26 agosto 2019

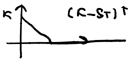
domenica 30 maggio 2021

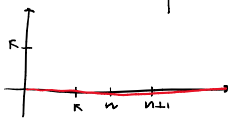
EX 1

i) $F(n,S_T) = (K-S_T)^+ \cdot \mathbb{1}_{n<S_T<n+1}$, $K>0$

Draw it $(K-S_T)^+ = (K-S_T)\mathbb{1}_{S_T<K}\,\mathbb{1}_{n<S_T<n+1}$

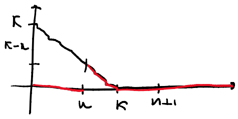
If $K<n$





$\equiv 0$, so price $\equiv 0$

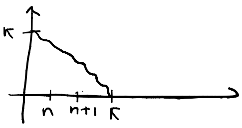
If $n<K<n+1$



$$\text{PAYOFF} = \underbrace{(K-n)\mathbb{1}_{S_T>n}}_{\text{digital option}} - (S_T-n)^+ + (S_T-K)^+$$

$$\text{price}_t = (K-n)e^{-r(T-t)}\Phi(d_2^n) - S_t\Phi(d_1^n) + ne^{-r(T-t)}\Phi(d_2^n) + S_t\Phi(d_1^K) - Ke^{-r(T-t)}\Phi(d_2^K)$$

If $K>n+1$



$$\text{payOFF} = (K-n)\mathbb{1}_{S_T>n} - (S_T-n)^+ + (n+1-K)\mathbb{1}_{S_T>n+1} + (S_T-(n+1))^+$$

$$\text{price}_t = (K-n)e^{-r(T-t)}\Phi(d_2^n) - S_t\Phi(d_1^n) + Ke^{-r(T-t)}\Phi(d_2^n) + (n+1-K)e^{-r(T-t)}\Phi(d_2^{n+1}) + S_t\Phi(d_1^{n+1}) - Ke^{-r(T-t)}\Phi(d_2^{n+1})$$

Limit. Intuitively $\to 0$ since for K fixed, $\exists\, \bar{N}$ s.t. $\forall n>\bar{N}$, $K<n<n+1$
$\to$ price $\equiv 0$

PERFECT ✓

ii) Delta

1) $\Delta \equiv 0$ $\xrightarrow{\text{same reasoning}}$ $\Delta_n \xrightarrow{n\to+\infty} 0$

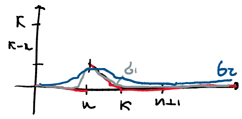
2) $\Delta_t = (K-n)\Delta^{dig} - \Delta call^{(n)} + \Delta call^{(K)}$ If I want explicit

3) $\Delta_t = (K-n)\Delta^{dig} - \Delta call^{(n)} + (n+1-K)\Delta^{Dig} + \Delta call^{(n+1)}$

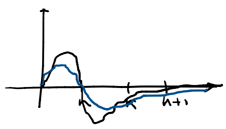
iii) Case 1: always 0

Case 2: $\sigma_1 \to \sigma_2$, $\sigma_2>\sigma_1$

Start with $\sigma_1<<\sigma_2$



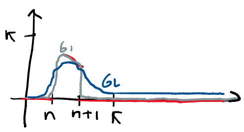
Δ



qualitative behaviour

Case iii)

$\sigma_1<<\sigma_2$



iv) $F(S_T) = \sum_{n=0}^{+\infty} F(n,S_T) = \sum_{n=0}^{+\infty}(K-S_T)^+\mathbb{1}_{n<S_T<n+1} =$
$= (K-S_T)^+$, just a PUT = CLASSIC PUT OPTION

$\text{price}_t(F(S_T)) = B\&S$ for put FIND

v) $F(S_T) = (K-S_T)^+$

$\Delta^F = \Phi(d_1^K) - 1$

$V^{TOT} = F + \alpha\, Call(1)$

$\Delta^V = 0$

$\longrightarrow 0 = \Phi(d_1^K) - 1 + \alpha\Phi(d_1^1) \to \alpha = \dfrac{-\Phi(d_1^K)+1}{\Phi(d_1^1)}$

EX 2

Exercise 2 *(6 points)*

In the Black-Scholes model, find the price at time $t \le T$ of an UP-AND-OUT contract

Exercice 2 *(6 points)*

In the Black-Scholes model, find the price at time $t \leq T$ of an UP-AND-OUT contract where the owner receives the payoff

$$F(n, S_T) = (K - S_T)^+ * 1_{n<S_T<n+1},$$

at the maturity T only if the asset has never reached the upper barrier $L > 0$. Provide the price of the contract when $n \to +\infty$. Finally, find the Delta of the contract.

