STAT 210
Applied Statistics and Data Analysis
Week 6 - Summary

Joaquin Ortega

King Abdullah University of Science and Technology

Announcements

- First exam will be on Saturday, October 22, 9:00 - 12:00 am, Room 2322.
- The exam is based on R. You will need to bring your computers.
- You can use the notes, presentations, books and exercises we have solved in the class, but you are not allowed to use resources from the internet outside KAUST.
- The exam will be posted in Blackboard at 9:00 am and you have to submit your solution through Blackboard at 12:00 noon.
- You need to submit two documents, a pdf with your answers, and a script with the R code. The script can be a Rmarkdown file.

Video 19: Tables in R

Tabular summaries of data are a frequent starting point for statistical analysis.

A **contingency table** in Statistics is a table that displays the multivariate frequency distribution of two or more variables.

The entries in the cells of a two-way table are the frequency counts or relative frequencies, and the table is usually presented as a matrix.

Karl Pearson first used the name in 1904.

Before looking at the usual statistical techniques for analysis of contingency tables, let us review some of the available tools for producing them.

The function `cut()` divides observations according to the values of a continuous variable.

```
cut(data, breaks, labels = NULL, right = TRUE)
```

- `breaks` defines the break points of each level or class
- `labels` specifies the value to use when an observation falls in one class
- `right` specifies the type of interval: `right = F` is for $[a, b)$ and `right = T` is for $(a, b]$.

Functions for Producing Tables

The function `table()` builds a contingency table of the counts at each combination of factor levels.

```
(table1 <- with(mtcars, table(cyl, gear)))
```

```
##    gear
## cyl  3  4  5
##   4  1  8  2
##   6  2  4  1
##   8 12  0  2
```

The function `prop.table()` produces tables of relative frequencies.

```
prop.table(data, margin)
```

Functions for Producing Tables

```
(table2 <- round(prop.table(table1,1),3))
```

```
##    gear
## cyl     3     4     5
##   4 0.091 0.727 0.182
##   6 0.286 0.571 0.143
##   8 0.857 0.000 0.143
```

```
rowSums(table2)
```

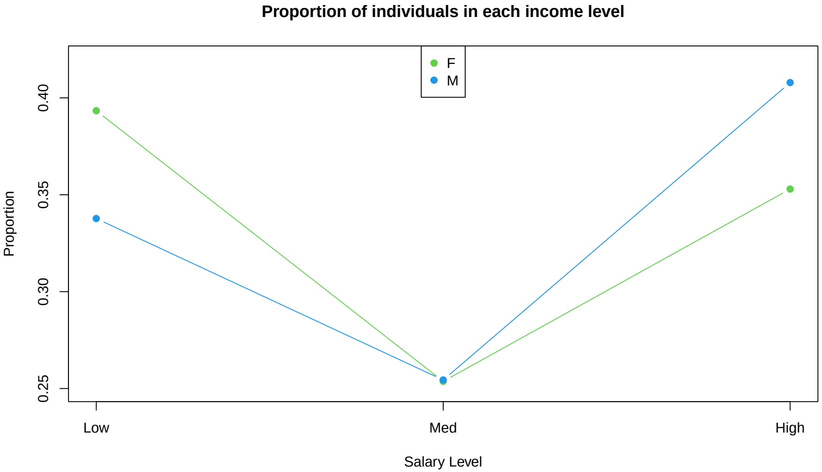
```
## 4 6 8
## 1 1 1
```

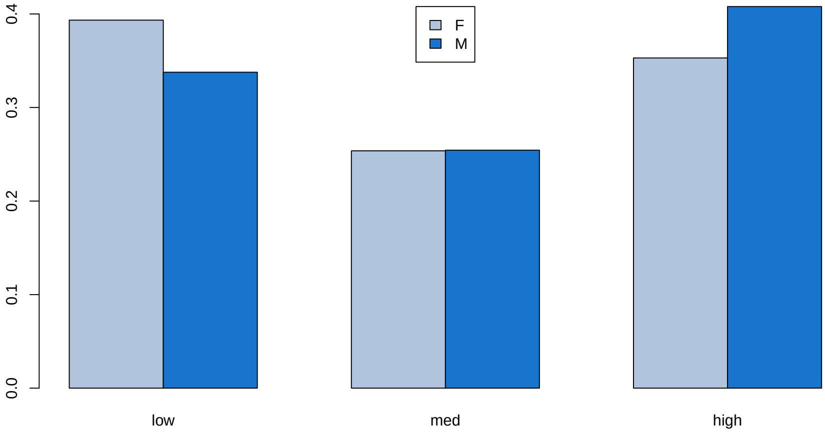
```
(table3 <- round(prop.table(table1,2),3))
```

```
##    gear
## cyl     3     4     5
##   4 0.067 0.667 0.400
##   6 0.133 0.333 0.200
##   8 0.800 0.000 0.400
```

