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Introduction

The object of this investigation is to look at the usefulness of *discrete nonlinear maps* as generators of musical design. The procedure used is an heuristic exploration of the maps' solution space. Sound is generated by direct computer synthesis, with the map output used to control pitch selection, "interonset time," envelope attack time, dynamics, tempo, textural density, and section length. A series of taped examples is discussed that illustrates typical map features. (Please contact the author directly to obtain a copy of this tape. Send a bank check or money order for $7 (US) made payable to the Music Department, La Trobe University. A cassette will be returned via airmail.) The work here falls short of a fully systematic investigation. However, the results do suggest that cognitive and musical processes can be modeled with some success by such a procedure.

There is by now an extensive literature on nonlinear maps, both for continuous and discrete variables (May 1976; Mandelbrot 1980; Bai-lin 1984; Holden 1986). There is also growing activity in the production of nonlinear (e.g., *fractal*) visual art by computer (Norton 1982; Peitgen and Richter 1986), and at least one composer has integrated sound with animation on this basis (Evans 1987). The general features that make these equations potentially interesting as generators of musical design include the presence of such phenomena as *fixed points, limit cycles, bifurcations, chaos,* and *strange attractors.* These terms are discussed in more detail later, but an example illustrates the possibilities. Musical development or variation can be viewed as the transformation or distortion of a simple entity (a motive), often followed by some sort of return to the original motive. When certain values are chosen for the input parameters to these equations, very similar behavior can be obtained from them. Thus a series of solutions can act like a repeated group of n notes for a number of steps in the iteration process, and then break away to more unpredictable (quasi-chaotic) behavior before eventually returning to the original n-note group, perhaps somewhat altered.

Most of the maps used here are related to the simple but surprisingly rich *logistic* map, defined over the domain [0,1]:

$$x_{n+1} = ax_n(1 - x_n),\ 0 < a \leq 4. \qquad (1)$$

The behavior of this function is as follows. For $0 \leq a \leq 1$, all iterations converge on the fixed point $x = 0$. For $1 < a < 3$ a fixed limit point of value $1 - 1a$ attracts all initial values of x. For $a \geq 3$, this fixed point still exists, but becomes unstable, so that x points in its neighborhood are not attracted but rather repelled. This same range sees the birth of an attractive two-member *limit cycle.* A limit cycle simply means that there exist two values p and q such that substitution of p in the equation yields q, and substitution of the value q yields p. At $a = 3.449499$ the 2-cycle bifurcates to a 4-cycle, at $a = 3.544090$ this in turn splits into an 8-cycle, and so on indefinitely, creating a so-called *harmonic cascade.* This region can be shown to correspond to such physical phenomena as the onset of turbulence in a fluid (Feigenbaum 1978). This process converges by 3.569946, beyond which there are cycles of all sizes, which show further bifurcations and cascades, as well as regions of "chaos," where the output looks random, or at least highly unpredictable, even though of course it is completely deterministic. Space does not allow a full characterization of the complex properties of this map. But it is useful to refer to Table 1, adapted from (May 1976), which lists all k-cycles up to k=6.

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Table 1. Cycle information for the logistic map (Eq. 1)

Period of basic cycle (k)	*a-value* Basic cycle first appears	Basic cycle becomes unstable	Harmonic cascade becomes unstable
1	1.0000	3.0000	3.5700
3	3.8284	3.8415	3.8495
4	3.9601	3.9608	3.9612
5	3.7382	3.7411	3.7430
5	3.9056	3.9061	3.9065
5	3.99026	3.99030	3.99032
6	3.6265	3.6304	3.6327
6	3.937516	3.937596	3.937649
6	3.977760	3.977784	3.977800
6	3.997583	3.997585	3.997586

Each basic cycle operates over a limited range, and then bifurcates to produce "harmonics" of frequency $2k$, $4k$, $8k$, . . . etc. From $k = 5$ on up, the basic cycles are degenerate (occur at more than one value of a). It can be seen that the range over which a given cycle is operative soon becomes very small, and above $a = 3.5$ the behavior of Eq. (1) is extremely sensitive to the value of a. Figure 1 shows a characteristic approximation to the long-term behavior of Eq. (1) for the range $2.9 < a < 3.9$ (Lauwerier 1986a). It was obtained by beginning with $x_0 = 1/2$, and plotting iterations in the range $250 \leq n \leq 400$. The left side of the figure shows focused convergence to fixed points and limit cycles; this is the bifurcation process for $k = 1$. On the right the 3-cycle, and one 5- and one 6-cycle, can be just seen at this resolution, but other cycles are obscured by chaotic and quasi-chaotic behavior.

