## Repeated eigenvalues

Example: find the eigenvalues of

$$A = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Determinant of matrix $A-\lambda I$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-9-\lambda) - (-18)(2) = 0$$

$$\Rightarrow -27 - 3\lambda + 9\lambda + \lambda^2 + 36 = 0$$

$$\Rightarrow \lambda^2 + 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda + 3)^2 = 0$$

$\therefore \lambda = -3$ twice

$\lambda_1 = \lambda_2 = -3$ is a root of multiplicity two. For this value, we find the single eigenvector.

$A\underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = \underline{0}$

$$\begin{pmatrix} 3-(-3) & -18 \\ 2 & -9-(-3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$6x_1 - 18x_2 = 0 \Rightarrow 6x_1 = 18x_2$
$x_1 = 3x_2$

If $x_2 = 1$, then $x_1 = 3$.

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow$ is the eigenvector corresponding to the eigenvalue $\lambda_1 = \lambda_2 = -3$.

Complex eigenvalues

If $\lambda_1 = p + iq$ & $\lambda_2 = p - iq$ where $i = \sqrt{-1}$ and $q > 0$ are complex eigenvalues of a coefficients matrix A, then we can expect their eigenvectors to also have complex entries.

Example:

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$$

$$|A - \lambda I| = 0 \implies \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$$\implies (6-\lambda)(4-\lambda) + 5 = 0$$
$$24 - 10\lambda + \lambda^2 + 5 = 0$$
$$\lambda^2 - 10\lambda + 29 = 0$$

Using quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 1,\ b = -10 \ \& \ c = 29.$$

$$\frac{10 \pm \sqrt{100 - 4(29)}}{2} = \frac{10 \pm \sqrt{-16}}{2}$$

$$\frac{10 \pm 4i}{2} = 5 \pm 2i$$

$\therefore \lambda_1 = 5 + 2i$ and $\lambda_2 = 5 - 2i$

For $\lambda_1 = 5 + 2i$

$$A\underline{x} = \underline{0}$$

$$\begin{pmatrix} 6-(5+2i) & -1 \\ 5 & 4-(5+2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-2i & -1 \\ 5 & -(1+2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1-2i)x_1 - x_2 = 0$

$(1-2i)x_1 = x_2$

If $x_1 = 1$, then $x_2 = 1-2i$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$ is the eigenvector corresponding to $\lambda_1 = 5+2i$

For $\lambda_2 = 5-2i$

$$\begin{pmatrix} 6-(5-2i) & -1 \\ 5 & 4-(5-2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1+2i & -1 \\ 5 & -1+2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1+2i)x_1 - x_2 = 0$

$(1+2i)x_1 = x_2$

If $x_1 = 1$, then $x_2 = 1+2i$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+2i \end{pmatrix}$ is the eigenvector corresponding to $\lambda_2 = 5-2i$

# Solutions to linear systems

Theorem 4.6: If $A \in \mathbb{R}^{n \times n}$ has a distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$, with corresponding eigenvectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m$ $(m \leq n)$, then the set of vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m$ is linearly independent.

$$(\underline{u}_1, \underline{u}_2, \dots \underline{u}_m) = M$$

Theorem 4.7: Any square matrix with all its eigenvalues distinct can be put into diagonal form by a change of basis. More specifically if $A \in \mathbb{R}^{n \times n}$ with $n$ distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$, and corresponding eigenvectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$, then

$$\Lambda = M^{-1} A M$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \cdots & & \lambda_n \end{pmatrix}$$

and $M = (\underline{u}_1 \; \underline{u}_2 \; \dots \; \underline{u}_n) \longrightarrow$ Matrix of the eigenvectors.

Homogeneous, constant coefficient linear systems.

Given the system

$$\dot{\underline{x}} = A \underline{x}, \quad x(t_0) \text{ given}$$

Decoupling or diagonalization of A

If $A \in \mathbb{R}^{n \times n}$ and A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, A can be diagonalized by changing from the standard basis to an eigenvector basis.

