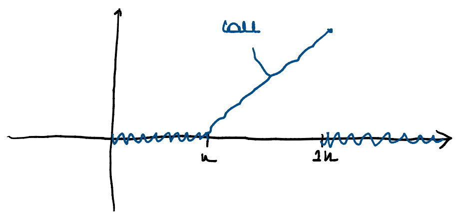
25 June 2020

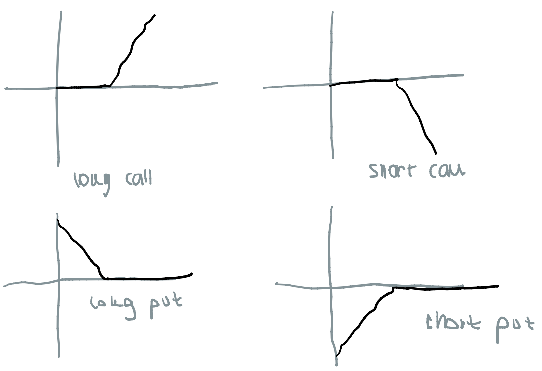
sabato 29 maggio 2021 15:13

E 1 1

- Fix n and try to find which is the price

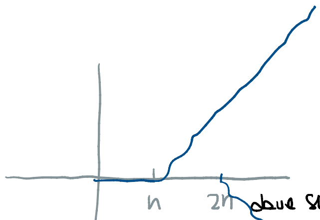


We want to write it as things I know



$$= \underbrace{(S_T - n)^+}_{\text{short position}} - (S_T - 2n)^+ - n\,\mathbb{1}_{S_T > 2n}$$

$(S_T - n)^+$



deve succedere qualcosa

$$(x-n)^+ + K = 0$$

$$K = -(x-n)^+ = -\begin{cases} x-n & x \geq 2n \\ 0 & \text{other} \end{cases} = \begin{cases} -n+x & x \geq 2n \\ 0 & \text{other} \end{cases}$$

Idea $\quad (S_T - n)^+ - (S_T - n)^+ \mathbb{1}_{S_T > 2n} =$ need to do that or it makes no sense

$$= (S_T - n)^+ - \underbrace{(S_T - 2n)\mathbb{1}_{S_T > 2n}}_{(S_T - 2n)^+} - n\,\mathbb{1}_{S_T > 2n}$$

i/ii) Price

We have a decomposition

$$\text{price}_t^n = \underbrace{S_t \Phi(d_1^n) - e^{-r(T-t)} n\, \Phi(d_2^n)}_{\text{CALL}} \underbrace{- S_t \Phi(d_1^{2n}) + e^{-r(T-t)} 2n\, \Phi(d_2^{2n})}_{-\text{CALL}} \underbrace{- n e^{-r(T-t)} \Phi(d_2^{2n})}_{\text{DIGITAL}}$$

$$\Delta_t^n = \Phi(d_1^n) - \Phi(d_1^{2n}) - \frac{n e^{-r(T-t)} e^{-\frac{(d_2^{2n})^2}{2}}}{\sqrt{2\pi}\, S \sigma \sqrt{T-t}}$$

Take limits: $\quad d_{1,2}^n = \dfrac{\ln S_t - \ln n + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

$$\text{price}_t = S_t\left[\Phi(d_1^n) - \Phi(d_1^{2n})\right] + e^{-r(T-t)} n \left[-\Phi(d_2^n)\right] + \Phi(d_2^{2n})\right] \longrightarrow 0$$

$$\int_{d_1^n}^{d_1^{2n}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy \sim \frac{e^{-\frac{(\ln n)^2}{2}}}{\sqrt{2\pi}} \left(\ln\left(\frac{2n}{n}\right)\right) \xrightarrow{n \to +\infty} 0$$

$$n \int_{d_1^n}^{d_1^{2n}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy \sim \frac{e^{-\frac{(\ln n)^2}{n} + \ln n}}{\sqrt{2\pi}} \ln\left(\frac{2n}{n}\right) \longrightarrow 0$$