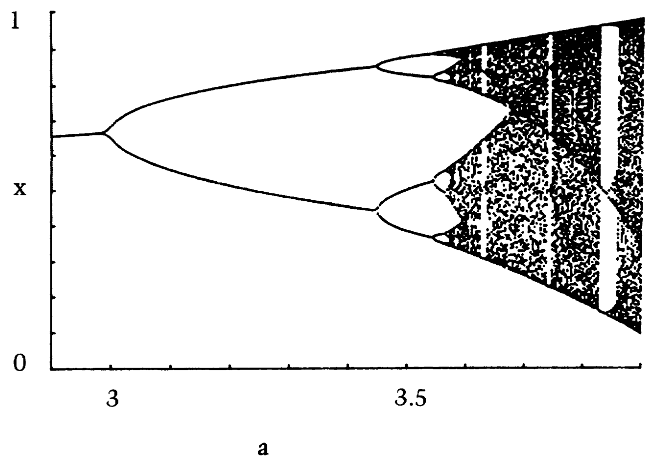
Of considerable interest for the concept of musical variation are the regions just before the onset of a new cycle. These regions are classified as chaotic, but only intermittently. In other words they are typically chaotic, then quasi-periodic, chaotic, then quasi-periodic, and so on indefinitely. The output shows unpredictability, but also some traces of the nearby cyclic behavior. The approach to the onset of pure cyclicity can be finely tuned by varying a, and can be shown to satisfy a limiting power law relation as a function of the distance from the basic cycle onset value (Hirsch, Huberman, and Scalapino 1982; Hirsch, Nauenberg, and Scalapino 1982). When this is translated into musical terms, the result can be something rather like some twentieth-century variation techniques.

An Implementation of Nonlinear Maps in Software

To actually produce sound with Eq. (1), the music synthesis language Csound, developed at M.I.T., was used. The same software synthesis instrument was employed for all examples, to allow ready audibility of map effects, minimally clouded by compositional shaping. The instrument was formed by mixing several sine tones with a plucked-string algorithm (Karplus and Strong 1983). These were gated by a linear attack-sustain-release (ASR) envelope. Score files were produced by a program written in the C language. Since all the values of x from Eq. (1) fall in the interval [0,1], it was necessary to convert this output to a range appropriate for the musical parameter being controlled. Frequency was converted as follows:

$$F = 2^{(cx+d)}, \tag{2}$$

Fig. 1. *Global bifurcation diagram for logistic map (Eq. 1) (Reproduced with permission from A. V. Holden, ed. 1986.* Chaos. *Princeton University Press.)*



where the constant c equals the range in octaves of the derived pitch set, and 2^d is the lowest pitch produced (in Hertz). For much of the work here, $c = 3.0$ and $d = 6.0$, so that the melody was limited to a three-octave range from two octaves below middle C (64 Hz) to an octave above (512 Hz). Quantization to tempered (or other) tuning norms could be applied, although for the most part this was not done, as it seemed to yield less interesting results.

Note interonset time (the time between notes) was used directly, since the range 0–1.0 expressed in beats provided a natural fit. Likewise dynamics were readily applicable after multiplication by a simple scale factor (10000 with one voice, somewhat less for more voices). The envelope attack time was computed by squaring the output of the logistic map equation, since otherwise small values that were judged interesting for the plucked component of the sound were too infrequent. Occasionally this was also then divided by a small constant in the range [2,3]. The envelope attack time was also scaled to the overall note duration (typically 0.8 beats) to promote a more unified timbre and to avoid unwanted clicks. Section size was delineated simply by using a logistic map equation to define the size (number of notes) of each section, which was governed by a newly computed tempo. Each tempo was computed by sampling the attack equation using the formula: *new tempo* $= 5 = 595x$.

In the first series of musical examples, primarily monophonic music was produced. A different logistic map equation with associated a-value was used for each parameter involved. By informed guesswork, musically interesting values of the relevant parameters were found.

Sound Example 1 uses the map to generate frequency and note time only. The chosen a-values are 3.8280 (for frequency) and 3.8283 (for note time). We designate the music sample so generated by (3.8280, 3.8283). These values are chosen to be just below the onset of the 3-cycle, which sets in at 3.8284, as can be seen from Table 1. (As mentioned previously, this and other musical examples are available on a tape from the author. A listing of the sound examples cited in this text is provided in the Appendix.) Here on the printed page a rough idea of the nature of the resulting musical design can be given by the following listing of the pitches produced from 150 iterations of Eq. (1) (with $a = 3.8280$). This list, and subsequent ones, are read left to right, with variable line lengths used to make repeating structures easier to see.

468.21	88.81	184.06	
467.97	88.96	184.93	
467.81	89.07	185.48	
467.69	89.15	185.91	
467.59	89.21	186.28	
467.50	89.28	186.62	
467.41	89.34	186.96	
467.31	89.40	187.31	
467.20	89.48	187.70	
467.07	89.56	188.17	
466.92	89.67	188.75	
466.70	89.81	189.52	
466.39	90.02	190.65	
465.90	90.35	192.48	
464.97	90.98	195.91	
462.85	92.43	203.81	
456.24	97.10	229.25	
422.50	124.76	362.81	192.29
465.07	90.91	195.52	
463.12	92.25	202.81	
457.20	96.41	225.51	
428.38	119.46	340.66	224.30
430.23	117.83	333.46	235.53

412.14	134.66	398.53	148.77			
436.18	112.74	309.86	275.07	339.21	226.52	
426.81	120.85	346.67	215.23			
443.08	107.11	282.02	326.03	247.54	390.95	157.20
451.36	100.71	248.62	388.99	159.45		
454.53	98.35	235.98				
411.37	135.42					
400.97	146.14					
430.35	117.72	333.00	236.27			
410.87	135.92					
402.53	144.49					
426.41	121.21	348.21	212.94			
446.01	104.80	270.14	348.58	212.41		
446.67	104.28	267.46	353.68	205.01		
455.05	97.97	233.95				
414.80	132.06	389.87	158.44			
453.15	99.37	241.46				
401.86	145.19					
428.11	119.69	341.68	222.73			
432.57	115.80	324.25	250.49	385.56	163.45	459.31

