Stochastic Methods for Finance

Exercises

Exercice 1 *Consider a standard Brownian motion B_t.*

i) Find the real parameter α such that the process $\ln(1+\alpha B_t^2)$ is a martingale with respect to the natural filtration;

ii) Find the real parameter α such that the process $\exp(1+\alpha B_t^2)$ is a martingale with respect to the natural filtration;

iii) Compute the expected value at time $t=2$ of the process X satisfying $dX_t = -X_t dt + 4dB_t$, $X_0 = 1$;

iv) Compute the variance of the random variable X_2;

v) Compute the quadratic covariation between the processes X and B;

vi) Show that the process $Y = B^4 - B^2$ is a sub-martingale;

vii) Find the Doob-Meyer decomposition of the process Y.

Exercice 2 *Consider a Black&Scholes market and a derivative contract with payoff at the maturity T given by*

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$$\begin{array}{cl} 0 & if\ S_T < K_1; \\ K_2 - S_T & if\ K_1 < S_T < K_2; \\ K_2 - K_1 & if\ S_T > K_2, \end{array}$$

where $0 < K_1 < K_2$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 3 *Dans le modéle de Black-Scholes, on considère un actif risqué S:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Soit t une date fixé, $0 \leq t \leq T$; une option Forward-Starting donne à la maturité T le payoff

$$Payoff_T^{FwCall} = \left(\frac{S_T}{S_t} - K\right)^+$$

i) Déterminez à la date 0 le prix de cette option dans le cas où le taux d'intéret est constant, puis déterminer la stratégie de couverture.

ii) Me͡me question dans le cas où le taux d'intéret est stochastique. Expliciter le prix si le taux est du type Vasicek.

Exercice 4 *Consider the process $X_t = \int_0^t e^{\pi} dB_s$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*

i) Find the probability measure $\mathbb{P}^1$ under which the process X becomes a martingale;

ii) Consider now the process $Y_t = \exp(X_t)$: show that it is not a martingale under the probability measure $\mathbb{P}$;

iii) Find the probability measure $\mathbb{P}^2$ under which Y becomes a $\mathbb{P}^2$-martingale;

iv) Compute $\mathbb{E}^{\mathbb{P}^1}[X_2]$;

v) Compute $\mathbb{E}^{\mathbb{P}^2}[X_2]$.

Exercice 5 *Consider a Black&Scholes market and a derivative contract with payoff at the maturity T given by*

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$$\begin{cases} a & if\ S_T < K; \\ K + a - S_T & if\ K < S_T < K + a; \\ b & if\ S_T > K + a, \end{cases}$$

where $0 < a < b < K$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility.

Exercice 6 *A risky asset S in a 2-period binomial model (one period =1 year) evolves according to an increasing factor of $u_1 = 1,1$ (resp. decreasing factor of $d_1 = 0,9$) for the first period, and $u_1 = 1,05$ (resp. decreasing factor of $d_1 = 0,95$) for the second one, starting from the initial price $S_0 = 100$. The riskless interest rate is flat at $r = 1\%$ per year.*

i) Find the initial price and the hedging strategy of a European PUT option on S with maturity $T =$ 2 years and strike price $K = 95$;

ii) Find the initial price and the hedging strategy of an American PUT option on S with maturity $T =$ 2 years and strike price $K = 99$;

iii) Same question at point ii) when the interest rate is $r_1 = 1\%$ per year for the first period and on the period $[0, 2years]$ is $r_2 = 1,5\%$ per year.

NO

Exercice 7 *Dans le modéle de Black-Scholes, on considère un actif risqué S:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,.$$

et un taux d'intéret constant r. Déterminez le prix d'une stratégie STRADDLE down-and-out, où on a le droit au payoff si l'actif ne touche pas la barrière L, puis le comparer avec le prix du contrat corréspondent sans la présence de la barrière.

