Exercises

venerdì 28 maggio 2021 18:04

Ex 1

Exercice 1 *Consider a standard Brownian motion B_t.*

i) Find the real parameter α such that the process $\ln(1+\alpha B_t^2)$ is a martingale with respect to the natural filtration;

ii) Find the real parameter α such that the process $\exp(1+\alpha B_t^2)$ is a martingale with respect to the natural filtration;

iii) Compute the expected value at time $t=2$ of the process X satisfying $dX_t = -X_t dt + 4dB_t$, $X_0 = 1$;

iv) Compute the variance of the random variable X_2;

v) Compute the quadratic covariation between the processes X and B;

vi) Show that the process $Y = B^4 - B^2$ is a sub-martingale;

vii) Find the Doob-Meyer decomposition of the process Y.

• B_t standard Brownian motion

i) Find α st $\ln(1+\alpha B_t^2)$ mg wrt natural filtration

ITO'S FORMULA

$$dX = a(x,t)\,dt + b(x,t)\,dW_t$$

$$df(x(t),t) = \left(a(x,t)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}(b(x,t))^2\frac{\partial^2 f}{\partial x^2}\right)dt + b(x,t)\frac{\partial f}{\partial x}dW_t$$

$B_t \to dB_t$, $a=0$, $b=1$

$f(B_t) = \ln(1+\alpha B_t^2)$

$$f(x) = \ln(1+\alpha x^2), \quad \frac{\partial f}{\partial x} = \frac{2x\alpha}{1+\alpha x^2}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2\alpha(1+\alpha x^2) - 2x\cdot 2x\alpha}{(1+\alpha x^2)^2} = \frac{\alpha^2 2x^2 + 2\alpha - 4x^2\alpha^2}{(1+\alpha x^2)^2} = \frac{-2x^2\alpha^2 + 2\alpha}{(1+x^2)^2}$$

$$df(x) = \frac{2B_t\alpha}{1+\alpha B_t^2}dB_t + \frac{1}{2}\left(\frac{-2\alpha^2 B_t^2 + 2\alpha}{(1+\alpha x^2)^2}\right)dt$$

$\longrightarrow \alpha(-\alpha B_t^2 + 1) = 0 \longrightarrow \alpha = 0,$ $\alpha = 1/B_t^2$ can't depend on B_t

$\longrightarrow \alpha = 0$

ii) $\exp(1+\alpha B_t^2)$

$f(x) = \exp(1+\alpha x^2)$

$$\frac{\partial f}{\partial x} = 2\alpha x\exp(1+\alpha x^2), \quad \frac{\partial^2 f}{\partial x^2} = 2\alpha\exp(1+\alpha x^2) + (2x\alpha)^2\exp(1+\alpha x^2)$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \alpha\exp[1+\alpha B_t^2] + 2B_t^2\alpha^2\exp(1+\alpha B_t^2) = 0$$

$$\alpha\exp[1+\alpha B_t^2](1+2B_t^2\alpha) = 0 \longrightarrow \alpha = 0, \; \alpha = -1/2B_t^2 \quad \longrightarrow \alpha = 0$$

iii) $t=2$, $dX_t = -X_t\,dt + 4\,dB_t$

$X_0 = 1$

1) $E[dX_t] = d\,E[X_t] = -E[X_t]\,dt + 0$

$dx = -x\,dt$, $x(t) = e^{-t} + c$

$E[X_t] = e^{-t} + c$

$1 = E[X_0] = E[1] = 1 + c \longrightarrow c = 0$

$E[X_t] = e^{-t}$, $E[X_2] = e^{-2}$

$1 = E[X_0] = E[1] = 1 + C \longrightarrow C = 0$

$E[X_t] = e^{-t}$, $E[X_2] = e^{-2}$

2) $\tilde{X}_t = X_t e^{+t}$

$d\tilde{X}_t = dX_t e^{+t} + X_t e^{+t} dt + 0$

$\overset{!}{=} -X_t e^{+t} dt + 4 dB_t e^t + X_t e^{+t} dt =$

$= 4 dB_t e^t$

$\tilde{X}_t - 1 = 4 \int_0^t e^s dB_s$

$X_2 e^2 - 1 = 4 \int_0^2 e^s dB_s$, $E[X_2] e^2 = 1$, $E[X_2] = e^{-2}$ ✓

iv)

1) NO

$f(x) = x^2$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial^2 f}{\partial x^2} = 2$

$dX_t^2 = 2X_t dX_t + \frac{1}{2} \cdot 2 \cdot 16 dt$

$= -2X_t^2 dt + 8 dB_t + 16 dt$

$d E[X_t^2] = -2 E[X_t^2] dt + 16 dt$

$dy = -2y dt + 16 dt$

$\overset{!}{=} -2 (y - 8) dt$

$d(y-8) = -2(y-8) dt$

$y - 8 = e^{-2t} + C$

$y = e^{-2t} + 8 + C$

$y(0) = 1 \longrightarrow 1 + 8 + C = 1 \longrightarrow C - 8$

$E[X_t^2] = e^{-2t}$

$dy = -2y dt + 16$

$y(0) = 1$

$e^{-2t} + 16t$

$de^{-2t} = -2e^{-2t} dt$

2) $X_t = 1 + 4 \int_0^t e^s dB_s$

$X_t^2 = 1 + 16 \left(\int_0^t e^s dB_s \right)^2 + 8 \int_0^t e^s dB_s$

$E[X_t^2] = 1 + \underbrace{16 \left[\int_0^t e^{2s} ds \right]}_{8(e^{2t} - 1)} + 0$

$var(X_2) = E[X_2^2] - E[X_2]^2 =$?

