Ex class

sabato 5 giugno 2021 11:11

17 marzo 2021, FEYNMANN-KAČ

Example, VERSION 1

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} = 0 \\ F(T,x) = x^2 \end{cases}$$

$\mu = 0$, $\sigma = \sigma$, $dx_t = \sigma dW_t$

$\longrightarrow x_T = x_t + \sigma(W_T - W_t)$

If $\left(\sigma \frac{\partial F}{\partial x}\right) \in \mathcal{H} \longrightarrow F(t, x_t) = \mathbb{E}_t[x_T^2] = \mathbb{E}_t\left[x_t^2 + \sigma^2 (W_T - W_t)^2 + 2x_t\sigma(W_T - W_t)\right] =$

$= x_t^2 + \sigma^2(T-t)$

$\longrightarrow F(t,x) = x^2 + \sigma^2(T-t)$

Check 1: $\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} = 0 \\ F(T,x) = x^2 \end{cases}$, $\begin{cases} -\sigma^2 + \frac{2}{2}\sigma^2 = 0 \\ x^2 + \sigma^2(T-T) = x^2 \end{cases}$

Check 2: $\frac{\partial F}{\partial x} = 2x$, $\mathbb{E}\int_0^T \sigma^2 \cdot 4 x_s^2 ds = \int_0^T \sigma^2 4\, \mathbb{E}[x_s^2] ds =$

$x_s^2 = x_0^2 + \sigma^2 W_s^2 + 2x_0\sigma W_s$

$= \sigma^2 4 \int_0^T x_0^2 + \sigma^2 s\, ds < +\infty$

21 marzo, EXAMPLE, Version 3

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}x^2 \frac{\partial^2 F}{\partial x^2} + x = 0 \\ F(T,x) = \ln x^2 \end{cases}$$

GBM $X_s = X_t \exp\left(-\frac{1}{2}(s-t) + (W_s - W_t)\right)$

$K = 0$, $H = X \longrightarrow dX = X dW \longrightarrow X_T = X_t \exp\left(-\frac{1}{2}(T-t) + (W_T - W_t)\right)$

$F(t,x) = \mathbb{E}_{t,x}[\ln x_T^2] + \int_t^T \mathbb{E}_{t,x}[X_s] ds =$

$= \mathbb{E}_{t,x}\left[\ln x_t^2 - (T-t) + 2(W_T - W_t)\right] + \int_t^T x_t \exp\left(-\frac{1}{2}(s-t)\right)\exp\left(\frac{1}{2}(s-t)\right) ds =$

$= \ln x_t^2 - (T-t) + x_t(T-t)$

$F(t,x) = 2\ln x - (T-t) + x(T-t) = 2\ln x + (T-t)(x-1)$

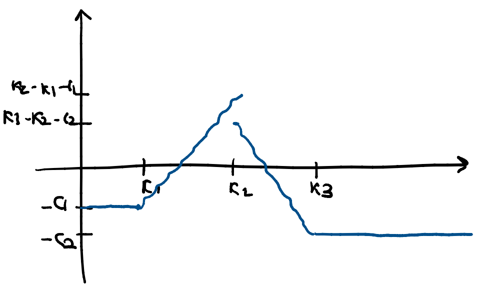
Check 1: $\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}x^2 \frac{\partial^2 F}{\partial x^2} + x = 0 \\ F(T,x) = \ln x^2 \end{cases}$ $\begin{cases} -(x-1) - \frac{\cancel{2}}{\cancel{x^2}} \cdot \frac{1}{\cancel{2}} x^2 + x = 0 \\ F(T,x) = \ln x^2 \end{cases}$ ✓

Check 2: $\frac{\partial F}{\partial x} = \frac{2}{x} + (T-t)$ $x\frac{\partial F}{\partial x} = 2 + x(T-t)$

$\int_0^T \mathbb{E}\left(4 + X_s^2 s^2 + 4X_s s\right) ds < +\infty$ ✓

26 marzo 2021

$$\text{PAYOFF}_T = \begin{cases} -c_1 & S_T \in (0, K_1) \\ S_T - K_1 - c_1 & (K_1, K_2) \\ K_3 - S_T - c_2 & (K_2, K_3) \\ -c_2 & (K_3, +\infty) \end{cases}$$



$\text{PAYOFF}_T =$

$= -c_1 + (S_T - K_1)^+ - 2(S_T - K_2)^+ + (K_3 + K_1 - c_2 + c_1 - 2K_2)\mathbb{1}_{S_T > K_2} + (S_T - K_3)^+$

$\text{price}_t(\text{PAYOFF}_T) = -c_1 e^{-r(T-t)} + S_t\Phi(d_1^{K_1}) - e^{-r(T-t)} K_1 \Phi(d_2^{K_1}) +$
$-2\Phi(d_1^{K_2}) + 2K_2 e^{-r(T-t)}\Phi(d_2^{K_2}) - \dots$

Delta$^{\text{CONTRACT}}$ = ..

