Lecture 14 Eigenvalues, eigenvectors and diagonalization

Let A be a square matrix. let v be a nonzero vector.

Av and v are unrelated, in general.

When they do relate i.e., Av is a scalar multiple of v,

then v is an eigenvector of A. Precisely,

Def. Let A be an $n \times n$ matrix. Let $v \in \mathbb{R}^n$ be a non-zero vector.

v is an eigenvector of A if

$$Av = \lambda v \quad \text{for some } \lambda \in \mathbb{R}.$$

In this case, λ is the corresponding eigenvalue of A.

Example Consider the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

Which of the following vectors are eigenvectors of A? Find the corresponding eigenvalue

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution

$$A v_1 = \begin{bmatrix} 3 & -2 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix} \quad \cdots\cdots \quad \overset{v_1}{\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}}$$

$$= 2 \cdot \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$v_1 \neq 0$

$\therefore$ v_1 is an eigenvector of A with eigenvalue $2 (=\lambda)$.

$$A v_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \quad \cdots \quad \overset{v_2 \,=}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$= 3 v_2$

$v_2 \neq 0$

$\therefore$ v_2 is an eigenvector of A with eigenvalue 3.

$$A v_3 = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \quad \cdots \quad \overset{v_3 \,=}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Not a scalar multiple of v_3,

hence v_3 is NOT an eigenvector of A.

$A v_4 = 0 = v_4$

$v_4 \neq 0$ Condition fails $\therefore$ $\underset{=0}{v_4}$ is NOT an eigenvector of A.

Example Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{bmatrix}$

Find the eigenvectors corresponding to the eigenvalue $\lambda = 2$.

Solution: So we are looking for vectors $v \in \mathbb{R}^3$ s.t.

- $Av = \lambda v$, i.e., $(A - \lambda I)v = 0$
- $v \neq 0$.

Say $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$,

$$A - \lambda I = A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & -2 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

all nonzero vectors of the above form are eigenvectors of A of eigenvalue $\lambda = 2$.

$\boxtimes$

As indicated in the above example, all eigenvectors of a given eigenvalue λ plus the zero vector, form a subspace of $\mathbb{R}^n$. Called the <u>eigenspace of λ</u>

<u>Def</u> Let A be an $n \times n$ matrix. let λ be an eigenvalue of A. The eigenspace of λ is the set

$$E_\lambda = \{ v \in \mathbb{R}^n \mid Av = \lambda v \}.$$

It is a subspace of $\mathbb{R}^n$.

<u>RMK</u>. Sometimes V_λ is used.

<u>Example</u> Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & -2 & 0 \end{bmatrix}$ Find a basis for each eigensp.

The matrix A has eigenvalues $\lambda = 1$ and $\lambda = 2$.

<u>Solution</u>. For $\lambda = 2$, we know from the previous computation

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } E_2.$$

For $\lambda = 1$, all eigenvectors of eigenvalue 1 must enjoy the property

$$Av = \lambda v = v \quad \text{i.e., } (A - I)v = 0$$

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\therefore \quad v = t \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore \quad E_\lambda \overset{\lambda=1}{=} E_1 \quad \text{has a basis} \left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}. \qquad \boxtimes$$

Exercise 8.1.1, 8.1.2, 8.1.5.

(II) How to find eigenvalues?

Here is the reduction:
Let A be an $n \times n$ matrix.
If λ is an eigenvalue of A,

then $\exists\ v \neq 0$ s.t. $Av = \lambda v$, i.e.,

$(A - \lambda I)v = 0$, i.e.,

$(A - \lambda I)x = 0$ has non-trivial solution(s), i.e.,

the matrix $A - \lambda I$ is <u>NOT</u> invertible, i.e.,

$\det(A - \lambda I) = 0$.

