all $0 < \zeta \leq 1$, we immediately have

$$
\begin{aligned}
\lim_{n\longrightarrow\infty} X[n,\zeta] &= \lim_{n\longrightarrow\infty} \cos^n\zeta \\
&= \begin{cases} 1, & \zeta = 0, \\ 0, & 0 < \zeta \leq 1. \end{cases}
\end{aligned}
$$

Hence this sequence converges a.e. (a.s.). Going on to check m.s. convergence, we have

$$
\begin{aligned}
E[|\cos^n\zeta - 0|^2] &= E[\cos^{2n}\zeta|] \\
&\longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned}
$$

hence it converges to 0 in the m.s. sense too.

(c) i)

$$X[n,\zeta] = e^{-n\zeta} \longrightarrow_{n\longrightarrow\infty} 0 \quad \text{(a.s.)}$$

ii)

$$X[n,\zeta] = \sin\left(\zeta + \frac{1}{n}\right) \longrightarrow_{n\longrightarrow\infty} \sin\zeta$$

iii)

$$X[n,\zeta] = \cos^n\zeta \longrightarrow_{n\longrightarrow\infty} 0 \quad \text{(a.s.)}.$$

48. If $X[n] \longrightarrow X$ in the m.s. sense, then X is independent of $X[n]$ since $X[n]$ is an independent random sequence, i.e. a sequence of independent RVs. Thus, and also since $X[n]$ is real valued,

$$
\begin{aligned}
E[|X[n] - X|^2] &= E[X^2[n]] - 2E[X[n]X] + E[X^2] \\
&= E[X^2[n]] - 2E[X[n]]E[X] + E[X^2].
\end{aligned}
$$

Also, from the given pdf information, we see that

$$f_X(x;n) \longrightarrow_{n\longrightarrow\infty} N(\sigma,\sigma^2),$$

thus X will be Gaussian distributed with mean $E[X]=\sigma$ and mean-square $E[X^2]=\sigma^2+\sigma^2=2\sigma^2$. So continuing with the m.s. calculation above, we get

$$\begin{aligned} E[|X[n]-X|^2] &= E[X^2[n]]-2E[X[n]]E[X]+E[X^2] \\ &\longrightarrow 2\sigma^2-2\sigma^2+2\sigma^2 \qquad \text{as } n\longrightarrow\infty, \\ &= 2\sigma^2 \neq 0, \end{aligned}$$

(i) thus there is no convergence in the m.s. sense here.

(ii) There is no convergence in probability either.

(iii) We have shown above that $X[n]$ converges in density (pdf), hence also in distribution (CDF), with the limiting pdf, the pdf of X given as

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x-\sigma)^2\right), \qquad ,-\infty<x<\infty.$$

49. (a) By integration, we have

$$\begin{aligned} f_X(\beta;n) &= \int_{-\infty}^{+\infty}\frac{mn}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(m^2\alpha^2-2\rho mn\alpha\beta+n^2\beta^2\right)d\alpha \\ &= \frac{n}{\sqrt{2\pi}}\exp\left(-\frac{n^2}{2}\beta^2\right), \quad \text{by known property of joint Gaussian pdf,} \end{aligned}$$

therefore $X[n]$ is Gaussian distributed as $N(0,\frac{1}{n^2})$. Guessing that the limit RV is 0, we compute

$$\begin{aligned} \lim_{n\longrightarrow\infty} E[|X[n]-0|^2] &= \lim_{n\longrightarrow\infty} E[X^2[n]] \\ &= \lim_{n\longrightarrow\infty} \sigma_X^2[n]] \\ &= \lim_{n\longrightarrow\infty}\frac{1}{n^2} \\ &= 0. \end{aligned}$$

Thus the random sequence $X[n]$ converges to $X=0$ in the mean-square sense.

(b) Since we have the degenerate RV $X=0$, it means that its CDF is $F_X(x)=u(x)$, the unit step function.

50. Since convergence in m.s. implies convergence in distribution, the limiting distribution will be Gaussian so long as the means converge to a finite number $-\infty<\mu<+\infty$ and the variances converge to a *positive* finite value $0<\sigma^2<\infty$. The case where $\sigma^2=0$ might be called 'degenerate' Gaussian with $p_X(x)=\delta(x-\mu)$.