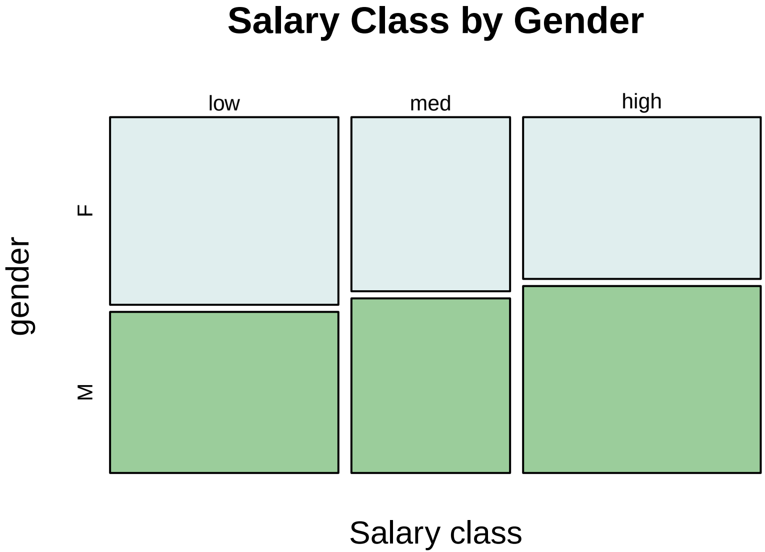
```
colSums(table3)
```

```
## 3 4 5
## 1 1 1
```

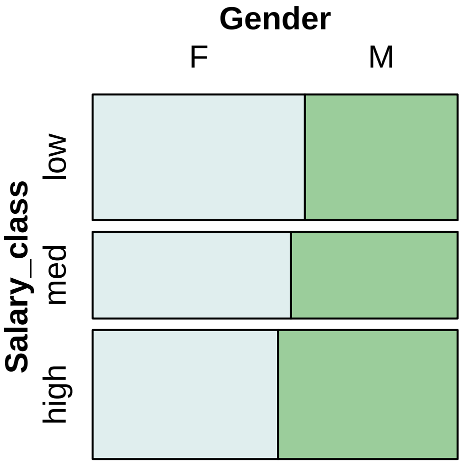

Proportion of individuals in each income level
F
M
0.25
0.30
0.35
0.40
Proportion
Low
Med
High
Salary Level


0.0
0.1
0.2
0.3
0.4
F
M
low
med
high

```
mosaicplot(freq_gender_sal1, xlab='Salary class', ylab='gender',
    main = 'Salary Class by Gender', col = c('azure2','darkseagreen3'))
```



Salary Class by Gender

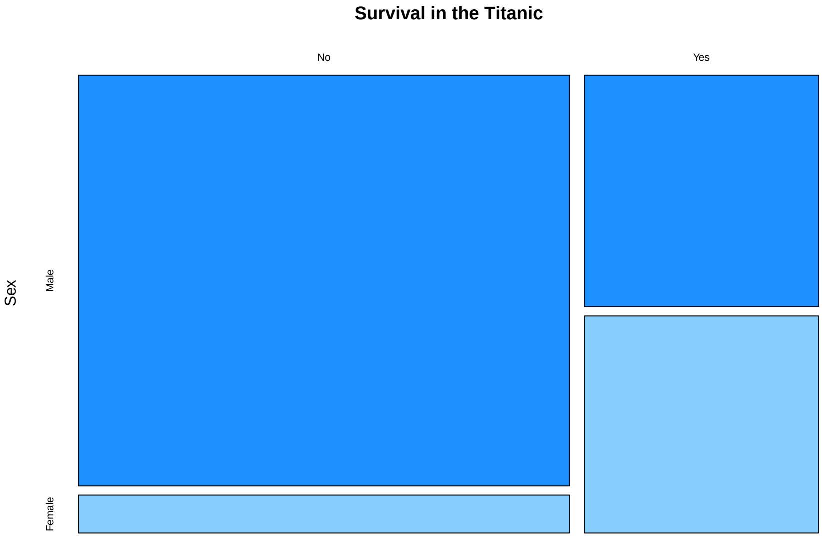


Video 20: Contingency Tables

```r
library(vcd)
data("Titanic")
titanic.table <- apply(Titanic, c(4, 2), sum)
(titanic.table <- addmargins(titanic.table))
```

```
##         Sex
## Survived Male Female  Sum
##      No  1364    126 1490
##      Yes  367    344  711
##      Sum 1731    470 2201
```

```
mosaicplot(titanic.table[1:2,1:2],
           col = c('dodgerblue','skyblue1'),
           main = 'Survival in the Titanic')
```