$= 1 + 8e^4 - 8 - e^{-4} =$

$= -7 + 8e^4 - e^{-4}$

v) $\langle X, B \rangle_t = \int_0^t d\langle X, B \rangle_t = \int_0^t +4 ds = 4t$

1) $dX dB = (-X dt + 4 dB)(dB) = +4 ds$

2) $\langle X, B \rangle_t = \frac{1}{2} \left[\langle X + B \rangle_t - \langle X \rangle_t - \langle B \rangle_t \right] = \frac{1}{2} 8t = 4t$

$\langle X + B \rangle_t = 25t$

$\langle X \rangle_t = 16t$

$\langle B \rangle_t = t$

vi) $Y = B^4 - B^2$

$f(x) = x^n \quad , \quad \frac{\partial f}{\partial x} = n x^{n-1} \quad , \quad \frac{\partial^2 f}{\partial x^2} = n(n-1)x^{n-2}$

$dB_t^4 = 4B_t^3 dB_t + 6B_t^2 dt$

$dB_t^2 = 2B_t dB_t + dt$

$dY = (4B_t^3 + 2B_t)dB_t + \underbrace{(6B_t^2+1)dt}_{\text{increasing}}$

mg + increasing, SUB MG, Doob decomposition

EX 2

Exercice 2 *Consider a Black&Scholes market and a derivative contract with payoff at the maturity T given by*

0	*if* $S_T < K_1$;
$K_2 - S_T$	*if* $K_1 < S_T < K_2$;
$K_2 - K_1$	*if* $S_T > K_2$,

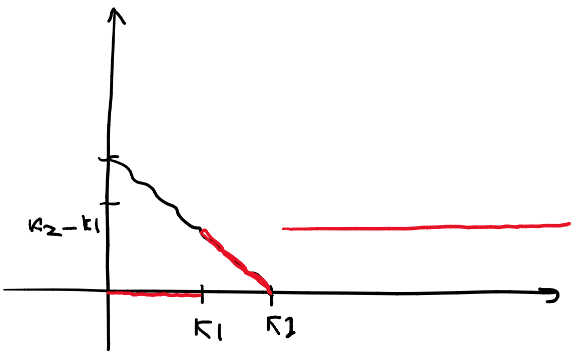
where $0 < K_1 < K_2$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Draw



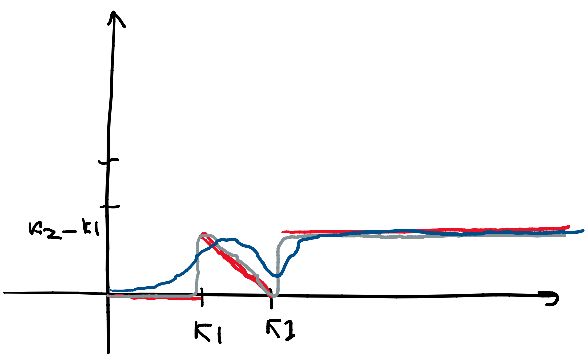
$K_2 - S_T$

i) $\text{PAYOFF}_T = (K_2 - K_1)\mathbb{1}_{S_T > K_1} - (S_T - K_1)^+ + (S_T - K_2)^+ + (K_2 - K_1)\mathbb{1}_{S_T > K_2}$

$\text{price}_t = (K_2 - K_1)e^{-r(T-t)}\Phi(d_2^{K_1}) - S_t\Phi(d_1^{K_1}) + e^{-r(T-t)}K_1\Phi(d_2^{K_1}) + S_t\Phi(d_1^{K_2}) - K_2 e^{-r(T-t)}\Phi(d_2^{K_2}) + (K_2 - K_1)e^{-r(T-t)}\Phi(d_2^{K_2})$

ii) $\text{Delta}_t = (K_2 - K_1)\Delta^{dig} - \Phi(d_1^{K_1}) + \Phi(d_1^{K_2}) + (K_2 - K_1)\Delta^{dig}$

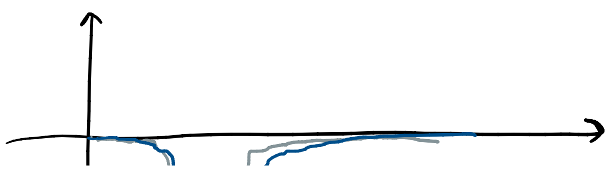
iii)

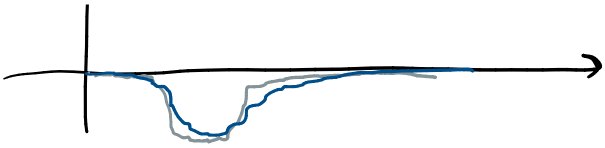


price

$\sigma_1 << \sigma_2$

Δ





EX 3)

Exercice 3 *Dans le modéle de Black-Scholes, on considère un actif risqué S:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Soit t une date fixé, 0≤ t ≤ T; une option Forward-Starting donne à la maturité T le payoff

$$Payoff_T^{FwCall} = \left(\frac{S_T}{S_t} - K\right)^+$$

i) Déterminez à la date 0 le prix de cette option dans le cas où le taux d'intéret est constant, puis déterminer la stratégie de couverture.

ii) Même question dans le cas où le taux d'intéret est stochastique. Expliciter le prix si le taux est du type Vasicek.

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i)

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t^{P} \\ dS_t = r S_t dt + \sigma S_t dW_t^{Q} \end{cases}$$

$$e^{-r(T-t)} \mathbb{E}_t^Q\left[\left(\frac{S_T}{S_t} - K\right)^+\right] = \cancel{\frac{S_t}{S_t}} 1\, \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) = price_t \longrightarrow \text{Sol. } t=0 \text{ BS}$$

$$d_{1,2} = \frac{\ln\left(\frac{1}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$price_0 = e^{-Tr} \mathbb{E}_0^Q\left[\left(\frac{S_T}{S_t} - K\right)^+\right] =$$

$$= e^{-Tr} \mathbb{E}_0^Q\left[\mathbb{E}_t^Q\left(\frac{S_T}{S_t} - K\right)^+\right] =$$

$$= e^{-Tr} \mathbb{E}_0^Q\left[e^{+r(T-t)}\Phi(d_1) - K\Phi(d_2)\right] =$$

$$= e^{-rt}\Phi(d_1) - K e^{-Tr}\Phi(d_2)$$

ii) Vasiček model

$$\underset{\sigma_Z}{\underset{\|}{Vol(Z)}} = Vol\left(\frac{S_t}{B(t,T)}\right) = \sigma - \frac{e^{-b(T-t)} - 1}{b}\sigma_r \quad \text{deterministic}$$

