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FIRST EVALUATED HOMEWORK

UNIVERSITÀ DEGLI STUDI DI PADOVA
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HOMEWORK 1

Exercise 1: Schmidt decomposition

Schmidt decomposition refers to a particular way of expressing a generic pure state in a joint system $A \otimes B$. Given two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimension d_A and d_B ($d_A \leq d_B$ for simplicity) and orthonormal basis $\{|a\rangle_A\}$ and $\{|b\rangle_B\}$, a generic state in the joint space is given by $|\psi\rangle_{AB} = \sum_{a=0}^{d_A-1} \sum_{b=0}^{d_B-1} \psi_{ab}|a\rangle_A|b\rangle_B$.

1. Demonstrate that, if in the system A we use the orthonormal basis $|v_a\rangle_A$ that diagonalizes the reduced state $\hat{\rho}_A$, the original state $|\psi\rangle_{AB}$ can be expressed with a single sum as
$$|\psi\rangle_{AB} = \sum_{a=0}^{d_A-1} \sqrt{p_a}|v_a\rangle_A|w_a\rangle_B \tag{1}$$
with p_a the eigenvalues of $\hat{\rho}_A$ and $|w_a\rangle_B$ orthonormal states. The decomposition written in eq. (1) is called **Schmidt decomposition**.

2. For the two-qubit state
$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B - \frac{3i}{5}|1\rangle_A|0\rangle_B - \frac{4i}{5}|1\rangle_A|1\rangle_B\right] \tag{2}$$
determine the reduced density matrices $\hat{\rho}_A$ and $\hat{\rho}_B$ on the A and B subsystem and their eigenvalues and eigenvectors. Is the pure state $|\phi\rangle_{AB}$ entangled?

3. Determine the Schmidt decomposition of the $|\phi\rangle_{AB}$ state.

Exercise 2

Consider a two-level system, A, coupled with an ancillary system B. The system B is a *qutrit*, namely a three-level system spanned by $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$. The ancillary system B is initially prepared into $|0\rangle_B$ and the A and B systems interact by the following unitary operator:

$$\begin{aligned}\mathcal{U}|0\rangle_A|0\rangle_B &= \frac{1}{\sqrt{3}}(|0\rangle_A|1\rangle_B + \sqrt{2}|0\rangle_A|2\rangle_B)\\ \mathcal{U}|1\rangle_A|0\rangle_B &= \frac{1}{2}(|1\rangle_A|0\rangle_B + \sqrt{2}|0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B)\end{aligned} \tag{3}$$

1. Calculate the resulting evolution in term of Krauss operators on the subsystem A.
2. Calculate the generalized measurement operators acting on A when the system B is measured in the "computational" basis $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$.
3. Calculate the corresponding POVM elements. What is their rank?
4. Calculate the output probabilities of the previous POVM when the input state is the mixed state
$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{8}(|0\rangle\langle 1| + |1\rangle\langle 0|) \tag{4}$$
5. Suppose that the system A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the post-measurement state, averaging over all measurement results?

Exercise 3

Consider arbitrary "spin" observables $\hat{A} = a_x\hat{\sigma}_x + a_z\hat{\sigma}_z$ and $\hat{B} = b_x\hat{\sigma}_x + b_z\hat{\sigma}_z$ with $a_x^2 + a_z^2 = b_x^2 + b_z^2 = 1$, $\hat{\sigma}_{x,z}$ the Pauli matrices. Let's also consider the two-qubit Bell state $|\Psi(\theta)\rangle_{AB} = \cos\frac{\theta}{2}|0\rangle_A|1\rangle_B - \sin\frac{\theta}{2}|1\rangle_A|0\rangle_B$.

1. Demonstrate that the eigenvalues of $\hat{A}$ and $\hat{B}$ are ± 1.

2. Evaluate the action of the possible combinations $\hat{\sigma}_i \otimes \hat{\sigma}_j$ with $i, j = x$ or z on the state $|\Psi(\theta)\rangle_{AB}$, i.e.

$$\hat{\sigma}_i \otimes \hat{\sigma}_j|\Psi(\theta)\rangle_{AB}\,, \qquad \forall i, j = x, z \tag{5}$$

3. Use the above result to calculate following expectation value

$$\langle\hat{A} \otimes \hat{B}\rangle_\theta = \langle\Psi(\theta)_{AB}|\hat{A} \otimes \hat{B}|\Psi(\theta)\rangle_{AB} \tag{6}$$

4. Consider the four operators $\hat{A}_k = \vec{a}_k \cdot \vec{\sigma}$, $\hat{B}_k = \vec{b}_k \cdot \vec{\sigma}$ with $k = 0, 1$. Define $\vec{a}_0$, $\vec{a}_1$, $\vec{b}_0$ and $\vec{b}_1$ the vectors in the (x, z) plane that allow to maximally violate the CHSH inequality with the state $|\Psi(\frac{\pi}{2})\rangle_{AB}$. Determine for which value of θ it is possible to violate the CHSH inequality with the same observables.

5 (extra) With a generic θ it is possible to find better observables to increase the violation? If yes, determine such observables in function of θ.

EXERCISE 1

1.1)

$d_A \neq d_B$

$A \otimes B$ joint system.

Let's start by writing $|\psi\rangle_{AB}$ into the two general orthonormal bases (ON) $\{|a\rangle_A\}$ and $\{|b\rangle_B\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$|\psi\rangle_{AB} = \sum_{a=0}^{d_A-1}\sum_{b=0}^{d_B-1} \psi_{ab}\,|a\rangle_A|b\rangle_B = \sum_{a=0}^{d_A-1}|a\rangle_A\underbrace{\left(\sum_{b=0}^{d_B-1}\psi_{ab}|b\rangle_B\right)}_{|\tilde{w}_a\rangle} = \sum_{a=0}^{d_A-1}|a\rangle_A|\tilde{w}_a\rangle_B$$

The problem is that $\{|\tilde{w}_a\rangle\}$ is not, in general, an orthonormal base for $\mathcal{H}_B$. In fact $|\tilde{w}_a\rangle$ depend on ψ_{ab} which depend on the choice of $|a\rangle$, so we can manage the latter one in order to get the desired property.

In fact it is possible to see that if we choose the base $\{|v_a\rangle\}$ that diagonalize the reduced state $\hat{\rho}_A$ rather than choosing a general $\{|a\rangle\}$ orthonormal base for $\mathcal{H}_A$, automatically $\{|\tilde{w}_a\rangle\}$ is orthogonal. Let's see how:

Let be $\{|v_a\rangle\}$ the base of $\mathcal{H}_A$ in which $\hat{\rho}_A$ is diagonal:

$$\hat{\rho}_A = \sum_{a=0}^{d_A-1} p_a|v_a\rangle_A\langle v_a|_A$$

Computing $\hat{\rho}_A$ starting from the definition:

$$\hat{\rho}_A = \mathrm{Tr}_B\,|\psi\rangle_{AB}\langle\psi|_{AB} = \mathrm{Tr}_B\left(\sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}|a\rangle_A|\tilde{w}_a\rangle_B\langle a'|_A\langle\tilde{w}_{a'}|_B\right) =$$

$$= \sum_{k=0}^{d_B-1}\langle k|_B\left(\sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}|a\rangle_A|\tilde{w}_a\rangle_B\langle a'|_A\langle\tilde{w}_{a'}|_B\right)|k\rangle_B =$$

$$= \sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}|a\rangle_A\langle a'|_A\left(\sum_{k=0}^{d_B-1}\langle k|\tilde{w}_a\rangle\langle\tilde{w}_{a'}|k\rangle\right) =$$

$$= \sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}|a\rangle_A\langle a'|_A\left(\sum_{k=0}^{d_B-1}\langle\tilde{w}_{a'}\underbrace{|k\rangle\langle k|}_{\mathbb{I}}\tilde{w}_a\rangle\right) =$$

$$= \sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}\langle\tilde{w}_{a'}|\tilde{w}_a\rangle\,|a\rangle_A\langle a'|_A$$

Where in * we use a generic orthonormal base $\{|k\rangle_B\}$ for $\mathcal{H}_B$ in order to compute the Trace (that is the same for each choice of the base) and in * we use Dirac's Completness for the orthonormal base $\{|k\rangle_B\}$.

