Lecture 15

III) Geometric interpretation of eigenvectors

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

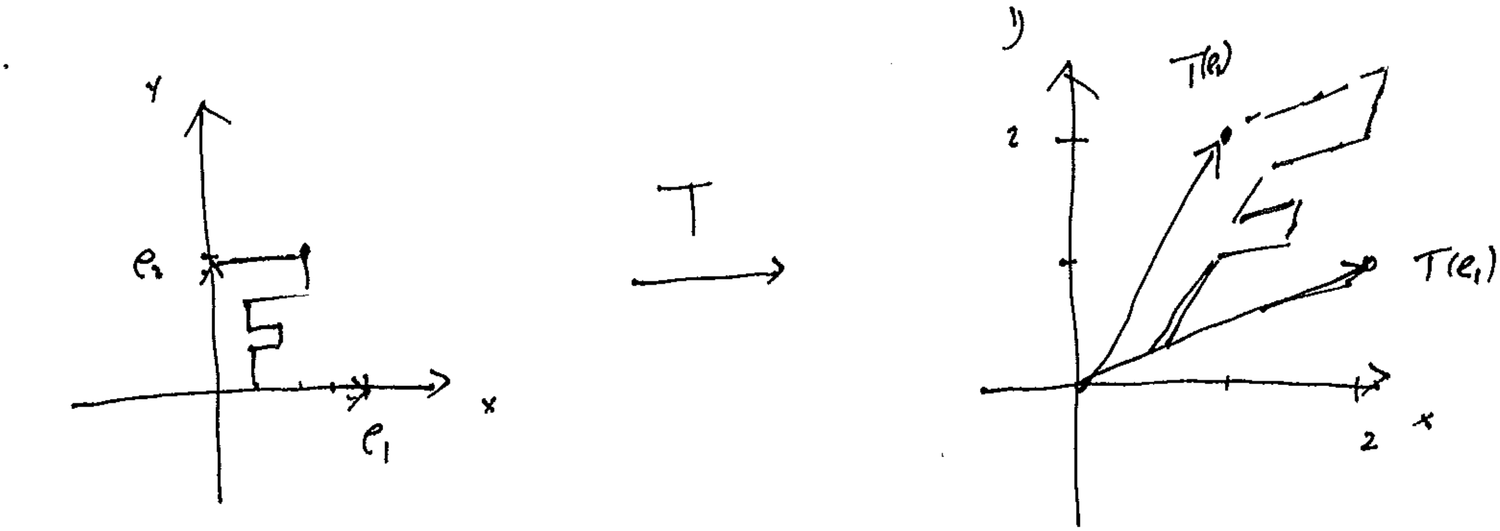
it defines a linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

In particular

$$T(e_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Pictorially, the before-and-after picture for this transformation



The letter "F" is being distorted. A short calculation shows that the basic eigenvectors are

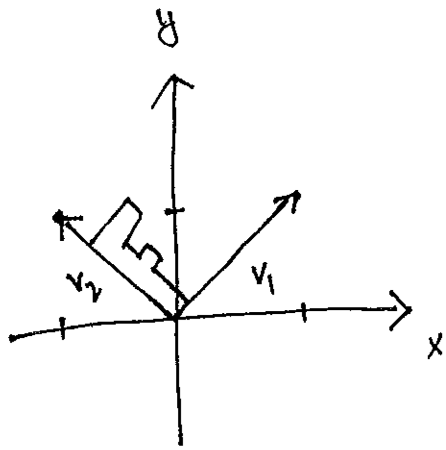
$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

