

## hw4

### Coding the Matrix, Summer 2013

Please fill out the stencil file named “hw4.py”. While we encourage you to complete the Ungraded Problems, they do not require any entry into your stencil file.

### Span of vectors over $\mathbb{R}$

**Problem 1:** Let  $\mathcal{V} = \text{Span} \{[2, 0, 4, 0], [0, 1, 0, 1], [0, 0, -1, -1]\}$ . For each of the following vectors, show it belongs to  $\mathcal{V}$  by writing it as a linear combination of the generators of  $\mathcal{V}$ .

- (a)  $[2, 1, 4, 1]$
- (b)  $[1, 1, 1, 0]$
- (c)  $[0, 1, 1, 2]$

**Problem 2:** Let  $\mathcal{V} = \text{Span} \{[0, 0, 1], [2, 0, 1], [4, 1, 2]\}$ . For each of the following vectors, show it belongs to  $\mathcal{V}$  by writing it as a linear combination of the generators of  $\mathcal{V}$ .

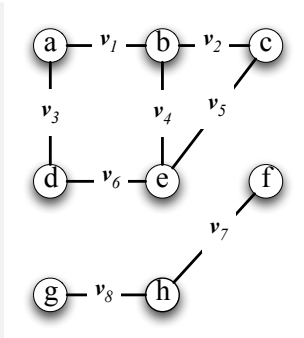
- (a)  $[2, 1, 4]$
- (b)  $[1, 1, 1]$
- (c)  $[5, 4, 3]$
- (d)  $[0, 1, 1]$

### Span of Vectors over $GF(2)$

**Problem 3:** Let  $\mathcal{V} = \text{Span} \{[0, 1, 0, 1], [0, 0, 1, 0], [1, 0, 0, 1], [1, 1, 1, 1]\}$  where the vectors are over  $GF(2)$ . For each of the following vectors over  $GF(2)$ , show it belongs to  $\mathcal{V}$  by writing it as a linear combination of the generators of  $\mathcal{V}$ .

- (a)  $[1, 1, 0, 0]$
- (b)  $[1, 0, 1, 0]$
- (c)  $[1, 0, 0, 0]$

**Problem 4:** The vectors over  $GF(2)$  representing the graph



are

	a	b	c	d	e	f	g	h
$v_1$	1	1						
$v_2$		1	1					
$v_3$	1			1				
$v_4$		1			1			
$v_5$			1		1			
$v_6$				1	1			
$v_7$						1		1
$v_8$							1	1

For each of the following vectors over  $GF(2)$ , show it belongs to the span of the above vectors by writing it as a linear combination of the above vectors.

- (a)  $[0, 0, 1, 1, 0, 0, 0, 0]$
- (b)  $[0, 0, 0, 0, 0, 1, 1, 0]$
- (c)  $[1, 0, 0, 0, 1, 0, 0, 0]$
- (d)  $[0, 1, 0, 1, 0, 0, 0, 0]$

## Linear Dependence over $\mathbb{R}$

**Problem 5:** For each of the parts below, show the given vectors over  $\mathbb{R}$  are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

- (a)  $[1, 2, 0], [2, 4, 1], [0, 0, -1]$
- (b)  $[2, 4, 0], [8, 16, 4], [0, 0, 7]$
- (c)  $[0, 0, 5], [1, 34, 2], [123, 456, 789], [-3, -6, 0], [1, 2, 0.5]$

**Problem 6:** For each of the parts below, show the given vectors over  $\mathbb{R}$  are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

- (a)  $[1, 2, 3], [4, 5, 6], [1, 1, 1]$
- (b)  $[0, -1, 0, -1], [\pi, \pi, \pi, \pi], [-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \sqrt{2}]$

(c)  $[1, -1, 0, 0, 0], [0, 1, -1, 0, 0], [0, 0, 1, -1, 0], [0, 0, 0, 1, -1], [-1, 0, 0, 0, 1]$

