Quantum Schubert Calculus

Aaron Bertram[1]

1. Introduction: Classical Schubert calculus, by which I mean the formulas of Giambelli and Pieri encoding the product structure of the cohomology ring of a complex Grassmannian, has been an essential tool in enumerative algebraic geometry for over a century.

String theorists (notably Witten [W]) recently introduced the notion of a "quantum" deformation of the cohomology ring of a smooth projective variety X. This quantum deformation, or quantum cohomology ring, as it is often called, is an algebra over a formal-power-series ring which specializes to the ordinary cohomology ring, and which is defined in terms of intersection data (the Gromov-Witten invariants) on all the spaces of holomorphic maps from pointed curves of genus zero to X.

A rigorous definition of the Gromov-Witten invariants, together with a verification of the algebra structure of these quantum deformations, has been established by two schools, namely the symplectic school of Ruan-Tian ([RT]) and the algebro-geometric school of Kontsevich-Manin ([KM]). One interesting variant of the quantum deformation is a "small" deformation of the cohomology ring (terminology taken from [F2]) which is an algebra over a polynomial ring (hence of finite-type over $\mathbf{C}$) sitting between the full quantum deformation and the cohomology ring itself. This "small" quantum cohomology ring can be defined independently, and we will do so in the Grassmannian case, where it turns out to be an algebra over a polynomial ring in one variable. We'll let q stand for the variable.

In this paper, the rules for the Schubert calculus are modified so that they are valid in the small quantum cohomology ring. In other words, whereas the Giambelli and Pieri formulas are valid in the cohomology ring of a Grassmannian, higher order terms (in q) may appear when the corresponding products are taken in this ring. The main result here is the computation of these higher order terms. Our computation relies on the recursive properties of a particular smooth compactification (the Grothendieck quot scheme) of the space of holomorphic maps of a fixed degree from $\mathbf{P}^1$ to a Grassmannian.

[1]Partially supported by NSF grant DMS-9218215 and a Sloan fellowship

In order to fix notation and refresh the reader's memory, we begin with an overview (following [GH]) of the classical Schubert calculus before continuing with the introduction.

Let: V be a vector space over $\mathbf{C}$ of dimension n,
$0 = V_0 \subset V_1 \subset ... \subset V_n = V$ be a full flag for V,
$G := G(n-k, n)$ be the Grassmannian of $n-k$-dim'l subspaces of V,
$\Lambda_x \subset V$ be the subspace corresponding to a point $x \in G$.

Given an $n-k$-tuple of integers $\vec{a} := (a_1, ..., a_{n-k})$ satisfying the inequalities $k \geq a_1 \geq ... \geq a_{n-k} \geq 0$, let:

$$W_{\vec{a}} = \{x \in G |\ \dim(\Lambda_x \cap V_{k+i-a_i}) \geq i\}.$$

Then $W_{\vec{a}}$ is a subvariety of G of complex codimension $|\vec{a}| := \sum_{i=1}^{n-k} a_i$. Let

$$\sigma_{\vec{a}} \in \ \mathrm{H}^{2|\vec{a}|}(G, \mathbf{C})$$

be the corresponding element in cohomology.

One calls $W_{\vec{a}}$ the Schubert variety associated to $\vec{a}$ (and the given flag). The cohomology classes $\sigma_{\vec{a}}$ produce a vector-space basis for $\mathrm{H}^*(G, \mathbf{C})$ as the $\vec{a} = (a_1, ..., a_{n-k})$ range over all tuples of integers with the given constraints.

The Schubert varieties $W_a := W_{(a,0,...,0)}$ are called special Schubert varieties. The corresponding cohomology classes σ_a coincide with the image in cohomology of the chern classes $c_a(Q)$ where Q is the universal quotient bundle on G. These special cohomology classes generate the cohomology ring of the Grassmannian as an **algebra** over $\mathbf{C}$ via the following determinantal formula:

Giambelli's Formula: By convention, let $\sigma_a = 0$ if $a < 0$ or $a > k$. Then:

$$\sigma_{\vec{a}} = \Delta_{\vec{a}}(\sigma_*) := \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+n-k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+n-k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & & \\ \vdots & & & & \vdots \\ \sigma_{a_{n-k}-(n-k)+1} & & \cdots & & \sigma_{a_{n-k}} \end{vmatrix}$$

The other "Italian" formula explicitly computes the intersections of a special cohomolgy class and a general one:

Pieri's Formula: The product of σ_a and $\sigma_{\vec{a}}$ in $\mathrm{H}^*(G, \mathbf{C})$ is:

$$\sigma_a \cdot \sigma_{\vec{a}} = p_{a,\vec{a}}(\sigma_{\vec{*}}) := \sum_{\vec{b}} \sigma_{\vec{b}}$$

where the $\vec{b}$ vary over all $n-k$-tuples satisfying:

$|\vec{b}| = a + |\vec{a}|$ and $k \geq b_1 \geq a_1 \geq ... \geq b_{n-k} \geq a_{n-k} \geq 0$.

The two Italian formulas determine the ring structure on the cohomology ring $\mathrm{H}^*(G, \mathbf{C})$. Indeed, (see [BT], Proposition 23.2) one obtains a convenient presentation of the cohomology ring as:

$$(*) : \mathbf{C}[X_1, ..., X_k]/(Y_{n-k+1}(X_*), ..., Y_n(X_*)) \cong \mathrm{H}^*(G, \mathbf{C}), \quad X_a \mapsto \sigma_a$$

where $Y_i(X_*)$ is the coefficient of t^i in the formal-power-series inverse of the polynomial $1 + X_1 t + ... + X_k t^k$.

Finally, recall that if we set $\vec{a}^c := (k - a_{n-k}, k - a_{n-k-1}, ..., k - a_1)$, then $\sigma_{\vec{a}}$ and $\sigma_{\vec{a}^c}$ are Poincaré dual. Equivalently, if we let $\langle W_{\vec{a}}, W_{\vec{b}} \rangle$ denote the intersection number in G of general translates of $W_{\vec{a}}$ and $W_{\vec{b}}$ (which is set to zero if $|\vec{a}| + |\vec{b}| \neq \dim(G) = k(n-k)$), then

$$\langle W_{\vec{a}}, W_{\vec{b}} \rangle = \begin{cases} 1 & \text{if } \vec{b} = \vec{a}^c \\ 0 & \text{otherwise} \end{cases}$$

The multiplication on $\mathrm{H}^*(G, \mathbf{C})$ can be understood solely in terms of intersection numbers as follows. If $\sigma_{\vec{a}_1}, ..., \sigma_{\vec{a}_N}$ are cohomology classes corresponding to Schubert varieties, let $\langle W_{\vec{a}_1}, ..., W_{\vec{a}_N} \rangle$ be the intersection number of general translates of the Schubert varieties, set to zero, as above, if $\sum_{i=1}^N |\vec{a}_i| \neq \dim(G)$. Then in the cohomology ring of G,

$$\sigma_{\vec{a}_1} \cdot ... \cdot \sigma_{\vec{a}_N} = \sum_{\vec{a}} \langle W_{\vec{a}}, W_{\vec{a}_1}, ..., W_{\vec{a}_N} \rangle \; \sigma_{\vec{a}^c}.$$

This is an immediate consequence of the fact that the cohomology classes $\sigma_{\vec{a}}$ satisfy the Poincaré duality property above.