UP and OUT contract

UP and OUT

$$(*)\ \text{price}_t^{LO} = \left[\text{price}(t, s, \phi^L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \text{price}\left(t, \frac{L^2}{s}, \phi^L\right)\right] 1\!\!1_{s<L}$$

$$\tilde{r} = r - \frac{1}{2}\sigma^2 \quad , \quad \phi^L(x) = \phi(x)\, 1\!\!1_{x<L}$$

$$\phi^L(s) = (K-s)^+\, 1\!\!1_{n<s<n+1}\, 1\!\!1_{s<L}$$

In case of UP and OUT we should distinct two possibilities

$\boxed{K<L}$, presence of barrier is redondant

$K - S > 0 \iff K > S$

If $K>L$, then S never reaches L

$$(K-S)\, 1\!\!1_{S<K}\, 1\!\!1_{n<S<n+1}$$

$K>L \rightarrow$ payoff cut before K

$$(K-S)\, 1\!\!1_{S<K}\, 1\!\!1_{n<S<n+1}$$

$$(K-S+L-L)\, 1\!\!1_{S<L}\, 1\!\!1_{n<S<n+1}$$

$$\Big(\underbrace{(L-S)\, 1\!\!1_{S<L}}_{(L-S)^+} + (K-L)\, 1\!\!1_{S<L}\Big)\, 1\!\!1_{n<S<n+1}$$

(A) $K<L$, 3 more cases

$\underline{A.1}$, $n \geq K \longrightarrow$ price $= 0$ ($\lim_{n\to+\infty}$ price $= 0$)

$\underline{A.2}$ $n<K<n+1 \longrightarrow$

$$\underbrace{(K-n)\,\text{Dig}(n) - (S-n)^+ + (S-K)^+}_{\text{plug into the general formula } (*)}$$

(A.3) $n+1 \leq K \longrightarrow (K-n)\,\text{Dig}(n) - (S-n)^+ + (S-(n+1))^+ - (K-(n+1))\,\text{Dig}(n+1)$

No presence of L, so plug into $*$

(B) $\boxed{K>L}$ I have

$$\phi^L = \big((K-L)\, 1\!\!1_{S<L} + (L-S)\, 1\!\!1_{S<L}\big)\, 1\!\!1_{n<S<n+1}$$

$$= \underbrace{(K-L)(1 - 1\!\!1_{S>L})\, 1\!\!1_{n<S<n+1}}_{\text{write it as follows}} + \underbrace{(L-S)\, 1\!\!1_{S<L}\, 1\!\!1_{n<S<n+1}}_{\text{AS BEFORE}}$$

$$\underbrace{(K-L)\, 1\!\!1_{n<S<n+1}}_{\text{Difference of two digital options}} - (K-L)\underbrace{1\!\!1_{S>L}\, 1\!\!1_{n<S<n+1}}_{\text{we can procede as before}}$$

We plug all this expression into $(*)$ and we get the price.

Finally, in order to get the DELTAs, we differentiate wrt S or, more conveniently, get the Delta of the con building blocks (ie Delta of a Digital, Delta of a Call, etc, etc)

14:29

ex 3 Check video

$$\begin{cases} \frac{\partial F}{\partial t} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0 \\ F(T, x, y, z) = x z e^y \end{cases}$$

- We want to use multi-dimensional Feymann-Kač
- There is no discounting
- No function dep on x, y, z

The solution is

$$F(t, x, y, z) = \mathbb{E}_{t,x}\left[x_T z_T e^{y_T}\right]$$

assuming that

$$\left(\frac{\partial F}{\partial x} H_x\right) \in \mathcal{H} \quad \left(\frac{\partial F}{\partial y} H_y\right) \in \mathcal{H} \quad , \left(\frac{\partial F}{\partial z} H_z\right) \in \mathcal{H}$$

