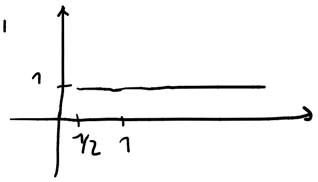
24 giugno 2019

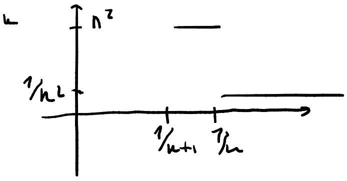
Ex 1

$F(n,S_T) = n^2 \mathbb{1}_{\frac{1}{n+1} < S_T < \frac{1}{n}} + \frac{1}{n^2}\mathbb{1}_{S_T \geq 1/n}$

i)

n=1





i) $F(n,S_T) = n^2 \mathbb{1}_{S_T > \frac{1}{n+1}} + \frac{1-n^4}{n^2}\mathbb{1}_{S_T > \frac{1}{n}}$ sum of dig

$price_t(F(n,S_T)) = n^2 e^{-r(T-t)}\Phi(d_2^{1/n+1}) + \frac{1-n^4}{n^2} e^{-r(T-t)}\Phi(d_2^{1/n}) =$

$= n^2 e^{-r(T-t)}\left[\Phi(d_2^{1/n+1}) - \Phi(d_2^{1/n})\right] + \frac{1}{n^2}e^{-r(T-t)}\Phi(d_2^{1/n}) \xrightarrow{n\to+\infty} 0$

$\int_{d_1^{1/n}}^{d_1^{1/n+1}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\left(d_1^{1/n+1} - d_1^{1/n}\right)$ $\xi \in \left(d_1^{1/n+1}, d_1^{1/n}\right)$

$n^2 \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} \sim n^2 e^{-(\ln n)^2/2} = e^{-\frac{(\ln n)^2}{2} + \ln n^2} = e^{-\frac{\ln n^2}{2} + 2\ln n} \xrightarrow{n\to+\infty} 0$

$\Phi(d_2^{1/n}) \xrightarrow{n\to+\infty} 1$ $\frac{1}{n^2}\to 0$ $d_1^{1/n} = \frac{\ln(S\cdot n) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$price_t \longrightarrow 0$

ii) Delta $n^2\Delta^{Dig}(1/n+1) + \frac{1-n^4}{n^2}\Delta^{Dig}(1/n) =$

$= n^2\left(\Delta^{Dig}(1/n+1) - \Delta^{Dig}(1/n)\right) + \frac{1}{n^2}\Delta^{Dig}(1/n) \longrightarrow 0$

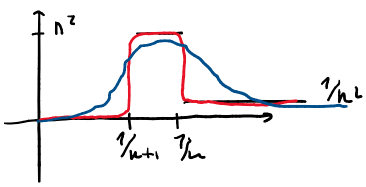
$Delta^{DIG} = \frac{e^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{S\sigma\sqrt{T-t}}$

$\sim n^2\frac{e^{-r(T-t)}}{S\sigma\sqrt{T-t}}\left(e^{-\frac{(\ln(n+1))^2}{2}} - e^{-\frac{(\ln(n))^2}{2}}\right) = c\, e^{-\frac{(\ln(n+1))^2}{2}+2\ln n} - e^{-\frac{(\ln n)^2}{2}+2\ln n} \longrightarrow 0$

$\frac{1}{n^2}\Delta^{dig} = \frac{e^{-r(T-t)}e^{-\frac{(\ln n)^2}{2}}}{S\sigma\sqrt{T-t}}\frac{1}{n^2} \longrightarrow 0$

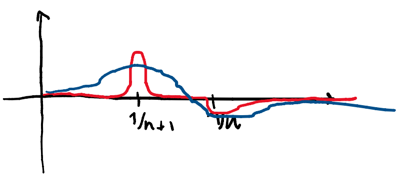
iii)

price



$\sigma_1 << \sigma_2$

Δ



una cosa così?

