Lecture 9

Method of variation of parameters

Thus far, we have use an approach in the search for particular solutions of non homogeneous linear constant coefficient equations.

Next, we give a more general way of finding particular solutions of ODEs which need not to be constant coefficients

For the sake of simplicity, we will restrict all considerations to scalar second-order linear ODEs; although the same techniques apply to higher order ODE as well.

Consider the second-order linear ODE

$$y'' + a(x)\, y' + b(x)\, y = c(x) \quad \text{———} \quad (1)$$

in which a, b, and c are continuous functions of x on an interval I over which (1) is defined.

Suppose that y_1 and y_2 are linearly independent solutions of the homogeneous equation associated with (1), then by the principle of superposition,

$$y_c = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are arbitrary constants. The general solution of (1) is

$$y = y_c + y_p,$$

where y_p is a particular solution of (1)

Now assume $\boxed{y_p = u(x)\,y_1 + v(x)\,y_2,}$

then

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

In order to avoid second derivatives of u and v in y'' we set

$$u'y_1 + v'y_2 = 0 \qquad \text{——— (2)}$$

So that

$$y_p' = uy_1' + vy_2'$$

and $\quad y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$

Substitute y_p, y_p' and y_p'' into (1)

$u'y_1' + uy_1'' + v'y_2' + vy_2'' + a(x)\,[uy_1' + vy_2']$
$+ b(x)\,[uy_1 + vy_2] = c(x)$

Opening the bracket and grouping gives

$u[y_1'' + a(x)y_1' + b(x)y_1] + v[y_2'' + a(x)y_2' + b(x)y_2]$
$+ u'y_1' + v'y_2' = c(x)$

Now both y_1 and y_2 are solutions to the homogeneous part of equation (1), ie y_1 and y_2 are solutions to

$$y'' + a(x)\,y' + b(x)\,y = 0,$$

then the terms $u[y_1'' + a(x)\,y_1' + b(x)y_1]$ and $v[y_2'' + a(x)\,y_2' + b(x)\,y_2]$ becomes zero

$$u'y_1' + v'y_2' = c(x) \qquad \text{——— (3)}$$

Solving 2 and 3 for u' and v', we get

$$u'y_1 + v'y_2 = 0 \quad \text{———} \quad *$$
$$u'y_1' + v'y_2' = c(x) \quad \text{———} \quad **$$

from *

$$u' = -\frac{v'y_2}{y_1} \quad \text{———} \quad ***$$

Substituting in **

$$\left(\frac{-v'y_2}{y_1}\right)y_1' + v'y_2' = c(x)$$

$$v'\left(y_2' - \frac{y_2y_1'}{y_1}\right) = c(x)$$

$$v'\left(\frac{y_1y_2' - y_2y_1'}{y_1}\right) = c(x)$$

$$v' = \frac{y_1 c(x)}{y_1y_2' - y_2y_1'} \quad \text{———} \quad ****$$

Plugging **** into *** gives

$$u' = \left(\frac{-\cancel{y_1} c(x)}{y_1y_2' - y_2y_1'}\right)\frac{y_2}{\cancel{y_1}}$$

$$u' = -\frac{y_2 c(x)}{y_1y_2' - y_2y_1'} = \frac{-y_2 c(x)}{W}$$

Solving for v' also gives

$$v' = \frac{y_1 c(x)}{y_1y_2' - y_2y_1'} = \frac{y_1 c(x)}{W}$$

$\therefore \quad u' = -\dfrac{y_2 c(x)}{W}, \quad v' = \dfrac{y_1 c(x)}{W}$ ——— (4)

To calculate W, find the determinant of the linearly independent solutions y_1 and y_2 and the derivatives y_1' and y_2', that is solve for

$$\boxed{W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'}$$

Where $W = y_1 y_2' - y_2 y_1'$, $W \neq 0$ and W is called the Wronskian.

Integrating (4), we can choose

$$y_p = u(x) y_1 + v(x) y_2$$

$$y_p = -y_1 \left[\int \frac{y_2 c(x)}{W} dx \right] + y_2 \left[\int \frac{y_1 c(x)}{W} dx \right]$$ ——— (5)

Finally, the general solution to (1) is

$y = y_c + y_p$,

Therefore,

$$y = C_1 y_1 + C_2 y_2 - y_1 \int \frac{y_2 c(x)}{W} dx + y_2 \int \frac{y_1 c(x)}{W} dx$$ ——— (6)

The method described here is a long and complicated one to use each time we wish to solve a differential equation. It is more efficient to solve using equation (4).

