Lecture 11

Introduction to Systems

4.1 Some Definitions

Notation 4.1: Vectors will be denoted by lower case letters underlined: $\underline{x}$ $\underline{y}$ e.t.c.

Notation 4.2: Matrices will be denoted by uppercase letters A, B, C, e.t.c

Notation 4.3: Scalars will be denoted by lowercase without underlying; e.g $k \in \mathbb{R}$, $c \in \mathbb{R}$.

Definition 4.1.1 System

A group of components which interact and operate in a dependent way, for example

- Life systems: Genes, Population, cells.
- Physical or chemical systems: Solids, liquids, gases.
- Social systems: Nations or Communities.
- Economic systems: companies, producers.

Definition 4.1.2 Dynamic System

A system that depends on time, for example:

- Launching a satellite
- Solar-heating system
- Transference of genes
- Family system.

It is important to bear in mind the time scale on which changes within the system occur in relation to the system itself.
On a scale of 50 years, the family system may appear constant, yet on a scale of 200 years, changes may become evident.

Definition 4.1.3 Continuous System

A system that depends continuously on time e.g., heat exchange between bodies. Such systems are usually modelled by differential equations.

Definition 4.1.4 Discrete System

A system that depends discretely on time, e.g., seasonal insect population
or monthly / yearly budgets. Such systems are usually described by difference equations.
Note that require a mixture of differential and difference equations also exist.

Definition 4.1.5 Parameters

Characteristic values associated with a system. Parameters may be constant in time or time-dependent and are usually measured experimentally or "guessed" by local experts.

Definition 4.1.6 State Variables

Dependent variables that totally describe the state of a system at a given time.

Definition 4.1.7 State Vector

A vector representation of the state of a system at a given time. If x_i, $i = 1, 2, \ldots, n$ are variable of a system, then the state vector is written as

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{with} \quad x \in \mathbb{R}^n$$

Definition 4.1.8 Linear Systems

Systems that can be represented (ie modelled) by linear equations (i.e., equations with no product or power terms of the independent variables and its derivatives). Linear systems are important because

- they are simple and there is a compact way of representing them.
- there is a large body of linear algebra to rely on.
- they are easier (than non-linear system) to implement numerically

Definition 4.1.9 First Order differential equations

Differential equations involving only first derivatives of the variables and the variables themselves

$$y' = f(x, y) \;; \qquad \boxed{\dot{x} = h(t, x)}$$

$$y' = \frac{dy}{dx} \qquad\qquad \boxed{\dot{x} = \frac{dx}{dt}}$$

From henceforth, we will reduce higher order ODEs to a system of first order ODEs.

4.2 Higher-order ODEs

$$y^{(n)} = f(x, y, y', \dots y^{(n-1)}) \quad n > 1$$

y - dependent variable,
x - independent variable

$$x^{(n)} = f(t, x, \dot{x} \dots, x^{(n-1)}) \quad n > 1$$

x - dependent variable,
t - independent variable

Any ODE of order higher than one can be reduced to a lower order.

For example, to reduce $\ddot{x} = \dot{x} - t$ to first order

let $x = x_1$ $\qquad\qquad \dot{x} = \frac{dx}{dt}$

$\dot{x} = \dot{x}_1$

$\ddot{x}_1 = \dot{x}_1 - t$ $\qquad\qquad \ddot{x}_1 = \dot{x}_2$

let $x_2 = \dot{x}_1$

$\dot{x}_2 = x_2 - -$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 - t \end{aligned}\right\} \text{ forms a first order differential equation system}$$

To reduce an nth order ODE

$$x^{(n)} = f(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}) \quad n > 1 \qquad (1)$$

to a first order system, define new set of variables (n of them) such that

$$x = x_1$$
$$\dot{x} = \dot{x}_1 = x_2$$
$$\ddot{x} = \ddot{x}_1 = \dot{x}_2 = x_3$$
$$\vdots$$
$$x^{(n-1)} = \dot{x}_1^{(n-1)} \cdots = x_n$$

so that we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$x^{(n-1)} = x_n$$

$$x^n = f(t, x_1, x_2, \ldots x_n) \qquad (2)$$

Equation (2) is equivalent to equation (1)

Higher order difference equation

Similarly, any ordinary difference equation of order higher than one can be represented by a system of first order difference equations

If

$$x_{k+n} = f(k, x_k, x_{k+1}, \ldots, x_{k+n-1}), \; n > 1 \quad \text{——— (3)}$$

define

$$x_k = u_k$$

$$x_{k+1} = u_{k+1} = v_k$$

$$x_{k+2} = u_{k+2} = v_{k+1} = w_k$$

$$\vdots$$

$$x_{k+n-1} = \cdots\cdots\cdots = z_k$$

So that we obtain

$$u_{k+1} = v_k$$
$$v_{k+1} = w_k$$
$$\vdots$$
$$z_{k+1} = z_k$$

$$z_{k+1} = f(k, u_k, v_k, \ldots, z_k) \quad \text{——— (4)}$$

Equation (4) represents a system of first order difference equation.

