Statistical Foundations of Data Science

Combinatorics Part 2

University of the Witwatersrand

2022

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$

Lesson Plan

- Review Question

Lesson Plan

- Review Question
- Combinations and binomial coefficients

Lesson Plan

- Review Question
- Combinations and binomial coefficients
- Combinatorial identities

Lesson Plan

- Review Question
- Combinations and binomial coefficients
- Combinatorial identities
- Derangements

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$
- Base case $n = 1$ both sides are equal to 1.

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$
- Base case $n = 1$ both sides are equal to 1.
- Assume true for $n = k$

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$
- Base case $n = 1$ both sides are equal to 1.
- Assume true for $n = k$
- $1^3 + 2^3 + 3^3 + .. + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$
- Base case $n = 1$ both sides are equal to 1.
- Assume true for $n = k$
- $1^3 + 2^3 + 3^3 + .. + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$
- $= \frac{(k+2)^2(k+1)^2}{4}$

Review Question

- Prove by induction that $1^3 + 2^3 + 3^3 + .. + n^3 = \frac{n^2(n+1)^2}{4}$
- Base case $n = 1$ both sides are equal to 1.
- Assume true for $n = k$
- $1^3 + 2^3 + 3^3 + .. + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$
- $= \frac{(k+2)^2(k+1)^2}{4}$
- So true by induction!

Combinations

- We have n prisoners and are selecting k for execution! How many ways can we select these k.

Combinations

- We have n prisoners and are selecting k for execution! How many ways can we select these k.
- Alternatively we are selecting a team of k players from n who tried out.

Combinations

- We have n prisoners and are selecting k for execution! How many ways can we select these k.
- Alternatively we are selecting a team of k players from n who tried out.
- In either case we'll call the answer $\binom{n}{k}$, it's also occasionally called nCk.

Combinations - Properties

- Before deriving a formula let's talk about some useful identities.

Combinations - Properties

- Before deriving a formula let's talk about some useful identities.
- $\binom{n}{1} = n$

Combinations - Properties

- Before deriving a formula let's talk about some useful identities.
- $\binom{n}{1} = n$
- $\binom{n}{0} = 1$

Combinations - Properties

- Before deriving a formula let's talk about some useful identities.
- $\binom{n}{1} = n$
- $\binom{n}{0} = 1$
- $\binom{n}{k} = \binom{n}{n-k}$

Combinations - Properties

- Before deriving a formula let's talk about some useful identities.
- $\binom{n}{1} = n$
- $\binom{n}{0} = 1$
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.
- Pick first $k!$.

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.
- Pick first $k!$.
- Now we can reorder the $n-k$ not picked in $(n-k)!$ ways to get $nPk = \frac{n!}{(n-k)!}$

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.
- Pick first $k!$.
- Now we can reorder the $n-k$ not picked in $(n-k)!$ ways to get $nPk = \frac{n!}{(n-k)!}$
- As we don't care about the order we execute them in $nCk = \binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.
- Pick first $k!$.
- Now we can reorder the $n-k$ not picked in $(n-k)!$ ways to get $nPk = \frac{n!}{(n-k)!}$
- As we don't care about the order we execute them in $nCk = \binom{n}{k} = \frac{n!}{(n-k)!k!}$.
- Properties hold $\binom{n}{k} = \binom{n}{n-k}$ is obvious from the formula.

Combinations - Formula

- Line up the n prisoners, in order of how much you feel like executing them. $n!$ ways.
- Pick first $k!$.
- Now we can reorder the $n-k$ not picked in $(n-k)!$ ways to get $nPk = \frac{n!}{(n-k)!}$
- As we don't care about the order we execute them in $nCk = \binom{n}{k} = \frac{n!}{(n-k)!k!}$.
- Properties hold $\binom{n}{k} = \binom{n}{n-k}$ is obvious from the formula.
- $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ is algebra.