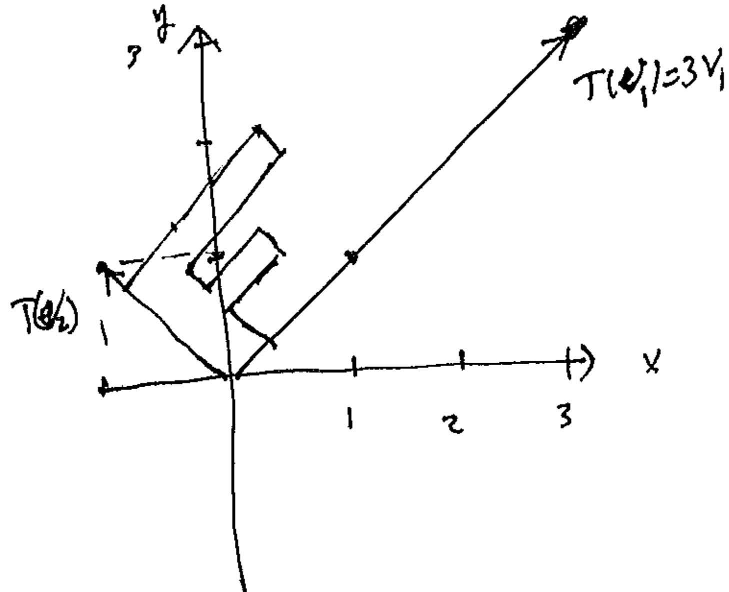
With corresponding eigenvalues $\lambda_1 = 3$ $\lambda_2 = 1$

So $\quad T(v_1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3v_1$

$$T(v_2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = v_2$$

So a more useful before-and-after picture is as follows





So the linear transformation described by A is revealed to be just a rescaling by a factor of 3 along the the direction of $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In Summary the geometric meaning of an eigen vector is that it is mapped to a multiple of itself. Thus when viewed from the point of view of its action on eigenvectors, a linear transformation behaves like a scaling of each eigenvectors.

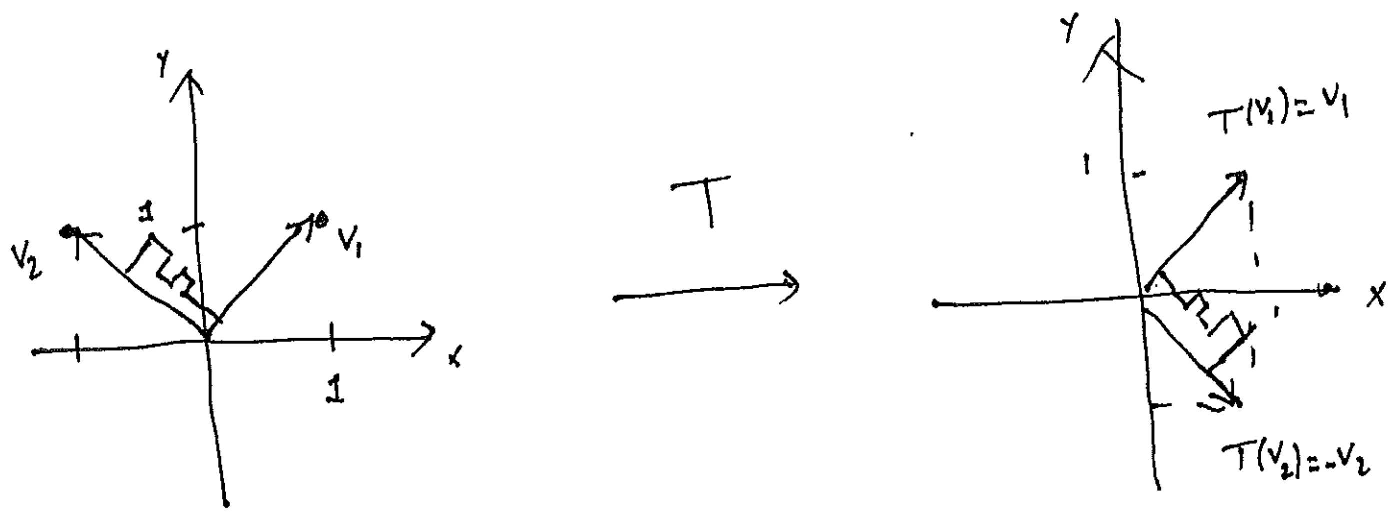
Example. Visualize the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution. 1. eigenvalue : $\det(\lambda I - A) = \begin{vmatrix} \lambda - 0 & -1 \\ -1 & \lambda - 0 \end{vmatrix} = \lambda^2 - 1 = 0$

