Integrating Factors

Question: Suppose we are required to find the integral of

$$y\,dx - x\,dy = 0$$

$$y\,dx = x\,dy$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + \ln|c|$$

$$\ln|y| = \ln|xc|$$

$$y = x\,c$$

$$c = y/x$$

$$y\,dx - x\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$$M_y \neq N_x \quad \text{is not exact}$$

$$M = y\,, \qquad N = -x$$

$$\frac{\partial M}{\partial y} = M_y = 1\,, \qquad \frac{\partial N}{\partial x} = N_x = -1$$

If we can make this equation exact by multiplying it by a function $I(x,y)$, then $I(x,y)$ is called an integrating factor.

Suppose that

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{--- (1)}$$

is not exact such that

$$M_y \neq N_x$$

We can make (1) exact by multiplying it by the function $I(x,y)$ to obtain

$$(IM)_y = (IN)_x$$

$I \cdot M$ & differentiate wrt y = $I \cdot N$ and differentiate wrt x

If such $I(x,y)$ exists, then equation (1) can be expressed as

$$(IM)dx + (IN)dy = 0$$

$$(IM)_y = (IN)_x \quad \text{--- (2)}$$

Using product rule on (2),

$$M\frac{\partial I}{\partial y} + I\underbrace{\frac{\partial M}{\partial y}}_{M_y} = N\frac{\partial I}{\partial x} + I\underbrace{\frac{\partial N}{\partial x}}_{N_x}$$

$$M\frac{\partial I}{\partial y} - N\frac{\partial I}{\partial x} = IN_x - IM_y.$$

$$= I(N_x - M_y) \quad \text{--- (3)}$$

In solving equation (3), two case are possible

Case 1:

Let I be a function of x only

then $\frac{\partial I}{\partial y} = 0 \Rightarrow M\frac{\partial I}{\partial y} = 0$

$$\frac{\partial I}{\partial x} = \frac{dI}{dx} \Big\} \Rightarrow \text{Now a function of one variable}$$

$$-N\frac{dI}{dx} = IN_x - IM_y$$

$$= I(N_x - M_y)$$

$$-N\frac{dI}{dx} = I(N_x - M_y)$$

Divide through by N

$$-\frac{dI}{dx} = \frac{I}{N}(N_x - M_y)$$

$$-dI = \frac{I}{N}(N_x - M_y)\,dx$$

$$\frac{-dI}{I(N_x - M_y)} = \frac{1}{N}\,dx$$

$$\frac{dI}{I(N_x - M_y)} = -\frac{1}{N}\,dx \qquad (4)$$

If equation (4) can be simplified as much as possible so that it depends solely on x, then the equation is a first order ODE and will become separable.

Case 2:

Let I be a function of y only then

$$\frac{\partial I}{\partial x} = 0 \quad , \quad \frac{N \partial I}{\partial x} = 0 \quad \& \quad \frac{\partial I}{\partial y} = \frac{dI}{dy}$$

$$M \frac{\partial I}{\partial y} - \cancel{N \frac{\partial I}{\partial x}}^{0} = I N_x - I M_y$$

$$M \frac{dI}{dy} = I(N_x - M_y)$$

$$\frac{dI}{I(N_x - M_y)} = \frac{1}{M} dy \qquad \text{——— (5)}$$

which can be simplified to become a separable equation.

Equation (4) or (5) may be solved for $I(x,y)$.

Going back to the example $y\,dx - x\,dy = 0$,

$$y\,dx - x\,dy = 0 \Rightarrow M(x,y)\,dx + N(x,y)\,dy$$

The above example is not exact since $M_y \neq N_x$.

Let us make the equation exact by determining an I.F. $I(x,y)$. to do this, we need

M, N, M_y & N_x

$M = y$ $\qquad$ $N = -x$

$M_y = 1$ $\qquad$ $N_x = -1$

$$M \frac{\partial I}{\partial y} - N \frac{\partial I}{\partial x} = I N_x - I M_y$$

$$= I(N_x - M_y).$$

$$y\frac{\partial I}{\partial y} + x\frac{\partial I}{\partial x} = I(-1-1)$$

$$y\frac{\partial I}{\partial y} + x\frac{\partial I}{\partial x} = -2I$$

Let us assume that I is a function of x only:

$$y\frac{\partial I}{\partial y} = 0, \qquad \frac{\partial I}{\partial x} = \frac{dI}{dx}$$

$$x\frac{dI}{dx} = -2I$$

$$\int \frac{dI}{I} = \int -\frac{2}{x}\,dx$$

$$\ln|I| = -2\ln|x| + \ln|k|$$

where k is an arbitrary constant

$$\ln|I| = \underbrace{\ln|x^{-2}|}_{\text{log rule}} + \ln|k|$$

$$\ln|I| = \ln\left|\frac{1}{x^2}\right| + \ln|k|$$

$$= \ln\left|\frac{k}{x^2}\right|$$

Assume $k=1$, then

$$\ln|I| = \ln\left|\frac{1}{x^2}\right|$$

$I = \frac{1}{x^2}$ is the I.F.

$$(IM)\,dx + (IN)\,dy = 0$$

and $(IM)_y = (IN)_x$ for equation ① to be exact

$\therefore$ from eqn ①, $\overset{M}{y}\,dx - \overset{N}{x}\,dy = 0$

$(IM)\,dx + (IN)\,dy = 0$ gives, $I = \dfrac{1}{x^2}$

$$\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0$$

$$(IM)_y = \frac{\partial}{\partial y}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}$$

$$(IN)_x = \frac{\partial}{\partial x}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$$

Hence the equation is now exact since $(IM)_y = (IN)_x$.