Using now $\{|v_a\rangle_A\}$ basis instead of $\{|a\rangle_A\}$ we can:

$$\sum_{a=0}^{d_A-1}\sum_{a'=0}^{d_B-1}\langle\tilde{w}_{a'}|\tilde{w}_a\rangle\,|v_a\rangle_A\langle v_a|_A=\sum_{a=0}^{d_A-1}p_a\,|v_a\rangle_A\langle v_a|_A$$

$$\Updownarrow$$

$$\langle\tilde{w}_{a'}|\tilde{w}_a\rangle=p_a\,\delta_{a'a}$$

So, using for $\mathcal{H}_A$ the base $\{|v_a\rangle\}$ that diagonalizes $\hat{\rho}_A$ the decomposition in * produces an orthogonal base $\{|\tilde{w}_a\rangle\}$. In order to get an orthonormal one we can normalize:

$$|w_a\rangle=\frac{1}{\sqrt{p_a}}|\tilde{w}_a\rangle \Rightarrow |\tilde{w}_a\rangle=\sqrt{p_a}\,|w_a\rangle \Rightarrow \langle w_a|w_{a'}\rangle=\delta_{aa'}$$

So starting from * and considering $|a\rangle=|v_a\rangle$ and $|\tilde{w}_a\rangle=\sqrt{p_a}\,|w_a\rangle$ we get:

$$|\psi\rangle_{AB}=\sum_{a=0}^{d_A-1}\sqrt{p_a}\,|v_a\rangle_A\,|w_a\rangle_B$$

∎

1.2) given $|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B - \frac{3i}{5}|1\rangle_A|0\rangle_B - \frac{4i}{5}|1\rangle_A|1\rangle_B\right]$

is it normalize?

$$_{AB}\langle\phi|\phi\rangle_{AB} = \frac{1}{2}\left({}_A\langle 0|\,{}_B\langle 0| + \frac{3i}{5}\,{}_A\langle 1|\,{}_B\langle 0| + \frac{4i}{5}\,{}_A\langle 1|\,{}_B\langle 1|\right)\left(|0\rangle_A|0\rangle_B - \frac{3i}{5}|1\rangle_A|0\rangle_B - \frac{4i}{5}|1\rangle_A|1\rangle_B\right)$$

$$= \frac{1}{2}\left(1 + \frac{9}{25} + \frac{16}{25}\right) = 1 \implies |\phi\rangle_{AB} \text{ is normalized!}$$

$$\hat{\rho}_{AB} = |\phi\rangle_{AB}\langle\phi|$$

$$\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho}_{AB}) = \sum_{j=0}^{1} {}_B\langle j|\hat{\rho}_{AB}|j\rangle_B = {}_B\langle 0|\hat{\rho}_{AB}|0\rangle_B + {}_B\langle 1|\hat{\rho}_{AB}|1\rangle_B =$$

$$= {}_B\langle 0|\phi\rangle_{AB}\langle\phi|0\rangle_B + {}_B\langle 1|\phi\rangle_{AB}\langle\phi|1\rangle_B$$

$${}_B\langle 0|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A - \frac{3i}{5}|1\rangle_A\right) \rightsquigarrow {}_{AB}\langle\phi|0\rangle_B = \frac{1}{\sqrt{2}}\left({}_A\langle 0| + \frac{3i}{5}\,{}_A\langle 1|\right)$$

$${}_B\langle 1|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(-\frac{4i}{5}|1\rangle_A\right) \rightsquigarrow {}_{AB}\langle\phi|1\rangle_B = \frac{4i}{5\sqrt{2}}\,{}_A\langle 1|$$

$$\implies \hat{\rho}_A = \frac{1}{2}\left(|0\rangle_A - \frac{3i}{5}|1\rangle_A\right)\left({}_A\langle 0| + \frac{3i}{5}\,{}_A\langle 1|\right) + \frac{1}{2}\left(-\frac{4i}{5}|1\rangle_A\right)\left(\frac{4i}{5}\,{}_A\langle 1|\right) =$$

$$= \frac{1}{2}|0\rangle_A{}_A\langle 0| + \frac{3i}{10}|0\rangle_A\langle 1| - \frac{3i}{10}|1\rangle_A\langle 0| + \frac{9}{50}|1\rangle_A\langle 1| + \frac{16}{50}|1\rangle_A\langle 1| =$$

$$= \frac{1}{2}|0\rangle_A\langle 0| + \frac{3i}{10}|0\rangle_A\langle 1| - \frac{3i}{10}|1\rangle_A\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| =$$

$$= \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{3i}{10}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} - \frac{3i}{10}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{3i}{10}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} - \frac{3i}{10}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{1}{2} & \frac{3i}{10}\\ -\frac{3i}{10} & \frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & \frac{3}{5}i\\ -\frac{3}{5}i & 1\end{pmatrix}$$

$$\hat{\rho}_B = Tr_A(\hat{\rho}_{AB}) = \sum_{j=0}^{1} {}_A\langle j|\hat{\rho}_{AB}|j\rangle_A = {}_A\langle 0|\hat{\rho}_{AB}|0\rangle_A + {}_A\langle 1|\hat{\rho}_{AB}|1\rangle_A =$$

$$= {}_A\langle 0|\phi\rangle_{AB}\langle\phi|0\rangle_A + {}_A\langle 1|\phi\rangle_{AB}\langle\phi|1\rangle_A$$

$${}_A\langle 0|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle_B \rightsquigarrow {}_{AB}\langle\phi|0\rangle_A = \frac{1}{\sqrt{2}}\,{}_B\langle 0|$$

$${}_A\langle 1|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(-\frac{3i}{5}|0\rangle_B - \frac{4i}{5}|1\rangle_B\right) \rightsquigarrow {}_{AB}\langle\phi|0\rangle_A = \frac{1}{\sqrt{2}}\left(\frac{3i}{5}\,{}_B\langle 0| + \frac{4i}{5}\,{}_B\langle 1|\right)$$

$$\Rightarrow \hat{\rho}_B = \frac{1}{2}|0\rangle_B\langle 0| + \frac{1}{2}\left(-\frac{3i}{5}|0\rangle_B - \frac{4i}{5}|1\rangle_B\right)\left(\frac{3i}{5}\,{}_B\langle 0| + \frac{4i}{5}\,{}_B\langle 1|\right) =$$

$$= \frac{1}{2}|0\rangle_B\langle 0| + \frac{1}{2}\left(\frac{9}{25}|0\rangle_B\langle 0| + \frac{12}{25}|0\rangle_B\langle 1| + \frac{12}{25}|1\rangle_B\langle 0| + \frac{16}{25}|1\rangle_B\langle 1|\right) =$$

$$= \frac{34}{50}|0\rangle_B\langle 0| + \frac{12}{50}|0\rangle_B\langle 1| + \frac{12}{50}|1\rangle_B\langle 0| + \frac{16}{50}|1\rangle_B\langle 1| =$$

$$= \frac{34}{50}\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \frac{12}{50}\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \frac{12}{50}\begin{pmatrix}0\\1\end{pmatrix}(1\ 0) + \frac{16}{50}\begin{pmatrix}0\\1\end{pmatrix}(0\ 1) =$$

$$= \frac{34}{50}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \frac{12}{50}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + \frac{12}{50}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \frac{16}{50}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{34}{50} & \frac{12}{50}\\ \frac{12}{50} & \frac{16}{50}\end{pmatrix} = \frac{2}{50}\begin{pmatrix}17 & 6\\6 & 8\end{pmatrix} = \frac{1}{25}\begin{pmatrix}17 & 6\\6 & 8\end{pmatrix}$$

Eingeinvalues and eingenvectors:

a) $\hat{\rho}_A$:

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{3}{10}i \\ -\frac{3}{10}i & \frac{1}{2}-\lambda \end{vmatrix} = \left(\frac{1}{2}-\lambda\right)^2 - \frac{9}{100} = \lambda^2-\lambda+\frac{1}{4}-\frac{9}{100} = \lambda^2-\lambda+\frac{\cancel{16}^{\,4}}{\cancel{100}_{\,25}} \overset{!}{=} 0$$

$$\lambda^A_{1,2} = \frac{+1 \pm \sqrt{1-\frac{16}{25}}}{2} = \frac{1 \pm \frac{3}{5}}{2} \quad \begin{matrix} \nearrow \frac{1}{5} = \lambda_1^A \\ \searrow \frac{4}{5} = \lambda_2^A \end{matrix}$$

$$\hat{\rho}_A |\lambda_1\rangle_A = \lambda_1^A |\lambda_1\rangle_A \rightsquigarrow (\hat{\rho}_A - \lambda_1^A \mathbb{1})|\lambda_1\rangle_A = 0 \qquad |\lambda_1\rangle_A = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}-\frac{1}{5} & \frac{3}{10}i \\ -\frac{3i}{10} & \frac{1}{2}-\frac{1}{5} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} \frac{3}{10}a + \frac{3}{10}i\,b = 0 \\ -\frac{3i}{10}a + \frac{3}{10}b = 0 \end{cases}$$

$$\begin{cases} a + ib = 0 \\ -ia + b = 0 \end{cases} \rightsquigarrow b = ia \rightsquigarrow |\lambda_1\rangle_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle_A + i|1\rangle_A)$$

Normalized

$$\hat{\rho}_A |\lambda_2\rangle_A = \lambda_2^A |\lambda_2\rangle_A \rightsquigarrow (\hat{\rho}_A - \lambda_2^A \mathbb{1})|\lambda_2\rangle_A = 0 \qquad |\lambda_2\rangle_A = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}-\frac{4}{5} & \frac{3}{10}i \\ -\frac{3}{10}i & \frac{1}{2}-\frac{4}{5} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} -\frac{3}{10}c + \frac{3}{10}i\,d = 0 \\ -\frac{3i}{10}c - \frac{3}{10}d = 0 \end{cases}$$

$$\begin{cases} -c + id = 0 \\ -ic - d = 0 \end{cases} \rightsquigarrow c = id \rightsquigarrow |\lambda_2\rangle_A = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(i|0\rangle_A + |1\rangle_A)$$

b) $\hat{\rho}_B$:

$$\begin{vmatrix} \frac{17}{25}-\lambda & \frac{6}{25} \\ \frac{6}{25} & \frac{8}{25}-\lambda \end{vmatrix} = \left(\frac{17}{25}-\lambda\right)\left(\frac{8}{25}-\lambda\right) - \frac{36}{625} = +\lambda^2 - \lambda + \frac{136}{625} - \frac{36}{625} = \lambda^2 - \lambda + \frac{4}{25} \overset{!}{=} 0$$

$$\lambda^B_{1,2} = \frac{1 \pm \sqrt{1 - \frac{16}{25}}}{2} = \begin{cases} \frac{4}{5} = \lambda^B_2 \\ \frac{1}{5} = \lambda^B_1 \end{cases}$$

$$\hat{\rho}_B |\lambda_1\rangle_B = \lambda^B_1 |\lambda_1\rangle_B \rightsquigarrow (\hat{\rho}_B - \lambda^B_1 \mathbb{1}) |\lambda_1\rangle_B = 0 \qquad |\lambda_1\rangle_B = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \frac{17}{25} - \frac{1}{5} & \frac{6}{25} \\ \frac{6}{25} & \frac{8}{25} - \frac{1}{5} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} \frac{12}{25} a + \frac{6}{25} b = 0 \\ +\frac{6}{25} a + \frac{3}{25} b = 0 \end{cases}$$

$$\begin{cases} 2a + b = 0 \\ 2a + b = 0 \end{cases} \rightsquigarrow b = -2a \rightsquigarrow |\lambda_1\rangle_B = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{5}} (|0\rangle_B - 2|1\rangle_B)$$