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal's Triangle

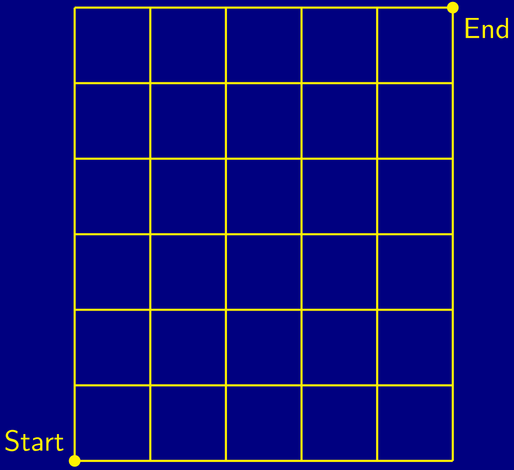
$$
\begin{array}{ccccccccccc}
 & & & & & 1=\binom{0}{0} & & & & & \\
 & & & & 1=\binom{1}{0} & & 1=\binom{1}{1} & & & & \\
 & & & 1=\binom{2}{0} & & 2=\binom{2}{1} & & 1=\binom{2}{2} & & & \\
 & & 1=\binom{3}{0} & & 3=\binom{3}{1} & & 3=\binom{3}{2} & & 1=\binom{3}{3} & & \\
 & 1=\binom{4}{0} & & 4=\binom{4}{1} & & 6=\binom{4}{2} & & 4=\binom{4}{3} & & 1=\binom{4}{4} & \\
1=\binom{5}{0} & & 5=\binom{5}{1} & & 10=\binom{5}{2} & & 10=\binom{5}{3} & & 5=\binom{5}{4} & & 1=\binom{5}{5}
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{8}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}
$$

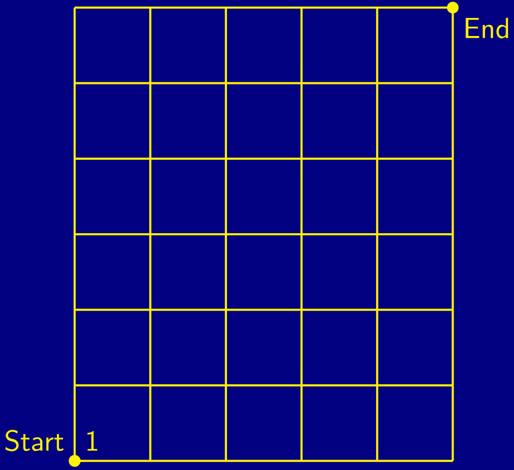
Applications of binomial coefficients

How many ways from start to end?



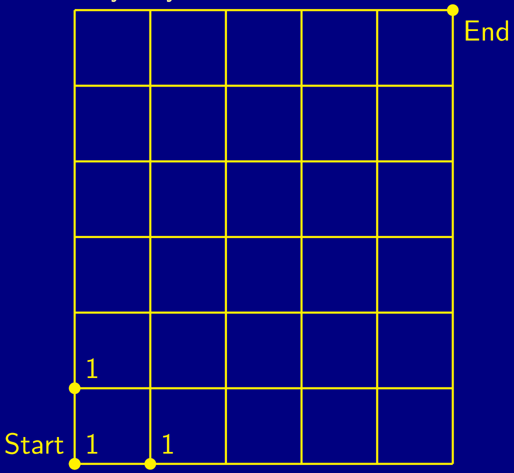
Applications of binomial coefficients

How many ways from start to end?



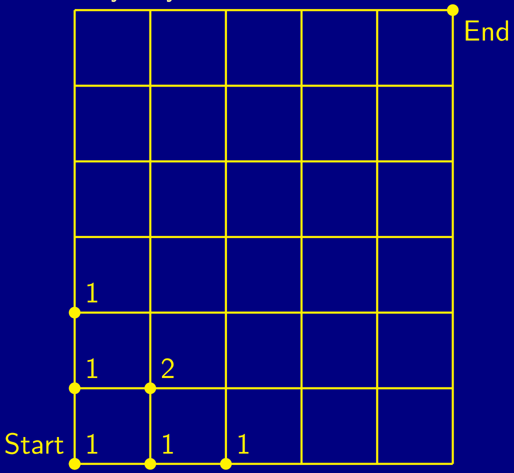
Applications of binomial coefficients

How many ways from start to end?