$$\Delta_n^t \longrightarrow 0$$

iii) $\quad \Gamma = \dfrac{\partial \text{price}}{\partial S^2}$

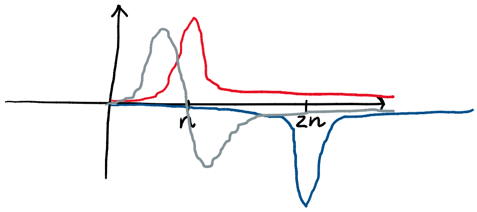
$$\Gamma_t^n = \frac{e^{-\frac{d_1^2 n}{2}}}{S\sigma\sqrt{T-t}\sqrt{2\pi}} - \frac{e^{-\frac{d_1^2 2n}{2}}}{S\sigma\sqrt{T-t}\sqrt{2\pi}} -$$

$$\left[n e^{-r(T-t)} \left(e^{-\frac{(d_2^{2n})^2}{2}}\right)\right]' = \text{const} \cdot \left[-1 \quad e^{-\frac{(d_2^{2n})^2}{2}} \quad + e^{-\frac{(d_2^{2n})^2}{2}} \quad - d_2^{2n} \quad \right]$$

$\overline{S\sigma\sqrt{T-t}\sqrt{2\pi}} - \overline{S\sigma\sqrt{T-t}\sqrt{2\pi}}$

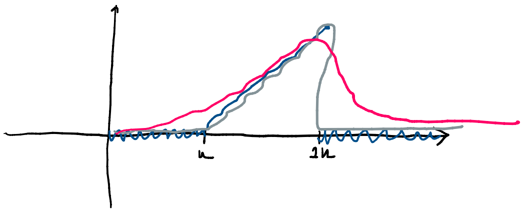
$$\left[\frac{n\,e^{-r(T-t)}}{\sigma\sqrt{T-t}\sqrt{2\pi}}\left(\frac{e^{-\frac{{d_2^{2n}}^2}{2}}}{S}\right)\right]' = \text{Const}\cdot\left[-\frac{1}{S^2}\,e^{-\frac{(d_2^{2n})^2}{2}} + \frac{1}{S}\,e^{-\frac{(d_2^{2n})^2}{2}}\cdot -d_2^{2n}\cdot\frac{1}{S\sqrt{2\pi}\,\sigma\sqrt{T-t}}\right]$$

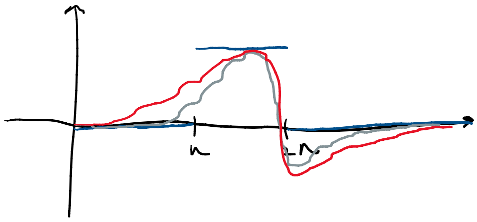
Π:



It might be negative

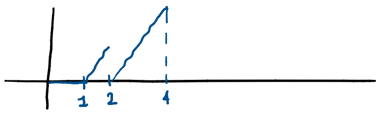
iv)





TO THINK

V) $F(S_T) = \sum_{n=1}^{2} F(n, S_T) = (S_T - 1)^+ \mathbb{1}_{1<S_T<2} + (S_T-2)^+ \mathbb{1}_{2<S_T<4}$



$F(S_T) = (S_T-1)^+ + (1-\mathbb{1}_{S_T>2})(S_T-1) \quad + (S_T-2)^+ \;+ (S_T-2)^+(1-\mathbb{1}_{S_T>4}) =$

$= 2(S_T-1)^+ + (S_T-2)^+ - \mathbb{1}_{S_T>2} - (S_T-4)^+ - 2\,\mathbb{1}_{S_T>4}$

price ... BUILDING BLOCKS

VI) Put/call strike $K=1$ one has to buy/sell in order to get a Delta/Vega neutral (global) portfolio

