Lecture 10

Systems of linear ODEs

Solving by method of elimination

$$\left.\begin{aligned} 2x + 5y &= -1 \\ x - y &= 0 \end{aligned}\right\} \begin{aligned} &\times 1 \\ &\times 2 \end{aligned}$$

y – dependent
x – independent*

$$\begin{aligned} 2x + 5y &= -1 \\ - \quad 2x - 2y &= 0 \\ \hline 7y &= -1 \end{aligned}$$

$\therefore \; y = -1/7$ method of elimination

Systems of linear differential equation are simultaneous ordinary differential equations that involve two or more equations that contains the derivatives of two or more dependent variables (unknown function) with respect to a single independent variable.

The "D" operator

$$y = 3x^2 - 1 \quad , \quad \frac{dy}{dx} = 6x$$

Find $D(3x^2 - 1) \Rightarrow D(3x^2) - D(1)$

$$6x - 0 = 6x$$

The scalar linear nth order ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

can be written using the D operator as:

$$a_n(x) D^n y + a_{n-1}(x) D^{n-1} y + \dots + a_1(x) Dy + a_0(x) y = b(x)$$

This operator will be used to express the systems in a more compact form and also for ease of solution.

For example if we have the systems as:

$$\left.\begin{aligned} u'' - 3u + v'' + 2v' - v &= \cos t \\ u' - 2u + v' - 4v &= e^t \end{aligned}\right\} \text{———} (1)$$

then using the D operator, the system (1) is same as

$$D^2 u - 3u + D^2 v + 2Dv - v = \cos t$$
$$Du - 2u + Dv - 4v = e^t$$

$$\left.\begin{aligned} (D^2 - 3)u + (D^2 + 2D - 1)v &= \cos t \\ (D - 2)u + (D - 4)4 &= e^t \end{aligned}\right\} (2)$$

Method of Solution of a system

Example:- Consider the system

1. $\frac{dx}{dt} = 3y$

$\frac{dy}{dt} = 2x$

2. $x' - 4x + y'' = t^2$

$x' + x + y' = 0$

Solve the systems using the method of elimination.

1. $\frac{dx}{dt} = 3y \implies \frac{dx}{dt} - 3y = 0$

$\frac{dy}{dt} = 2x \qquad -2x + \frac{dy}{dt} = 0$

Using the D operator

$Dx - 3y = 0$ —— (3) $\times -2$

$-2x + Dy = 0$ —— (4) $\times D$

$$
\begin{aligned}
- \quad -2\cancel{Dx} + 6y &= 0 \\
-2\cancel{Dx} + D^2y &= 0 \\
\hline
6y - D^2y &= 0 \\
(6 - D^2)y &= 0
\end{aligned}
$$

Using the substitution $y = e^{\lambda t}$,

$y = e^{\lambda t}$

$D = y' = \lambda e^{\lambda t}$

$D^2 = y'' = \lambda^2 e^{\lambda t}$

The characteristic equation is

$(6 - \lambda^2)e^{\lambda t} = 0$

$6 - \lambda^2 = 0$

$6 = \lambda^2$

$\lambda = \pm\sqrt{6}$

$y_1 = e^{\lambda_1 t} = e^{\sqrt{6}t}, \quad y_2 = e^{\lambda_2 t} = e^{-\sqrt{6}t}$

$$y(t) = C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}$$

Going back to equations (3) and (4) to eliminate y,

$$Dx - 3y = 0 \text{ ——— } (3) \quad \times D$$
$$-2x + Dy = 0 \text{ ——— } (4) \quad \times -3$$

$$D^2x - 3D\cancel{y} = 0$$
$$-\ 6x - \cancel{3}Dy = 0$$

$$(D^2 - 6)x = 0$$

Using the substitution $x = e^{\lambda t}$

$D = x' = \lambda e^{\lambda t}$

$D^2 = x'' = \lambda^2 e^{\lambda t}$

The characteristic equation is

$$(\lambda^2 - 6)e^{\lambda t} = 0$$

$$\lambda^2 - 6 = 0$$
$$\lambda = \pm\sqrt{6}$$

$$x_1 = e^{\lambda_1 t} = e^{\sqrt{6}t}, \quad x_2 = e^{\lambda_2 t} = e^{-\sqrt{6}t}$$