With this information, we want to explore if survival is related to gender.

The proportion of surviving individuals in the male population is

$$\pi_1 = \frac{367}{1731} = 0.212$$

while for the female population, it is

$$\pi_2 = \frac{344}{470} = 0.732$$

We want to test

$$H_0 : \pi_1 = \pi_2 \quad \text{vs} \quad H_A : \pi_1 \neq \pi_2.$$

In general, the simple case of a 2×2 contingency table can be described as follows: We have two populations or groups, and we want to study whether the presence of some characteristic occurs in the same proportion.

Let us call P1 and P2 the two populations and p_1 and p_2 the proportions of the given trait in each of them.

We take samples of sizes n_1 (from P1) and n_2 (from P2) and $s_i, i = 1, 2$ represent how many trials in each sample were successful, i.e., the individuals have the characteristic.

With these results, we build a contingency table.

Table 1: Observed values

	P1	P2	Total
Success	s_1	s_2	s
Failure	$n_1 - s_1$	$n_2 - s_2$	$n - s$
Total	n_1	n_2	n

Here $s = s_1 + s_2$ is the total number of successes.

Let $d = p_1 - p_2$. We want to use the information in the table to test

$$H_0 : d = 0 \qquad \text{vs} \qquad H_A : d \neq 0$$

Use the data to estimate the proportions:

$$\pi_1 = \frac{s_1}{n_1}, \qquad \pi_2 = \frac{s_2}{n_2}.$$

Under the null hypothesis $p_1 = p_2 = p$.

To estimate p, pool all the information:

$$\pi = \frac{n_1}{n}\pi_1 + \frac{n_2}{n}\pi_2 \Big(= \frac{s_1 + s_2}{n} \Big)$$

If p is the true proportion for both samples, we would expect to have $n_i \times p$ successes and $n_i \times (1 - p)$ failures in sample $i = 1, 2$.

Use π instead of p and create a table of expected values.

How many successes do we **expect** in each population?

	P1	P2	Total
Success			
Failure			
Total	n_1	n_2	n

How many successes do we **expect** in each population?

	P1	P2	Total
Success	$\pi \times n_1$	$\pi \times n_2$	
Failure			
Total	n_1	n_2	n

How many successes do we **expect** in each population?

	P1	P2	Total
Success	$\pi \times n_1$	$\pi \times n_2$	$\pi \times n$
Failure			
Total	n_1	n_2	n

How many failures do we **expect** in each population?

	P1	P2	Total
Success	$\pi \times n_1$	$\pi \times n_2$	$\pi \times n$
Failure	$(1 - \pi) \times n_1$	$(1 - \pi) \times n_2$	
Total	n_1	n_2	n

How many failures do we **expect** in each population?

	P1	P2	Total
Success	$\pi \times n_1$	$\pi \times n_2$	$\pi \times n$
Failure	(1 - π)$\times n_1$	(1 - π)$\times n_2$	(1 - π)$\times n$
Total	n_1	n_2	n

Compare expected values with observed.

If the difference is large, we will question the null hypothesis.

The statistic for Pearson's test is

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

where O stands for observed, and E for expected and the sum runs over all cases.

Under the null hypothesis, this statistic has approximately a χ^2 distribution with $(r-1)(c-1)$ degrees of freedom, where r and c stand for the number of rows and columns of the table.

In R:

```
chisq.test(titanic.table[1:2,1:2],correct = FALSE)
```

```
##
##  Pearson's Chi-squared test
##
## data:  titanic.table[1:2, 1:2]
## X-squared = 456.87, df = 1, p-value < 2.2e-16
```

```
chisq.test(titanic.table[1:2,1:2])
```

```
##
##  Pearson's Chi-squared test with Yates' continuity corre
##
## data:  titanic.table[1:2, 1:2]
## X-squared = 454.5, df = 1, p-value < 2.2e-16
```

Contingency Tables: Independence

The χ^2 test can also be used to test for independence of categorical variables in contingency tables.

