Lecture 15.5

VI. Properties of eigenvectors and eigenvalues

<u>Prop</u>. Let A be a square matrix. Suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ with corresponding eigenvectors $v_1, \dots, v_k$. Then $v_1, \dots, v_k$ are linearly independent.

<u>Proof</u>. Suppose, for the sake of obtaining a contradiction, that $v_1, \dots, v_k$ are linearly independent. Let m be the smallest index s.t. v_m is redundant, i.e.,

$$v_m = a_1 v_1 + \dots + a_{m-1} v_{m-1}. \qquad \cdots (*)$$

Apply A.

$$A v_m = a_1 A v_1 + \dots + a_m A v_{m-1}$$

$$\lambda_m v_m = a_1 \lambda_1 v_1 + \dots + a_m \lambda_{m-1} v_{m-1}. \qquad \cdots (**)$$

Subtract $\lambda_m (*)$ from $(**)$, we get

$$0 = a_1 (\lambda_m - \lambda_1) v_1 + \dots + a_{m-1} (\lambda_m - \lambda_{m-1}) v_{m-1} = 0$$

$\because$ $\lambda_1, \dots, \lambda_m$ are distinct, $\lambda_m - \lambda_i \neq 0 \ \forall 1 \le i \le m-1$

$\because$ $v_1, \dots, v_{m-1}$ are linearly indep.

$$\therefore \quad a_1(\lambda_m - \lambda_1) = 0, \quad \dots, \quad a_{m-1}(\lambda_m - \lambda_{m-1}) = 0$$

$\therefore \quad a_1 = 0, \ \dots, \ a_{m-1} = 0$

$\therefore \quad V_m = a_1 v_1 + \dots + a_{m-1} v_{m-1} = 0$

$\nrightarrow$ V_m an eigenvector.

$\boxtimes$

<u>Corollary</u> Let A be an $n \times n$ matrix, and suppose that it has n distinct eigenvalues, then A is diagonalizable.

If a is an eigenvalue of A. the <u>algebraic multiplicity</u> of a is the number of factors $(\lambda - a)$ appearing in the characteristic polynomial of A. The <u>geometric</u> multiplicity is the dimension of the associated eigenspace.

<u>Prop</u>. Let a be an eigenvalue of A.

Alg. Mult. of a $\geq$ geom. mult. of $a \geq 1$

Prop if $\lambda_1, \dots, \lambda_k$ are all distinct eig. values of A

If A is an $n \times n$ matrix, then A is diagonalizable

$\Longleftrightarrow$

geom. mult. of λ_1 + ... + geom. mult. of $\lambda_k = n$.

Exercise 8.9.1 (d), 8.9.2 (b)

VII. Complex eigenvectors & eigenvalues.

As observed previously, a matrix may not have eigenvalues in real #s. For example, consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Then its ~~eige~~ characteristic poly is

$$\lambda^2+1.$$

Hence A does not have real eigenvalues. But A does have two eigenvalues in complex numbers: $\lambda = \pm\sqrt{-1}$.

The fundamental theorem of algebra says that any non-constant polynomial has a root in complex numbers. So eigenvalues in complex #s always exist.

Ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Diagonalize A.

$\lambda = \pm\sqrt{-1} = \pm i$

$\lambda = i$: $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = -i$: $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} i & \\ & -i \end{bmatrix}$

$A = PDP^{-1}$.

Ex. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ over the complex numbers. Diagonalize A if possible.

i) $ch_A = \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2+1$

$\therefore$ $\lambda = 1 \pm i$

ii) $\lambda = 1+i$, $(\lambda I - A)X = 0$ has solution $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = 1-i$ $(\lambda I - A)X = 0$ . . . $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\therefore \quad A = PDP^{-1}, \quad P = [v_1 \ v_2] = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1+i & \\ & 1-i \end{bmatrix}.$

Prop. Let A be a square matrix, whose entries are real numbers. If λ is an eigenvalue of A, then so is $\bar{\lambda}$, its complex conjugate.

Proof. If v is an eig. vector of A of eigvalue λ

then

$$Av = \lambda v$$

$$\therefore \quad \overline{Av} = \overline{\lambda v}$$

$$\therefore \quad \bar{A} \cdot \bar{v} = \bar{\lambda} \cdot \bar{v}$$

$$\therefore \quad A \cdot \bar{v} = \bar{\lambda} \cdot \bar{v} \qquad \because \bar{A} = A.$$

$\boxtimes$

Prop. A square matrix A is diagonalizable over the complex number if and only if the geom. mult. of each eigenvalue = the alg. mult.

$\boxtimes$

Ex. Consider the sequence of numbers defined by the recurrence

$$f_0 = 1$$

$$f_1 = 3$$

$$f_{n+2} = 2f_{n+1} - 2f_n \qquad \text{for all } n \geq 0.$$

Solve the recurrence, i.e., find a closed formula for f_n.

Solution

Set $\quad V_n = \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$

$$V_{n+1} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ 2f_{n+1} - 2f_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix}$$

$$V_{n+1} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} V_n$$

set $A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$.

$$ch_A = \lambda^2 - 2\lambda + 2.$$

$$\lambda = \frac{2 \pm 2i}{2} = 1 \pm i$$

For $\lambda_1 = 1+i$, $\quad V = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

For $\lambda_2 = 1-i = \overline{\lambda_1}$. $\quad \overline{V} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

So $P = [V \ \overline{V}] = \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1+i & \\ & 1-i \end{bmatrix}$

$$P^{-1} = \frac{1}{4} \begin{bmatrix} 2i & 1-i \\ -2i & 1+i \end{bmatrix}$$

$$\begin{aligned}
\therefore \ f_n &= [1 \ 0] \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} \\
&= [1 \ 0] \ V_n \\
&= [1 \ 0] \ A^n V_0 \\
&= [1 \ 0] \ P D^n P^{-1} \cdot V_0 \\
&= [1 \ 0] \begin{bmatrix} 1-i & 1+i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} (1+i)^n & \\ & (1-i)^n \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2i & 1-i \\ -2i & 1+i \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\end{aligned}$$

$$= \frac{1}{4} [1-i \quad 1+i] \begin{bmatrix} (1+i)^n & \\ & (1-i)^n \end{bmatrix} \begin{bmatrix} 3-i \\ 3+i \end{bmatrix}$$

$$= \frac{1}{4} \left[\underline{(1-i)} (1+i)^n (\underline{3-i}) + (1+i)(1-i)^n (3+i) \right]$$

$$= \frac{1}{4} \left[(2-4i)(1+i)^n + (2+4i)(1-i)^n \right]$$

$$= \frac{1}{2} \left[(1-2i)(1+i)^n + (1+2i)(1-i)^n \right]$$

$\boxtimes$

Exercise. 8.11.1 B, 8.11.3.
