Lecture 13 Linear transformations in $\mathbb{R}^n$.

Recall from calculus, a <u>function</u> $f : \mathbb{R} \to \mathbb{R}$ is a rule that maps a real number $x \in \mathbb{R}$ to a real number $f(x) \in \mathbb{R}$

$$x \longrightarrow \boxed{f} \longrightarrow f(x).$$

More generally,

A (<u>vector</u>) function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that inputs an n-dimensional vector $v \in \mathbb{R}^n$ and outputs an m-dimensional vector $T(v) \in \mathbb{R}^m$

$$\underset{\substack{\cap \\ \mathbb{R}^n}}{v} \longrightarrow \boxed{T} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^m}}{T(v)}$$

Ex.

$$T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x^2 \\ x+y \\ y^2 \end{bmatrix}$$

The study of these functions is the socalled Multivariable Calculus.

In this ~~caurse~~ course, we are only interested in the linear ones.

<u>Definition</u> A vector function $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transform-ation, if it satisfies the following two conditions

1) T preserves addition, i.e., for all $v, w \in \mathbb{R}^n$,

$$T(v+w) = T(v) + T(w)$$

2) T preserves scalar multiplication, that is $\forall\ v \in \mathbb{R}^n,\ k \in \mathbb{R}$,

$$T(k.v) = k.T(v).$$

<u>EX</u> $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ is linear (transformation)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ x+y+z \\ 0 \end{bmatrix}$$

1) $T_2(v+w) \stackrel{?}{=} T_2(v) + T_2(w) \Leftarrow \begin{bmatrix} v_1+v_2 \\ v_1+v_2+v_3 \\ 0 \end{bmatrix} + \begin{bmatrix} w_1+w_2 \\ w_1+w_2+w_3 \\ 0 \end{bmatrix}$

$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad$ $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$T_2\left(\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{bmatrix}\right) = \begin{bmatrix} (v_1+w_1)+(v_2+w_2) \\ (v_1+w_1)+(v_2+w_2)+(v_3+w_3) \\ 0 \end{bmatrix}$$

$// \checkmark$

2). $T_2(k.v) \overset{?}{=} k.T_2(v)$

$$\begin{bmatrix} kv_1+kv_2 \\ kv_1+kv_2+kv_3 \\ 0 \end{bmatrix} = k.\begin{bmatrix} v_1+v_2 \\ v_1+v_2+v_3 \\ 0 \end{bmatrix}$$

√

$\therefore$ T_2 is a linear transformation.

Ex T_1 is NOT linear.

Note $T(0)=0$ $\because$ $T(0)=T(0+0)=T(0)+T(0)$

$\therefore T(0)=0$

Prop A vector function $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if

$\forall\ v, w \in \mathbb{R}^n,\ a, b \in \mathbb{R}$

$$T(av+bw) = a.T(v) + b.T(w)$$

Exercise 6.1.1.

II) The matrix of a linear transformation.

Prop. Let A be an $m \times n$ matrix, and consider the vector function

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ defined by}$$

$$Tv = Av.$$

Then T is linear.

Proof. (1) $T(v+w) = A(v+w) = Av + Aw = Tv + Tw$

(2) $T(kv) = A(kv) = k.Av = kT(v)$.

$\boxtimes$

THM. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Then there exists an $m \times n$ matrix A such that for all $v \in \mathbb{R}^n$,

$$Tv = Av.$$

In other words, any linear transformation is a matrix transformation.

"Proof." Recall that $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

For each i, Let $u_i = T(e_i)$, and let

$$A = [u_1 \; \ldots \; u_n] = [T(e_1) \; T(e_2) \; \ldots T(e_n)]$$

Then A is an $m \times n$ matrix. We claim that

$$T(v) = Av.$$

For $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \cdots + x_n e_n$

$$\begin{aligned} \therefore \quad T(v) &= T(x_1 e_1 + \cdots + x_n e_n) \\ &= T(x_1 e_1) + \cdots + T(x_n e_n) \\ &= x_1 T(e_1) + \cdots + x_n T(e_n) \\ &= \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A \cdot v \end{aligned}$$

$\boxtimes$

Example Suppose that $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation where

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}9\\-3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix},$$

then
$$A = [T(e_1), T(e_2), T(e_3)]$$
$$= \begin{bmatrix}1 & 9 & 1\\2 & -3 & 1\end{bmatrix}.$$

EX. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x+y\\x+2y-z\end{bmatrix}$$

for all $x, y, z \in \mathbb{R}$. Find the matrix of this linear transformation.