Consider as an example the set `survey` in the `MASS` package that has the responses of 237 Statistics students to a series of questions.

We consider

- `Smoke`,

  a factor with four levels: `Heavy`, `Regul` (regularly), `Occas` (occasionally), `Never`, and

- `Exer`,

  how frequently the student exercises, with levels `Freq` (frequently), `Some`, `None`.

We use `table` to produce the contingency table for these two variables.

```
library(MASS)
stdt.tab <- with(survey,table(Smoke,Exer))
stdt.tot <- cbind(stdt.tab,
                  Total = apply(stdt.tab, 1, sum))
(stdt.tot <- rbind(stdt.tot,
                  Total = apply(stdt.tot, 2, sum)))
```

```
##        Freq None Some Total
## Heavy     7    1    3    11
## Never    87   18   84   189
## Occas    12    3    4    19
## Regul     9    1    7    17
## Total   115   23   98   236
```

We want to compare (categorical) variables X and Y with values

$$x_1, \ldots, x_m, \quad \text{and} \quad y_1, \ldots, y_n$$

and probability functions

$$p_1, \ldots, p_m, \quad \text{and} \quad q_1, \ldots, q_n.$$

If the variables are independent

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) = p_i q_j$$

for any $1 \leq i \leq m, 1 \leq j \leq n$.

If the total sample is of size N, we would expect

$$N p_i q_j$$

individuals to be in the ij-th cell of the contingency table.

Since p_i and q_j are unknown, we estimate them by the corresponding proportions.

Use the row totals divided by N to estimate the p_i's and the column totals divided by N to estimate the q_j's.

Let n_{ij} be the number in the ij-th cell for $1 \leq i \leq m, 1 \leq j \leq n$. Introduce the notation:

$$n_{\bullet j} = \sum_{i=1}^{m} n_{ij} \qquad n_{i\bullet} = \sum_{j=1}^{n} n_{ij}$$

$$n_{\bullet\bullet} = \sum_{i=1}^{m} \sum_{j=1}^{n} n_{ij} = N.$$

Then

$$\hat{p}_i = \frac{n_{i\bullet}}{n_{\bullet\bullet}}, \qquad \hat{q}_j = \frac{n_{\bullet j}}{n_{\bullet\bullet}}.$$

The expected value for the number in the ij-th cell is

$$E_{ij} = N\hat{p}_i\hat{q}_j = n_{\bullet\bullet}\frac{n_{i\bullet}}{n_{\bullet\bullet}}\frac{n_{\bullet j}}{n_{\bullet\bullet}} = \frac{n_{i\bullet}n_{\bullet j}}{n_{\bullet\bullet}}.$$

We use the same statistic as before

$$\chi^2 = \sum_{ij} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

where O stands for observed, E for expected, and the sum runs over all cases.

This statistic has a χ^2_ν distribution with

$$\nu = (m-1)(n-1)$$

degrees of freedom.

```
chisq.test(stdt.tab)
```

```
## Warning in chisq.test(stdt.tab): Chi-squared approximati
##
##  Pearson's Chi-squared test
##
## data:  stdt.tab
## X-squared = 5.4885, df = 6, p-value = 0.4828
```

Small Samples: Fisher's Exact Test

The Chi-square distribution approximation requires that the expected value for each cell be at least 5. When this is not satisfied, results can be incorrect.

Under the assumption that the margins (totals) in the contingency table are fixed, it is possible to calculate an exact value for the significance of the deviation from the null hypothesis.

Fisher's exact test is mostly used for 2×2 tables and small samples, but in principle can be extended to general contingency tables, although for large tables, the calculation may be complicated.

For 2×2 tables, the calculation uses the hypergeometric distribution.

Consider a population of size N with K individuals of type A.