! iv) $F(S_T) = \sum_{n=1}^{+\infty} F(n,S_T) = \sum_n n^2 \mathbb{1}_{1/n+1 < S_T < 1/n} + 1/n^2\, \mathbb{1}_{S_T > 1/n}$

$price_t = \sum_{n=1}^{+\infty} n^2 e^{-r(T-t)}\left[\Phi(d_2^{1/n+1}) - \Phi(d_2^{1/n})\right] + \frac{1}{n^2}e^{-r(T-t)}\Phi(d_2^{1/n})$

v) $F + x\,\mathrm{Call}(1/2)$

$$0 = \Delta_F + x\,\Delta_{\mathrm{call}}(1/2)$$

$$\longrightarrow x = \frac{-\Delta F}{\Delta\mathrm{call}(1/2)}$$

EX2) $(S_T^3 - K)^+$

i) 1) $\frac{dS}{S} = r\,dt + \sigma\,dW^Q$

$$dS^3 = 3S^2 dS + \frac{1}{2} 3\cdot 2 S\sigma^2 S^2$$

$$= 3S^3(r\,dt + \sigma\,dW^Q) + 3S^3\sigma^2 dt$$

$$\frac{dS^3}{S^3} = 3(r+\sigma^2)\,dt + 3\sigma\,dW^Q =$$

$$= [r + \underbrace{(2r+3\sigma^2)}_{-q \text{ dividend}}]dt + 3\sigma\,dW^Q$$

$$\mathrm{Call}_t = S_t^3 e^{+(2r+3\sigma^2)(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

$$d_{1,2} = \frac{\ln\left(\frac{S_t^3}{K}\right) + \left(3r + 3\sigma^2 \pm \frac{9}{2}\sigma^2\right)(T-t)}{3\sigma\sqrt{T-t}} = \begin{cases} \dfrac{\ln\left(\frac{S_t^3}{K}\right) + \left(3r + \frac{15}{2}\sigma^2\right)(T-t)}{3\sigma\sqrt{T-t}} \\[2ex] \dfrac{\ln\left(\frac{S_t^3}{K}\right) + \left(3r - \frac{3}{2}\sigma^2\right)(T-t)}{3\sigma\sqrt{T-t}} \end{cases}$$

2) Price (cube call) $= \int_{-\infty}^{+\infty} (S_T^3 - K)^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy =$

$$= \int_{\hat y}^{+\infty} S_0^3 \exp\left(3\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y\right)\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy - Ke^{-rT}\Phi(d_2)$$

$$S_0^3\exp(3(\quad)) > K$$

$$\exp(\quad) > \frac{K}{S_0^3}$$

$$\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y > \ln\left(\frac{K}{S_0^3}\right)\frac{1}{3}$$

$$y > \frac{\ln\left(\frac{K}{S_0^3}\right)\frac{1}{3} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\frac{1}{3}\left(\ln\left(\frac{K}{S_0^3}\right) - 3\left(r - \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}} =$$

$$= \frac{\ln\left(\frac{K}{S_0^3}\right) - \left(3r - \frac{3\sigma^2}{2}\right)T}{3\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\left(\frac{S_0^3}{K}\right) + \left(3r - \frac{3\sigma^2}{2}\right)T}{3\sigma\sqrt{T}}$$

$$= S_0^3 \exp\left[\left(3r - \frac{3\sigma^2}{2}\right)T\right]\int_{\hat y}^{+\infty}\frac{e^{-\frac{1}{2}(6\sigma\sqrt{T}y - y^2)}}{\sqrt{2\pi}}\,dy = S_0^3 e^{\left(3r - \frac{3\sigma^2}{2}\right)T} e^{\frac{9\sigma^2 T}{2}}\int_{\hat y}^{+\infty}\frac{e^{\overbrace{-\frac{1}{2}\left(9\sigma^2 T - 6\sigma\sqrt{T}y + y^2\right)}^{-\frac{1}{2}(3\sigma\sqrt{T} - y)^2}}}{\sqrt{2\pi}}\,dy =$$

$$= S_0^3 e^{(3r + 3\sigma^2)T}\int_{\hat y}^{+\infty}\frac{e^{-\frac{1}{2}(3\sigma\sqrt{T} - y)^2}}{\sqrt{2\pi}}\,dy =$$

