# Scalar nth-order ODE with Constant Coefficients

A linear ODE has constant coefficients if

$$y^{(n)} + a_{n-1}y^{n-1} + \dots + a_1(x)y' + a_0(x)y = b(x) \to 0 \quad — (1)$$

all the coefficients $a_i(x)$ in equation (1) are constants. If one or more of these coefficients are not constant, then equation (1) is said to have variable coefficients.

NB: Constant coefficients is the coefficients not attached to variables in an expression.

## Homogeneous Equations

Consider the homogeneous equation

$$y^n + a_{n-1}(x)y^{n-1} + \dots + a_1(x)y' + a_0(x)y = 0 \quad — (2)$$

Let $y = e^{\lambda x}$ be the solution of equation (2), then

$$y' = \lambda e^{\lambda x}, \; y'' = \lambda^2 e^{\lambda x} \dots y^{(n)} = \lambda^n e^{\lambda x} \quad — (3)$$

Substitute equation (3) into equation (2) to obtain:

$$\lambda^n e^{\lambda x} + a_{n-1}(x)\lambda^{n-1}e^{\lambda x} + \dots + a_1(x)\lambda e^{\lambda x} + a_0(x)e^{\lambda x} = 0$$

Divide all through by $e^{\lambda x}$ to get

$$\lambda^n + a_{n-1}(x)\lambda^{n-1} + \dots + a_1(x)\lambda + a_0(x) = 0 \quad — (4)$$

Equation (4) is called the characteristic or auxiliary equation.

Example: The characteristic equation of the DE

a. $y''' + 3y'' + 2y' + y = 0$ is

$$\lambda^3 e^{\lambda x} + 3\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0$$

$\lambda^3 + 3\lambda^2 + 2\lambda + 1 \longrightarrow$ The characteristic equation

b. $y'' - 2y' + y = 0$

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0$$

$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow$ The characteristic equation

Exercise:
What is the characteristic equation of:

1. $y'''' - 3y''' + 2y'' - y = 0$

$$\lambda^4 e^{\lambda x} - 3\lambda^3 e^{\lambda x} + 2\lambda^2 e^{\lambda x} - e^{\lambda x} = 0$$
$$\lambda^4 - 3\lambda^3 + 2\lambda^2 - 1 = 0$$

2. $\dfrac{d^5y}{dx^5} - \dfrac{3d^3y}{dx^3} + 5\dfrac{dy}{dx} - 7y = 0$

$$\lambda^5 e^{\lambda x} - 3\lambda^3 e^{\lambda x} + 5\lambda e^{\lambda x} - 7e^{\lambda x} = 0$$
$$\lambda^5 - 3\lambda^3 + 5\lambda - 7 = 0$$

Note: You can only define characteristic equation for linear homogeneous equations with constant coefficients.

# General Solution for nth homogeneous ODE

1. If all the roots of equation 4 are real and distinct, then the n linearly n linearly independent solutions are

$$y_1(x) = e^{\lambda_1 x}, \quad y_2(x) = e^{\lambda_2 x}, \quad \ldots, \quad y_n(x) = e^{\lambda_n x}$$

A linear combination, (superposition principle) of these n solution for the homogeneous equation.

Hence for real and distinct roots, the general solution is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

where $c_i$'s are arbitrary constants.

2. If some of the roots are equal say $\lambda_1 = \lambda_2$ and the rest are distinct, the general solution is:

$$y = (c_1 + c_2 x) e^{\lambda_1 x} + c_3 e^{\lambda_3 x} + \cdots + c_n e^{\lambda_n x}$$

OR $$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + c_3 e^{\lambda_3 x} + \cdots + c_n e^{\lambda_n x}$$

In general, if $\lambda$ is a root of multiplicity $p > 1$, it will induce the following p linearly independent solutions

$$x^i e^{\lambda x}, \quad i = 0, 1, \ldots p-1$$

For example if $\lambda$ is repeated 5 times, then

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{\lambda_1 x} + \cdots$$

N.B: By saying an equation has real roots, we mean that the solutions (or roots) of the equation are equal to one another. Moreover, by saying that the equation has distinct roots, we mean that all the roots (solutions) of the equations are not equal to one another.