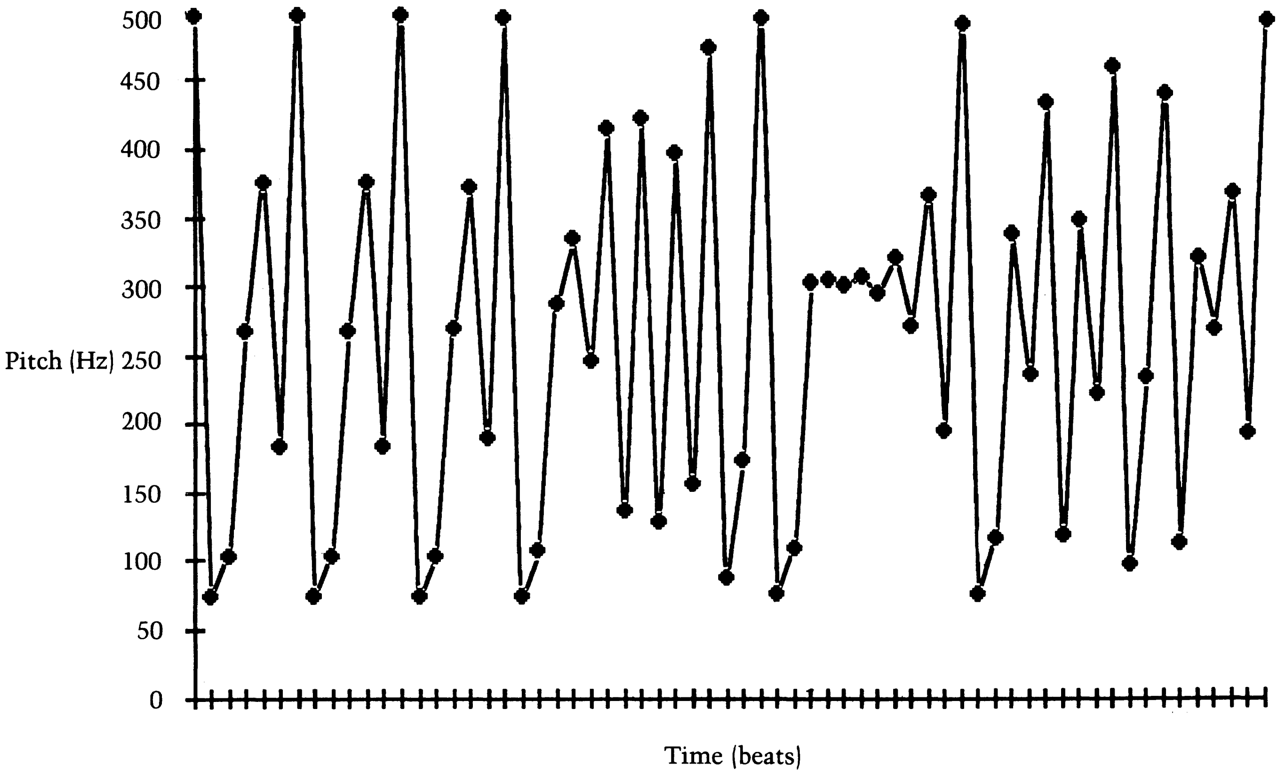
Note that the repetitious 3-cycle gives way by steady microtonal shifting to pitch groups of different lengths, varying here between 2 and 7 notes. Groupings have been taken to occur each time the frequency exceeds 400 Hz. Further, each group with n notes shares a characteristic pitch contour (frequency rank order) with all other n-note groups. A cursory look at other comparable a-values suggests that this property is almost always true, though there seems to be no mention of it in the mathematical literature. It is certainly audible. In musical terms, the overall effect is like a variation technique that inserts and removes material from a motive undergoing mildly erratic pitch transformations, in the style of an adventurous but development-oriented free jazz player, perhaps. An impression of unfamilarity lingers, since the stable cycles normally do not settle on simple integer ratios characteristic of traditional pitch and rhythm structures. Repeated listening makes the structure increasingly audible; initially it tends to be categorically perceived in terms of traditional values for pitch and rhythm.

Sound Example 2 shows the same phenomenon in the neighborhood of a 6-cycle (3.93744, 3.93740, 3.97776), where the third parameter controls attack time and is a literal 6-cycle. In this case the pitch contour cycles occur in the following order: 6,6,6,6, 2,2,2,3,12,5,4,3,6,7,5,9,6,6,6,5,4,3,3,2,2,3,6,8,3,5,... Figure 2 shows a pitch-versus-time plot of the beginning of this example, exhibiting 6-cycle variations as well as chaotic and fixed-point-like behavior.

Neighborhoods of 4- and 5- cycles are found in Sound Example (3, 3.95, 3.96, 3.90 [10]), where [10] indicates the number of notes in a section. A more satisfactory form can be produced by sectionalization, sometimes at the expense of overall melodic unity.