The key idea behind the quantum deformations is to introduce "higher order terms" into the product by considering a sequence of intersection numbers, starting with the intersections on G itself. To be more precise, here is a definition for the small quantum deformation.

For each integer $d \geq 0$, let $\langle W_{\vec{a}_1}, ...W_{\vec{a}_N} \rangle_d \in \mathbf{Z}$ be the "Gromov-Witten" intersection number defined as follows. Choose general points $p_1, ..., p_N \in \mathbf{P}^1$ and general translates of the $W_{\vec{a}_i}$. Then $\langle W_{\vec{a}_1}, ..., W_{\vec{a}_N} \rangle_d$ is "by definition" the number of holomorphic maps $f : \mathbf{P}^1 \to G$ of degree d with the property that $f(p_i) \in W_{\vec{a}_i}$ for all $i = 1, ..., N$ (and zero if the sum of the $|\vec{a}_i|$ is such that one expects this number not to be finite). In §2, we make rigorous sense out of this definition and reinterpret it as an intersection of generalized Schubert cohomology classes in a Grothendieck quot scheme, which in this case happens to be a smooth, projective variety of dimension $nd +$ dim(G). Notice that in particular, the Gromov-Witten number $\langle W_{\vec{a}_1}, ...W_{\vec{a}_N} \rangle_0$ is the original intersection number in G.

The "quantum" product of $\sigma_{\vec{a}_1}, ..., \sigma_{\vec{a}_N}$, which we will denote with asterisks as $\sigma_{\vec{a}_1} * ... * \sigma_{\vec{a}_N}$, is defined as follows. Let q be a formal variable. Then:

$$\sigma_{\vec{a}_1} * ... * \sigma_{\vec{a}_N} = \sum_{d \geq 0} q^d (\sum_{\vec{a}} \langle W_{\vec{a}}, W_{\vec{a}_1}, ..., W_{\vec{a}_N} \rangle_d \ \sigma_{\vec{a}^c})$$

Notice that setting $q = 0$, one recovers the original product. Notice also that this sum is finite, because the dimensions of the spaces of holomorphic maps from $\mathbf{P}^1$ to G increase with d.

The really surprising aspect of quantum cohomology is the following:

Associativity Theorem: Extend the quantum product to a product on elements of $H^*(G, \mathbf{C})[q]$ by linearity and by setting

$$(\sigma_{\vec{a}_1} q^{d_1}) * ... * (\sigma_{\vec{a}_N} q^{d_N}) = (\sigma_{\vec{a}_1} * ... * \sigma_{\vec{a}_N}) q^{(d_1 + ... + d_N)}.$$

Then the pairwise quantum product is associative and gives $\mathrm{H}^*(G, \mathbf{C})[q]$ the structure of a $\mathbf{C}[q]$-algebra. The quantum product of more than two terms agrees with the product in this ring.

It would be confusing to refer to this ring as $H^*(G, \mathbf{C})[q]$ because the quantum product is not the same as the natural product on this polynomial ring, so we make the following:

Definition (of the small quantum ring): The small quantum cohomology ring $QH^*(G)$ is by definition the vector space $\mathrm{H}^*(G, \mathbf{C})[q]$ equipped with the extended quantum product.

As I said earlier, this theorem is a special case of more general associativity results in Ruan-Tian [RT] or Kontsevich-Manin [KM]. It is also a very powerful theorem, as it tells us that all quantum products are determined by the pairwise products(!) For example, Siebert and Tian ([ST]), following ideas of Witten, use this idea to reduce the proof of the following presentation for $QH^*(G)$ to a single computation for degree one maps:

$$(*)_q : \mathbf{C}[X_1, ..., X_k, q]/(Y_{n-k+1}(X_*), ..., Y_n(X_*) - (-1)^{k-1}q) \cong \ QH^*(G),$$

where $X_a \mapsto \sigma_a, q \mapsto q$ and the $Y_i(X_*)$ are defined as in $(*)$.

In this paper, we will compute versions of the Italian formulas where the ordinary multiplication is replaced by quantum multiplication. Unlike the presentation for the quantum cohomology ring above, it seems that the best way to approach this problem is not by invoking the associativity theorem, even though the quantum product is, of course, determined by $(*)_q$. Rather, both formulas follow rather quickly from a theorem of Kempf and Laksov ([KL]) once we have analyzed the relevant Grothendieck quot scheme in §3. To be precise, we will prove the following formulas in §4:

Quantum Giambelli:

$$\sigma_{\vec{a}} = \Delta_{\vec{a}}(\sigma_*),$$

when the determinant is evaluated in $QH^*(G)$ using the quantum product. In other words, no higher order terms arise from the Giambelli determinant!

Quantum Pieri:

$$\sigma_a * \sigma_{\vec{a}} = p_{a,\vec{a}}(\sigma_{\vec{*}}) + q(\sum_{\vec{c}} \sigma_{\vec{c}})$$

where the $\vec{c}$ range over all $n-k$-tuples satisfying:

$$|\vec{c}| = a + |\vec{a}| - n \text{ and } a_1 - 1 \geq c_1 \geq a_2 - 1 \geq ... \geq a_{n-k} - 1 \geq c_{n-k} \geq 0.$$

Notice that as is the case with the classical Giambelli and Pieri formulas, the quantum versions determine all the quantum products.

In §5, as a quick application of quantum Giambelli, we see that a residue formula of Vafa and Intriligator computing the Gromov-Witten numbers for special Schubert varieties can readily be modified to compute all the Gromov-Witten numbers.

Final Remarks: By substituting the Giambelli determinant, one of course has the identity: $\sigma_{\vec{a}} * \sigma_{\vec{a}_1} * ... * \sigma_{\vec{a}_N} = \Delta_{\vec{a}}(\sigma_*) * \sigma_{\vec{a}_1} * ... * \sigma_{\vec{a}_N}$ in $QH^*(G)$ for any Schubert cohomology classes $\sigma_{\vec{a}}$ and $\sigma_{\vec{a}_1}, ..., \sigma_{\vec{a}_N}$. Similarly one can substitute for a product $\sigma_a * \sigma_{\vec{a}}$ using quantum Pieri. This is obvious once the associativity theorem is established. But it can be (and was orignally) proved directly using the methods of this paper without appealing to the quantum ring, and can indeed be used to obtain an independent proof of the associativity theorem in this context. (Quantum Giambelli implies $\mathrm{H}^*(G, \mathbf{C})[q]$ is a quotient of the polynomial ring $\mathbf{C}[x_1, ..., x_k, q]$ and quantum Pieri implies that the kernel is an ideal, putting the quotient ring structure, which is the quantum product, on $\mathrm{H}^*(G, \mathbf{C})[q]$.)

Finally, it is possible to derive quantum Pieri from quantum Giambelli and the presentation $(*)_q$ of the quantum cohomology ring, bypassing the delicate geometric arguments in §4. Quantum Giambelli itself, however, seems to require a special proof.

