We consider that the probability of at least two people in a group of n people sharing a birthday, $\mathbb{P}(s)$, is the the same as 1 minus the probability of no shared birthdays in a group of n people, $\mathbb{P}(n)$

$$\mathbb{P}(s) = 1 - \mathbb{P}(n)$$

We know there are, in the simple case, 365 possible birthdays. So we see for the case of 2 people:

$$\mathbb{P}(2) = \frac{365}{365} \times \frac{364}{365}$$

Due to there being 364 possible days in which person 1 and person 2 cannot share a birthday. Therefore for a group of n people:

$$\mathbb{P}(n) = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-(n-1)}{365}$$

$$\mathbb{P}(n) = \frac{365 \times 364 \times \cdots \times (365-(n-1))}{365^n}$$

This can be rewritten using factorials:

$$365 \times 364 \times \cdots \times (365-(n-1)) = \frac{365!}{(365-n)!}$$

Therefore $\mathbb{P}(n)$ can be written as:

$$\mathbb{P}(n) = \frac{365!}{(365-n)!\,365^n}$$

Finally the probability of at least two people sharing a birthday in a group of n people is:

$$\mathbb{P}(s) = 1 - \frac{365!}{(365-n)!\,365^n}$$

Furthermore, to find the minimum group size such that $\mathbb{P}(s) \geq 0.5$ we find that $n = 23$. $n = 23$, $\mathbb{P}(s) \simeq 0.507297$

Notice the expression for $\mathbb{P}(n)$ is similar to the Binomial Coefficient, therefore we can rewrite:

using: $$\binom{k}{x} = \frac{k!}{x!\,(k-x)!}$$

so $$\mathbb{P}(n) = \frac{n!}{365^n}\binom{365}{n}$$

and $$\mathbb{P}(s) = 1 - \left[\frac{n!}{365^n}\binom{365}{n}\right]$$

and for a general number of days in a year; d:

$$\mathbb{P}(s) = 1 - \left[\frac{n!}{d^n}\binom{d}{n}\right]$$

A second way to define $\mathbb{P}(n)$ is to note that in a room of n people there are $\binom{n}{2}$ pairs and the probability of a given pair not sharing a birthday is $\frac{364}{365}$, so picking with replacement, (key assumption) we say that $\mathbb{P}(n)$ becomes:

$$\mathbb{P}(n) = \left(\frac{364}{365}\right)^{\binom{n}{2}}$$

and for the general d days case:

$$\mathbb{P}(n) = \left(\frac{d-1}{d}\right)^{\binom{n}{2}}$$

Finally $\mathbb{P}(s)$ becomes:

$$\mathbb{P}(s) = 1 - \left(\frac{d-1}{d}\right)^{\binom{n}{2}}$$

This formula is used in the analytical solution of the extension.

For the case of a specific birthday we now have n comparisons, not $\binom{n}{2}$ so:

$$\mathbb{P}(s) = 1 - \left(\frac{d-1}{d}\right)^{n}$$