26 june 2018

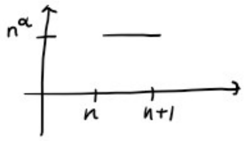
mercoledì 9 giugno 2021 10:19

Ex 1

B-S market, derivative contract

$F(n, S_T) = n^\alpha \cdot \mathbb{1}_{n<S_T<n+1}$, $\alpha>0$, $n\in\mathbb{N}$

i)



$F(n, S_T) = n^\alpha \mathbb{1}_{S_T>n} - n^\alpha \mathbb{1}_{S_T>n+1}$, difference of digitals

$$\text{price}_t(F(n,S_T)) = n^\alpha e^{-r(T-t)}\Phi(d_2^n) - n^\alpha e^{-r(T-t)}\Phi(d_2^{n+1})$$

$$\phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \quad , \quad d_2^n = \frac{\ln\left(\frac{S_t}{n}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

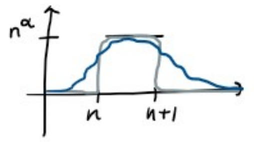
$$e^{-r(T-t)} n^\alpha \left(\int_{d_2^{n+1}}^{d_2^n} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy\right) =$$

$$= e^{-r(T-t)} n^\alpha \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\left(d_2^n - d_2^{n+1}\right)$$

$$\underbrace{e^{-\frac{(\ln n)^2}{2} + \alpha \ln n}}_{\substack{n\to+\infty \\ \downarrow\ \alpha>0 \\ 0}} \overbrace{\left(\ln\left(\frac{n+1}{n}\right)\right)}^{\substack{0 \\ n\to+\infty}} \xrightarrow[n\to+\infty]{} 0$$

ii) $$\text{Delta} = n^\alpha \frac{e^{-r(T-t)} e^{-\frac{{d_2^n}^2}{2}}}{S\sigma\sqrt{T-t}} - \frac{n^\alpha e^{-r(T-t)} e^{-{d_2^{n+1}}^2}}{S\sigma\sqrt{T-t}} \xrightarrow{n\to+\infty} 0$$

iii)



$\sigma_2 >> \sigma_1$

iv) $$F(S_T) = \sum_{n=1}^{+\infty} F(n, S_T) = \sum_{n=1}^{+\infty} n^\alpha \mathbb{1}_{n<S_T<n+1} =$$

$$= \sum_{n=1}^{+\infty} n^\alpha\left(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1}\right)$$

$$\text{price}_t(F_T) = \sum n^\alpha e^{-r(T-t)}\Phi(d_2^n) - n^\alpha e^{-r(T-t)}\Phi(d_2^{n+1}) =$$

$$= e^{-r(T-t)}\sum n^\alpha\left(\Phi(d_2^n) - \Phi(d_2^{n+1})\right)$$ meglio di così?

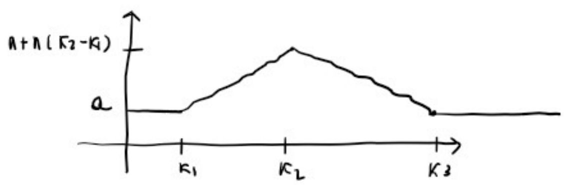
v) $0 = \Delta^F + x\Delta^{call}$, $x = -\frac{\Delta^F}{\Delta^{call}}$

vi)

Exercise 2

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \quad \sim r dt + \sigma dW_t^Q$$

$S_0 = s$



i) $\text{PAYOFF}_T = a + n(S_T - K_1)^+ - 2n(S_T - K_2)^+ + n(S_T - 2K_2 + K_1)^+$

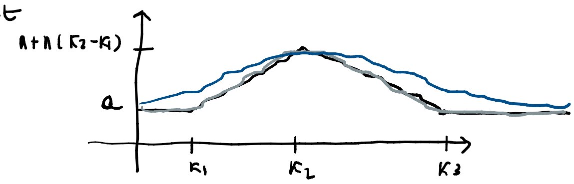
$\text{price}_t = e^{-r(T-t)} a + n\,\text{Call}_t^{K_1} - 2n\,\text{Call}_t^{K_2} + n\,\text{Call}_t^{2K_2 - K_1}$

ii) $\Delta_t = \dots$

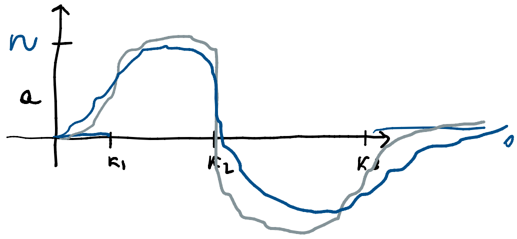
$\Gamma_t = \dots$

iii) always the same

price_t



Δ_t

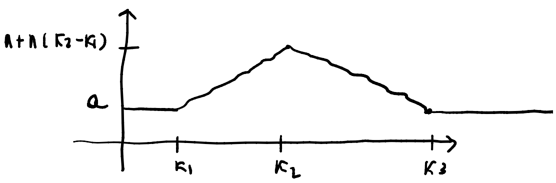


EX 3 DOWN and OUT

$$\text{price}_{L_0}(t, S, G) = \left[\text{price}(t, S, G_L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \text{price}\left(t, \frac{L^2}{S}, G_L\right)\right] \mathbb{1}_{S > L}$$

$\tilde{r} = r - \frac{1}{2}\sigma^2, \qquad G_L(x) = G(x)\,\mathbb{1}_{x > L}$

Compute G_L



$$\begin{cases} a & if\ S_T < K_1; \\ a + nS_T - nK_1 & if\ K_1 < S_T < K_2; \\ a - nS_T + n(2K_2 - K_1) & if\ K_2 < S_T < K_3; \\ a & if\ S_T > K_3, \end{cases}$$