Thus to solve (i)

1. We first solve for the homogeneous equation and find the complementary solution

$$y_c = C_1 y_1 + C_2 y_2$$

2. We compute the wronskian $W = y_1 y_2' - y_2 y_1'$

3. Using $c(x)$, we find $u(x)$ and $v(x)$ by integrating (4)

4. Obtain the particular solution y_p using (5)

5. The general solution is then

$$y = y_c + y_p \, .$$

Example 1.

Solve $y'' - 3y' + 2y = 4e^{3x}$,

Solution:

$a(x) = -3$, $b(x) = 2$, $c(x) = 4e^{3x}$

The complementary solution is

$$\lambda^2 - 3\lambda + 2 = 0 \, ,$$

so $\lambda = 2$ or 1

$$y_c = C_1 e^{2x} + C_2 e^{x}$$

hence $y_1(x) = e^{2x}$ & $y_2(x) = e^{x}$

Compute the Wronskian (W)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2e^{2x} & e^{x} \end{vmatrix}$$

$$x^a \cdot x^b = x^{a+b}$$

$$= e^{2x} \cdot e^{x} - 2e^{2x} \cdot e^{x}$$
$$= e^{3x} - 2e^{3x}$$
$$W = -e^{3x}$$

$$u' = \frac{-y_2\, c(x)}{W}$$
$$= \frac{\cancel{-}\, e^{x} \cdot 4\cancel{e^{3x}}}{\cancel{-}\,\cancel{e^{3x}}}$$

$$u' = 4e^{x}$$
$$\int u'\,dx = \int 4e^{x}\,dx$$
$$\boxed{u = 4e^{x}}$$

$$v' = \frac{y_1\, c(x)}{W}$$
$$= \frac{e^{2x} \cdot 4\cancel{e^{3x}}}{-\cancel{e^{3x}}}$$
$$v' = -4e^{2x}$$
$$\int v'\,dx = \int -4e^{2x}\,dx$$
$$\boxed{v = -2e^{2x}}$$

$$y_p = u(x)\, y_1 + v(x)\, y_2$$
$$= 4e^{x} \cdot e^{2x} + (-2e^{2x}) \cdot e^{x}$$
$$= 4e^{3x} - 2e^{3x}$$
$$= 2e^{3x}$$

$$y = y_c + y_p$$
$$\boxed{y = c_1 e^{2x} + c_2 e^{x} + 2e^{3x}}$$

Example 2:

Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

Solution:

$a(x) = -4$, $b(x) = 4$, $c(x) = (x+1)\, e^{2x}$

The complementary solution is

$$\lambda^2 - 4\lambda + 4 = 0$$
$$(\lambda-2)^2 = 0$$
so $\lambda = 2$ twice

hence $y_1 = e^{2x}$ and $y_2 = xe^{2x}$
and
$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{2x}(e^{2x} + 2xe^{2x}) - (xe^{2x})(2e^{2x})$$

$$= e^{4x} + 2\cancel{xe^{4x}} - 2\cancel{xe^{4x}}$$

$$W = e^{4x}$$

Here $c(x) = (x+1)e^{2x}$

$$\therefore\ u = -\int \frac{y_2 c(x)\,dx}{W} = -\int \frac{xe^{2x}(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= -\int \frac{xe^{2x}(xe^{2x} + e^{2x})}{e^{4x}}\,dx$$

$$= -\int \frac{x^2e^{4x} + xe^{4x}}{e^{4x}}\,dx$$

$$= -\int \frac{(x^2 + x)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx$$

$$u = -\int x^2 + x \, dx = -\left(\frac{x^3}{3} + \frac{x^2}{2}\right)$$

$$\boxed{u = -\left(\frac{x^3}{3} + \frac{x^2}{2}\right)}$$

and $v = \int \frac{y_1 \, c(x)}{W} dx$

$$= \int \frac{e^{2x} (x+1) e^{2x}}{e^{4x}} dx$$

$$= \int \frac{x e^{4x} + e^{4x}}{e^{4x}} dx$$

$$= \int \frac{\cancel{e^{4x}} (x+1)}{\cancel{e^{4x}}} dx$$

$$= \int (x+1) \, dx = \frac{x^2}{2} + x$$

$$\boxed{v = \frac{x^2}{2} + x}$$

Hence using 5

$$y_p = y_1 u(x) + y_2 v(x)$$

$$= e^{2x}\left(\frac{-x^3}{3} - \frac{x^2}{2}\right) + x e^{2x}\left(\frac{x^2}{2} + x\right)$$