Exercice 8 *Soient*

- $X(t)$: *la valeur á l'instant t en unité monétaire domestique d'une unité monétaire étrangère,*
- r_d: *le taux d'intérêt domestique,*
- r_f: *le taux d'intérêt étranger.*

On suppose que la dynamique de X, sous la probabilité réelle, est donnée par

$$dX(t) = X(t)\mu dt + X(t)\sigma dW_t$$

où W est un mouvement brownien standard et où μ et σ sont deux constantes positives.

1. *Déterminez la dynamique du taux de change X sous la probabilité risque-neutre.*
2. *Trouvez le prix d'un contrat STRADDLE sur X, de maturité T, et de prix d'exercice K.*

Exercice 9 *Consider the process $X_t = \exp(B_t - t)$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*
i) Show that the process X is not a martingale under $\mathbb{P}$;
ii) Find the probability measure $\mathbb{P}^1$ under which the process X becomes a martingale;
iii) Compute $\mathbb{E}^{\mathbb{P}^1}[X_2]$;
iv) Consider now the process $Y_t = \exp(\int_0^t f(u)du + B_t)$, where f is a function of time: show that it is not a martingale under the probability measure $\mathbb{P}$;
v) Find the condition under which there exixts a probability measure $\mathbb{P}^2$ such that the process Y becomes a $\mathbb{P}^2$-martingale;
vi) Compute $\mathbb{E}^{\mathbb{P}^2}[Y_2]$ when $f(t) = t$.

Exercice 10 *Résoudre l'EDP suivante*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2\alpha^2\frac{\partial^2 F}{\partial x^2} + \beta F + \gamma x^\delta &= 0 \\ F(T,x) &= 2x^3, \end{aligned}$$

et vérifier que la solution satisfait l'EDP dans le cas $\delta = 1$.

Exercice 11 *Soit S le prix d'un actif dont la dynamique sous la probabilité risque neutre $\mathbb{Q}$ est la suivante*

$$\frac{dS(t)}{S(t)} = r_t dt + \sigma dW_t,$$

où le taux d'intérêt r_t est stochastique et suive la dynamique (risque neutre) de Vasicek:

$$dr_t = (a - br_t)\,dt + \sigma_r dW_t,$$

où W est le même Brownien de l'actif. Le but de cet exercice est d'évaluer le prix d'un contrat Butterfly Spread dans ce contexte.

1. *Décomposez le payoff d'un butterfly spread a l'aide des contrats CALL*

2. *Introduisez la probabilité Forward associée aux maturités concernées*

3. *Determinez le prix du contrat en utilisant la formule de Black et Scholes d'aprés un changement de mesure.*

Exercice 12 *(Quanto Option) Soient $X(t)$ la valeur á l'instant t en unité monétaire domestique d'une unité monétaire étrangère, r_d le taux d'intérêt domestique, r_f le taux d'intérêt étranger. On suppose que la dynamique de X, sous la probabilité risque neutre domestique, est donnée par*

$$\frac{dX(t)}{X(t)} = (r_d - r_f)\,dt + \sigma_x dW_t^1$$

où W^1 est un mouvement brownien standard et σ est une constante positive. Soit aussi S^f le prix d'un actif étranger suivant

$$\frac{dS^f(t)}{S^f(t)} = r_f dt + \sigma_f dW_t^2,$$

où W^1 est un mouvement brownien standard indépendant de W^2.

1. *Déterminez la dynamique du taux de change $\frac{1}{X}$ sous la probabilité risque-neutre étrangère.*

2. *Trouvez le prix en unité monétaire étrangère d'un contrat CALL sur S^f de maturité T , où le prix d'exercice K est exprimé en unité monétaire domestique.*

NO

Exercice 13 *(Option barrière) Dans le modèle de Black-Scholes, en utilisant les propriétés de symétrie et parité pour les contrats barrière, déterminez le prix d'une option UIP (Up and In Put dont le prix d'exercice est K), où on a le droit au payoff seulement si l'actif touche la barrière L.*