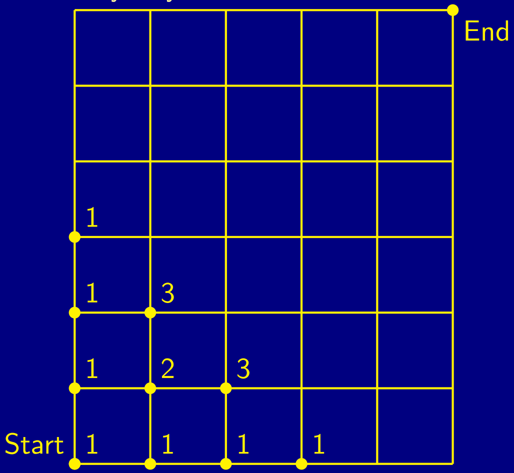
Applications of binomial coefficients

How many ways from start to end?



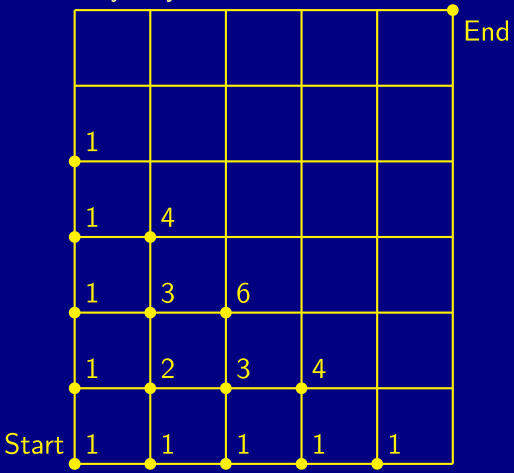
Applications of binomial coefficients

How many ways from start to end?



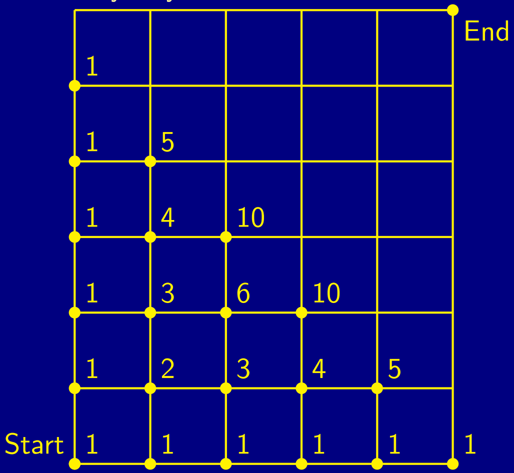
Applications of binomial coefficients

How many ways from start to end?



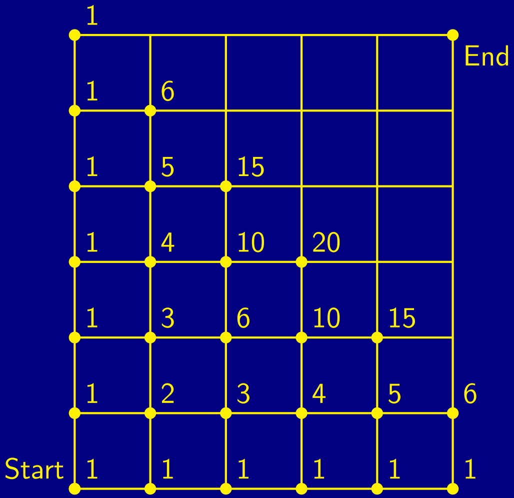
Applications of binomial coefficients

How many ways from start to end?



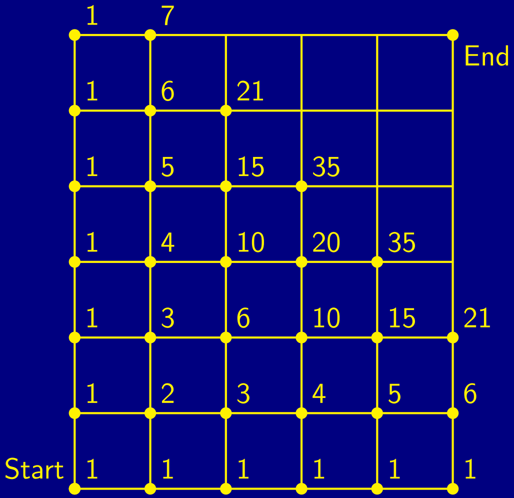
Applications of binomial coefficients

How many ways from start to end?



