Lecture 7

Bernoulli Equation (Nonlinear Equation)

The equation

$$y' + a(x)\, y = b(x)\, y^r \qquad r \neq 0, 1 \quad r \in \mathbb{R}. \quad \text{——— (1)}$$

is called the Bernoulli equation.

To integrate equation (1), divide through by y^r

$$\frac{y'}{y^r} + a(x) \frac{y}{y^r} = b(x) \frac{y^r}{y^r}$$

$$\frac{y'}{y^r} + a(x)\, y \cdot y^{-r} = b(x)$$

$$\frac{y'}{y^r} + a(x)\, y^{1-r} = b(x) \quad \text{——— (2)}$$

$$\frac{d}{dx}\left(y^{1-r}\right) = 1-r\, y^{1-r-1} \cdot y'$$

$$= (1-r)\, y^{-r} \cdot y' \quad \text{– Implicit differentiation}$$

$$\frac{1}{1-r} \frac{d}{dx}\left(y^{1-r}\right) = y^{-r} y' = \frac{y'}{y^r}$$

Since $\dfrac{y'}{y^r} = \dfrac{1}{1-r} \dfrac{d}{dx}\left(y^{1-r}\right)$

then equation ② becomes

$$\frac{1}{1-r}\frac{d}{dx}\left(y^{1-r}\right) + a(x)\,y^{1-r} = b(x)$$

Make the change of variable

$Z = y^{1-r}$, then

$$\frac{1}{1-r}\frac{dz}{dx} + a(x)\,z = b(x)$$

or in standard form

$$\frac{dz}{dx} + a(x)(1-r)z = b(x)(1-r) \quad \text{— ④}$$

Equation ④ is now a linear equation and can be solved as such.

Exercise: from equation ①, let $Z = y^{1-r}$ and use this substitution to obtain equation ④.

Solution:

$$y' + a(x)\,y = b(x)\,y^r$$

Let $Z = y^{1-r}$, $\frac{dz}{dx} = (1-r)y^{1-r-1}\,y'$

$$= (1-r)\,y^{-r}\,y'$$

$$\boxed{\begin{aligned} Z &= \frac{y}{y^r} \\ y &= Zy^r \end{aligned}} \Rightarrow$$

OR

$$y' = \frac{y^r}{1-r}\frac{dz}{dx}$$

So that equation ① becomes

$$\frac{y^r}{1-r}\frac{dz}{dx} + a(x)\,Z\,y^r = b(x)\,y^r$$

$$\frac{y^r}{1-r}\frac{dz}{dx} + a(x) z y^r = b(x) y^r$$

$$\frac{1}{1-r}\frac{dz}{dx} + a(x) z = b(x) .$$

Example:
Solve $y - 2x\frac{dy}{dx} = x(x+1)y^3$

first write the equation in the standard

form

$$y' + a(x) y = b(x) y^r$$

or

$$\frac{dy}{dx} + a(x) y = b(x) y^r$$

Divide all through by $-2x$

$$-\frac{y}{2x} + \frac{2x}{2x}\frac{dy}{dx} = -\frac{x(x+1)y^3}{2x}$$

$$\frac{dy}{dx} - \frac{y}{2x} = -\frac{(x+1)y^3}{2} \quad \text{——— } **$$

Divide all through by y^3

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{y}{2x}\cdot\frac{1}{y^3} = -\frac{(x+1)}{2}y^3\cdot\frac{1}{y^3}$$

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x}y^{-2} = -\frac{(x+1)}{2}$$

Make your substitution

Let $z = y^{1-r}$, here $r = 3$

$z = y^{1-3} = y^{-2}$

$\therefore \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

Multiply ~~**~~ all through by $-2y^{-3}$

$-2y^{-3}\frac{dy}{dx} - \frac{y}{2x} \cdot \frac{-2}{y^3} = -\frac{(x+1)y^3}{2} \cdot \frac{-2}{y^3}$