The solution $x(t)$ is

$$x(t) = C_3 e^{\sqrt{6}t} + C_4 e^{-\sqrt{6}t}$$

Next, we solve for the constants C_3 \& C_4, in terms of C_1 and C_2. To solve, we substitute x and y into equations (3) and (4)

$$Dx - 3y = 0 \quad \text{——— } (3)$$
$$-2x + Dy = 0 \quad \text{——— } (4)$$

$$D\left(C_3 e^{\sqrt{6}t} + C_4 e^{-\sqrt{6}t}\right) - 3\left(C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}\right] = 0$$

$$-2\left(C_3 e^{\sqrt{6}t} + C_4 e^{-\sqrt{6}t}\right) + D\left(C_1 e^{\sqrt{6}t} + C_2 e^{-\sqrt{6}t}\right) = 0$$

Expanding

$$\sqrt{6}C_3 e^{\sqrt{6}t} - \sqrt{6}C_4 e^{-\sqrt{6}t} - 3C_1 e^{\sqrt{6}t} - 3C_2 e^{-\sqrt{6}t} = 0$$

$$-2C_3 e^{\sqrt{6}t} - 2C_4 e^{-\sqrt{6}t} + \sqrt{6}C_1 e^{\sqrt{6}t} - \sqrt{6}C_2 e^{-\sqrt{6}t} = 0$$

Factorising:

$$e^{\sqrt{6}t}\left[\sqrt{6}C_3 - 3C_1\right] + e^{-\sqrt{6}t}\left[-\sqrt{6}C_4 - 3C_2\right] = 0$$

$$e^{\sqrt{6}t}\left[-2C_3 + \sqrt{6}C_1\right] + e^{-\sqrt{6}t}\left[-2C_4 - \sqrt{6}C_2\right] = 0$$

Since these expressions has to be zero for all values of t, we must have

$$\sqrt{6}C_3 - 3C_1 = 0 \quad \text{and} \quad -\sqrt{6}C_4 - 3C_2 = 0$$

and

$$-2C_3 + \sqrt{6}C_1 = 0 \quad \text{and} \quad -2C_4 - \sqrt{6}C_2 = 0$$

$$\sqrt{6}C_3 = 3C_1 \qquad \text{and} \qquad -\sqrt{6}C_4 = 3C_2$$

$$\therefore \; C_3 = \frac{\sqrt{3}}{6}C_1 \qquad\qquad \therefore \; C_4 = -\frac{3}{\sqrt{6}}C_2$$

Hence

$$y(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$$

$$x(t) = \frac{3}{\sqrt{6}} c_1 e^{\sqrt{6}t} - \frac{3}{\sqrt{6}} c_2 e^{-\sqrt{6}t}$$

Using $-2c_3 + \sqrt{6}c_1 = 0$ and $-2c_4 - \sqrt{6}c_2 = 0$

$$c_3 = \frac{\sqrt{6}c_1}{2} \qquad \text{and} \qquad c_4 = \frac{-\sqrt{6}c_2}{2}$$

which when you rationalise $\frac{3c_1}{\sqrt{6}}$, you get $\frac{\sqrt{6}}{2}c_1$

and rationalising $\frac{-3}{\sqrt{6}}c_2$, you get $-\frac{\sqrt{6}c_2}{2}$.

$\therefore$ Using either equation, you get the same result.

2. $x' - 4x + y'' = t^2$
$x' + x + y' = 0$

Using the D-operator:

$$(D-4)x + D^2y = t^2 \quad \times (D+1) \qquad \text{—— (1)}$$
$$(D+1)x + Dy = 0 \quad \times (D-4) \qquad \text{—— (2)}$$

$$(D-4)(D+1)x + (D+1)D^2y = (D+1)t^2$$
$$-$$
$$(D+1)(D-4)x + D(D-4)y = 0$$

$$[D^2(D+1) - D(D-4)]y = (D+1)t^2$$

$$(D^3 + D^2 - D^2 + 4D)y = Dt^2 + t^2$$

$$(D^3 + 4D)y = 2t + t^2$$

This is a nonhomogeneous equation, hence, we solve the homogeneous part first.

$$D^3 + 4D = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda = 0$$

$$\lambda(\lambda^2 + 4) = 0 \implies \lambda = 0 \quad \text{or} \quad \lambda^2 + 4 = 0$$