E.g the equation

$$x^2 - 2^2 = 0 \Rightarrow (x-2)(x+2) = 0$$ has two distinct roots which are 2 and $-2$.

Whereas the equation

$$x^2 - 2x + 1 = 0 \Rightarrow (x-1)(x-1) = 0$$

has two real roots, but which are equal and are $-1$ and $-1$.

3. If there are complex conjugate roots, say $\lambda_1 = u + iv$ and $\lambda_2 = u - iv$, where $i = \sqrt{-1}$ and the remaining roots are distinct, then

$$y = c_1 e^{(u+iv)x} + c_2 e^{(u-iv)} + c_3 e^{\lambda_3 x} + \dots + c_n e^{\lambda_n x}$$

$$e^{u+iv} = e^{u} \cdot e^{iv}$$

$$y = e^{ux}\left(c_1 e^{ivx} + c_2 e^{-ivx}\right) + c_3 e^{\lambda_3 x} + \dots + c_n e^{\lambda_n x}$$

In real form:

$$y = e^{ux}\left(\bar{c}_1 \cos vx + \bar{c}_2 \sin vx\right) + c_3 e^{\lambda_3 x} + \dots + c_n e^{\lambda_n x}$$

is the general solution of the homogeneous equation.

Note that

$$e^{ivx} = \cos vx + i \sin vx$$

$$e^{-ivx} = \cos vx - \sin vx$$

$$e^{ux}\left(c_1 e^{ivx} + c_2 e^{-ivx}\right) \text{ is}$$

$$e^{ux}\left(c_1 \cos vx + c_1 \sin vx + c_2 \cos vx - c_2 \sin vx\right)$$

$$e^{ux}\left(\underbrace{(c_1 + c_2)}_{\bar{c}_1} \cos vx + \underbrace{(c_1 - c_2)}_{\bar{c}_2} \sin vx\right)$$

$$\therefore\ y = e^{ux}\left(\bar{c}_1 \cos vx + \bar{c}_2 \sin vx\right) + c_3 e^{\lambda_3 x} + \dots + c_n e^{\lambda_n x}$$

Examples:

1. $y'' - y' - 2y = 0$

characteristic equation

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0$$

$$\lambda^2 - \lambda - 2$$

factorising: $(\lambda - 2)(\lambda + 1) = 0$

$$\lambda - 2 = 0 \text{ or } \lambda + 1 = 0$$

$$\lambda = 2 \text{ or } \lambda = -1$$

$$y_1 = e^{2x}, \quad y_2 = e^{-x}$$

$$\therefore\ y = c_1 e^{2x} + c_2 e^{-x}$$

which is the general solution to the second order ODE.

2. $y'' - 8y' + 16y = 0$

characteristic equation

$$\lambda^2 e^{\lambda x} - 8\lambda e^{\lambda x} + 16 e^{\lambda x} = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

Factorising: $(\lambda-4)^2 = 0$
$\lambda = 4$ twice

$\lambda_1 = \lambda_2 = 4$
$y_1 = e^{4x}$ , $y_2 = xe^{4x}$

$$y = c_1 e^{4x} + c_2 x e^{4x}$$

3. $y''' + y = 0$
characteristic equation:
$\lambda^3 e^{\lambda x} + e^{\lambda x} = 0$
$\lambda^3 + 1 = 0$
factorising: $(\lambda+1)(\lambda^2-\lambda+1) = 0$
using quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} , \quad b=-1,\ a=1,\ c=1$$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \qquad i = \sqrt{-1}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$\lambda_1 = -1$, $\lambda_2 = \frac{1+\sqrt{3}i}{2}$ , $\lambda_3 = \frac{1-\sqrt{3}i}{2}$

$y_1 = e^{-x}$ , $y_2 = e^{\left(\frac{1+\sqrt{3}i}{2}\right)x}$ , $y_3 = e^{\left(\frac{i-\sqrt{3}i}{2}\right)x}$

For which the general solution is

$$y = c_1 e^{-x} + c_2 e^{\frac{1+\sqrt{3}}{2}ix} + c_3 e^{\frac{1-\sqrt{3}i}{2}x}$$

or $y = c_1 e^{-x} + e^{1/2x}\left(\bar{c}_2 \cos\frac{\sqrt{3}}{2}x + \bar{c}_3 \sin\frac{\sqrt{3}}{2}x\right)$

Exercise

Try

a. $y'' - 2y' + y = 0$

b. $4y'' + y' = 0$

c. $y'' + 8y' + 16 = 0$

## Method of Undetermined Coefficients

Obtain the general solution for a non-homogeneous linear ODE; $b(x) \neq 0$, but we

1. First solve the associated homogeneous equation

2. Try a particular solution that looks like the right hand side $b(x)$.

Note that this method works for constant coefficients linear equations only.

