22 Sept 2017

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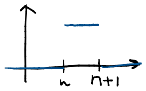
Exercise 1

$F(n,S_T) = \underbrace{\mathbb{1}_{n < S_T < n+1}}$ indicator function

Fixed n



i) $\text{PAYOFF}_T = \underbrace{\mathbb{1}_{S_T > n}}_{\text{digital}} - \mathbb{1}_{S_T > n+1}$ 2 digital

$\text{Dig}_t = e^{-r(T-t)} E^Q[\mathbb{1}_{S_T > K}] = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} \mathbb{1}_{S_T > K}\, dy$

$S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}\, y\right) > K$

$\rightarrow y = \dfrac{\ln\left(\frac{K}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

$d_2 = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

$\text{price}_t = e^{-r(T-t)} \Phi(d_2^n) - e^{-r(T-t)} \Phi(d_2^{n+1}) =$

$= e^{-r(T-t)} \left[\Phi(d_2^n) - \Phi(d_2^{n+1})\right] =$

$= e^{-r(T-t)} \int_{d_2^{n+1}}^{d_2^n} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy \longrightarrow 0$

$d_2^{n, n+1} = \dfrac{\ln\left(\frac{S_t}{n, n+1}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \underset{\sim}{\overset{n \to +\infty}{}} -\ln n \cdot \text{const}(r, \sigma, T, t)$

$\int_{d_2^{n+1}}^{d_2^n} \sim = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} (d_2^n - d_2^{n+1}) \sim \frac{e^{-(\log n)^2/2}}{\sqrt{2\pi}} (\ln(n+1) - \ln(n)) =$

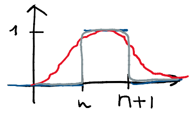
$= \frac{e^{-(\log n)^2/2}}{\sqrt{2\pi}} \log\left(\frac{n+1}{n}\right) \xrightarrow{n \to +\infty} 0$

Multiplication by a constant doesn't change the limit

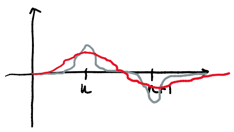
ii) $\text{Delta}^{DIG} = e^{-r(T-t)} \frac{e^{-\frac{(d_2^n)^2}{2}}}{S\sigma\sqrt{T-t}} - e^{-r(T-t)} \frac{e^{-\frac{(d_2^{n+1})^2}{2}}}{S\sigma\sqrt{T-t}}$

$= \frac{e^{-r(T-t)}}{S\sigma\sqrt{T-t}} \left[e^{-\frac{(d_2^n)^2}{2}} - e^{-\frac{(d_2^{n+1})^2}{2}}\right] \xrightarrow[\text{def } d_2]{} 0$

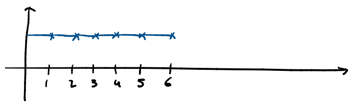
iii) price_t $\sigma_2 >> \sigma_1$



delta



iv) $F(S_T) = \sum_{n=0}^{+\infty} F(n, S_T) = 1$



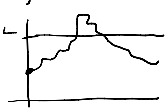
$\text{price}_t(F(S_T)) = e^{-r(T-t)}$

v) F initial portfolio
F + x Call (1)

$0 = \Delta^F + x\,\Delta^{Call(1)}$

$\Delta^F = 0 \rightarrow x\Delta^{Call(1)} = 0 \rightarrow x = 0$

EX 2) UP and IN CONTRACT



$F(n, S_T) = \mathbb{1}_{n < S_T < n+1}$

$F(n, S_T) = \mathbb{1}_{S_T > n} - \mathbb{1}_{S_T > n+1}$

$\overset{!}{=} H(S_T, n) - H(S_T, n+1) = \phi(n, S_T)$

$\text{price}^{LI} = \text{price} - \text{price}^{LO}$

$$\text{price}^{LI}(t,S,\phi) = \text{price}(t,S,\phi_L) + \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \text{price}\left(t, \frac{L^2}{S}, \phi^L\right)$$

$$\phi_L = \left(\mathbb{1}_{n<S_T<n+1}\right)\mathbb{1}_{S_T>L}, \qquad \Phi^L = \left(\mathbb{1}_{n<S_T<n+1}\right)\mathbb{1}_{S_T<L}$$