Let M be a matrix whose columns is formed by the eigenvectors corresponding to the distinct eigenvalues, then by theorem 4.7,

$$\Lambda = M^{-1} A M.$$

To solve the matrix equation

$$\underline{\dot{x}} = A \underline{x} \text{ ——— } (1)$$

Let $\underline{p} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{pmatrix}$

be a new basis from which we form a solution

$$\underline{x} = M \underline{p} \text{ — } (2)$$

To solve for $\underline{p}$, differentiate equation (2)

$$\underline{\dot{x}} = M \underline{\dot{p}} \text{ ——— } (3)$$

$M^{-1}M = I \dot{p}$
$= \dot{p}$

Substitute (1) into (3)

but $\underline{\dot{x}} = A\underline{x}$ from (1)

then $A\underline{x} = M\underline{\dot{p}}$

premultiply equation (4) by $M^{-1}$

$$M^{-1}A\underline{x} = M^{-1}M\dot{\underline{p}}$$

$$M^{-1}A\underline{x} = \dot{\underline{p}} \qquad \text{———} \quad (5)$$

Substitute equation (2) into (5)

$$M^{-1}AM\underline{p} = \dot{\underline{p}}$$

$$\Lambda\underline{p} = \dot{\underline{p}} \qquad \text{———} \quad (6)$$

Hence if we can solve for P from equation (6) then the solution to the system

$$\dot{\underline{x}} = A\underline{x}$$

is

$$\underline{x} = M\underline{p}.$$

This method is called the diagonalization or decoupling method.

Example: Given the system $\dot{x} = Ax$ in the form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

find the general solution using the diagolisation method.

Solution

$$\underline{x} = M\underline{p}$$

$$\dot{\underline{p}} = \Lambda\underline{p}$$

Since $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda-5)(\lambda+1) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = -1.$$

$$\Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues forms the diagonal of the matrice and other entries are zero.

For $\lambda_1 = 5$ $\quad A \quad \underline{x} = 0$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-4x_1 + 2x_2 = 0$$

$$-4x_1 = -2x_2$$

$$2x_1 = x_2$$

If $x_2 = 2$, then $x_1 = 1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the eigenvector corresponding to $\lambda_1 = 5$

for $\lambda = -1$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 2x_2 = 0$$
$$2x_1 = -2x_2$$
$$x_1 = -x_2$$

If $x_1 = -1$, then $x_2 = 1$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the eigenvectors corresponding to $\lambda_2 = -1$.

$$M = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

but $\dot{\underline{x}} = M\dot{\underline{P}}$

$$A\underline{x} = M\dot{P}$$

$M^{-1}A\underline{x} = \dot{\underline{P}}$ or $M^{-1}AM\underline{P} = \dot{\underline{P}}$

$$\Lambda\underline{P} = \dot{\underline{P}}$$

So that $\dot{\underline{P}} = \Lambda\underline{P}$

$$\Lambda\underline{P} = \overset{\Lambda}{\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}} \overset{\underline{P}}{\begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix}}$$

$$\overset{\dot{\underline{P}}}{\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \end{pmatrix}} = \overset{\Lambda}{\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}} \overset{\underline{P}}{\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}}$$

$$\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 5P_1 \\ -P_2 \end{pmatrix} \qquad \dot{P}_1 = \frac{dP_1}{dt}$$

$$\dot{P}_1 = 5P_1$$

$$\frac{dP_1}{dt} = 5P_1 \implies \int \frac{1}{P_1} dP_1 = 5\int dt$$

$$\ln|P_1| = 5t + c$$

$$e^{\ln|P_1|} = e^{5t+c}$$

$$P_1 = C_1 e^{5t}$$

$$\frac{dP_2}{dt} = -P_2 \implies \int \frac{1}{P_2} dP_2 = -\int dt$$

$$= \ln|P_2| = -t + c$$

$$P_2 = C_2 e^{-t}$$

$$\therefore \begin{pmatrix} P_1(t) \\ P_2(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}$$

$$\underline{x} = \underline{M}P$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{5t} \\ C_2 e^{-t} \end{pmatrix}$$

$$= \begin{matrix} C_1 e^{5t} - C_2 e^{-t} \\ 2C_1 e^{5t} + C_2 e^{-t} \end{matrix}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

Try: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ -5/2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Use diagonalisation to find the solution.