Exercice 14 *Solve the following Partial Differential Equation:*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial x^2} + 1 &= 0 \\ F(T,x) &= x^2, \end{aligned}$$

Exercice 15 *In the Black-Scholes model, find the price at time $t \leq T$ of a Digital Call UP-AND-OUT with strike price K, where the owner receives a unitary payoff at the maturity T if and only if the asset has not reached the barrier L. Compare this price with the one of a Digital Call (without barrier). Finally, find the Delta of both options*

NO

Exercice 16 *Let $X(t)$ be the value at time t in domestic currency of one monetary unit of the foreign currency, r_d the domestic interest rate, r_f the foreign interest rate.*

Suppose that the dynamics of X, under the real world probability measure, is given by

$$dX(t) = X(t)\mu dt + X(t)\sigma dW_t$$

where W is a standard Brownian motion and μ and σ are positive constants.

i) Find the price at time t of a Digital Call option expiring at time $T \geq t$ with strike K written on X.

ii) Find the price at time t of a Digital Call option expiring at time $T \geq t$ with strike K written on $1/X$.

iii) Give an interpretation of $1/X$ in terms of X.

Exercice 17 *A risky asset S in a 2-period binomial model (one period =1 year) evolves according to an increasing rate of 10% (resp. decreasing rate of −10%), starting from the initial price $S_0 = 100$. The riskless interest rate is $r = 1\%$. The trader has a portfolio consisting in*

i) 5 short positions in a European Call on S with maturity $T = 2$ years and strike price $K_1 = 95$;

ii) 10 short positions in a American Put on S with maturity $T = 2$ years and strike price $K_2 = 100$;

iii) 5 short positions in a European Call on S with maturity $T = 1$ year and strike price $K_3 = 98$.

Find the position that the trader has to take in a European Call with maturity $T = 1$ year and strike price $K_4 = 90$ in order to obtain a Delta-neutral portfolio at time 0.

Exercice 18 *Consider a standard Brownian motion B_t.*

i) Find the real parameter α such that the process $\exp(1 + \alpha B_t^2 + B_t(\alpha - 1))$ is a submartingale with respect to the natural filtration;

ii) For the values of α of point i) find the corresponding Doob-Meyer decomposition; chiedere

iii) Compute the expected value at time $t = 2$ of the process $Y = X^3$, where X satisfies the SDE $dX_t = -X_t dt + \sigma X_t dB_t$, $X_0 = 1$ ($\sigma > 0$); CliCK

iv) Compute the variance of the random variable Y_2;

v) Compute the quadratic covariation between the processes X and Y;

vi) Compute the values of σ for which the process Y is a submartingale and find the corresponding Doob-Meyer decomposition.

Exercice 19 *In a Black&Scholes market let consider a risky asset evolving according to the risk neutral dynamics:*

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t$$
$$S_0 = s,$$

Consider the option paying the following payoff: $Payoff_T = min[S_T, K]$.

i) Find the price of this contract at a generic time $t \leq T$;
ii) Compute the Delta and the Vega of this contract;
iii) Give an illustration of the Delta and the Gamma for the contract when the volatility parameter has an upward shock from σ to $\hat{\sigma} > \sigma$.

Exercice 20 *Consider a Black&Scholes market as in the previous exercise. Assume that $c_1, c_2 > 0$, $K_1 < K_2 < K_3$ and consider the following payoff at the maturity T:*

$$\begin{cases} c_1 & for\ S_T \in [0, K_1); \\ K_1 - S_T + c_1 & for\ S_T \in [K_1, K_2]; \\ -c_2 & for\ S_T \in (K_2, K_3]; \\ -c_2 + S_T - K_3 & for\ S_t \in (K_3, +\infty) \end{cases}$$

i) Compute the price of the contract at any time $t \in [0, +\infty)$; ii) Compute the Delta and the Gamma of the contract;
iii) Give an illustration of the shape of the price and the Delta of the contract for an upward shock of the volatility from σ to $\hat{\sigma} > \sigma$.