Normalized

$$\hat{\rho}_B |\lambda_2\rangle_B = \lambda^B_2 |\lambda_2\rangle_B \rightsquigarrow (\hat{\rho}_B - \lambda^B_2 \mathbb{1}) |\lambda_2\rangle_B = 0 \qquad |\lambda_2\rangle_B = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} \frac{17}{25} - \frac{4}{5} & \frac{6}{25} \\ \frac{6}{25} & \frac{8}{25} - \frac{4}{5} \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} -\frac{3}{25} a + \frac{6}{25} b = 0 \\ \frac{6}{25} a - \frac{12}{25} b = 0 \end{cases}$$

$$\begin{cases} -a + 2b = 0 \\ a - 2b = 0 \end{cases} \rightsquigarrow a = 2b \rightsquigarrow |\lambda_2\rangle_B = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} (2|0\rangle_B + |1\rangle_B)$$

Normalized

In this case $|\phi\rangle_{AB}$ is a pure state and it is entangled $\Leftrightarrow Tr(\hat{\rho}_A^2) < 1$

$$\hat{\rho}_A^2 = \frac{1}{4}\begin{pmatrix} 1 & \frac{3}{5}i \\ -\frac{3}{5}i & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{3}{5}i \\ \frac{-3i}{5} & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \frac{34}{25} & \frac{6i}{5} \\ -\frac{6}{5}i & \frac{34}{25} \end{pmatrix} \rightsquigarrow Tr(\hat{\rho}_A^2) = \frac{17}{50} + \frac{17}{50} = \frac{17}{25} < 1$$

The same holds for $\hat{\rho}_B$:

$$\hat{\rho}_B^2 = \frac{1}{625}\begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix} = \frac{1}{625}\begin{pmatrix} 325 & 150 \\ 150 & 100 \end{pmatrix} \rightsquigarrow Tr(\hat{\rho}_B^2) = \frac{325}{625} + \frac{100}{625} = \frac{17}{25} < 1$$

$\Rightarrow |\phi\rangle_{AB}$ is an entangled state.

1.3) From first point, it is possible to use the orthonormal basis $|\lambda_1\rangle_A, |\lambda_2\rangle_A$ that diagonalizes $\hat{\rho}_A$ and $|\phi\rangle_{AB}$ can be expressed as:

$$|\phi\rangle_{AB} = \sqrt{\lambda_1^A}\,|\lambda_1\rangle_A|\lambda_1\rangle_B + \sqrt{\lambda_2^A}\,|\lambda_2\rangle_A|\lambda_2\rangle_B$$

$$|\lambda_1\rangle_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle_A + i|1\rangle_A) \qquad \lambda_1^A = \frac{1}{5}$$

$$|\lambda_2\rangle_A = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(i|0\rangle_A + |1\rangle_A) \qquad \lambda_2^A = \frac{4}{5}$$

$$|\lambda_1\rangle_B = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{5}}(|0\rangle_B - 2|1\rangle_B)$$

$$|\lambda_2\rangle_B = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}(2|0\rangle_B + |1\rangle_B)$$

orthogonality check (above vectors are already normalize):

$${}_A\langle\lambda_1|\lambda_2\rangle_A = \frac{1}{2}({}_A\langle 0| - i\,{}_A\langle 1|)(i|0\rangle_A + |1\rangle_A) = \frac{1}{2}(i - i) = 0$$

$${}_B\langle\lambda_1|\lambda_2\rangle_B = \frac{1}{5}({}_B\langle 0| - 2\,{}_B\langle 1|)(2|0\rangle_B + |1\rangle_B) = \frac{1}{5}(2-2) = 0$$

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{5}}|\lambda_1\rangle_A\left[\frac{1}{\sqrt{5}}(|0\rangle_B - 2|1\rangle_B)\right] + \frac{2}{\sqrt{5}}|\lambda_2\rangle_A\left[\frac{1}{\sqrt{5}}(2|0\rangle_B + |1\rangle_B)\right] =$$

$$= \frac{1}{5}|\lambda_1\rangle_A|0\rangle_B - \frac{2}{5}|\lambda_1\rangle_A|1\rangle_B + \frac{4}{5}|\lambda_2\rangle_A|0\rangle_B + \frac{2}{5}|\lambda_2\rangle_A|1\rangle_B$$

EX 2.

2.1) Let be: $\rho_{AB}(0)$ a separable state $\Rightarrow \rho_A \otimes \rho_B = \rho_{AB}(0)$

ρ_B pure state $\Rightarrow \rho_B = |0\rangle_B\langle 0|$ with $|0\rangle_B \in \mathcal{H}_B$

↑ sub-system 2.

So the initial state can be re-written as: $\rho_{AB}(0) = \rho_A \otimes |0\rangle_B\langle 0|$

So the temporal evolution is: $\rho_{AB} = U(t)\,\rho_{AB}(0)\,U^\dagger(t)$

Defined $\rho_A(t)$ the reduced matrix:

$$\rho_A(t) = \mathrm{Tr}_B\left(\rho_{AB}(t)\right) = \mathrm{Tr}_B\left[U\left(\rho_A \otimes |0\rangle_B\langle 0|\right)U^\dagger\right] = \sum_{k=1}^{\dim\mathcal{H}_B} \underbrace{\langle k|_B U |0\rangle_B}_{E_k}\, \rho_A\, \underbrace{\langle 0|_B U^\dagger |k\rangle_B}_{E_k^\dagger}$$

$$= \sum_{k=1}^{\dim\mathcal{H}_B} E_k\, \rho_A\, E_k^\dagger$$

where $\{|k\rangle_B\}$ is an orthonormal base for $\mathcal{H}_B$.

The operators $\{E_k : \mathcal{H}_A \to \mathcal{H}_A\}$, called Krauss operators, act on $\mathcal{H}_A$'s states and they satisfy $\sum_{k=1}^{\dim\mathcal{H}_B} E_k = \mathbb{1}_A$.

Now let's look for the evolution:

Let be: - $\mathcal{H}_A, \mathcal{H}_B$ Hilbert spaces.

- B is prepared in $|0\rangle_B \Rightarrow \rho_B = |0\rangle_B\langle 0|$
- A and B interact as:

$$U|0\rangle_A|0\rangle_B = \frac{1}{\sqrt{3}}\left(|0\rangle_A|1\rangle_B + \sqrt{2}\,|0\rangle_A|2\rangle_B\right)$$

$$U|1\rangle_A|0\rangle_B = \frac{1}{2}\left(|1\rangle_A|0\rangle_B + \sqrt{2}\,|0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right)$$

- B ancillary system is a qutrit $\{|0\rangle_B, |1\rangle_B, |2\rangle_B\}$

- $\rho_A = |\alpha|^2|0\rangle_A\langle 0| + |\beta|^2|1\rangle_A\langle 1| + \alpha\bar{\beta}|0\rangle_A\langle 1| + \bar{\alpha}\beta|1\rangle_A\langle 0|$

s.t. $|\alpha|^2 + |\beta|^2 = 1$ with $\alpha, \beta \in \mathbb{C}$

$$\rho_A(t) = \sum_{K=0}^{d_B-1} \langle K|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger |K\rangle_B =$$

$$= \underbrace{\langle 0|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger|0\rangle_B}_{A} + \underbrace{\langle 1|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger|1\rangle_B}_{B} + \underbrace{\langle 2|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger|2\rangle_B}_{C}$$

$$A = \langle 0|_B U|0\rangle_B \left(|\alpha|^2|0\rangle_A\langle 0| + |\beta|^2|1\rangle_A\langle 1| + \alpha\bar\beta|0\rangle_A\langle 1| + \bar\alpha\beta|1\rangle_A\langle 0|\right)\langle 0|_B U^\dagger|0\rangle_B =$$

$$= \langle 0|_B \Big[\frac{|\alpha|^2}{\sqrt3}\left(|0\rangle_A|1\rangle_B + \sqrt2\,|0\rangle_A|2\rangle_B\right)\langle 0|_A + \frac{|\beta|^2}{2}\left(|1\rangle_A|0\rangle_B + \sqrt2\,|0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right)\langle 1|_A +$$

$$+ \frac{\alpha\bar\beta}{\sqrt3}\left(|0\rangle_A|1\rangle_B + \sqrt2\,|0\rangle_A|2\rangle_B\right)\langle 1|_A + \frac{\bar\alpha\beta}{2}\left(|1\rangle_A|0\rangle_B + \sqrt2\,|0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right)\langle 0|_A \Big] \cdot$$

$$\cdot \langle 0|_B U^\dagger|0\rangle_B = \left[\frac{|\beta|^2}{2}|1\rangle_A\langle 1| + \frac{\bar\alpha\beta}{2}|1\rangle_A\langle 0|\right]\langle 0|_B U^\dagger|0\rangle_B =$$