$$= \frac{-x^3 e^{2x}}{3} - \frac{x^2 e^{2x}}{2} + \frac{x^3 e^{2x}}{2} + x^2 e^{2x}$$

$$= x e^{2x}\left[\frac{-x^2}{3} - \frac{x}{2} + \frac{x^2}{2} + x\right]$$

$$= xe^{2x}\left[\frac{-2x^2 - 3x + 3x^2 + 6x}{6}\right]$$

$$= xe^{2x}\left[\frac{x^2}{6} - \frac{x}{2}\right]$$

and the general solution is $y = y_c + y_p$

$$\therefore \quad y = C_1e^{2x} + C_2xe^{2x} + xe^{2x}\left[\frac{x^2}{6} + \frac{x}{2}\right].$$

Exercise: Solve the differential equation by variation of parameters.

$$y'' - 9y = 9xe^{-3x}$$

Solution: $y = C_1e^{3x} + C_2e^{-3x} - \frac{x}{4}e^{-3x}\left[1 + 3x\right] - \frac{e^{-3x}}{24}.$

Reduction of order

A second order homogeneous equation can be reduced to a linear first order ODE by means of a substitution of a known solution y_1.

Suppose we know one independent solution say. $y_1(x)$ of the homogeneous equation associated with

$$y'' + a(x)y' + b(x)y = 0 \quad \text{—— } (*)$$

The second linearly independent solution can be obtained by setting $y_2 = u(x)y_1$

$$y_2(x) = u(x)\, y_1(x) = u y_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$= u'' y_1 + 2u' y_1' + u y_1''$$

Substitute into

$$y'' + a(x) y' + b(x) y = 0$$

$$u'' y_1 + 2u' y_1' + u y_1'' + a(x)\left[u' y_1 + u y_1'\right] + b(x) u y_1 = 0$$

$$u'' y_1 + 2u' y_1' + u y_1'' + a(x) u' y_1 + a(x) u y_1' + b(x) u y_1 = 0$$

$$u'' y_1 + \left(2y_1' + a(x) y_1\right) u' + u y_1'' + a(x) u y_1' + b(x) u y_1 = 0$$

but $u y_1'' + a(x) u y_1' + b(x) u y_1 = 0$

Since $u y_1$ is a known solution

$$u'' y_1 + \left[2y_1' + a(x) y_1\right] u' = 0$$

or

$$\boxed{u'' + \left[\frac{2y_1'}{y_1} + a(x)\right] u' = 0}$$

This is equation (4) in the next slide.

It has been shown that $u(x)$ satisfies the linear second-order equation

$$u'' + \left(\frac{2y_1'}{y_1} + a(x)\right)u' = 0 \quad \text{—— (7)}$$

The form of (7) allows for a reduction of order. Thus, in order to solve (9), we set

$$P = u' \quad \text{—— (8)}$$

and deduce that (7) reduces to

$$P' + \left(\frac{2y_1'}{y_1} + a(x)\right)P = 0 \quad \text{—— (9)}$$

Solve (9) for $P(x)$, then substitute the expression for $P(x)$ into (8) to obtain $u(x)$. The second independent solution $y_2(x)$ can be can then be determined.

Let $y = u(x)e^{-\int \frac{a(x)}{2}dx}$. Substitute this into the homogeneous equation and find the differential equation that u satisfies.

Example: Given that $y_1 = e^x$ is a solution of $y'' - y = 0$, use reduction of order to find the second solution $y_2(x)$.

$$\text{Let } y = u(x)\,y_1$$
$$= u(x)e^x \Rightarrow ue^x$$

$$y' = u'y_1 + uy_1'$$
$$= u'e^x + ue^x$$

$$y'' = u''e^x + u'e^x + u'e^x + ue^x$$
$$= u''e^x + 2u'e^x + ue^x$$

$y'' - y = 0$, gives

$$u''e^x + 2u'e^x + \cancel{ue^x} - \cancel{ue^x} = 0$$
$$(u'' + 2u')e^x = 0$$

$$u'' + 2u' = 0$$
$$e^x \neq 0$$

Let $p = u'$ & $p' = u''$

$$p' + 2p = 0$$
$$y' + a(x)y = b(x)$$

Now solve using integrating factor method.