Exercice 21 *Consider a standard Brownian motion B_t.*
i) Find the real parameter α such that the process $\ln(1 + \alpha B_t)$ is a martingale with respect to the natural filtration;
ii) Find the real parameter α such that the process $\exp(1 + \alpha B_t)$ is a martingale with respect to the natural filtration;
iii) Compute the expected value of the process X satisfying $dX_t = -X_t dt + 4dB_t$, $X_0 = 1$;
iv) Compute the variance of the random variable X_2;
v) Compute the quadratic covariation between the processes X and B;
vi) Show that the process $Y = B^4 - B^2$ is a sub-martingale;
vii) Find the Doob-Meyer decomposition of the process Y.

Exercice 22 *Consider the process $\exp(2B_t)$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*
i) Show that the process $\exp(2B_t)$ is not a martingale under $\mathbb{P}$;
ii) Find the probability measure $\mathbb{P}^1$ under which the process $\exp(2B_t)$ becomes a martingale;
iii) Compute $\mathbb{E}^{\mathbb{P}^1}[\exp(2B_t)]$;
iv) Consider now the process $\exp(\int_0^t u dB_u)$: show that it is not a martingale under the probability measure $\mathbb{P}$;
v) Find if there exixts a probability measure $\mathbb{P}^2$ under which the process $\exp(\int_0^t u dB_u)$ becomes a martingale.

Exercice 23 *Consider a Black&Scholes market where a risky asset evolves according to*

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_0 = s,$$

and a riskless asset is associated to the risk free rate r. Consider a Static Butterfly option, that is a derivative contract with payoff at the maturity T given by

$$\begin{array}{ll} a & if\ S_T < K_1; \\ b & if\ K_1 < S_T < K_2; \\ a & if\ S_T > K_2, \end{array}$$

where $0 < a < b$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$;

ii) Compute the Delta and the Gamma of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility σ.

Exercice 24 *Consider a standard Brownian motion B_t.*

i) Find the real parameter α such that the process $\exp(2B_t - \alpha t)$ is a martingale with respect to the natural filtration;

ii) Find the real parameter α such that the process $\exp(2B_t - \alpha t)$ is a supermartingale with respect to the natural filtration;

iii) Compute the expected value of the process $\exp(\int_0^t B_u du)$;

iv) Compute the variance of the process $\exp(\int_0^t B_u du)$ for t fixed;

v) Compute the quadratic covariation between the processes $\exp(\int_0^t B_u du)$ and B_t; ?

vi) Find the Doob-Meyer decomposition of the submartingale $B_t^2 + t$.

Exercice 25 *Consider the process $\exp(B_t)$ where B_t is a standard Brownian motion under the probability measure $\mathbb{P}$.*

i) Show that the process $\exp(B_t)$ is not a martingale under $\mathbb{P}$;

ii) Find the probability measure $\mathbb{P}^1$ under which the process $\exp(B_t)$ becomes a martingale;

iii) Compute $\mathbb{E}^{\mathbb{P}^1}[\exp(B_t)]$;

iv) Consider now the process $\exp(\int_0^t B_u du)$: show that it is not a martingale under the probability measure $\mathbb{P}$;

v) Find if there exixts a probability measure $\mathbb{P}^2$ under which the process $\exp(\int_0^t B_u du)$ becomes a martingale.

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Exercice 26 *Consider a Black&Scholes market where a risky asset evolves according to*

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$
$$S_0 = s,$$

and a riskless asset is associated to the risk free rate r. Consider a Purple Collar option, that is a derivative contract with payoff at the maturity T given by

$$\begin{array}{ll} a & if\ S_T < K_1; \\ S_T & if\ K_1 < S_T < K_2; \\ K_2 & if\ S_T > K_2, \end{array}$$

where $K_1 < a < K_2$.

i) Compute the price of the contract at any time $t \in [0, +\infty)$*;*

ii) Compute the Delta and the Gamma of the contract;

iii) Illustrate graphically the change of price and Delta for a upward shift of the volatility σ.