ϕ_L

$L>n+1$, $\text{price}(t,S,\phi_L)=0$
$L<n$, $\text{price}(t,S,\phi_L) - \text{price}(t,S,\phi)$

$n<L<n+1$, $\mathbb{1}_{L<S_T<n+1} \rightarrow$

$\mathbb{1}_{S_T>L} - \mathbb{1}_{S_T>n+1}$

$\text{price}(t,S,\phi_L) = H(t,S,L) - H(t,S,n+1)$

- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T>L} = \mathbb{1}_{S_T>L} - \mathbb{1}_{S_T>L} = 0$
- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T>L} = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1}$
- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T>L} = \mathbb{1}_{S_T>L} - \mathbb{1}_{S_T>n+1}$

$\phi^L, \phi\mathbb{1}_{x<L}$ $\qquad \text{price}(t,x,\phi^L)$

$n>L$, $\text{price} \equiv 0$
$n+1<L$, $\text{price}(t,x,\phi)$

$n<L<n+1$, $\mathbb{1}_{n<x<L} = \mathbb{1}_{x>n} - \mathbb{1}_{x>L}$ $\qquad H(t,x,n) - H(t,x,L)$

UP

- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T<L} = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1} - (\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1}) = 0$
- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T<L} = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1} - 0 = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1}$
- $(\mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1})\mathbb{1}_{S_T<L} = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>n+1} - (\mathbb{1}_{S_T>L} - \mathbb{1}_{S_T>n+1}) = \mathbb{1}_{S_T>n} - \mathbb{1}_{S_T>L}$

$$\text{price}^{LI}(t,S,\phi) = \text{price}(t,S,\phi_L) + \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \text{price}\left(t, \frac{L^2}{S}, \phi^L\right)$$

- $L>n+1$ $\quad \text{price}^{LI}(t,S,\phi) = \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}}\left(H\left(t,\frac{L^2}{S},n\right) - H\left(t,\frac{L^2}{S},n+1\right)\right)$ *
- $L<n$ $\quad \text{price}^{LI}(t,S,\phi) = H(t,S,n) - H(t,S,n+1)$
- $n<L<n+1$ $\quad \text{price}^{LI}(t,S,\phi) = H(t,S,L) - H(t,S,n+1) + \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}}\left(H\left(t,\frac{L^2}{S},n\right) - H\left(t,\frac{L^2}{S},L\right)\right)$

If $n \to +\infty$, then surely $L<n+1 \longrightarrow$ * goes to 0

Delta: differentiate under S the building blocks

EX 3)

$$\begin{cases} \frac{\partial F}{\partial t} + x^2\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + x^n = 0 \\ F(T,x,y) = yx^2 \end{cases}$$

- $dx = \sqrt{2}\, x\, dW_1$ $\quad W_1 \perp\!\!\!\perp W_2$
  $dy = \sqrt{2}\, dW_2$

GBM
$X_T = x_t \exp\left(-(T-t) + (W_T^1 - W_t^1)\sqrt{2}\right)$
$Y_T = \sqrt{2}(W_T^2 - W_t^2) + y_t$

under right assumptions

$$F(t,x,y) = E_t[Y_T X_T^2] + \int_t^T E_t[X_s^n]\,ds$$

$$E_t[Y_T]\,E_t[X_T^2] = y_t\left[x_t^2 e^{-2(T-t)} E\left[e^{\overbrace{2\sqrt{2}(W_T^1 - W_t^1)}^{8T-t}}\right]\right] =$$
$$= y_t x_t^2 e^{-2(T-t)} e^{4(T-t)} = y_t x_t^2 e^{2(T-t)}$$

$$\int_t^T x_s^n e^{-n(s-t)+n^2(s-t)}\,ds = \int_t^T x_s^n e^{(s-t)(n^2-n)}\,ds = x_s^n \left.\frac{e^{(s-t)(n^2-n)}}{n^2-n}\right|_t^T = \frac{x^n}{n^2-n}\left[e^{(T-t)(n^2-n)} - 1\right]$$

$$F(t,x,y) = yx^2 e^{2(T-t)} + \frac{x^n}{n^2-n}\left[e^{(T-t)(n^2-n)} - 1\right]$$