51.

$$
\begin{aligned}
X[n] &= X[n-1] + W[n], \qquad n \geq 1, \\
&= X[n-2] + W[n-1] + W[n], \qquad n \geq 2, \\
&\vdots \\
&= 0 + W[1] + W[2] + \cdots + W[n] \\
&= \sum_{k=1}^{n} W[k], \qquad \text{a sum of independent RVs.}
\end{aligned}
$$

(a) For the mean,

$$
\begin{aligned}
\mu_X[n] &\triangleq E[X[n]] \\
&= E\left[\sum_{k=1}^{n} W[k]\right] \\
&= \sum_{k=1}^{n} E[W[k]] \\
&= \sum_{k=1}^{n} \eta \\
&= n\eta.
\end{aligned}
$$

For the variance,

$$
\begin{aligned}
\sigma_X^2 &\triangleq \operatorname{Var}[X[n]] \\
&= \operatorname{Var}\left[\sum_{k=1}^{n} W[k]\right] \\
&= \sum_{k=1}^{n} \operatorname{Var}[W[k]], \text{ since the } W[k] \text{ are uncorrelated,} \\
&= \sum_{k=1}^{n} \sigma^2 \\
&= n\sigma^2.
\end{aligned}
$$

(b) We apply Chebyshev's inequality to the RV $X[n]/n$, with mean

$$
\begin{aligned}
E\left[\frac{X[n]}{n}\right] &= \frac{1}{n} n\eta \\
&= \eta,
\end{aligned}
$$

and variance

$$
\begin{aligned}
\operatorname{Var}\left[\frac{X[n]}{n}\right] &= \frac{1}{n^2} n\sigma^2 \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

Then, by Chebyshev,

$$\begin{aligned} P\left[\left|\frac{X[n]}{n} - \eta\right| > \epsilon\right] &\leq \frac{\sigma^2}{n\epsilon} \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

This is just the condition for convergence in probability, which is expressed notationally as

$$\frac{X[n]}{n} \longrightarrow \eta \qquad (p).$$

52. In this problem, we assume real-valued random variables.

(a) Let $A \triangleq \{X[n] \leq x\}, B \triangleq \{|X[n]-X| \geq \epsilon\}$, and $C \triangleq \{X \leq x-\epsilon\}$, then $A^c \cap C \subset B$, which implies that $P[A^c \cap C] \leq P[B]$. But also, $(A^c \cap C) \cup A \supset C$, which implies $P[A^c \cap C] + P[A] \geq P[C]$, or equivalently $P[A^c \cap C] \geq P[C] - P[A]$. Combining these two upper and lower bounds on $P[A^c \cap C]$, we get $P[C] - P[A] \leq P[B]$ or equivalently $P[C] \leq P[A] + P[B]$, the statement to be proved, i.e. $P[X \leq x - \epsilon] \leq P[X[n] \leq x] + P[|X[n] - X| \geq \epsilon]$.

(b) Here we let $A \triangleq \{X[n] > x\}, B \triangleq \{|X[n]-X| \geq \epsilon\}$, and $C \triangleq \{X > x+\epsilon\}$, then $A^c \cap C \subset B$, since $A^c \cap C \Longrightarrow B$. Therefore $P[A^c \cap C] \leq P[B]$. Also, we have $P[A^c \cap C] \geq P[C] - P[A]$, hence $P[C] \leq P[A] + P[B]$, which is $P[X > x + \epsilon] \leq P[X[n] > x] + P[|X[n] - X| \geq \epsilon]$.

