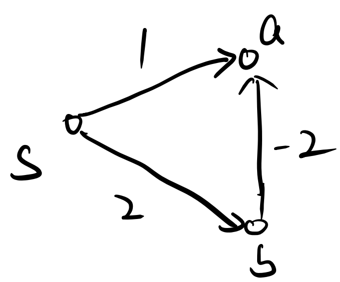
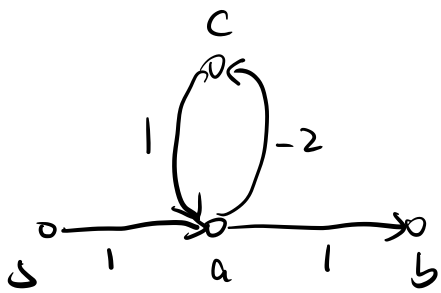
Input: a directed graph $G=(V,E)$, a edge cost $c(e)$ for each edge e,
and a source $s \in V$.

Output: 1. the shortest-path distance $dist(s,v)$ for every $v \in V$.
if G has no negative cycle.
2. report "G has a negative cycle" if G has a negative cycle.

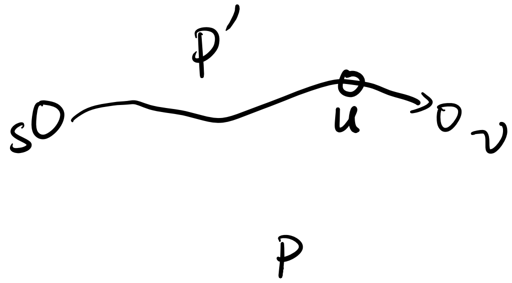
Dijkstra's $O(|V|\log|V| + |E|)$ assuming $c(e) \geq 0$



$dist(s,s)=0$
$dist(s,a)=1$
$dist(s,b)=2$



Bellman-Ford algorithm DP



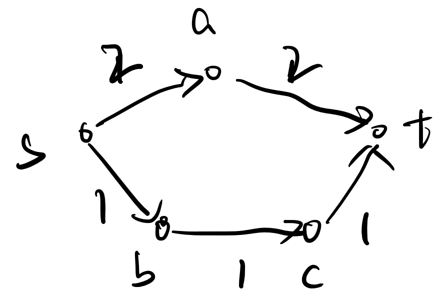
1. P' is a shortest path from s to u.

2. P' contains fewer edges than P.

subproblem

for $i \geq 0$, for $v \in V$,

let C[v][i] be the distance of the shortest path with *at most* i edges from s to v.
(C[v][i] = +∞ if no ~~path~~ such path exists)



C[t][0] = +∞
C[t][1] = +∞
C[t][2] = 4
C[t][3] = 3

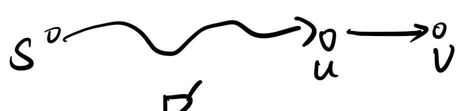
Recurrence

$P^* \longrightarrow C[v][i]$

case 1. P^* contains at most i-1 edge

$P^* \longrightarrow C[v][i-1]$

case 2. P^* contains exactly i edges



$P' \longrightarrow C[u][i-1]$

for $i \geq 1$, for $v \in V$

→ case 2

$$C[v][i] = \min \left\{ C[v][i-1],\ \min_{(u,v)\in E} \{C[u][i-1] + c(u,v)\} \right\}$$

base case $i=0$

$C[s][0]=0$

$C[v][0]=+\infty$ for $v\neq s$

BF(G):

1. $C[s][0]=0$
2. $C[v][0]=+\infty$ for $v\neq s$
3. for $i=1,2,3\ldots n$
4. for $v\in V$
5. $C[v][i]=\min\left\{\begin{array}{l}C[v][i-1]\\ \min\limits_{(u,v)\in E}\{C[u][i-1]+c(u,v)\}\end{array}\right\}$
6. If $C[v][n-1]=C[v][n]$ for all $v\in V$

   return $\{C[v][n-1]\}_{v\in V}$

7. else

   report "G has a negative cycle"

step 5 $O[1+\text{in-deg}(v)]$

Step 4~5 $\sum\limits_{v\in V} O(1+\text{in-deg}(v))$

$\simeq O(|V|+|E|)$

$\geq O(|E|)$

Lemma (Stopping Criterion)

If for some $k\geq 0$

$C[v][k+1]=C[v][k]$ for any $v\in V$

	round $k+1$	round $k+2$
Input	$C[v][k]$	$C[v][k+1]$

Output $C[v][k+1]$ $C[v][k+2]$

1. $C[v][i] = C[v][k]$ for every $i > k$ and every $v \in V$

2. $C[v][k] = \text{dist}(s,v)$ for every $v \in V$.