$F(T,x,y) = yx^2$ ✓

$$\frac{\partial F}{\partial t} = -yx^2 2e^{(T-t)} - x^n e^{(T-t)(n^2-n)}$$

$$\frac{\partial F^2}{\partial x^2} = 2ye^{2(T-t)} - \frac{\cancel{n(n-1)}}{\cancel{n^2-n}}\, x^{n-2}\left[1 - e^{(T-t)(n^2-n)}\right]$$

$$\frac{\partial F^2}{\partial y^2} = 0$$

$$\rightarrow -2yx^2\cancel{e^{(T-t)}} - x^n\cancel{e^{(T-t)(n^2-n)}} + \cancel{2yx^2e^{2(T-t)}} - x^n + \cancel{x^n e^{\sim}} + x^n = 0 \quad ✓$$

to check

$$\left(\frac{\partial F}{\partial x}\sqrt{2}x\right) \in \mathcal{H} \longleftrightarrow \left(\frac{\partial F}{\partial x}x\right) \in \mathcal{H}, \qquad \frac{\partial F}{\partial x} = 2yxe^{2(T-t)} + \frac{x^{n-1}}{n-1}\left[e^{(T-t)(n^2-n)} - 1\right]$$

$$\frac{\partial F}{\partial x}(s,X_s)\cdot X_s = 2y_s X_s^2 e^{2(T-s)} + \frac{X_s^{n-1}}{n-1}\left[e^{(T-s)(n^2-n)} - 1\right]$$

$$\left(\frac{\partial F}{\partial x}(s,X_s)\cdot X_s\right)^2 = 4y_s^2 X_s^4 e^{4(T-s)} + X_s^{2(n-1)}\left[e^{2(T-s)(n^2-n)} + 1 - 2e^{(T-s)(n^2-n)}\right] +$$

$$\frac{\partial F}{\partial x}(s, x_s) \cdot x_s = 2 y_s x_s^2 e^{2(T-s)} + \frac{x_s^{n-1}}{n-1}\left[e^{(T-s)(n^2-n)} - 1\right]$$

$$\left(\frac{\partial F}{\partial x}(s, x_s) \cdot x_s\right)^2 = 4 y_s^2 x_s^4 e^{4(T-s)} + \frac{x_s^{2(n-1)}}{(n-1)^2}\left[e^{2(T-s)(n^2-n)} + 1 - 2e^{(T-s)(n^2-n)}\right] +$$
$$+ \frac{4 y_s x_s^{n+1}}{n-1} e^{2(T-s)}\left[e^{(T-s)(n^2-n)} - 1\right]$$

$$\int_0^T 4 E[y_s^2] E[x_s^4] e^{4(T-s)} + \frac{E[x_s^{2(n-1)}]}{(n-1)^2}\left[e^{2(T-s)(n^2-n)} + 1 - 2 \cdots\right] +$$
$$+ 4 E[y_s] \frac{E[x_s^{n+1}]}{n-1} e^{2(T-s)} e\left[e^{(T-s)(n^2-n)} - 1\right] =$$
$$= \int_0^T \text{const} \cdot e^{s \cdot k} \, ds < +\infty$$

$$\left(\frac{\partial F}{\partial y} \sqrt{2}\right) \in \mathcal{H} \longleftrightarrow \frac{\partial F}{\partial y} \in \mathcal{H}$$

$$\left(\frac{\partial F}{\partial y}(s, x_s)\right)^2 = \left(x_s^2 e^{2(T-t)}\right)^2 = x_s^4 e^{4(T-s)}$$

$$\int_0^T E[x_s^4] e^{4(T-s)} ds = \int_0^T e^{s \cdot \text{const}} e^{4(T-s)} ds < +\infty$$