$$IF = e^{\int a(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x}(p' + 2p) = e^{2x} \cdot 0$$

$$\frac{d}{dx}\Big(\overbrace{e^{2x}p}^{IF \times p}\Big) = 0$$

$\frac{d}{dx}(e^{2x}p)$

$2e^{2x}p + p'e^{2x}$

$e^{2x}(p' + 2p)$

$$\int \cancel{\frac{d}{dx}}(e^{2x}p)\,\cancel{dx} = \int 0\,dx$$

$$pe^{2x} = c$$
$$p = ce^{-2x}$$

But $p = u'$

$$\int u'\,dx = \int ce^{-2x}\,dx$$

$$u(x) = \frac{-ce^{-2x}}{2} + c_1$$

$$y_2 = u(x)y_1$$
$$= ue^x$$

$$\therefore y_2(x) = \left[\frac{-ce^{-2x}}{2} + c_1\right]e^x$$

$$y_2(x) = \frac{-ce^{-x}}{2} + c_1 e^x$$

Exercise: The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find a second solution $y_2(x)$ using reduction of order.

Ans: $y_2(x) = cx^2 \ln(x) + c_1 x^2$

Euler or Cauchy equation

General form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = 0 \quad \text{--- (1)}$$

Note the powers of x and the order of the derivatives match, so we examine the general solution for the homogeneous second-order equation.

$$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0 \quad \text{--- (2)}$$

and then extend this to the non-homogeneous case

$$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = c(x) \quad \text{--- (3)}$$

Methods of Solution:

There are two methods of solution.

1. Make the change of variable $\boxed{x = e^t}$ to obtain the new equation

$$\frac{d^2 y}{dt^2} + (a-1)\frac{dy}{dt} + by = c(e^t) \quad \text{--- (4)}$$

How the new equation (4) is obtained

$x = e^t \quad , \quad t = \ln(x)$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \Rightarrow \text{chain rule}$$

$$= \boxed{\frac{dy}{dt} \cdot \frac{1}{x} \Rightarrow \frac{dy}{dt} \cdot \frac{1}{e^t}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] \Rightarrow \frac{d}{dx}\left[\frac{dy}{dt} \cdot \frac{dt}{dx}\right] \Rightarrow \text{Product Rule}$$

$$\frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{dt}{dx}\right) = \boxed{\frac{d}{dx}\left[\frac{dy}{dt}\right] \cdot \frac{dt}{dx} + \frac{d}{dx}\left[\frac{dt}{dx}\right] \cdot \frac{dy}{dt}}$$

$$= \frac{d}{dx}\left[\frac{dy}{dt}\right] \cdot \frac{dt}{dx} + \frac{d^2t}{dx^2} \cdot \frac{dy}{dt}$$

$$= \frac{d}{dt}\left[\frac{dy}{dt}\right] \cdot \frac{dt}{dx} \cdot \frac{dt}{dx} + \frac{d^2t}{dx^2} \cdot \frac{dy}{dt}$$

$$= \frac{d^2y}{dt^2} \cdot \left(\frac{dt}{dx}\right)^2 + \frac{d^2t}{dx^2} \cdot \frac{dy}{dt}$$

but $\frac{dt}{dx} = \frac{1}{x} = \frac{1}{e^t} \Rightarrow$ since $x = e^t$

$$\therefore \left(\frac{dt}{dx}\right)^2 = \left(\frac{1}{x}\right)^2 = \left(\frac{1}{e^t}\right)^2 = \frac{1}{e^{2t}}$$

Also, $\frac{d^2t}{dx^2} = \frac{d}{dx}\left(\frac{dt}{dx}\right)$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2} = -\frac{1}{e^{2t}}$$

$\therefore x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = c(x)$, gives

$$(e^t)^2 \frac{d^2y}{dx^2} + a e^t \frac{dy}{dx} + by = ce^t$$

$$\cancel{e^{2t}}\left[\frac{d^2y}{dt^2} \cdot \frac{1}{\cancel{e^{2t}}} + \frac{dy}{dt} \cdot -\frac{1}{\cancel{e^{2t}}}\right] + a\cancel{e^t} \frac{dy}{dt} \cdot \frac{1}{\cancel{e^t}} + by = ce^t$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + a\frac{dy}{dt} + by = ce^t$$

$$\boxed{\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = ce^t}$$

$$\Downarrow$$

This gives equation 4

2. Try a solution of the form $\boxed{y = x^r}$ where r is to be determined

Three cases are considered when solving for r. The roots might be real and distinct, real and equal or complex.

