Statistical Foundations of Data Science

Combinatorics Part 1

University of the Witwatersrand

2022

Review Question

- Find the coefficent of x^5 in the expansion of $(2 + x)^7$

Lesson Plan

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- Pigeonhole Principle

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- $\binom{7}{5}2^2 = 21 \times 4 = 84$

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- I haven't counted the hairs on every head in Randburg and neither has anyone else
- Still we know that essentially no one has over 200000 hairs on their head and the population of Randburg is a bit over 300000 people.

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- There are 740 students in a school. Show that three of them have birthdays on the same day.

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- In the Wits maths sciences there are 2022 undergrad students. Some of them are FB friends, others are not. Friendship is symmetric (if I'm friends with you you're friends with me). Show that there are two students who're friends with the same number of other students.

Basic Counting

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- Product example. There are 30 CSAM faculty members and 150 students. No one is both faculty and a student. How many ways can we have a faculty member meet with a student? Answer 4500 by multiplication.

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- $26 \times 25 \times 24 \times 23 \times 22$

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- How many ways to write the numbers 1 through 5 in some order? $5! = 120$. e.g $1, 2, 3, 4, 5$ or $4, 3, 2, 1, 5$ or $2, 4, 3, 5, 1$.

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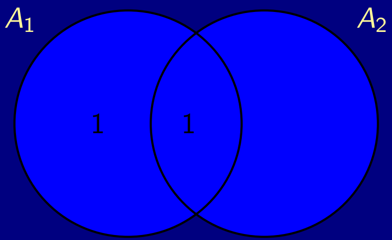
- $10 \times 9 \times 8 = \frac{10!}{7!} = 10P3$.

Inclusion Exclusion

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_2 \cap A_2|$

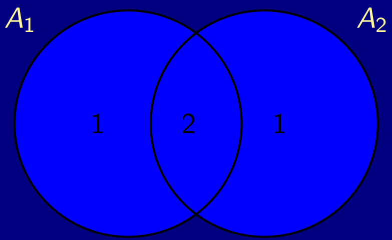
Inclusion Exclusion

A_1



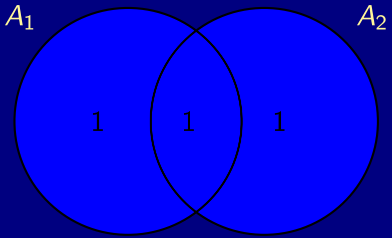
Inclusion Exclusion

$$A_1 + A_2$$



Inclusion Exclusion

$$A_1 + A_2 - A_1 \cap A_2$$



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- $$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |(A_1 \cup A_2) \cup A_3| \\ &= |(A_1 \cup A_2)| + |A_3| \\ &- |(A_1 \cup A_2) \cap A_3| \\ &= |(A_1 \cup A_2)| + |A_3| \\ &- |(A_1 \cap A_3) \cup (A_2 \cap A_3)| \end{aligned}$$

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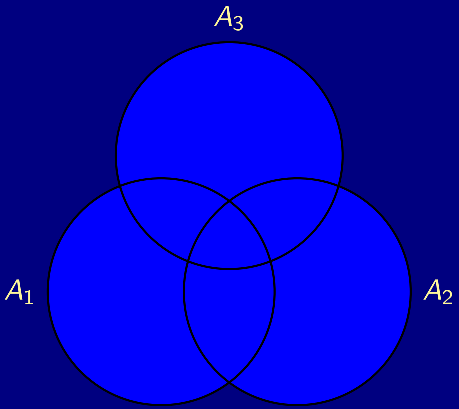

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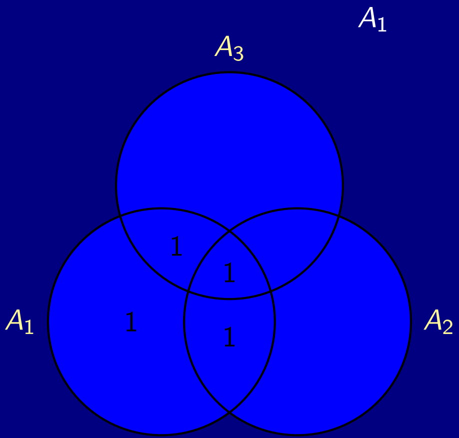