5 maggio 2021

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) + \dfrac{\partial^2 F}{\partial x^2}(t,x) + F(t,x) + e^x = 0 \\ F(T,x) = x \end{cases}$$

• Mixed case

$$d(F(t,x)e^t) = F(t,x)e^t dt + e^t\left(\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}K dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}H^2 dt + \frac{\partial F}{\partial x}H dW\right) =$$

$K = 0$, $H = \sqrt{2}$, $dX_t = \sqrt{2}\, dW_t$

$$= F(t,x)e^t dt + e^t\left(\frac{\partial F}{\partial t}dt + \frac{\partial^2 F}{\partial x^2}dt + \frac{\partial F}{\partial x}\sqrt{2}\, dW\right) = -e^x e^t dt + e^t \frac{\partial F}{\partial x}\sqrt{2}\, dW$$

assume $\left(\frac{\partial F}{\partial x} e^t \sqrt{2}\right) \in \mathcal{H}$

$$\to F(t,x_t) = e^{(T-t)} F(t,X_T) - e^{-t}\int_t^T e^{(x+s)} ds + e^{-t}\int_t^T e^s \frac{\partial F}{\partial x}\sqrt{2}\, dW$$

$$F(t,x_t) = e^{(T-t)}\mathbb{E}_t[X_T] - e^{-t}\int_t^T \mathbb{E}_t[e^{X_s+s}]ds$$

$$\mathbb{E}_t[X_T] = \mathbb{E}_t[X_t + \sqrt{2}(W_T - W_t)] = X_t$$

$$e^s\mathbb{E}_t[e^{X_t+\sqrt{2}(W_s-W_t)}] = e^s e^{X_t+(s-t)} = e^{X_t+2s-t}$$

$$e^{-t}\int_t^T e^{X_t+2s-t} ds = e^{-2t+X_t}\frac{e^{2s}}{2}\Big|_t^T = \frac{e^{X_t}e^{-2t}}{2}\left[e^{2T}-e^{2t}\right] = \frac{e^{X_t}}{2}\left[e^{2(T-t)}-1\right]$$

$$\to F(t,x) = e^{(T-t)}x + \frac{e^x}{2}\left[e^{2(T-t)}-1\right]$$

Check

$$\frac{\partial F}{\partial t} = -e^{(T-t)}x - e^x e^{2(T-t)}$$

$$\frac{\partial F}{\partial x} = e^{(T-t)} + \frac{e^x}{2}\left[e^{2(T-t)}-1\right]$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{e^x}{2}\left[e^{2(T-t)}-1\right]$$

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) + \dfrac{\partial^2 F}{\partial x^2}(t,x) + F(t,x) + e^x = 0 \\ F(T,x) = x \end{cases}$$

$$\begin{cases} -\cancel{e^{(T-t)}}x - e^x e^{2(T-t)} + \dfrac{e^x}{2}\left[e^{2(T-t)}-1\right] + \cancel{e^{(T-t)}}x + \dfrac{e^x}{2}\left[e^{2(T-t)}-1\right] + e^x = 0 \\ e^{(T-T)}x + \dfrac{e^x}{2}\left[e^{2(T-T)}-1\right] = x \end{cases}$$

Check 2: $\frac{\partial F}{\partial x}(s, X_s) \in \mathcal{H}$

$$\mathbb{E}\left[e^{(T-s)} + \frac{e^{x_0+\sqrt{2}W_s}}{2}\left[e^{2(T-s)}-1\right]\right] = e^{(T-s)} + \frac{e^{x_0+s}}{2}\left[e^{2(T-s)}-1\right]$$