4.3 Motivation for algebraic notation.

The System of equations

$$\dot{x}_1 = 2x_1 + 3x_2 + x_3$$
$$\dot{x}_2 = -x_2 + x_3$$
$$\dot{x}_3 = 5x_1 + 7x_3$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 5 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Can neatly and compactly be expressed as the system

$$\dot{\underline{x}} = A \underline{x}$$

where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$, $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 5 & 0 & 7 \end{pmatrix} \in \mathbb{R}^{3\times 3}$

Algebraic notation is compact, and we can replace many equations by a simple equation.

4.4 General system of equations

Any system of first-order ODEs can be written in the form

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + \underline{g}(\underline{x}(t), t) + \underline{f}(t), \quad \text{— } (4.5a)$$

where

$\underline{x}: \mathbb{R} \longrightarrow \mathbb{R}^n$ is the vector-valued state function

$A(t)\underline{x}(t) \in \mathbb{R}^n$ is the linear term if any;

$A: \mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$ is a matrix-valued function,

$\underline{g}: \mathbb{R}^{n\times n} \times \mathbb{R} \longrightarrow \mathbb{R}^n$ represents the non-linear terms if any,

$\underline{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$ is the term involving the independent variable, (often time)

- If $\underline{g} \equiv 0$, then the system is linear otherwise it is nonlinear.

- If $\underline{g} \equiv 0$ and $\underline{f} \equiv 0$, then the system is a homogeneous linear system.

For example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2 - t$$

$$\underline{\dot{x}}(t) = A(t)\,\underline{\dot{x}}(t) + \underline{g}(x(t), t) + \underline{f}(t)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -t \end{pmatrix}$$

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + \underline{f}(t)$$

$$g(x(t), t) = 0$$

- If $\underline{g} \equiv 0$ and $\underline{f} \not\equiv 0$ then the system is a

non-homogeneous linear system

- If A is independent of t and $g \equiv f \equiv 0$, then the system is a linear homogeneous system with constant coefficients.

- If the system is not explicitly dependent on t, then it is said to be autonomous, otherwise it is non-autonomous

Any solution of first order, difference equation can be written in the form

$$\underline{X}_{k+1} = A(k)\,\underline{X}_k + \underline{g}(X_k, k) + \underline{f}(k)$$

Example: Express the following using the matrix vector algebraic notation.

1. $\ddot{x} + 2\dot{x} - 8x = e^t$ $\qquad x(0) = 1$, $\quad \dot{x}(0) = 4$

2. $\dddot{x} - 2\ddot{x} + \dot{x} = 0$

Solution

1. $\ddot{x} + 2\dot{x} - 8x = e^t$

Let $x = x_1$, so that

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_2 + 8x_1 + e^t$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} .$$

The initial conditions are

$x(0) = 1$ $\qquad$ $\dot{x}(0) = 4$

$x_1(0) = 1$ $\qquad$ $x_2(0) = 4$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} .$$

2. $\dddot{x} - 2\ddot{x} + \dot{x} = 0 \implies \dddot{x} = 2\ddot{x} - \dot{x}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = 2x_3 - x_2$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3. $x_{k+2} = 5x_{k+1} - 4x_k$

Let $x_k = u_k$
$u_{k+1} = v_k$
$v_{k+1} = 5v_k - 4u_k$.

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

4.5 Linear algebra

4.5.1 – 4.5.11 (Read-up!)

4.5.12 – 24

4.5 Linear algebral

Theorem 4.1

A matrix A is non-singular if and only if $|A| \neq 0$

Theorem 4.2

If A is an nxn matrix, the homogeneous system $A\underline{x} = 0$ has a non-trivial (non zero) solution if and only if A is singular, ie if $|A| = 0$.