- The dynamics of x, y, z can be deduced by the PDE

$$\begin{aligned} dx &= x\sqrt{2}\, dW_1 \\ dy &= \sqrt{2}\, dW_2 \\ dz &= \sqrt{2}\, dW_3 \end{aligned} \quad , W_1, W_2, W_3 \text{ indep. BM}$$

ex 3 TYPO

$$\begin{cases} \frac{\partial F}{\partial t} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0 \\ F(T, x, y, z) = x^n z e^y \quad \text{for } n = 0, 1, 2, .. \end{cases}$$

- We want to use multi-dimensional Feymann-Kač
- There is no discounting
- No function dep on x, y, z

The solution is

$$F(t, x, y, z) = \mathbb{E}_{t,x}\left[x_T^n z_T e^{y_T}\right]$$

assuming that

$$\left(\frac{\partial F}{\partial x} H_x\right) \in \mathcal{H} \quad \left(\frac{\partial F}{\partial y} H_y\right) \in \mathcal{H} \quad , \left(\frac{\partial F}{\partial z} H_z\right) \in \mathcal{H}$$

- The dynamics of x, y, z can be deduced by the PDE

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Notice that the dynamic of X_t is the one of a GBM

$dX = X\sqrt{2}\, dW \quad \to \sigma = \sqrt{2}$

$$X_T = X_t \exp\left(-(T-t) + (W_T^1 - W_t^1)\sqrt{2}\right)$$
$$Y_T = \sqrt{2}\,(W_T^2 - W_t^2) + Y_t$$
$$Z_T = Z_t + \sqrt{2}\,(W_T^3 - W_t^3)$$

$$e^{Y_T} = e^{Y_t} e^{\sqrt{2}(W_T^2 - W_t^2)} \quad \sim N(0, 2(T-t))$$

$$X_T Y_T e^{Y_T} = e^{Y_t}\Big(e^{\overbrace{\sqrt{2}(W_T^2 - W_t^2)}}\Big) \cdot X_t e^{\left(-(T-t) + \overbrace{(W_T^1 - W_t^1)\sqrt{2}}^{\substack{N(0,\, 2(T-t)) \\ Z}}\right)} \cdot \left[Z_t + \sqrt{2}\,(W_T^3 - W_t^3)\right]$$

$$E_t(X_T Y_T e^{Y_T}) \underset{\text{indip}}{=} E_t[X_T]\, E_t[Z_T]\, E_t[e^{Y_T}]$$

$$E_t[X_T] = x_t e^{-(T-t)} E_t\Big[\exp\big(\overbrace{\sqrt{2}\, W_T^1 - W_t^1}^{N(0,2(T-t))}\big)\Big]$$
$$= x_t e^{-(T-t)} e^{\frac{1}{2}\cdot 2(T-t)} = x_t$$

$$E_t[Z_T] = Z_t$$

$$E_t[Y_T] = e^{Y_t} E\left[e^{\sqrt{2}(W_T^2 - W_t^2)}\right] = e^{Y_t} e^{\frac{2}{2}(T-t)} = e^{Y_t + (T-t)}$$

$$F(t,x,y,z) = xz e^{y + (T-t)}$$

1st check) It satisfies the PDE

$$\frac{\partial F}{\partial t} = -xz e^y e^{(T-t)}$$
$$\frac{\partial F}{\partial x} = z e^y e^{(T-t)} \to \frac{\partial^2 F}{\partial x^2} = 0$$
$$\frac{\partial F}{\partial y} = xz e^y e^{(T-t)} \to \frac{\partial^2 F}{\partial y^2} = xz e^{y + (T-t)}$$
$$\frac{\partial^2 F}{\partial^2 z} = 0$$