$\lambda = \pm 1$

2. $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ eig. vectors of eig. values

$\lambda_1 = 1$ $\quad \lambda_2 = -1$



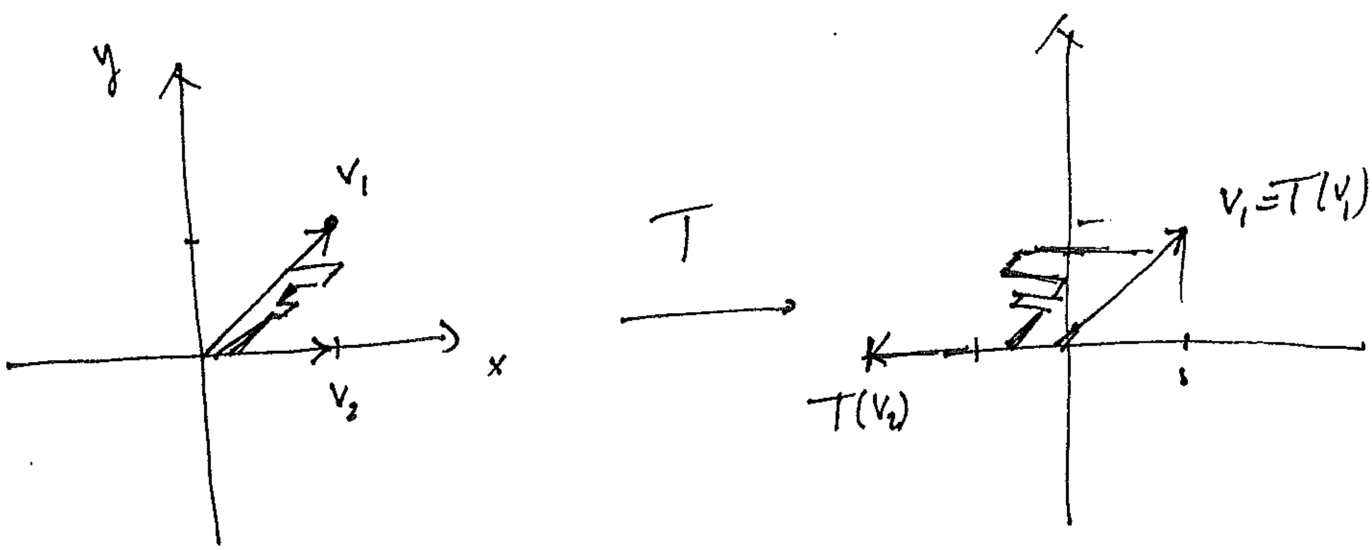
So T is a reflection along the vector v_1.

Ex Visualize the linear tranformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix}$$

1. eigenvalue $\lambda_1 = 1$, $\lambda_2 = -2$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$



a slanted reflection with scaling

Exercise : 8.3.1 (C). 8.3.3.

IV) Diagonalization.

A diagonal matrix is a square matrix whose off-diagonal entries are zero. Typically,

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{bmatrix},$$

or $D = (d_{ij})$, $d_{ij} = 0$ if $i \neq j$.

What is good about a diagonal matrix?

1. eigenvalues : diagonal entries
2. basic eigenvectors: $e_1, \ldots, e_n$.
3. closed under matrix addition, matrix multiplication, i.e.,

$$D, D' \text{ diag.} \Rightarrow \begin{array}{l} D+D' \text{ diagonal} \\ D \cdot D' \text{ diagonal} \end{array}$$

Ex. $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $D' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$DD' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Diagonalization is a process of "reducing" a matrix to a diagonal matrix.

<u>Definition</u>. (i) Let A, B be two square matrices. We say that A & B are <u>similar</u> if $\exists$ an invertible matrix P s.t.

$$B = P^{-1} A P.$$

(ii) A matrix is <u>diagonalizable</u> if it is similar to a diagonal matrix.

