Solution to Practice Midterm 2

1. (a), (b)

See text book.

1. (c) Since (x_n) is bounded, it has a convergent subseq (x_{n_k}) by Bolzano-Weierstrass Thm. Then the seq

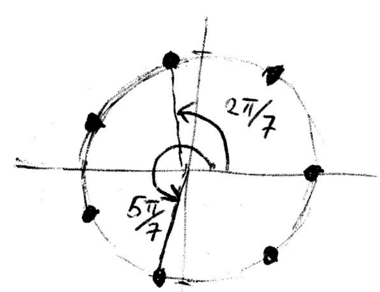
$$\left(f(x_{n_k})\right)_{k=1}^{\infty}$$

is also bounded because f is a bounded function. Again by B-W Theorem, $\left(f(x_{n_k})\right)_{k=1}^{\infty}$ has a convergent subseq. Let's called it $(f(y_k))$. Then (y_k) is also a subseq of (x_{n_k}). So (y_k) conv. too.

2. (a) (i) $\left\{0, \pm\frac{\sqrt{2}}{2}, \pm 1\right\}$. (ii) $\limsup = 1$, $\liminf = -1$.
(iii) limit does not exist

(b) (i) $\left\{\sin\frac{\pi}{7}, \sin\frac{2\pi}{7}, \sin\frac{3\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{5\pi}{7}, \sin\frac{6\pi}{7}, 0\right\}$.

(ii) $\limsup = \sin\frac{2\pi}{7}$
$\liminf = \sin\frac{5\pi}{7}$.



(iii) does not exist.

(c) (i) $\{0\}$. (ii) 0, 0. (iii) 0.

Since $\left|\sin\frac{n\pi}{7}\right| < \sin\frac{3\pi}{7} < 1$.

3(a) For any $N > 0$,

$$\sup \underbrace{\{ s_{n_k} : k > N \}}_{A_N} \leq \sup \underbrace{\{ s_n : n > N \}}_{B_N}$$

because A_N is a subset of B_N.

Both sequences $(\sup A_N)_{N=1}^{\infty}$ and $(\sup B_N)_{N=1}^{\infty}$ has a limit and

$$\lim(\sup A_N) \leq \lim(\sup B_N).$$

So $\limsup s_{n_k} \leq \limsup s_n$.

(b). If (s_n) is decreasing then $\lim s_n$ exists ($\in \mathbb{R}$ OR $\pm\infty$). Therefore

$\lim s_{n_k} = \lim s_n$ for any subseq (s_{n_k}).

4. Let $s_n = \dfrac{(2n)!!}{n^n}$. Then

$$\left| \frac{s_{n+1}}{s_n} \right| = \frac{(2n+2)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n)!!} = \frac{(2n+2)\cancel{(2n+1)}}{n+1} \cdot \frac{n^n}{(n+1)^n}$$

$$= 2 \cdot \frac{1}{(1+\frac{1}{n})^n} \to \frac{2}{e}.$$

Hence $\lim a_n = \lim |s_n|^{1/n} = \lim \left| \frac{s_{n+1}}{s_n} \right| = \frac{2}{e}$.

5(a) There exists $N > 0$, s.t

$$\ln n < n^{1/2}, \quad \text{for } n > N.$$

So $$\frac{\ln n}{n^2} < \frac{1}{n^{3/2}}.$$

By Comparison Theorem, $\sum \frac{\ln n}{n^2}$ converges since $\sum \frac{1}{n^{3/2}}$ converges.

(b) $$\left| \frac{\sin n}{n^2+n+1} \right| \leq \frac{1}{n^2}.$$

By C.T. $\sum \frac{\sin n}{n^2+n+1}$ converges (absolutely) since $\sum \frac{1}{n^2}$ converges.

(c). $$\sum (-1)^n \frac{1+2^n}{3^n} = \sum \left[\left(-\frac{1}{3}\right)^n + \left(-\frac{2}{3}\right)^n \right]$$

$$= \sum \left(-\frac{1}{3}\right)^n + \sum \left(-\frac{2}{3}\right)^2$$ converges since both series converge (Geometric series with $|r| < 1$.)

(d) Let $a_n = (-1)^n \frac{n}{n+1}$. Then

$$\lim a_{2m} = 1, \quad \lim a_{2m+1} = -1.$$

So (a_n) diverges and hence Σa_n diverges.

(e) Let $a_n = (-1)^n \left(\frac{n}{n+1}\right)^n$. Then

$$\lim a_{2m} = \frac{1}{e}, \quad \lim a_{2m+1} = -\frac{1}{e}.$$

So (a_n) diverges and hence Σa_n diverges.

5(f). Let $f(x) = \dfrac{1}{x(\ln x)^{1/2}}$.

Then $f(x)$ is >0, continuous and decreasing (since $x(\ln x)^{1/2}$ is increasing) on $[2, \infty)$.