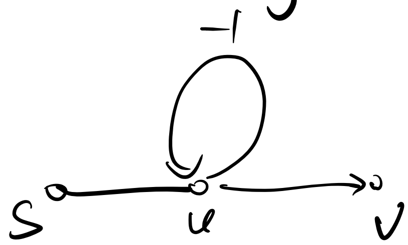
   if $C[v][k] > \text{dist}(s,v)$,

   $\exists\, P^*$, $\text{cost}(P^*) < C[v][k]$

   $|P^*| > k$

   $\exists\, i = |P^*| > k$, $C[v][i] < C[v][k]$ contradiction!

3. G has no negative cycle.



Lemma

G has no negative cycle if and only if

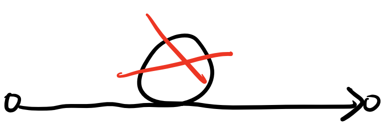
$C[v][n-1] = C[v][n]$ for all $v \in V$

where $n = |V|$.

Proof:

$\Leftarrow$) trivial

$\Rightarrow$) G has no negative cycle



$\Downarrow$

for every $v \in V$, at least one shortest path from s to v has no cycle.

$\Downarrow$

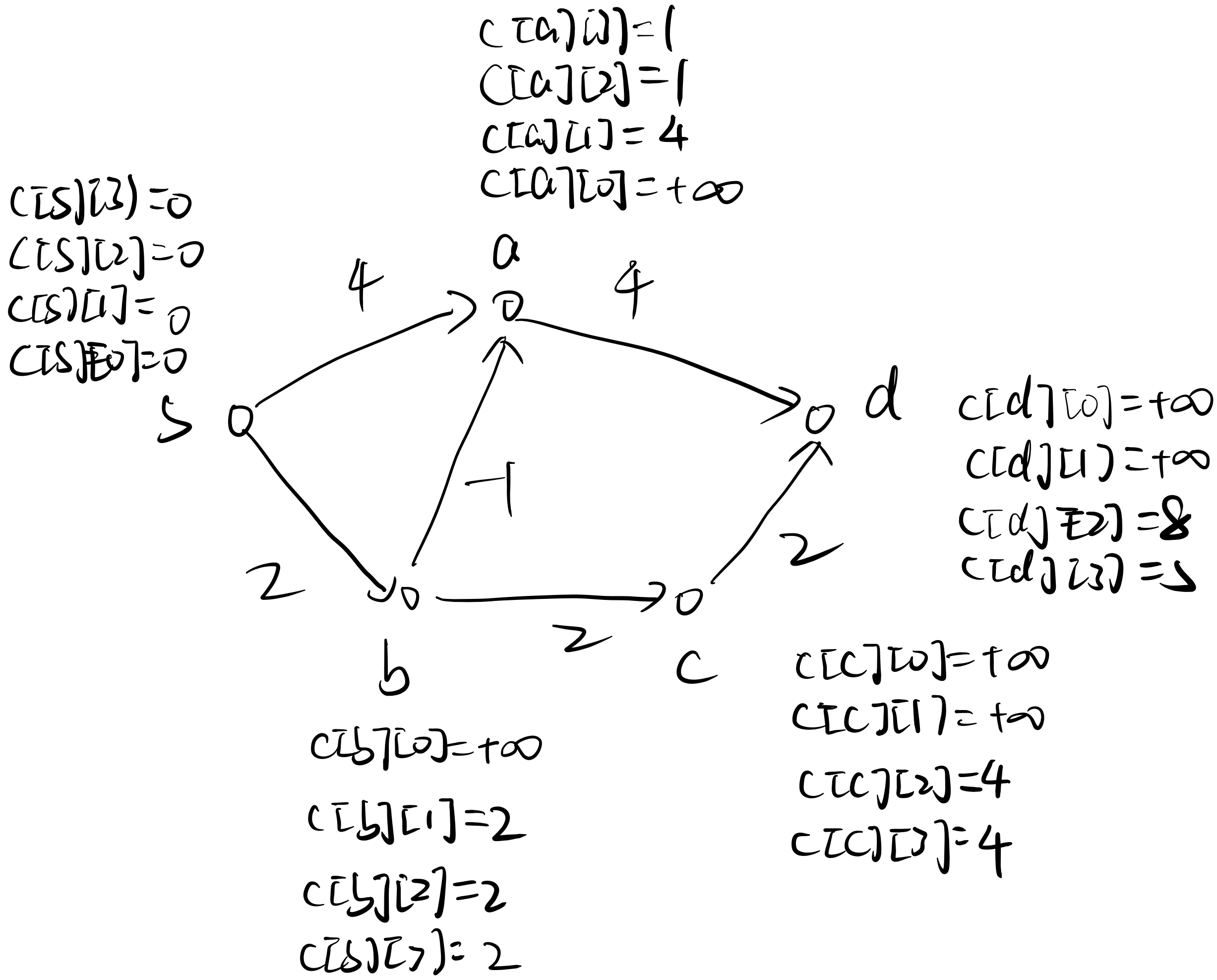
has at most $n-1$ edges.