$$price_t(call) = price_t\left(\left(\frac{S_T}{S_t} - K\right)^+\right) = \Phi(d_1) - K B(t,T)\Phi(d_2)$$

$$d_2 = \frac{\ln\left(\frac{1}{K B(t,T)}\right) - \frac{1}{2}\int_t^T \sigma_Z^2 ds}{\sqrt{\int \sigma_Z^2 ds}} \qquad d_1 = d_2 + \sqrt{\int_t^T \sigma_Z ds}$$

$$price_t = \mathbb{E}_t^Q\left[e^{-\int_t^T r_s ds}\left(\frac{S_T}{S_t} - K\right)^+\right] =$$

$$= \mathbb{E}_t^Q\left[e^{-\int_t^T r_s ds}\left(\frac{S_T}{S_t} - K\right)\mathbb{1}_{\frac{S_T}{S_t} \geq K}\right] =$$

$$= \mathbb{E}_t^{Q}\left[e^{-\int_t^1 r_s ds} \left(\frac{S_T}{S_t} - K \right) \mathbb{1}_{\frac{S_T}{S_t} \geq K} \right] =$$

$$= \frac{\cancel{S_t}}{\cancel{S_t}} \mathbb{E}_t^{Q}\left[\frac{S_T}{S_t} \geq K \right] - K B(t,T)\, Q_t^T\left(\frac{S_T}{S_t} \geq K \right)$$

$$Q_t^T\left(\frac{S_T}{S_t} \geq K \right) = Q_t^T\left(\frac{S_T}{B(T,T)} \geq K S_t \right) = Q_t^T(Z_T \geq K) = \Phi(d_2)$$

$$d_2 = \frac{\ln \frac{S_t}{\cancel{S_t} K B(t,T)} - \frac{1}{2}\int_t^T \|\sigma_z\|^2 dt}{\sqrt{\int_t^T \|\sigma_z\|^2 dt}} \quad \ldots \quad \text{analogous}$$

$$\text{price}_0 = E_0^{Q}\left[e^{-\int_0^T} \left(\frac{S_T}{S_t} - K \right)^+ \right] =$$

$$= E_0^{Q}\left[E_t^{Q}\left[e^{-\int_0^T} (\quad)^+ \right] \right] =$$

$$= E_0^{Q}\left[e^{-\int_0^t} E_t^{Q}\left[e^{-\int_t^T} (\quad)^+ \right] \right] =$$

EX 4

Exercice 4 *Consider the process $X_t = \int_0^t e^{\pi} dB_s$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*

i) Find the probability measure $\mathbb{P}^1$ under which the process X becomes a martingale;

ii) Consider now the process $Y_t = \exp(X_t)$: show that it is not a martingale under the probability measure $\mathbb{P}$;

iii) Find the probability measure $\mathbb{P}^2$ under which Y becomes a $\mathbb{P}^2$-martingale;

iv) Compute $\mathbb{E}^{\mathbb{P}^1}[X_2]$;

v) Compute $\mathbb{E}^{\mathbb{P}^2}[X_2]$.

i) $d\left[\int_0^t e^{\pi} dB_s \right] = e^{\pi} dB_t$, already a mg under $\mathbb{P}$

$$e^{\pi} \int_0^t dB_s = e^{\pi}(B_t - B_0) = e^{\pi} B_t$$

ii) $Y_t = \exp(X_t)$

$$dY_t = \exp(X_t)\, dX_t + \frac{\exp(X_t)}{2} e^{2\pi} dt =$$

$$= \exp(X_t) e^{\pi} \underbrace{\left[dB_t + \frac{e^{\pi}}{2} dt \right]}_{d\tilde{B}} \qquad k_s = -\frac{e^{\pi}}{2}$$

$$\left.\frac{d\mathbb{P}^2}{d\mathbb{P}}\right|_t = e^{-\int_0^t \frac{e^{\pi}}{2} dB^{\mathbb{P}} - \frac{1}{2}\int_0^t \frac{e^{2\pi}}{4} dt} = e^{-\frac{e^{\pi}}{2} B_t - \frac{1}{8} e^{2\pi} t} \quad \checkmark \quad \text{TRUE mg} \quad k = -\frac{e^{\pi}}{2}$$

iii) ↑

iv) $E^{P_1}[X_2] \overset{mg}{=} X_0 = 0$

v) $$E^{P_2}[X_2] = E^{\mathbb{P}^2}\left[\int_0^2 e^{\pi} dB_s \right] =$$

$$= \mathbb{E}^{\mathbb{P}^2}\left[e^{\pi} B_2 \right] = e^{\pi} E^{P_2}\left[\tilde{B}_2 - e^{\pi} \right] = -e^{2\pi}$$

EX 5

Exercice 5 *Consider a Black&Scholes market and a derivative contract with payoff at the maturity T given by*

Exercice 5 *Consider a Black&Scholes market and a derivative contract with payoff at the maturity T given by*

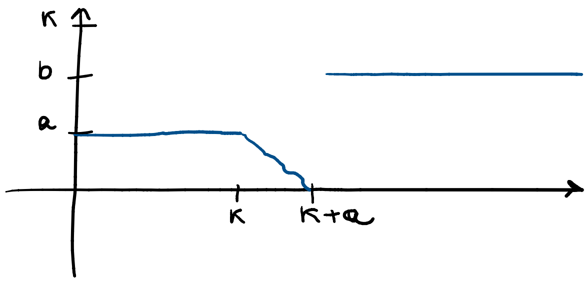
$$\begin{array}{rl} a & if\ S_T < K; \\ K + a - S_T & if\ K < S_T < K + a; \\ b & if\ S_T > K + a, \end{array}$$

where $0 < a < b < K$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.