Acknowledgements: I was introduced to quantum cohomology and the formula of Vafa and Intriligator by Richard Wentworth (see [BDW] or [RRW] for a completely different way of computing the intersection numbers!). I also profited from conversations with Youngbin Ruan, Henri Shahrouz and Ionut Ciocan-Fontanine on the papers mentioned above, as well as many much-appreciated comments from Bill Fulton.

Finally, I would like to thank Lowell Abrams, who pointed out an error in an earlier formulation of quantum Pieri. While correcting that error, I decided to revise the paper and "quantize" the title, which was formerly "Modular Schubert Calculus."

2. Intersections on the Space of Maps: In this section, we will rigorously define the intersection of Schubert varieties on the moduli spaces $\mathcal{M}_d$ of holomorphic maps of degree d from $\mathbf{P}^1$ to G. We will first prove a moving lemma, stating that the Schubert varieties can be made to intersect in points when they ought to. Then we will prove a cohomological lemma, interpreting the intersection as the intersection of cohomology classes in a given smooth, *projective* variety.

We begin with $\mathcal{M}_d$ itself. The usual way to prove that the space of maps is represented by a quasiprojective scheme is to embed it as an open set in a Hilbert scheme based on the product $\mathbf{P}^1 \times G$. However, there is another moduli space available which contains $\mathcal{M}_d$ as an open subscheme, namely Grothendieck's quot scheme, which will be our compactification of choice.

Recall that a map $f : \mathbf{P}^1 \to G$ is equivalent to specifying a quotient vector bundle $V \otimes \mathcal{O}_{\mathbf{P}^1} \to Q$, or dually, a subbundle $S = Q^* \hookrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ (modulo the action of $\mathrm{GL}(V)$) where S has degree $-d$ and rank k. The quot scheme will parametrize maps $S \hookrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ that are injective as maps of *sheaves*. In other words, the cokernel $\mathcal{F}$ of such a map is not required to be a vector bundle. Specifically, let $\chi(m) = (m+1)(n-k) + d$. That is, χ is the Hilbert polynomial of a vector bundle of rank $n-k$ and degree d on $\mathbf{P}^1$. Then:

Grothendieck's Theorem: The functor $Quot_\chi(V^*/\mathbf{P}^1)$ parametrizing flat families of quotients $V^* \otimes \mathcal{O}_{\mathbf{P}^1} \to \mathcal{F}$ of Hilbert polynomial χ is representable by a smooth projective variety of dimension $nd + \dim(G)$. Moreover, if we let $\mathcal{Q}_d$ denote this fine moduli space, then $\mathcal{M}_d$ is an open subscheme of $\mathcal{Q}_d$ via the canonical inclusion.

We will call $\mathcal{Q}_d$ the quot scheme compactification of $\mathcal{M}_d$. ([Gro] is the standard reference for the proof of Grothendieck's Theorem.)

Since $\mathcal{Q}_d$ is a fine moduli space, there is by definition a universal exact sequence:

$$0 \to \mathcal{S}_d \to V^* \otimes \mathcal{O} \to \mathcal{T}_d \to 0,$$

on $\mathbf{P}^1 \times \mathcal{Q}_d$ which is flat over $\mathcal{Q}_d$. The sheaf $\mathcal{T}_d$ is certainly not usually a vector bundle, but it is an easy consequence of flatness that the *kernel* $\mathcal{S}_d$ is a vector bundle.

We will be very interested in the dual (nonsurjective!) universal map:

$$u : V \otimes \mathcal{O} \to \mathcal{S}_d^*$$

Next, we define the pull-back of Schubert varieties to $\mathcal{M}_d$.

Definition 2.1: If $p \in \mathbf{P}^1$ and $W_{\vec{a}} \subset G$ is a Schubert variety, then let:

$$W_{\vec{a}}(p) = \{f \in \mathcal{M}_d \mid f(p) \in W_{\vec{a}}\}$$

We put a scheme structure on $W_{\vec{a}}(p)$ via the universal evaluation map $ev : \mathbf{P}^1 \times \mathcal{M}_d \to G$ by redefining:

$$W_{\vec{a}}(p) := ev^{-1}(W_{\vec{a}}) \cap \{\{p\} \times \mathcal{M}_d\}\,.$$

We may extend $W_{\vec{a}}(p)$ as a degeneracy locus to the quot scheme:

Definition 2.1A: Recall the flag of subspaces $0 = V_0 \subset V_1 \subset ... \subset V_n = V$. For each $i = 1, ..., n-k$, let $D_{i,a_i} \subset \mathbf{P}^1 \times \mathcal{Q}_d$ be the largest subscheme on which the dimension of the kernel of $u : V_{k-i+a_i} \otimes \mathcal{O} \to \mathcal{S}_d^*$ is at least i, and let $D_{i,a_i}(p)$ be the intersection: $D_{i,a_i} \cap \{\{p\} \times \mathcal{Q}_d\}$ thought of as a subscheme of $\mathcal{Q}_d$. Then exactly as in the definition of $W_{\vec{a}}$, we define:

$$\overline{W}_{\vec{a}}(p) := D_{1,a_1}(p) \cap ... \cap D_{n-k,a_{n-k}}(p).$$

Suppose now that $\vec{a}_1, ..., \vec{a}_N$ are $(n-k)$-tuples describing Schubert varieties, and let $A = \sum_{j=1}^N |\vec{a_j}|$. Then:

Moving Lemma 2.2: For any points $p_1, ..., p_N \in \mathbf{P}^1$, corresponding general translates of the $W_{\vec{a}_j} \subset G$, and a fixed subvariety $Z \subset \mathcal{M}_d$, the intersection: $W_{\vec{a}_1}(p_1) \cap ... \cap W_{\vec{a}_N}(p_N) \cap Z$ is either empty, or has pure codimension A in Z.

Proof: It suffices by induction to prove that given a subvariety $Z \subset \mathcal{M}_d$, a point $p \in \mathbf{P}^1$, and a general translate of $W_{\vec{a}}$, the intersection $Z \cap W_{\vec{a}}(p)$ is empty or has codimension $|\vec{a}|$ in Z. But if we let $T \subset G$ be the image of $Z \subset \{\{p\} \times \mathcal{M}_d\}$ under the evaluation map ev, then by an argument of Kleiman (see [H], III.10.8), a general translate of $W_{\vec{a}}$ intersects T in codimension $|\vec{a}|$. More generally, the general translate intersects each locus in T over which $ev|_Z$ has constant fiber dimension in codimension $|\vec{a}|$. Since $W_{\vec{a}}(p) \cap Z = ev^{-1}(W_{\vec{a}} \cap T) \cap Z$, the lemma follows.