$-2y^{-3}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y^2} = x+1$

$-2y^{-3}\frac{dy}{dx} + \frac{1}{x}y^{-2} = x+1$

ie. $\frac{dz}{dx} + \frac{1}{x}z = x+1$

Solve using integrating factor

$IF = e^{\int a(x)\,dx}$, here $a(x) = \frac{1}{x}$

$= \int a(x)\,dx = \int \frac{1}{x}dx = \ln x$

$\therefore IF = e^{\ln x} = x$

Multiply through by the IF

$x\frac{dz}{dx} + \frac{1}{x}z \cdot x = (x+1) \cdot x$

$x\left(z' + \frac{z}{x}\right) = x^2 + x$

$$\frac{d}{dx}(zx) = x^2 + x \qquad \text{OR} \quad \frac{d}{dx}(I{\cdot}F \times z) = x^2 + x$$

$$\frac{d}{dx}(zx) = z'x + z = \frac{dz}{dx}x + z$$

Integrate both sides

$$\int \cancel{\frac{d}{dx}}(zx)\,\cancel{dx} = \int x^2 + x$$

$$zx = \frac{x^3}{3} + \frac{x^2}{2} + c$$

but $z = y^{-2}$

$$\frac{1}{y^2}x = \frac{x^3}{3} + \frac{x^2}{2} + c$$

$$\frac{x}{y^2} = \frac{2x^3 + 3x^2 + 6c}{6}$$

$$\frac{x}{y^2} = \frac{2x^3 + 3x^2 + A}{6}, \quad \text{where } A = 6c$$

$$y^2 = \frac{6x}{2x^3 + 3x^2 + A}$$

$$\therefore \; y = \left(\frac{6x}{2x^3 + 3x^2 + A}\right)^{1/2}$$

Exercise: Solve $2y - 3\frac{dy}{dx} = y^4 e^{3x}$

Ans: $y^3 = \dfrac{5e^{2x}}{e^{5x} + A}$ OR $y = \left(\dfrac{5e^{2x}}{e^{5x} + A}\right)^{1/3}$ where $A = 5c$.

Ricatti equation

J.F. Riccati investigated the equation

$$y' = ay^2 + bx^m$$

where $a \neq 0$ and b, m are constants

Later on d'Alembert investigated the more general equation

$$y' = a(x)y^2 + b(x)y + c(x), \quad ***$$

where $a(x) \neq 0$

Equation *** is referred to as Riccati equation.

Some interesting properties of Riccati equation:

1. If $a=0$, the equation becomes linear and may be written in standard form to become:

$$y' - b(x)y = c(x).$$

2. If $c=0$, the equation is a special case of the Bernoulli equation, ie the equation becomes

$$y' - b(x)y = a(x)y^2$$

From now on, we assume that $a, c \neq 0$.

$$y' = a(x)y^2 + b(x)y + c(x) \quad \text{——— } (1)$$

To solve equation (1), we assume

$y_1(x)$ is a particular solution and then perform the substitution

$$y = y_1 + \frac{1}{v} \quad \text{———} \quad (2)$$

$$y = y_1 + v^{-1}$$
$$y' = y_1' - v^{-2} \cdot v'$$
$$y' = y_1' - \frac{1}{v^2} \cdot v'$$
$$= y_1' - \frac{v'}{v^2} \quad \text{———} \quad (3)$$

Substitute (2) and (3) in (1) to obtain

$$y_1' - \frac{v'}{v^2} = a(x)\left(y_1 + \frac{1}{v}\right)^2 + b(x)\left(y_1 + \frac{1}{v}\right) + c(x)$$

$$y_1' - \frac{v'}{v^2} = a(x)\left(y_1^2 + \frac{2y_1}{v} + \frac{1}{v^2}\right) + b(x)y_1 + \frac{b(x)}{v} + c(x)$$

$$= a(x)y_1^2 + \frac{2a(x)y_1}{v} + \frac{a(x)}{v^2} + b(x)y_1 + \frac{b(x)}{v} + c(x)$$

$$v^2y_1' - v' = a(x)(vy_1)^2 + 2a(x)(vy_1) + a(x) + b(x)v^2y_1 + b(x)v + c(x)v^2$$

$$= v^2y_1(a(x)y_1 + b(x)) + v(2a(x)y_1 + b(x)) + c(x)v^2 + a(x)$$

Equating coefficients

$$y_1' = a(x)y_1^2 + b(x)y_1 + c(x) \Rightarrow \text{coefficient of } v^2$$
$$-v' = (2a(x)y_1 + b(x))v + a(x)$$

$$v' + (2a(x)y_1 + b(x))v = -a(x) \quad \text{———} \quad (4)$$

Equation (4) is now a linear equation in v.