Solution: The matrix is given by $A = [T(e_1), T(e_2), T(e_3)]$

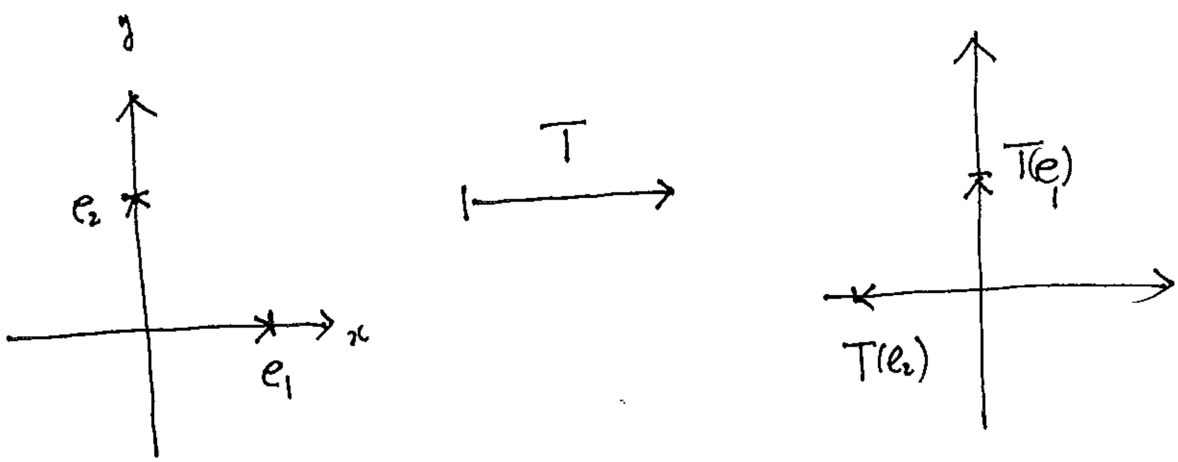
$$T(e_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1+0\\1+2\times0-0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$
$$T(e_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0+1\\0+2\times1-0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$
$$T(e_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0+0\\0+2\times0-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$$

So $A = \begin{bmatrix}1 & 1 & 0\\1 & 2 & -1\end{bmatrix}$ $\boxtimes$

Exercise 6.2.1, 6.2.3.

III) Geometric interpretation of linear transformations

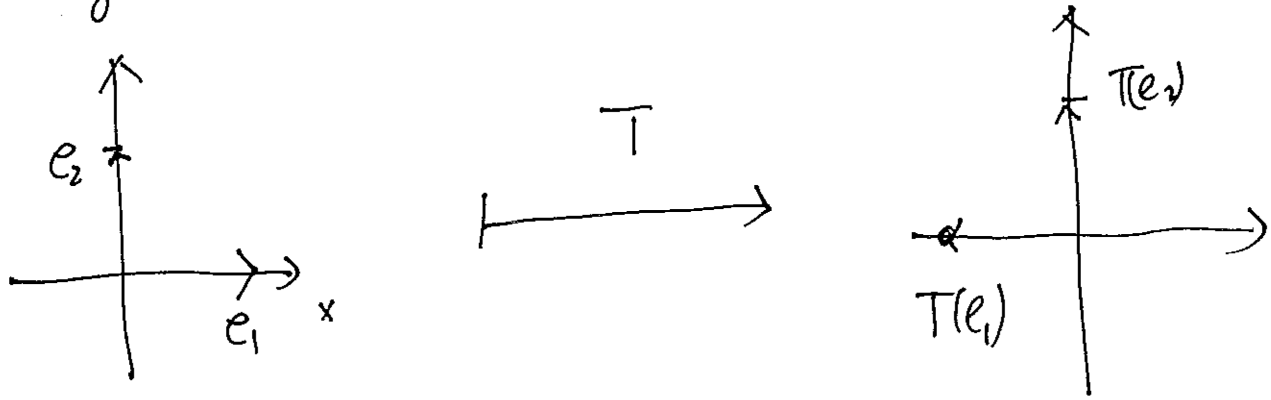
Example Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that is given by a counterclockwise rotation by $90°$. Find the matrix A corresponding to this linear transformation. Find a formula for T.



$\Rightarrow$ The matrix is $[T(e_1), T(e_2)] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