The lemma implies that when $A = \dim(\mathcal{M}_d)$, the schemes $W_{\vec{a}_i}(p_i)$ can be chosen to intersect in (reduced) points. Recall that the Gromov-Witten intersection number is defined as the number of these points when the p_i are in general position. However, it turns out that it suffices for them to be distinct:

Moving Lemma 2.2A: If $p_1, ..., p_N \in \mathbf{P}^1$ are *distinct* points, then for general choices of N full flags on V, an intersection $\overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N)$ of generalized Schubert varieties is either empty, or has pure codimension A in $\mathcal{Q}_d$. Moreover, the intersection $W_{\vec{a}_1}(p_1) \cap ... \cap W_{\vec{a}_N}(p_N)$ is Zariski dense in $\overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N)$. In particular, if $A = \dim(\mathcal{Q}_d)$, then

$$\overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N) = W_{\vec{a}_1}(p_1) \cap ... \cap W_{\vec{a}_N}(p_N).$$

In order to prove this lemma, we will need to analyze the structure of the boundary $\mathcal{B}_d := \mathcal{Q}_d - \mathcal{M}_d$. This we will do in the next section. For now, we list the main consequences of the lemma.

Corollary 2.3: The cohomology class $\sigma_{\vec{a}} \in \mathrm{H}^{2|\vec{a}|}(\mathcal{Q}_d, \mathbf{C})$ associated to $\overline{W}_{\vec{a}}(p)$ is independent of $p \in \mathbf{P}^1$ and the choice of flag on V. We will call this the generalized cohomology class associated to the Schubert variety $W_{\vec{a}}$.

Proof: It follows immediately from Definition 2.1A that the $\overline{W}_{\vec{a}}(p)$ are fibers of a morphism from a subscheme $X \subset \mathbf{P}^1 \times F \times \mathcal{Q}_d$ to $\mathbf{P}^1 \times F$, where F is the full flag variety associated to V.

Because the automorphism groups of $\mathbf{P}^1$ and F are transitive, the map $X \to \mathbf{P}^1 \times F$ is even a fiber bundle, of fiber codimension A by the Lemma, and the Corollary follows. (The reader may check that X is not empty!)

Corollary 2.4: If $A = \dim(\mathcal{M}_d)$, then the total degree of the intersection in Lemma 2.2 is independent of the (general) translates of the $W_{\vec{a}_j}$ and the points $p_j \in \mathbf{P}^1$ as long as the p_j are distinct.

Proof: If the p_j are distinct, then the latter part of Lemma 2.2A applies and the intersection number may be interpreted as the degree of the product of the $\sigma_{\vec{a}_j}$ in the cohomology ring of the quot scheme.

It is very important that the points are distinct in Corollary 2.4. If the Corollary were true for any collection of points, then the quantum Schubert calculus would be trivial! (But see quantum Giambelli in §4.)

Conclusion: If $\sum_{i=1}^{N} |\vec{a}_i| = \dim(\mathcal{M}_d)$, then the Gromov-Witten number $\langle W_{\vec{a}_1}, ..., W_{\vec{a}_N} \rangle_d$ from the introduction is well-defined, and coincides with the degree of the product of the generalized Schubert cohomology classes $\sigma_{\vec{a}_1}, ..., \sigma_{\vec{a}_N}$ in the cohomology ring of $\mathcal{Q}_d$.

Remark: In §4, we will use this conclusion to extend the definition of the Gromov-Witten numbers to situations where the cohomological interpretation is the correct one, and the naive definition from §1 is not correct.

Thus, the definition of the quantum product is now secure, and the Gromov-Witten numbers have a cohomological interpretation. As another application of the moving lemmas, we prove the following formula for the "trivial" quantum product.

Lemma 2.5: $\sigma_{\vec{b}} = \sum_{d \geq 0} q^d (\sum_{\vec{a}} \langle W_{\vec{a}}, W_{\vec{b}} \rangle_d \ \sigma_{\vec{a}^c})$

Proof: We need to prove that $\langle W_{\vec{a}}, W_{\vec{b}} \rangle_d = 0$ for all pairs of Schubert varieties $W_{\vec{a}}$ and $W_{\vec{b}}$, and all *positive* values of d.

The Gromov-Witten number is zero if $|\vec{a}| + |\vec{b}| \neq \dim(\mathcal{M}_d)$ by definition. So we assume equality, and by the Moving Lemmas, we know that the intersection number is realized as the degree of $\overline{W}_{\vec{a}}(p) \cap \overline{W}_{\vec{b}}(o)$ for distinct points $o, p \in \mathbf{P}^1$, and general translates of $W_{\vec{a}}$ and $W_{\vec{b}}$. Moreover, we know that the intersection is contained in $\mathcal{M}_d$.

Now suppose that the intersection is nonempty. Then we have just seen that there is a map $f : \mathbf{P}^1 \to G$ of degree d such that $f(p) \in W_{\vec{a}}$ and $f(o) \in W_{\vec{b}}$. But there are an entire $\mathbf{C}^*$ of automorphisms $\lambda : \mathbf{P}^1 \to \mathbf{P}^1$ which fix p, o, and the compositions $f \circ \lambda$ all produce *different* elements of $\overline{W}_{\vec{a}}(p) \cap \overline{W}_{\vec{b}}(o)$. Since the intersection was proven to be finite, we get a contradiction.

(Notice that there is no contradiction in case $d = 0$ because if f is a *constant* map, then all the $f \circ \lambda$ are the same!)

3. The Recursive Structure of the Quot Scheme: Recall from §2 the definition of the boundary of $\mathcal{Q}_d$:

$$\mathcal{B}_d := \mathcal{Q}_d - \mathcal{M}_d.$$

As we noted in §2, the universal quotient sheaf $\mathcal{T}_d$ on $\mathbf{P}^1 \times \mathcal{Q}_d$ is not locally free. In fact, $\mathcal{M}_d$ is the largest subset U of the quot scheme with the property that $\mathcal{T}_d$ has constant rank $n-k$ on $\mathbf{P}^1 \times U$. In the following theorem, we obtain precise information about a stratification of the boundary determined by the loci where $\mathcal{T}_d$ has rank at least $n-k+r$.

Theorem 3.1 (Structure Theorem for the Quot Scheme):
For all positive integers $r \leq k$, let $\pi_r : \mathcal{G}_{d,r} \to \mathbf{P}^1 \times \mathcal{Q}_{d-r}$ be the Grassmann bundle of r-dimensional quotients of $\mathcal{S}_{d-r}$ on $\mathbf{P}^1 \times \mathcal{Q}_{d-r}$, and let $\hat{\mathcal{S}}_{d-r}$ be the kernel of the tautological quotient $\pi_r^* \mathcal{S}_{d-r} \to Q$. Then there are maps $\beta_r : \mathcal{G}_{d,r} \to \mathcal{Q}_d$ satisfying:

(i) If $\mathcal{T}_d$ has rank at least $n-k+r$ at a point $(p,x) \in \mathbf{P}^1 \times \mathcal{Q}_d$, then x is in the image of β_r.

(ii) The restriction of β_r to $\pi_r^{-1}(\mathbf{P}^1 \times \mathcal{M}_{d-r})$ is an embedding.

