ECM 5100 CFD I Project 4: A Newton Based Methodology for the Numerical Solution of the Invicid Burgers Equation

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Scheme Derivation

Preliminaries Starting with the implicit Euler time differenced finite volume discretization of the Burgers Equation written in functional form as:

$$F\left(u^{n+1}\right) = u_i^{n+1} - u_i^n + \frac{\triangle t}{\triangle x}\left(f_{i+1/2}^{n+1} - f_{i-1/2}^{n+1}\right) \tag{1}$$

where $f(u) = \frac{u^2}{2}$.

First write down the simple upwind flux extrapolation for a finite volume as:

$$f_{i+1/2}^{n+1} = f\left(u_i^{n+1}\right) \tag{2}$$

$$f_{i-1/2}^{n+1} = f\left(u_{i-1}^{n+1}\right) \tag{3}$$

Where, for example, a cell volume is centered at i, the forward face of cell i is located at $+1/2$, while the aft face is located at $-1/2$. Using this flux definition, we can say

$$F\left(u^{n+1}\right) = u_i^{n+1} - u_i^n + \frac{\triangle t}{\triangle x}\left(f\left(u_i^{n+1}\right) - f\left(u_{i-1}^{n+1}\right)\right) \tag{4}$$

Newton Methodology Examining 4, we find that our functional contains two unknowns:

$$F\left(u^{n+1}\right) = F\left(u_i^{n+1}, u_{i-1}^{n+1}\right) \tag{5}$$

Therefore perform a Taylor's expansion of this functional and truncate as

$$F\left(u^{n+1,m+1}\right) = F\left(u_i^{n+1,m}, u_{i-1}^{n+1,m}\right) + \left(u_i^{n+1,m+1} - u_i^{n+1,m}\right)\left(\frac{\partial F}{\partial u_i^{n+1,m}}\right) + \left(u_{i-1}^{n+1,m+1} - u_{i-1}^{n+1,m}\right)\left(\frac{\partial F}{\partial u_{i-1}^{n+1,m}}\right) \quad (6)$$

We wish to find values of u^{m+1} that result in $F\left(u^{n+1,m+1}\right) = 0$. Accordingly, set the LHS term of 6 to zero.

$$0 = F\left(u_i^{n+1,m}, u_{i-1}^{n+1,m}\right) + \left(u_i^{n+1,m+1} - u_i^{n+1,m}\right)\left(\frac{\partial F}{\partial u_i^{n+1,m}}\right) + \left(u_{i-1}^{n+1,m+1} - u_{i-1}^{n+1,m}\right)\left(\frac{\partial F}{\partial u_{i-1}^{n+1,m}}\right) \quad (7)$$

Or, rearrange as

$$\left(u_i^{n+1,m+1} - u_i^{n+1,m}\right)\left(\frac{\partial F}{\partial u_i^{n+1,m}}\right) + \left(u_{i-1}^{n+1,m+1} - u_{i-1}^{n+1,m}\right)\left(\frac{\partial F}{\partial u_{i-1}^{n+1,m}}\right) = - F\left(u_i^{n+1,m}, u_{i-1}^{n+1,m}\right) \quad (8)$$

Simplify the LHS of 8 by defining

$$\triangle u^{n+1,m} = \left(u^{n+1,m+1} - u^{n+1,m}\right) \quad (9)$$

So we have

$$\triangle u_i^{n+1,m}\left(\frac{\partial F}{\partial u_i^{n+1,m}}\right) + \triangle u_{i-1}^{n+1,m}\left(\frac{\partial F}{\partial u_{i-1}^{n+1,m}}\right) = -F\left(u_i^{n+1,m}, u_{i-1}^{n+1,m}\right) \quad (10)$$

The RHS of 10 contains our Burger's equation physics. On the LHS of 10, use the flux defined in 2 to evaluate the partial derivatives of F:

$$\frac{\partial F^{n+1,m}}{\partial u_i^{n+1,m}} = 1 + \frac{\triangle t}{\triangle x} u_i^{n+1,m} \quad (11)$$

$$\frac{\partial F^{n+1,m}}{\partial u_{i-1}^{n+1,m}} = -\frac{\triangle t}{\triangle x} u_{i-1}^{n+1,m} \tag{12}$$

Back substitute to complete the scheme.

$$\triangle u_i^{n+1,m} \left(1 + \frac{\triangle t}{\triangle x} u_i^{n+1,m} \right) - \triangle u_{i-1}^{n+1,m} \left(\frac{\triangle t}{\triangle x} u_{i-1}^{n+1,m} \right) = -F^m \tag{13}$$

Where on the RHS we have

$$F^m = u_i^{n+1,m} - u_i^n + \frac{\triangle t}{\triangle x} \left(\frac{\left(u_i^{n+1,m}\right)^2}{2} - \frac{\left(u_{i-1}^{n+1,m}\right)^2}{2} \right)$$

The RHS contains all the Burger's physics. The LHS forms a tri-diagonal matrix of knowns. The super diagonal is zero. The ith terms make up the main diagonal. The ith-1 terms make up the sub diagonal. The $\triangle u$ vector makes up the unkowns. Solve for $\triangle u$ and update the equations via 9 until $\triangle u$ is driven to zero. At that point, all that remains is the physics on the RHS - now updated for the next timestep.