$$= \Big[\frac{|\beta|^2}{2}|1\rangle_A\Big[\frac{1}{2}\left(\langle 1|_A\langle 0|_B + \sqrt2\,\langle 0|_A\langle 1|_B - \langle 0|_A\langle 2|_B\right]\Big] +$$

$$+ \frac{\bar\alpha\beta}{2}|1\rangle_A\left[\langle 0|_A\langle 1|_B + \sqrt2\,\langle 0|_A\langle 2|_B\right]\Big]|0\rangle_B =$$

$$= \frac{|\beta|^2}{4}|1\rangle_A\langle 1|$$

Performing the same type of calculations for B and C we can find that:

$$B = \langle 1|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger|1\rangle_B = \frac{|\alpha|^2}{3}|0\rangle_A\langle 0| + \frac{|\beta|^2}{2}|0\rangle_A\langle 0| + \frac{\alpha\bar\beta+\bar\alpha\beta}{\sqrt6}|0\rangle_A\langle 0|$$

$$C = \langle 2|_B U|0\rangle_B \, \rho_A \, \langle 0|_B U^\dagger|2\rangle_B = \frac{2}{3}|\alpha|^2|0\rangle_A\langle 0| + \frac{|\beta|^2}{4}|0\rangle_A\langle 0| - \frac{\alpha\bar\beta+\bar\alpha\beta}{\sqrt6}|0\rangle_A\langle 0|$$

So the temporal evolution for the state is:

$$\rho_A(t) = A + B + C = \frac{|\beta|^2}{4}|1\rangle_A\langle 1| + \frac{|\alpha|^2}{3}|0\rangle_A\langle 0| + \frac{|\beta|^2}{2}|0\rangle_A\langle 0| + \frac{\alpha\bar\beta+\bar\alpha\beta}{\sqrt6}|0\rangle_A\langle 0| +$$

$$+ \frac{2}{3}|\alpha|^2|0\rangle_A\langle 0| + \frac{|\beta|^2}{4}|0\rangle_A\langle 0| - \frac{\alpha\bar\beta+\bar\alpha\beta}{\sqrt6}|0\rangle_A\langle 0| =$$

$$= \left(|\alpha|^2 + \frac{3}{4}|\beta|^2\right)|0\rangle_A\langle 0| + \frac{|\beta|^2}{4}|1\rangle_A\langle 1|$$

So $\rho_A(t) = \sum_{K=0}^{d_B-1} E_K \rho_A E_K^\dagger = \left(|\alpha|^2 + \frac{3|\beta|^2}{4}\right)|0\rangle_A\langle 0| + \frac{|\beta|^2}{4}|1\rangle_A\langle 1|$ *

we can now recognize also the Krauss operators $\{E_K\}_{K=0}^{2}$:

In fact knowing $E_K = \langle K|_B U|0\rangle$, $U|\psi\rangle_A|0\rangle_B = \sum_{K=0}^{2} E_K|\psi\rangle_A|K\rangle_B$

with $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_B$, rearranging in the following way:

$\langle k'|k\rangle = \delta_{k'k}$ since we have ON bases.

$$\hat{U}|0\rangle_A|0\rangle_B = \frac{1}{\sqrt{3}}|0\rangle_A\underbrace{\langle 0|0\rangle_A}_{=1}|1\rangle_B + \sqrt{\frac{2}{3}}|0\rangle_A\langle 0|0\rangle_A|2\rangle_B =$$

$$K_0|0\rangle_A|1\rangle_B + K_1|0\rangle_A|2\rangle_B$$

where $\hat{K}_0 = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0|$ $\quad \hat{K}_1 = \sqrt{\frac{2}{3}}|0\rangle_A\langle 0|$

$$\hat{U}|1\rangle_A|0\rangle_B = \frac{1}{2}|1\rangle_A\langle 1|1\rangle_A|0\rangle_B + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|1\rangle_A|1\rangle_B - \frac{1}{2}|0\rangle_A\langle 1|1\rangle_A|2\rangle_B =$$

$$= K_2|1\rangle_A|0\rangle_B + K_3|1\rangle_A|1\rangle_B + K_4|1\rangle_A|2\rangle_B$$

where $\hat{K}_2 = \frac{1}{2}|1\rangle_A\langle 1|$, $\hat{K}_3 = \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|$, $\hat{K}_4 = -\frac{1}{2}|0\rangle_A\langle 1|$

We obtain that the Krauss operators are:

$$\hat{E}_0 = \hat{K}_2 = \frac{1}{2}|1\rangle_A\langle 1| = {}_B\langle 0|\hat{U}|0\rangle_B$$

$$\hat{E}_1 = \hat{K}_0 + \hat{K}_3 = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1| = {}_B\langle 1|\hat{U}|0\rangle_B$$

$$\hat{E}_2 = \hat{K}_1 + \hat{K}_4 = \sqrt{\frac{2}{3}}|0\rangle_A\langle 0| - \frac{1}{2}|0\rangle_A\langle 1| = {}_B\langle 2|\hat{U}|0\rangle_B$$

N.B. In * I have found the evolution of subsystem A.
If I want to find the evolution of the total system AB
I can apply: $\hat{U}_{AB}|\psi\rangle_A|0\rangle_B = \sum_{K=0}^{d_B-1}\hat{E}_K|\psi\rangle_A|K\rangle_B$
with the Krauss operators found above:

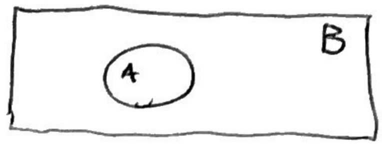
$$\hat{U}_{AB}|\psi_A\rangle|0\rangle_B = \hat{U}_{AB}(\alpha|0\rangle_A + \beta|1\rangle_A)|0\rangle_B = \sum_{K=0}^{d_B-1}\hat{E}_K(\alpha|0\rangle_A + \beta|1\rangle_A)|K\rangle_B =$$

$$= \hat{E}_0(\alpha|0\rangle_A + \beta|1\rangle_A)|0\rangle_B + \hat{E}_1(\alpha|0\rangle_A + \beta|1\rangle_A)|1\rangle_B + \hat{E}_2(\alpha|0\rangle_A + \beta|1\rangle_A)|2\rangle_B =$$

$$= \frac{\beta}{2}|1\rangle_A|0\rangle_B + \left(\frac{\alpha}{\sqrt{3}}|0\rangle_A|1\rangle_B + \beta\frac{\sqrt{2}}{2}|0\rangle_A|1\rangle_B\right) + \sqrt{\frac{2}{3}}\alpha|0\rangle_A|2\rangle_B - \frac{\beta}{2}|0\rangle_A|2\rangle_B =$$

$$= \beta/2\,|1\rangle_A|0\rangle_B + \left(\frac{\alpha}{\sqrt{3}} + \beta\frac{\sqrt{2}}{2}\right)|0\rangle_A|1\rangle_B + \left(\sqrt{\frac{2}{3}}\alpha - \beta/2\right)|0\rangle_A|2\rangle_B$$

2.2) Given the system we have that



$$|\psi\rangle_A|0\rangle_B = \hat{U}_{AB}|\psi\rangle_A|0\rangle_B = \sum_{k=0}^{2} \hat{M}_k^A |\psi\rangle_A |k\rangle_B$$

Let's find $\hat{M}_k^A$ for $k=0,1,2$ given $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$ with $\alpha, \beta \in \mathbb{C}$ as in previous point.

$$\hat{U}_{AB}|\psi\rangle_A|0\rangle_B = \hat{U}_{AB}(\alpha|0\rangle_A + \beta|1\rangle_A)|0\rangle_B =$$

$$= \hat{U}_{AB}\,\alpha|0\rangle_A|0\rangle_B + \hat{U}_{AB}\,\beta|1\rangle_A|0\rangle_B =$$

$$= \frac{\alpha}{\sqrt{3}}|0\rangle_A|1\rangle_B + \sqrt{\frac{2}{3}}\,\alpha|0\rangle_A|2\rangle_B + \frac{\beta}{2}|1\rangle_A|0\rangle_B + \frac{\sqrt{2}}{2}\alpha|0\rangle_A|1\rangle_B + - \frac{\alpha}{2}|0\rangle_A|2\rangle_B$$

writing in the form $\sum_{k=0}^{2}\hat{M}_k^A|\psi_A\rangle|k\rangle_B$

$$= \frac{\beta}{2}|1\rangle_A|0\rangle_B + \left(\frac{\alpha}{\sqrt{3}}|0\rangle_A + \frac{\sqrt{2}}{2}\beta|0\rangle_A\right)|1\rangle_B + \left(\sqrt{\frac{2}{3}}\alpha|0\rangle_A - \frac{\beta}{2}|0\rangle_A\right)|2\rangle_B$$

Now:

$$\frac{1}{2}\beta|1\rangle_A|0\rangle_B \Leftrightarrow \hat{M}_0^A|\psi\rangle_A|0\rangle_B \Rightarrow \hat{M}_0^A = \frac{1}{2}|1\rangle_A\langle 1|$$

$$\left(\frac{\alpha}{\sqrt{3}}|0\rangle_A + \frac{\sqrt{2}}{2}\beta|0\rangle_A\right)|1\rangle_B \Leftrightarrow \hat{M}_1^A|\psi\rangle_A|1\rangle_B \Rightarrow \hat{M}_1^A = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|$$

$$\left(\sqrt{\frac{2}{3}}\alpha|0\rangle_A - \frac{\beta}{2}|0\rangle_A\right)|2\rangle_B \Leftrightarrow \hat{M}_2^A|\psi\rangle_A|2\rangle_B \Rightarrow \hat{M}_2^A = \sqrt{\frac{2}{3}}|0\rangle_A\langle 0| - \frac{1}{2}|0\rangle_A\langle 1|$$