(c) By result of part (a)

$$F_X(x - \epsilon) \leq F_X(x; n) + P[|X[n] - X| \geq \epsilon].$$

Then, since $X[n] \longrightarrow_{n \longrightarrow \infty} X \ \ (p)$, i.e. convergence in probability, taking the limit as $n \longrightarrow \infty$ of this equation yields $F_X(x - \epsilon) \leq \lim_{n \longrightarrow \infty} F_X(x; n)$. Since this is true for any $\epsilon > 0$, we must have

$$F_X(x) \leq \lim_{n \longrightarrow \infty} F_X(x; n) \tag{7}$$

Then by part (b), we have $1 - F_X(x + \epsilon) \leq 1 - \lim_{n \longrightarrow \infty} F_X(x; n)$. Since this must be true for any $\epsilon > 0$, we must have $1 - F_X(x) \leq 1 - \lim_{n \longrightarrow \infty} F_X(x; n)$, or equivalently

$$F_X(x) \geq \lim_{n \longrightarrow \infty} F_X(x; n). \tag{8}$$

Comparing (7) and (8), we can then conclude that

$$\lim_{n \longrightarrow \infty} F_X(x; n) = F_X(x).$$

53. (a) Define $Y_N[n] \triangleq \sum_{k=-N}^{+N} h[k]X[n-k]$ and then consider the Cauchy convergence criterion. Let M and N be positive integers,

$$\begin{aligned} & E[|Y_N[n] - Y_M[n]|^2] \\ = \; & E[|Y_N[n]|^2] - E[Y_N[n]Y_M^*[n]] - E[Y_M[n]Y_N^*[n]] + E[|Y_M[n]|^2]. \end{aligned} \tag{9}$$

A general term is the second one $E[Y_N[n]Y_M^*[n]]$, so we investigate its convergence

$$\begin{aligned} E[Y_N[n]Y_M^*[n]] &= E\left[\sum_{k=-N}^{+N} \sum_{l=-M}^{+M} h[k]X[n-k]h^*[l]X^*[n-l]\right] \\ &= \sum_{k=-N}^{+N} \sum_{l=-M}^{+M} h[k]h^*[l]E[X[n-k]X^*[n-l]] \\ &= \sum_{k=-N}^{+N} \sum_{l=-M}^{+M} h[k]h^*[l]R_{XX}[n-k, n-l]. \end{aligned}$$

Since this is a general term, if the limiting sum exists, i.e.

$$\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} h[k]h^*[l]R_{XX}[n-k, n-l] < \infty,$$

then all four terms in (9) are finite and equal, and because of the equal number of plus and minus signs, add up to zero, thus

$$\lim_{N,M \longrightarrow \infty} E[|Y_N[n] - Y_M[n]|^2] = 0.$$

So, we say that $Y[n] \triangleq \sum_{k=-\infty}^{+\infty} h[k]X[n-k]$ exists in the sense of a m.s. limit (by the Cauchy criterion) and write

$$Y[n] = \sum_{k=-\infty}^{+\infty} h[k]X[n-k] \qquad \text{(m.s.)}.$$

(b) Here the correlation function $R_{XX}[m+n, n] = R_{XX}[m]$, so the above Cauchy convergence condition becomes

$$\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} h[k]h^*[l]R_{XX}[l-k] < \infty,$$

or equivalently

$$h[m] * h^*[-m] * R_{XX}[m]|_{m=0} < \infty.$$

(c) Here we further specialize to $R_{XX}[m] = \sigma^2\delta[m]$ and then the m.-s. existence condition becomes

$$\sigma^2 \sum_{k=-\infty}^{+\infty} |h[m]|^2 < \infty,$$

in words we would say that, for finite σ^2, that the impulse response h must be square summable.

54. Proof by mathematical induction: At $n = 0$, it is obviously true, and at $n = 1$,

$E[X[m+1]|X[m], X[m-1], \cdots, X[0]] = X[m]$ by the Martingale definition.

Now for the general step in mathematical induction: Take n an arbitrary positive integer and assume that we have established the result for this n, i.e. that for all positive integers m that

$$E[X[m+n]|X[m], X[m-1], \cdots, X[0]] = X[m],$$

then we must show it is also true at $n+1$. But by definition of a Martingale, we also have

$$E[X[m+n+1]|X[m+n], X[m+n-1], \cdots, X[0]] = X[m+n],$$

which implies

$$\begin{aligned} & E\left[E[X[m+n+1]|X[m+n], X[m+n-1], \cdots, X[0]]|X[m], \cdots, X[0]\right] \\ = \;& E[X[m+n]|X[m], \cdots, X[0]] \\ = \;& X[m]. \end{aligned}$$