$$\lambda^2 = -4 \implies \lambda = \pm 2i$$

$$\boxed{\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \pm i \cdot 2}$$

$$\lambda = 0, \quad \lambda = 2i, \quad \lambda = -2i$$

$$y_c = C_1 + C_2 e^{2it} + C_3 e^{-2it}$$

$$= C_1 + C_2 \cos 2t + C_3 \sin 2t.$$

$$(D^3 + 4D)y = 2t + t^2$$

$$y_p = At^3 + Bt^2 + Ct$$

$$y_p' = 3At^2 + 2Bt + C$$

$$y_p'' = 6At + 2Bt$$

$$y_p''' = 6A$$

$$y_p''' + 4y_p' = 2t + t^2$$

Substitute into

$(D^3 + 4D)y = 2t + t^2$

$6A + 4(3At^2 + 2Bt + C) = 2t + t^2$

$6A + 12At^2 + 8Bt + 4C = 2t + t^2$

$t^2: \; 12A = 1 \implies A = 1/12$

$t: \; 8B = 2 \implies B = 2/8 = 1/4$

$\therefore 6A + 4C = 0 \implies 6A = -4C$

$6(1/12) = -4C$

$\frac{1}{2} = -4C$

$\therefore C = -\frac{1}{8}$

$y_p = \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t$

$y(t) = y_c + y_p$

$= C_1 + C_2\cos 2t + C_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t$

Exercise: Use the same steps to solve for $x(t)$ and show that

$x(t) = x_c + x_p$

$x(t) = C_4 \cos 2t + C_5 \sin 2t - \frac{1}{4}t^2 + \frac{1}{8}$

and solving for C_4 and C_5 gives the final solution:

$x(t) = -\frac{1}{5}(4C_2 + 2C_3)\cos 2t + \frac{1}{5}(2C_2 - 4C_3)\sin 2t$

$- \frac{1}{4}t^2 + \frac{1}{8}.$

Difference Equation

Definitions

Consider a mapping

$$y : \mathbb{Z} \longrightarrow \mathbb{F},$$

where $\mathbb{Z}$ is a ring of integers and $\mathbb{F}$ is a field of real (or complex) numbers.

y is called a sequence of real (or complex) numbers.

Notations (Kth term): The number $y(k)$ is called the kth term and denoted as y_k.

Difference equation: Any equation relating to the terms $y_k, y_{k+1}, \ldots$ of a sequence (y_j) is called a difference equation.

Example: The following are examples of difference equations:

1. $y_{k+n} = y_k \, y_{k+n-1}$

$$y(k) = y_k \quad , \qquad k = 0, 1, 2, \ldots$$

$$y(x) = f(x, y, y' \ldots)$$

$x \to$ independent variable

$y \to$ dependent variable

For difference equation, K is the independent variable while y remains the dependent variable.

2. $y_{k+1} = 2y_k^2$

order: $k+1-k=1$ order 1

3. $y_{k+1} = 3y_k$

order: $k+1-k = 1$ order 1

Order of a difference equation: The order of a difference equation is the difference between highest and the lowest indices appearing in the equation.

Example: The difference equation

$$y_{k+3} + k y_{k+1}^2 + 3_{k-1} = 0$$

has order 4.

How?

$$k+3-(k-1) = \cancel{k}+3-\cancel{k}+1 = 4$$

$$y_{k+2} - 4 + y_{k-3} + 5y_{k-5}$$

$$k+2-(k-5) = 7$$

Linear difference equations

A linear difference equation of order n has the form:

$$a_n(k)y_{k+n} + a_{n-1}(k)y_{k+n-1} + \dots + a_1(k)y_{k+1} + a_0(k)y_k = g(k) \quad \text{——— } (1)$$

where the a_i's and g are sequences ie functions of $\mathbb{Z}$

Homogeneous difference equation

If $g(k)=0$, then equation ① is a homogeneous difference equation. Otherwise it is a non-homogeneous difference equation.

Solution of a linear difference equation.

In order to obtain the general solution of equation ①,

1. Solve the homogeneous part of the equation first.
2. Find a particular solution of the non-homogeneous part.
3. The general solution is the sum of the solution of the homogeneous equation and the particular solution of the non-homogeneous equation

The principle of superposition holds for homogeneous linear difference equation ie Any linear combination of solutions to a homogeneous linear difference equation is also a solution.

If there are n linearly independent solutions, this forms a fundamental set of solutions.

Example:- Consider the first order linear difference equation (geometric model)

$$y_{k+1} = a y_k$$

where a is a constant.

This has solution

$$y_k = C a^k$$

where c is an arbitrary constant.

Since

$$y_{k+1} = ca^{k+1}$$
$$= ca^k \cdot a$$
$$y_{k+1} = y_k \cdot a = a y_k$$

For each value of c, we have a solution. Thus there is one-parameter family of solutions.

A difference equation can have one solution, for example, the equation

$$y^2_{k+1} + y^2_k = 0$$

has the trivial solution

$$y_k = 0$$

as the only solution

finally, it is possible for no real solution to exist.