Ans: $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{t}$

OR $c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}$.

## Existence and Uniqueness theorem

Theorem 5.1

Suppose we have a linear system $\dot{x} = A\underline{x}$, $\underline{x}(t_0) = x_0$, with $A$ a constant matrix having $n$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\underline{u_1}, \underline{u_2}, \dots, \underline{u_n}$. Then the solution

$$* \quad \underline{x}(t) = \alpha_1 \underline{u_1} e^{\lambda_1(t-t_0)} + \alpha_2 u_2 e^{\lambda_2(t-t_0)} + \dots + \alpha_n \underline{u_n} e^{\lambda_n(t-t_0)}$$

If $x(t_0) = x_0$, then

$$\underline{x_0} = \alpha_1 \underline{u_1} + \alpha_2 \underline{u_2} + \dots + \alpha_n \underline{u_n}$$

Existence: Differentiate $\underline{x}(t)$ to see if the solution satisfies $\dot{x} = Ax$

Uniqueness: Assume that we have two solutions

$$\underline{x}_1(t) = \alpha_1 \underline{u}_1 e^{\lambda_1(t-t_0)} + \alpha_2 \underline{u}_2 e^{\lambda_2(t-t_0)} + \dots +$$
$$\alpha_n \underline{u}_n e^{\lambda_n(t-t_0)}$$

and

$$\underline{x}_2(t) = \beta_1 \underline{u}_1 e^{\lambda_1(t-t_0)} + \beta_2 \underline{u}_2 e^{\lambda_2(t-t_0)} + \dots +$$
$$\beta_n \underline{u}_n e^{\lambda_n(t-t_0)}$$

Since the initial conditions are the same, ie
$\underline{x}_1(t_0) = \underline{x}_2(t_0) \Rightarrow \underline{x}_1(t_0) - \underline{x}_2(t_0) = 0$

$$(\alpha_1 - \beta_1)\underline{u}_1 + (\alpha_2 - \beta_2)\underline{u}_2 + \dots + (\alpha_n - \beta_n)\underline{u}_n = 0.$$

This implies that $\alpha_i = \beta_i$, $i = 1, \dots, n$
since the vectors $\underline{u}_i$ are linearly independent.

Hence $\underline{x}_1 = \underline{x}_2$

Example: Solve the following systems using the above theorem.

$$\dot{x}_1 = x_1 + 12x_2 \qquad x_1(0) = 0$$
$$\dot{x}_2 = 3x_1 + x_2 \qquad x_2(0) = 1.$$

Solution

$$\dot{\underline{x}} = A\underline{x}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 12 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{x}(t) = \alpha_1 \underline{u_1} e^{\lambda_1 (t-t_0)} + \alpha_2 \underline{u_2} e^{\lambda_2 (t-t_0)}$$

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \qquad \begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}$$

$$|A - \lambda \underline{I}| \Rightarrow \begin{pmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{pmatrix}$$

$$\Rightarrow (1-\lambda)(1-\lambda) - 36$$
$$\Rightarrow 1 - \lambda - \lambda + \lambda^2 - 36$$
$$\Rightarrow \lambda^2 + 2\lambda - 35$$
$$\Rightarrow \lambda_1 = 7 \,, \quad \lambda_2 = -5$$

for $\lambda_1 = 7$

$$(A - \lambda_1 I)\underline{x} = 0$$

$$\begin{pmatrix} -6 & 12 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -6x_1 + 12x_2 = 0$$
$$-6x_1 = -12x_2$$
$$x_1 = 2x_2$$

Let $x_1 = 2$, then $x_2 = 1$

$\therefore$ The eigenvector corresponding to $\lambda_1 = 7$ is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{u_1}$$

For $\lambda_2 = -5$

$(A - \lambda_2 I)\underline{x} = 0$

$$\begin{pmatrix} 6 & 12 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$6x_1 + 12x_2 = 0$

