Stochastic Methods for Finance

Exam July 5, 2021

Exercice 1 *(for everybody)*

Consider a Black-Scholes market and a derivative contract with payoff $F(n, S_T)$ at the maturity T given by

$$
\begin{array}{rl}
0 & if\ 0 < S_T < n; \\
S_T - n & if\ n < S_T < 2n; \\
2n & if\ 2n < S_T < 3n; \\
4n - S_T & if\ 3n < S_T < 4n; \\
0 & if\ S_T > 4n.
\end{array}
$$

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, ...$ and the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

Exercice 2 *(for everybody)*

Solve the following PDE for $t \le T$:

$$
\begin{aligned}
\frac{\partial F}{\partial t} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{1}{2} \frac{\partial^2 F}{\partial y^2} + 2x &= 0 \\
F(T, x, y) &= x^2 e^y.
\end{aligned}
$$

Exercice 3 *(for 7 and 9 ETCS, not for Data Science)*

In the Black-Scholes model, find the price at time $t \le T$ for a contract where the owner receives at the maturity T the payoff

$$F(S_T) = \sum_{n=1}^{2} F(n, S_T) - (S_T - 2)^+;$$

provided that the underlying asset did reach the upper barrier L, where $F(n, S_T)$ is the function defined in Exercise 1. Find the Delta of the contract.

Exercice 4 *(for Data Science and 9 ECTS)*

Consider a (stationary) binomial model for the evolution of a risky asset S, starting from the initial price $S_0 = 100$, and increasing (resp. decreasing) factor $u = 1,2$ (resp. $d = 0,8$). The interest rate is flat at $0,5\%$ per period (1 period = 1 year).

i) Price 10 long positions in a European Call on S with maturity $T = 2$ years and strike price $K_1 = 100$;

ii) Find the price of 5 long positions in a American Put on S with maturity $T = 3$ years and strike price $K_2 = 110$;

iv) Find the position that the trader has to take in a European Call with maturity $T = 1$ year and strike price $K_3 = 95$ in order to obtain a Delta-neutral portfolio at time 0 involving the positions at points i) and ii).

v) Find the amount of Forwards with maturity $T = 3$ years the trader has to buy/sell at time $T = 2$ years in order to Delta-neutralise the global position also at time $T = 2$ years.