Stochastic Methods for Finance

Exam June 15, 2021

Exercice 1 *(for everybody)*

Consider a Black-Scholes market and a derivative contract with payoff $F(n, S_T)$ at the maturity T given by

$$\begin{array}{rl} n + S_T & if\ 0 < S_T < n; \\ n & if\ n < S_T < 2n; \\ S_T - n & if\ S_T > 2n. \end{array}$$

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, ...$ and the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

Exercice 2 *(for everybody)*

Solve the following PDE for $t \leq T$:

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x \partial y} &= 0 \\ F(T, x, y) &= xy^2. \end{aligned}$$

Exercice 3 *(for 7 and 9 ETCS, not for Data Science)*

In the Black-Scholes model, find the price at time $t \leq T$ for a contract where the owner receives at the maturity T the payoff

$$F(S_T) = \sum_{n=1}^{2} F(n, S_T) + (S_T - K)^+;$$

provided that the underlying asset did not reach the lower barrier L, where $F(n, S_T)$ is the function defined in Exercise 1. Find the Delta of the contract.

Exercice 4 *(for Data Science and 9 ECTS)*

A risky asset S, starting from the initial price $S_0 = 100$, has an estimated historical volatility $\sigma = 20\%$ per year. There are zero coupon bonds (with notional 100 euros) with maturities 3 months and 9 months, quoted respectively $100,2$ and $99,8$ euros.

i) Build up a binomial model with 2 periods (first period 3 months, second period 9 months) and find the risk neutral probability measure ;

ii) Find the price of 2 long positions in a European Call on S with maturity $T = 3$ months and strike price $K_1 = 99$;

iii) Find the price of 3 long positions in a American Put on S with maturity $T = 9$ months and strike price $K_2 = 100$;

iv) Find the position that the trader has to take in a European Call with maturity $T = 3$ months and strike price $K_3 = 90$ in order to obtain a Delta-neutral portfolio at time 0 involving the positions at points ii) and iii).

v) Find the number of Forward with maturity $T = 9$ months the trader has to buy/sell at time $T = 3$ in order to Delta-neutralise the global position also at time $T = 3$.