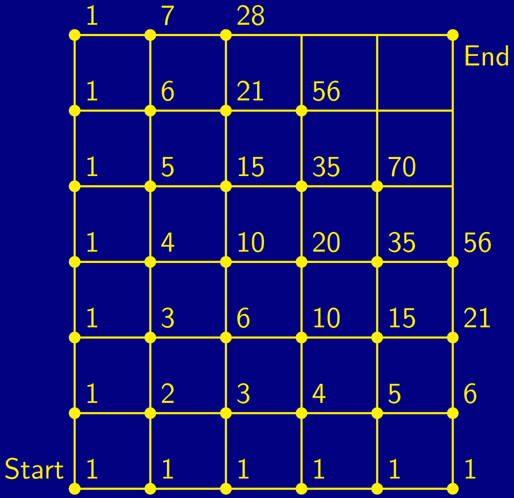
Applications of binomial coefficients

How many ways from start to end?



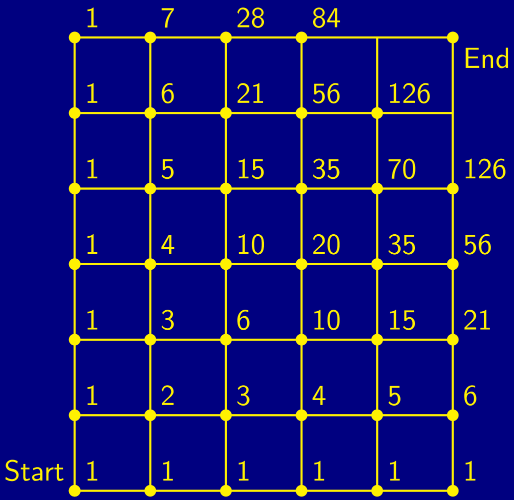
Applications of binomial coefficients

How many ways from start to end?



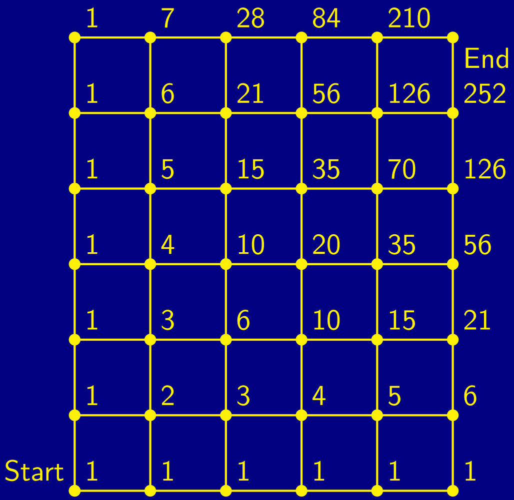
Applications of binomial coefficients

How many ways from start to end?



Applications of binomial coefficients

How many ways from start to end?



Applications of binomial coefficients

How many ways from start to end?

Applications of binomial coefficients

- Alternative solution. We'll make 11 steps. 6 up and 5 right. Choose 6 of the 11 steps to be "up steps. $\binom{11}{6}$

Applications of binomial coefficients

- Alternative solution. We'll make 11 steps. 6 up and 5 right. Choose 6 of the 11 steps to be "up steps. $\binom{11}{6}$
- $\binom{11}{6} = \frac{11!}{5!6!}$

Applications of binomial coefficients

- Alternative solution. We'll make 11 steps. 6 up and 5 right. Choose 6 of the 11 steps to be "up steps. $\binom{11}{6}$
- $\binom{11}{6} = \frac{11!}{5!6!}$
- $= \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1}$

Applications of binomial coefficients

- Alternative solution. We'll make 11 steps. 6 up and 5 right. Choose 6 of the 11 steps to be "up steps. $\binom{11}{6}$
- $\binom{11}{6} = \frac{11!}{5!6!}$
- $= \frac{11\times 10\times 9\times 8\times 7}{5\times 4\times 3\times 2\times 1}$
- $= \frac{11\times 3\times 2\times 7}{1} = 462$

Binomial Theorem and it's relation to choosing

- $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Binomial Theorem and it's relation to choosing

- $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$
- To see this write out $(x+y)(x+y)...(x+y)$.

Binomial Theorem and it's relation to choosing

- $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$
- To see this write out $(x + y)(x + y)...(x + y)$.
- coefficent of $x^i y^{n-i}$, well choose i of the n to have the x and the other $n - i$ to have the y

Choosing problems

- You are given a 5 card poker hand. How many ways can you get a full house?