Δ-$\mathcal{V}$ neutral

F initial port
$F + x\,\text{call}(1) + y\,\text{put}(1)$

We look for the amount, so we want to find x and y

$0 = \Delta^F + x\,\Delta^{\text{call}(1)} + y\,\Delta^{\text{Put}(1)}$
$0 = \mathcal{V}^F + x\,\mathcal{V}^{\text{call}(1)} + y\,\mathcal{V}^{\text{Put}(1)}$ $\quad\mathcal{V}$ is the same for put and call
$= \mathcal{V}^F + (x+y)\,\mathcal{V}^{\text{call}(1)}$

$x+y = \dfrac{-\mathcal{V}^F}{\mathcal{V}^{\text{call}(1)}}$, $\quad x = -y - \dfrac{\mathcal{V}^F}{\mathcal{V}^{\text{call}(1)}}$

$0 = \Delta^F - y\,\Delta^{\text{call}(1)} - \dfrac{\mathcal{V}^F}{\mathcal{V}^{\text{call}(1)}}\Delta^{\text{call}(1)} + y\,\Delta^{\text{Put}(1)}$

$y = \dfrac{\Delta^F - \mathcal{V}^F\Delta^{\text{call}}/\mathcal{V}^{\text{call}}}{\Delta^{\text{call}(1)} - \Delta^{\text{Put}(1)}} = \Delta^F - \mathcal{V}^F\Delta^{\text{call}}/\mathcal{V}^{\text{call}}$

EX 2

$(K - S_T^n)^+$

- Show S_t^n Geometric Brownian motion

$$dS_t^n = n S_t^{n-1}\, dS_t + \frac{1}{2} n(n-1) S^{n-2}\, S^2 \sigma^2\, dt =$$
$$= n S_t^n r\, dt + n S_t^n \sigma dW + \frac{1}{2} n(n-1) S^n \sigma^2 dt$$
$$\frac{dS_t^n}{S_t^n} = \left(nr + \frac{n}{2}(n-1)\sigma^2\right) dt + n\sigma dW$$
$$= \Big[r + \underbrace{\left((n-1)r + \frac{n}{2}(n-1)\sigma^2\right)}_{-q}\Big] dt + n\sigma dW$$

$$Put_t = Ke^{-r(T-t)}\left(1-\Phi(d_2)\right) - S_t^n e^{\left((n-1)r + \frac{n}{2}(n-1)\sigma^2\right)(T-t)}\left(1-\Phi(d_1)\right)$$

$$d_1 = \frac{\lg\left(\frac{S_t^n}{K}\right) + \left(nr + \frac{n}{2}(n-1)\sigma^2 \pm \frac{n^2}{2}\sigma^2\right)(T-t)}{n\sigma\sqrt{T-t}} \quad , \; d_1 = d_2 + n\sigma\sqrt{T-t}$$

$$Call_t = S_t^n e^{\left((n-1)r + \frac{n}{2}(n-1)\sigma^2\right)(T-t)}\, \Phi(d_1) - Ke^{-r(T-t)}\, \Phi(d_2)$$

$$Call_t - Put_t = S_t^n e^{-q(T-t)} - Ke^{-r(T-t)}$$

ii) Delta , Like June 2019

$$\Delta_t^{Put} = Ke^{-r(T-t)}\left(-\frac{\partial \Phi(d_2)}{\partial S}\right) - n S_t^{n-1} e^{(\quad)}\left(1-\Phi(d_1)\right) +$$
$$+ S_t^n e^{(\quad)} \frac{\partial \Phi(d_1)}{\partial S} = -n S_t^{n-1} e^{(\quad)}\left(1-\Phi(d_1)\right)$$