Example: for the ODE

$$(2y^2 + 3x)\,dx + 2xy\,dy = 0$$

Find an appropriate I.F. and solve

$M = 2y^2 + 3x$, $M_y = \dfrac{\partial M}{\partial y} = 4x$

$N = 2xy$, $N_x = \dfrac{\partial N}{\partial x} = 2y$

$$M\frac{\partial I}{\partial y} - N\frac{\partial I}{\partial x} = IN_x - IM_y$$
$$= I(N_x - M_y)$$

$$(2y^2+3x)\frac{\partial I}{\partial y} - 2xy\frac{\partial I}{\partial x} = I(2y-4y)$$

$$(2y^2+3x)\frac{\partial I}{\partial y} - 2xy\frac{\partial I}{\partial x} = I(-2y)$$

If I is a function of x only,

$\frac{\partial I}{\partial y} = 0$, $\frac{\partial I}{\partial x} = \frac{dI}{dx}$

$$(2y^2+3x)\frac{\partial I}{\partial y} = 0$$

$$-2xy\frac{dI}{dx} = I(-2y)$$

Multiply both sides by $\frac{dx}{-2xy}$

$$dI = \frac{I(-2y)}{-2xy}dx$$

Divide both sides by I

$$\frac{dI}{I} = \frac{-\cancel{2}y}{-\cancel{2}xy}dx$$

$$\frac{dI}{I} = \frac{1}{x}dx$$

$$\int\frac{dI}{I} = \int\frac{1}{x}dx$$

$$\ln|I| = \ln|x| + \ln|k|$$

where k is an arbitrary constant

$$\ln|I| = \ln|xk|$$

$$I = xk$$

assume $k=1$,

then $I = x$ is the I.F.

$$(2y^2+3x)dx + 2xy\,dy = 0$$

To make the above exact, we multiply by the I.F. $I = x$ to get

$$\overset{IM}{(2xy^2+3x^2)}dx + \overset{IN}{2x^2y}\,dy = 0$$

$$\frac{\partial IM}{\partial y} = (IM)_y = 4xy, \qquad \frac{\partial IN}{\partial x} = (IN)_x = 4xy$$

$$IM = 2xy^2 + 3x^2 = \frac{\partial u}{\partial x}$$

Integrate the above wrt x

$$u = x^2y^2 + x^3 + g(y)$$

where $g(y)$ is an arbitrary function

$$\frac{\partial u}{\partial y} = 2x^2y + g'(y) \quad \text{———} \quad (*)$$

$$IN = 2x^2y = \frac{\partial u}{\partial y} \quad \text{———} \quad (**)$$

Equate $(*)$ and $(**)$

$$\cancel{2x^2y} + g'(y) = \cancel{2x^2y}$$

$$\therefore \quad g'(y) = 0$$

$g'(y) = 0$

$\int g'(y)\,dy = c$

$g(y) = c$

$u = x^2y^2 + x^3 + c = c_1$

$\therefore\ u = x^2y^2 + x^3 = c_2$

where $c_2 = c_1 - c$

Linear Equations

The most general scalar first order linear equation is

$$y' + a(x)\,y = b(x) \quad \text{———} \quad (1)$$

where a and b are functions defined on the real interval $\mathbb{R}$

Equation (1) is said to be in standard form for first order linear ODEs.

A first order ODE which is not of this form is said to be nonlinear.

When $b(x) = 0$, the equation is homogeneous otherwise the equation is non homogeneous.

To solve equation (1), we use the variation of parameters method.

$$y' + a(x)\,y = b(x) \quad \text{———} \quad (1)$$

First we solve the associated homogeneous equation which is separable

$$y' + a(x)y = 0 \quad \text{———} \quad (2)$$

Integrating equation (2)

$$\frac{dy}{dx} = -a(x)y$$

Separating variables

$$\frac{dy}{y} = -a(x)dx$$

$$\int \frac{1}{y}dy = \int(-a(x))dx$$

$$\ln|y| = \int(-a(x))dx + c$$

$$y = e^{-\int a(x)dx + c}$$

$$y = Ce^{-\int a(x)dx} \quad \text{———} \quad (3)$$

Now that we have a solution for the homogeneous part, we solve the non-homogeneous part of equation (1) ie

$$y' + a(x)y = b(x)$$

To solve, we assume that from (3)