$\text{PAYOFF}_T = a + n(S_T - K_1)^+ - 2n(S_T - K_2)^+ + n(S_T - 2K_2 + K_1)^+$

$a\,\mathbb{1}_{S_T > L} + n(S_T - K_1)\mathbb{1}_{S_T > K_1}\mathbb{1}_{S_T > L} - 2n(S_T - K_2)\mathbb{1}_{S_T > K_2}\mathbb{1}_{S_T > L} + n(S_T - 2K_2 + K_1)\mathbb{1}_{S_T > K_3}\mathbb{1}_{S_T > L}$

If $L < K_1 < K_2 < K_3 \longrightarrow a\,\mathbb{1}_{S_T > L} + n(S_T - K_1)^+ - 2n(S_T - K_2)^+ + n(S_T - 2K_2 + K_1)^+$

$\rightarrow$ plug in

$K_1 < L < K_2 < K_3 \longrightarrow a\,\mathbb{1}_{S_T > L} + n(S_T - L)^+ + n(L - K_1)\mathbb{1}_{S_T > L} - 2n(S_T - K_2)^+ + n(S_T - 2K_2 + K_1)\mathbb{1}_{S_T > K_3}$

$\rightarrow$ plug in

$K_1 < K_2 < L < K_3 \longrightarrow a\,\mathbb{1}_{S_T > L} + n(S_T - L)^+ + n(L - K_1)\mathbb{1}_{S_T > L} - 2n(S_T - L)^+ + 2n(K_2 - L)\mathbb{1}_{S_T > L} + n(S_T - 2K_2 + K_1)^+$

$\rightarrow$ plug in

$K_1 < K_2 < K_3 < L \longrightarrow a\,\mathbb{1}_{S_T > L} + n(S_T - L)^+ + n(L - K_1)\mathbb{1}_{S_T > L} - 2n(S_T - L)^+ + 2n(K_2 - L)\mathbb{1}_{S_T > L} + n(S_T - L)^+ + n(L - K_3)\mathbb{1}_{S_T > L}$

$\rightarrow$ plug in

- Delta computing building blocks   LIMIT, missing

• Delta computing Building blocks UNIT, missing

EX 4

Exercice 4 *(5 points)*
Solve the following PDE for $t \leq T$:

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} + xy &= 0 \\ F(T,x,y) &= x. \end{aligned}$$

The solution is

$$F(t,x,y) = \mathbb{E}_{t,x,y}[X_T] + \int_t^T \mathbb{E}_{t,x,y}(X_s Y_s)\,ds$$

Provided $dX_t = X\,dW^1$, $dY_t = dW^2$, $W^1 \perp\!\!\!\perp W_2$, $\left(X\frac{\partial F}{\partial x}\right) \in H$, $\left(\frac{\partial F}{\partial y}\right) \in H$

$$X_T = X_t \exp\left(-\frac{1}{2}(T-t) + \overbrace{(W^1_T - W^1_t)}^{N(0,T-t)}\right), \quad X_s = X_t \exp\left(-\frac{1}{2}(s-t) + (W^1_s - W^1_t)\right)$$

$$Y_T = Y_t + (W^2_T - W^2_t), \quad Y_s = Y_t + (W^2_s - W^2_t)$$

$$\mathbb{E}_t(X_T) = \mathbb{E}_t\left[X_t \exp\left(-\frac{1}{2}(T-t) + (W_T - W_t)\right)\right] = X_t e^{-\frac{1}{2}(T-t)} e^{\frac{1}{2}(T-t)} = X_t$$

$$\mathbb{E}_t(X_s Y_s) \overset{\perp\!\!\!\perp}{=} \mathbb{E}_t[X_s]\,\mathbb{E}_t(Y_s] = X_t e^{-\frac{1}{2}(s-t)} e^{\frac{1}{2}(s-t)} \cdot [Y_t + 0] = X_t Y_t$$

$$\int_t^T X_t Y_t\,ds = X_t Y_t (T-t)$$

$$F(t,x,y) = x + xy(T-t)$$

$$\frac{\partial F}{\partial t} = -xy, \quad \frac{\partial F}{\partial x} = 1 + y(T-t), \quad \frac{\partial^2 F}{\partial x^2} = 0$$

$$\frac{\partial F}{\partial y} = x(T-t), \quad \frac{\partial^2 F}{\partial y^2} = 0$$

$$\longrightarrow \quad -xy + xy = 0 \; ✓$$
$$F(T,x,y) = x \; ✓$$

Check $\left(\frac{\partial F}{\partial x} x\right) \in H$

$$\mathbb{E}\left[\int_0^T (1 + y_s(T-s))^2 x_s^2\,ds\right]$$
$$= \int_0^T \mathbb{E}[1 + y_s^2(T-s)^2 + 2y_s(T-s)]\,\mathbb{E}[x_s^2]\,ds =$$
$$= \int_0^T \left(1 + (T-s)^2\underbrace{\mathbb{E}[y_s^2]}_{[y_0^2 + s]} + 2(T-s)\underbrace{\mathbb{E}[y_s]}_{y_0}\right)\underbrace{\mathbb{E}[x_s^2]}_{x_0 e^s}\,ds \quad < +\infty$$

$\left(\frac{\partial F}{\partial y}\right) \in H$

$$\mathbb{E}\left[\int_0^T x_s^2 (T-s)^2\,ds\right] = \int_0^T x_0 e^s (T-s)^2\,ds \quad < +\infty$$