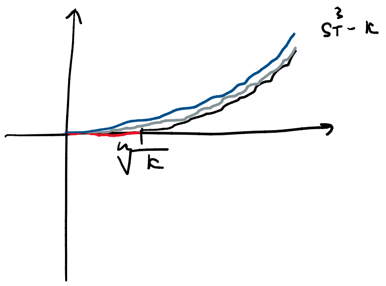
$$\frac{\partial \Phi(d_1)}{\partial S} = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}\,\cancel{n}\sigma\sqrt{T-t}}\; \frac{1}{S_t^n}\, \cancel{n} S_t^{n-1} = \frac{e^{-d_1^2/2}}{\sqrt{2\pi}\sigma\sqrt{T-t}}\; \frac{1}{S_t}$$

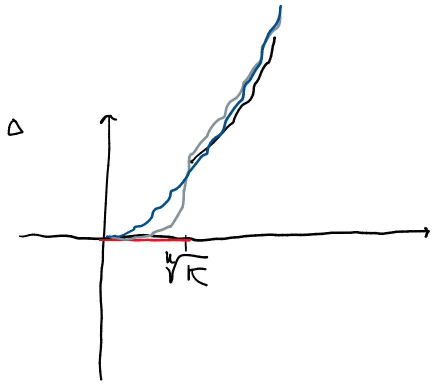
$$\frac{\partial \Phi(d_2)}{\partial S} = \frac{e^{-d_2^2/2}}{\sqrt{2\pi}\,\cancel{n}\sigma\sqrt{T-t}}\; \frac{1}{S_t^n}\, \cancel{n} S_t^{n-1} = \frac{e^{-d_2^2/2}}{\sqrt{2\pi}\sigma\sqrt{T-t}}\; \frac{1}{S_t}$$

$$-\frac{d_1^2}{2} = -\frac{d_2^2}{2} - \frac{n^2\sigma(T-t)}{2} - n\sigma\sqrt{T-t}\, d_2 = -\frac{d_2^2}{2} - \frac{n^2\sigma(T-t)}{2} - \lg\left(\frac{S_t^n}{K}\right) - \left[\left(nr - \frac{n}{2}\sigma^2\right)(T-t)\right]$$

$$\frac{e^{-d_2^2/2}}{\sqrt{2\pi}\,\sigma\sqrt{T-t}}\; \cancel{S_t^n}\, e^{\left((n-1)r + \frac{n}{2}(n-1)\sigma^2\right)(T-t)}\; e^{-\frac{n^2}{2}\sigma(T-t)}\; \frac{K}{\cancel{S_t^n}}\; e^{-nr(T-t)} =$$
$$= \frac{e^{-d_2^2/2}}{\sqrt{2\pi}\,\sigma\sqrt{T-t}}\; e^{-r(T-t)} K\, \frac{1}{S_t}$$

iii) price





EX 3)

$$price^{U}(t,S,G) = price(t,S,G_L) + \left(\frac{L}{S}\right)^{\frac{2\tilde r}{\sigma^2}} price\left(L, \frac{L^2}{S}, G^L\right)$$
$$(K - S_T^n)^+ = (K - S_T^n)\,\mathbb{1}_{\{K - S_T^n > 0\}} = (K - S_T^n)\,\mathbb{1}_{S_T < \sqrt[n]{K}}$$

$G_L = G\,\mathbb{1}_{x > L}$, $(K - S_T^n)\,\mathbb{1}_{S_T < \sqrt[n]{K}}\,\mathbb{1}_{S_T > L}$

$\sqrt[n]{K} < L$ $\qquad 0 \longrightarrow price(t,S,G_L) = 0$

$\sqrt[n]{K} > L$ $\qquad (K - S_T^n)\,\mathbb{1}_{S_T > L} - (K - S_T^n)\,\mathbb{1}_{S_T > \sqrt[n]{K}}\,\mathbb{1}_{S_T > L} =$

$$= -(S_T^n - L^n)^+ + (K - L^n)\,\mathbb{1}_{S_T > L} - (K - S_T^n)^+$$

$G^L = G\,\mathbb{1}_{x < L}$, $(K - S_T^n)\,\mathbb{1}_{S_T < \sqrt[n]{K}}\,\mathbb{1}_{S_T < L}$