(iii) The preimage of Schubert varieties in $\mathcal{G}_{d,r}$ is given by

$$\beta_r^{-1}(\overline{W}_{\vec{a}}(p)) = \pi_r^{-1}(\mathbf{P}^1 \times \overline{W}_{\vec{a}}(p)) \cup \widehat{W}_{\vec{a}-\vec{r}}(p)$$

where $\vec{r} = (r, r, ..., r)$, $\widehat{W}_{\vec{a}-\vec{r}}(p) = \cap_{i=1}^{n-k} \hat{D}_{i,a_i-r}(p)$ and $\hat{D}_{i,a}(p)$ is the degeneracy locus inside $\pi_r^{-1}(p \times \mathcal{Q}_{d-r})$ where the kernel of $V_{k-r+i-a} \otimes \mathcal{O} \to \hat{\mathcal{S}}_{d-r}^*$ has rank at least i.

Proof of the Structure Theorem: To construct the maps β_r, we need to find bundles $\mathcal{E}_{d,r} \hookrightarrow V^* \otimes \mathcal{O}$ on $\mathbf{P}^1 \times \mathcal{G}_{d,r}$ which have rank k and relative degree $-d$ over $\mathcal{G}_{d,r}$. We obtain these from the $\pi_r^* \mathcal{S}_{d-r}$ by elementary modifications. Namely, let $\pi_\Delta^* \mathcal{S}_{d-r} \to \pi_\Delta^* Q$ be the pull-back of the tautological quotient to the preimage of $\Delta \times \mathcal{Q}_{d-r}$ in $\mathbf{P}^1 \times \mathcal{G}_{d,r}$. ($\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$ is the diagonal.) Let $\pi^* \mathcal{S}_{d-r}$ be the pull-back of $\mathcal{S}_{d-r}$ to $\mathbf{P}^1 \times \mathcal{G}_{d,r}$, and consider the composition:

$$f_{d,r} : \pi^* \mathcal{S}_{d-r} \to \pi_\Delta^* \mathcal{S}_{d-r} \to \pi_\Delta^* Q.$$

Since the quotient is a vector bundle of rank r supported on a divisor which intersects each fiber of the projection $\mathbf{P}^1 \times \mathcal{G}_{d,r} \to \mathcal{G}_{d,r}$ in a point, the kernel of $f_{d,r}$ is a vector bundle $\mathcal{E}_{d,r}$ with the desired properties. Since the quot scheme is a fine moduli space, moreover, we know that $(\mathrm{id}, \beta_r)^*\mathcal{S}_d = \mathcal{E}_{d,r}$.

It may be more illuminating to think of the maps β_r pointwise. Namely, if $S \hookrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ is a vector bundle subsheaf of rank k and degree $-d+r$, then a point $p \in \mathbf{P}^1$ and rank r quotient $S(p) \to \mathbf{C}^r(p)$ determine a point $x \in \mathcal{G}_{d,r}$. The kernel of the map $S \to \mathbf{C}^r(p)$ is a new vector bundle E of rank r and degree $-d$ which becomes a subsheaf of $V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ via its inclusion as a subsheaf of S. The resulting subheaf $E \hookrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ is the image $\beta_r(x)$.

Now, suppose that $V^* \otimes \mathcal{O}_{\mathbf{P}^1} \to T$ is a quotient with $\chi(\mathbf{P}^1, T(m)) = \chi$, and that the rank of T at $p \in \mathbf{P}^1$ is at least $n-k+r$. Then let $i : E \hookrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}^1}$ be the kernel, and consider the dual map i^*. The fact that T has rank $n-k+r$ at p implies that at p, the map on fibers: $i^*(p) : V(p) \to E^*(p)$ has a cokernel of rank at least r. Thus, we may choose a quotient $E^* \to \mathbf{C}^r(p)$ such that i^* factorizes through the kernel, S^*, which proves (i). Moreover, if the sheaf T has rank exactly $n-k+r$ at exactly one point $p \in \mathbf{P}^1$, then the bundle S^* is uniquely determined, which proves (ii) on the level of sets.

To prove (ii) completely, we observe that the map β_r may be inverted on the image of $\pi_r^{-1}(\mathbf{P}^1 \times \mathcal{M}_{d-r})$ by globalizing the previous paragraph. Namely, on this image, the cokernel N of the map $u : V \otimes \mathcal{O} \to \mathcal{S}_d^*$ is torsion, supported on a section Z of $\mathbf{P}^1 \times \mathcal{Q}_d$ over $\mathcal{Q}_d$, and of rank r on its support. The projection of Z to $\mathbf{P}^1$, kernel $\mathcal{E}^*$ of the map $\mathcal{S}_d^* \to N$ (which is a bundle!), and the cokernel of the map $\mathcal{S}_d|_Z \to \mathcal{E}|_Z$ will give us the inverse to β_r.

Finally, reconsider the maps $V \otimes \mathcal{O} \to \pi^*\mathcal{S}_{d-r}^* \to \mathcal{E}_{d,r}^* = (\mathrm{id}, \beta_r)^*\mathcal{S}_d^*$ of vector bundles on $\mathbf{P}^1 \times \mathcal{G}_{d,r}$. The latter map is an isomorphism off of the preimage of $\Delta \times \mathcal{Q}_{d-r}$, and when restricted to the preimage of $\Delta \times \mathcal{Q}_{d-r}$, it factors through $\hat{S}_{d-r}^*$. Thus the degeneracy locus where $V_{k+i-a_i} \otimes \mathcal{O} \to \mathcal{E}_{d,r}^*$ has kernel of rank i is the union of the same degeneracy loci for $\pi^*\mathcal{S}_{d-r}$ generically and for $\hat{S}_{d-r}^*$ on the preimage of $\Delta \times \mathcal{Q}_{d-r}$. Since the rank of $\hat{S}_{d-r}^*$ is $k-r$, we get (iii) when we restrict the degeneracy loci to $p \times \mathcal{G}_{d,r}$.

As our first application of the structure theorem, we will prove the second moving lemma.

Proof of Moving Lemma 2.2A: Note that the Lemma is identical to Lemma 2.2 in case $d = 0$, and anyway it is easy in case $d = 0$ because $\mathcal{Q}_0 = \mathcal{M}_0 = G$. We prove the Lemma in general by induction on the degree.

Notice first of all that the codimension of the intersection cannot be *larger* than A because each $\overline{W}_{\vec{a}}(p)$ has codimension at most $|\vec{a}|$, by [F1], Theorem 14.3(b). Since Lemma 2.2 already takes care of the restriction to $\mathcal{M}_d$, it therefore suffices to show that

$$\overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N) \cap \mathcal{B}_d$$

has codimension greater than A in $\mathcal{Q}_d$.

By the structure theorem, it suffices to show that:

$$\cap_{j=1}^{N} \left\{ \pi_r^{-1}(\mathbf{P}^1 \times \overline{W}_{\vec{a}_j}(p_j)) \cup \widehat{W}_{\vec{a}_j - \vec{r}}(p_j) \right\}$$

has codimension greater than $A - (\dim(\mathcal{Q}_d) - \dim(\mathcal{G}_{d,r}))$ in each $\mathcal{G}_{d,r}$. (In fact, it suffices to show this for $r = 1$, but we will need the other cases below.)