By integral test,

$$\sum \frac{1}{n(\ln n)^{1/2}} \text{ conv. iff } \int_2^\infty f(x)\,dx < \infty.$$

$$\int_2^\infty f(x)\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^{1/2}}\,du \qquad \left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array} \right)$$

$$= +\infty.$$

So $\sum \frac{1}{n(\ln n)^{1/2}}$ diverges.

(g). Let $a_n = (-1)^n \dfrac{(2n)!}{n^{2n}}$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{n^{2n}}{(n+1)^{2n}}$$

$$\to 4 \cdot \left(\frac{1}{e}\right)^2 = \frac{4}{e^2} < 1.$$

So $\sum a_n$ converges (absolutely) by ratio test.

6. $a_n > 0 \Rightarrow \dfrac{a_n}{1+a_n} < 1 \Rightarrow \dfrac{a_n^2}{1+a_n} < a_n$.

By C.T, $\sum \dfrac{a_n^2}{1+a_n}$ converges since $\sum a_n$ does.

7 (a) See textbook.

7(b) Suppose

$$|b_n| \leq M \text{ , for all } n.$$

Given any $\varepsilon > 0$, let $\varepsilon_1 = \varepsilon / M$. Then there exist $N > 0$ s.t.

$$\text{" } m \geq n > N \Rightarrow \sum_{k=n}^{m} |a_k| < \varepsilon_1 = \varepsilon/M \text{ ".}$$

Since Σa_n converges absolutely

Thus,

$$m \geq n > N$$

$$\Rightarrow \sum_{k=n}^{m} |a_k| < \varepsilon_1$$

$$\Rightarrow \left| \sum_{k=n}^{m} a_k b_k \right| \leq \sum_{k=n}^{m} |a_k b_k| \quad \text{(triangle inequality)}$$

$$\leq \sum_{k=n}^{m} |a_k| \cdot M$$

$$< M \cdot \varepsilon_1 = \varepsilon.$$

Therefore, by Cauchy Criterion, $\Sigma a_n b_n$ Converges.

8. ① See textbook.

② Counterexample: $\Sigma a_n = \Sigma \frac{1}{n}$ div. but

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

9. See textbook

10. See textbook.

11 (a). For any $\varepsilon > 0$, let $\delta = \varepsilon$. Then

$|x - \overset{0}{x_0}| = |x| < \delta$ implies that

$$|f(x) - \underset{0}{f(0)}| = \begin{cases} |x \cos \frac{1}{x}|, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\leq |x| \quad (\text{since } |\cos \tfrac{1}{x}| \leq 1.)$$

$$< \delta = \varepsilon.$$

So by ε-δ property $f(x)$ is cont. at $x_0 = 0$.

(b) Take a sequence $x_n = \frac{1}{2n\pi}$ $(n \in \mathbb{N})$. Then

$$\lim x_n = \lim \frac{1}{2n\pi} = 0. \quad \text{and}$$

$$g(x_n) = \cos(2n\pi) = 1$$

So $\lim g(x_n) = 1 \neq f(0)$.

By def'n, $g(x)$ is discontinuous at 0.

12. (a) Suppose x_0 is irrational, then $f(x_0) = 0$.

Consider the set

$$S_1 = \left\{ \frac{p}{q} \in (x_0, x_0+1) : p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p, q) = 1, \overbrace{\frac{1}{q} \geq \varepsilon}^{\text{i.e. } q \leq \frac{1}{\varepsilon}} \right\}.$$

There are finitely many $q \in \mathbb{N}$ with $q \leq \frac{1}{\varepsilon}$ (namely, $[\frac{1}{\varepsilon}]$.)

and for each given $q \in \mathbb{N}$, there are finitely many fractions with denominator q in (x_0, x_0+1). (At most $\lceil \frac{1}{q} \rceil$)

So for a given $\varepsilon > 0$, the set S_1 is finite, and has a minimum m. $(m > x_0)$

Similarly,

$$S_2 = \left\{ \frac{p}{q} \in (x_0 - 1, x_0) : p \in \mathbb{Z},\ q \in \mathbb{N},\ \gcd(p, q) = 1,\ \frac{1}{q} \geq \frac{1}{\varepsilon} \right\}$$

is also finite and has a maximum M. $(M < x_0)$

Thus, for any $\varepsilon > 0$, let

$$\delta = \min\{ m - x_0,\ x_0 - M \} > 0.$$

Then $|x - x_0| < \delta$ implies that

$$x_0 - \delta < x < x_0 + \delta$$

$$\Rightarrow x_0 - (x_0 - M) < x < x_0 + (m - x_0)$$

$$\Rightarrow \quad M < x < m.$$

By def'n of M and m, (i) if $x = \frac{p}{q} \in \mathbb{Q}$, $\gcd(p, q) = 1$, then

$$\frac{1}{q} < \varepsilon \Rightarrow f(x) < \frac{1}{\varepsilon}.$$

OR (ii) $x \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow f(x) = 0$.