Consider different cases for $b(x)$

1. If $b(x)$ is a polynomial of degree $P$,

then $b(x) = K_p x^p + K_{p-1} x^{p-1} + \dots + K_1 x + K_0$
assume a particular solution which is also a polynomial of degree $P$ of the form:

$$y_p = A_p x^p + A_{p-1} x^{p-1} + \dots + A_1 x + A_0$$

2. If $b(x)$ is an exponential form; then $b(x) = Ke^{wx}$

Try a particular solution of the form
$y_p = Ae^{wx}$

3. If $b(x)$ is a trigonometric form,

then $b(x) = K\sin wx$ or $b(x) = K\cos wx$

try a particular solution of the form

$y_p = A\cos wx + B\sin wx$

4. If $b(x)$ is a sum of the forms given above, try a particular solution which is a sum of the above forms

5. If $b(x)$ is a product of the forms given above, try a particular solution which is a product of the above forms.

Examples:

1. $y'' + y = 2x^2$

Solution:

Solve the homogeneous part first:

$y'' + y = 0$

$\lambda^2 + 1 = 0$ $\qquad i = \sqrt{-1}$

$\lambda = \pm\sqrt{-1}$ ,

$\lambda = \pm i$

or $\lambda_1 = i$ , $\lambda_2 = -i$

$y_c = c_1 \cos x + c_2 \sin x$

Solve the non-homogeneous part.

$y'' + y = 2x^2$

Since $b(x) = 2x^2$, then assume

$y_p = A_2 x^2 + A_1 x + A_0$

$y_p' = 2A_2 x + A_1$

$y_p'' = 2A_2$

Substitute into the non-homogeneous equation $y'' + y = 2x^2$

$2A_2 + A_2 x^2 + A_1 x + A_0 = 2x^2$

Equate coefficients

$A_2 = 2$

$A_1 = 0$

$2A_2 + A_0 = 0$

$2A_2 + A_0 = 0 \Rightarrow 4 + A_0 = 0 \therefore A_0 = -4$

$y_p = 2x^2 - 4$

Then the general solution is

$y = y_c + y_p$

$\therefore y = c_1 \cos x + c_2 \sin x + 2x^2 - 4$.

Try:

$$y'' - 3y' + 2y = 4e^{3x}$$

Solve the homogeneous part first

$$y'' - 3y' + 2y = 0$$

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda_1 = 2 \quad \& \quad \lambda_2 = 1$$

$$y_c = c_1 e^{2x} + c_2 e^{x}$$

Solve the non-homogeneous Part $y'' - 3y' + 2y = 4e^{3x}$

$$y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y'' - 3y' + 2y = 4e^{3x}$$

$$9Ae^{3x} - 3(3Ae^{3x}) + 2(Ae^{3x}) = 4e^{3x}$$

$$\cancel{9Ae^{3x}} - \cancel{9Ae^{3x}} + 2Ae^{3x} = 4e^{3x}$$

$$2A\cancel{e^{3x}} = 4\cancel{e^{3x}}$$

$$2A = 4, \quad A = 2$$

$$\therefore \ y_p = 2e^{3x}$$

$$\therefore \ y = y_c + y_p$$

$$y = c_1 e^{2x} + c_2 e^{x} + 2e^{3x}.$$

Thus far we have solved for higher order ODEs with constant coefficients using particular solution of the non-homogeneous equations.

Next, we give a more general way of finding particular solutions of ODEs which need not to be constant coefficients

Exercise:

Solve $y'' - y' - 2y = 3\sin x$.

Solution

$$y = C_1 e^{-x} + C_2 e^{2x} + \frac{3}{10}\cos x - \frac{9}{10}\sin x.$$