Stochastic Methods for Finance

Exam August 26, 2019

Exercice 1 *(~~10~~ points) Consider a Black-Scholes market and a derivative contract with payoff at the maturity T given by*

$$F(n, S_T) = (K - S_T)^+ * 1_{n<S_T<n+1},$$

where $K > 0$ and 1_A denotes the indicator function of the event A.

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 0, 1, 2, ...$ Compute the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for an upward shift of the volatility ;

iv) Compute the price of the portfolio F given by

$$F(S_T) = \sum_{n=0}^{\infty} F(n, S_T);$$

v) Compute the amount of Call options with strike price $K = 1$ one has to buy/sell in order to get a Delta neutral (global) portfolio.

Exercice 2 *(6 points)*

In the Black-Scholes model, find the price at time $t \leq T$ of an UP-AND-OUT contract where the owner receives the payoff

$$F(n, S_T) = (K - S_T)^+ * 1_{n<S_T<n+1},$$

at the maturity T only if the asset has never reached the upper barrier $L > 0$. Provide the price of the contract when $n \to +\infty$. Finally, find the Delta of the contract.

Exercice 3 *(8 points)*

Solve for any $n = 1, 2, ...$ the following PDE for $t \leq T$:

$$\begin{aligned} \frac{\partial F}{\partial t} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} &= 0 \\ F(T, x, y, z) &= xze^y. \end{aligned}$$

Exercice 4 *(6 points)*

Questions on the theory.

i) Show the backward equation satisfied by the optimal value of an American option in a discrete time model and prove it.

ii) State and prove the PDE satisfied by the price of a European derivative in a Black-Scholes model.

iii) Find the moment generating function of the first hitting time for the Brownian motion