Scalar nth order ordinary differential equations Cont'd

The degree of a differential equation is the highest power of the highest derivative in the equation

$$5\left(\frac{d^4 b}{dp^4}\right)^5 + 7\left(\frac{db}{dp}\right)^{10} + b^7 - b^5 = p$$

4th order ODE
An ODE of degree 5

Example

$$\frac{d^2 y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^3 + y^4 = 0$$

2nd order ODE and degree 1

$$\left(\frac{d^3 y}{dx^3}\right)^2 + x^2\frac{dy}{dx} - 2y = 5x$$

3rd order ODE and degree of 2

Linear ordinary differential equations: An nth order ODE is given as

$$F(x, y\, y', \dots, y^n) \quad \text{—— (1)}$$

is said to be linear if F is linear in $y, y', \dots y^{(n)}$

Equation (i) is linear when it can be expressed as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x) \quad \text{— (ii)}$$

If equation (ii) is reduced to first and second order ODEs, we have

$$a_1(x)y' + a_0(x)y = g(x)$$

Linear 1st order ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \quad \text{— (iii)}$$

Linear 2nd order ODE

The two properties of linear ODEs are:

1. The dependent variable y and all its derivatives are of first degree

$$\underline{(y'')^2} + 2xy' + y = e^x \longrightarrow \text{Nonlinear ODE (NL)}$$

NL term

$$y'' + 2xy' + \underline{y^2} = e^x \longrightarrow \text{Nonlinear ODE}$$

NL term

2. The coefficient $a_0, a_1, \dots a_n$ of $y, y' \dots y^{(n)}$ depend at most on the independent variable x

$$y'' + 2xy' + y = e^x \text{ — Linear ODE}$$

using $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$$a_2 = 1, \quad a_1 = 2x, \quad a_0 = 1, \quad g(x) = e^x$$

$$\underline{yy''} + 2xy' + \underline{yy'} = e^x \longrightarrow \text{Nonlinear ODE}$$

Nonlinear ODEs: A nonlinear ODE is an ODE that is not linear. These types of ODEs contain nonlinear functions or a product of the dependent variable or its derivatives.

Examples of nonlinear functions: e^y, $\cos(y)$.

Examples: Identify the linearity or non linearity of the following ODEs:

1. $y - x + 4xy' = 0$ $\Rightarrow$ Linear 1st order ODE

2. $y'' - 2y' + y = -8x$ $\rightarrow$ Linear 2nd order ODE

3. $\underbrace{(1-y)y'}_{\text{Nonlinear term (Product of } y \text{ and } y')} + 2y = e^x$ $\rightarrow$ Nonlinear

$(y)^4 \neq y^4$

$(y)^4$: $y \times y \times y \times y$

y^4: Fourth derivative

4. $\dfrac{d^2y}{dx^2} + \underbrace{\sin y}_{\text{Nonlinear term (A nonlinear function)}} = 0$ $\rightarrow$ Nonlinear

5. $y^{(4)} + \underbrace{y^2}_{\text{Nonlinear term (Power not 1)}} = 0$ $\rightarrow$ Nonlinear

6. $y^{(4)} + y = 0$ $\rightarrow$ Linear

If $y(t)$ is the displacement of a particle moving at time t,

$$y' = f(t, y)$$

where $y'(t)$ is the velocity of the particle at time t

The second order D.E is given as

$$y'' = f(t, y, y')$$

where $y''(t)$ is the acceleration

Application to a falling object problem

Consider an object falling down from a height y_0,

Let $v(t)$ be the velocity of the object at time t,

According to Newton's second law

$$F_{net} = Ma$$

since $a = \frac{dv}{dt}$

$$= m\frac{dv}{dt}$$

$$= m\frac{d^2 s}{dt^2}$$ in terms of distance covered

$$F_{net} = m\frac{dv}{dt} \quad \text{———} \quad (1)$$

F_{net} is a constant force and g is the acceleration due to gravity

$$F_{net} = mg \quad \text{———} \quad (2)$$

From (1) $F_{net} = m\frac{dv}{dt}$ and from (ii), $F_{net} = mg$

$\therefore$ from (1) $F_{net} = m\frac{dv}{dt}$

$$mg = m\frac{dv}{dt} \quad \text{———} \quad ***$$

Dividing both sides of (***) by m, we have

$$g = \frac{dv}{dt}$$

Here we have taken the downward direction as the positive direction.