Definition 4.5.12 Differentiation and integration

If $A(t) = [a_{ij}(t)]$, where the a_{ij}'s are functions of t, then

$$\frac{d}{dt} A(t) = \left[\frac{d}{dt} a_{ij}(t)\right],$$

ie $$\dot{A}(t) = [\dot{a}_{ij}(t)],$$

and
$$\int_{t_0}^{t_1} A(t)\, dt = \left[\int_{t_0}^{t_1} a_{ij}(t)\, dt\right]$$

Example: Given $A(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ e^{-t} & e^{t} \end{pmatrix}$, then

$$\frac{d}{dt}\left(A(t)\right) = \dot{A}(t) = \begin{pmatrix} 2e^{2t} & 3e^{3t} \\ -e^{-t} & e^{t} \end{pmatrix}$$ while

$$\int_{t_0}^{t_1} A(t)\, dt = ?$$

Definition 4.5.13 Linear dependence

A set of vectors in $\mathbb{R}^n$, $\{\underline{x}_1, \underline{x}_2 \cdots \underline{x}_n\}$ is said to be a linearly set of vectors if there exist constant $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n = 0 \qquad (4.5)$$

If the only solution to (4.5) is for all the constants $c_i\ (i = 1, 2, \ldots n)$ to be zero, then the set of vectors is said to be linearly independent.

Definition 4.5.14 Vector Space

A set of vectors that obey certain properties of multiplication, addition, etc, for example $\mathbb{R}^n$ is a vector space.

Definition 4.5.15 Basis of Vector Space

A set of linearly independent vectors in the vector space with the property that any vector in the vector space is a linear combination of the vectors in the basis $\{e_i\}$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{v} = c_1 e_1 + c_2 e_2 \qquad \mathbb{R} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Definition 4.5.16 Standard basis

This is a basis that contains unit vectors that are orthogonal

The standard basis of $\mathbb{R}^n$ is

$$\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{e_2}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

Theorem 4.3

A vector $a_1, a_2, a_3, \ldots, a_n$ comprising the columns of $A \in \mathbb{R}^{n \times n}$ are linearly independent if and only if the matrix A is non-singular (ie, invertible $\Rightarrow |A| \neq 0$)

Proof: A linear combination of the vectors $a_1, a_2, \ldots, a_n$ with respective weights $x_1, x_2, \ldots x_n$ can be represented as Ax, where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

By theorem 4.2, $Ax=0$ has a non-zero solution if and only if A is singular. Thus $Ax=0$ has only the zero solution if and only if A is non-singular.

Definition 4.5.17 Linear transformation

A linear function $T: V \longrightarrow V$ that maps a vector v to itself, with the property that for any vectors x and y in V

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for arbitrary scalars α and β is called a linear transformation

The above definition for a linear transformation is equivalent to showing the linear function satisfies

$$T(x+y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

Definition 4.5.18 change of basis on a vector representation

If x is given with respect to the standard basis $e_1, e_2, \ldots, e_n$ then

$$\underline{x} = x_1 e_1 + x_2 e_2 + x_n e_n = (\overset{\Rightarrow \text{ standard basis}}{e_1}\ e_2 \ldots e_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underset{\substack{\downarrow \\ \text{matrix} \\ I_n}}{(e_1\ e_2 \ldots e_n)}$$

$$\underline{x} = I_n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

To obtain the representation of x with respect to a new basis $\underline{p_1}, \underline{p_2}, \ldots, \underline{p_n}$, we need $\underline{y}_1$, $\underline{y}_2, \ldots, \underline{y}_n$ such that

$$\underline{x} = y_1 p_1 + y_2 p_2 + \cdots + y_n p_n \qquad A^{-1}A = I$$

$$= \underset{\substack{\Downarrow \\ \underline{P} \text{ is a matrix}}}{(p_1\ p_2 \ldots p_n)} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$x = \underline{P} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \text{ say } \Rightarrow P^{-1}x = \underline{P^{-1}P} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

P has columns formed by the basis vectors $p_1, p_2, \ldots, p_n$ (which are linearly independent), this implies P is non singular and P^{-1} exists (by theorem 4.3)

Thus $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = P^{-1}\underline{x}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

OR $\underline{y} = P^{-1}\underline{x}$

Example:

Show that the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ wrt the standard basis, is represented as

$\begin{array}{cc} \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} & \begin{array}{c} y_1 \\ y_2 \end{array} \end{array}$ wrt the basis $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

Solution

$\underline{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{P_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{P_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\underline{y} = P^{-1}\underline{x}$

$$P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 1/4 \\ -1/4 & 3/4 \end{pmatrix}$$

$$\underline{y} = \begin{pmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

Definition 4.5.19 Matrix representation of linear transformations

Theorem 4.4

Every linear transformation in $\mathbb{R}^n$, say $T: \mathbb{R}^n \to \mathbb{R}^n$ can be expressed by matrix multiplication, such that

$$T(x) = Ax,$$

where $x \in \mathbb{R}^n$ is with respect to the standard basis $e_1, \ldots, e_n$.

