Stochastic Methods for Finance

Exam July 13, 2020

Exercice 1 *Consider a Black-Scholes market and a derivative contract with payoff at the maturity T given by*

$$F(n, S_T) = (2n - S_T)^+ \times 1_{n<S_T<2n},$$

where 1_A denotes the indicator function of the event A.

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, ...$ and the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Compute the Gamma of the contract $F(n, S_T)$ and provide evidence of potential issues in hedging the contract ; [HINT : prove that the Gamma may be negative]

iv) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

v) Compute the price of the portfolio F given by

$$F(S_T) = \sum_{n=1}^{3} F(n, S_T);$$

vi) Compute the amount of Call and/or Put options with strike price $K = 1$ one has to buy/sell in order to get a Delta-Vega neutral (global) portfolio.

Exercice 2 *A firm has an exposure on the US market due to some material that will be sold in dollars at a future time T. Let $X(t) = EUR/USD(t)$ be the value at time t of one euro denominated in US dollars, while r_{euro} and r_{US} denote the interest rates in the respective zones.*

Assume the dynamics of X, under the reference statistical probability, is given by

$$dX(t) = X(t)\mu dt + X(t)\sigma dW_t$$

where W is a standard Brownian motion and μ and σ are positive constants.

The firm aims at neutralising the variability of the exchange rate by using different hedging strategies, involving linear contracts and/or options.

i) Find the forward price of a FORWARD on X with maturity T. Determine (the sign of) the position on a FORWARD that neutralises the variability of the exchange rate. That is, show which is the dangerous side of the exposure for the firm.

ii) Find the price of a ATM CALL on X, that is with strike equal to the forward price. Illustrate pros and cons wrt the solution proposed at point i).

iii) Find the price of a SYNTHETIC FORWARD at any $t \leq T$, consisting in a short position in a ATM PUT and a long position in a ATM CALL, that is with strike equal to the forward price. Show pros and cons of this solution with respect to the ones proposed at points i) and ii).

iv) Find the price of a COLLAR on X, associated to a short position in a ITM PUT with strike K_1 and a long position in a OTM CALL with strike $K_2 > K_1$. Prove that the COLLAR does not provide any hedging of the exposure in the range (K_1, K_2) and show pros and cons wrt the previous solutions. [HINT : take into account the price of the structure.]

v) Compute the (market) probability of no-hedging in the solution provided by a COLLAR at point iv) and compute the limits for $K_1 \to 0$ and $K_2 \to +\infty$ first jointly, then separately.

vi) Consider a COLLAR with symmetric strikes, that is centered at ATM (forward) and width ϵ. Find ϵ corresponding to a market probability of no hedging equal to 50%.

Exercice 3 *In the Black-Scholes model, find the price at time $t \leq T$ for a contract where the owner receives at the maturity T the payoff*

$$F(S_T) = \sum_{n=1}^{2} F(n, S_T);$$

provided that the underlying asset reached the lower barrier L, where $F(n, S_T)$ is the function defined in Exercise 1. Find the Delta of the contract.

Exercice 4 *Solve the following PDE for $t \leq T$:*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + \frac{1}{2}\frac{\partial^2 F}{\partial x \partial y} + y &= 0 \\ F(T, x, y) &= e^{x+y}. \end{aligned}$$

Exercice 5 *(FOR 9 ECTS EXAM)*

A risky asset S, starting from the initial price $S_0 = 100$, has an estimated historical volatility $\sigma = 20\%$ per year. There are zero coupon bonds (with notional 100 euros) with maturities 3 months and 6 months, quoted respectively $100,2$ and $99,8$ euros.

i) Build up a binomial model with 2 periods (one period = 3 months) and find the risk neutral probability measure ;

ii) Find the price of 5 long positions in a European Call on S with maturity $T = 6$ months and strike price $K_1 = 98$;

iii) Find the price of 3 short positions in a American Put on S with maturity $T = 6$ months and strike price $K_2 = 100$;

iv) Find the position that the trader has to take in a European Call with maturity $T = 3$ months and strike price $K_3 = 95$ in order to obtain a Delta-neutral portfolio at time 0 involving the positions at points ii) and iii).