Since the points are *distinct* and $\widehat{W}_{\vec{a}-\vec{r}}(p)$ is concentrated in $\pi_r^{-1}(p \times \mathcal{Q}_{d-r})$, it follows that the only nonempty intersections admit one or zero occurrances of an $\widehat{W}_{\vec{a}_j - \vec{r}}(p_j)$. Moreover, since we are assuming the Lemma for lower degree, we find that the intersection $\cap_{j=1}^{N} \pi_r^{-1}(\mathbf{P}^1 \times \overline{W}_{\vec{a}_j}(p_j))$ has codimension exactly A in $\mathcal{G}_{d,r}$, and since $\dim(\mathcal{G}_{d,r}) < \dim(\mathcal{Q}_d)$ (either by the structure theorem or a dimension count), we only have to prove (rearranging indices!) that intersections of the form:

$$(\dagger) \quad \pi_r^{-1}(\mathbf{P}^1 \times \cap_{j=1}^{N-1} \overline{W}_{\vec{a}_j}(p_j)) \cap \widehat{W}_{\vec{a}_N - \vec{r}}(p_N)$$

are of codimension greater than $A - (\dim(\mathcal{Q}_d) - \dim(\mathcal{G}_{d,r}))$ in $\mathcal{G}_{d,r}$.

Now consider the (largest) open subscheme $U_{d,r}(p_N) \subset \pi_r^{-1}(p_N \times \mathcal{Q}_{d-r})$ over which the restriction of the map $V \otimes \mathcal{O} \to \hat{\mathcal{S}}_{d-r}^*$ to $p_N \times \mathcal{G}_{d,r}$ is surjective. This restriction determines a map (which we may call evaluation at p_N) $ev_{p_N} : U_{d,r}(p_N) \to G(n-k+r, n)$. By the same argument as Lemma 2.2, one concludes that for any $Z \subset U_{d,r}(p_N)$, the intersection $Z \cap \widehat{W}_{\vec{a}_N - \vec{r}}(p_N)$ has codimension at least $|\vec{a}_N| - r(n-k)$ (and greater if $(a_N)_{n-k} - r < 0$) in Z. If we let Z be the (codimension $A - |\vec{a}_N|$) intersection of the $\mathbf{P}^1 \times \overline{W}_{\vec{a}_j}(p_j)$ with $U_{d,r}(p_N)$, then the open subset of $(\dagger)$ obtained by restricting to $U_{d,r}(p_N)$ has codimension at least $A - r(n-k) + 1$ in $\mathcal{G}_{d,r}$, and from the dimension count:

$$\begin{aligned} \dim(\mathcal{Q}_d) - \dim(\mathcal{G}_{d,r}) &= dn - [(d-r)n + 1 + r(k-r)] \\ &= r(n-k) + r^2 - 1 \end{aligned}$$

we get the desired result for the restriction of (†) to $U_{d,r}(p_N)$.

On the other hand, by (i) of Theorem 3.1, if $x \in \mathcal{Q}_d$ is in the image of $\pi_r^{-1}(p_N \times \mathcal{Q}_{d-r})$ but not in the image of $U_{d,r}(p_N)$ and $r < k$, then $\mathcal{T}_d$ has rank at least $n-k+r+1$ at (p_N, x), so x is in the image of $\mathcal{G}_{d,r+1}$.

Recall that we needed to prove the codimension estimate for (†) on $\mathcal{G}_{d,1}$ (since this surjects birationally onto the boundary). We could get the estimate for the open intersection with $U_{d,1}(p_N)$, and observed that the complement maps to the image of $\mathcal{G}_{d,2}$ (which is birational to $\mathcal{G}_{d,2}$). By the same reasoning and induction on r, we are therefore reduced to proving the codimension estimate for $\mathcal{G}_{d,k}$. (In other words, we still need to consider the case where p_N is a base point.)

But in this case, we have:

$$\begin{array}{lcl} \mathcal{G}_{d,k} & = & \mathbf{P}^1 \times \mathcal{Q}_{d-k}, \\ \widehat{W}_{\vec{a}_N - \vec{k}}(p_N) & = & p_N \times \mathcal{Q}_{d-k} \end{array}$$

and the sum $\sum_{j=1}^{N-1} |\vec{a}_j| = A - |\vec{a}_N|$ is certainly at least $A - k(n-k)$. This implies that the codimension of (†) in $\mathcal{G}_{d,k}$ is at least $A - k(n-k) + 1$, by induction on the degree, and we obtain the last case by the same dimension count as before with $r = k$.

Remark 3.2: If $\vec{b} = (b_1, ..., b_{n-k+1})$ is an $(n-k+1)$-tuple of integers satisfying $k \geq b_1 \geq ... \geq b_{n-k+1} \geq 0$, then we define $\overline{W}_{\vec{b}}(o)$ as in 2.1A. If $b_{n-k+1} = 0$, then $\overline{W}_{\vec{b}}(o) = \overline{W}_{\vec{b}^t}(o)$, where $\vec{b}^t = (b_1, ..., b_{n-k})$. However if $b_{n-k+1} \neq 0$, then $\overline{W}_{\vec{b}}(o) \subset \mathcal{B}_d$ and Theorem 3.1(iii) applies here, too, to give:

$$\beta_1^{-1}(\overline{W}_{\vec{b}}(o)) = \pi_1^{-1}(\mathbf{P}^1 \times \overline{W}_{\vec{b}}(o)) \cup \widehat{W}_{\vec{b}-\vec{1}}(o)$$

The proof of Moving Lemma 2.2A can be applied to distinct points $o, p_1, ..., p_N \in \mathbf{P}^1$, $\overline{W}_{\vec{b}}(o)$ $(b_{n-k+1} \neq 0)$, and "ordinary" Schubert varieties $\overline{W}_{\vec{a}_1}(p_1), ..., \overline{W}_{\vec{a}_N}(p_N)$. In this case, It tells us that the intersection has the expected codimension $|\vec{b}| + \sum |\vec{a}_i|$ in $\mathcal{Q}_d$, and that the image under β_1 of:

$$\widehat{W}_{\vec{b}-\vec{1}}(o) \cap \pi_1^{-1}(\{o\} \times \overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N) \cap \mathcal{M}_{d-1})$$

is Zariski dense in that intersection. But β_1 is an embedding when restricted to $\pi_1^{-1}(\{o\} \times \mathcal{M}_{d-1})$, so when the intersection consists of distinct points, they may be counted in $\mathcal{Q}_d$ or in $\pi_1^{-1}(\{o\} \times \mathcal{M}_{d-1})$, or even in $\pi_1^{-1}(\{o\} \times \mathcal{Q}_{d-1})$ (any extra points in the intersection $\widehat{W}_{\vec{b}-\vec{1}}(o) \cap \pi_1^{-1}(\{o\} \times \overline{W}_{\vec{a}_1}(p_1) \cap ... \cap \overline{W}_{\vec{a}_N}(p_N))$ would map to extra intersection points in $\mathcal{Q}_d$).

4. Quantum Schubert Calculus: In this section, we use Theorem 3.1 to prove the quantum versions of Giambelli and Pieri as stated in §1.

