Lecture 11

I) Column space, row space, and null space of a matrix.

i) ~~Let A mxm~~ Let A be an $m \times n$ matrix. The column space of A, written $\mathrm{col}(A)$, is the span of the columns.

$$\mathrm{col}(A) = \mathrm{Span}\{a_1, \dots, a_n\}, \quad \text{if } A = [a_1, \dots, a_n].$$

ii) The row space of A, written $\mathrm{row}(A)$, is the span of the rows:

$$\mathrm{row}(A) = \mathrm{Span}\{b_1, \dots, b_m\} \quad \text{if } A = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

iii) The null space of A, written as $\mathrm{null}(A)$, is the set

$$\mathrm{Null}(A) = \{x \mid Ax = 0\}.$$

FACT:

$\mathrm{Col}(A) \subseteq \mathbb{R}^m$

$\mathrm{Null}(A) \subseteq \mathbb{R}^n$

$\mathrm{row}(A)$ "$\subseteq$" $\mathbb{R}^n$ if we regard row vectors in $\mathbb{R}^n$.

EXAMPLE $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix}1\\4\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix} \right\} \quad (= \mathbb{R}^2)$$

$$\text{Row}\ \cancel{\text{Null}}\ (A) = \text{span}\left\{ [1\ 2\ 3], [4\ 5\ 6] \right\}$$

$$\text{Null}(A) = \text{Span}\{ x \mid Ax = 0 \}$$

<u>Prop</u> If A & B are row equivalet, then

$$\text{row}(A) = \text{row}(B) \quad \& \quad \text{Null}(A) = \text{Null}(B).$$

"Proof". Elem. row operations do NOT change the solution set.

Hence $\text{Null}(A) = \text{Null}(B)$.

· Elem. row operations lead to linear comb. of the row vectors

$$\text{row}(A) = \text{row}(B)$$

<u>EXAMPLE</u> Find a basis for the column space, row space & null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

i) <u>Column space</u> : $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\3\\7\end{bmatrix}, \begin{bmatrix}1\\6\\8\end{bmatrix}, \begin{bmatrix}3\\0\\6\end{bmatrix}, \begin{bmatrix}2\\2\\6\end{bmatrix} \right\}$

$$A \xrightarrow[\text{equiv.}]{\text{row}} \begin{bmatrix} \boxed{1} & 0 & -9 & 9 & 2 \\ 0 & \boxed{1} & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\3\\7\end{bmatrix} \right\}$ is a basis of $\mathrm{col}(A)$.

ii) row space : $\mathrm{row}(A) = \mathrm{Span}\{ [1,2,1,3,2], [1,3,6,0,2], [3,7,8,6,6] \}$

Now write the vectors column-wise (taking transpose)

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 7 \\ 1 & 6 & 8 \\ 3 & 0 & 6 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow[\text{eq.}]{\text{row}} ??? \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 5 & 0 \\ 9 & -3 & 0 \\ 2 & 0 & 0 \end{bmatrix}}$$

We can apply prop. that if $A \xrightarrow[\text{eq.}]{\text{row}} B$, then $\mathrm{row}(A) = \mathrm{row}(B)$

Take B to the reduced ech. form of A, we have

$\Rightarrow$ Basis of $\mathrm{row}(A) = \{ [1,0,-9,9,2], [0,1,5,-3,0] \}$

iii)

Finally, Null(A), we have

$$X = r\begin{bmatrix} 9 \\ -5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Null(A) has a basis $\left\{ \begin{bmatrix} 9 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\boxtimes$

<u>Prop</u> Let A be an $m \times n$ matrix. Then we have

$$\dim(\mathrm{col}(A)) = \mathrm{rank}(A) := \#\{\text{pivot columns}\}$$

$$\dim(\mathrm{row}(A)) = \mathrm{rank}(A)$$

$$\dim(\mathrm{Null}(A)) = n - \mathrm{rank}(A) = \#\{\text{free variable columns}\}.$$

Nullity of A : dim Null(A).

<u>THM</u>

$$\text{Ran}(K) + \text{Nullity}(A) = n.$$

<u>Example</u>. Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}$$

<u>Solution</u> $$\text{REF}(A) = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 13/2 \\ 0 & \boxed{1} & 0 & 2 & -5/2 \\ 0 & 0 & \boxed{1} & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank(A) = 3

Nullity(A) = 2

$3 + 2 = 5 =$ # of columns of A.

$\boxtimes$

<u>THM</u> The following statements are equivalent for an $m\times n$ matrix A.

(i) $\text{rank}(A) = n$

(2) $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$

(3) The columns of A are linearly independent in $\mathbb{R}^m$ (hence $m \geq n$)

(4) The $n\times n$ matrix $A^T A$ is invertible.

(5) A is left invertible, i.e., $\exists$ B s.t. $BA = I$

(6) The system $Ax = 0$ has only the trivial solution.

<u>Proof</u> Skipped.

THM The following are equivalent for an $m\times n$ matrix A

(1) $\text{rank}(A) = m$

(2) $\text{col}(A) = \mathbb{R}^m$

(3) The rows ~~space row~~ of A are linearly indep. in $\mathbb{R}^n$

(4) The $m\times m$ matrix AA^T is invertible

(5) A is right invertible

(6) $Ax = b$ is consistent for every $b \in \mathbb{R}^m$.

<u>Exercise:</u>

5.5.1 (b)