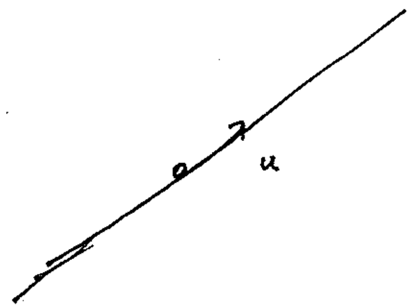
Lecture 10 · Subspaces of $\mathbb{R}^n$
· Basis & dimension

I) Subspaces

Geometrically:

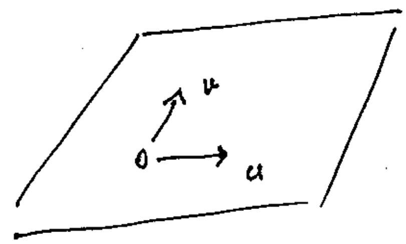
Span of 0 vector: $\{0\}$ — a point

Span of one vector: $\{a.u\}_{a\in\mathbb{R}}$



a line

Span of two vectors Span$\{u,v\}$



a plane

(corresponding to 0-dimensional subspace, 1-dimensional subspace, 2-dim subsp. of $\mathbb{R}^n$. More generally,

Def. A subset V of $\mathbb{R}^n$ is called a <u>subspace</u> of $\mathbb{R}^n$ if

1. $0 \in V$: V is nonempty
2. $\forall\ u, v \in V,\quad u+v \in V$: V is closed under addition
3. $\forall\ k\in\mathbb{R},\ u\in V,\quad ku \in V$: V is closed under scalar multiplication.

<u>Ex</u>. · $V=\{0\}$ the zero subspace.

· Every line through origin is a subspace

- $\mathbb{R}^n$ itself is a subspace of $\mathbb{R}^n$

- Any subspace of $\mathbb{R}^n \neq \mathbb{R}^n$ is called a proper subspace.

Prop Spans are subspaces : More precisely,

Let $u_1, u_2, \dots, u_p$ be vectors in $\mathbb{R}^n$. Then span $\{u_1, \dots, u_p\}$ is a subspace of $\mathbb{R}^n$.

"Proof" (i) $0 = 0u_1 + \dots + 0u_p \in \text{Span}\{u_1, \dots, u_p\}$

(ii) $(a_1 u_1 + \dots + a_p u_p) + (b_1 u_1 + \dots + b_p u_p)$

$= (a_1+b_1) u_1 + \dots + (a_p+b_p) u_p \in \text{Span}\{u_1, \dots, u_p\}$

(iii) $k.(a_1 u_1 + \dots + a_p u_p) = (ka_1) u_1 + \dots + (ka_p) u_p \in \text{Span}\{u_1, \dots, u_p\}.$

$\boxtimes$

Conversely any subspace is a span

Example A line in $\mathbb{R}^3$ through origin

In $\mathbb{R}^3$, Let L be the line through the origin that is parallel to the vector

$$d = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}.$$

Show that L is a subspace of $\mathbb{R}^3$.

Proof $\because$ $L = \text{Span}\{d\}$

$\therefore$ L is a subspace by the previous proposition. $\boxtimes$

Solution space of a homogeneous system of linear equations:

Prop Solution space of a homogeneous system of linear equations: Let A be an $m\times n$ matrix. Consider the matrix equation $AX=0$.

The solution set $V = \{x \in \mathbb{R}^n \mid Ax=0\}$ is a subspace of $\mathbb{R}^n$, called the solution space of $AX=0$.

Proof $0\in V$: $A0 = 0$

if $u, v \in V$, then $u+v\in V$: $A(u+v) = Au + Av = 0+0 = 0$

if $k\in\mathbb{R}$, $u\in V$, then $ku \in V$: $A(ku) = k(Au) = k.0 = 0$. $\boxtimes$

Ex. The plane $2x+3y-z$ is a subspace of $\mathbb{R}^3$.

Ex. (1) the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is NOT a subspace of $\mathbb{R}^3$

(ii) the plane $2x+3y-z=5$ is Not a subsp. of $\mathbb{R}^3$

(3) the set of vectors

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x, y, z \geq 0 \right\}$$

is Not a subspace of $\mathbb{R}^3$.