So resuming the generalized operators are:

$$\hat{M}_0^A = \frac{1}{2}|1\rangle_A\langle 1| \;;\; \hat{M}_1^A = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1| \;;\; \hat{M}_2^A = \sqrt{\frac{2}{3}}|0\rangle_A\langle 0| - \frac{1}{2}|0\rangle_A\langle 1|$$

and it is possible to verify that:

$$\hat{U}_{AB}|\psi\rangle_A|0\rangle_B = \hat{M}_0^A|\psi\rangle_A|0\rangle_B + \hat{M}_1^A|\psi\rangle_A|1\rangle_B + \hat{M}_2^B|\psi\rangle_A|2\rangle_B.$$

2.3) Let's calculate the POVM elements and their rank:

The POVM are defined as $\{\hat{F}_K\} \mid F_K \equiv \hat{M}_K^\dagger M_K$

- So for $K=0$: $F_0 \equiv \hat{M}_0^\dagger \hat{M}_0$

$\hat{M}_0 = \frac{1}{2}|1\rangle_A\langle 1| \Rightarrow \hat{M}_0^\dagger = \frac{1}{2}|1\rangle_A\langle 1| \Rightarrow F_0 = \left(\frac{1}{2}|1\rangle_A\langle 1|\right)\left(\frac{1}{2}|1\rangle_A\langle 1|\right) = \frac{1}{4}|1\rangle_A\langle 1|$

$$F_0 = \frac{1}{4}|1\rangle_A\langle 1| = \frac{1}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

$\det(F_0) = 0 \Rightarrow \text{rank}(F_0) < 2 \Rightarrow$ since $\begin{vmatrix} 0-\lambda & 0 \\ 0 & \frac{1}{4}-\lambda \end{vmatrix} = (-\lambda)(\frac{1}{4}-\lambda) = 0$

we have $\lambda_1^0 = 0 \quad \lambda_2^0 = \frac{1}{4}$, so one of the eingenvalues $\neq 0$

So $\text{rank}(F_0) = \underline{1}$

- for $K=1$

$$\hat{M}_1 = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1| \qquad M_1^\dagger = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|1\rangle_A\langle 0|$$

$$F_1 = \hat{M}_1^\dagger \hat{M}_1 = \left(\frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|1\rangle_A\langle 0|\right)\left(\frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|\right) =$$

$$= \frac{1}{3}|0\rangle_A\langle 0| + \frac{1}{\sqrt{6}}|0\rangle_A\langle 1| + \frac{1}{\sqrt{6}}|1\rangle_A\langle 0| + \frac{1}{2}|1\rangle_A\langle 1|$$

$$F_1 = \frac{1}{3}|0\rangle_A\langle 0| + \frac{1}{\sqrt{6}}|0\rangle_A\langle 1| + \frac{1}{\sqrt{6}}|1\rangle_A\langle 0| + \frac{1}{2}|1\rangle_A\langle 1| = \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} +$$

$$+ \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{6}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}$$

$\det(F_1) = 0 \Rightarrow \text{rank}(F_1) < 2 \Rightarrow$ since $\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2}-\lambda \end{vmatrix} = (\frac{1}{3}-\lambda)(\frac{1}{2}-\lambda) - \frac{1}{6} = 0$

So $\frac{1}{6} - \frac{1}{2}\lambda - \frac{1}{3}\lambda + \lambda^2 - \frac{1}{6} = 0 \Rightarrow \lambda^2 - \frac{5}{6}\lambda = 0 \quad \lambda_1^1 = 0 \quad \lambda_1^2 = \frac{5}{6}$

so one of the eingeinvalues is $\neq 0$ so $\text{rank}(F_1) = \underline{1}$

- $k=2 \qquad F_2 = \hat{M}_2^\dagger \hat{M}_2$

$$\hat{M}_2 = \sqrt{\frac{2}{3}}\,|0\rangle_A\langle 0| - \frac{1}{2}\,|0\rangle_A\langle 1|$$

$$\hat{M}_2^\dagger = \sqrt{\frac{2}{3}}\,|0\rangle_A\langle 0| - \frac{1}{2}\,|1\rangle_A\langle 0|$$

$$\hat{M}_2^\dagger \hat{M}_2 = \left(\sqrt{\frac{2}{3}}\,|0\rangle_A\langle 0| - \frac{1}{2}\,|1\rangle_A\langle 0|\right)\left(\sqrt{\frac{2}{3}}\,|0\rangle_A\langle 0| - \frac{1}{2}\,|0\rangle_A\langle 1|\right) =$$

$$= \frac{2}{3}\,|0\rangle_A\langle 0| - \frac{1}{\sqrt{6}}\,|0\rangle_A\langle 1| - \sqrt{\frac{1}{6}}\,|1\rangle_A\langle 0| + \frac{1}{4}\,|1\rangle_A\langle 1|$$

So $F_2 = \frac{2}{3}|0\rangle_A\langle 0| - \frac{1}{\sqrt{6}}|0\rangle_A\langle 1| - \sqrt{\frac{1}{6}}|1\rangle_A\langle 0| + \frac{1}{4}|1\rangle_A\langle 1| =$

$$= \frac{2}{3}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} - \frac{1}{\sqrt{6}}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{2}{3} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\frac{1}{\sqrt{6}}\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ -\frac{1}{\sqrt{6}} & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{2}{3} & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{4}\end{pmatrix}$$

$\det(F_2) = 0 \Rightarrow \mathrm{rank}(F_2) < 2 \Rightarrow$ since $\begin{vmatrix}\frac{2}{3}-\lambda & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{4}-\lambda\end{vmatrix} = \left(\frac{2}{3}-\lambda\right)\left(\frac{1}{4}-\lambda\right) - \frac{1}{6} = 0$

So $\lambda^2 - \frac{2}{3}\lambda - \frac{1}{4}\lambda = \lambda^2 - \frac{11}{12}\lambda = 0 \Rightarrow \lambda_1^2 = 0 \quad \lambda_2^2 = \frac{11}{12}$

so one of the eingeinvalues is $\neq 0$ so $\mathrm{rank}(F_2) = \underline{1}$.

Resuming:

$$F_0 = \begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}\end{pmatrix}, \quad F_1 = \begin{pmatrix}\frac{1}{3} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}} & \frac{1}{2}\end{pmatrix}, \quad F_2 = \begin{pmatrix}\frac{2}{3} & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{4}\end{pmatrix}$$

and it is possible to see that:

$$\sum_{k=0}^{2} F_k = \begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}\end{pmatrix} + \begin{pmatrix}\frac{1}{3} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}} & \frac{1}{2}\end{pmatrix} + \begin{pmatrix}\frac{2}{3} & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{4}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbb{1}_{2\times 2}$$

as expected.

2.4) $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{8}|0\rangle\langle 1| + \frac{1}{8}|1\rangle\langle 0|$

We know that in the case of ρ mixed state the i-th probability is given by: $p_i = Tr(F_i \rho)$

So re-writing ρ in matrix form:

$$\rho = \frac{3}{4}\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \frac{1}{4}\begin{pmatrix}0\\1\end{pmatrix}(0\ 1) + \frac{1}{8}\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \frac{1}{8}\begin{pmatrix}0\\1\end{pmatrix}(1\ 0) =$$

$$= \begin{pmatrix}\frac{3}{4} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{8}\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{1}{8} & 0\end{pmatrix} = \begin{pmatrix}3/4 & 1/8\\ 1/8 & 1/4\end{pmatrix}$$

Let's compute the p_i for F_i;

$$F_0 : p_0 = Tr(F_0\rho) = Tr\left[\begin{pmatrix}0 & 0\\ 0 & \frac{1}{4}\end{pmatrix}\begin{pmatrix}3/4 & 1/8\\ 1/8 & 1/4\end{pmatrix}\right] = Tr\left[\begin{pmatrix}0 & 0\\ 1/32 & 1/16\end{pmatrix}\right] = \frac{1}{16}$$

$$F_1 : p_1 = Tr(F_1\rho) = Tr\left[\begin{pmatrix}\frac{1}{3} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}3/4 & 1/8\\ 1/8 & 1/4\end{pmatrix}\right] = Tr\left[\begin{pmatrix}\frac{2\sqrt{6}+1}{8\sqrt{6}} & \frac{1+\sqrt{6}}{24}\\ \frac{4\sqrt{3}+\sqrt{2}}{16\sqrt{2}} & \frac{1+\sqrt{6}}{8\sqrt{6}}\end{pmatrix}\right] = \frac{2+3\sqrt{6}}{8\sqrt{6}}$$

$$F_2 : p_2 = Tr(F_2\rho) = Tr\left[\begin{pmatrix}\frac{2}{3} & -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & 1/4\end{pmatrix}\begin{pmatrix}3/4 & 1/8\\ 1/8 & 1/4\end{pmatrix}\right] = Tr\left[\begin{pmatrix}\frac{4\sqrt{6}-1}{8\sqrt{6}} & \frac{\sqrt{6}-3}{12\sqrt{6}}\\ \frac{-8\sqrt{3}+\sqrt{2}}{32\sqrt{2}} & \frac{-2+\sqrt{6}}{16\sqrt{6}}\end{pmatrix}\right] =$$

$$= \frac{4\sqrt{6}-1}{8\sqrt{6}} + \frac{-2+\sqrt{6}}{16\sqrt{6}} = \frac{9\sqrt{6}-4}{16\sqrt{6}}$$

So the output probabilities are: $p_0 = 1/16$

$p_1 = \frac{2+3\sqrt{6}}{8\sqrt{6}}$ and we can see that $\sum_{i=0}^{2} p_i = 1$

$p_2 = \frac{9\sqrt{6}-4}{16\sqrt{6}}$

2.5) Let's consider a measurament process with an uncertain result such that the final stat is an element $\{|\psi_i\rangle\}_{i=1,..N}$ chosen with probability p_i.