$0 < a < b < K$

$$\text{PAYOFF} = a - (S_T - K)^+ + b\,\mathbb{1}_{S_T > K+a} + (S_T - K - a)^+$$

$$a - S_T + K + y = b$$

$$y = S_T - K - a + b$$

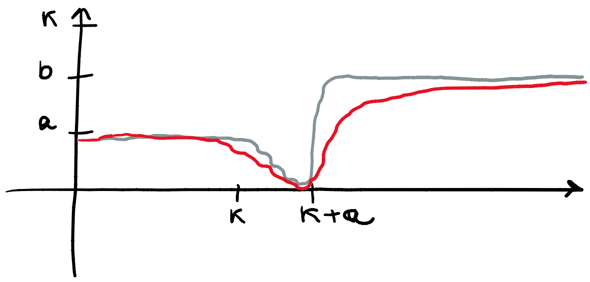
$$y = (S_T - K - a)^+ + b\,\mathbb{1}_{S_T > K+a}$$

$$\text{price}_t = a e^{-r(T-t)} - \text{Call}_t^K + b\ \text{Dig}^{K+a} + \text{Call}^{K+a}$$

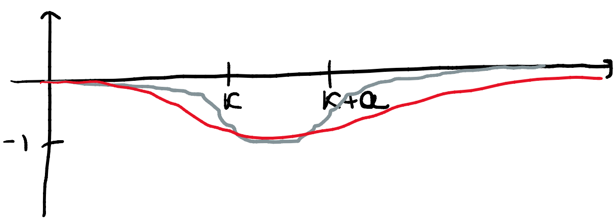
$$\Delta_t = 0 - \Delta\text{call}^K + b\,\Delta\text{Dig}^{K+a} - \Delta\text{call}^{K+a}$$

iii) price_t

$\sigma_1 << \sigma_2$



Delta



EX 10

Exercice 10 *Résoudre l'EDP suivante*

$$\begin{aligned}\frac{\partial F}{\partial t}+\frac{1}{2}x^2\alpha^2\frac{\partial^2 F}{\partial x^2}+\beta F+\gamma x^\delta &= 0\\ F(T,x) &= 2x^3,\end{aligned}$$

et vérifier que la solution satisfait l'EDP dans le cas $\delta = 1$.

$$d(Fe^{\beta t}) = dF e^{\beta t} + \beta F e^{\beta t} dt \qquad dX_t = X\alpha dW_t$$

$$= \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} X\alpha dW_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} X^2\alpha^2 dt\right)e^{\beta t} + \beta F e^{\beta t} dt =$$

$$= \underbrace{\frac{\partial F}{\partial x} X\alpha dW_t e^{\beta t}}_{\text{assume mg}} \quad \cancel{-\beta F e^{\beta t}} - \gamma x^\delta dt + \cancel{\beta F e^{\beta t}}$$

$$F(T,X_T)e^{\beta T} - F(t,X_t)e^{\beta t} = \int_t^T \frac{\partial F}{\partial x} X\alpha dW_t e^{\beta t} - \int_t^T \gamma X_s^\delta ds$$

If $\frac{\partial F}{\partial x} X\alpha e^{\beta t} \in \mathcal{M}$

$$F(t,X_t) = e^{-\beta t}\left(\mathbb{E}_t^{Q}\left[F(T,X_T)e^{\beta T}\right] - \int_t^T \gamma\, E_t^{Q}\left[X_s^\delta\right] ds\right)$$

$$X_T = X_t e^{-\frac{1}{2}\alpha^2(T-t) + \underbrace{\alpha(W_T - W_t)}_{\alpha\sqrt{T-t}}}$$

$$2\cdot X_t^3 e^{-\frac{3}{2}\alpha^2(T-t)} e^{\frac{9}{2}\alpha^2(T-t)} = 2\cdot X_t^3 e^{3\alpha^2(T-t)} +$$

$$\gamma\int_t^T X_t^\delta e^{-\frac{\delta}{2}\alpha^2(s-t)} e^{\frac{\delta^2}{2}\alpha^2(s-t)} ds =$$

$$+ \quad = \frac{\gamma X^\delta}{\alpha^2}\frac{1}{\frac{\delta}{2}(-1+\delta)}\left[e^{\left(-\frac{\delta}{2}+\frac{\delta^2}{2}\right)\alpha^2(T-t)} - 1\right]$$

Per $\delta = 1$ $\quad \gamma X_t$

$$e^{(\beta+3\alpha^2)(T-t)}\, 2\cdot X_t^3 \qquad + e^{\beta t}\gamma X_t(T-t) = F(t,T)$$

$$\frac{\partial F}{\partial t} = -(-\beta+3\alpha^2)e^{(-\beta+3\alpha^2)(T-t)}2x^3 \underbrace{- e^{\beta t}\gamma x}_{?} + \beta e^{\beta t} x(T-t)$$

$$\frac{\partial F}{\partial x} = e^{(\ )(T-t)}6x_t^2 + e^{\beta t}\gamma(T-t)$$

$$\frac{1}{2}\alpha^2 x^2 \frac{\partial F}{\partial x^2} = 6\alpha^2 x^3 e^{(\ )(T-t)}$$

conti ?

EX 9

Exercice 9 *Consider the process $X_t = \exp(B_t - t)$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*

i) Show that the process X is not a martingale under $\mathbb{P}$;

ii) Find the probability measure $\mathbb{P}^1$ under which the process X becomes a martingale;

iii) Compute $\mathbb{E}^{\mathbb{P}^1}[X_2]$;

iv) Consider now the process $Y_t = \exp(\int_0^t f(u)du + B_t)$, where f is a function of time: show that it is not a martingale under the probability measure $\mathbb{P}$;

v) Find the condition under which there exixts a probability measure $\mathbb{P}^2$ such that the process Y becomes a $\mathbb{P}^2$-martingale;

vi) Compute $\mathbb{E}^{\mathbb{P}^2}[Y_2]$ when $f(t) = t$.