Exercise : 5.3.1 (c), (d); 5.3.2, $\boxed{5.3.6}$ extra credit.

↑

Recall $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

$$w \cdot u = w_1 u_1 + w_2 u_2 + w_3 u_3 + w_4 u_4$$

II) Basis :

Recall that I claimed previous that

Prop Any subspace in $\mathbb{R}^n$ is a span.

Proof Let V be a subspace of $\mathbb{R}^n$.

Let $\mathrm{Span}(V)$ be the span of all vectors in V.

Then $V = \mathrm{Span}(V)$, and hence a span.

- $V \subseteq \mathrm{span}(V)$ is obvious.
- $\mathrm{Span}(V) \subset V$ because V is a subspace.

$\boxtimes$

This proposition can be refined as follows.

THM. Let V be a subspace of $\mathbb{R}^n$. If V is <u>Not</u> a zero space, then $\exists$ linearly independent vectors $u_1, \dots, u_k$ in V s.t.

$$V = \mathrm{span}\{u_1, \dots, u_k\}.$$

Proof. $\because V \neq \{0\}$

$\therefore \exists\ u_1 \neq 0$

Let $V_1 = \text{span}\{u_1\}$. If $V_1 = V$, we are done.

o.w. $\exists \; u_2 \in V - V_1$ $\qquad$ $V - V_1 = \{u \in V \mid u \notin V_1\} \subseteq V$

Let $V_2 = \text{span}\{u_1, u_2\}$. If $V_2 = V$, we are done.

Contine in this manner, at the p-step ~~until we have a list of vectors,~~

$u_1, u_2, \dots, u_p$

s.t. $\quad V_p = \text{span}\{u_1, \dots, u_p\}$

Note that $u_1, \dots, u_p$ are linearly independent because there is no redundant vector.

Note that by a previous proposition, there are <u>at most</u> n linearly independent vectors in $\mathbb{R}^n$. Therefore the above process must stop after k steps for some $k \leq n$.

Then we have $\quad V = \text{span}\{u_1, u_2, \dots, u_k\}$, as desired.

$\boxtimes$

$\{u_1, \dots, u_k\}$ is ~~the~~ a skeleton of V: basis.

Definition Let V be a subspace of $\mathbb{R}^n$.

Then $\{u_1, \dots, u_k\}$ is a basis of V if

(1) $\text{Span}\{u_1, \dots, u_k\} = V$

(2) $u_1, \dots, u_k$ are linearly independent.

Example. Recall i, j, k from $\mathbb{R}^3$: $\{i, j, k\}$ is a basis of $\mathbb{R}^3$.

More generally let e_i be the vector in $\mathbb{R}^n$ whose ith component $=1$, 0 o.w.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th comp.} \quad \dots \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis

Proof. (1) all vectors are linear comb. of $e_1, \dots, e_n$.

(2) $[e_1 \; e_2 \dots e_n] = I_n$, REF, no free variable, hence linearly indep.

<u>EX</u> basis is NOT unique.

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is a basis of $\mathbb{R}^3$.

First of all $A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{2} \end{bmatrix}$ $\therefore$ u_1, u_2, u_3 linearly indep.

For any $b \in \mathbb{R}^3$, $Ax = b$ has a solution. $\because$ $A^{-1} \cdot Ax = b$
$\therefore$ $x = A^{-1}b$

$$\begin{array}{ccc} Ax & = & b \\ \| & & \| \\ [u_1 \; u_2 \; u_3]\, x & = & x_1 u_1 + x_2 u_2 + x_3 u_3 \end{array}$$

$\therefore$ any vector $b \in \mathbb{R}^3$ is a linear comb. of u_1, u_2, u_3, i.e.,

$$\operatorname{Span}\{u_1, u_2, u_3\} = \mathbb{R}^3$$

$\therefore$ $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$. $\boxtimes$

Prop Invertible matrices and bases of $\mathbb{R}^n$

Let A be an $n\times n$ matrix. Write $A=[a_1, \dots, a_n]$

Then A is invertible $\iff$ $a_1, a_2, \dots, a_n$ is linearly indep.