$$= S_0^3 e^{(3r+3\sigma^2)T}\,\Phi(-\hat y + 3\sigma\sqrt{T})$$

$$-3\sigma\sqrt{T} + y = z$$

$$\rightarrow d_1 = d_2 + 3\sigma\sqrt{T}$$

$$d_1 = \frac{\ln\left(\frac{S_0^3}{K}\right) + \left(3r - \frac{3\sigma^2}{2}\right)T}{3\sigma\sqrt{T}} + 3\sigma\sqrt{T} =$$

$$= \frac{\ln\left(\frac{S_0^3}{K}\right) + \left(3r - \frac{3\sigma^2}{2}T\right) + 9\sigma^2 T}{3\sigma\sqrt{T}} =$$

$$= \frac{\ln\left(\frac{S_0^3}{K}\right) + \left(3r + \frac{15\sigma^2}{2}T\right)}{3\sigma\sqrt{T}}$$

ii) $\mathrm{Call}_t = S_t^3 e^{+(2r+3\sigma^2)(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$

$$d_1 = \frac{\ln\left(\frac{S_t^3}{K}\right) + \left(3r + \frac{15}{2}\sigma^2\right)T-t}{3\sigma\sqrt{T-t}}, \qquad d_2 = \frac{\ln\left(\frac{S_t^3}{K}\right) + \left(3r - 3\frac{\sigma^2}{2}\right)(T-t)}{3\sigma\sqrt{T-t}}$$

Cube ... $2\,(2r+3\sigma^2)(T-t)$ ... $(2r+3\sigma^2)(T-t)$ ... d_1^2

$3\sigma\sqrt{T-t}$ $3\sigma\sqrt{T-t}$

$$\Delta^{cube}_{call} = 3S_t^2 e^{(2r+3\sigma^2)(T-t)}\Phi(d_1) + S_t^3 e^{(2r+3\sigma^2)(T-t)} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \frac{1}{\cancel{3}\sigma\sqrt{T-t}} \frac{1}{S_t^{\cancel{3}}} \cancel{3}\cancel{S_t^2} +$$

$$- Ke^{-r(T-t)} \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \frac{1}{3\sigma\sqrt{T-t}} \frac{1}{S_t^{3}} \cancel{3S_t^2} = 3S_t^2 e^{(2r+3\sigma^2)(T-t)}\Phi(d_1) + \dots$$

$$d_1 = d_2 + 3\sigma\sqrt{T-t}$$

$$d_1^2 = d_2^2 + 9\sigma^2(T-t) + 6\sigma d_2\sqrt{T-t}$$

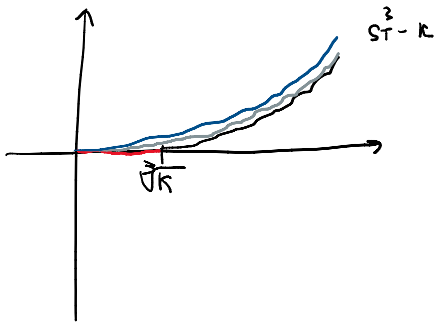
$$-\frac{d_1^2}{2} = -\frac{d_2^2}{2} - \frac{9\sigma^2(T-t)}{2} - 3\sigma d_2\sqrt{T-t} = -\frac{d_2^2}{2} - \frac{9\sigma^2}{2}(T-t) - \ln\left(\frac{S_t^3}{K}\right) - \left(3r - \frac{3}{2}\sigma^2\right)(T-t) =$$

$$= -\frac{d_2^2}{2} - 3\sigma^2(T-t) - \ln\left(\frac{S_t^3}{K}\right) - 3r(T-t)$$

$$\dots + \frac{S_t^2 e^{(2r+3\sigma^2)(T-t)}}{\sqrt{2\pi}\,\sigma(T-t)} e^{-\frac{d_2^2}{2}} e^{-3\sigma^2(T-t)} \frac{K}{S_t^3} e^{-3r(T-t)} - \frac{Ke^{-r(T-t)}}{S_t} \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{T-t}} =$$

$$= 3S_t^2 e^{(2r+3\sigma^2)(T-t)}\Phi(d_1)$$

iii) price



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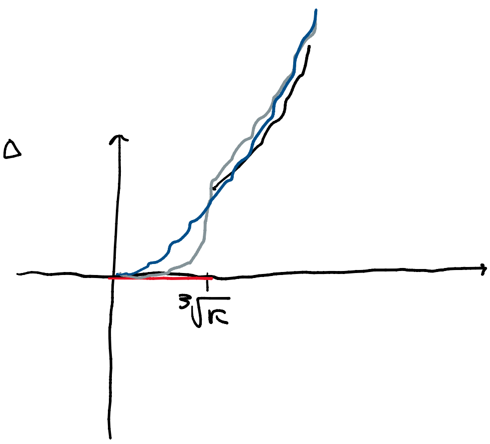

