5.4.4)

$$V_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad V = a_1 u_1 + a_2 u_2 + a_3 u_3$$

$$V = 2u_1 + 1u_2 + (-3)u_3$$

$$2\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 7 \\ -7 \end{bmatrix}}$$

5.4.5) $V = c_1 u_1 + c_2 u_2 + c_3 u_3$

$$\begin{cases} c_1 + 0c_2 + 2c_3 = 4 \\ 0c_1 + c_2 + 2c_3 = 3 \\ 3c_1 + c_2 + c_3 = 8 \end{cases}$$

$$3(4-2c_3) + (3-2c_3) + c_3 = 8$$

$$12 - 6c_3 + 3 - 2c_3 + c_3 = 8 \qquad V_B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$15 - 7c_3 = 8$$

$w = d_1 u_1 + d_2 u_2 + d_3 u_3$

$$\begin{cases} d_1 + 2d_3 = -1 \\ d_2 + 2d_3 = -1 \\ 3d_1 + d_2 + d_3 = 3 \end{cases}$$

$$3(-1-2d_3) + (-1-2d_3) + d_3 = 3$$

$$-3 - 6d_3 - 1 - 2d_3 + d_3 = 3$$

$$-4 - 7d_3 = 3$$

$$d_3 = -1$$

$$w_B = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

5.4.6)

$$a\begin{bmatrix}3\\3\\-6\end{bmatrix}+b\begin{bmatrix}0\\-1\\p\end{bmatrix}=\begin{bmatrix}3a\\3a-b\\-6a+pb\end{bmatrix}$$

$$z=-6\left(\frac{x}{3}\right)+p(x-y)=-2x+px-py=(p-2)x-py$$

$$\begin{bmatrix}3&0&0\\3&-1&0\\-6&p&1\end{bmatrix}\qquad\boxed{\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

5.4.7) $u_1 = 1+1+0+2(-1) = 0$

$u_2 = -2+(-2)+(-2)+2(3) = 0$

$u_3 = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$ $\boxed{\left\{\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}-2\\-2\\-2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\right\}}$

$A\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}$ $\begin{bmatrix}-1 & 2 & -1\\2 & -1 & -1\\-1 & -1 & 2\end{bmatrix}$ $\begin{matrix}R_1\cdot -1\\R_2-2R_1\\R_3+R_1\end{matrix}$ $\begin{bmatrix}1 & -2 & 1\\0 & 3 & -3\\0 & -3 & 3\end{bmatrix}$

$R_3+R_2\begin{bmatrix}1 & -2 & 1\\0 & 3 & -3\\0 & 0 & 0\end{bmatrix}$ Linearly dependent

NO

5.4.15) No, as a 5-dimensional space, $\mathbb{R}^5$ is spanned by 5 different vectors, anything that is linearly independent can have at most 5 in a set.

5.5.1) b)

$\begin{bmatrix}1 & 3 & 0 & 2\\3 & 9 & 1 & 7\\1 & 3 & 1 & 3\end{bmatrix}$ $\begin{matrix}R_2-3R_1\\R_3-R_1\end{matrix}$ $\begin{bmatrix}1 & 3 & 0 & 2\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}$ R_3-R_2 $\begin{bmatrix}1 & 3 & 0 & 2\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$

Rank: 2

nulity: 4-2 = 2

$\begin{bmatrix}1\\3\\1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\begin{bmatrix}1 & 3 & 0 & 2\end{bmatrix}$ $\begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}$

$\left\{\begin{matrix}x_1+3x_2+0x_3+2x_4=0\\x_3+x_4=0\end{matrix}\right\}$ $x = \begin{bmatrix}-3t-2s\\t\\-s\\s\end{bmatrix} = t\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}$

Rank = 2

Nullity: 2

Row(B) = span$\{[1\ 3\ 0\ 2], [0\ 0\ 1\ 1]\}$

Col(B) = span$\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ Null(B) = span$\left\{\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}\right\}$

7.1.1 (b)) $B = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$ $\det = ad - bc$

$(0)(2) - (3)(0) = \boxed{0}$

7.1.2(c))

$C = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ $\det = 3 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}$

$3 \cdot (-3) - 4(-1) + 1(1) = \boxed{-4}$

7.2.1

$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$ $R_1 = 1, 2, 4$ $R_1 = 1, 4$

$R_2 = 0, 1, 3$ $R_2 = 0, 3$

$R_3 = -2, 5, 1$ $\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} (1)(3) - (4)(0) = 3$

$\boxed{M_{32} = 3}$

$A = \begin{bmatrix} 0 & -1 & 3 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 3 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ $R_1 : 0, 3, 1$ $R_2 : 1, 2, 2$ $R_4 : 1, 0, 1$ $B = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \cdot (1) - 2 \cdot (1) = -1$

$0 - 3(-1) + 1(-2) = 0 + 3 - 2 = 1$

$\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1(0) - 2(1) = -2$

$\boxed{M_{32} = 1}$

$C_{32} = (-1)(M_{32}) = (-1)(1) = -1$ $\boxed{C_{32} = -1}$

7.3.2(b))

$\begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix}$ $M_{21} = (-2)(1) - (2)(3) = -2 - 6 = -8$

