Stochastic Methods for Finance

Exam June, 24, 2019

Exercice 1 *(8 points) Consider a Black-Scholes market and a derivative contract with payoff at the maturity T given by*

$$F(n, S_T) = n^2 * 1_{1/(n+1)<S_T<1/n} + 1/n^2 * 1_{S_T \geq 1/n},$$

where 1_A denotes the indicator function of the event A.

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, ...$ and the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

iv) Compute the price of the portfolio F given by

$$F(S_T) = \sum_{n=1}^{\infty} F(n, S_T);$$

v) Compute the amount of Call options with strike price $K = 1/2$ one has to buy/sell in order to get a Delta neutral (global) portfolio.

Exercice 2 *(5 points) Consider a Black-Scholes market and a Cube-Call, that is a derivative with payoff at the maturity T given by*

$$(S_T^3 - K)^+$$

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta of the contract ;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 3 *(5 points)*

In the Black-Scholes model, find the price at time $t \leq T$ of a DOCC (DOWN-AND-OUT-CUBE-CALL) contract where the owner receives at the maturity T the payoff

$$F(S_T) = (S_T^3 - K)^+$$

provided that the underlying asset has never reached the lower barrier L. Find the Delta of the contract.

Exercice 4 *(6 points)*

Solve for any $n = 1, 2, ...$ *the following PDE for* $t \le T$ *:*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + y x^n &= 0 \\ F(T, x, y) &= y x^2. \end{aligned}$$

Exercice 5 *(8 points)*

Questions on the theory.

i) Show that the Forward Price is a martingale under the forward risk neutral measure while the Futures Price is a martingale under the risk neutral measure

ii) Provide the price of a CALL option in the extended Black-Scholes model where interest rates are Gaussian and independent of the Brownian motion driving the asset returns.

iii) Show that the optimal value problem associated to the pricing of an American option is the Snell envelope of the payoff process.

iv) Show that the barrier pricing function is linear in the payoff.