$X_t = \exp(B_t - t)$ $\quad B_t$ standard Brownian motion

i) $dX_t = -\exp(B_t - t)dt + \exp(B_t - t)dB_t + \frac{1}{2}\exp(B_t - t)dt$

$= \underbrace{-\frac{1}{2}\exp(B_t - t)dt}_{\text{drift} \neq 0,\ \text{not a mg}} + \exp(B_t - t)dB_t$

ii) $dx_t = 0\,dt + \exp(B_t - t)\underbrace{\left(dB^{\mathbb{P}} - \frac{1}{2}dt\right)}_{dB^{\tilde{\mathbb{P}}}}$

$\left.\frac{\tilde{\mathbb{P}}}{\mathbb{P}}\right|_t = \exp^{-\frac{1}{2}\int_0^t dB_s - \frac{1}{2}\frac{1}{4}\int_0^t ds} = \exp^{-\frac{1}{2}B_t - \frac{1}{8}t}$ true-mg ✓

iii) $E^{P^1}[X_2] \overset{mg}{=} \exp(B_0 - 0) = 1$

iv) $dY_t = Y_t f(t)dt + Y_t dB_t + \frac{1}{2}Y_t dt =$

$= Y_t\left(f(t) + \frac{1}{2}\right)dt + Y_t dB_t$ $\quad$ not a mg unless $f(t) = -\frac{1}{2}$

v) $\quad \kappa_s = -f(t) - \frac{1}{2}$

NOVIKOV COND

$E\left[e^{\frac{1}{2}\int_0^T (f(t)+\frac{1}{2})^2 dt}\right] < +\infty$ $\quad$??

vi) $\mathbb{E}^{\mathbb{P}^2}[Y_2] \overset{mg}{=} Y_0 = 1$

check martingality condition:

$\exp\left[\int_0^t s + \frac{1}{2}\,dW - \frac{1}{2}\int_0^t \left(s + \frac{1}{2}\right)^2 ds\right]$

Exercice 16 *Let $X(t)$ be the value at time t in domestic currency of one monetary unit of the foreign currency, r_d the domestic interest rate, r_f the foreign interest rate.*

Suppose that the dynamics of X, under the real world probability measure, is given by

$$dX(t) = X(t)\mu dt + X(t)\sigma dW_t$$

where W is a standard Brownian motion and μ and σ are positive constants.

i) Find the price at time t of a Digital Call option expiring at time $T \geq t$ with strike K written on X.

ii) Find the price at time t of a Digital Call option expiring at time $T \geq t$ with strike K written on $1/X$.

iii) Give an interpretation of $1/X$ in terms of X.

i) $F(X_T) = \mathbb{1}_{X_T > K}$

Let be Q riskless measure

$$\text{price}_t(\ ) = e^{-r_d(T-t)} \mathbb{E}_t^Q(\mathbb{1}_{X_T > K}) =$$

$$\frac{dX_t}{X_t} = (r_d - r_f)\,dt + \sigma\, dW^Q$$

$$= e^{-r_d(T-t)} Q(X_T > K) = e^{-r_d(T-t)} \Phi(d_2)$$

$$d_2 = \frac{\ln\left(\frac{X}{K}\right) + \left(r_d - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

ii) $F(X_T) = \mathbb{1}_{\frac{1}{X_T} > K}$

Let be Q riskless measure

$$\text{price}_t(\ ) = e^{-r_d(T-t)} \mathbb{E}_t^Q(\mathbb{1}_{\frac{1}{X_T} > K}) = \ ...$$

$$\frac{dX_t}{X_t} = (r_d - r_f)\,dt + \sigma\, dW^Q$$

$$d\left(\frac{1}{X_t}\right) = -\frac{1}{X_t^2}\,dX_t + \frac{\cancel{2}}{\cancel{2}}\frac{1}{X_t^{\cancel{3}}}\,\sigma^2 \cancel{X_t^2}\,dt$$

$$= -\frac{(r_d - r_f)}{X_t}dt - \frac{\sigma \cancel{X_t}}{X_t^{\cancel{2}}}dW^Q + \frac{\sigma^2 dt}{X_t}$$

$$\frac{d\left(\frac{1}{X_t}\right)}{1/X_t} = \left(-(r_d - r_f) + \sigma^2\right)dt - \sigma\, dW^Q = \Big(r_d + \overbrace{\left(-2r_d + r_f + \sigma^2\right)}^{-q\text{, dividend}} dt\Big) - \sigma\, dW^Q$$

$$\dots = e^{-r(T-t)} \Phi(d_2)$$

$$d_2 = \frac{\ln\left(\frac{1}{XK}\right) + \left(\sigma^2 - (r_d - r_f) - \frac{1}{2}\sigma^2\right)(T-t)}{-\sigma\sqrt{T-t}}$$

iii) ?

EX 19

Exercice 19 *In a Black&Scholes market let consider a risky asset evolving according to the risk neutral dynamics:*

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t$$
$$S_0 = s,$$

Consider the option paying the following payoff: $Payoff_T = \min[S_T, K]$.

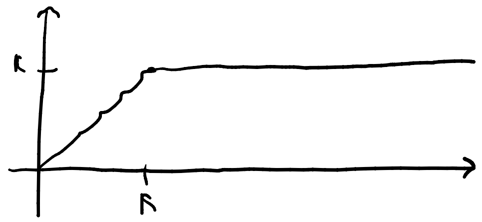
Exercice 19 *In a Black&Scholes market let consider a risky asset evolving according to the risk neutral dynamics:*

$$\frac{dS_t}{S_t} = r dt + \sigma dB_t$$
$$S_0 = s,$$

Consider the option paying the following payoff: $Payoff_T = min[S_T, K]$.

i) Find the price of this contract at a generic time $t \leq T$;
ii) Compute the Delta and the Vega of this contract;
iii) Give an illustration of the Delta and the Gamma for the contract when the volatility parameter has an upward shock from σ to $\hat{\sigma} > \sigma$.

i)