**Problem 7:** Show that one of the vectors is superfluous by expressing it as a linear combination of the other two.

$$\begin{aligned} \mathbf{u} &= [3, 9, 6, 5, 5] \\ \mathbf{v} &= [4, 10, 6, 6, 8] \\ \mathbf{w} &= [1, 1, 0, 1, 3] \end{aligned}$$

**Problem 8:** Give four vectors that are linearly dependent but such that any three are linearly independent.

## Linear Dependence over $GF(2)$

**Problem 9:** For each of the parts below, show the given vectors over  $GF(2)$  are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

- (a)  $[1, 1, 1, 1], [1, 0, 1, 0], [0, 1, 1, 0], [0, 1, 0, 1]$
- (b)  $[0, 0, 0, 1], [0, 0, 1, 0], [1, 1, 0, 1], [1, 1, 1, 1]$
- (c)  $[1, 1, 0, 1, 1], [0, 0, 1, 0, 0], [0, 0, 1, 1, 1], [1, 0, 1, 1, 1], [1, 1, 1, 1, 1]$

**Problem 10:** Each of the parts below specifies some of the vectors over  $GF(2)$  specified in Problem 4. Show that these vectors are linearly dependent by giving a subset of those vectors whose sum is the zero vector. (Hint: Looking at the graph will help.)

- (a)  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$
- (b)  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_7, \mathbf{v}_8\}$
- (c)  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_6\}$
- (d)  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8\}$

## Exchange Lemma for Vectors over $\mathbb{R}$

**Problem 11:** Let  $S = \{[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]\}$ , and let  $A = \{[1, 0, 0, 0, 0], [0, 1, 0, 0, 0]\}$ . For each of the following vectors  $\mathbf{z}$ , find a vector  $\mathbf{w}$  in  $S - A$  such that  $\text{Span } S = \text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$ .

- (a)  $\mathbf{z} = [1, 1, 1, 1, 1]$
- (b)  $\mathbf{z} = [0, 1, 0, 1, 0]$
- (c)  $\mathbf{z} = [1, 0, 1, 0, 1]$

## Exchange Lemma for Vectors over GF(2)

**Problem 12:** Let

	a	b	c	d	e	f	g	h
$v_1$	1	1						
$v_2$		1	1					
$v_3$	1			1				
$v_4$		1			1			

Let  $S = \{v_1, v_2, v_3, v_4\}$ . Each of the following parts specifies a subset  $A$  of  $S$  and a vector  $z$  such that  $A \cup \{z\}$  is linearly independent. For each part, specify a vector  $w$  in  $S - A$  such that  $\text{Span } S = \text{Span } (S \cup \{z\} - \{w\})$ . (Hint: Drawing subgraphs of the graph will help.)

(a)  $A = \{v_1, v_4\}$  and  $z$  is

a	b	c	d	e	f	g	h
			1	1			

(b)  $A = \{v_2, v_3\}$  and  $z$  is

a	b	c	d	e	f	g	h
			1	1			

(c)  $A = \{v_2, v_3\}$  and  $z$  is

a	b	c	d	e	f	g	h
1				1			

**Problem 13:** Write and test a procedure `rep2vec(u, veclist)` with the following spec:

- *input:* a vector  $u$  and a list `veclist` of Vectors  $[a_0, \dots, a_{n-1}]$ . The domain of  $u$  should be  $\{0, 1, 2, n-1\}$  where  $n$  is the length of `veclist`.
- *output:* the vector  $v$  such that  $u$  is the coordinate representation of  $v$  with respect to  $a_0, \dots, a_{n-1}$ , where entry  $i$  of  $u$  is the coefficient of  $a_i$  for  $i = 0, 1, 2, \dots, n-1$ .

Your procedure should not use any loops or comprehensions but of course can use the operations on instances of `Mat` and `Vec` and can also use procedures from the `matutil` module. Note that the procedures `coldict2mat` and `rowdict2mat` (defined in `matutil`) can accept lists, not just dictionaries.

In the following problem and others to come, you should make use of the `solve` procedure of the `solver` module.