$C_{21} = -(-8) = 8$

$\det(B) = 3 \cdot 8 = \boxed{24}$

c) $\det(C) = 4C_{31} + C_{33}$

$\begin{bmatrix} 2 & -2 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix}$ $2(2 \cdot 2 - 3 \cdot 1) - (-2)(3 \cdot 2 - 3 \cdot 2) + 2(3 \cdot 1 - 2 \cdot 2) = 0$

$M_{31} = 0$ $C_{31} = 0$

7.5.10)

$C_{33} = (-1)^{3+3} M_{33} = M_{33}$

$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ $M_{33} = 0$ $C_{33} = 0$ $\det = 4 \cdot 0 + 1 \cdot 0 = \boxed{0}$

$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & a^2-1 \\ 0 & b-1 & b^2-1 \end{vmatrix}$

$\det(A) = 1 \cdot \begin{vmatrix} a-1 & a^2-1 \\ b-1 & b^2-1 \end{vmatrix} = \begin{vmatrix} a-1 & (a-1)(a+1) \\ b-1 & (b-1)(b+1) \end{vmatrix}$

$(a-1)(b^2-1) - (a^2-1)(b-1)$

$(a-1)(b-1)(b+1) - (a-1)(a+1)(b-1)$

$(a-1)(b-1)((b+1) - (a+1))$

$\boxed{(a-1)(b-1)(b-a)}$

7.5.13

a) False $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

b) True, two columns are equal, determinant $= 0$

c) False $A = I_2, B = I_2$

$\det(A+B) = \det(2I) = 4$

$\det(A) + \det(B) = 1 + 1 = 2$

d) False

e) True, $AA^{-1} = I$ $\det(A)\det(A^{-1}) = 1$ so $\det(A^{-1}) = 1/\det(A)$

f) True, multiplying one row by scalar multiplies determinant by scalar

g) True, $-A = (-1)A$, so $\det(-A) = (-1)^n \det(A)$

h) True, $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \geq 0$

i) True, $A^k=0$ then $\det(A^k)=(\det(A))^k=0$ then $\det(A)=0$

j) True, existence of nonzero solution $Ax=0$ means columns are linearly dependent, $\det(A)=0$

7.6.1) B) $M_{11}=\begin{vmatrix}1 & -2\\ -1 & 3\end{vmatrix}=1\cdot3-(-2)(-1)=1$

$C_{11}=(1)(1)=1$

$M_{12}=\begin{vmatrix}-1 & -2\\ 2 & 3\end{vmatrix}=(-1)(3)-(-2)(2)=1$

$C_{12}=(-1)(1)=-1$

$M_{13}=\begin{vmatrix}-1 & 1\\ 2 & -1\end{vmatrix}=(-1)(-1)-(1)(2)=-1$

$C_{13}=(1)(-1)=-1$

$M_{21}=\begin{vmatrix}0 & 1\\ -1 & 3\end{vmatrix}=0\cdot3-1(-1)=1$

$C_{21}=(-1)(1)=-1$

$M_{22}\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}=1\cdot3-1\cdot2=1$

$C_{22}=(1)(1)=1$

$M_{23}=\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix}=1(-1)-0(2)=-1$

$C_{23}=(-1)(-1)=1$

$M_{31}\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}=0(-2)-1\cdot1=-1$

$C_{31}=(1)(-1)=-1$

$M_{32}=\begin{vmatrix}1 & 1\\ -1 & -2\end{vmatrix}=1(-2)-1(-1)=-1$

$C_{32}=(-1)(-1)=1$

$M_{33}=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}=1(1)-0(-1)=1$

$C_{33}=(1)(1)=1$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \qquad adj(B) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

7.6.4)

$$\det(A) = 3 \cdot \begin{vmatrix} 2 & -3 \\ 4 & -3 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ -5 & 4 \end{vmatrix}$$

$$3((2)(-3) - (-3)(4)) + 3((-1)(4) - (2)(-5))$$

$$\det(A) = 36$$

$$C_{11} = (1) \cdot \begin{vmatrix} 2 & -3 \\ 4 & -3 \end{vmatrix} = 6$$

$$C_{12} = (-1) \cdot \begin{vmatrix} -1 & -3 \\ -5 & -3 \end{vmatrix} = 12$$

$$C_{13} = (1) \cdot \begin{vmatrix} -1 & -2 \\ -5 & 4 \end{vmatrix} = 6$$

$$C_{21} = (-1) \begin{vmatrix} 0 & 3 \\ 4 & -3 \end{vmatrix} = 12$$

$$C_{22} = (1) \begin{vmatrix} 3 & 3 \\ -5 & -3 \end{vmatrix} = 6$$

$$C_{23} = (-1) \begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix} = -12$$

$$C_{31} = (1) \begin{vmatrix} 0 & 3 \\ 2 & -3 \end{vmatrix} = -6$$

$$C_{32} = (-1) \begin{vmatrix} 3 & 3 \\ -1 & -3 \end{vmatrix} = 6$$

$$C_{33} = (1) \begin{vmatrix} 3 & 0 \\ -1 & 2 \end{vmatrix} = 6$$

$$C = \begin{bmatrix} 6 & 12 & 6 \\ 12 & 6 & -12 \\ -6 & 6 & 6 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} 6 & 12 & -6 \\ 12 & 6 & 6 \\ 6 & -12 & 6 \end{bmatrix}$$

$$\boxed{A^{-1} = 1/36 \begin{bmatrix} 6 & 12 & -6 \\ 12 & 6 & 6 \\ 6 & -12 & 6 \end{bmatrix}}$$