$6x_1 = -12x_2$

$x_1 = -2x_2$

Let $x_1 = -2, \quad x_2 = 1$

$\therefore$ The eigenvector corresponding to $\lambda_2 = -5$ is

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \underline{u_2}$$

$$\underline{x}(t) = \alpha_1 \underline{u_1} e^{\lambda_1 (t - t_0)} + \alpha_2 \underline{u_2} e^{\lambda_2 (t - t_0)}$$

$$\underline{x}(t) = \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{7t} + \alpha_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x(0) = \alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$= 2\alpha_1 - 2\alpha_2 = 0$

$\alpha_1 + \alpha_2 = 1$

$$\Rightarrow \alpha_1 = \alpha_2 = \frac{1}{2}$$

$$\underline{x}(t) = \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{7t} + \frac{1}{2}\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$$

$$\underline{x}(t) = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{7t} + \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} e^{-5t}$$

$$\left.\begin{aligned} \dot{x}(t) &= ? \\ A\underline{x} &= ? \end{aligned}\right\}$$

Exponential Matrix

Given a system $\dot{\underline{x}} = A\underline{x}$, using an exponential matrix method of solution, the solution to the system is

$$\underline{x} = e^{A(t-t_0)}\,\underline{c}$$

$e^x$ is a series expansion given as:

$$e^x = 1 + x + 1/2!\,x^2 + 1/3!\,x^3 + \dots$$

$$= \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

If $A \in \mathbb{R}^{n\times n}$ is a square matrix, then

$$e^A = A^0 + \frac{A^1}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$A^0 = I_n$$

$$e^A = I_n + A + 1/2!\,A^2 + \frac{1}{3!}A^3 + \dots$$

$$= \sum_{r=0}^{\infty} \frac{A^r}{r!}$$

$$e^{At} = I_n + At + \frac{1}{2!}(At)^2 + \dots$$

$$= \sum_{r=0}^{\infty} \frac{(At)^r}{r!}$$

Example: Given a diagonal matrix A as:

$$A = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

Show that the exponential matrix of A is

$$e^A = \begin{pmatrix} e^p & 0 \\ 0 & e^q \end{pmatrix}$$

Solution

$$e^A = I_2 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^3 + \dots$$

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^2 = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

$$= \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix}$$

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^3 = \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

$$= \begin{pmatrix} p^3 & 0 \\ 0 & q^3 \end{pmatrix}$$

$\vdots$

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^2 + \frac{1}{3!} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^3 + \cdots$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} p^3 & 0 \\ 0 & q^3 \end{pmatrix} + \cdots$$

Adding we obtain:

$$= \begin{pmatrix} 1+p+\frac{1}{2!}p^2+\frac{1}{3!}p^3+\dots & 0 \\ 0 & 1+q+\frac{1}{2!}q^2+\frac{1}{3!}q^3+\dots \end{pmatrix}$$

$$= \begin{pmatrix} e^p & 0 \\ 0 & e^q \end{pmatrix}$$

Hence, the exponential matrix of any diagonal matrix $A \in \mathbb{R}^{n \times n}$ can be represented in the above form.

If $A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, $e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^5 \end{pmatrix}$

Recall that if $A$ has $n$ distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$, then the modal matrix is formed by using the eigenvectors as columns of the matrix ie

$$M = [\underline{u}_1, \underline{u}_2 \dots \underline{u}_n]; \quad \Lambda = \begin{pmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

then

$\Lambda = M^{-1} A M$

if we premultiply by $M$

$$M\Lambda = MM^{-1}AM = I_n AM$$
$$= AM$$

and postmultiply by $M^{-1}$,

$$M\Lambda M^{-1} = A\underbrace{MM^{-1}}_{I_n} = A$$

$$\therefore \; A = M\Lambda M^{-1}$$

$$e^{A} = Me^{\Lambda}M^{-1}$$

$$\boxed{e^{At} = Me^{\Lambda t}M^{-1}}$$

$$e^{At} = I_n + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$
$$= \sum_{r=0}^{\infty} \frac{(At)^r}{r!}$$
$$= \sum_{r=0}^{\infty} \frac{(M\Lambda M^{-1}t)^r}{r!}$$

for $r \geqslant 2$,

$$(M\Lambda M^{-1})^2 = M\Lambda \underbrace{M^{-1} \cdot M}_{I_n}\Lambda M^{-1}$$
$$= M\Lambda^2 M^{-1}$$