The integral is finite ✓

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Example

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 F}{\partial x^2} + \dfrac{1}{2}\gamma^2\dfrac{\partial^2 F}{\partial y^2} = 0 \\ F(T,x,y) = xy^2 \end{cases}$$

$$\begin{aligned} dX_t &= \sigma dW_t^1 \\ dY_t &= \gamma dW_t^2 \end{aligned} \quad \to \quad \begin{aligned} X_T &= X_t + \sigma(W_T^1 - W_t^1) \\ Y_T &= Y_t + \gamma(W_T^2 - W_t^2) \end{aligned}$$

$$F(t,x) = \mathbb{E}_{t,x}[X_T Y_T^2]$$

$\mathbb{E}_t[x_t] = x_t$, $\mathbb{E}_t[y_T^2] = y_t^2 + \gamma^2(T-t)$

$F(t,x) = x(y^2 + \gamma^2(T-t))$

Check 1 $\begin{cases} \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\gamma^2 \frac{\partial^2 F}{\partial y^2} = 0 \\ F(T,x,y) = xy^2 \end{cases}$ $\begin{cases} -x\gamma^2 + \gamma^2 x = 0 \\ F(T,x,y) = xy^2 \end{cases}$

Check 2 $\cdot$ $\frac{\partial F}{\partial x}(s, X_s, Y_s) = Y_s + \gamma^2(T-s)$ $\longrightarrow$ CONTI...

$\frac{\partial F}{\partial y}(s, X_s, Y_s) = 2X_s Y_s$

EX (GIRSANOV)

$X_t = e^{B_t - t/2}$, $dX_t = X_t dB_t$

1) X is a $\mathbb{P}$-mg.
$X_t = e^{\alpha B_t - \frac{\alpha^2}{2}t}$, $\alpha = 1$, so it is a simple exponential martingale

2) Show X^3 is not a mg

$X_t^3 = e^{3B_t - \frac{3}{2}t}$, not a mg since it does not respect $e^{\alpha B_t - \frac{\alpha^2}{2}t}$

3) Find $\tilde{\mathbb{P}}$ under which X^3 is a $\tilde{\mathbb{P}}$-mg

$f(x) = x^3$, $dX_t^3 = 3X_t^2 dX_t + 3X_t \cdot X_t^2 dt = 3X_t^3 dB_t^{\mathbb{P}} + 3X_t^3 dt =$
$= 3X_t^3 (\underbrace{dB_t^{\mathbb{P}} + dt}_{dB_t^{\tilde{\mathbb{P}}}})$

$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_t = e^{-\int_0^t dB^{\mathbb{P}} - \frac{1}{2}\int_0^t ds} = e^{-B_t^{\mathbb{P}} - \frac{t}{2}}$, ok, it is a mg

4) Compute $\mathbb{E}^{\tilde{\mathbb{P}}}(X_t)$
$\mathbb{E}^{\tilde{\mathbb{P}}}[X] = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{B - \frac{1}{2}t}] = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{B^{\tilde{\mathbb{P}}} - t - \frac{t}{2}}] = e^{-t}\,\mathbb{E}^{\tilde{\mathbb{P}}}[e^{B^{\tilde{\mathbb{P}}} - \frac{t}{2}}] = e^{-t}$

5) Compute $\mathbb{E}^{\tilde{\mathbb{P}}}(X_t^3) = 1$

6) Compute $\mathbb{E}^{\mathbb{P}}(X_t) = 1$

7) Compute $\mathbb{E}^{\mathbb{P}}(X_t^3)$
$\mathbb{E}^{\mathbb{P}}(X_t^3) = \mathbb{E}^{\mathbb{P}}[e^{3B^{\mathbb{P}} - \frac{3}{2}t}] = \mathbb{E}^{\mathbb{P}}[e^{3B^{\mathbb{P}} - \frac{9}{2}t + 3t}] = e^{3t}$

7 Maggio 2021

i) Find $\alpha \in \mathbb{R}$ s.t. $(1 + \alpha B_t^2)^2$ is a mg wrt the natural filtration

$X_t = (1 + \alpha B_t^2)^2 = f(B_t)$

$dX_t = \underbrace{2(1+\alpha B_t^2) \cdot 2\alpha B_t}_{2\alpha B_t + 2\alpha^2 B_t^3} dB_t + \left[2\alpha + 6\alpha^2 B_t^2\right] dt =$

$= (\quad) dB + 2\alpha \underbrace{(1 + 3\alpha B_t^2)}_{=0} dt$

if $\alpha = 0$ or $\alpha = -\frac{1}{3B_t^2}$

ii) Expected value at time $t=3$ of the process satisfying $dX_t = X_t dt - 2dB_t$, $X_0 = 0$