$$\to -xz e^y e^{(T-t)} + xz e^y e^{(T-t)} = 0 \quad ✓$$
$$F(T,x,y,z) = xz e^y$$

2 check) $\frac{\partial F}{\partial x} H_x \in \mathcal{H}$, ..., ecc

$$\frac{\partial F}{\partial x} = z e^y e^{(T-t)}, \quad H_x = x\sqrt{2}$$

$$E\left[2\int_0^T X_s^2 Z_s^2 e^{2Y_s} e^{2(T-t)}\, ds\right]$$

$\underbrace{2E[X_s^2] < +\infty}_{?\text{TRUE}?}$, $E[Z_s^2] < +\infty$, $E[e^{2Y_s}] < +\infty$

$$\frac{\partial F}{\partial y} = xz e^y e^{(T-t)} \quad \text{same}$$
$$\frac{\partial F}{\partial z} = x e^y e^{(T-t)} \quad \text{same}$$

Notice that the dynamic of X_t is the one of a GBM

$dX = X\sqrt{2}\, dW \quad \to \sigma = \sqrt{2}$

$$X_T = X_t \exp\left(-(T-t) + (W_T^1 - W_t^1)\sqrt{2}\right)$$
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$$e^{Y_T} = e^{Y_t} e^{\sqrt{2}(W_T^2 - W_t^2)} \quad \sim N(0, 2(T-t))$$

$$X_T^n Y_T e^{Y_T} = e^{Y_t}\Big(e^{\overbrace{\sqrt{2}(W_T^2 - W_t^2)}}\Big) \cdot X_t^n e^{\left(-n(T-t) + \overbrace{(W_T^1 - W_t^1)\sqrt{2}\,n}^{\substack{N(0,\, n^2 2(T-t)) \\ Z}}\right)} \cdot \left[Z_t + \sqrt{2}\,(W_T^3 - W_t^3)\right]$$

$$E_t(X_T^n Y_T e^{Y_T}) \underset{\text{indip}}{=} E_t[X_T^n]\, E_t[Z_T]\, E_t[e^{Y_T}]$$

$$E_t[X_T] = x_t^n e^{-n(T-t)} E_t\Big[\exp\big(\overbrace{\sqrt{2}\, n W_T^1 - W_t^1}^{N(0,2(T-t)n^2)}\big)\Big]$$
$$= x_t e^{-n(T-t)} e^{\frac{1}{2}\cdot 2(T-t)n^2} = x_t^n e^{n(T-t)(n-1)}$$

$$E_t[Z_T] = Z_t$$

$$E_t[Y_T] = e^{Y_t} E\left[e^{\sqrt{2}(W_T^2 - W_t^2)}\right] = e^{Y_t} e^{\frac{2}{2}(T-t)} = e^{Y_t + (T-t)}$$

$$F(t,x,y,z) = x^n z e^{y + (T-t)} e^{(n^2-n)(T-t)}$$ candidate solution

1st check) It satisfies the PDE

$$\frac{\partial F}{\partial t} = -x^n z e^y e^{(T-t)(n^2-n+1)} (n^2-n+1) = F(-1+n-n^2)$$
$$\frac{\partial F}{\partial x} = n x^{n-1} z e^y e^{(T-t)} \to \frac{\partial^2 F}{\partial x^2} = n(n-1)\, x^{n-2} z e^y e^{(T-t)(n^2-n+1)}$$
$$\frac{\partial F}{\partial y} = xz e^y e^{(T-t)(n^2-n+1)} \to \frac{\partial^2 F}{\partial y^2} = xz e^{y + (T-t)(n^2-n+1)}$$
$$\frac{\partial^2 F}{\partial^2 z} = 0$$

$$\to -x^n (n^2-n+1) z e^y e^{(T-t)} + (n^2-n+1)\, x^n z e^y e^{(T-t)} = 0 \quad ✓$$
$$F(T,x,y,z) = x^n z e^y$$

2 check) $\frac{\partial F}{\partial x} H_x \in \mathcal{H}$, ..., ecc

$$\frac{\partial F}{\partial x} = n x^{n-1} z e^y e^{(T-t)(n^2-n+1)}, \quad H_x = x\sqrt{2}$$

$$E\left[2n^2\int_0^T X_s^{2n} Z_s^2 e^{2Y_s} e^{2(T-t)(n^2-n+1)}\, ds\right] \longrightarrow \int_0^T e^{t \cdot \text{const}}\, dt < +\infty$$