For example:

Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$ (**)

Using the substitution $x = e^t$.

$x^2 y'' + axy' + by = C(x)$

Remember $x = e^t$, $t = \ln(x)$

$\rightarrow \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = C(e^t)$

from (**), $a = -2$, $b = -4$, $c = 0$

$\frac{d^2y}{dt^2} + (-2-1)\frac{dy}{dt} - 4y = 0$

$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$ $\Rightarrow$ second order ODE with constant coefficients

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 4$ or $\lambda = -1$

$y = c_1 e^{4t} + c_2 e^{-t}$

but $t = \ln x$

$y = c_1 e^{4\ln x} + c_2 e^{-\ln x}$

$$y = c_1 e^{\ln x^4} + c_2 e^{\ln\left(\frac{1}{x}\right)}$$

$$y = c_1 x^4 + c_2 x^{-1}$$

Examples:

Using the substitution $y = x^r$, solve

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

Solution:

$$y = x^r$$

$$\frac{dy}{dx} = r x^{r-1}, \quad \frac{d^2y}{dx^2} = r(r-1)x^{r-2}$$

Substituting these into $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$

gives

$$x^2 \cdot \left[r(r-1)x^{r-2}\right] - 2x\left[r x^{r-1}\right] - 4x^r = 0$$

$$x^r\left[r(r-1)\right] - 2\left[r x^r\right] - 4x^r = 0$$

$$x^r\left[r(r-1) - 2r - 4\right] = 0$$

$$x^r\left[r^2 - r - 2r - 4\right] = 0$$

$$x^r\left[r^2 - 3r - 4\right] = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r+1)(r-4) = 0, \quad r = -1 \text{ or } 4$$

$$\therefore \quad y = c_1 x^{-1} + c_2 x^4.$$

Equations explicitly independent of x

Equations that are explicitly independent of x are called autonomous equations. That is your independent variable x does not appear explicitly in your equation, ie it is not forms in your equation.

Consider the equation

$$y'' = h(y, y') \quad \text{———} \quad (11)$$

Note that by the virtue of the chain rule

$$y'' = \frac{dy'}{dy} \cdot \frac{dy}{dx}$$

$\Longrightarrow$ Since the equation is explicitly independent of x

$$= \frac{dy'}{dy} \cdot y'$$

So (11) becomes

$$y' \frac{dy'}{dy} = h(y, y') \quad \text{———} \quad (12)$$

Now let $u = y'$, then (12) reads

$$u \frac{du}{dy} = h(y, u) \quad \text{———} \quad (13)$$

which is a first-order ODE. Suppose 13 has the general solution

$$y' = u = S(y, c),$$

where c is an arbitrary constant, then

$$\frac{dy}{dx} = S(y,c)$$

$$\frac{dy}{S(y,c)} = dx$$

which is now separable and has the general solution

$$\int \frac{dy}{S(y,c)} = \int dx$$

$$\int \frac{dy}{S(y,c)} = x + K,$$

where K is a further constant of integration.

Example: Solve $y y'' = (y')^2$

Solution:

$$y \cdot \underbrace{y' \frac{dy'}{dy}}_{y''} = (y')^2$$

$$y \cdot y' \frac{dy'}{dy} = (y')^2$$

Divide through by y' since its common to both sides

$$y \frac{dy'}{dy} = y'$$

Let $u = y'$

$$y \frac{du}{dy} = u$$

Seperate variables

$$\frac{du}{u} = \frac{dy}{y}$$

$$\int \frac{1}{u} \, du = \int \frac{1}{y} \, dy$$

$$\ln|u| = \ln|y| + c$$

$$\ln|u| = \ln|y| + \ln|c|$$

$$\ln|u| = \ln|yc|$$

$$u = yc$$

but $u = y'$

$$u = yc$$

$$y' = yc$$

$$\frac{dy}{dx} = yc$$

$$\frac{dy}{y} = c\,dx$$

$$\int \frac{dy}{y} = \int c \, dx$$

$$\ln|y| = cx + c_1$$

$$y = e^{cx + c_1}$$

$$= c_2 e^{cx} \quad , \text{ where } c_2 = e^{c_1}$$

$$\therefore \; y = c_2 e^{cx}$$

Equations explicitly independent of y

The most general second-order ODE not containing y explicitly is

$$y'' = g(x, y')$$

The equation can be written as

$$\frac{dy'}{dx} = g(x, y') \qquad \text{(T4)}$$

Make the change

$$u = y'$$

and obtain the first order ODE

$$\frac{du}{dx} = g(x, u)$$

From here on, proceed as in the previous section.