Consider the equation

$$y^2_{k+1} + 1 = 0$$

Linear difference equation with constant coefficients

Consider the linear difference equation with constant coefficients:

$$a_n y_{k+n} + a_{n-1} y_{k+n-1} + \dots + a_1 y_{k+1} + a_0 y_k = f(k) \quad (2)$$

where a_i's are constant and $f(k)$ is a function.

Introduce the operator E, such that

$$E y_k = y_{k+1}$$

$$E^2 y_k = E(\dot{y}_k) = E(y_{k+1}) = y_{k+2}$$

$$E^3 y_k \Rightarrow y_{k+3}$$

$$\vdots$$

in general $E^n y_k \Rightarrow y_{k+n}$ —— (3)

in general,

$$E^s y_k = y_{k+s} \quad \text{—— (4)}$$

then we can write equation (2) as:

$$P(E y_k) = (a_n E^n + a_{n-1} E^{n-1} + \dots + a_1 E^n + a_0) y_k = f(k) \quad \text{—— (5)}$$

To solve, assume that
$y_k = \lambda^k$, is a solution of the homogeneous equation

For $y_k = \lambda^k$ to be a solution of the homogeneous equation associated with equation (5), it is necessary and sufficient that

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0 \quad \text{—— (6)}$$

Equation (6) is called the characteristic or auxilliary equation and three cases are possible

Case 1:

If all roots of equation (6) are real and distinct, then

$$y_K^c = C_1\lambda_1^K + C_2\lambda_2^K + \cdots + C_n\lambda_n^K \quad \text{——— (7)}$$

is the general solution of the homogeneous equation

Case 2:

If some roots are equal say $\lambda_1 = \lambda_2$ and the rest are distinct, then

$$y_K^c = (C_1 + C_2 K)\lambda_1^K + C_3\lambda_3^K + \cdots + C_n\lambda_n^K \quad \text{—— (8)}$$

In general, if there is a root λ of multiplicity P, then the combination of this root will be

$$(C_1 + C_2 K + \cdots + C_p K^{P-1})\lambda^K \quad \text{——— (9)}$$

Case 3:

If some root are complex, say $\lambda_1 = P - iq$ and $\lambda_2 = P + iq$, then

$$y_K^c = C_1(P-iq)^K + C_2(P+iq)^K + C_3\lambda_3^K + \cdots + C_n\lambda_n^K \quad \text{—— (10)}$$

$$y_K^c = (\bar{C}_1 \cos K\theta + \bar{C}_2 \sin K\theta)\rho^K + C_3\lambda_3^K + \cdots + C_n\lambda_n^K \quad \text{——— (11)}$$

where $(p, q) = \rho(\cos\theta \ \sin\theta)$ and $\rho = \sqrt{p^2+q^2}$

Examples: The following examples illustrate the various situations that can arise:

1. Distinct root: Consider the linear difference equation

$$y_{k+2} - y_{k+1} - 2y_k = 0$$

This equation can be written as:

$$(E^2 - E - 2)\, y_k = 0$$

and the associated characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \implies (\lambda+1)(\lambda-2)$$

So $\lambda = -1$ or $\lambda = 2$

and the general solution is then

$$y_k = (-1)^k C_1 + 2^k C_2.$$

2. Repeated roots: Consider the difference equation

$$y_{k+2} - 6y_{k+1} + 9y_k = 0$$

In terms of the E-operator:

$$(E^2 - 6E + 9)\, y_k = 0$$

So that the characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

$$\lambda = 3 \text{ twice}$$

The general solution is

$$y_k = (C_1 + C_2 k)(3)^k$$

3. Complex roots: Consider the linear difference equation

$$y_{k+2} + 9y_k = 0$$

This equation can be written as:

$$(E^2 + 9) y_k = 0$$

The characteristic equation is

$$\lambda^2 + 9 = 0$$
$$\lambda^2 = -9$$
$$\lambda = \pm 3i$$

In complex domain:

$$y_k = C_1 (3i)^k + C_2 (-3i)^k$$

To obtain y_k in the real form, we rewrite it as

$$y_k = 3^k (C_1 e^{\pi/2 ik} + C_2 e^{-\pi/2 ik})$$

ie $y_k = 3^k (\bar{C}_1 \cos \pi/2 k) + \bar{C}_2 \sin \pi/2 k)$

For the non-homogeneous cases:

In order to find a particular solution to the non-homogeneous equation, we use the method of undetermined coefficients. This consists of trying a particular solution which looks like the RHS in equation (2).

