Stochastic Methods for Finance

Exam June 26, 2018

Exercice 1 *(10 points) Consider a Black-Scholes market and a derivative contract with payoff at the maturity T given by*

$$F(n, S_T) = n^\alpha * 1_{n<S_T<n+1}, \qquad \alpha > 0, n \in \mathbb{N}.$$

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, ...$ and the limit of the price for $n \to \infty$. [HINT : compare the cumulated Gaussian distribution with the behavior of the power function $n^\alpha, \alpha > 0$.];

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

iv) Compute the price of the portfolio F given by

$$F(S_T) = \sum_{n=1}^{\infty} F(n, S_T);$$

v) Compute the amount of Call options with strike price $K = 1$ one has to buy/sell in order to get a Delta neutral (global) portfolio including F.

vi) Compute the amount of Call options with strike price $K = 1$ one has to buy/sell in order to get a Delta-Gamma neutral (global) portfolio including F.

Exercice 2 *(5 points)*

Consider a Black&Scholes market where a risky asset evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_0 = s,$$

and a riskless asset is associated to the risk free rate r. Consider a Super-Butterfly option, that is a derivative contract with payoff at the maturity T given by

$$\begin{array}{rl} a & if\ S_T < K_1; \\ a + nS_T - nK_1 & if\ K_1 < S_T < K_2; \\ a - nS_T + n(2K_2 - K_1) & if\ K_2 < S_T < K_3; \\ a & if\ S_T > K_3, \end{array}$$

where $n \in \mathbb{N}, a > 0$ and $K_3 = 2K_2 - K_1$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta and the Gamma of the contract ;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 3 *(5 points)*

In the Black-Scholes model, find the price at time $t \leq T$ of a DOWN-AND-OUT contract where the owner receives the Super-Butterfly payoff (as in the previous exercise)

$$\begin{aligned} a \quad & if\ S_T < K_1; \\ a + nS_T - nK_1 \quad & if\ K_1 < S_T < K_2; \\ a - nS_T + n(2K_2 - K_1) \quad & if\ K_2 < S_T < K_3; \\ a \quad & if\ S_T > K_3, \end{aligned}$$

(where $n \in \mathbb{N}, a > 0$ and $K_3 = 2K_2 - K_1$) at the maturity T only if the asset has not reached the lower barrier L. Provide the price of the contract when $n \to +\infty$. Finally, find the Delta of the contract.

Exercice 4 *(5 points)*

Solve the following PDE for $t \leq T$:

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + xy &= 0 \\ F(T, x, y) &= x. \end{aligned}$$

Exercice 5 *(8 points)*

Questions on the theory.

i) State and prove the (first version of the) Feynman-Kac formula.

ii) Provide the price of a CALL option in the Black-Scholes model where all parameters are deterministic

iii) Provide the price of a DOC and DOP barrier contract iv) Find the distribution function of the running maximum for the Brownian motion with (constant) drift.