Sound Example 4 adds dynamics to the controlled parameters, listed fourth in (3.6785, 3.99943, 3.99943, 3.99943 [12]). This example shows the effect of choosing a starting value of x that generates an x-value near an unstable fixed point, $x = 1 - 1/a$ for $a > 3.0$. The result is an oscillating divergence from the fixed point whose exact trajectory depends on the starting distance from the fixed point and the computer's arithmetic rounding protocols. The phenomenon is also briefly visible in Fig. 2. The limitations of finite decimals mean that the exact values of fixed points can never really be achieved, except for those rare cases where $1/a$ has a decimal expansion of a length that is less than or equal to the maximum precision of floating-point arithmetic in the machine. The effect of such a trajectory in pitch space is roughly that of microtonally varying ostinato that expands intervallically into pointillism. Here the fixed point for pitch is $0.728150061 = 1 - 1/3.6785$, which is frequency $2^{3(.728150061)+6} = 290.914$ Hz. The third value of the relevant pitch output listed below, at 290.96 Hz, initiates the described expanding "ostinato," which lasts at least 18 notes. As can be seen, there are further recurrences of this feature. And remarkably, the entire list repeats to very high accuracy after 144 steps, as indicated by the two * positions. Within this large cycle, there are a number of quasi-4-cycles, indicated by !, and a quasi-12-cycle in the largest block of 4-cycles.

Fig. 2. Pitch versus time plot of Sound Example 2.



*433.20, 112.65, 290.96, 290.84, 291.05, 290.69, 291.28, 290.29, 291.96, 289.17, 293.85, 285.99, 299.21, 277.12, 314.32, 252.79, 355.64, 193.34, 429.88, 115.34, 302.41, 271.86, 323.30, 238.92, 378.02, 166.01, 427.49, 117.32, 310.58, 258.71, 345.72, 206.57, !420.04, 123.69, 335.18, 221.37, !403.26, 139.25, 383.55, 159.78, !421.42, 122.48, 330.72, 227.85, 394.47, 148.10, 403.39, 139.12, 383.21, 160.16, 421.86, 122.11, 329.30, 229.93, 391.50, 151.20, !409.06, 133.68, 368.19, 177.59, 432.91, 112.87, 291.94, 289.20, 293.80, 286.09, 299.05, 277.37, 313.88, 253.48, 354.49, 194.84, 429.07, 116.01, 305.21, 267.31, 331.08, 227.31, 395.22, 147.32, !401.87, 140.62, 386.97, 156.04, !416.62, 126.72, 345.94, 206.27, !420.32, 123.44, 334.27, 222.68, !401.54, 140.95, 387.78, 155.17, !415.37, 127.85, 349.76, 201.10, !424.80, 119.58, 319.62, 244.55, 369.11, 176.48, 432.70, 113.05, 292.68, 287.95, 295.90, 282.59, 304.98, 267.69, 330.45, 228.24, 393.91, 148.67, 404.50, 138.04, 380.41, 163.30, 425.14, 119.30, 318.49, 246.29, 366.31, 179.88, 433.17, 112.67, 291.07, 290.66, 291.34, 290.19, 292.13, 288.88, 294.33, 285.20, 300.56, 274.90, 318.11, 246.88, 365.36, 181.05, *433.20, 112.65, 290.96, 290.84, 291.05, 290.69

The value 3.99943 chosen for the other parameters produces some quite small x-values that, largely due to drastic effects on tempo, give this example a uniquely through-composed quality. It also has the property of oscillating divergence from a fixed point, as can be seen from the start of the time data which is read from left to right:

0.74989, 0.75011, 0.74968, 0.75053, 0.74882, 0.75224, 0.74540, 0.75901, 0.73155, 0.78543, 0.67403, 0.87872, 0.42621, 0.97808, 0.08573, 0.31349, 0.86074, 0.47941, 0.99816, 0.00734

Sound Example 5 shows the effect of adding a second voice every nth note, where n is determined from yet another logistic map equation by taking the integer part of $12x$. This example is not sectionalized and has the description (3.00, 2.5, 3.9999, 3.84, 3.80) where the fifth value refers to second voice times. Voice two's other parameters are identical to those of voice one, although there are modest changes in the performing instrument to allow the two voices to be readily distinguished (sine tones have been deleted from the sound). Hence the result is a mensuration canon. Pitch slowly converges by oscillation to a fixed value of 256 Hz, rhythm rapidly assumes a constant value, while other values are basically chaotic. The effect is rather like a modernistic bluesy guitar ostinato with the second voice introducing syncopations. Sound Example 6, (3.6785, 3.6785, 3.99943, 3.99943, 3.6785) uses the same set-up as Sound Example 5, producing a triplet-like expanding ostinato motive. The realization uses two separate audio channels and four voices. The channels are isogestural, differing only in pitch content due to different starting values.

Sound Example 7 looks at further textural development. The real logistic map is used again, but with six voices. Each voice was limited to a two octave range, and they were separated by octaves, with voice entries staggered by 7.5 beats. Each part follows (3.55, 3.6785, 3.97, 3.97) (no sectionalization), but with different initial x-values. The pitch selection produces an 8-cycle by double bifurcation of a 2-cycle, and the overall effect is of overlaid shifting ostinati. Finally, Sound Example 8 shows a single voice following (3.99, 3.99, 3.99, 3.99) but where the formula $x_{n+1} = ax_n^{1/3}(1 - x_n^{1/3})$ has been used for pitch and time. This *cube root logistic map* produces a jazzlike melody and walking bass-line, rather surprisingly. The apparent effect of introducing such fractional exponents into the equation is to "smooth out" chaotic effects.