Ex4) Price $t \le T$

$F(S_T) = (S_T^3 - K)^+$ DOWN and OUT

$$price_{LO}(t,S,F) = \left[price(t,S,G_L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} price\left(t,\frac{L^2}{S},G_L\right)\right]\mathbb{1}_{S>L}$$

$$\tilde{r} = r - \frac{1}{2}\sigma^2, \quad G_L(x) = G(x)\mathbb{1}_{x>L}$$

$$G_L = (S_T^3 - K)^+\mathbb{1}_{x>L} = (S_T^3-K)\mathbb{1}_{S^3>K}\mathbb{1}_{S>L} = (S_T^3-K)\mathbb{1}_{S>\sqrt[3]{K}}\mathbb{1}_{S>L}$$

$L < \sqrt[3]{K}$

$$(S_T^3 - K)\mathbb{1}_{S>\sqrt[3]{K}} = (S_T^3-K)^+$$

$L > \sqrt[3]{K}$

$$(S_T^3-K)\mathbb{1}_{S>\sqrt[3]{K}}\mathbb{1}_{S>L} =$$

$$= (S_T^3-K)\mathbb{1}_{S>L} = (S_T^3-L^3)^+ + (L^3-K)\mathbb{1}_{S>L}$$

$L < \sqrt[3]{K}$

$$Call_{LO}(t,S,K) = \left[Call^{cube}(t,S,K) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} Call^{cube}\left(t,\frac{L^2}{S},K\right)\right]$$

$L > \sqrt[3]{K}$

$$Call_{LO}(t,S,K) = \Big[Call^{cube}(t,S,L^3) + (L^3-K)H(t,S,L) +$$

$$-\left(\frac{L}{S}\right)^{\frac{2\hat{r}}{\sigma^2}}\left[Call(t,L,L^3) + (L^3-K)H(t,L^2,L)\right]\mathbb{1}$$

$\text{Call}_{DO}(t,S,L) = \left[\text{Call}\quad (t,S,L) + (L-K)\,H(t,S,L) + \right.$

$$-\left(\frac{L}{S}\right)^{\frac{2\hat{\lambda}}{\sigma^2}}\left[\text{Call}\left(t,\frac{L}{S},L^3\right) + (L^3-K)\,H\left(t,\frac{L^2}{S},L\right)\right]\mathbb{1}_{S_T>L}$$

Delta : differentiate blocks

EX4

$$\begin{cases} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} + x^2\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + y x^n = 0 \\ F(T,x,y) = y x^2 \end{cases}$$

- No mixed derivatives
- No discounting
- Multidimensional Feynmann-Kač

The solution is

$$F(t,x,y) = E_t[Y_T X_T^2] + \int_t^T E_t[Y_s X_s^n]\,ds$$

provided that if

$dX_t = \mu_x dt + \sigma_x dW_1$, $\left(\sigma_x \frac{\partial F}{\partial x}\right) \in \mathcal{H}$

$dY_t = \mu_y dt + \sigma_y dW_2$, $\left(\sigma_y \frac{\partial F}{\partial y}\right) \in \mathcal{H}$

W_1, W_2 B.m.