But there is a general property of conditional expectation that for two random vectors $\mathbf{Y}$ and $\mathbf{Z}$ satisfying $\mathbf{Z} = \mathbf{g}(\mathbf{Y})$ that

$$E[[E[X|\mathbf{Y}]|\mathbf{Z}] = E[X|\mathbf{Z}].$$

Applying this general result here, we take $\mathbf{Z}^T = (X[m], ..., X[0])$ and $\mathbf{Y}^T = (X[m+n], X[m+n-1], ..., X[0])$. Then, we get

$$\begin{aligned} E[X[m+n+1]|X[m], ..., X[0] &= E[X|Z] \\ &= E[E[X|Y]|Z] \end{aligned}$$

$$\begin{aligned} &= E[E[X[m+n+1]|X[m+n], X[m+n-1], ..., X[0]]|X[m], ..., X[0]] \\ &= E[X[m+n]|X[m], ..., X[0]] \\ &= X[m]. \end{aligned}$$

This completes the general step in math induction. So we have established that, for all $n \geq 0$,

$$E[X[m+n]|X[m], X[m-1], ..., X[0]] = X[m].$$

55. We use the same property of conditional expectation as in problem 8.54, i.e. $E[[E[X|\mathbf{Y}]|\mathbf{Z}] = E[X|\mathbf{Z}]$, valid in the case where $\mathbf{Z} = \mathbf{g}(\mathbf{Y})$ to conclude

$$\begin{aligned} E[E[X|Y[0], ..., Y[n+1]|Y[0], ..., Y[n]] &= E[X|Y[0], ..., Y[n]] \quad (10) \\ &= G[n]. \quad (11) \end{aligned}$$

But

$$E[E[G[n+1]|Y[0], ..., Y[n]]|G[0], ..., G[n]] = E[G[n+1]|G[0], ..., G[n]]. \quad (12)$$

From (10) we know

$$\begin{aligned} E[E[G[n+1]|Y[0], ..., Y[n]]|G[0], ..., G[n]] &= E[G[n]|G[0], ..., G[n]] \\ &= G[n]. \quad (13) \end{aligned}$$

Finally, comparing (12) and (13), we have

$$E[G[n+1]|G[0], ..., G[n]] = G[n].$$

Thus $G[n]$ is a Martingale.

56. (a) Proof: For $0 < j < n$, define the mutually exclusive events:

$$A_j \triangleq \{|G[k] \geq \epsilon \text{ for first time at } j\}.$$

Then the event $\{\max_{0 \leq k \leq n} |G[k]| \geq \epsilon\}$ is just the union of these disjoint events. Also define the random variables

$$I_j \triangleq \begin{cases} 1, & \text{if } A_j \text{ occurs,} \\ 0, & \text{else,} \end{cases}$$

called the *indicators* of the events A_j. Then, since $\sum_{j=0}^{n} I_j \leq 1$,

$$E[G^2[n]] \geq \sum_{j=0}^{n} E[G^2[n] I_j]. \quad (14)$$

Upon writing $G^2[n] = (G[j] + G[n] - G[j])^2$, expanding it, and inserting into (14), we get

$$
\begin{aligned}
E[G^2[n]] &\geq \sum_{j=0}^{n} E[G^2[j]I_j] + 2\sum_{j=0}^{n} E[G[j](G[n] - G[j])I_j] \\
&\qquad + \sum_{j=0}^{n} E[(G[n] - G[j])^2 \, I_j] \\
&\geq \sum_{j=0}^{n} E[G^2[j]I_j] + 2\sum_{j=0}^{n} E[G[j](G[n] - G[j])I_j]. \qquad (15)
\end{aligned}
$$

Letting $Z_j \triangleq G[j]I_j$, we can write the second term in (15) as $E[Z_j(G[n] - G[j])]$ and noting that Z_j depends only on $X[0], X[1], ..., X[j]$, we then have

$$
\begin{aligned}
E[Z_j(G[n] - G[j])] &= E[E[Z_j(G[n] - G[j])|X[0], ..., X[j]]] \\
&= E[Z_j E[G[n] - G[j]|X[0], ..., X[j]]] \\
&= E[Z_j(G[j] - G[j])] \\
&= 0.
\end{aligned}
$$

