Lecture 12 Determinants

If A is an $n \times n$ square matrix, its determinant $\det(A)$ is a number.

I). If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\det(A) = ad - bc$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad \det(A) = 1 \times 4 - 2 \times 3 = 4 - 6 = -2.$

II) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

It can be memorized via the following picture

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$\boxed{a_{31}a_{22}a_{13} \quad a_{32}a_{23}a_{11} \quad a_{33}a_{21}a_{12}}$ $\quad \boxed{\underbrace{a_{11}a_{22}a_{33},\ a_{12}a_{23}a_{31},\ a_{13}a_{21}a_{32}}_{+}}$

Ex. $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

$$A = \left[\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{array}\right] \begin{array}{cc} 0 & 1 \\ 3 & 1 \\ 1 & 1 \end{array}$$

$2 = 2\times1\times1 \quad 0 \quad -3 = -1\times3\times1 \qquad 0 \quad 0 \quad 6 = 2\times3\times1.$

$$\det(A) = 0+0+6 - (2+0+(-3))$$
$$= 6 - (-1)$$
$$= 7$$

Exercise: 7.1.1. (b)

7.1.2 (c)

(Before defining det(A) for a general A, we introduce

III) Minors and cofactors

i) Let A be an $n \times n$ matrix. The (i,j) th **minor** of A, denoted by M_{ij}, is the determinant of the $(n-1)\times(n-1)$ matrix that is obtained by deleting the ith row and the j th column of A.

Ex Find the minors of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

M_{11} : $\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 3 & 2 \\ \cancel{3} & 2 & 1 \end{bmatrix}$ $\quad M_{11} = \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = 3\times 1 - 2\times 2 = 3-4 = -1$

M_{12} : $\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{3} & 2 \\ 3 & \cancel{2} & 1 \end{bmatrix}$ $\quad M_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = 4\times 1 - 2\times 3 = 4-6 = -2$

M_{13} : $\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 3 & \cancel{2} \\ 3 & 2 & \cancel{1} \end{bmatrix}$ $\quad M_{13} = \det \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = 4\times 2 - 3\times 3 = 8-9 = -1$

M_{21} : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad M_{21} = \det \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = 2\times 1 - 3\times 2 = 2-6 = -4$

M_{22} : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad M_{22} = \det \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = 1-9 = -8$

M_{23} : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & \end{bmatrix}$ $\quad M_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = 1\times 2 - 2\times 3 = 2-6 = -4$

M_{31} $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad M_{31} = \det \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = 4-9 = -5$

M_{32} $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad M_{32} = \det \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = 2-12 = -10$

M_{33} $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\quad M_{33} = \det \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3-8 = -5$

ii) <u>Def</u>. Let A be an $n \times n$ matrix, the (i,j) cofactor, denoted by C_{ij} is defined to be $C_{ij} = (-1)^{i+j} M_{ij}$

<u>Sign</u> of $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}_{4\times4} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3}$$

Example cont'd

M_{ij} $$\begin{bmatrix} -1 & -2 & -1 \\ -4 & -8 & -4 \\ -5 & -10 & -5 \end{bmatrix}$$

$(-1)^{i+j}$ $$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

C_{ij} $$\begin{bmatrix} -1 & 2 & -1 \\ 4 & -8 & 4 \\ -5 & 10 & -5 \end{bmatrix}$$

iii) Let $A = (a_{ij})$ be an $n \times n$ matrix. The determinant $\det(A)$ of A can be computed as follows.

(1) Pick a row, say i, and add all (i,j)th cofactor, multiply by a_{ij} for various j.

$$\det(A) = a_{i1} \cdot c_{i1} + a_{i2} c_{i2} + \cdots + a_{in} c_{in}$$

or

(2) Pick a column, say j, add all (ij) cofactor multiply by a_{ij} for various i.

$$\det(A) = a_{1j} c_{1j} + a_{2j} c_{2j} + \cdots + a_{nj} c_{nj}.$$

EX. continued : use the new def to compute the determinant of the matrix A in previous example.

row 1 :

a_{1j}	1	2	3
C_{1j}	-1	2	-1

$-1 + 4 + -3 = 0$

row 2 :

a_{2j}	4	3	2
C_{2j}	4	-8	4

$16 + -24 + 8 = 0$

row 3

a_{3j}	3	2	1
C_{3j}	-5	10	-5

$-15 + 20 + -5 = 0$

Column 1

$$
\begin{array}{rrrrl}
a_{i1} & 1 & 4 & 3 & \\
C_{i1} & -1 & 4 & -5 & \\
\hline
 & -1 + & 16 + & -15 & = 0
\end{array}
$$

Column 2

$$
\begin{array}{rrrrl}
a_{i2} & 2 & 3 & 2 & \\
C_{i2} & 2 & -8 & 10 & \\
\hline
 & 4 + & -24 + & 20 & = 0
\end{array}
$$

Column 3

$$
\begin{array}{rrrrl}
a_{i3} & 3 & 2 & 1 & \\
C_{i3} & -1 & 4 & -5 & \\
\hline
 & -3 + & 8 + & -5 & = 0
\end{array}
$$

Original def.

$$
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}
\begin{matrix} 1 & 2 \\ 4 & 3 \\ 3 & 2 \end{matrix}
$$

$$\underbrace{27 + 4 + 8}_{= 39} \qquad \underbrace{3 + 12 + 24}_{= 39}$$

$\therefore \det(A) = 39 - 39 = 0.$

Why the various computations lead to the same answer?

