Stochastic Methods for Finance

Exam July 19, 2018

Consider a Black&Scholes market where a risky asset evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_0 = s,$$

and a riskless asset is associated to the risk free rate r.

Exercice 1 *(5 points) We want to approximatively hedge a digital option with maturity T with vanillas. Consider the following static strategy for any $h > 0$: $1/h * (Call(S, K) - Call(S, K + h))$.*

i) Show that the price of this contract tends to the price of the digital contract with payoff $1_{S_T > K}$ for $h \to 0$.

ii) Find the relation between the Delta of the digital option and the Delta of this static strategy for any $h > 0$ and in the limit for $h \to 0$.

Exercice 2 *(5 points)*

Consider a contract giving the following payoff at the maturity T

$$\begin{array}{rl} a & if\ S_T < K_1; \\ a + nS_T & if\ K_1 < S_T < K_2; \\ a - nS_T + n & if\ K_2 < S_T < K_3; \\ a & if\ S_T > K_3, \end{array}$$

where $n \in \mathbb{N}, a > 0$ and $K_3 > K_2 > K_1$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta and the Gamma of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 3 *(8 points)*

In a Black-Scholes model, find the price at time $t \leq T$ of a DOWN-AND-OUT contract where the owner receives the exchange option payoff

$$(S_1(T) - S_2(T))^+$$

provided that the first asset has been always above the 50% of the second one, that is $S_1(t) > 0,5 * S_2(t)$ for all $t \leq T$. Find the Delta of this contract.

Exercice 4 *(8 points)*

Solve the following PDE for $t \leq T$:

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + xe^y &= 0 \\ F(T, x, y) &= xy. \end{aligned}$$

Exercice 5 *(4 points)*

Questions on the theory.

i) Show that under a suitable parameter assumption the binomial model converges to the Black-Scholes asset price dynamics.

ii) State and prove the Second Fundamental theorem of asset pricing for a 2-periods finite-dimensional market model