Two-Dimensional Maps for Increased Coordination

At this point a different direction was taken. One obvious musical limitation of the previous examples is that the control of musical parameters is too independent. Even if identical a-values are chosen for all parameters, the effect is parallel rather than interactive. The line can become too unpredictable, too information-laden, one could almost say "hyperexpressive," when all parameters vary independently rather than support each other in the service of coordinated musical effect.

One way to address this problem is by the use of two-dimensional maps, so that the mathematical cross-terms introduce variable correlations. Several such maps were investigated. For example, the following map was developed,

$$x_{n+1} = ax_n e^{\frac{1-\sqrt{(1+y_n)}}{b}} \tag{3}$$

$$y_{n+1} = a^2 x_n - x_{n+1}$$

based on a modification of the *Metz map*, which is known to produce some striking visual patterns (Lauwerier 1986b). Characteristic of this map was the measured approach from chaos to a fixed point or limit cycle. Because the output y values can be very large, different transformations had to be applied to these values to bring them within musical parameter range. The time values were transformed by the relation $1/4y^{1/2}$ and the dynamics by $1/2y^{1/4}$.

Sound Example 9 shows the result of the following parameter choices: pitch and time $a = 3.2$, $b = 1.0$, attack time and dynamics $a = 4.0$, $b = 0.5$. The aural result is a leaping motive that quickly constricts to an ostinato in pitch, resulting in a shift of attention to the timbral and dynamic variation, which is quasicyclic.

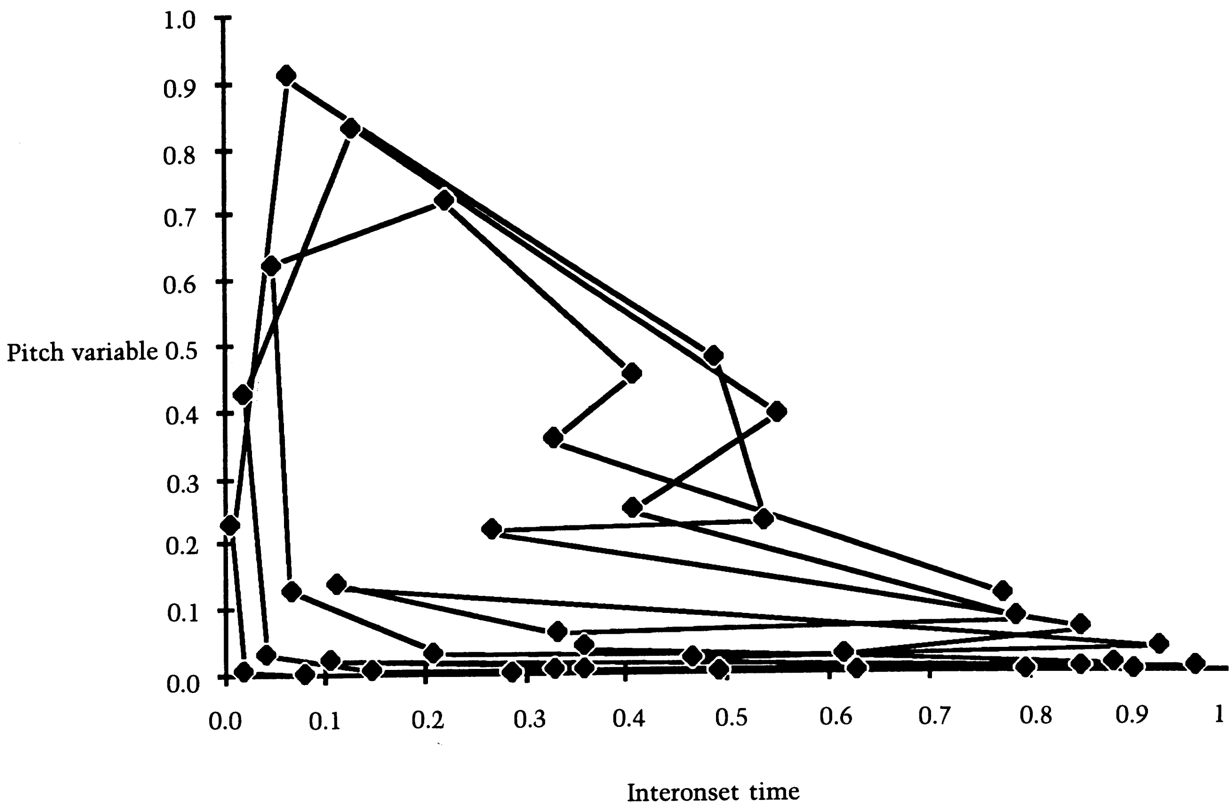
To avoid problems of scaling and divergences, work then centered on maps whose range did not vary very far from [0,1]. The following map proved useful (Lauwerier 1986b):

$$x_{n+1} = ax_n(1 - x_n - y_n)$$

$$y_{n+1} = ax_n y_n$$

It arises in predator-prey modeling, with $2 < a \leq 4$. Variables were assigned as follows for Sound Example 10: $a = 3.50$ for x = pitch, y = interonset time, $a = 3.50$ for x = attack time, y = dynamics, and three-note sections. The sectionalization introduces a strong rhythmic element.