We will be concerned with f's which are polynomial in k or of the form Aa^k or a sum of these.

- If $f(k)$ is a polynomial in k, try a particular solution which is also a polynomial of the same order as f.

- If $f(k) = Aa^k$, look for y_k^p in the form
$$y_k^p = Ba^k.$$

- If $f(k)$ is a combination of the above forms, look for a particular solution in the same form.

Remark: Note that there is no (similar) analogue of variation of parameters method for linear difference equations.

Example: Consider the equation

$$2y_{k+2} + 5y_{k+1} + 2y_k = k.$$

Solution
Solving the homogeneous part first

Let $y_k = \lambda^k$

$$2\lambda^{k+2} + 5\lambda^{k+1} + 2\lambda^k = 0$$
$$2\lambda^2 + 5\lambda + 2 = 0$$

and the characteristic equation is

$$2\lambda^2 + 5\lambda + 2 = 0 \Rightarrow (2\lambda+1)(\lambda+2)$$

So $\lambda = -1/2$ or $\lambda = -2$

The solution of the associated homogeneous equation is
$$y_k^c = C_1\left(-1/2\right)^k + C_2(-2)^k.$$

To solve for y_k^p, look for a particular solution in the form

$$y_k^p = Ak + B$$

Then

$$y_{k+1}^p = A(k+1) + B$$

$$y_{k+2}^p = A(k+2) + B$$

Substituting,

$$2y_{k+2} + 5y_{k+1} + 2y_k = k$$

$$2(A(k+2)+B) + 5(A(k+1)+B) + 2(Ak+B) = k.$$

$$2(Ak+2A+B) + 5(Ak+A+B) + 2Ak+2B = k$$

$$2Ak + 4A + 2B + 5Ak + 5A + 5B + 2Ak + 2B = k$$

$$9Ak + 9A + 9B = k$$

k: $9A = 1$

$$A = 1/9$$

$$9A + 9B = 0$$

$$9(1/9) + 9B = 0$$

$$1 + 9B = 0$$

$$\therefore \; B = -1/9.$$

$$y_k^p = 1/9\, k - 1/9.$$

$$y_k = y_k^c + y_k^p$$

$$= C_1(-1/2)^k + C_2(-2)^k + \frac{1}{9}k - \frac{1}{9}.$$

Exercise:

2. Solve the difference equation:

$$y_{k+2} + 2y_{k+1} + y_k = 2(3^k)$$

Solution:

Solving the homogeneous part first

Let $y_k = \lambda^k$

$$\lambda^{k+2} + 2\lambda^{k+1} + \lambda^k = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$
$$(\lambda+1)^2 = 0$$
$$\lambda = -1 \text{ twice}$$

The solution of the associated homogeneous equation is

$$y_k^c = c_1(-1)^k + c_2 k(-1)^k$$

To solve the non-homogeneous part, look for a particular solution in the form

$$y_k^p = A(3^k)$$
$$y_{k+1}^p = A(3^{k+1}) = 3A(3^k)$$
$$y_{k+2}^p = A(3^{k+2}) = 3^2A(3^k) \Rightarrow 9A(3^k)$$

Substituting

$$9A(3^k) + 2(3A(3^k)) + A(3^k) = 2(3^k)$$
$$9A(3^k) + 6A(3^k) + A(3^k) = 2(3^k)$$
$$16A(\cancel{3^k}) = 2(\cancel{3^k})$$

$$A = 2/16 = 1/8$$

$$\therefore y_k^p = 1/8(3^k)$$

The general solution is

$$y_k^c + y_k^p$$
$$= c_1(-1)^k + c_2k(-1)^k + \frac{1}{8}(3^k)$$

Exercise 1:

Find the general solution to the non-homogeneous second order difference equation

$$y_{k+2} - 2y_{k+1} + 5y_k = k$$

Answer:

$$y = y_k^c + y_k^p$$
$$= (\sqrt{5})^k (C_1 \cos k\theta + C_2 \sin k\theta) + 1/4\, k.$$

Exercise 2:

Given the second order homogeneous difference equation

$$y_{k+2} - 6y_{k+1} + 8y_k = 0$$

i) Find the general solution to the difference equation.

ii) If $y_0 = 3$ and $y_1 = 2$, determine the particular solution of the difference equation.

iii) Hence or otherwise, find y_3.

Answers:

i) $y_k = C_1(2^k) + C_2(4^k)$.

ii) $y_k = 5(2^k) - 2(4^k)$ is the particular solution when $y_0 = 3$ & $y_1 = 2$

iii) $y_3 = -88$