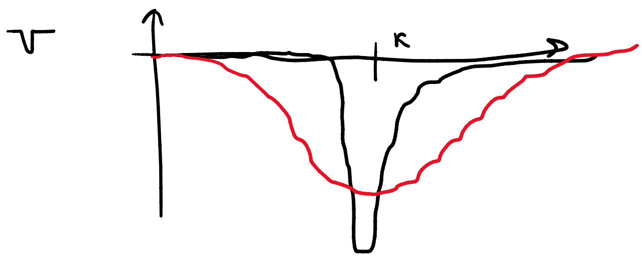
$$\text{PAYOFF} = \min(S_T, K) = S_T - (S_T - K)^+ = S_T - [S_T - K] \mathbb{1}_{S_T > K}$$

$$\text{price}_t(S_T) = e^{-r(T-t)} E^{Q}_t [S_T] = e^{+rt} E_t[e^{-rT} S_T] =$$
$$= e^{rt} e^{-rt} S_t = S_t$$

$$\text{price}_t = S_t - S_t \Phi(d_1^K) + e^{-r(T-t)} K \Phi(d_2^K)$$

ii) $\Delta_t = 1 - \Phi(d_1^K)$

$$\mathcal{V} = 0 - \frac{S_t \sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$



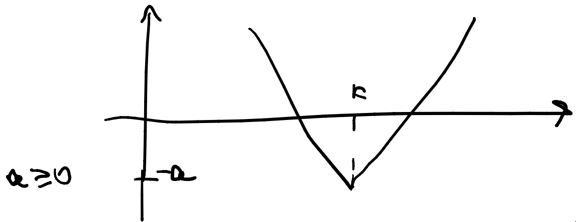
Exercice 7 *Dans le modéle de Black-Scholes, on considère un actif risqué S:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,.$$

et un taux d'intéret constant r. Déterminez le prix d'une stratégie STRADDLE down-and-out, où on a le droit au payoff si l'actif ne touche pas la barrière L, puis le comparer avec le prix du contrat corréspondent sans la présence de la barrière.

• STRADDLE down and out

· STRADDLE down and out



$$\text{PAYOFF}_T = (-S_T + K - a)\mathbb{1}_{S_T<K} + (K - a + S_T)\mathbb{1}_{S_T>K}$$

$$= -(1-\mathbb{1}_{S_T>K})(S_T-K+a) + (S_T-K)\mathbb{1}_{S_T>K} - a\mathbb{1}_{S_T>K} =$$

$$= (K-a-S_T) + (S_T-K)\mathbb{1}_{S_T>K} + \cancel{a\mathbb{1}_{S_T>K}} + (S_T-K)\mathbb{1}_{S_T>K} - \cancel{a\mathbb{1}_{S_T>K}}$$

$$= 2(S_T-K)^+ + K - a - S_T$$

$$\text{price}_{L_0}(t,S,G) = \left[\text{price}(t,S,G_L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \text{price}\left(t,\frac{L^2}{S},G_L\right)\right]\mathbb{1}_{S>L}$$ plug in

$$\tilde{r} = r - \frac{\sigma^2}{2}, \qquad G_L(x) = G(x)\,\mathbb{1}_{x>L}$$

$$G_L(x) = 2(S_T-K)\mathbb{1}_{S_T>K}\mathbb{1}_{S_T>L} + (K-a)\mathbb{1}_{S_T>L} - S_T\mathbb{1}_{S_T>L}$$

$$2(S_T-K)^+ \quad + (K-a)\mathbb{1}_{S_T>L} - (S_T-L)^+ \quad - L\mathbb{1}_{S_T>L}$$

$L<K$ $\quad 2(S_T-K)\mathbb{1}_{S_T>K} + (K-a)\mathbb{1}_{S_T>L} - (S_T-L)\mathbb{1}_{S_T>L} - L\mathbb{1}_{S_T>L}$

$$- (S_T-L)^+$$

$L>K$ $\quad 2(S_T-L)^+ + 2(L-K)\mathbb{1}_{S_T>L} + (K-a)\mathbb{1}_{S_T>L} - (S_T-L)\mathbb{1}_{S_T>L} - L\mathbb{1}_{S_T>L}$

$$\text{price}(t,S,G) = 2\,\text{Call}^K + (K-a)e^{-r(T-t)} + S_t$$

$$\text{Call}^K = S_t\Phi(d_1^K) - Ke^{-r(T-t)}\Phi(d_2^K)$$

EX 8) Soient

- $X(t)$: la valeur á l'instant t en unité monétaire domestique d'une unité monétaire étrangère,
- r_d: le taux d'intérêt domestique,
- r_f: le taux d'intérêt étranger.

On suppose que la dynamique de X, sous la probabilité réelle, est donnée par

$$dX(t) = X(t)\mu dt + X(t)\sigma dW_t$$

où W est un mouvement brownien standard et où μ et σ sont deux constantes positives.

1. Déterminez la dynamique du taux de change X sous la probabilité risque-neutre.
2. Trouvez le prix d'un contrat STRADDLE sur X, de maturité T, et de prix d'exercice K.

Want to work with risk free probability

$\frac{dB_d}{B_d} = r_d\,dt, \quad \frac{dB_f}{B_f} = r_f\,dt$

$\Rightarrow B_d = e^{r_d t}$

want $\tilde{B_f} = X_t B_f$ to be a Q mg divided for B_d

$$d\left(\frac{\tilde{B_f}}{B_d}\right) = \frac{d\tilde{B_f}}{e^{r_d t}} - r_d\frac{\tilde{B_f}}{e^{r_d t}}dt = e^{-r_d t}(dX_t\,B_f + dB_f\,X_t) - r_d e^{-r_d t}X_t B_f\,dt =$$

$$= e^{-r_d t}(X_t B_f \mu_X dt + X_t B_f \sigma_X dW_t + r_f B_f\,dt) - r_d e^{-r_d t}X_t B_f\,dt =$$

$$= e^{-r_d t}X_t B_f(\mu_X + r_f - r_d)dt + e^{-r_d t}X_t B_f \sigma_X dW_t$$

$$= e^{-r_d t} X_t B_f \left(\mu_X + r_f - r_d\right) dt + e^{-r_d t} X_t B_f \sigma_X dW_t$$

$$dW^P = dW^Q - \frac{\mu_X + r_f - r_d}{\sigma_X} dt$$

$$\frac{dX_t}{X_t} = \mu_X dt + \sigma_X dW^Q - (\mu_X + r_f - r_d) dt = (r_d - r_f) dt + \sigma_X dW^Q$$

dividend

2) STRADDLE

$$PAYOFF_X = 2(X_T - K)^+ + K - a - X_T$$

$$price_t(STRADDLE) = 2\left[X_t e^{-r_f(T-t)} \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)\right] + (K-a) e^{-r_d(T-t)} + X_t$$

$$d_{1,2} = \frac{\log\left(\frac{X_t}{K}\right) + \left(r_d - r_f \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d\left(X_t e^{-r_d t}\right) = \left(X_t r_d dt + X_t \sigma_X dW\right) e^{-r_d t} - r_d e^{-r_d t} X_t dt \checkmark$$