Fig. 3. Pitch versus interonset time for Sound Example 11.



Sound Example 11 shows a reversal of the x and y assignments, for $a = 3.888$, without sectionalization. The result is a variation on a descending strummed string motive. Figure 3 shows a plot of the first 39 points of the trajectory of the map in the Interonset time(x)/Pitch(y) plane. The preferential filling of musical space is apparent.

The third two-dimensional map used was simply Eq. (1), using complex values for x and a. For this purpose the equation was recast into its more common complex form:

$$z_{n+1} = z_n^2 + c, \qquad (5)$$

with $z = x + iy$. Here the choice of starting value can be critical in determining the behavior, in a way that did not occur on the real line. Many choices of c cause z either to converge rapidly to a fixed point or limit cycle, or rapidly diverge to infinity. More interesting behavior occurred for c values such as $.35 - .35i$, $- .505 - .505i$, $-.7 + .25i$, $0 + .63i$, and $.320 + .043i$. Since z values are largely within the unit circle (or else they nearly always soon diverge to infinity), so that real and imaginary parts fall in the range $[-1,1]$, they were converted to the $[0,1]$ interval by the simple operations $(1 + x)/2$ and $(1 + y)/2$.

Sound Example 12 uses the values $(- .505 - .505i, - .51 - .51i)$ referring respectively to pitch (x), time(y), attack(x), dynamics(y), without sectionization. The result is strongly rhythmical and emphasizes a 5-cycle whose range slowly compresses.

A different technique was then adopted. To minimize the trouble created by divergences, it was found convenient to erect a *reflecting barrier* enclosing the origin. This was implemented so that whenever the absolute value of either component of z exceeded a fixed amount, the value of that component was reflected back towards the origin.

Typical forms for this were:

$$if\,|\,x\,|>1.5\ then\ x=\pm\frac{1}{3}\sqrt{x}\,,$$
$$if\,|\,x\,|>5.0\ then\ x=\frac{1}{3}\sqrt{\sqrt{x}}\,.$$

This, of course, superimposes on the map's intrinsic structure another restoring force. But the characteristic types of map behavior seem to be preserved, while additional phenomena arise. Sound Example 13 shows the result of parameter values $(-1-i, -1-i)$. In this and in most succeeding examples the reflecting barrier allows interonset time to sometimes assume negative values, which means that the next note generated sometimes falls before the "previous" one. This negative time reflection produces note bunching, so that sound masses of highly variable density can result.

Higher Dimensional Maps

Since the system primarily used here has four main control variables (pitch, time, attack time, and dynamics), it is natural to consider extending the parametric correlations by using four-dimensional maps. Study in this area is far from intensive, but work by Rössler (1979, 1983) on four-dimensional continuous maps shows that qualitatively new behavior occurs, which has been dubbed *hyperchaos.* Full graphical depiction of such systems is difficult (but see Norton 1982), and probably the ear can follow four dimensions more readily. There is potential here for the development of a new field of audible mathematics (many historical precedents notwithstanding).

The extension used here was developed by again using Eq. (5), $z_{n+1} = z_n^2 + c$, but with z and c as *quaternions* rather than complex numbers. Since quaternions remain a special topic, their properties will be briefly summarized. A quaternion is basically a scalar-vector pair that corresponds to a double rotation. Invented by Sir William Hamilton in 1843 (though previously derived but not published by K. F. Gauss), they form one of only three possible *division algebras* (Pontryagin 1966). A quaternion is four-dimensional and can be expressed in terms of its orthonormal unit vectors, **1, i, j, k,** so that we may write $\mathbf{z} = \mathbf{x1} + \mathbf{yi} + \mathbf{wj} + \mathbf{tk}$, or $\mathbf{z} = (\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{t})$. Multiplication of quaternions is derived from the unit vector multiplication table (Altmann 1986), shown as Table 2. Note that the nonreal unit vectors are anticommutative. From Table 2 it follows that if $\mathbf{c}$ is a constant equal to $c + c_i\mathbf{i} + c_j\mathbf{j} + c_k\mathbf{k}$, the quaternion equation $\mathbf{z}_{n+1} = \mathbf{z}_n^2 + \mathbf{c}$ is equivalent to

Table 2. Quaternion unit vector multiplication table

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

$$(x, y, w, t) \leftarrow (x^2 - y^2 - w^2 - t^2 + c, \quad 2xy + c_i, 2xw + c_j, 2xt + c_k), \tag{6}$$

which includes the complex map version as the special case where $\mathbf{z}_0$ is in the x/y plane and c_j and c_k are zero. The quaternion map was then implemented with basically the same parameter conversion scheme as used for the complex logistic map, with a reflective barrier enclosing the origin. The rebound characteristics of the barrier were varied more extensively than above.

Sound Example 14 uses the **c**-value $-.9 - .7\mathbf{i} - 6\mathbf{j} + .5\mathbf{k}$, and produced a clear motive with bass accompaniment. Figure 4 shows a transfer of the first two-thirds of this example into approximate musical notation. Pitch is represented to the nearest quarter-tone, with small arrows adjacent to noteheads indicating quarter-tone sharp (up) or flat (down). Accidentals apply only to the notes they immediately precede. Rhythm has been quantized using only compounded duplet and triplet divisions. Accent marks are used to indicate short attack times (< ca. 20 msec), and have no dynamic implications. Repeating treble and bass motives are clearly visible.

