Stochastic Methods for Finance

Exam June, 25, 2020

Exercice 1 *Consider a Black-Scholes market and a derivative contract with payoff at the maturity T given by*

$$F(n, S_T) = (S_T - n)^+ \times 1_{n<S_T<2n},$$

where 1_A denotes the indicator function of the event A.

i) Compute the price of the contract $F(n, S_T)$ at any time $t \in [0, T)$ and any $n = 1, 2, \ldots$ and the limit of the price for $n \to \infty$;

ii) Compute the Delta of the contract $F(n, S_T)$ and the limit of the Delta for $n \to \infty$;

iii) Compute the Gamma of the contract $F(n, S_T)$ and provide evidence of potential issues in hedging the contract ; [HINT : show that the Gamma may be negative]

iv) Illustrate graphically the change of price and Delta of $F(n, S_T)$ for a upward shift of the volatility ;

v) Compute the price of the portfolio F given by

$$F(S_T) = \sum_{n=1}^{2} F(n, S_T);$$

vi) Compute the amount of Call/Put options with strike price $K = 1$ one has to buy/sell in order to get a Delta-Vega neutral (global) portfolio ;

Exercice 2 *Consider a Black-Scholes market and a power PUT, that is a derivative with payoff at the maturity T given by*

$$(K - S_T^n)^+$$

i) Compute the price of the contract at any time $t \in [0, +\infty)$ for any $n = 0, 1, \ldots$;

ii) Compute the Delta of the contract ;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 3 *In the Black-Scholes model, find the price at time $t \leq T$ for any $n = 0, 1, \ldots$ of a UIPPP (UP-AND-IN-POWER-PUT) contract where the owner receives at the maturity T the payoff*

$$(K - S_T^n)^+$$

provided that the underlying asset reached the upper barrier L. Find the Delta of the contract.

Exercice 4 *Solve for any $n = 1, 2, \ldots$ the following PDE for $t \leq T$:*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} + x^2 \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + zyx^n &= 0 \\ F(T, x, y, z) &= zyx^2. \end{aligned}$$

Exercice 5 *(FOR 9 ECTS EXAM) A risky asset S in a 2-period binomial model (one period =1 year) evolves according to an increasing factor of $u_1 = 1,1$ (resp. decreasing factor of $d_1 = 0,9$) for the first period, and $u_1 = 1,05$ (resp. decreasing factor of $d_1 = 0,95$) for the second one, starting from the initial price $S_0 = 100$. The riskless interest rate is flat at zero.*

i) Find the initial price and the hedging strategy of a European CALL option on S with maturity $T =$ 2 years and strike price $K = 95$;

ii) Find the initial price and the hedging strategy of an American PUT option on S with maturity $T =$ 2 years and strike price $K = 95$;

iii) Same question at point ii) when the interest rate is still zero in the first year but it is equal to 1% per year on the whole period $[0, 2years]$;

iv) Provide the hedging strategy for the whole portfolio consisting in the two positions at the points i) and ii).