Exercice 14 *Solve the following Partial Differential Equation:*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial x^2} + 1 &= 0 \\ F(T, x) &= x^2, \end{aligned}$$

$$dX_t = dt + \sqrt{2}\, dW$$

$$X_T = (T-t) + \sqrt{2}(W_T - W_t) + X_t$$

If $\left(\sqrt{2} \frac{\partial F}{\partial x}\right) \in \mathcal{H}$

$$\longrightarrow F(t,x) = E_t[X_T^2] + \int_t^T E_t(1) ds$$

$$= E_t\left[X_t^2 + (T-t)^2 + 2(W_T - W_t)^2 + \sim + \sim + 2X_t(T-t)\right] + (T-t)$$

$$= X_t^2 + (T-t)^2 + 2(T-t) + 2X_t(T-t) + (T-t) =$$

$$= x(x + 2(T-t)) + (T-t)^2 + 3(T-t)$$

$$F(T,x) = x^2$$

$$\frac{\partial F}{\partial t} = -2x - 2(T-t) - 3$$

$$\frac{\partial F}{\partial x} = 2x + 2(T-t) \qquad \frac{\partial F}{\partial x^2} = 2$$

$$\longrightarrow \cancel{-2x} - \cancel{2(T-t)} - \cancel{3} + \cancel{2x} + \cancel{2(T-t)} + \cancel{2} + \cancel{1} = 0$$

$$\frac{\partial F}{\partial x} = 2x + 2(T-t)$$

$$\int_0^T 4E[X_t^2] + 8t\, E[X_t] + 4t^2 dt < +\infty \leftrightarrow$$

$$\int_0 4t\,[X_t^2] + 8t\; t\,[X_t] + 4\; t\; dt < +\infty \iff$$

$$\int_0^T 4\left(x_0^2 + \quad t^2 + 2\; t\; x_0 + 4t\right) dt < +\infty \checkmark$$

$$\int_0^T (T-t)\left[x_0 + t\right] dt < +\infty \checkmark$$

Exercice 12 *(Quanto Option) Soient $X(t)$ la valeur á l'instant t en unité monétaire domestique d'une unité monétaire étrangère, r_d le taux d'intérêt domestique, r_f le taux d'intérêt étranger. On suppose que la dynamique de X, sous la probabilité risque neutre domestique, est donnée par*

$$\frac{dX(t)}{X(t)} = (r_d - r_f)\,dt + \sigma_x dW_t^1$$

où W^1 est un mouvement brownien standard et σ est une constante positive. Soit aussi S^f le prix d'un actif étranger suivant

$$\frac{dS^f(t)}{S^f(t)} = r_f dt + \sigma_f dW_t^2,$$

où W^1 est un mouvement brownien standard indépendant de W^2.

1. *Déterminez la dynamique du taux de change $\frac{1}{X}$ sous la probabilité risque-neutre étrangère.*
2. *Trouvez le prix en unité monétaire étrangère d'un contrat CALL sur S^f de maturité T , où le prix d'exercice K est exprimé en unité monétaire domestique.*

$$d\left(\frac{1}{X_t}\right) = -\frac{1}{X_t^2}\,dX_t + \frac{1}{2}\cdot(-2)-\frac{1}{X_t^3}\;\sigma_x^2 X_t^2\,dt$$

$$= -\frac{1}{X_t^2}\left(X_t(r_d - r_f)\,dt + \sigma_x X_t\,dW_t^1\right) + \frac{1}{X_t}\,\sigma_x^2\,dt =$$

$$= -\frac{1}{X_t}(r_d - r_f)\,dt - \frac{1}{X_t}\sigma_x\,dW_t^1 + \frac{1}{X_t}\,\sigma_x^2\,dx$$

$$\frac{\left(d\frac{1}{X_t}\right)}{1/X_t} = (r_f - r_d)\,dt + \sigma_x^2\,dt - \sigma_x\,dW_t^1$$

$$= r_f\,dt - \sigma_x\underbrace{\left(dW^1 + \left(\frac{r_d}{\sigma_x} - \sigma_x\right)dt\right)}_{dW_t^2}$$

$$d\left(\frac{e^{-r_f t}}{X_t}\right) = d\left(\frac{1}{X_t}\right)e^{-r_f t} - r_f\,\frac{e^{-r_f t}}{X_t}\,dt$$

$$= \left[-\frac{1}{X_t}(r_d - r_f)\,dt - \frac{1}{X_t}\sigma_x\,dW_t^1 + \frac{1}{X_t}\,\sigma_x^2\,dx\right]e^{-r_f t} - r_f\,\frac{e^{-r_f t}}{X_t}\,dt =$$

$$= \left[-\frac{1}{X_t}r_d\,dt - \frac{1}{X_t}\sigma_x\,dW_t^1 + \frac{1}{X_t}\,\sigma_x^2\,dx\right]e^{-r_f t}$$

2. $(\tilde{S_f} - K)^+$

$$\text{price}_t = \tilde{S_f}\,\Phi(d_1) - Ke^{-r_d(T-t)}\,\Phi(d_2)$$

$$\frac{d\tilde{S_f}}{\tilde{S_f}} = r_d\,dt + (\sigma_f + \sigma_x)\,dW^Q$$

$$d_1 = \frac{\ln\left(\frac{\tilde{S_f}}{K}\right) + \left(r_d + \frac{1}{2}\|\sigma_f + \sigma_x\|^2(T-t)\right)}{\|\sigma_f + \sigma_x\|\sqrt{T-t}}$$