<u>THM</u> Let A be a square matrix. Let $\lambda \in \mathbb{R}$. Then

λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$.
$\iff \det(\lambda I - A) = 0$

<u>Example</u> Find the eigenvalues of $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda + 5 & -2 \\ 7 & \lambda - 4 \end{vmatrix}$$

$$= (\lambda+5)(\lambda-4) - (7)(-2)$$

$$= \lambda^2 + \lambda - 20 + 14$$

$$= \lambda^2 + \lambda - 6$$

$$= (\lambda+3)(\lambda-2)$$

$\begin{matrix} 1 & & 3 \\ & \times & \\ 1 & & -2 \end{matrix}$

Set $\quad \det(\lambda I - A) = 0$, i.e.,

$$(\lambda+3)(\lambda-2) = 0$$

$\therefore \quad \lambda = 2$ & $\lambda = -3$ are eigenvalues of A. $\boxtimes$

<u>Example</u> Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & -4 & 4 \\ 2 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & 4 & -4 \\ -2 & \lambda+1 & -2 \\ 0 & 0 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-2) \cdot \begin{vmatrix} \lambda-5 & 4 \\ -2 & \lambda+1 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-5)(\lambda+1) - (-2)\cdot 4\right]$$

$$= (\lambda-2)\left[\lambda^2 - 4\lambda - 5 + 8\right]$$

$$= (\lambda-2)\left[\lambda^2 - 4\lambda + 3\right]$$

$$= (\lambda-2)(\lambda-1)(\lambda-3)$$

$\therefore \quad \lambda = 1,\ \lambda = 2,\ \lambda = 3$ are eigenvalues of A.

Example Find the eigenvalue of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

set $\lambda^2+1=0$, we get

$$\lambda = \pm i, \qquad i = \sqrt{-1}. \qquad \boxtimes$$

$\therefore$ A has no real eigenvalues.

Def. A polynomial in variable λ is an expression of the form

$$p(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Def. The characteristic polynomial of a sq. matrix A is

$$p(\lambda) = \det(\lambda I - A).$$

or $ch_A \equiv ch_A(\lambda)$

Example. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}$$

$$p(\lambda) = \det(\lambda I - A) = \begin{bmatrix} \lambda-3 & 0 & -2 \\ -6 & \lambda-4 & -3 \\ 4 & 0 & \lambda+3 \end{bmatrix}$$

$$= (\lambda-4)\begin{vmatrix} \lambda-3 & -2 \\ 4 & \lambda+3 \end{vmatrix} = (\lambda-4)\left[(\lambda-3)(\lambda+3)+8\right]$$

$$= (\lambda-4)(\lambda^2-1)$$

$$= (\lambda-4)(\lambda+1)(\lambda-1)$$

Summing up, we have a procedure to find the eigenvalues and eigenvectors of A:

1. Compute $\det(\lambda I - A)$

2. set $\det(\lambda I - A) = 0$, and find its roots, i.e., eig.value

3. For each eigenvalue λ, find the solutions to

$$(A - \lambda I)v = 0 \qquad (\text{or } (\lambda I - A)v = 0)$$

these are eig vectors.

EX Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}$

Note $ch_A = (\lambda - 4)(\lambda + 1)(\lambda - 1)$, so

eigenvalues are $\lambda = -1, 1, 4$.

For $\lambda = -1$, we solve $(A + I)v = 0$, we get

$$v = s \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

For $\lambda = 1$, we solve $(A - I)v = 0$, we get

$$v = s \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 4$, We solve $(A - 4I)v = 0$, we get

$$v = s \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\boxtimes$

<u>Ex</u>. Find eigenvalues & eig. vectors of $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\det(\cancel{A - \lambda I}\ \lambda I - A) = \lambda^3 - 6\lambda^2 + 8\lambda = \lambda(\lambda^2 - 6\lambda + 8)$$
$$= \lambda(\lambda - 2)(\lambda - 4)$$

$\therefore$ $\lambda = 0, 2, 4$ are eig. values.

(So yes $\lambda = 0$ can be an eigenvalue.)

For $\lambda = 0$, We solve $Av = 0$,

$$v = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\vdots$

If $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$, then

$$\det(\lambda I - A) = (\lambda - 1)(\lambda - 4)(\lambda - 6),$$

i.e., $\lambda = 1, 4, 6$ are eig. values of A

<u>Prop</u>. If A is an upper/lower triangular matrix, then the diagonal entries are the eigenvalues of A.

<u>Exercise</u> 8.2.2., 8.2.4, 8.2.6.