Thus (15) becomes

$$
\begin{aligned}
E[G^2[n]] &\geq \sum_{j=0}^{n} E[G^2[j]I_j] \\
&\geq \epsilon^2 E\left[\sum_{j=0}^{n} I_j\right] \\
&= \epsilon^2 P\left[\bigcup_{j=0}^{n} A_j\right] \\
&= \epsilon^2 P\left[\max_{0\leq k\leq n} |G[k]| \geq \epsilon\right].
\end{aligned}
$$

(b) Proof: Let $m \geq 0$ and define $Y[n] \triangleq G[n+m] - G[m]$ for $n \geq 0$. Then $Y[n]$ is a Martingale, so by the result in part (a)

$$
P\left[\max_{0\leq k\leq n} |G[m+k] - G[m]| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} E[Y^2[n]],
$$

where

$$
\begin{aligned}
E[Y^2[n]] &= E[(G[n+m] - G[m])^2] \\
&= E[G^2[n+m]] + 2E[G[n+m]G[m]] + E[G^2[m]].
\end{aligned}
$$

Then, upon rewriting the middle term, we have

$$
\begin{aligned}
E[G[n+m]G[m]] &= E[G[m]E[G[n+m]|X[m], ..., X[0]]] \\
&= E[G[m]G[m]] \\
&= E[G^2[m]],
\end{aligned}
$$

since G is a Martingale wrt X. So,

$$\begin{aligned} E[Y^2[n]] &= E[G^2[n+m]] - E[G^2[m]] \\ &\geq 0 \quad \text{for all } m, n \geq 0. \end{aligned} \tag{16}$$

Therefore, $E[G^2[n]]$ must be monotone non-decreasing. Since it is bounded from above by $C < \infty$, it thus must converge to a limit as $n \longrightarrow \infty$. Since it has a limit, then by (16), the $E[Y^2[n]] \longrightarrow 0$ as m and $n \longrightarrow \infty$. Thus

$$\lim_{m \longrightarrow \infty} P[\max_{k \geq 0} |G[m+k] - G[m]| > \epsilon] = 0,$$

which implies $P[\lim_{m \longrightarrow \infty} \max_{k \geq 0} |G[m+k] - G[m]| > \epsilon$ by the continuity of probability measure. Finally by the Cauchy convergence criteria, there exists a random variable G such that

$$G[n] \longrightarrow G \qquad \text{(a.s.)}.$$

57. We proceed taking expectations as

$$\begin{aligned} & E[L_X[n+1] | X[0], ..., X[n], H_0] \\ = & \int_{-\infty}^{+\infty} \frac{f_X(X[0], ..., X[n], x[n+1] | H_1)}{f_X(X[0], ..., X[n], x[n+1] | H_0)} f_X(x[n+1] | X[0], ..., X[n], H_0) dx[n+1] \\ = & \int_{-\infty}^{+\infty} \frac{f_X(X[0], ..., X[n], x[n+1] | H_1)}{f_X(X[0], ..., X[n] | H_0)} \frac{f_X(x[n+1] | X[0], ..., X[n], H_0)}{f_X(x[n+1] | X[0], ..., X[n], H_0)} dx[n+1] \\ = & \int_{-\infty}^{+\infty} \frac{f_X(X[0], ..., X[n], x[n+1] | H_1)}{f_X(X[0], ..., X[n] | H_0)} dx[n+1] \\ = & \frac{1}{f_X(X[0], ..., X[n] | H_0)} \int_{-\infty}^{+\infty} f_X(X[0], ..., X[n], x[n+1] | H_1) dx[n+1] \\ = & \frac{f_X(X[0], ..., X[n] | H_1)}{f_X(X[0], ..., X[n] | H_0)} \\ = & L_X[n]. \end{aligned}$$

Thus, by the definition, the likelihood ratio $L_X[n]$ is a Martingale wrt $X[n]$ under hypothesis H_0.