Proof of Quantum Giambelli: Suppose $M(W_{\vec{*}}) = c\prod_{\vec{a}} W_{\vec{a}}^{n_{\vec{a}}}$ is some monomial in the Schubert varieties. We define the Gromov-Witten invariants of M in the obvious way, by setting

$$\langle M(W_{\vec{a}})\rangle_d := c\langle ..., W_{\vec{a}}, ..., W_{\vec{a}}, ...\rangle_d.$$

where each $W_{\vec{a}}$ appears $n_{\vec{a}}$ times on the right. This definition extends in the obvious way to define Gromov-Witten invariants of any collection $P_1(W_{\vec{*}}), ..., P_N(W_{\vec{*}})$ of polynomials in the Schubert varieties. Also, the Conclusion following Corollary 2.4 applies to show that the Gromov-Witten invariant defined in this way coincides with the degree of the product of the $P_i(\sigma_{\vec{*}})$, thought of as polynomials in the generalized Schubert cohomology classes (see Corollary 2.3) when evaluated in the cohomology ring of the quot schemes $\mathcal{Q}_d$.

Thus, for example, the Giambelli determinantal formula for $W_{\vec{a}}$, evaluated with a quantum product, becomes:

$$\sum_{d\geq 0} q^d (\sum_{\vec{b}} \langle W_{\vec{b}}, \Delta_{\vec{a}}(W_*)\rangle_d \ \sigma_{\vec{b}}^c).$$

If we put this together with Lemma 2.5, then the quantum Giambelli formula is equivalent to the statement:

$$\langle W_{\vec{b}}, W_{\vec{a}}\rangle_d = \langle W_{\vec{b}}, \Delta_{\vec{a}}(W_*)\rangle_d \text{ for all } d \geq 0 \text{ and Schubert classes } W_{\vec{b}}.$$

But because these invariants are just the evaluations of the corresponding cohomology classes in the corresponding quot scheme, quantum Giambelli follows from the (even stronger!) assertion:

$$\sigma_{\vec{a}} = \Delta_{\vec{a}}(\sigma_*) \text{ in the cohomology ring of each } \mathcal{Q}_d.$$

Choose $p \in \mathbf{P}^1$. Then since $\sigma_{\vec{a}}$ is the image in cohomology of $\overline{W}_{\vec{a}}(p)$, which by the moving lemma has pure codimension $|\vec{a}|$ in $\mathcal{Q}_d$, this statement is a direct application of a theorem of Kempf and Laksov ([KL]) to the universal map $V \otimes \mathcal{O} \to \mathcal{S}_d^*$ of vector bundles on $\mathbf{P}^1 \times \mathcal{Q}_d$, or rather, to the restriction of the universal map to $\{p\} \times \mathcal{Q}_d$.

Proof of Quantum Pieri: There is a *polynomial* identity:

$$\sigma_a \Delta_{\vec{a}}(\sigma_*) = \sum_{\vec{b}} \Delta_{\vec{b}}(\sigma_*)$$

where $\Delta_{\vec{a}}$ and the $\Delta_{\vec{b}}$ are the Giambelli determinants and $\vec{b}$ varies over all $(n-k+1)$-tuples $(b_1, ..., b_{n-k+1})$ with $k \geq b_1 \geq a_1 \geq b_2 \geq ... \geq b_{n-k+1} \geq 0$. (See Lemma A.9.4 of [F1] for a proof of this.)

Note that the $\vec{b}$ are not $(n-k)$-tuples! In the classical Schubert calculus, any $\vec{b}$ with nonzero b_{n-k+1} gives rise to the empty variety in the Grassmannian, so if one sets those $\sigma_{\vec{b}}$ to zero, then the classical Pieri's formula results. However, when we evaluate them in the cohomology ring of $\mathcal{Q}_d$ for positive d, we have seen in Remark 3.2 that such $\vec{b}$ may give rise to nonzero varieties. In fact, I claim that if $d > 0$ and $W_{\vec{a}}$ is any Schubert variety, then:

$$\langle \Delta_{\vec{b}}(W_*), W_{\vec{a}} \rangle_d = \begin{cases} 0 \text{ if } b_{n-k+1} > 0 \text{ and } b_1 < k \\ \langle W_{(b_2-1,...,b_{n-k+1}-1)}, W_{\vec{a}} \rangle_{d-1} \text{ if } b_{n-k+1} > 0 \text{ and } b_1 = k \end{cases}$$

We'll prove this claim later. Let's first see how quantum Pieri follows.

The polynomial identity above, together with quantum Giambelli and classical Pieri, gives the following identity among quantum products:

$$(\dagger)\ \sigma_a * \sigma_{\vec{a}} = \sigma_a * \Delta_{\vec{a}}(\sigma_*) = p_{a,\vec{a}}(\sigma_{\vec{*}}) + \sum_{\{\vec{b}\ |\ b_{n-k+1} \neq 0\}} \Delta_{\vec{b}}(\sigma_*)$$

Multiply the formula in Lemma 2.5 by q to get:

$$q\sigma_{\vec{c}} = \sum_{d>0} q^d (\sum_{\vec{a}} \langle W_{\vec{a}}, W_{\vec{c}} \rangle_{d-1}\ \sigma_{\vec{a}^c})$$

for any Schubert cohomology class $\sigma_{\vec{c}}$. Using this formula applied to the Schubert cohomology class $\sigma_{(b_2-1,...,b_{n-k+1}-1)}$ together with the claim, we see that the last terms in $(\dagger)$ evaluate under the quantum product as follows:

$$\Delta_{\vec{b}}(\sigma_*) = \begin{cases} 0 \text{ if } b_1 < k \text{ and} \\ q\sigma_{(b_2-1,...,b_{n-k+1}-1)} \text{ if } b_1 = k \end{cases}$$

Putting this together with $(\dagger)$ gives the quantum Pieri formula.

Proof of the claim: Suppose that $\vec{b} = (b_1, ..., b_{n-k+1})$ satisfies the inequalities $k \geq b_1 \geq b_2 ... \geq b_{n-k+1} > 0$. Then by the theorem of Kempf-Laksov again, together with Remark 3.2, we know that when evaluated in the cohomology ring of $\mathcal{Q}_d$, the Giambelli determinant $\Delta_{\vec{b}}(\sigma_*)$ is equal to $\sigma_{\vec{b}}$, the image in cohomology of the degeneracy locus $\overline{W}_{\vec{b}}(p)$.

For such $\vec{b}$, let's **define** the Gromov-Witten invariants $\langle W_{\vec{b}}, W_{\vec{a}} \rangle_d$ to be the degree in the cohomology ring of $\mathcal{Q}_d$ of the product of $\sigma_{\vec{b}}$ and $\sigma_{\vec{a}}$. Equivalently, these Gromov-Witten invariants are the number of points **in the quot scheme** (as opposed to $\mathcal{M}_d$, which would trivially give zero) in $\overline{W}_{\vec{b}}(o) \cap \overline{W}_{\vec{a}}(p)$ for general translates of the flags and distinct points $o, p \in \mathbf{P}^1$. Then with this definition, the claim is equivalent to the equalities:

$$\langle W_{\vec{b}}, W_{\vec{a}} \rangle_d = \begin{cases} 0 \text{ if } b_{n-k+1} > 0 \text{ and } b_1 < k \\ \langle W_{(b_2-1,\ldots,b_{n-k+1}-1)}, W_{\vec{a}} \rangle_{d-1} \text{ if } b_{n-k+1} > 0 \text{ and } b_1 = k \end{cases}$$

for all $d > 0$ and $W_{\vec{a}}$.