Example: Solve $y'' = 2x(y')^2$

$$\frac{dy'}{dx} = 2x(y')^2$$

Solution:

Let $u = y'$, $u' = y''$

$$u' = 2xu^2$$

$$\frac{du}{dx} = 2xu^2$$

$$\frac{du}{u^2} = 2x\,dx$$

$$\int \frac{du}{u^2} = \int 2x\,dx$$

$$\int u^{-2}\,du = x^2 + C$$

$$-u^{-1} = x^2 + c$$

$$-\frac{1}{u} = x^2 + c \implies u = -\frac{1}{x^2+c}$$

but $u = y'$

$$y' = -\frac{1}{x^2+c}$$

$$\frac{dy}{dx} = -\frac{1}{x^2+c}$$

$$dy = -\frac{1}{x^2+c}\,dx$$

$$\int dy = \int -\frac{1}{x^2+c}\,dx$$

$$y = -\frac{1}{\sqrt{c}} \tan^{-1}\left(\frac{x}{\sqrt{c}}\right) + c_1$$

where c and c_1 are arbitrary constants.

Existence and uniqueness of solutions of first order ODE.

It should be noted that an ordinary differential equation may have no solution, unique solution or infinitely many solutions. Hence in this section, we give a general solution criterion to determine if a given ordinary differential equation has a solution.

We also discuss what additional conditions are required to ensure that the solution to the ordinary differential equation is unique.

Consider the IVP for the general first-order ODE given by

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0 \quad \text{—— (15)}$$

Before now, it has been assumed that once we specify the value of the unknown, at a point (x_0, y_0), then we will have a unique solution to the first order ODE. However, this is not always so. Hence for the general first order ordinary differential equation in (15), we try to answer the following questions:

1. Under what condition can we be sure that a solution to (15) exists?

   That is among all the solution to $\frac{dy}{dx} = f(x,y)$, do any of the solutions pass through the point (x_0, y_0)?