A characterization for a matrix being diagonalizable is

THM An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover,

let $P = [v_1 \ v_2 \ \cdots \ v_n]$, $\{v_1, \cdots, v_n\}$ linearly indep. eigenvector of A

let $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_1 \ldots \lambda_n$ corresp. eigenvalues.

Then $D = P^{-1} \cdot A P$.

"Proof"

$$Av_i = \lambda_i v_i$$

$$A\,[v_1\ v_2 \cdots v_n] = [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$$

$$= [v_1 \cdots v_n] \cdot D$$

$\therefore\ P = [v_1 \cdots v_n]$

$$A\ P = P \cdot D \quad \text{i.e.,}$$

$$P^{-1} A P = D.$$

<u>Example</u> Diagonalize the matrix $A = \begin{bmatrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ -4 & 0 & -3 \end{bmatrix}$, i.e.,

find an invertible matrix P, s.t. $P^{-1}AP = D$
and a diagonal matrix D

<u>Solution</u>. 1) compute the eigenvalues of A. : $\lambda_1 = 1$ $\lambda_2 = -1$ $\lambda_3 = 4$

2. Compute the respective ~~basic~~ eigenvector of A: $v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

3. Form the matrices P & D

$$P = [v_1\ v_2\ v_3] = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 4 \end{bmatrix}$$

$\Rightarrow$ $P^{-1}AP = D$.

Ex. Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$

Solution (i) Find eigenvalues of A

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-4 & 1 \\ 2 & 4 & \lambda-4 \end{vmatrix} = (\lambda-2) \begin{vmatrix} \lambda-4 & 1 \\ 4 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-4)^2 - 4 \right] = (\lambda-2)\left[\lambda^2 - 8\lambda + \overset{12}{\cancel{4}} \right]$$

$$= (\lambda-2)(\lambda-2)(\lambda-6)$$

$\therefore$ $\lambda = 2$, $\lambda = 6$ eigen. values of A

(ii) $\lambda = 2$. $(2I - A)v = 0$

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(iii) $\lambda = 6$, $(6I - A)v = 0$

$$v_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$P = [v_1\ v_2\ v_3] = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 6 \end{bmatrix}$$

$$P^{-1}AP = D$$

EX. Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ Cannot be diagonalized.

1. eigenvalue: $\lambda = 1$

2. eigenvectors:

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

There is no enough basic ~~eigenvectors~~ for A to be diagonalized

Exercise: 8.4.3, 8.4.5.

V). Application: matrix powers

Suppose we have a square matrix A, we want to compute A^{50}.

One can compute it via brute force. But it would take a lot of work!

Instead, if A is diagonalizable, say $P^{-1}AP = D$,

then $$A = PDP^{-1}$$

$$\therefore \quad A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1}$$
$$= P \cdot D^2 \cdot P^{-1}$$

$$A^3 = (P \cdot D \cdot P^{-1})(P \cdot D^2 \cdot P^{-1}) = P \cdot D^3 \cdot P^{-1}$$

$$\vdots$$

$$A^{50} = P \cdot D^{50} \cdot P^{-1}$$

D^{50} is easier to compute!

In general, $\boxed{A^n = PD^nP^{-1}.}$

Ex $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. Find A^{50}.

Solution. 1. Find eigenvalues of A : $\lambda = 1$, $\lambda = 2$

2. Find eigenvectors of A $\quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

3. $P = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$

$\therefore \quad A = PDP^{-1}$

$A^{50} = PD^{50}P^{-1}$ $\qquad D^{50} = \begin{bmatrix} 1^{50} & & \\ & 1^{50} & \\ & & 2^{50} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2^{50} \end{bmatrix}$

$$A^{50} = PDP^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{bmatrix}.$$

$\boxtimes$

EXAMPLE Let $A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 5 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ Find a square of A, i.e., find a matrix B such that $B^2 = A$.

Solution. (i) Find eigenvalues of A : $\lambda = 1$, $\lambda = 2$, $\lambda = 4$

($\det(\lambda I - A) = \lambda^3 - 7\lambda^2 + 14\lambda - 8$.