Since $y = Ce^{-\int a(x)dx}$, we let

$$\begin{cases} y_1 = e^{-\int a(x)dx} \\ y = Cy_1 \end{cases}$$

then $y = Cy_1$. So that we assume

the equation of the form
$$y = y_1 v(x).$$

Here we want to find a function $v(x)$ so that
$$y = e^{-\int a(x)\,dx}\, v(x) \quad \text{—— } (4)$$
forms a solution to equation (1)

If $y = y_1 v(x)$, then
$$y' = y_1 v'(x) + y_1' v(x) \Longrightarrow \text{Product rule}$$

$$y' = y_1 v'(x) - a(x) e^{-\int a(x)\,dx} v(x) \text{ — **}$$
Substitute ** into (1) $\Rightarrow$ $y' + a(x) y = b(x)$

$$y_1 v'(x) - a(x) v(x) \underbrace{e^{-\int a(x)\,dx}}_{y_1} + a(x) y_1 v(x) = b(x)$$

$$y_1 v'(x) - a(x) v(x) y_1 + a(x) y_1 v(x) = b(x)$$

$$y_1 v'(x) + a(x)\left(-v(x) y_1 + y_1 v(x)\right) = b(x)$$

$$y_1 v'(x) = b(x)$$

$$v'(x) = \frac{b(x)}{y_1}$$

$$v'(x) = \frac{b(x)}{e^{-\int a(x)\,dx}} = b(x) e^{\int a(x)\,dx}$$

Integrating:
$$v(x) = \int b(x) e^{\int a(x)\,dx}\,dx + A$$

where A is an arbitrary constant

then Since $y = y_1 v(x)$

then the Solution is

$$y = e^{-\int a(x)\,dx} \cdot \left(\int b(x) e^{\int a(x)\,dx} + A \right) \quad \text{--- } (5)$$

Steps to Solving linear equations

1. Write the linear equation in the standard form

$$y' + a(x) y = b(x).$$

2. From the standard form, identify $a(x)$ and then find an integrating factor $e^{\int a(x)\,dx}$

3. Multiply the standard form of the equation by the integrating factor $e^{\int a(x)\,dx}$ to obtain

$$e^{\int a(x)\,dx} \left(y' + a(x) y \right) = b(x) e^{\int a(x)\,dx}$$

4. The left hand side of (3) is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\Big(\underbrace{e^{\int a(x)\,dx} y}_{\text{derive using product rule}}\Big) = \left(y' + a(x) y \right) e^{\int a(x)\,dx}$$

So that

$$\frac{d}{dx}\left(e^{\int a(x)\,dx} y \right) = b(x) e^{\int a(x)\,dx}$$

5. Integrate both sides of the equation

ie $\int \frac{d}{dx}\left(e^{\int a(x)\,dx} y\right) dx = \int b(x)\, e^{\int a(x)\,dx}\, dx$

Since you are integrating the derivative of a function,

$\int \frac{d}{dx}$ \$ dx will cancel out

$$e^{\int a(x)\,dx} y = \int b(x)\, e^{\int a(x)\,dx}\, dx + A$$

$$y = e^{-\int a(x)\,dx} \left(\int b(x)\, e^{\int a(x)\,dx}\, dx + A\right.$$

The above is the same as equation 5.

Example: Solve $\frac{dy}{dx} - 3y = 6$

Solution

Using the steps above:

1. Write the linear equation in the standard form.

   $\frac{dy}{dx} - 3y = 6$ is already in the standard form $y' + a(x)y = b(x)$

2. Identify $a(x)$ and find the integrating $e^{\int a(x)\,dx}$.

   Here $a(x) = -3$ and $b(x) = 6$

   $\therefore$ Integrating factor is

   $$e^{\int (-3)\,dx} = e^{-3x} = \text{I.F.}$$

3. Multiply the standard form of the equation by the integrating factor $e^{\int a(x)\,dx}$.

In this example, we are multiply by e^{-3x}

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$e^{-3x}(y' - 3y) = 6e^{-3x}$$

4. The left hand side of 3 is automatically the derivative of the integrating factor and y.

ie $\frac{d}{dx}\left(e^{-3x}y\right) = 6e^{-3x}$

5. Integrate both sides of equation in 4

$$\int \frac{d}{\cancel{dx}}\left(e^{-3x}y\right)\cancel{dx} = \int 6e^{-3x}\,dx$$

$\int \frac{d}{dx}$ and dx will cancel out since we are integrating the derivative of a function. We now have

$$e^{-3x}y = \int 6e^{-3x}\,dx$$

$$e^{-3x}y = \frac{6e^{-3x}}{-3} + c$$

$$e^{-3x}y = -2e^{-3x} + c$$

Divide all through by e^{-3x}

$$y = -2 + \frac{c}{e^{-3x}}$$

$$y = -2 + ce^{3x}$$ $\Rightarrow$ This is our solution

Exercise

Solve using the I.F. method

$$x\frac{dy}{dx} + y = x^3$$

Ans $\Rightarrow$ $y = \frac{x^3}{4} + \frac{c}{x}$

OR $y = \frac{x^3}{4} + cx^{-1}$