Fig. 4. Quaternion music: Sound Example 14.

Sound Example 15 uses the input value $1 - \mathbf{i} - \mathbf{j} + \mathbf{k}$, where the strength of rebound from the barrier has been increased, producing a texture of active sound clouds. Sound Example 16 expands the pitch range and reflection properties, using **c**-value $0 - \mathrm{i} - \mathbf{j} - \mathbf{k}$, producing an ascending bass motive with treble accompaniment. Sound Example 17 gives a stereo version of $\mathbf{z}_{n+1} = (1.475 + 0.906\mathbf{i})\,\mathbf{z}_n\,(1 - \mathbf{z}_n)$, which is just Eq. (1) for quaternions. The parameter input value was suggested by Alan Norton.

Finally, a type of quaternionlike four-dimensional object was produced by making Table 2 symmetric, to define a different multiplication scheme shown in Table 3. The transformation corresponding to $\mathbf{z}_{n+1} = \mathbf{z}_n^2 + c$ is just

$$(x, a, b, c) \leftarrow (x^2 - y^2 - w^2 - t^2 + c, \quad (7)$$
$$2(xy - wt) + c_i,\ 2(xw - yt) + c_j,\ 2(xt - yw) + c_k).$$

Here the input constant $.9 - \mathbf{i} - .7\mathbf{j} + \mathbf{k}$ produces, in Sound Example 18, erratically cascading note

Table 3. Symmetric four-dimensional unit vector multiplication table

	1	i	j	k
1	1	i	j	k
i	i	−1	−k	−j
j	j	−k	−1	−i
k	k	−j	−i	−1

groups, while .9 + .4**i** + .3**j** + **k**[1], due to the individual note sectionization, results in a quirkily expressive melodic line (Sound Example 19).

Conclusion

The usefulness of these maps as generators of musical design is to a considerable degree a matter for individual judgment. I can here only state my attempts at objective evaluation. The produced musical examples are idiosyncratic, but show a listenable degree of structural consistency. Repeated listenings allow for improved perception of the map structures. While the maps share certain global structural features, clear musically valent differences between them are apparent. Two-dimensional maps offer more integrated parametric control of variables but do not seem to have better potential for producing more traditional musical structures than one-dimensional maps. (Although musical control variables are better linked in two-dimensional maps, these links are not necessarily musically well-founded.) The introduction of the reflecting barrier about the origin produced an improved flexibility of musical design, by removing the divergences associated with certain parameter ranges. Four-dimensional maps based on quaternions and quaternionlike objects were also used with the reflecting barrier technique and showed rich promise as more general generators of musical design. The examples in general exhibit a fairly wide range of musical "styles," despite their lack of explicit historical or cultural inputs. The most obvious parallels are to certain kinds of folk music, free jazz, and European "cloud" music (Xenakis, Ligeti, Penderecki), insofar as such simple analogies are useful.

The capacity of the maps to generate variation or paraphraselike alteration of specified groups of events seems their most interesting cognitive property. Human thinking processes almost certainly involve nonlinearities (Pressing 1987) and so such a connection is plausible, though inadequately specific. It is clear that substantial shaping and selection of such nonlinear processes by learned or intrinsic goal-directed design is necessary to produce aesthetically viable music on a larger scale, and that the results make musical sense only from a twentieth-century musical perspective. The maps can certainly be used as compositional aids, and more exploration might find maps of more general musical utility. Further development could take such forms as generalizing the above procedures by changing a- or c-values in coordinated fashion during the course of a piece, quantizing all variables to promote traditionalist music perception, linking parameters between voices in a more meticulous fashion to clarify contrapuntal effects, using functional inverses to effect time reversal, and pursuing mathematical extensions such as continuous variables, fractional exponents, and rational functions of polynomials. Some of these extensions have already been implemented. Other extensions would be to use a larger number of control variables, controlling such things as reverberation, spatial panning, partial amplitudes and phases, modulation index, frequency of modulator, filtering, phase vocoding parameters, and so forth. There is no obvious limit to the technique's scope of applicability.

On the other hand, at least two types of aesthetic limitations are apparent. Procedurally, the method used here could be described as "found process," by analogy to the found object approach that underlies photography and has contributed to sculpture, painting, music, and other fields in recent times. While the limitations of found process are probably less than for found objects, since tunable parameters are already built into the method, it is no good trying to look for snowballs on the coast of Florida. Some things may have to be built rather than found. A more serious objection arises by asking to what extent musical and mathematical order are or can be congruent. As Howard Gardner (Gardner 1983) and others have convincingly argued, it may be best to model the two fields as completely separate intelligences. Certainly music and mathematics have their own separate historical traditions, their own distinct logical designs, their own separate aesthetic guidelines. Yet there are too many suggestive parallels between the fields to convincingly support their total cognitive independence. These parallels cannot be surveyed adequately in this short paper, but they include the mathematics/music correlations of the ancient Greeks and Chinese and medieval and Renaissance periods of Europe, recent group theoretic interpretations of pitch and rhythm (Balzano 1980; Pressing 1983) and important works and compositional philosophies by such twentieth-century composers as Schönberg, Bartok, Babbitt, and Xenakis, which are characterized by increasing mathematical sophistication. To this can be added the similarity between the disciplines that arises from their being the most common loci for exceptionally rapid early development in children: the phenomenon of prodigies. Presumably this is due to the abstractness of the two fields, so that a large fund of worldly experience is not required, simply aptitude. But it is also conceivable that one faculty partially underlies both, reaching expression in a form shaped by the development of ancillary abilities—aural discrimination, spatial visualization or motor control, for example. The question seems open.