58.

$$
\begin{aligned}
E\left[\left|\sum_{n=-N}^{+N} X_e[n]e^{-j\omega n}\right|^2\right] &= \sum_{n=-N}^{+N}\sum_{m=-N}^{+N} E[X_e[n]X_e^*[m]]e^{-j\omega(n-m)} \\
&= \sum_{n,m \text{ even}} E[X[\frac{n}{2}]X^*[\frac{m}{2}]]e^{-j\omega(n-m)} \\
&= \sum_{k,l=-N/2}^{+N/2} E[X[k]X^*[l]]e^{-j\omega 2(k-l)}, \quad \text{with } k \triangleq \frac{n}{2}, l \triangleq \frac{m}{2}, \\
& \sum_{k,l=-N/2}^{+N/2} R_{XX}[k-l]e^{-j\omega 2(k-l)} \\
&= \sum_{k,l=-\infty}^{+\infty} R_{XX}[k-l]\text{rect}(\frac{k}{N})\text{rect}(\frac{l}{N})e^{-j\omega 2(k-l)} \\
&= \sum_{k',l=-\infty}^{+\infty} R_{XX}[k']\text{rect}(\frac{k'+l}{N})\text{rect}(\frac{l}{N})e^{-j\omega 2k'}, \quad \text{with} \quad k' \triangleq k-l, \\
&= \sum_{k'=-\infty}^{+\infty} R_{XX}[k'] \sum_{l=-\infty}^{+\infty}\left(\text{rect}(\frac{k'+l}{N})\text{rect}(\frac{l}{N})\right)e^{-j\omega 2k'} \\
&= \sum_{k'=-N}^{+N} R_{XX}[k']\left(N-|k'|\right)e^{-j\omega 2k'}.
\end{aligned}
$$

Finally, we divide this result by $2N+1$ to get

$$
\frac{N}{2N+1}\sum_{k'=-N}^{+N} R_{XX}[k']\left(1-\frac{|k'|}{N}\right)e^{-j\omega 2k'}.
$$

and then taking the limit as $N \to \infty$ yields

$$
S_{X_eX_e}(\omega) = \frac{1}{2}S_{XX}(2\omega).
$$

59. The randomized sequence that is expanded by 2 is denoted as $X_e^{(r)}[n]$. Then

$$
\begin{aligned}
E[X_e^{(r)}[m]X_e^{(r)}[m+k]] &= E\left[X\left[\frac{m+\Theta}{2}\right]X\left[\frac{m+k+\Theta}{2}\right]\right] \\
&= E\left[E_\Theta\left[X\left[\frac{m+\Theta}{2}\right]X\left[\frac{m+k+\Theta}{2}\right]\right]\right],
\end{aligned}
$$

where the inner expectation is taken only over Θ. Taking this inner expectation, we then get

$$
E\left[E_\Theta\left[X\left[\frac{m+\Theta}{2}\right]X\left[\frac{m+k+\Theta}{2}\right]\right]\right] = \frac{1}{2}E\left[X\left[\frac{m}{2}\right]X\left[\frac{m+k}{2}\right]\right]+\frac{1}{2}E\left[X\left[\frac{m+1}{2}\right]X\left[\frac{m+k+1}{2}\right]\right].
$$

We the observe that when m, k =odd,even, only the first term only is non-zero, and when m, k =even,even, only the second term is non-zero. Hence

$$E[X_e^{(r)}[m]X_e^{(r)}[m+k]] = \begin{cases} \frac{1}{2}R_{XX}[k/2], & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Likewise, for the randomized version of the original WSS random sequence $X[n]$, we denote it as $X^{(r)}[n] = X[n+\Theta]$. Then we similarly get

$$\begin{aligned} E[E[X^{(r)}[m]X^{(r)}[m+k]] &= \frac{1}{2}E\left[X\left[m\right]X\left[m+k\right]\right] + \frac{1}{2}E\left[X\left[m+1\right]X\left[m+k+1\right]\right] \\ &= \frac{1}{2}R_{XX}[k] + \frac{1}{2}R_{XX}[k], \quad \text{by WSS property}, \\ &= R_{XX}[k]. \end{aligned}$$