Proof

$$a_1, \dots, a_n \text{ linearly indep.}$$
$$\Updownarrow$$
$$\text{Ref}\left(A=[a_1 \dots a_n]\right) \text{ has no free variable column.}$$
$$\Updownarrow$$
$$\text{Ref } A = I$$
$$\Updownarrow$$
$$A \text{ invertible.}$$

$\boxtimes$

<u>Question</u> : How to find bases for subspaces of $\mathbb{R}^n$?

1. Basis of a span : easy : using casting-out algorithm

<u>Ex:</u>
Let $u_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $u_4 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$, $u_5 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

Find a basis of $\text{span}\{u_1, \dots, u_5\}$.

linearly indep.

<u>Solution</u>. $$[u_1, \dots, u_5] \xrightarrow[\text{equivalent}]{\text{row}} \begin{bmatrix} \boxed{2} & -1 & 1 & 3 & -1 \\ 0 & 0 & \boxed{3} & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

redundant vector.

$\therefore \{u_1, u_3\}$ a basis of $\text{span}\{u_1, \dots, u_5\}$.

Ex: Find a basis for the solution space of the system of equation

$$\begin{cases} x+y-z+3w-2v = 0 \\ x+y+z-11w+8v = 0 \\ 4x+4y-3z+5w-3v = 0 \end{cases}$$

The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 3 & -2 & 0 \\ 1 & 1 & 1 & -11 & 8 & 0 \\ 4 & 4 & -3 & 5 & -3 & 0 \end{array}\right] \xrightarrow[\text{equiv.}]{\text{row}} \begin{array}{c} \begin{array}{ccccc|c} x & y & z & w & v & \end{array} \\ \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & -4 & 3 & 0 \\ 0 & 0 & \boxed{1} & -7 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\therefore \quad \begin{array}{l} \boxed{x} + y - 4w + 3v = 0 \\ \boxed{z} - 7w + 5v = 0 \end{array}$$

$$\left\{ \begin{array}{l} x = -y + 4w - 3v \\ y = y \\ z = 7w - 5v \\ w = w \\ v = v \end{array} \right.$$

$$\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} -y + 4w - 3v \\ y \\ 7w - 5v \\ \cancel{w} \quad w \\ v \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 4 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis of the solution space.

Exercise: 5.4.1 (b),

III) Bases and coordinate systems

<u>Principle</u> : To give a basis of a subspace V is the same as to give a coordinate system for V. More preceisely,

Let $\{u_1, u_2, \dots, u_p\} = \mathcal{B}$ be a basis of V, i.e., $u_1, \dots, u_p$ is linearly independent and span V. Because they span V, any vector $v \in V$ is a linear combination of $u_1, \dots, u_p$, say

$$v = a_1 u_1 + \dots + a_p u_p, \qquad a_1, \dots, a_p \in \mathbb{R}.$$

Moreover since $u_1, \dots, u_p$ are linearly indep, $a_1, \dots, a_p$ are unique, they are called the <u>coordinate</u> of v with respect to the basis $\mathcal{B}$.

We write

$$[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}.$$

<u>Ex</u>. Let $\mathcal{B} = \{u_1, u_2, u_3\}$, $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

It is a basis of $\mathbb{R}^3$.

Let $[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Find v.

$$V = 1.u_1 - 1u_2 + 2u_3$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

This is the natural generalization of the coordinate under the standard basis

EX. Find the vector v that has coordinates

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Where $\mathcal{B}$ is the standard basis of $\mathbb{R}^3$

Solution $\quad V = 1 \cdot i - j + 2k = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$

Conversely, we can ask for the coordinates of a given vector under a given basis.

<u>Ex</u> Find the coordinate of the vector $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{u_1, u_2, u_3\}$ of $\mathbb{R}^3$, where

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

<u>Solution</u>. $[v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. if

$$v = a_1 u_1 + a_2 u_2 + a u_3$$

the associated augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\text{equiv}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \quad [v]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Exercise 5.4.1 (b), (c), 5.4.5

IV) Dimension :

Given a subspace V of $\mathbb{R}^n$. Suppose that we have two bases of V, say $\mathcal{B} = \{u_1, u_2, \dots, u_p\}$, $\mathcal{C} = \{w_1, w_2, \dots, w_q\}$

One is interested in seeing if

$$p = q \ ?$$

Example $\mathcal{B} = \{i, j, k\}$ is a basis of $\mathbb{R}^3$.