**Problem 14:** Write and test a procedure `vec2rep(veclist, v)` with the following spec:

- *input:* a list `veclist` of vectors  $[a_0, \dots, a_{n-1}]$ , and a vector  $v$  with the domain  $\{0, 1, 2, \dots, n-1\}$  where  $n$  is the length of `veclist`. You can assume  $v$  is in  $\text{Span } \{a_0, \dots, a_{n-1}\}$ .
- *output:* the vector  $u$  whose coordinate representation with respect to  $a_0, \dots, a_{n-1}$  is  $v$ .

As in Problem 13, your procedure should use no loops or comprehensions directly but can use procedures defined in `matutil`.

**Problem 15:** Write and test a procedure `is_superfluous(L, i)` with the following spec:

- *input:* a list  $L$  of vectors, and an integer  $i$  in  $\{0, 1, \dots, n-1\}$  where  $n = \text{len}(L)$

- *output*: True if the span of the vectors in  $L$  equals the span of

$$L[0], L[1], \dots, L[i-1], L[i+1], \dots, L[n-1]$$

Your procedure should not use loops or comprehensions but can use procedures defined in the module `matutil` and can use the procedure `solve(A,b)` defined in `solver` module. Your procedure will most likely need a special case for the case where `len(L)` is 1.

Note that the `solve(A,b)` always returns a vector  $\mathbf{u}$ . It is up to you to check that  $\mathbf{u}$  is in fact a solution to the equation  $A\mathbf{x} = \mathbf{b}$ . Moreover, over  $\mathbb{R}$ , even if a solution exists, the solution returned by `solve` is approximate due to roundoff error. To check whether the vector  $\mathbf{u}$  returned is a solution, you should compute the residual  $\mathbf{b} - A * \mathbf{u}$ , and test if it is close to the zero vector:

```
>>> residual = b - A*u
>>> residual * residual
1.819555009546577e-25
```

If the sum of squares of the entries of the residual (the dot-product of the residual with itself) is less than, say  $10^{-14}$ , it is pretty safe to conclude that  $\mathbf{u}$  is indeed a solution.

**Problem 16:** Write and test a procedure `is_independent(L)` with the following spec:

- *input*: a list  $L$  of vectors
- *output*: True if the vectors form a linearly independent list.

Your algorithm for this procedure should be based on the Span Lemma. You can use as a subroutine any one of the following:

- the procedure `is_superfluous(L, b)` from Problem 15, or
- the `solve(A,b)` procedure from the `solver` module (but see the provisos in Problem 15).

You will need a loop or comprehension for this procedure.

**Problem 17:** Write and test a procedure `superset_basis(S, L)` with the following spec:

- *input*: a linearly independent list  $S$  of vectors, and a list  $L$  of vectors such that every vector in  $S$  is in the span of  $L$ .
- *output*: a linearly independent list  $T$  containing all vectors in  $S$  such that the span of  $T$  equals the span of  $L$  (i.e.  $T$  is a basis for the span of  $L$ ).

Your procedure should be based on either a version of the Grow algorithm or a version of the Shrink algorithm. Think about each one to see which is easier for you. You will need a loop or comprehension for this procedure. You can use as a subroutine any one of the following:

- the procedure `is_superfluous(L, b)` from Problem 15, or
- the procedure `is_independent(L)` from Problem 16, or
- the procedure `solve(A,b)` from the `solver` module (but see the provisos in Problem 15).

**Problem 18:** Write and test a procedure `exchange(S, A, z)` with the following spec:

- *input:* A list  $S$  of vectors, a list  $A$  of vectors that are all in  $S$  (such that  $\text{len}(A) < \text{len}(S)$ ), and a vector  $z$  such that  $A + [z]$  is linearly independent
- *output:* a vector  $w$  in  $S$  but not in  $A$  such that

$$\text{Span } S = \text{Span } (\{z\} \cup S - \{w\})$$

Your procedure should follow the proof of the Exchange Lemma. You should use the `solver` module or the procedure `vec2rep(veclist, u)` from Problem 14. You can test whether a vector is in a list using the expression `v in L`.