$$(M\Lambda M^{-1})^3 = M\Lambda^2 M^{-1} \cdot M\Lambda M^{-1}$$
$$= M\Lambda^3 \underbrace{M^{-1} \cdot M}_{I}\Lambda M^{-1}$$
$$= M\Lambda^3 M^{-1}$$

hence $M$ and $M^{-1}$ are not dependent on $r$

$$= \sum_{r=0}^{\infty} \frac{(M \Lambda M^{-1} t)^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{M \Lambda^r M^{-1} t^r}{r!} = M \sum_{r=0}^{\infty} \frac{(\Lambda t)^r}{r!} M^{-1}$$

$$= M e^{\Lambda t} M^{-1}$$

$$\therefore e^{At} = M e^{\Lambda t} M^{-1}$$

For any constant vector $\underline{u} \in \mathbb{R}^n$,

$x(t) = e^{At} \underline{u}$
is a solution of $\underline{\dot{x}} = Ax$

Given that $x(t_0) = x_0$ is an initial value to the system $\underline{\dot{x}} = A\underline{x}$, then if $\underline{x}(t) = e^{At} \underline{u}$ is a solution to the system we obtain:

$\underline{x}(t_0) = \underline{x_0}$ into $\underline{x}(t) = e^{At} \cdot \underline{u}$

$$x_0 = e^{At_0} \underline{u}$$

OR $\underline{u} = x_0 e^{-At_0}$

$$\underline{x}(t) = e^{At} \cdot \underline{x_0} e^{-At_0}$$

$$\underline{x}(t) = e^{A(t-t_0)} \cdot \underline{x_0} ;$$

is the solution to the system $\underline{\dot{x}} = A\underline{x}$ ;

when $\underline{x}(t_0) = \underline{x}_0$.

Example:- Consider the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$,

Solve the system if $\underline{x}(t_0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e^{At} = M e^{\Lambda t} M^{-1}$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$

Verify!!!

The eigenvector corresponding to $\lambda_1 = 0$ is

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

while the eigenvector corresponding to $\lambda_2 = 1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

values of $\lambda_1$ and $\lambda_2$ on the main diagonal

Other entries are zero

$$e^{\Lambda t} = \begin{pmatrix} e^{0\cdot t} & 0 \\ 0 & e^{1\cdot t} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{t} \end{pmatrix}$$

$$M = \left( \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right)$$ values of eigenvectors

$$M = \begin{pmatrix} \overset{a}{0} & \overset{b}{1} \\ \overset{c}{1} & \overset{d}{1} \end{pmatrix} , \quad M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$M^{-1} = -1 \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Me^{\Lambda t}M^{-1} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{t} \end{pmatrix}} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{t} \\ 1 & e^{t} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Me^{\Lambda t}M^{-1} = \begin{pmatrix} e^{t} & 0 \\ e^{t}-1 & 1 \end{pmatrix}$$

but $\underline{x}(t_0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{x}_0$

$$\underline{x}(t) = e^{A(t-t_0)}\,\underline{x}_0$$

$$= e^{A(t-t_0)} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-t_0)} & 0 \\ e^{(t-t_0)} - 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha e^{(t-t_0)} \\ \alpha\left(e^{(t-t_0)} - 1\right) + \beta \end{pmatrix}$$

$$e^{At} = I_n + At + \frac{1}{2!}(At)^2 + \dots$$

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad$ $At = \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix}$

$e^{At} = e^{\begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix}}$ $\qquad$ $(At)^2 = \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix}^2$

$(At)^3 = \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix}^3$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & 0 \\ t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} t^2 & 0 \\ t^2 & 0 \end{pmatrix}$$

$$+ \frac{1}{3!}\begin{pmatrix} t^3 & 0 \\ t^3 & 0 \end{pmatrix} + \cdots$$

Adding:

$$= \begin{pmatrix} 1 + t + 1/2!\, t^2 + \frac{1}{3!} t^3 + \cdots & 0 \\ t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 \\ e^t - 1 & 1 \end{pmatrix}$$

Exercise: Find $e^{At}$ when $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}.$$