Linear Regression

University of the Witwatersrand

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Review Question

- Invert the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

Lesson Plan

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- We'll use p as the number of variables because n will be reserved for the number of observations.

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- To minimize this error we can take p partial derivatives

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- As with the single variable case this works just fine as a calculus problem but the assumption that the ϵ are distributed normally, independently and with common variance allows us to to inference!

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- This matters because if some β are zero it means that we can ignore those variables!

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  - Ridge/Lasso regression (more later).

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- Imagine we're trying to tell a child's age from his height and weight. This is pretty likely to be a predictive model because teenagers are taller and heavier than toddlers.
- On the other hand height and weight are really well correlated with each other.
- Maybe the true model is $Y = X_1 + X_2 + \epsilon$ but with X_1 and X_2 really similar this would make $\hat{Y} = 2X_1$ and $\hat{Y} = 2X_2$ nearly as good fits as $\hat{Y} = X_1 + X_2$, worse so is $hatY = 200X_1 - 199X_2$. Your estimates on β_i become larger

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- Overfitted_Data.png

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- We have criteria to deal with with AIC and BIC to favour lower dimensional models.

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- Contrasts with Ridge Regression:
  Minimize $\mathbb{E}[(y - \hat{y})^2] + \lambda\hat{\beta}^2$, which keeps coefficents small but rarely actually gets them to zero

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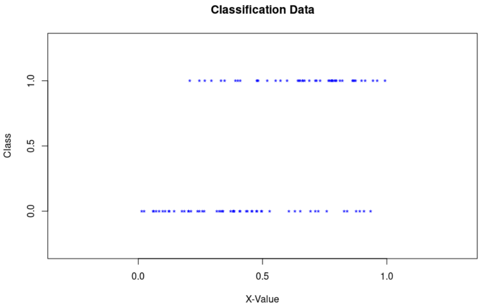
- Then we try to do regression on it.

A problem



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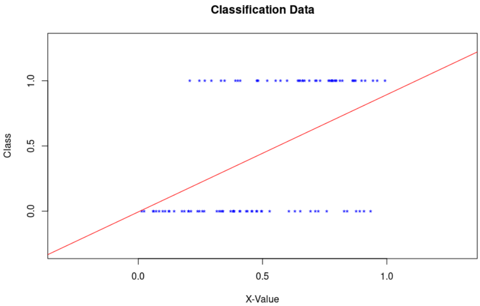

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- Could use a lot of functions. In machine learning we sometimes do. For logistic regression we use a sigmoid function

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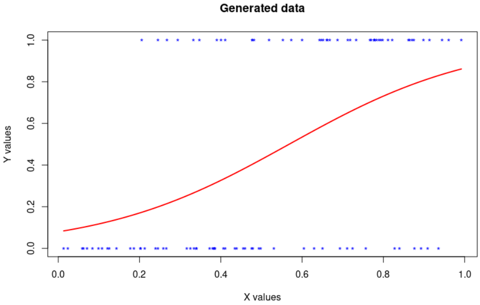

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$$\begin{aligned}
p(X) &= \frac{e^{\beta^T X}}{1 + e^{\beta^T X}} \\
p(X) &= \frac{Z}{1 + Z} \\
p(X)(1 + Z) &= Z \\
p(X) + Zp(X) &= Z
\end{aligned}$$

Sigmoid Function

- $$\begin{aligned} p(X) + Zp(X) &= Z \\ p(X) &= Z(1 - p(X)) \\ Z &= \frac{p(X)}{1 - p(X)} \\ \frac{p(X)}{1 - p(X)} &= e^{\beta^T X} \\ \ln \frac{p(X)}{1 - p(X)} &= \beta^T X \end{aligned}$$

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- That is to say that the "log odds" is modeled as a linear function of the X variables.

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- Due to similarities with Linear Regression we can do inference on β. We won't here but your favourite software package will have tests.

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- $$\begin{aligned} l(\beta) &= \sum_i y_i \ln p(x_i) + (1 - y_i) \ln[1 - p(x_i)] \\ &= \sum_i y_i \ln \frac{p(x_i)}{1 - p(x_i)} + \ln[1 - p(x_i)] \\ &= \sum_i y_i(X^T\beta) - \ln[1 + e^{X^T\beta}] \end{aligned}$$

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