We can use a larger class of operators $\{M_i\}$, not necessararly self adjoint, that map $|\psi\rangle$ state into the different possibilities $|\psi_i\rangle$:

$$|\psi_i\rangle = \frac{M_i|\psi\rangle}{\sqrt{\langle\psi|M_i^+ M_i|\psi\rangle}} \qquad p_i = \langle\psi|M_i^+ M_i|\psi\rangle \qquad i=1,..,N$$

we want that the different $|\psi_i\rangle$ exhaust all possible evolutions of $|\psi\rangle \Rightarrow$ so $\sum_{i=1}^{N} p_i = 1 \Leftrightarrow \langle\psi|\sum_{i=1}^{N} M_i^+ M_i|\psi\rangle = 1 \Leftrightarrow \sum_{i=1}^{N} M_i^+ M_i = \mathbb{1}$

So we have $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle)_A$

and $\hat{M}_0^A = \frac{1}{2}|1\rangle_A\langle 1|$, $\hat{M}_1^A = \frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|$, $\hat{M}_2^A = \sqrt{\frac{2}{3}}|0\rangle_A\langle 0| - \frac{1}{2}|0\rangle_A\langle 1|$

Let's start from $|\psi_0\rangle_A$:

$$\hat{M}_0^A|\psi\rangle_A = \frac{1}{2}|1\rangle_A\langle 1|\left(\frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\right) = \frac{1}{2\sqrt{2}}|1\rangle_A$$

$${}_A\langle\psi_0|\hat{M}_0^{A+}\hat{M}_0^A|\psi\rangle_A = {}_A\langle 1|\frac{1}{2\sqrt{2}} \cdot \frac{1}{2\sqrt{2}}|1\rangle_A = \frac{1}{8} = p_0$$

$$\text{So } |\psi_0\rangle_A = \frac{1/2\sqrt{2}\,|1\rangle_A}{\underbrace{\sqrt{1/8}}_{=1}} = |1\rangle_A$$

• $|\psi_1\rangle_A$: $\hat{M}_1^A|\psi_1\rangle_A = \left(\frac{1}{\sqrt{3}}|0\rangle_A\langle 0| + \frac{\sqrt{2}}{2}|0\rangle_A\langle 1|\right)\left(\frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\right) =$

$$= \frac{1}{\sqrt{6}}|0\rangle_A + \frac{1}{2}|0\rangle_A = \left(\frac{1}{\sqrt{6}} + \frac{1}{2}\right)|0\rangle_A = \frac{\sqrt{6}+3}{6}|0\rangle_A$$

$${}_A\langle\psi|\hat{M}_1^{A+}\hat{M}_1^A|\psi\rangle_A = {}_A\langle 0|\left(\frac{1}{\sqrt{6}} + \frac{1}{2}\right)\left(\frac{1}{\sqrt{6}} + \frac{1}{2}\right)|0\rangle = \frac{1}{6} + \frac{1}{\sqrt{6}} + \frac{1}{4} = \frac{5}{12} + \frac{1}{\sqrt{6}} = \frac{5+2\sqrt{6}}{12} = p_1$$

$$\text{So } |\psi_1\rangle_A = \frac{\frac{(\sqrt{6}+3)}{6}|0\rangle_A}{\sqrt{\frac{(5+2\sqrt{6})}{12}}} = \frac{\sqrt{6}+3}{\sqrt{3}\sqrt{5+2\sqrt{6}}}|0\rangle_A$$

• $|\psi_2\rangle_A$: $\hat{M}_2^A |\psi\rangle_A = \left(\sqrt{\frac{2}{3}}\, |0\rangle_A \langle 0| - \frac{1}{2} |0\rangle_A \langle 1|\right)\left(\frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right)\right) =$

$$= \frac{1}{\sqrt{3}} |0\rangle_A - \frac{1}{2\sqrt{2}} |0\rangle_A = \frac{8\sqrt{3} - 3\sqrt{8}}{24} |0\rangle_A$$

$${}_A\langle \psi | \hat{M}_2^{A\dagger} \hat{M}_2^A |\psi\rangle_A = {}_A\langle 0| \left(\frac{8\sqrt{3}}{24} - \frac{3\sqrt{8}}{24}\right)\left(\frac{8\sqrt{3}}{24} - \frac{3\sqrt{8}}{24}\right)|0\rangle_A =$$

$$= \frac{11 - 4\sqrt{6}}{24} = p_2$$

So $$|\psi_2\rangle_A = \frac{\frac{8\sqrt{3} - 3\sqrt{8}}{24}}{\sqrt{\frac{11 - 4\sqrt{6}}{24}}} |0\rangle_A = \underbrace{\frac{(2\sqrt{2} + \sqrt{3})\left(\sqrt{11 - 4\sqrt{6}}\right)}{5}}_{=1} |0\rangle_A = |0\rangle_A$$

Now usign the probabilities p_0, p_1, p_2 we can obtained the post measurament state, averaging over all measurament results:

$$\rho_{AM} = p_0 |\psi_0\rangle_A\langle\psi_0| + p_1 |\psi_1\rangle_A\langle\psi_1| + p_2 |\psi_2\rangle_A\langle\psi|$$

- $p_0|\psi_0\rangle_A\langle\psi| = \frac{1}{8}|1\rangle_A\langle 1| = \frac{1}{8}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1/8\end{pmatrix}$

- $p_1|\psi_1\rangle_A\langle\psi_1| = \left(\frac{5+2\sqrt{6}}{12}\right)\underbrace{\left(\frac{\sqrt{6}+3}{\sqrt{3}\sqrt{5+2\sqrt{6}}}\right)^2}_{=1}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}$

$$= \left(\frac{5+2\sqrt{6}}{12}\right)\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}\frac{5+2\sqrt{6}}{12} & 0\\ 0 & 0\end{pmatrix}$$

- $p_2|\psi_2\rangle_A\langle\psi_2| = \left(\frac{11-4\sqrt{6}}{24}\right)\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}\frac{11-4\sqrt{6}}{24} & 0\\ 0 & 0\end{pmatrix}$

So: $\rho_{AM} = \begin{pmatrix}0 & 0\\ 0 & 1/8\end{pmatrix} + \begin{pmatrix}\frac{5+2\sqrt{6}}{12} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\frac{11-4\sqrt{6}}{24} & 0\\ 0 & 0\end{pmatrix} =$

$$= \begin{pmatrix}\frac{5+2\sqrt{6}}{12} + \frac{11-4\sqrt{6}}{24} & 0\\ 0 & 1/8\end{pmatrix} = \begin{pmatrix}\frac{7}{8} & 0\\ 0 & 1/8\end{pmatrix} = \frac{7}{8}|0\rangle_A\langle 0| + \frac{1}{8}|1\rangle_A\langle 1|$$

So the post measurament state, averaging all measuraments is:

$$\rho_{AM} = \frac{7}{8}|0\rangle_A\langle 0| + \frac{1}{8}|1\rangle_A\langle 1| = \begin{pmatrix}\frac{7}{8} & 0\\ 0 & \frac{1}{8}\end{pmatrix}$$

EXERCISE 3

3.1) $\hat{A} = a_x \hat{\sigma}_x + a_z \hat{\sigma}_z$
$\hat{B} = b_x \hat{\sigma}_x + b_z \hat{\sigma}_z$ with $a_x^2 + a_z^2 = b_x^2 + b_z^2 = 1$

Knowing that $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

we have that $\hat{A} = \begin{pmatrix} 0 & a_x \\ a_x & 0 \end{pmatrix} + \begin{pmatrix} a_z & 0 \\ 0 & -a_z \end{pmatrix} = \begin{pmatrix} a_z & a_x \\ a_x & -a_z \end{pmatrix}$

Let's find the eingenvalues:

$$\det \begin{pmatrix} a_z - \lambda & a_x \\ a_x & -a_z - \lambda \end{pmatrix} = 0$$

$$(a_z - \lambda)(-a_z - \lambda) - a_x^2 = -a_z^2 - \lambda a_z + \lambda a_z + \lambda^2 - a_x^2$$
$$= -a_z^2 + \lambda^2 - a_x^2$$
knowing $a_x^2 + a_z^2 = 1$
$$= \lambda^2 - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Similarly for $\hat{B} = \begin{pmatrix} b_z & b_x \\ b_x & -b_z \end{pmatrix}$:

$$\det \begin{pmatrix} b_z - \lambda & b_x \\ b_x & -b_z - \lambda \end{pmatrix} = 0$$

$$(b_z - \lambda)(-b_z - \lambda) - b_x^2 = -b_z^2 + \lambda^2 - b_x^2$$
$b_x^2 + b_z^2 = 1$
$$= \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$\blacksquare$

3.2) Let be $|\Psi(\theta)\rangle_{AB} = \cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B$

($|\Psi(\theta)\rangle_{AB}$ is already normalized)

We know that using the $\hat{I}$ representation $\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1|$

The Pauli Matrices can be written as: $\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$

$\hat{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

So evaluating their action on $|0\rangle$ and $|1\rangle$:

$*$

$\hat{\sigma}_x|0\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)|0\rangle = |1\rangle\underbrace{\langle 0|0\rangle}_{=1} + |0\rangle\underbrace{\langle 1|0\rangle}_{=0} = |1\rangle$

$\hat{\sigma}_x|1\rangle = (|1\rangle\langle 0| + |0\rangle\langle 1|)|1\rangle = |1\rangle\underbrace{\langle 0|1\rangle}_{=0} + |0\rangle\underbrace{\langle 1|1\rangle}_{=1} = |0\rangle$

Similarly for $\hat{\sigma}_y$ and $\hat{\sigma}_z$ (without all the calculations):

$\hat{\sigma}_y|0\rangle = i|1\rangle$, $\hat{\sigma}_y|1\rangle = -i|0\rangle$, $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$

So given $\hat{\sigma}_i \otimes \hat{\sigma}_j$ $\forall i,j = z,x$ with $\hat{\sigma}_i \in \mathcal{H}_A$

$\hat{\sigma}_j \in \mathcal{H}_B$

The possible combinations are using the previous results:

- $(\hat{\sigma}_x \otimes \hat{\sigma}_x)|\Psi(\theta)\rangle_{AB} = (\hat{\sigma}_x \otimes \hat{\sigma}_x)\left(\cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B\right)$

  applied to $|\psi\rangle_A \in \mathcal{H}_A$ with $|\psi\rangle_A = \{|0\rangle_A, |1\rangle_A\}$

  applied to $|\psi\rangle_B \in \mathcal{H}_B$ with $|\psi\rangle_B = \{|0\rangle_B, |1\rangle_B\}$

  | using $*$

  $= \cos\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B - \sin\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B$

- $(\hat{\sigma}_x \otimes \hat{\sigma}_z)|\Psi(\theta)\rangle_{AB} = (\hat{\sigma}_x \otimes \hat{\sigma}_z)\left(\cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B\right)$

  | using $\hat{\sigma}_x|0\rangle = |1\rangle$, $\hat{\sigma}_x|1\rangle = 0$, $\hat{\sigma}_z|1\rangle = -|1\rangle$, $\hat{\sigma}_z|0\rangle = |0\rangle$ $*$

  $= -\cos\left(\frac{\theta}{2}\right)|1\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|0\rangle_A|0\rangle_B$

- $(\hat{\sigma}_z \otimes \hat{\sigma}_z)|\Psi(\theta)\rangle_{AB} = (\hat{\sigma}_z \otimes \hat{\sigma}_z)\left(\cos\frac{\theta}{2}|0\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B\right)$

  | using $\hat{\sigma}_z|0\rangle = |0\rangle$, $\hat{\sigma}_z|1\rangle = -|1\rangle$

  $= -\cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B + \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B$

- $(\hat{\sigma}_z \otimes \hat{\sigma}_x)|\Psi(\theta)\rangle_{AB} = (\hat{\sigma}_z \otimes \hat{\sigma}_x)\left(\cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B\right) - \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B)$

  | using $*$

  $= \cos\left(\frac{\theta}{2}\right)|0\rangle_A|0\rangle_B + \sin\left(\frac{\theta}{2}\right)|1\rangle_A|1\rangle_B$

Hence resuming:

$$(\hat{\sigma}_x \otimes \hat{\sigma}_x) \,|\Psi(\theta)\rangle_{AB} = \cos\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B - \sin\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B$$

$$(\hat{\sigma}_x \otimes \hat{\sigma}_z) \,|\Psi(\theta)\rangle_{AB} = -\cos\left(\frac{\theta}{2}\right)|1\rangle_A|1\rangle_B - \sin\left(\frac{\theta}{2}\right)|0\rangle_A|0\rangle_B$$

$$(\hat{\sigma}_z \otimes \hat{\sigma}_z) \,|\Psi(\theta)\rangle_{AB} = -\cos\left(\frac{\theta}{2}\right)|0\rangle_A|1\rangle_B + \sin\left(\frac{\theta}{2}\right)|1\rangle_A|0\rangle_B$$

$$(\hat{\sigma}_z \otimes \hat{\sigma}_x) \,|\Psi(\theta)\rangle_{AB} = \cos\left(\frac{\theta}{2}\right)|0\rangle_A|0\rangle_B + \sin\left(\frac{\theta}{2}\right)|1\rangle_A|1\rangle_B$$

3.3) we know that $\hat{A} = \hat{\sigma}_x a_x + a_z \hat{\sigma}_z$ and $\hat{B} = b_x \hat{\sigma}_x + b_z \hat{\sigma}_z$

So $\hat{A} \otimes \hat{B} = (\hat{\sigma}_x a_x + a_z \hat{\sigma}_z) \otimes (b_x \hat{\sigma}_x + b_z \hat{\sigma}_z) =$

$= a_x \hat{\sigma}_x \otimes b_x \hat{\sigma}_x + a_x \hat{\sigma}_x \otimes b_z \hat{\sigma}_z + a_z \hat{\sigma}_z \otimes b_x \hat{\sigma}_x + a_z \hat{\sigma}_z \otimes b_z \hat{\sigma}_z$

Now usign the results of previous points:

- $(a_x \hat{\sigma}_x \otimes b_x \hat{\sigma}_x) |\Psi(\theta)\rangle_{AB} = a_x \cdot b_x \cdot \left(\cos\left(\frac{\theta}{2}\right) |1\rangle_A |0\rangle_B - \sin\left(\frac{\theta}{2}\right) |0\rangle_A |1\rangle_B \right)$ (A)
- $(a_x \hat{\sigma}_x \otimes b_z \hat{\sigma}_z) |\Psi(\theta)\rangle_{AB} = a_x \cdot b_z \left(-\cos\left(\frac{\theta}{2}\right) |1\rangle_A |1\rangle_B - \sin\left(\frac{\theta}{2}\right) |0\rangle_A |0\rangle_B \right)$ (B)
- $(a_z \hat{\sigma}_z \otimes b_z \hat{\sigma}_z) |\Psi(\theta)\rangle_{AB} = a_z b_z \left(-\cos\left(\frac{\theta}{2}\right) |0\rangle_A |1\rangle_B + \sin\left(\frac{\theta}{2}\right) |1\rangle_A |0\rangle_B \right)$ (C)
- $(a_z \hat{\sigma}_z \otimes b_x \hat{\sigma}_x) |\Psi(\theta)\rangle_{AB} = a_z b_x \left(\cos\left(\frac{\theta}{2}\right) |0\rangle_A |0\rangle_B + \sin\left(\frac{\theta}{2}\right) |1\rangle_A |1\rangle_B \right)$ (D)

Now let's apply $\langle \Psi(\theta)|_{AB} = \cos\left(\frac{\theta}{2}\right) \langle 0|_A \langle 1|_B - \sin\left(\frac{\theta}{2}\right) \langle 1|_A \langle 0|_B$ to all the contributes:

A: $\langle \Psi(\theta)|_{AB}\, a_x \hat{\sigma}_x \otimes b_x \hat{\sigma}_x |\Psi(\theta)\rangle_{AB} = a_x b_x \left(\cos\left(\frac{\theta}{2}\right) \langle 0|_A \langle 1|_B - \sin\left(\frac{\theta}{2}\right) \langle 1|_A \langle 0|_B \right) \cdot$

$\cdot \left(\cos\left(\frac{\theta}{2}\right) |1\rangle_A |0\rangle_B - \sin\left(\frac{\theta}{2}\right) |0\rangle_A |1\rangle_B \right) =$

$= a_x b_x \left(-\sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \right) = -2 a_x b_x \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)$

B: $\langle a_x \hat{\sigma}_x \otimes b_z \hat{\sigma}_z \rangle_{(\theta)} = a_x b_z \left(\cos\left(\frac{\theta}{2}\right) \langle 0|_A \langle 1|_B - \sin\left(\frac{\theta}{2}\right) \langle 1|_A \langle 0|_B \right) \cdot$

$\cdot \left(-\cos\left(\frac{\theta}{2}\right) |1\rangle_A |1\rangle_B - \sin\left(\frac{\theta}{2}\right) |0\rangle_A |0\rangle_B \right)$

$= a_x b_z \cdot 0 = 0$

C: $\langle a_z \hat{\sigma}_z \otimes b_z \hat{\sigma}_z \rangle_\theta = a_z b_z \left(\cos\left(\frac{\theta}{2}\right) \langle 0|_A \langle 1|_B - \sin\left(\frac{\theta}{2}\right) \langle 1|_A \langle 0|_B \right) \cdot$

$\cdot \left(-\cos\left(\frac{\theta}{2}\right) |0\rangle_A |1\rangle_B + \sin\left(\frac{\theta}{2}\right) |1\rangle_A |0\rangle_B \right)$

$= a_z b_z \left(\underbrace{-\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}_{= -1} \right) = -a_z b_z$

D: $\langle a_z \hat{\sigma}_z \otimes b_x \hat{\sigma}_x \rangle = a_z b_x \cdot \left(\cos\left(\frac{\theta}{2}\right) \langle 0|_A \langle 1|_B - \sin\left(\frac{\theta}{2}\right) \langle 1|_A \langle 0|_B \right) \cdot$

$\cdot \left(\cos\left(\frac{\theta}{2}\right) |0\rangle_A |0\rangle_B + \sin\left(\frac{\theta}{2}\right) |1\rangle_A |1\rangle_B \right) = a_z \cdot b_x \cdot 0 = 0$