For example: Consider the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2) = (x_2, x_1 - x_2, 2x_1 + x_2)$$

First, we show that T is a linear transformation and as such, by theorem 4.4 must have a matrix representation.

For $x, y \in \mathbb{R}^2$

$$T(x+y) = T([x_1+y_1 ; x_2+y_2])$$

$$= (x_2+y_2, (x_1+y_1)-(x_2+y_2), 2(x_1+y_1)+(x_2+y_2))$$

$$= (x_2+y_2, x_1+y_1-x_2-y_2, 2x_1+2y_1+x_2+y_2)$$

$$= (x_2+y_2, x_1-x_2+y_1-y_2, 2x_1+x_2+2y_1+y_2)$$

$$= (x_2, x_1-x_2, 2x_1+x_2) + (y_2, y_1-y_2, 2y_1+y_2)$$

$$= T(x_1, x_2) + T(y_1, y_2)$$

$$= T(x) + T(y)$$

Thus vector addition is preserved.

Now

$$T(\alpha x) = T([\alpha x_1, \alpha x_2])$$

$$= (\alpha x_2, \alpha x_1 - \alpha x_2, 2\alpha x_1 + \alpha x_2)$$

$$= \alpha(x_2, x_1-x_2, 2x_1+x_2)$$

$$= \alpha T(x_1, x_2)$$

$$= \alpha T(x)$$

which shows that scalar multiplication by a vector is preserved by the transformation.

Hence T is a linear transformation and therefore has a matrix representation with respect to the standard basis e_1, e_2 spanning vector $\underline{x} \in \mathbb{R}^2$

$$\text{that is } T(x) = \begin{pmatrix} x_2 \\ x_1 - x_2 \\ 2x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\underline{x}$$

Thus matrix A is the transformation matrix with respect to the standard basis for the linear map T.

A theorem that highlights the relationship between bases in $\mathbb{R}^n$ and the matrix representation of linear transformation in $\mathbb{R}^n$ is given below:

Theorem 4.5

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let $P = \{P_1, P_2, \ldots, P_n\}$ be basis for $\mathbb{R}^n$. For any vector $\underline{x} \in \mathbb{R}^n$, the vector $T(x) = Ax$ is uniquely determined by the vectors $T(P_1), T(P_2), \ldots, T(P_n)$.

Before proving, note that the last sentence of the theorem implies that matrix A is uniquely determined by the vectors $T(P_1), T(P_2), \ldots, T(P_n)$

Definition 4.5.20 Change of basis on a matrix representation of a linear transformation.

Suppose that a particular linear transformation has a matrix representation A wrt the standard basis $e_1, e_2, \ldots, e_n$. Then we have $AX = Z$. We wish to find the matrix representation of the linear transformation B wrt a new basis $p_1, p_2, \ldots, p_n$. So we want the linear transform

$$By = w$$

Let $P = (p_1\ p_2, \ldots, p_n)$. From the previous result, X would have the representation $y = P^{-1}X$ wrt the basis $p_1, p_2, \ldots, p_n$ and Z would have the representation $w = P^{-1}Z$, say wrt the basis $p_1, p_2, \ldots, p_n$. B represents the same linear transformation as A, so

$$y = P^{-1}x \quad , \quad w = P^{-1}z$$

$$By = w \qquad \text{Since } AX = \underline{z}$$

$$B(P^{-1}x) = P^{-1}z = P^{-1}AX$$

$$BP^{-1} = P^{-1}A \quad , \text{ Post multiplying by } P$$

$$BP^{-1}P = P^{-1}AP$$

$$B = P^{-1}AP$$

Definition 4.5.21 Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n \times n}$. If there exist $\underline{x} \in \mathbb{R}^n$ such that $\boxed{\underline{x} \neq 0}$ and $A\underline{x} = \lambda \underline{x}$ for λ scalar, then λ is an eigenvalue of $\underline{A}$ with corresponding eigenvector $\underline{x}$.

$$A\underline{x} = \lambda \underline{x} \quad (\lambda \text{ is a scalar})$$

$$A\underline{x} - \lambda \underline{x} = 0$$

$$(A - \lambda I)\underline{x} = 0 \implies \text{If } \boxed{\det(A - \lambda I) = 0}$$

Eigen is often replaced by proper characteristic or latent in textbook and research papers. Geometrically x is a direction along which the linear transformation A expands or contracts, but does not rotate. Such directions have important physical significance for many systems in physics, mechanics, biology, chemical engineering, e.t.c.