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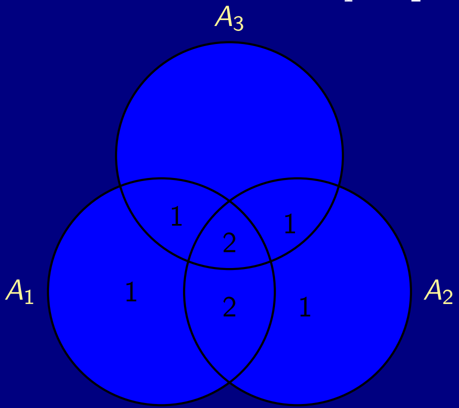
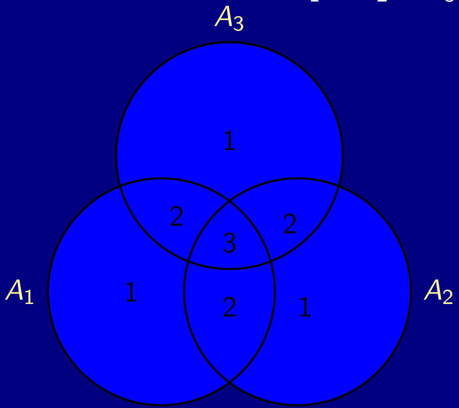
$A_1 + A_2$



Inclusion Exclusion

$A_1 + A_2 + A_3$



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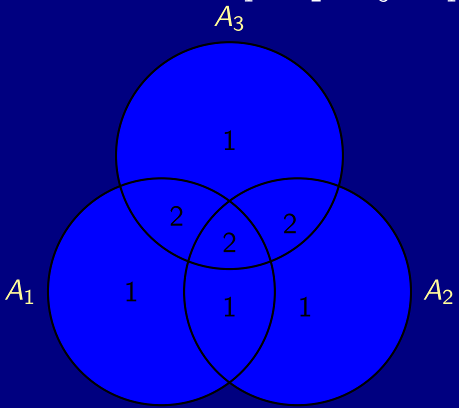
$$A_1 + A_2 + A_3 - A_1 \cap A_2$$



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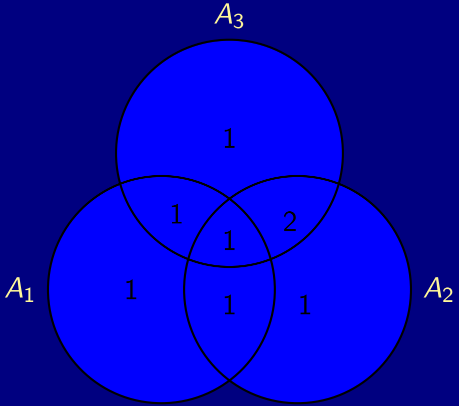

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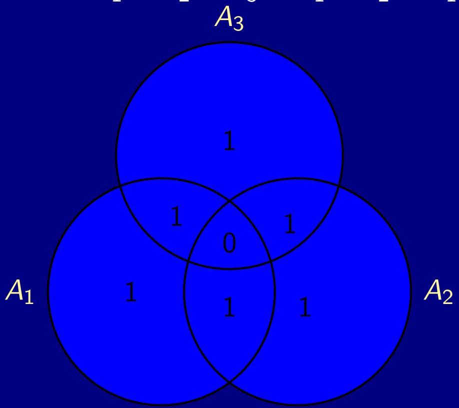

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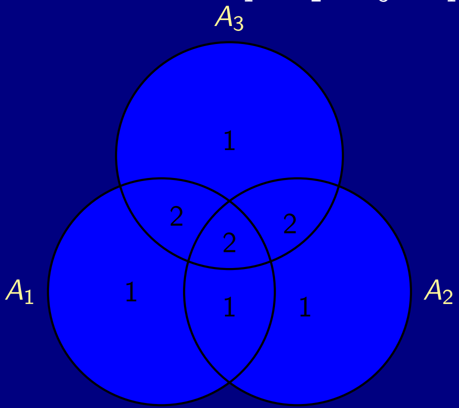

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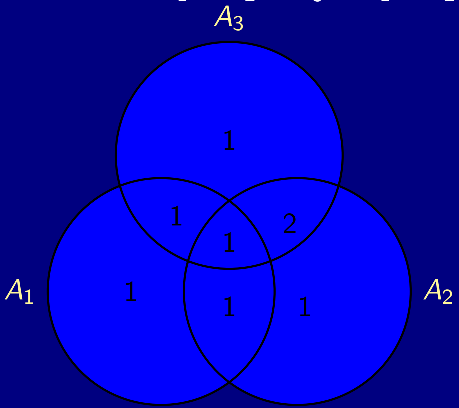

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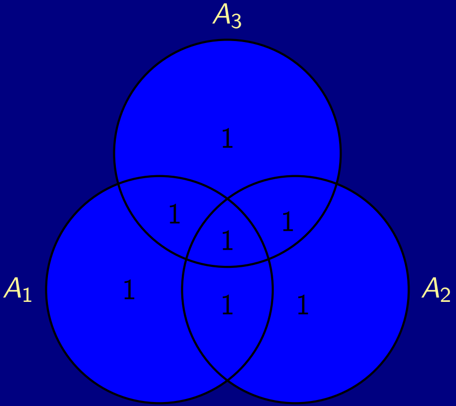

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- How many of the whole numbers from 1 to 100 are divisible by 2, 3 or 5.

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- $500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$

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- General formula is $|\cup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{t \in \text{k element sets}} |\cap_t A_t|$