It requires more advanced math beyond the scope of this course. Instead, we put this observation as a theorem.

THM Expanding/computing the determinant of an $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

Exercise 7.2.1 (c), (f), 7.2.2. M_{32}, $C_{3,2}$.

7.2.3 (b), (c).

IV) The determinant of a triangular matrix.

<u>Def</u>. A square matrix A is <u>upper triangular</u> if $a_{ij} = 0 \ \forall \ i > j$

It looks like

$$\begin{bmatrix} * & \cdots & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{bmatrix}$$

entries below the diagonal are zero

<u>THM</u> If A is upper triangular, then

$$\det(A) = \text{product of the diagonal entries}$$
$$= a_{11} a_{22} \cdots a_{nn}$$

<u>EX</u>

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 & 16 \\ 0 & \boxed{2} & 6 & -7 \\ 0 & 0 & \boxed{3} & 33 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix}$$

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

Similar one can define lower triangular and its determinant is still produc of diagonal entr

Exercise 7.3.1 (c).

V) Determinants and row operations

$$A \xrightarrow[\text{operation}]{\text{a row}} B$$

$$\det(A) \longleftrightarrow \det(B)$$

How to relate?

THM. (1) If B is obtained from A by switching two rows, then

$$-\det(A) = \det(B)$$

(2) If B is obtained from A by multiplying one row by a non zero scalar k, then

$$\det(B) = k\det(A)$$

(3) If B is obtained from A by adding a multiple of one row to another row, then

$$\det(B) = \det(A).$$

VI)

Properties of determinants

THM Let A be an $n\times n$ matrix. Then

A is invertible if and only if $\det(A) \neq 0$.

Corollary Let A be an $n\times n$ matrix. Then the homogeneous system $Ax=0$ has non-trivial solutions if and only if

$\det(A) = 0$.

THM Let A, B be $n\times n$ matrices. Then

$$\det(AB) = \det(A)\cdot\det(B).$$

Example $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $AB = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$

$\det(A) = 2-(-3)\times 2 = 8$ $\det(B) = 3-8 = -5$, $\det(AB) = -40$

$\det(A)\cdot\det(B) = 8\times(-5) = -40 = \det(AB)$.

Prop

- $\det(I) = 1$,
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
- $\det(A^k) = \det(A)^k$
- $\det(kA) = k^n \det(A)$, if A $n\times n$ matrix
- $\det(A^T) = \det(A)$.
- If A has a row/column consisting of only zeros, then $\det(A) = 0$.
- If A has two rows linearly dependent, then

$$\det(A) = 0.$$

Exercise. 7.5.10, 7.5.13.

Consider the cofactor matrix $\mathrm{Cof}(A)$ of A

THM

$$\mathrm{Cof}(A) = [C_{ij}] = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}$$

C_{ij} the $(i,j)^{th}$ cofactor of A

$$\| $$

$$(-1)^{i+j} M_{ij},$$

M_{ij} the ~~det~~ (i,j)-th minor of A.

Let $\mathrm{adj}(A) := \mathrm{cof}(A)^T$, the classical adjoint of A.

EX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \qquad \mathrm{Cof}(A) = \begin{bmatrix} -1 & 2 & -1 \\ 4 & -8 & 4 \\ -5 & 10 & -5 \end{bmatrix} \qquad \mathrm{adj}(A) = \begin{bmatrix} -1 & 4 & -5 \\ +2 & -8 & 10 \\ -1 & 4 & -5 \end{bmatrix}$$

EX

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & 6 \end{bmatrix} \qquad \mathrm{adj}(A) = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & 6 \end{bmatrix}$$

THM If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A).$$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & 6 \end{bmatrix}$ $\quad adj(A) = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & 6 \end{bmatrix}$

$$\det(A) = \begin{array}{c} 3 \quad 0 \quad 1 \\ 4 \quad -2 \quad 0 \\ \hline 12+0+0 \end{array} = 12 .$$

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A) = \frac{1}{12} \begin{bmatrix} -2 & 4 & 4 \\ -2 & -2 & 8 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Verify that $A \cdot A^{-1} = I$.

Exercise. 7.6.1. B, 7.6.4.

VIII) Cramer's rule: Self study

Exercise: 7.7.4.