$\Downarrow$

$C[v][n-1] = C[v][n]$

Time $O(|V||E|)$

Space $O(|V|^2) \to O(|V|)$



All pairs shortest path

Input: a directed $G=(V,E)$, an edge cost $c(e)$ for $e \in E$

Output: $dist(u,v)$ for every pair $(u,v) \in V \times V$.

If $c(e) \geq 0$,

Dijkstra's $\times |V|$ $\quad O(|V|^2 \log|V| + |E||V|)$

If $c(e) < 0$ for some e. (assuming no negative cycle)

Bellman-Ford $\times |V|$ $\quad O(|E||V|^2)$

Floyd-Warshall $\quad O(|V|^3)$ $\quad$ DP

$|V|$: $v_1, v_2, v_3, \ldots, v_n$

P:  rank(P) = largest index of internal nodes of P

(rank(P) = 0 if no internal node)

$v_9 \to v_2 \to v_5 \to v_1 \to v_{10}$ $\quad$ rank = 5

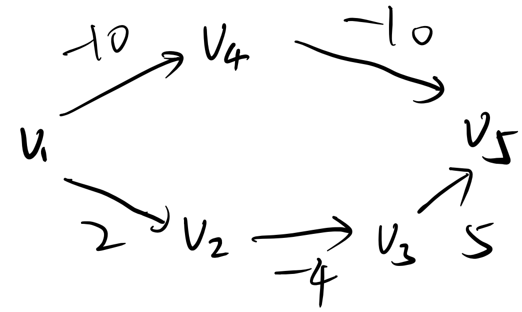
Subproblem

for $1 \leq i \leq n$, $1 \leq j \leq n$, $0 \leq k \leq n$

having no cycle

let $C[i][j][k]$ = length of the shortest simple path with rank* at most k^* from v_j to v_i.

($c[i][j][k] = +\infty$ if no such path exist)



$c[1][5][0] = +\infty$

$c[1][5][1] = +\infty$

$c[1][5][2] = +\infty$

$c[1][5][3] = 3$

$c[1][5][4] = -20$

$c[1][4][0] = -10$

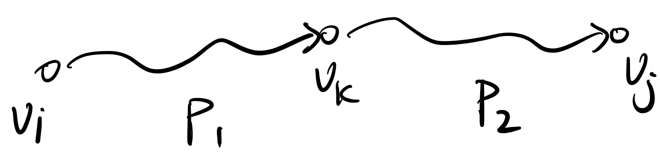
Recurrence

$P^* \rightarrow c[i][j][k]$

case 1. v_k is not an internal node of P^*

$P^* \rightarrow c[i][j][k-1]$

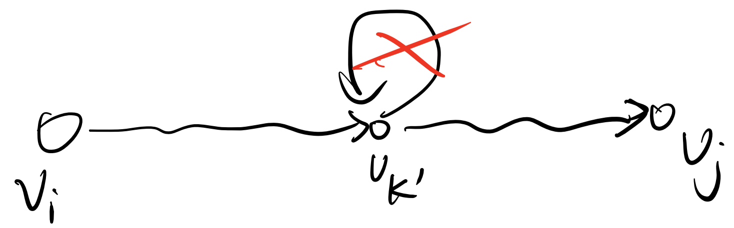
case 2. v_k is an internal node of P^*



$P_1 \rightarrow c[i][k][k-1]$

$P_2 \rightarrow c[k][j][k-1]$

$P_1 \cap P_2 = v_{k'}$



for $1 \le i \le n$ $\quad 1 \le j \le n$ $\quad 1 \le k \le n$

$$C[i][j][k] = \min\{ C[i][j][k-1],\ C[i][k][k-1] + C[k][j][k-1] \}$$

Base case $k=0$

$$C[i][j][0] = \begin{cases} 0 & \text{if } i=j \\ c(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ +\infty & \text{otherwise} \end{cases}$$

FW(G)

1. Initialize base case
2. for $k=1$ to n
3. for $i=1$ to n
4. for $j=1$ to n
5. $C[i][j][k] = \min\{ C[i][j][k-1],\ C[i][k][k-1] + C[k][j][k-1] \}$

Time $O(n^3)$

Space $O(n^3) \to O(n^2)$

Remark

G has a negative cycle
if and only if
$c[i][i][n] < 0$ for some i