(ii) Find basic eigenvectors of A : $v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(iii) $P = [v_1\, v_2\, v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 4 \end{bmatrix}$

$\therefore \quad A = PDP^{-1}$

Square of the diagonal matrix $\quad D^{\frac{1}{2}} = \begin{bmatrix} 1 & & \\ & \sqrt{2} & \\ & & 2 \end{bmatrix}$

$$B := P D^{\frac{1}{2}} P^{-1}$$

$$B^2 = (P \cdot D^{\frac{1}{2}} P^{-1})(P D^{\frac{1}{2}} P^{-1}) = P(D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}) P^{-1} = A.$$

$$B = P D^{\frac{1}{2}} P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \sqrt{2} & \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1-\sqrt{2} & 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & 1-\sqrt{2} & 1 \end{bmatrix}.$$

RMK: Note that B is NOT unique, because $D^{\frac{1}{2}}$ is Not unique

$$D^{\frac{1}{2}} = \begin{bmatrix} \pm 1 & & \\ & \pm\sqrt{2} & \\ & & \pm 2 \end{bmatrix} \qquad 8 \text{ choices.}$$

Exercise $\quad$ 8.5.2, 8.5.4.

VI. Application: Solving recurrences

Consider the sequence of integers, called the Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

The first two Fibonacci numbers are 0 & 1. Every subsequent Fibonacci number is the sum of the previous two numbers. Thus if we write F_n for the n^{th} Fibonacci number, then the Fibonacci sequence is given by the following conditions:

1. $F_0 = 0$
2. $F_1 = 1$
3. $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

The third condition is known as a <u>recurrence relation</u>, or simply as a recurrence. The first two conditions are known as the <u>base cases</u> of the recurrence.

<u>Ex</u> Compute F_{10}

$F_0 = 0$

$F_1 = 1$

$F_2 = F_1 + F_0 = 1+0 = 1$

$F_3 = F_2 + F_1 = 1+1 = 2$

$F_4 = F_3 + F_2 = 2+1 = 3$

$F_5 = F_4 + F_3 = 3+2 = 5$

$F_6 = F_5 + F_4 = 5+3 = 8$

$F_7 = F_6 + F_5 = 8+5 = 13$

$F_8 = F_7 + F_6 = 13+8 = 21$

$F_9 = F_8 + F_7 = 21+13 = 34$

$F_{10} = F_9 + F_8 = 34+21 = 55.$

Is it an easier way? to compute say F_{100}?

Consider $V_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$. Then

$$V_{n+1} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+1} + F_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} V_n.$$

Therefor to compute V_{n+1}, we only need V_n.

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$V_n = A^n V_0.$$

Now we diagonalize A.

(1) eigenvalues: $\det(\lambda I - A) = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1.$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad \frac{1-\sqrt{5}}{2} = \lambda_2$$

Note . $\lambda_1 + \lambda_2 = 1$ so $\lambda_2 = 1 - \lambda_1$

. $\lambda_1 \cdot \lambda_2 = -1$ or $\lambda_2 = -\frac{1}{\lambda_1}$

(2) eigenvectors: $v_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$\therefore$ $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ $D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{bmatrix}$

Note $F_n = [1 \ 0] V_n = [1 \ 0] \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$

$\therefore$

$$\begin{aligned} F_n &= [1 \ 0] V_n \\ &= [1 \ 0] A^n V_0 \\ &= [1 \ 0] P \cdot D^n P^{-1} V_0 \\ &= \underline{[1 \ 0] \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}} \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \underline{\frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \\ &= \frac{1}{\sqrt{5}} [1 \ 1] \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} [\lambda_1^n \ \lambda_2^n] \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n). \end{aligned}$$

So the n-th Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}}\left(\lambda_1^n - \lambda_2^n\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Ex $F_{100} = ?$

$$F_{100} = \frac{1}{\sqrt{5}}\left(\lambda_1^{100} - \lambda_2^{100}\right)$$

$$= 354224848179261915075.$$

$\boxtimes$

Exercise 86.1.