N.B. In The previous calculations we consider that $\langle 1|0\rangle_A = 0$, $\langle 1|_A 1\rangle_A = 1$, $\langle 0|_A 0\rangle_A = 1$, $\langle 0|_A 1\rangle_A = 0$, $\langle 0|_B 0\rangle_B = 1$, $\langle 1|_B 1\rangle_B = 1$, $\langle 1|_B 0\rangle_B = 0$, $\langle 0|_B 1\rangle_B = 0$

and the fact that:

$$(\langle \varphi|\langle \phi|)\cdot(|\varphi'\rangle|\phi'\rangle) = (\langle \varphi|\otimes\langle \phi|)(|\varphi'\rangle\otimes|\phi'\rangle) = \langle \varphi|\varphi'\rangle\cdot\langle \phi|\phi'\rangle$$

Now let's sum up all the terms:

$$\langle \hat{A}\otimes\hat{B}\rangle_\theta = \langle \Psi(\theta)|_{AB}\,\hat{A}\otimes\hat{B}\,|\Psi(\theta)\rangle_{AB} = A+B+C+D = -a_z b_z - 2a_x b_x \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

So
$$\langle \hat{A}\otimes\hat{B}\rangle_\theta = -2a_x b_x \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - a_z b_z = -a_x b_x \sin(\theta) - a_z b_z$$

3.4) Given $\hat{A}_K = \vec{a}_K \cdot \vec{\sigma}$, $\hat{B}_K = \vec{b}_K \cdot \vec{\sigma}$ with $K = 0, 1$

we have that

$$\hat{A}_0 = \vec{a}_0 \cdot \vec{\sigma} = a_{0x}\sigma_x + a_{0z}\sigma_z$$
$$\hat{A}_1 = \vec{a}_1 \cdot \vec{\sigma} = a_{1x}\sigma_x + a_{1z}\sigma_z$$
$$\hat{B}_0 = \vec{b}_0 \cdot \vec{\sigma} = b_{0x}\sigma_x + b_{0z}\sigma_z$$
$$\hat{B}_1 = \vec{b}_1 \cdot \vec{\sigma} = b_{1x}\sigma_x + b_{1z}\sigma_z$$

Let's consider the CHSH inequality:

$$|S| = | E(\hat{A}_0, \hat{B}_0) + E(\hat{A}_0, \hat{B}_1) + E(\hat{A}_1, \hat{B}_0) - E(\hat{A}_1, \hat{B}_1)| \leq 2$$

with $E(\hat{A}_i, \hat{B}_j) = \langle \Psi(\frac{\pi}{2}) | \hat{A}_i \otimes \hat{B}_j | \Psi(\frac{\pi}{2}) \rangle$ is the correlation function for the Bell state .

Evaluating the correlation function for $|\Psi(\frac{\pi}{2})\rangle_{AB} = |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$ the results are:

$$E(\hat{A}_0, \hat{B}_0) = -\vec{a}_0 \cdot \vec{b}_0 = -a_{0x}b_{0x} - a_{0z}b_{0z}$$
$$E(\hat{A}_1, \hat{B}_1) = -\vec{a}_1 \cdot \vec{b}_1 = -a_{1x}b_{1x} - a_{1z}b_{1z}$$
$$\vec{E}(\hat{A}_1, \hat{B}_0) = -\vec{a}_1 \cdot \vec{b}_0 = -a_{1x}b_{0x} - a_{1z}b_{0z}$$
$$\hat{E}(\hat{A}_0, \hat{B}_1) = -\vec{a}_0 \cdot \vec{b}_1 = -a_{0x}b_{1x} - a_{0z}b_{1z}$$

In order to maximize the violation of the CHSH inequality we need $\vec{a}_0, \vec{b}_0, \vec{a}_1, \vec{b}_1$ such that:

$$E(\hat{A}_0, \hat{B}_0) + E(\hat{A}_0, \hat{B}_1) = \sqrt{2}$$
$$E(\hat{A}_1, \hat{B}_0) - E(\hat{A}_1, \hat{B}_1) = \sqrt{2}$$

So the maximum violation is obtained for:

$$a_{0x}b_{0x} + a_{0z}b_{0z} = a_{0x}b_{1x} + a_{0z}b_{1z} = a_{1x}b_{0x} + a_{1z}b_{0z} = \frac{\sqrt{2}}{2}$$
$$a_{1x}b_{1x} + a_{1z}b_{0z} = \frac{\sqrt{2}}{2}$$

Hence,

$a_{0x} = 0$ $\quad b_{0x} = \frac{1}{\sqrt{2}}$

$a_{0z} = 1$ $\quad b_{0z} = \frac{1}{\sqrt{2}}$

$a_{1x} = 1$ $\quad b_{1x} = -\frac{1}{\sqrt{2}}$

$a_{1z} = 0$ $\quad b_{1z} = \frac{1}{\sqrt{2}}$

Now let's find Θ s.t. CHSH inequality is violated with the same observables as above.

Repeating what we have done before but with $|\Psi(\theta)\rangle_{AB}$ we obtained:

$E(\hat{A}_0, \hat{B}_0) = -\frac{\sqrt{2}}{2}$ $E(\hat{A}_0, \hat{B}_1) = -\frac{\sqrt{2}}{2}$ $E(\hat{A}_1, \hat{B}_0) = -\frac{\sqrt{2}}{2}\sin\Theta$ $E(\hat{A}_1, \hat{B}_1) = \frac{\sqrt{2}}{2}\sin\Theta$

So we have that the CHSH inequality becomes:

$$|-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\sin\Theta - \frac{\sqrt{2}}{2}\sin\Theta| \leq 2$$

$$|-\sqrt{2}(1+\sin\Theta)| \leq 2$$

which is true for $\Rightarrow$ $|(1+\sin\Theta)| \leq \sqrt{2}$

So the value of Θ must satisfy this, $\blacksquare$
otherwise CHSH inequality is violated for all Θ s.t. $|(1+\sin\Theta)| > \sqrt{2}$.

3.5) Let's assume that we choose a new set of Observables $\hat{A}'_K$ and $\hat{B}'_K$ that are different from $\hat{A}_K$ and $\hat{B}_K$ and that we compute their expectation values for the Bell state $|\Psi(\theta)\rangle_{AB}$. We can express the new observables as:

$$\hat{A}'_K = \vec{a}'_K \cdot \vec{\sigma} \quad \text{and} \quad \hat{B}'_K = \vec{b}'_K \cdot \vec{\sigma}$$ where $\vec{a}'_K$ and $\vec{b}'_K$ are the new vectors in The (x,z) plane.

We can the write the CHSH expression as:

$$|\langle A'_0 B'_0\rangle + \langle A'_0 B'_1\rangle + \langle A'_1 B'_0\rangle - \langle A'_1 B'_1\rangle| \leq 2$$

If we want to increase the violation of the CHSH inequality we need to find a set of observables that leads to a higher value of the CHSH expression than $2\sqrt{2}$. However let's show that it is not possible using the Bell state $|\Psi(\theta)\rangle_{AB}$ and any set of Observables $\hat{A}'_K$ and $\hat{B}'_K$ that satisfy the same conditions as $\hat{A}_K$ and $\hat{B}_K$ (i.e. $a_x^2 + a_z^2 = b_x^2 + b_z^2 = 1$).

In order to show that there are no other observables that can maximize the violation of CHSH for $|\Psi(\theta)\rangle_{AB}$, we can use the fact that we can write the observables as a linear combination of Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$.

Therefor we can write any observable $\hat{A}$ as:

$$\hat{A} = a_1\hat{\sigma}_x + a_2\hat{\sigma}_y + a_3\hat{\sigma}_z$$

The same for $\hat{B}$: $\quad \hat{B} = b_1\hat{\sigma}_x + b_2\hat{\sigma}_y + b_3\hat{\sigma}_z$

where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$.

So the CHSH expression for A'_K B'_K becomes:

$$|\langle A'_0 B'_0\rangle + \langle A'_1 B'_0\rangle + \langle A'_0 B'_1\rangle - \langle A'_1 B'_1\rangle| = 2|a_1 b_1 - a_2 b_2 - a_3 b_3|$$

Now to show that This new CHSH expression cannot be larger than $2\sqrt{2}$ we can use the fact that $|a_1| \leq 1$, $|a_2| \leq 1$, $|a_3| \leq 1$, $|b_1| \leq 1$, $|b_2| \leq 1$ and $|b_3| \leq 1$.

Therefore we have:

$$|a_1b_1 - a_2b_2 - a_3b_3| \leq |a_1b_1| + |a_2b_2| + |a_3b_3| \leq 1+1+1 = 3$$

Substituting this inequality into the CHSH expression, we obtain:

$$|\langle A_0' B_0' \rangle + \langle A_0' B_1' \rangle + \langle A_1' B_0' \rangle - \langle A_1' B_1' \rangle| = 2|a_1b_1 - a_2b_2 - a_3b_3| \leq 6$$

This means that the CHSH expression for $\hat{A}'_k$ and $\hat{B}'_k$ cannot be larger than 6, which is less than $2\sqrt{2} \approx 2.83$. Therefore, there are no other observables that can lead to higher violation of the CHSH inequality than $\hat{A}_k$ and $\hat{B}_k$ for the Bell state $|\Psi(\theta)\rangle_{AB}$.

Therefore, we conclude that for the Bell state $|\Psi(\theta)\rangle_{AB}$, the observables $\hat{A}_k$ and $\hat{B}_k$ corresponding to $\vec{a_0}, \vec{a_1}, \vec{b_0}, \vec{b_1}$ are the optimal observables for maximally violating the CHSH inequality and there are no other observables that can lead to an higher violation for any value of θ.