The probability that in a sample of size $n \leq N$ there are precisely $k \leq K$ individuals of type A when sampling **without replacement** is given by the **hypergeometric distribution**

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

for $1 \leq n \leq N$ and $0 \leq k \leq K \leq N$. Recall that

$$\binom{a}{b} = \frac{a!}{b!(a-b)!}.$$

Small Samples: Fisher's Exact Test

Titanic data

```
fisher.test(titanic.table[1:2,1:2])
```

```
##
##  Fisher's Exact Test for Count Data
##
## data:  titanic.table[1:2, 1:2]
## p-value < 2.2e-16
## alternative hypothesis: true odds ratio is not equal to
## 95 percent confidence interval:
##   7.97665 12.92916
## sample estimates:
## odds ratio
##    10.1319
```

Small Samples: Fisher's Exact Test

Student data

```
fisher.test(stdt.tab)
```

```
##
##  Fisher's Exact Test for Count Data
##
## data:  stdt.tab
## p-value = 0.4138
## alternative hypothesis: two.sided
```

Video 21: Comparing Proportions

The distribution of the number of individuals of type A, n_A, in the sample is binomial with parameters n and p:

$$P(n_A = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

The expected value and variance for this distribution are given by

$$E(n_A) = np, \qquad Var(n_A) = np(1-p).$$

A natural estimator for the (unknown) proportion p is the observed proportion of individuals of type A in the sample:

$$\pi = \frac{n_A}{n}.$$

It is unbiased with variance

$$Var(\pi) = \frac{1}{n^2} Var(n_A) = \frac{p(1-p)}{n}.$$

By the Central Limit Theorem, for n large, the binomial distribution can be approximated by a normal distribution.

Rule of Thumb
If n and p are such that $np \geq 5$ and $n(1-p) \geq 5$, the binomial distribution can be approximated by the normal distribution.

Thus, if $np \geq 5$ and $n(1-p) \geq 5$ the sampling density for the (sample) proportion π can be approximated by a normal distribution with parameters

$$p \qquad \text{and} \qquad \frac{p(1-p)}{n}.$$

One-sample problem for proportions.

The following data come from Kaye, D.H., *Statistical Evidence of Discrimination*, JASA (1982).

In a case about discrimination against blacks in grand jury selection in Alabama, the plaintiff argued that of the 1050 individuals called to serve as jurors, only 177 were black.

At the time, 25% of those eligible to serve were blacks.

Do the data support the claim of discrimination? We want to test

$$H_0 : p = 0.25 \quad vs \quad p < 0.25$$

and choose a level $\alpha = 0.01$.

```
prop.test(n.A,n,p_0)
```

```
##
##  1-sample proportions test with continuity correction
##
## data:  n.A out of n, null probability p_0
## X-squared = 36.698, df = 1, p-value = 1.379e-09
## alternative hypothesis: true p is not equal to 0.25
## 95 percent confidence interval:
##  0.1466952 0.1929145
## sample estimates:
##         p
## 0.1685714
```

Example

```
binom.test(n.A,n,p_0)
```

```
##
##  Exact binomial test
##
## data:  n.A and n
## number of successes = 177, number of trials = 1050, p-value = 2.454e-10
## alternative hypothesis: true probability of success is not equal to 0.25
## 95 percent confidence interval:
##  0.1464049 0.1926129
## sample estimates:
## probability of success
##              0.1685714
```

Two Independent Proportions

Assume now that we have two samples of sizes n_1 and n_2, respectively, with number of successes m_1 and m_2. The corresponding proportions are

$$\pi_i = \frac{m_i}{n_i}$$

for $i = 1, 2$, and we want to compare these two values.

We want to test

$$H_0 : \pi_1 = \pi_2 \quad \text{vs.} \quad H_A : \pi_1 \neq \pi_2$$

The normal approximation requires

$$\begin{aligned} n_i \times \pi_i &\geq 5 \\ n_i \times (1 - \pi_i) &\geq 5 \end{aligned}$$

for $i = 1, 2$.

The test can be carried out using `prop.test`.

Example

The following matrix corresponds to the number of patients involved in car accidents that survived or died. The use of seat belts is also reported in the data, which were registered at a hospital in North Carolina.

```
car.accidents <- data.frame(survived = c(1781,1443),
                            died=c(135,47))
rownames(car.accidents) <- c('nsb','sb')
car.accidents
```

```
##     survived died
## nsb     1781  135
## sb      1443   47
```

Example

To test whether the use of seat belts affected the rates of survival we compare the proportions using the function `prop.test`

```
prop.test(as.matrix(car.accidents))
```

```
##
##  2-sample test for equality of proportions with continuity correction
##
## data:  as.matrix(car.accidents)
## X-squared = 24.333, df = 1, p-value = 8.105e-07
## alternative hypothesis: two.sided
## 95 percent confidence interval:
##  -0.05400606 -0.02382527
## sample estimates:
##    prop 1    prop 2
## 0.9295407 0.9684564
```