In either case $|f(x) - f(x_0)| < \frac{1}{\varepsilon}$. So f is cont. at x_0

12(b). If $x_0 \in \mathbb{Q}$, $x_0 = \frac{p}{q}$, $\gcd(p, q) = 1$. Then

$f(x_0) = \frac{1}{q} > 0$.

However, by denseness of rationals, for each $n \in \mathbb{N}$, there is a rational y_n, s.t

$$x_0 - \sqrt{2} < y_n < x_0 + \frac{1}{n} - \sqrt{2}$$

$$\Rightarrow \quad x_0 < \underbrace{y_n + \sqrt{2}}_{= x_n} < x_0 + \frac{1}{n}.$$

So $\lim x_n = x_0$, and $x_n = y_n + \sqrt{2} \notin \mathbb{Q}$.

Thus $f(x_n) = 0$ and $\lim f(x_n) = 0 \neq f(x_0)$.

By def'n, $f(x)$ is discontinuous at $x_0 \in \mathbb{Q}$.

13. Case 1: $x_0 = 0$. Then $f(x_0) = 1$.

For each $\varepsilon > 0$, let $\delta = \min\{(\frac{1}{2}\varepsilon)^2, \sqrt{\varepsilon/2}\}$. Then

$$\delta^2 \leq (\sqrt{\varepsilon/2})^2 = \varepsilon/2; \quad \sqrt{\delta} \leq \sqrt{(\tfrac{1}{2}\varepsilon)^2} = \varepsilon/2.$$

So $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| = x^2 + \sqrt{x} < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Case 2: $x_0 > 0$. ~~[illegible]~~

For each $\varepsilon > 0$, let $\delta = \min\left\{1, \frac{3x_0}{4}, \frac{\varepsilon}{1 + 2|x_0| + \frac{2}{3\sqrt{x_0}}}\right\}$. Then

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| = |x^2 + \sqrt{x} - x_0^2 - \sqrt{x_0}|$$
$$\leq |x^2 - x_0^2| + |\sqrt{x} - \sqrt{x_0}|$$

$$\leq |x-x_0|\cdot|x+x_0| + \frac{|\sqrt{x}-\sqrt{x_0}|(\sqrt{x}+\sqrt{x_0})}{\sqrt{x}+\sqrt{x_0}} \qquad \left(|x-x_0|<\delta\leq\frac{3x_0}{4} \right.$$

$$\leq \delta\cdot\Big(\underbrace{|x-x_0|}_{\leq\delta\leq 1.}+2|x_0|\Big) + \frac{\delta}{\sqrt{x_0/4}+\sqrt{x_0}} \qquad \Rightarrow x > x_0-\frac{3x_0}{4} \quad \left. \Rightarrow x>\frac{x_0}{4}. \right)$$

$$\leq \delta\left[1+2|x_0|+\frac{2}{3\sqrt{x_0}}\right] \leq \varepsilon.$$

So by definition,

$f(x) = x^2+\sqrt{x}+1$ is continuous on its domain $\mathbb{R}^+=\{x\in\mathbb{R} : x\geq 0\}$

A typo in #13. f is not defined if $x<0$.

14 (a) See textbook.

(b). See "Solution to Homework 8." # 18.8.

15. (a) See textbook.

(b) For any $\varepsilon>0$, let $\delta = \varepsilon(\cos a)^2$.

Then "$|x-y|<\delta$, $x,y\in[0,a]$" implies that

$$|\tan x - \tan y| = \left|\frac{\sin x}{\cos x}-\frac{\sin y}{\cos y}\right|$$

$$= \frac{|\sin x\cos y-\cos x\sin y|}{|\cos x\cos y|}$$

$$= \frac{|\sin(x-y)|}{|\cos x|\cdot|\cos y|} \leq \frac{|x-y|}{(\cos a)^2} < \frac{\delta}{(\cos a)^2} = \varepsilon$$

Note that $x, y \in [0, a]$, $a < \frac{\pi}{2}$ implies that

$$|\cos x| \geq \cos a, \quad |\cos y| \geq \cos a.$$

So by definition,

$\tan x$ is unif. cont. on $[0, a]$.

(c). Let $\varepsilon_0 = 1/2$. For any $\delta > 0$, by Archemedian Property, there exists $n \in \mathbb{N}$ s.t.

$$(2\pi) \cdot n > \max\{1/\delta, 1\}$$

So $\frac{1}{(2\pi)n} < \delta$.

Let $x = \frac{1}{(2\pi)n}$, $y = \frac{1}{(2\pi)n + \pi/2}$. Then

$$0 < y < x < \delta, \quad y < x < 1.$$

$\Rightarrow$ $|x - y| < \delta$ and $x, y \in (0, 1)$.

But $|f(x) - f(y)| = \left|\sin(2\pi n) - \sin\left(2\pi n + \frac{\pi}{2}\right)\right|$

$$= |0 - 1| = 1 > \varepsilon_0 = 1/2.$$

So by defn, $\sin \frac{1}{x}$ is not unif. cont. on $(0, 1)$.