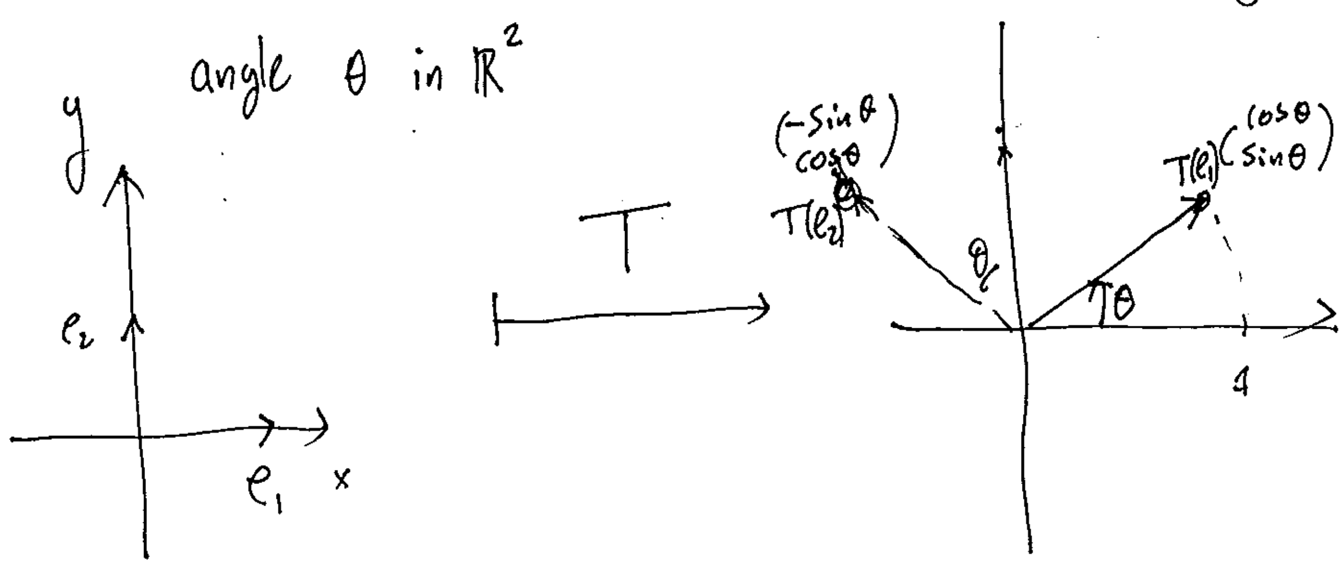
Example Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection about the y-axis. Find the matrix A corresponding to this linear transformation, and a formula for T



$$A = \left[T(e_1) \ T(e_2) \right] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Example. Find the matrix A for a counterclockwise rotation by angle θ in $\mathbb{R}^2$



$$A = \begin{bmatrix} Te_1 & Te_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Exercise 6.3.1

IV) Properties of linear transformations

<u>Proposition</u>: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

(1) T preserves the zero vector, i.e., $T(0) = 0$

(2) T preserves negation: $T(-v) = -T(v)$

(3) T preserves linear combination:

$$T(a_1 v_1 + a_2 v_2 + \dots + a_p v_p) = a_1 T(v_1) + a_2 T(v_2) + \dots + a_p T(v_p).$$

<u>Proof</u>

$$T(\underset{\mathbb{R}^n}{\vec{0}}) = T(\underset{\mathbb{R}}{0} \cdot \underset{\mathbb{R}^n}{\vec{0}}) = 0 \cdot T(\vec{0}) = \underset{\mathbb{R}^m}{\vec{0}} \qquad \boxtimes$$

<u>Example</u>: Linear combination

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear ~~combination~~ transformation such that

$$T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T\left(\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}\right)$

<u>Solution</u>: Try to write $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$, and then use the above proposition.

In particular, we have

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Thus,

$$T\left(\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2\cdot\begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 10 \\ -2 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$$

$\boxtimes$

Composition of Linear transformations

Let $S: \mathbb{R}^k \to \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations.

Then the composition of S and T is the linear transformation

$$T \circ S : \mathbb{R}^k \to \mathbb{R}^m$$

defined by $\big(T \circ S\big)(v) = T\big(S(v)\big) \quad \forall\, v \in \mathbb{R}^k$

Pictorially

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "S"] \arrow[rr, bend right, "T \circ S"'] & \mathbb{R}^n \arrow[r, "T"] & \mathbb{R}^m \end{tikzcd}$$

THM. Let $\mathbb{R}^k \xrightarrow{S} \mathbb{R}^n$, $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ be linear transformations. let A be the matrix corresponding to S, and let B be the matrix corresponding to T. Then the matrix corresponding to the composite $T \circ S$ is $B \cdot A$.

Proof $\quad T \circ S(v) = T(Sv) = T(Av) = B(Av) = (BA)v$. $\boxtimes$

Example Find the matrix for counterclockwise rotation by angle $\theta+\phi$ in two different ways, and compare.

Solution. Let A_θ be the matrix of a rotation by θ.

A_ϕ . . . . ϕ.

Then a rotation by the angle $\theta+\phi$ is given by the product of these two matrices

$$A_\theta A_\phi = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix}$$

$$\overset{(*)}{=} \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = A_{\theta+\phi},$$

where $(*)$ is due to

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Def. Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Suppose that for each $v \in \mathbb{R}^n$,

$$S \circ T(v) = v \quad \& \quad T \circ S(v) = v.$$

Then S is called the inverse of T, written as $S = T^{-1}$.

EX. what is the inverse of a counterclockwise rotation by the angle θ in $\mathbb{R}^2$?

Solution: the inverse is a clockwise rotation by the same angle.

THM. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. let A be the corresponding $n \times n$ matrix. Then T has an inverse if and only if the matrix A is invertible. In this case, the matrix of T^{-1} is A^{-1}.

Ex. Find the inverse of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 7x+4y \end{bmatrix}$.

Find the matrix $\overset{A}{\text{of}}$ T:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$\therefore \quad T^{-1}(v) = A^{-1} \cdot v$, i.e.,

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x-y \\ -7x+2y \end{bmatrix}.$$

$\boxtimes$

Exercise: 6.4.1., 6.4.3, 6.4.7, 6.4.10.