2. Under what condition can we be sure that there is purely one and only one solution (unique solution) to (15)?

An important theorem by Picard says that under fairly mild assumption on

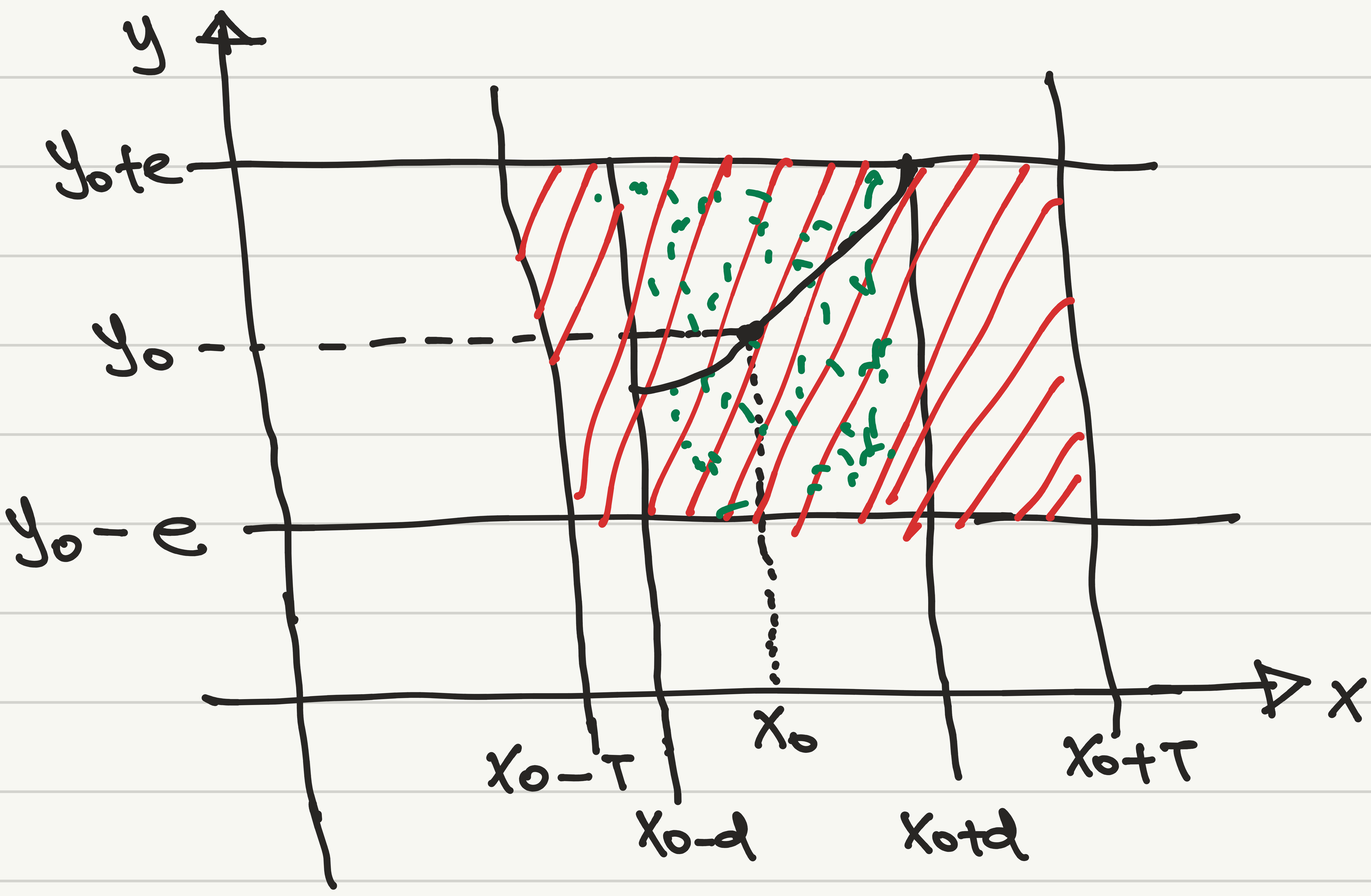
$f(x,y)$, initial value problems have unique solutions, at least locally

Solution of a differential equation.

Let I be an interval and $x_0 \in I$. We say that a differentiable function $y: I \to \mathbb{R}$ is a solution of (15) in the interval I if

$$\frac{dy}{dx} = f(x,y) \quad \forall \cdot x \in I \text{ and } y(x_0) = y_0$$

Existence theorem:



Consider the interval $I_T = [x_0-T, x_0+T]$ and $B_d = [y_0-e, y_0+e]$ for positive real numbers T and e.

Suppose that $f: I_T \times B_d \to \mathbb{R}$ is continuous. Then there is a d such that the IVP in (15) has a solution in the interval

$$I_d = [x_0-d, x_0+d]$$

Note here that for the existence theorem to hold, f must be continuous near (x_0, y_0), if this happens, then a solution exists.

Uniqueness theorem:

Suppose that $f : I_T \times B_d \rightarrow \mathbb{R}$ is continuous and that its partial derivative $\frac{\partial f}{\partial y} : I_T \times B_d \rightarrow \mathbb{R}$ is also continuous, then the solution to the IVP (15) described in the existence theorem in the interval $I_d = [x_0 - d, x_0 + d]$ is unique.

Hence both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are bounded and there exist constant K and L such that