All other bases of $\mathbb{R}^3$ we encountered have 3 elts.

This is NOT coincident.

Theorem : Retaining the above setting, we have $p = q$.

The "proof" of the Theorem is based on the following

Lemma. If $u_1, \dots, u_r$ is a list of linearly independent vectors in $\operatorname{Span}\{v_1, \dots, v_s\}$, then $r \le s$.

Proof of THM.

If $\mathcal{B}$ and $\mathcal{C}$ are as above bases of V, then

$u_1, \dots, u_p \in \operatorname{Span}\{w_1, \dots, w_q\}$ linearly indep.

$\therefore \ p \le q$.

Switching the roles of $\mathcal{B}$ and $\mathcal{C}$, we have

$$q \le p.$$

$$\therefore \quad p = q. \qquad \boxtimes$$

<u>Proof of the Lemma</u> $\{u_1, \dots, u_r\}$, $\{v_1, \dots, v_s\}$

$\because$ u_j is in Span $\{v_1, \dots, v_s\}$,

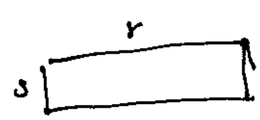
$$u_j = a_{1j} v_1 + \dots + a_{sj} v_s$$

Let $A = [a_{ij}]$ $s \times r$ $\qquad \begin{matrix} a_{1j} \\ \vdots \\ a_{sj} \end{matrix}$

(For the sake of contradiction.)

If $r > s$, we have r columns of pivot variables

But there are only s column. $\rightarrow\leftarrow$.

In other words, if $r > s$, ,

\# of pivot column $\le s < r$

$\Rightarrow$ there are free variables in the matrix eq. $Ax = 0$

$\Rightarrow$ $Ax = 0$ has non-trivial solutions. Pick one $x = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} \ne 0$.

So that $\quad a_{i1} x_1 + \dots + a_{ir} x_r = 0 \quad \forall\, i = 1, \dots, s$

Then

$$
\begin{aligned}
x_1 u_1 + \cdots + x_r u_r &= x_1 (a_{11} v_1 + \cdots + a_{s1} v_s) + \cdots + x_r (a_{1r} v_1 + \cdots + a_{sr} v_s) \\
&= (a_{11} x_1 + \cdots + a_{1r} x_r) v_1 + \cdots + (\cancel{a_{sr} v_1}\, a_{s1} x_1 + \cdots + a_{sr} x_r) v_s \\
&= 0 v_1 + \cdots + 0 v_s \\
&= 0
\end{aligned}
$$

$\therefore$ $u_1, \dots, u_r$ is linearly dependent $\rightarrow\!\leftarrow$ the assumption that $u_1, \dots, u_r$ being linearly independent.

So $r \leq s$.

$\boxtimes$

The Theorem leads to the notion of "dimension".

<u>Def</u>. Let V be a subspace of $\mathbb{R}^n$. Then the dimension of V, written $\dim(V)$, or $\dim V$, is defined to be the number of vector in a, and hence any, basis of V.

$$\dim\{0\} \overset{\text{def.}}{=} 0$$

<u>Ex</u>. $\dim \mathbb{R}^n = n$.

$\because \{e_1, \cdots, e_n\}$ is a basis of $\mathbb{R}^n$.

Ex. Let $V = \left\{ \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\in \mathbb{R}^3} \;\middle|\; x - y + 2z = 0 \right\}$

What is the dimension of V?

$$\begin{cases} x = y - 2z \\ y = y \\ z = z \end{cases} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad V = \operatorname{span} \underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{a basis of } V} \qquad \therefore \dim V = 2.$$

Example Let $W = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}}_{u_3}, \underbrace{\begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}}_{u_4}, \underbrace{\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{u_5}, \underbrace{\begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{u_6} \right\}$

What is the dimension of W?