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Bibliography

Altmann, S. 1986. *Rotations, Quaternions, and Double Groups.* Oxford: Clarendon Press.

Bai-lin, Hao. 1984. *Chaos.* Singapore: World Scientific.

Balzano, G. 1980. "The Group-theoretic Description of Twelvefold and Microtonal Pitch Systems." *Computer Music Journal* 4(4): 66–84.

Evans. B. 1987. "Integration of Music and Graphics through Algorithmic Congruence." In J. Beauchamp, ed. *Proceedings of the 1987 International Computer Music Conference.* San Francisco: Computer Music Association, pp. 17–24.

Feigenbaum, M. 1978. "Quantitative Universality for a Class of Nonlinear Transformations." *Journal of Statistical Physics* 19:25–35.

Gardner, H. 1983. *Frames of Mind: The Theory of Multiple Intelligences.* New York: Basic Books.

Hirsch, J. E., B. A. Huberman, and D. Scalapino. 1982. "Theory of Intermittency." *Physical Review* A25:519.

Hirsch, J. E., M. Nauenberg, and D. Scalapino. 1982. "Intermittency in the Presence of Noise: A Renormalization Group Formulation." *Physics Letters* 87A:391.

Holden, A. V., ed. 1986. *Chaos.* Princeton: Princeton University Press.

Karplus, K., and A. Strong. 1983. "Synthesis of Plucked-string and Drum Timbres." *Computer Music Journal* 7(2):43–55.

Lauwerier, H. 1986a. "One Dimensional Iterative Maps." In A. V. Holden, ed. *Chaos.* Princeton: Princeton University Press, pp. 39–57.

Lauwerier, H. 1986b. "Two Dimensional Iterative Maps." In A. V. Holden, ed. *Chaos.* Princeton: Princeton University Press, pp. 58–95.

Mandelbrot. B. 1980. "Fractal Aspects of the Iteration of $z \rightarrow \lambda z(1-z)$ for Complex λ and z." *Annals of the New York Academy of Sciences* 357:249–59.

May, R. 1976. "Simple Mathematical Models with Very Complicated Dynamics." *Nature* 261:459–67.

Norton, A. 1982. "Generation and Display of Geometric Fractals in 3-D." *Computer Graphics* 16(3):61–67.

Peitgen, H.-O., and P. H. Richter. 1986. *The Beauty of Fractals.* Berlin: Springer Verlag.

Pontryagin, L. 1966. *Topological Groups.* 2nd ed. A. Brown, trans. New York: Gordon and Breach.

Pressing, J. 1983. "Cognitive Isomorphisms Between Pitch and Rhythm in World Musics: West Africa, the Balkans, and Western Tonality." *Studies in Music* 17:38–61.

Pressing, J. 1987. "Improvisation: Methods and Models." in J. Sloboda, ed. *Generative Processes in Music.* Oxford: Oxford University Press.

Rössler, O. 1979. "An Equation for Hyperchaos." *Physics Letters* 71A:155–7.

Rössler, O. 1983. "The Chaotic Hierarchy." *Zeitschrift für Physik* 38a:788–801.

Appendix: Sound examples cited in the text

Example	Label	Comments
		Logistic map (1):
1	(3.8280, 3.8283)	Just before 3-cycle
2	(3.93744, 3.93740, 3.97776)	Just before 6-cycles
3	(3.95, 3.96, 3.90[10])	Just before 4- and 5-cycles
4	(3.6785, 3.99943, 3.99943, 3.99943[12])	Unstable fixed points
5	(3.00, 2.50, 3.9999, 3.84, 3.80)	Second voice added
6	(3.6785, 3.6785, 3.99943, 3.99943, 3.6785)	Four voices
7	(3.55, 3.6785, 3.97, 3.97)	Six voices
8	(3.99, 3.99, 3.99, 3.99)	Cube root logistic map
9	(3.20, 1.00, 4.00, 0.50)	Modified Metz map (3)
10	(3.50, 3.50)	Predator prey map (4)
11	(3.888, 3.888)	Pp map (4) reversed variables
12	$(-.505-.505i; -.51-.51i)$	Complex quadratic map
13	$(-1-i; -1-i)$	Complex quadratic map with barrier
14	$(-.9-.7\mathbf{i}-.6\mathbf{j}+.5\mathbf{k})$	Quaternion quadratic map with barrier
15	$(1-\mathbf{i}-\mathbf{j}+\mathbf{k})$	As 14
16	$(0-\mathbf{i}-\mathbf{j}-\mathbf{k})$	As 14
17	$(1.475+.906\mathbf{i})$	Quaternion logistic map, two voices
18	$(.9-\mathbf{i}-.7\mathbf{j}+\mathbf{k})$	Symmetrized quaternion with barrier
19	$(.9+.4\mathbf{i}+.3\mathbf{j}+\mathbf{k}[1])$	As 18 with one-note sections