In real life, falling objects experience some resistive force due to air

Case 1: If the resistive force is assumed to be proportional to the particle velocity, then

$$F_r = -hv$$

Total force = Force on the body + resistive force

$$F_1 + F_r$$

$$F = mg - hv$$

$$m\frac{dv}{dt} = mg - hv$$

$$\frac{dv}{dt} = g - \frac{h}{m}v \qquad \text{if } K = h/m$$

$$\frac{dv}{dt} = g - Kv$$

K is the resistant per unit mass which depends on the size and shape of the object

Model 1: For a falling object

$$\frac{dv}{dt} = g - Kv \quad \text{———} \quad (3)$$

Case 2: If the resistive force is proportional to the square of the particle velocity:

$$F_r = -hv^2$$

Then $$\frac{dv}{dt} = g - Kv^2$$

Model 2: for a falling object:

$$\frac{dv}{dt} = g - Kv^2 \quad \text{———} \quad (4)$$

If $v(t)$ reaches a limiting (or terminal) velocity which is constant say V_f, for which the acceleration is zero, then:

Model 1 becomes

$$\frac{dv}{dt} = g - Kv$$

$$0 = g - KV_f$$

$$g = KV_f$$

$$\therefore V_f = \frac{g}{K}$$

while in model 2

$$\frac{dv}{dt} = g - Kv^2$$

$$0 = g - KV_f^2$$

$$V_f = \sqrt{\frac{g}{K}}$$

Exercise:

Given that the particle begins to fall from rest;

1. Show that the solutions to equations (3) & (4) are:

$$v(t) = V_f\left(1 - e^{-Kt}\right) \quad \text{——— (5)}$$

$$v(t) = V_f \frac{\left(e^{\frac{2gt}{V_f}} - 1\right)}{e^{\frac{2gt}{V_f}} + 1} \quad \text{——— (6)}$$

3. Show that as $t \to \infty$ in equations 5 & 6 reduces to

$$v(t) = gt$$

Integration of ordinary differential equations (same definitions)

1. Implicit and Explicit forms

Consider the nth order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where $y^{(i)} = \dfrac{d^i y}{dx^i}$

This equation is said to be in the $\boxed{\text{implicit form}}$

for example:

$$y'' + 2xy' + y - e^x = 0$$

The explicit or normal form is:

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

for example:

$$y'' = e^x - 2xy' - y$$

2. Solution: To integrate or solve a differential equation of order n, we need to find all relations

$$f(x, y) = 0$$

Such that the values $y, y', \ldots, y^{(n)}$ obtained from $f(x, y)$ in terms of x satisfy the equation identically.

Examples: Verify that the indicated function is a solution of the given differential equation.

1. $\frac{dy}{dx} = xy^{1/2}$, $y = \frac{1}{16}x^4$ $\boxed{y^{1/2} = \frac{1}{4}x^2}$

If $y = \frac{1}{16}x^4$, then $\frac{dy}{dx} = \frac{4}{16}x^3$

$$\therefore \frac{dy}{dx} = \frac{1}{4}x^3 \checkmark$$
$$= \frac{1}{4}x^2 \cdot x$$
$$= y^{1/2} \cdot x$$
$$= xy^{1/2}$$

$$\frac{dy}{dx} = xy^{1/2} = x\left(\frac{1}{16}x^4\right)^{1/2}$$
$$= x \cdot \frac{1}{4}x^2 = \frac{1}{4}x^3 \checkmark$$

2. $y'' - 2y' + y = 0$ if $y = xe^x$

If $y = xe^x$
$$y' = xe^x + e^x$$
$$y'' = xe^x + e^x + e^x$$
$$= xe^x + 2e^x$$

$$xe^x + 2e^x - 2(xe^x + e^x) + xe^x = 0$$
$$xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$
$$\cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{2xe^x} = 0$$

then $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$

Exercise: Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$

(where c_1 & c_2 are arbitrary constants) a solution of $y'' + 4y = 0$?

3. General Solution: When an infinite set of solutions is grouped as

$$f(x, y, c_1, c_2, \ldots, c_n) = 0$$

Such solution is called a general solution

In the exercise above, $y = c_1 \sin 2x + c_2 \cos 2x$ is a general solution.

Let $c_1 = 5$ & $c_2 = -3$

$y = 5 \sin 2x - 3 \cos 2x$ → A particular solution.

4. Particular Solution: A particular solution is one that is obtained from the general solution by giving particular values to the arbitrary constants.

5. Initial value problems: If for an nth order ODE, the values $y(x_0), y'(x_0), \ldots, y^{(n)}(x_0)$ are specified at a given point x_0, we say we have an initial value problem (IVP).

$y' = 1$ and $y(0) = 1$ → This is an example of an IVP.

$\frac{dy}{dx} = 1$ or $\int dy = \int dx$

$\boxed{y = x + c}$ ⟶ General Solution

and substituting the initial condition $y(0) = 1$
x

$1 = 0 + c \Rightarrow c = 1$

$y = x + 1 \Rightarrow$ This is the solution to the IVP.