Choosing problems

- You are given a 5 card poker hand. How many ways can you get a full house?
- $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$

Choosing problems

- You are given a 5 card poker hand. How many ways can you get a full house?
- $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$
- The probability of drawing such a hand? $3744/\binom{52}{5} = 0.00144057623$

Multinomial Coefficients

- Suppose I have n prisoners and I want to choose s for a soccer team and g to guillotine!

Multinomial Coefficients

- Suppose I have n prisoners and I want to choose s for a soccer team and g to guillotine!
- $\binom{n}{s,g}\binom{n}{s}\binom{n-s}{g} = \frac{n!}{s!(n-s)!} \times \frac{(n-s)!}{g!(n-s-g)!} = \frac{n!}{s!g!(n-s-g)!}$

Multinomial Coefficients

- Suppose I have n prisoners and I want to choose s for a soccer team and g to guillotine!
- $\binom{n}{s,g}\binom{n}{s}\binom{n-s}{g} = \frac{n!}{s!(n-s)!} \times \frac{(n-s)!}{g!(n-s-g)!} = \frac{n!}{s!g!(n-s-g)!}$
- This can also be done combinatorially!

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.
- Example there are $5! = 120$ ways to arrange the five numbers 1 through 5. For example. $2, 4, 5, 1, 3$ and $1, 2, 3, 4, 5$.

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.
- Example there are $5! = 120$ ways to arrange the five numbers 1 through 5. For example. $2, 4, 5, 1, 3$ and $1, 2, 3, 4, 5$.
- How many ways are there to arrange these numbers if no number can remain in it's natural position?

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.
- Example there are $5! = 120$ ways to arrange the five numbers 1 through 5. For example. $2, 4, 5, 1, 3$ and $1, 2, 3, 4, 5$.
- How many ways are there to arrange these numbers if no number can remain in it's natural position?
- Example $\mathbf{1}, 4, 3, 5, 2$ wouldn't count because of the 1.

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.
- Example there are $5! = 120$ ways to arrange the five numbers 1 through 5. For example. $2, 4, 5, 1, 3$ and $1, 2, 3, 4, 5$.
- How many ways are there to arrange these numbers if no number can remain in it's natural position?
- Example $\mathbf{1}, 4, 3, 5, 2$ wouldn't count because of the 1.
- Example $5, 3, 4, 1, 2$ would count.

Derangements

- Recall that there are $n!$ ways to arrange the first n integers.
- Example there are $5! = 120$ ways to arrange the five numbers 1 through 5. For example. $2, 4, 5, 1, 3$ and $1, 2, 3, 4, 5$.
- How many ways are there to arrange these numbers if no number can remain in it's natural position?
- Example $\mathbf{1}, 4, 3, 5, 2$ wouldn't count because of the 1.
- Example $5, 3, 4, 1, 2$ would count.
- Example $3, \mathbf{2}, 4, 1, \mathbf{5}$ wouldn't coutn because of the 2 and the 5.

Solution

- Inclusion-exclusion!

Solution

- Inclusion-exclusion!
- There are $n!$ arrangements, we subtract off the $n \times (n-1)!$ that have each particular fixed point.

Solution

- Inclusion-exclusion!
- There are $n!$ arrangements, we subtract off the $n \times (n-1)!$ that have each particular fixed point.
- We then add back the ones that have each pair of 2 fixed points! There are $\binom{n}{2} \times (n-2)!$ of these.

Solution

- Inclusion-exclusion!
- There are $n!$ arrangements, we subtract off the $n \times (n-1)!$ that have each particular fixed point.
- We then add back the ones that have each pair of 2 fixed points! There are $\binom{n}{2} \times (n-2)!$ of these.
- Again we need to subtract off those taken 3 at a time and add back those taken 4 at a time and so on.

Solution

- Inclusion-exclusion!
- There are $n!$ arrangements, we subtract off the $n \times (n-1)!$ that have each particular fixed point.
- We then add back the ones that have each pair of 2 fixed points! There are $\binom{n}{2} \times (n-2)!$ of these.
- Again we need to subtract off those taken 3 at a time and add back those taken 4 at a time and so on.
- Finally we get $n!(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \ldots + \frac{(-1)^n}{n!}) \approx \frac{n!}{e}$