Definition 4.5.22 characteristic polynomial of a matrix.

The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ is the polynomial of degree n in $\underline{\lambda}$ obtained by expanding

$$|A - \lambda I_n|$$

The roots of the characteristic polynomial are the eigenvalues of A. Roots may be complex and may be repeated.

If A is a 2×2 matrix, then x would be quadratic
A 3×3 , then x would be cubic
⋮
A $n\times n$ matrix, then x would be a polynomial of degree n

Example: Determine the eigenvalues and eigenvectors of the matrix is given by

$$A = \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}, \quad A \text{ is a } 2\times2$$

$$\lambda I_2 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\left| \begin{pmatrix} 7 & 4 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 7-\lambda & 4 \\ 3 & 6-\lambda \end{vmatrix} = 0$$

$$= (7-\lambda)(6-\lambda) - 12 = 0$$
$$= 42 - 7\lambda - 6\lambda + \lambda^2 - 12 = 0$$
$$= \lambda^2 - 13\lambda + 30 = 0$$

* The characteristic equation *

$\lambda_1 = 3$ and $\lambda_2 = 10$.

λ_1 and λ_2 are the eigenvalue of A.

To every eigenvalue, there is a corresponding eigenvector, and $\underline{X}_1$ is associated with λ_1 and $\underline{X}_2$ is associated with λ_2. If $\underline{X}_1$ is an eigenvector for λ_1, then

$$\underline{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{for } \lambda_1 = 3 \qquad \begin{vmatrix} 7-\lambda_1 & 4 \\ 3 & 6-\lambda_1 \end{vmatrix}$$

$$(A - \lambda_1 I)\underline{X} = 0$$

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 4x_1 + 4x_2 &= 0 \\ 4x_1 &= -4x_2 \\ x_1 &= -x_2 \end{aligned}$$

Assume that $x_2 = 1$, so that $x_1 = -1$, then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So that for eigenvalue $\lambda_1 = 3$, the eigenvector is

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)\underline{x} = 0$$

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{for } \lambda_2 = 10 \qquad \begin{vmatrix} 7-\lambda_2 & 4 \\ 3 & 6-\lambda_2 \end{vmatrix}$$

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} -3x_1 + 4x_2 &= 0 \\ -3x_1 &= -4x_2 \end{aligned}$$

$$x_1 = \frac{4}{3} x_2$$

then assume $x_2 = 1$ and then $x_1 = 4/3$ so that for the eigenvalue $\lambda_2 = 10$, the eigen vector is

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_2 \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}.$$

Definition 4.5.23 Fundamental theorem of Algebra.

Every polynomial of degree $n \geq 1$ has at least one root and can be decomposed into first degree factors. (Thus every linear transformation has at least one eigenvalue and eigenvector).

Definition 4.5.24 Eigenvalues and Eigenvectors when the basis is changed.

Eigenvalues are independent of the basis used. If $A\underline{X} = \lambda \underline{X}$ and $B = P^{-1}AP$ is the representation of A wrt the new basis defined by P, then we have

$A\underline{x} = \lambda \underline{x}$

$P^{-1}(A\underline{x}) = P^{-1}(\lambda \underline{x})$ if $\underline{x} = P\underline{z}$ (As shown above)

then $P^{-1}(AP\underline{z}) = P^{-1}(\lambda P \underline{z})$

$B\underline{z} = \lambda P^{-1} P \underline{z}$

$\underline{x} = P\underline{y}$

$\underline{y} = P^{-1}\underline{x}$

$$\boxed{B\underline{z} = \lambda \underline{z}}$$

$\Rightarrow$ $\underline{z} = P^{-1}\underline{x}$ (the representation of $\underline{x}$ wrt the basis defined by P) is an eigenvector of B (the representation of A wrt) the basis defined by P), with eigenvalue λ.