Exercice 18 *Consider a standard Brownian motion B_t.*

i) Find the real parameter α such that the process $\exp(1 + \alpha B_t^2 + B_t(\alpha - 1))$ is a submartingale with respect to the natural filtration;

ii) For the values of α of point i) find the corresponding Doob-Meyer decomposition;

iii) Compute the expected value at time $t = 2$ of the process $Y = X^3$, where X satisfies

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iii) Compute the expected value at time $t = 2$ of the process $Y = X^3$, where X satisfies the SDE $dX_t = -X_t dt + \sigma X_t dB_t$, $X_0 = 1$ $(\sigma > 0)$;

iv) Compute the variance of the random variable Y_2;

v) Compute the quadratic covariation between the processes X and Y;

vi) Compute the values of σ for which the process Y is a submartingale and find the corresponding Doob-Meyer decomposition.

i) $Y_t = \exp(X_t)$

ii) $dY_t = \exp(X_t)\,dX_t + \frac{1}{2}\exp(X_t)\,\varphi^2 dt$

$X_t = 1 + \alpha B_t^2 + B_t(\alpha - 1)$

$dX_t = \alpha 2 B_t\, dB_t + \alpha dt + dB_t(\alpha - 1)$

mg

$dY_t = \exp(X_t)(3\alpha - 1)dB_t + \exp(X_t)\alpha dt +$

$+ \frac{1}{2}\exp(X_t)(3\alpha - 1)^2 dt$

$\exp(X_t)\left(\alpha + \frac{(3\alpha-1)^2}{2}\right)$ > 0

increasing if $\alpha + \frac{(3\alpha-1)^2}{2} > 0$

$2\alpha + 9\alpha^2 + 1 - 6\alpha > 0$

$9\alpha^2 - 4\alpha + 1 > 0$ $\Delta < 0$, always pos, always a submg

iii) $dY_t = 3X_t^2 dX_t + \frac{1}{2}\cdot 2\cdot 3X_t\, \sigma^2 X_t^2 dt$

$= -3X_t^3 dt + 3\sigma X_t^3 dX_t + 3X_t^3\sigma^2 dt$

$= X_t^3(-3 + \sigma^2)dt + 3\sigma X_t^3 dX_t$ $N\left(\left(-3+\sigma^2 - \frac{9\sigma^2}{2}\right)t,\ 9\sigma^2 t\right)$

$X_t^2 = X_0^3 \exp\left(\left(-3 + \sigma^2 - \frac{9\sigma^2}{2}\right)t + 3\sigma dW_t\right)$

$E[X_t^3] = X_0^3 \exp(-3t + \sigma^2 t)$

$E[X_2^2] = 1 \exp(-6 + \sigma^2 \cdot 2)$

Exercice 26 *Consider a Black&Scholes market where a risky asset evolves according to*

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_0 = s,$$

and a riskless asset is associated to the risk free rate r. Consider a Purple Collar option, that is a derivative contract with payoff at the maturity T given by

$$\begin{array}{ll} a & \text{if } S_T < K_1; \\ S_T & \text{if } K_1 < S_T < K_2; \\ K_2 & \text{if } S_T > K_2, \end{array}$$

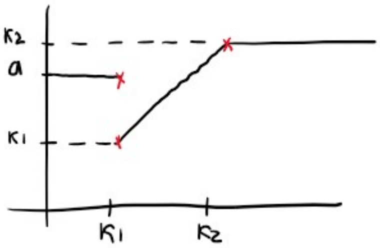
where $K_1 < a < K_2$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta and the Gamma of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility σ.

• Draw the PAYOFF $K_1 < a < K_2$



· I want to write it as sum of payoff I know

$$PAYOFF_T = a - (a-K_1)\mathbb{1}_{S_T>K_1} + (S_T-K_1)^+ - (S_T-K_2)^+$$

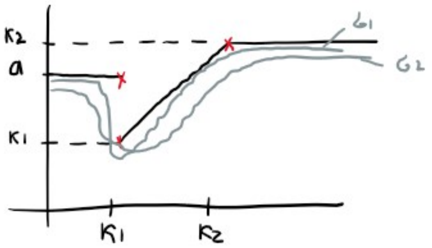
then at K_1 I go down to K_1 $a+y=K_1$, $y=K_1-a$

now I am at $\cancel{a}=\cancel{a}+K_1$, want to go up, call

then want to stab on K_2 $S_T+y=K_2$ $y=K_2-S_T$

$$\text{price}_t(\quad) = a e^{-r(T-t)} - (a-K_1)e^{-r(T-t)}\Phi(d_2^{K_1}) + S_t\Phi(d_1^{K_1}) - K_1 e^{-r(T-t)}\Phi(d_2^{K_1}) + - S_t\Phi(d_1^{K_2}) + e^{-r(T-t)}K_2\Phi(d_2^{K_2})$$

ii) $$\text{Delta}(\quad) = -(a-K_1)\frac{e^{-r(T-t)}e^{-\frac{{d_2^{K_1}}^2}{2}}}{S\sigma\sqrt{T-t}} + \Phi(d_1^{K_1}) - \Phi(d_1^{K_2})$$



$\sigma_2 > \sigma_1$

ii) $$\text{Gamma} = \text{Gamma}^{DIG}_{cerco} + \frac{1}{S\sigma\sqrt{T-t}}e^{-\frac{{d_1^{K_1}}^2}{2}} - \frac{1}{S\sigma\sqrt{T-t}}e^{-\frac{{d_1^{K_2}}^2}{2}}$$

The delta and gamma of a digital call option are (for $Q=1$):

$$\Delta = \frac{e^{-r\tau}n(d_2)}{S\sigma\sqrt{\tau}} \quad \text{and} \quad \Gamma = -\frac{e^{-r\tau}d_1 n(d_2)}{S^2\sigma^2\tau} \quad \text{with} \quad d_1 = d_2 + \sigma\sqrt{\tau}$$