1) $\tilde{X}_t = X_t e^{-t}$
$d\tilde{X}_t = dX_t e^{-t} - X_t e^{-t} dt =$
$= X_t e^{-t} dt - 2e^{-t} dB_t - X_t e^{-t} dt$

$d\tilde{X}_t = -2e^{-t} dB_t$ $\longrightarrow$ $\tilde{X}_t - \tilde{X}_0 = -2\int_0^t e^{-s} dB_s$

$X_3 e^{-3} - 0 = -2\int_0^3 e^{-s} dB_s$, $e^{-s} \in \mathcal{H}$, $\mathbb{E}\int_0^T e^{-2s} ds < +\infty$

$\longrightarrow \mathbb{E}[X_3] = 0$

2) $dX_t = X dt - 2dB$

$\rightarrow E[X_3]=0$

2) $dX_t = X\,dt - 2\,dB$
$d\mathbb{E}[X_t] = \mathbb{E}[X_t]\,dt - 0 \quad \rightarrow E[X_t]=0 \ \forall\ t$
$X_0 = 0$

iii) covariance of X_3

$$X_t = -2e^t \int_0^t e^{-s}\,dB_s$$

$$E[X_t^2] = 4e^{2t}\,\mathbb{E}\left[\left(\int_0^t e^{-s} dB_s\right)^2\right] =$$

$$= 4e^{2t}\,\mathbb{E}\left[\int_0^t e^{-2s}\,ds\right] =$$

$$= 4e^{2t}\left[-\frac{1}{2}\left[e^{-2t}-1\right]\right]$$

For $t=3$

iv) Find $\alpha \in \mathbb{R}$ $(\exp(\alpha t + B_t))$ is a mg with the natural filtration

$$e^{\sigma B - \frac{1}{2}\sigma^2 t} = mg \ \forall\ \sigma \rightarrow \alpha t = -\frac{1}{2}t \rightarrow \alpha = -\frac{1}{2}$$

v) Quadratic covariation of the process X and B^2

$$\langle X, B^2\rangle_t = \int_0^t d\langle X, B^2\rangle_s\, ds = \int_0^t -4B_s\,ds$$

$$dX\,dB^2 = (X_t dt - 2dB)(2B\,dB + dt) = -4B\,ds$$

vi) Show $Y = 2B^6 + B^2$ is a submartingale

$$dY = 12B^5\,dB + 30B^4 dt + 2B\,dB + \frac{1}{2}dt =$$

$$= \underbrace{(12B^5 + 2B)}_{mg}\,dB + \underbrace{\left(30B^4 + \frac{1}{2}\right)}_{\geq 0} dt$$

vii) expected value and variance of the r.v. $B_{-1} + B_1 + B_2$

$$\mathbb{E}[B_{-1} + B_1 + B_2] = B_{-1}$$

$$Var[B_{-1} + B_1 + B_2] = E[(B_{-1} + B_1 + B_2 - B_{-1})^2] = E[B_1^2 + B_2^2 + 2B_1B_2] = 1+2+2\cdot 1 = 5$$

EX compute the process $\exp(B_t)$, B_t B.m and it is a B.m. under $\mathbb{P}$

1) Show that $\exp(B_t)$ is not a mg under $\mathbb{P}$

$$d\exp(B_t) = \exp(B_t)\,dB_t + \frac{\exp(B_t)}{2}dt$$

2) Find $\mathbb{P}'$ under which the process is a $\mathbb{P}^1$-mg

$$dX_t = X_t\left(dB^{\mathbb{P}} + \frac{1}{2}dt\right)$$

$$\frac{d\mathbb{P}'}{d\mathbb{P}} = \exp\int_0^t -\frac{1}{2}dB^{\mathbb{P}} - \frac{1}{2}\int_0^t \frac{1}{4}\,ds = \exp\left(-\frac{1}{2}B^{\mathbb{P}}_t - \frac{1}{8}t\right)$$, true mg

3) $\mathbb{E}^{P'}[\exp(B_t)] = 1$

4) $\exp\left(\int_0^t B_s\,ds\right) = Y_t$, show it is not a mg

$Y_t = \exp(Z_t) \qquad dZ_t = B_t\,dt$
$dY_t = Y_t\,dZ_t + \frac{1}{2}Y_t\,d\langle Z_t\rangle = Y_t B_t\,dt$

5) Find $\mathbb{P}^2$ s.t. Y_t is a $\mathbb{P}^2$-mg
It does not exists