Notice that the dimensions work out(!) In other words, if $b_1 = k$, then:

$$|\vec{b}| + |\vec{a}| = \dim(\mathcal{Q}_d) \Leftrightarrow |(b_2 - 1, ..., b_{n-k+1} - 1)| + |\vec{a}| = \dim(\mathcal{Q}_{d-1})$$

We assume that this equality holds (otherwise the claim is trivial). Then by Remark 3.2,

$$\langle W_{\vec{b}}, W_{\vec{a}} \rangle_d = \text{ number of points in } \widehat{W}_{\vec{b}-\vec{1}}(o) \cap \pi_1^{-1}(o \times \overline{W}_{\vec{a}}(p))$$

where $\pi_1^{-1}(\{o\} \times \mathcal{Q}_{d-1}) \subset \mathcal{G}_{d,1}$ is the projectivization of the restriction $\mathcal{S}_d(o)$ of $\mathcal{S}_d$ to $\{o\} \times \mathcal{Q}_{d-1}$, and $\widehat{W}_{\vec{b}-\vec{1}}(o)$ is the degeneracy locus for the map from $V \otimes \mathcal{O}$ to $\widehat{\mathcal{S}}^*$, where $\widehat{\mathcal{S}}$ is the universal subbundle of $\pi_1^*\mathcal{S}_d(o)$.

Thus the claim follows if we can show that in the homology of the smooth varieties $\pi_1^{-1}(\{o\} \times \mathcal{Q}_{d-1}) \subset \mathcal{G}_{d,1}$ and $\mathcal{Q}_{d-1}$,

$$(\pi_1)_*([\widehat{W}_{\vec{b}-\vec{1}}(o)].\pi_1^*[\overline{W}_{\vec{a}}(p)]) = \begin{cases} 0 \text{ if } b_1 < k \text{ and} \\ \left[\overline{W}_{(b_2-1,\ldots,b_{n-k+1}-1)}(o)\right].[\overline{W}_{\vec{a}}(p)] \text{ if } b_1 = k \end{cases}$$

But Kempf-Laksov applied to $\widehat{W}_{\vec{b}-\vec{1}}(o)$ and $\overline{W}_{(b_2-1,\ldots)}(o)$ makes this a special case of a formula of Jósefiak, Lascoux and Pragacz (see Example 14.2.2 of [F1] and [JLP]).

5. The Formula of Vafa and Intriligator: There is a marvelous residue formula due to Vafa and Intriligator which uses the presentation $(*)_q$ of the quantum cohomology ring to compute the Gromov-Witten intersection numbers:

$$\langle W_{a_1}, ..., W_{a_N}\rangle_d$$

of **special** Schubert varieties. (See [I],[ST], [B].) The formula is the following:

(Vafa and Intriligator's) Formula: Fix ζ a primitive nth root of $(-1)^k$ and assume that $0 \leq a_i \leq k$ and $a_1 + ... + a_N = \dim(\mathcal{M}_d)$. Then:

$\langle W_{a_1}, ..., W_{a_N}\rangle_d =$

$$(-1)^{\binom{k}{2}} n^{-k} \sum_{i_1 > ... > i_k} \sigma_{a_1}(\zeta^I) \cdots \sigma_{a_N}(\zeta^I) \left(\frac{\prod_{j \neq l}(\zeta^{i_j} - \zeta^{i_l})}{\prod_{j=1}^k \zeta^{(n-1)i_j}} \right)$$

where $\zeta^I = (\zeta^{i_1}, ..., \zeta^{i_k})$ and σ_{a_i} are the elemetary symmetric polynomials in k variables (i.e. $\sigma_0(\zeta^I) = 1, \sigma_1(\zeta^I) = \zeta^{i_1} + ... + \zeta^{i_k}$, etc.)

The point I want to make is that because of quantum Giambelli, the same formula computes all the Gromov-Witten intersection numbers . That is:

Corollary (of quantum Giambelli): Assume $W_{\vec{a}_1}, ..., W_{\vec{a}_N}$ are Schubert varieties on G satisfying $|\vec{a}_1| + ... + |\vec{a}_N| = \dim(\mathcal{M}_d)$. Then the Gromov-Witten intersection number:

$$\langle W_{\vec{a}_1}, ..., W_{\vec{a}_N}\rangle_d$$

may be computed by the Vafa-Intriligator formula, where the elementary symmetric polynomials $\sigma_{a_i}(\zeta^I)$ are replaced by the Giambelli determinants $\Delta_{\vec{a}_i}(\sigma_*(\zeta^I))$ of the elementary symmetric polynomials.

References

[B] A. Bertram, *Computing Schubert's calculus with Severi residues*, Lecture Notes in Pure and Applied Math. **179**, (1996) pp. 1-10.

[BDW] A. Bertram, G.Daskalopoulos and R. Wentworth, *Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians*, Journal of the AMS **9** (1996) pp. 529-571.

[BT] R. Bott and L. Tu, Differential Forms in Algebraic Topology, GTM **82**, Springer-Verlag New York, Heidelberg, Berlin (1982).

[F1] W. Fulton, Intersection Theory, Springer-Verlag Berlin Heidelberg New York Tokyo (1984).

[F2] W. Fulton, *Enumerative geometry via quantum cohomology*, preprint (1995).

[G] D. Gepner, *Fusion rings and geometry*, Comm. in Math. Physics, **141**, (1991), p. 381.

[Gro] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géometrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki **221** (1960/61).

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, Wiley-Interscience (1978).

[H] R. Hartshorne, Algebraic Geometry, Springer-Verlag (1977).

[I] K. Intriligator, *Fusion residues*, Modern Physics Letters A **6** (1991), Number 38, pp. 3543-3556.

[JLP] T. Józefiak, A. Lascoux and P. Pragacz, *Classes of determinantal varieties associated with symmetric and skew-symmetric matrices*, Math. USSR Izv. **18** (1982), pp. 575-586.

[KL] G. Kempf and D. Laksov, *The determinantal formula of Schubert calculus*, Acta. Math. **132** (1974) pp. 153-162.

[KM] M. Kontsevich and Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Preprint (1994).

[RRW] M.S. Ravi, J. Rosenthal and X.Wang, *Degree of the generalized Plücker embedding of a Quot scheme and quantum cohomology*, Preprint (1993).

[RT] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, Journal of Diff. Geo. **42** (1995) pp. 259-367.

[S] H. Shahrouz, PhD Thesis, Harvard.

[ST] B.Siebert and G.Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Preprint (1994).

[Str] S.A. Strømme, *On parametrized rational curves in grassmann varieties*, Lecture Notes in Math, **1266** (1987), pp. 251-272.

[V] C. Vafa, *Topological mirrors and quantum rings*, Essays on Mirror Manifolds (S.T. Yau, ed.), International Press (1992).

[W] E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, Preprint (1993).

University of Utah, Salt Lake City, UT 84112

email: bertram@math.utah.edu