$$|f(x,y)| \leq K \quad \forall\, x, y \in I_T \times B_d \text{ ——— } (16)$$

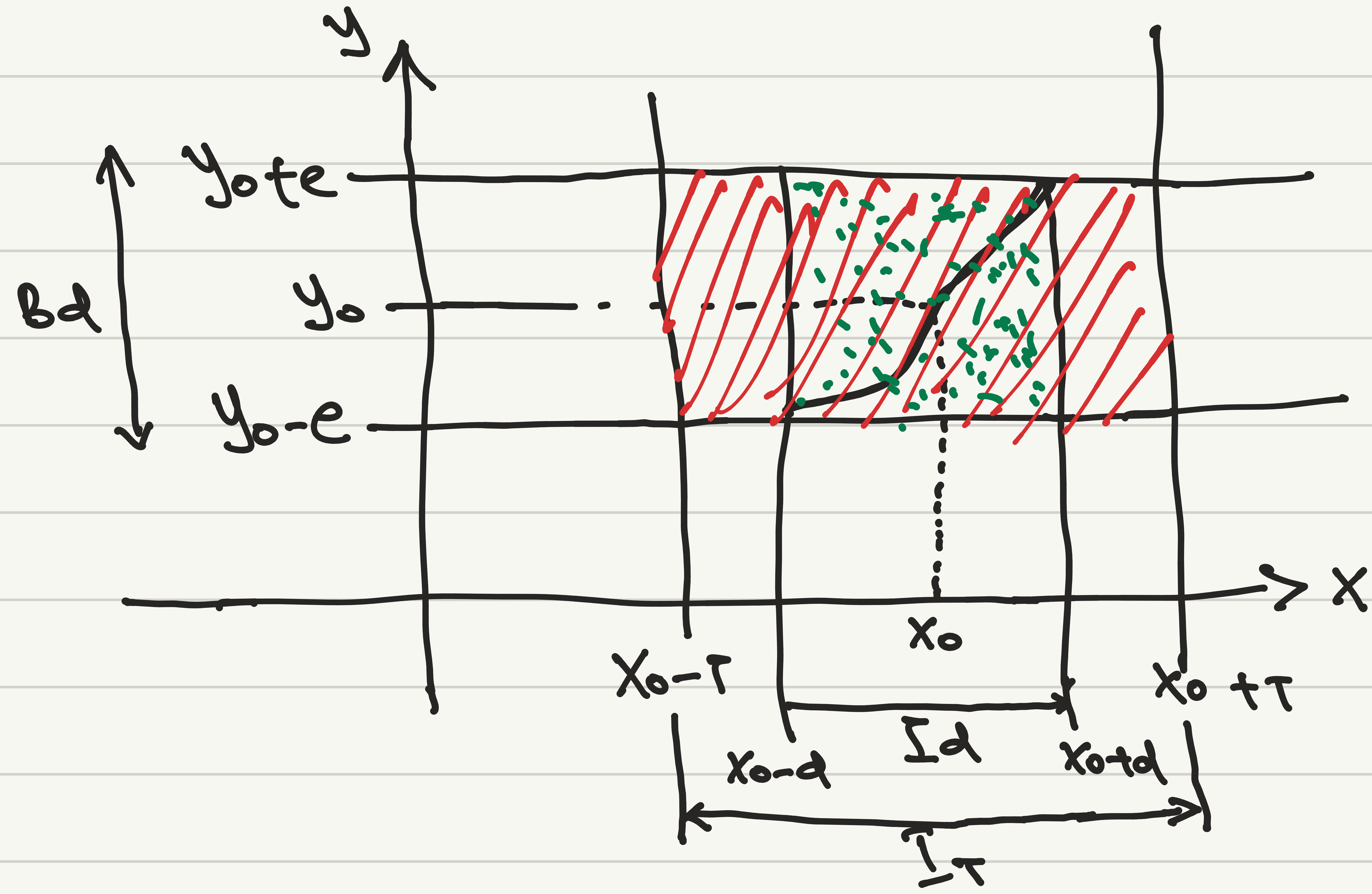
and

$$\left|\frac{\partial f}{\partial y}\right| \leq L \quad \forall\, x, y \in I_T \times B_d \text{ ——— } (17)$$

The condition in (17) can be replaced by the weaker condition that $f(x,y)$ satisfies the Lipschitz condition with respect to its second argument. Thus instead of the continuity of $\frac{\partial f}{\partial y}$, we require

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \; \forall\, x, y \in I_T \times B_d$$

Note that if $\frac{\partial f}{\partial y}$ exists and is bounded, then it necessarily satisfies the Lipschitz condition. On the other hand, a function $f(x,y)$ may be Lipschitz continuous but $\frac{\partial f}{\partial y}$ may not exist



$y(x_0) = y_0$

Points to note from the figure:

1. The region $I_T \times B_d$ is the diagonally shaded region and that is where the function $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous

2. The point (x_0, y_0) is contained in this region and the graph of the solution we seek must pass through this point.

3. The region for the smaller interval I_d which is $I_d \times B_d$ is, however the region where we expect to have a unique solution to the IVP.