From the PDE we get

$$dX = \overset{\sigma}{\sqrt{2}}\,x\,dW_1 \xrightarrow{GBB} X_T = X_t \exp\left(-(T-t) + \sqrt{2}(W_T{}' - W_t{}')\right)$$

$$dY = dt + \sqrt{2}\,dW_2 \longrightarrow Y_T = Y_t + (T-t) + \sqrt{2}(W_T{}^2 - W_t{}^2)$$

$$E[X_T^2 Y_T] \overset{\text{indip}}{=} E[X_T^2]\,E[Y_T]$$

$$E[X_T^2] = X_t^2 \exp(-2(T-t))\, E_t\left[\exp\left(\underbrace{2\sqrt{2}(W_T{}' - W_t{}')}_{N[0,\ 4\cdot 2(T-t)]}\right)\right] =$$

$$= X_t^2 \exp(-2(T-t))\exp(4(T-t)) = X_t^2\exp(2(T-t))$$

$$E[Y_T] \overset{\text{BM prop}}{=} y_t + (T-t)$$

$$X_s^n = X_t^n \exp\left(-(s-t)n + \overbrace{n\sqrt{2}(W_s{}' - W_t{}')}^{N(0,\,n^2 2(s-t))}\right)$$

$$E_t[X_s^n] = X_t^n \exp(-n(s-t))\exp(n^2(s-t))$$

$$E_t[Y_s] = y_t + (s-t)$$

$$\int_t^T E_t[X_t^n Y_s]\,ds = X_t^n\left[y_t\int_t^T \exp((s-t)(n^2-n))\,ds + \int_t^T (s-t)\exp((s-t)(n^2-n))\,ds\right] =$$

$$= \frac{x_t^n}{(n^2-n)}\left[y_t\left(\exp((T-t)(n^2-n))-1\right) + (T-t)\exp((T-t)(n^2-n)) - \frac{1}{n^2-n}\left(\exp((T-t)(n^2-n))-1\right)\right]$$

$$F(t,x,y) = x^2\exp(2(T-t))\,y + x^2\exp(2(T-t))\,y(T-t) +$$

$$+ \frac{x^n}{(n^2-n)}\left(y\exp((T-t)(n^2-n)) - 1 + (T-t)\exp((T-t)(n^2-n)) - \frac{1}{n^2-n}\left(\exp((T-t)(n^2-n))-1\right)\right]$$

- Satisfies PDE

$$\frac{\partial F}{\partial t} = -2x^2\exp(2(T-t))y - x^2\exp(2(T-t))y - 2x^2\exp(2(T-t))y(T-t) +$$

$$+ \frac{x^n}{n^2-n}\left[-y(n^2-n)\exp(T-t) - \exp((T-t)(n^2-n)) - (T-t)(n^2-n)\exp(\qquad) + \exp((T-t)(n^2-n))\right]$$

$$= x^n y\left[-\exp(T-t) - \frac{\exp((T-t)(n^2-n))}{(n^2-n)} - (T-t)\exp((T-t)(n^2-n)) + \frac{\exp((T-t)(n^2-n))}{n^2-n}\right]$$

$$\frac{\partial F}{\partial x} = 2x\exp(2(\quad))y + 2x\exp(\quad)\,y(T-t) + \frac{n x^{n-1} y}{(n^2-n)}\left[\qquad\right]$$

$$\frac{\partial^2 F}{\partial x^2} = 2\exp(\qquad)y + 2\exp(\quad)y(T-t) + \frac{\cancel{n(n-1)}\,x^{n-2}y}{\cancel{(n^2-n)}}\left[\qquad\right]$$

$$\frac{\partial^2 F}{\partial x^2} = 2\exp(\quad)y + 2\exp(\quad)y(T-t) + \frac{n(n-1)}{(n^2-n)} x^{n-2}y\Big[\quad\Big]$$

$$\frac{\partial F}{\partial y} = x^2\exp(\quad) + x^2\exp(\quad)(T-t) + \frac{x^n}{n^2-n}\Big[\quad\Big]$$

$$\frac{\partial F}{\partial y^2} = 0$$

CONTI NO! no idea OK

Satisfies the PDE ✓

$$\int_0^T \mathbb{E}\left(\frac{\partial F}{\partial x}(s, X_s, Y_s)\cdot X_s\right)^2 ds = \int_0^T k e^{s\cdot c}\, ds < +\infty$$

$$\int_0^T \mathbb{E}\left(\frac{\partial F}{\partial y}(s, X_s, Y_s)\right)^2 ds = \int_0^T k e^{s\cdot c}\, ds < +\infty$$