Solution

$$[u_1 \cdots u_6] = \begin{bmatrix} 1 & 1 & 8 & -6 & 1 & 1 \\ 2 & 3 & 19 & -15 & 3 & 5 \\ -1 & -1 & -8 & 6 & 0 & 0 \\ 1 & 1 & 8 & -6 & 1 & 1 \end{bmatrix} \xrightarrow[\text{equiv}]{\text{row}} \begin{bmatrix} \boxed{1} & 0 & 5 & -3 & 0 & -2 \\ 0 & \boxed{1} & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \{u_1, u_2, u_5\}$ is a basis of W.

$\therefore \quad \dim W = 3.$

More properties of bases and dimension

THM. Every subspace of $\mathbb{R}^n$ has a basis.

Proof. This is a restatement of the fact that every subspace of $\mathbb{R}^n$ is a span. $\boxtimes$

The following lemma is also a restatement of a process we used.

<u>LEMMA</u> Let V be a subspace of $\mathbb{R}^n$.

If $u_1, \ldots, u_\ell \in V$ are linearly independent, then it is possible to extend $\{u_1, \ldots, u_\ell\}$ to a basis of V.

I.e., $\exists\ w_1, \cdots, w_s \in V$ s.t.

$$\{u_1, \ldots, u_\ell, w_1, \cdots, w_s\} \text{ is a basis of } V.$$

<u>Proof</u>. If $\operatorname{Span}\{u_1, \ldots, u_\ell\} = V$, we are done.

Otherwise, $\exists\ w_1 \in V - \operatorname{span}\{u_1, \cdots, u_\ell\}$.

Consider $\operatorname{Span}\{u_1, \ldots, u_\ell, w_1\}$, repeat the above process until $\exists\ w_1, \cdots, w_s$ s.t.

$$\operatorname{Span}\{u_1, \cdots, u_\ell, w_1, \cdots, w_s\} = V. \qquad \boxtimes$$

<u>Example</u> Extend $\{u_1, u_2\}$ to a basis of $\mathbb{R}^4$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$$

Consider

$$\begin{bmatrix} u_1 & u_2 & e_1 & e_2 & e_3 & e_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Big\downarrow \text{ row equiv.}$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 1 \end{bmatrix}$$

$\therefore$ ~~Span~~ $\{u_1, u_2, e_2, e_3\}$ is a basis of $\mathbb{R}^4$.

Example Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \;\middle|\; x+2y+z-w=0 \right\}$

Note that $u_1, u_2 \in V$, where

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Extend $\{u_1, u_2\}$ to a basis of V.

Solution. (1) Find a ~~solut~~ basis of V. : $y=r,\ z=s,\ w=t$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2r-s+t \\ r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left\{ v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } V.$$

(2) Form the matrix $[u_1, u_2, v_1, v_2, v_3]$ and find its echelon form.

$$\begin{bmatrix} 1 & -2 & -2 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & -2 & -2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 1 & -1 \\ 0 & 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \{u_1, u_2, v_1\}$ is a basis of V.

LEMMA. Let V be a subspace of $\mathbb{R}^n$. Let $u_1, \dots, u_p$ be a set of vectors spanning V. Then it is possible to obtain a basis of V by "shrinking" the set, i.e., throwing out the redundant vectors.

<u>Prop</u>. Let be a k-dimensional subspace of $\mathbb{R}^n$. Then

(a) Every linearly independent set of vectors in V has at most k vectors

(b) Every spanning set of vectors in V has at least k vectors.

<u>Proof</u>: skipped.

Prop Let V be a k-dimensional subspace of $\mathbb{R}^n$. Consider k vectors $u_1, \dots, u_k$ in V.

- If $u_1, \dots, u_k$ are linearly indep., then they form a basis for V
- If $u_1, \dots, u_k$ span V, then they form a basis for V.

Ex. Do the vectors

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution: $\because \dim \mathbb{R}^3 = 3$

$\therefore \{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$ if they are linearly ind.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-1} & -1 \\ 0 & 0 & \boxed{-1} \end{bmatrix}$$

$\therefore u_1, u_2, u_3$ linear indep. $\therefore$ they form a basis of $\mathbb{R}^3$.

Prop. Let V and W be subsp. of $\mathbb{R}^n$.

If $V \subseteq W$, then $\dim V \leq \dim V$.

If $\dim V = \dim W$, then $V = W$.

Exercise. 5.4.6, 5.4.7, 5.4.10, 5.4.15.