$L > \sqrt[n]{K}$, $(K - S_T^n)^+$

$L < \sqrt[n]{K}$, $(K - S_T^n)\,\mathbb{1}_{S_T < L} = (L^n - S_T^n)\,\mathbb{1}_{S_T < L} + (K - L^n)\,\mathbb{1}_{S_T < L} =$

$$= (L^n - S_T^n)^+ + (K - L^n) - (K - L^n)\,\mathbb{1}_{S_T > L}$$

$L < \sqrt[n]{K}$, $(K - S_T^n)\,\mathbb{1}_{S_T < L} = (L^n - S_T^n)\,\mathbb{1}_{S_T < L} + (K - L^n)\,\mathbb{1}_{S_T < L} =$

$= (L^n - S_T^n)^+ + (K - L^n) - (K - L^n)\,\mathbb{1}_{S_T > L}$

Plug in

- compute Delta using building blocks

Ex4

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} + x^2\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + zyx^n = 0$$
$$F(T,x,y,z) = zyx^2.$$

- No mixed derivative but function dep on x, y, z
- But first derivative

$$\begin{aligned} dx &= K_x dt + \gamma_x dW_1 &&= \sqrt{2}\,x\,dW_1 \\ dy &= K_y dt + \gamma_y dW_2 &&= \sqrt{2}\,dW_2 + dt \\ dz &= K_z dt + \gamma_z dW_3 &&= \sqrt{2}\,dW_3 \end{aligned}$$

$\xrightarrow{\text{GBM}}$

$$\begin{aligned} X_T &= x_t \exp\left(-(T-t) + \sqrt{2}\,(W_1(T) - W_1(t))\right) \\ Y_T &= y_t + \sqrt{2}\,(W_2(T) - W_2(t)) + (T-t) \\ Z_T &= z_t + \sqrt{2}\,(W_3(T) - W_2(t)) \end{aligned}$$

GBM $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$
$\rightarrow S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

$x^2 = \frac{H^2}{2} \longrightarrow H = \sqrt{2}\,x$

$$E_t[\underbrace{Z_T Y_T X_T^2}_{\perp\!\!\!\perp}] + \int_t^T E_{t,x}[Z_s Y_s X_s^n]\,ds$$

asking $\int_t^T \left(\frac{\partial F}{\partial x} x\right)^2 ds < +\infty$

$\int_t^T \left(\frac{\partial F}{\partial y}\right)^2 ds < +\infty \qquad \int_t^T \left(\frac{\partial F}{\partial z}\right)^2 ds < +\infty$

$$E_t[Z_T Y_T X_T^2] \overset{\perp\!\!\!\perp}{=} E_t[Z_T]\,E_t[Y_T]\,E_t[X_T^2] = z_t(y_t + (T-t))\,x_t^2 \exp\left(-(T-t) + (T-t)\right) =$$
$$= z_t x_t^2 (y_t + (T-t))$$

$$E_t[Z_s]\,E_t[Y_s]\,E_t[X_s^n] = z_t(y_t + (s-t))\,x_t^n \exp\left(\underbrace{-n(s-t) + n^2(s-t)}_{(s-t)(n^2-n)}\right)$$

$$z_t y_t x_t^n \int_t^T \exp((s-t)(n^2-n))\,ds + z_t y_t x_t^n \int_t^T (s-t)\exp((s-t)(n^2-n))\,ds =$$
$$= \frac{z_t y_t x_t^n}{(n^2-n)}\left[\exp((T-t)(n^2-n)) - 1\right] + z_t y_t x_t^n \left[\frac{(T-t)}{(n^2-n)}\exp((T-t)(n^2-n)) - \frac{1}{(n^2-n)^2}\left[\exp((T-t)(n^2-n)) - 1\right]\right]$$

Check: Boundary conditions: ✓

PDE: $\frac{\partial F}{\partial t} = -z x^2 y$, ...