Note that the I.F. in equation (4) is

$$e^{\int (2a(x) y_1 + b(x))\, dx}$$

and the first order linear equation in (4) has the general solution :

$$v = e^{-\int (2a(x) y_1 + b(x))\, dx} \left[\int -a(x) \cdot e^{\int (2a(x) y_1 + b(x)) dx} + A \right]$$

Example:

Given that $y_1 = 2/x$ is a known solution of the Riccati equation

$$y' = -\frac{4}{x^2} - \frac{1}{x} y + y^2,$$

find the general solution of the Riccati equation.

Solution:

$$y_1 = \frac{2}{x} = 2x^{-1} \quad , \quad y_1' = -2x^{-2} = \frac{-2}{x^2}$$

$$y' = \frac{-4}{x^2} - \frac{1}{x} y + y^2$$

Let $y = y_1 + \frac{1}{v} = \frac{2}{x} + \frac{1}{v}$

$$y' = y_1' - \frac{v'}{v^2}$$

$$= \frac{-2}{x^2} - \frac{v'}{v^2}$$

Substituting in $y' = -\frac{4}{x^2} - \frac{1}{x}y + y^2$, gives

$$-\frac{2}{x^2} - \frac{v'}{v^2} = -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + \frac{1}{v}\right) + \left(\frac{2}{x} + \frac{1}{v}\right)^2$$

$$\cancel{-\frac{2}{x^2}} - \frac{v'}{v^2} = \cancel{-\frac{4}{x^2}} \cancel{- \frac{2}{x^2}} - \frac{1}{xv} + \cancel{\frac{4}{x^2}} + \frac{4}{xv} + \frac{1}{v^2}$$

$$-\frac{v'}{v^2} = \frac{3}{xv} + \frac{1}{v^2}$$

or

$$-v' = \frac{3}{x}v + 1$$

$$v' + \frac{3}{x}v = -1 \quad ** \Rightarrow$$ This is now a linear equation

$$\text{I.F.} = e^{\int 3/x \, dx}$$

$$\int \frac{3}{x} dx = 3\ln(x)$$

$$= 3\ln(x)$$

$$e^{\int 3/x \, dx} = e^{3\ln(x)}$$

$$= e^{\ln x^3}$$

$$= x^3$$

$$\therefore \text{I.F.} = x^3$$

Multiply through ~~**~~ by the I.F.

$$x^3 v' + \frac{3}{x} v x^3 = -x^3$$

$$x^3\left(v' + \frac{3}{x}v\right) = -x^3$$

OR

$$\frac{d}{dx}\left(I.F. \times v\right) = -x^3$$

$$\frac{d}{dx}\left(x^3 v\right) = -x^3$$

$$\frac{d}{dx}\left(x^3 v\right) =$$

$$3x^2 v + v' x^3$$

Divide through by x^3

$$\frac{3x^2}{x^3} v + v'$$

$$\frac{3}{x} v + v'$$

Integrating

$$x^3 v = \frac{-x^4}{4} + A$$

$$v = \frac{-x}{4} + \frac{A}{x^3}$$

but $y = y_1 + \frac{1}{v}$

Since $y_1 = \frac{2}{x}$

$$\frac{1}{v} = y - y_1 = y - \frac{2}{x}$$

$$y - \frac{2}{x} = \left[\frac{A}{x^3} - \frac{x}{4}\right]^{-1}$$

$$y = \frac{2}{x} + \left[\frac{A}{x^3} - \frac{x}{4}\right]^{-1}.$$

Abel equation of the Second Kind

The following equation

$$y y' + a(x) y = b(x) \qquad \text{———} \quad (5)$$

appears in different fields of science, for example, in fluid mechanics and general relativity.

Equation (5) is a first order nonlinear equation called the Abel equation of the second kind.

It is solveable only for certain forms of $a(x)$ and $b(x)$.

Consider for instance, the equation

$$y y' + A x^{-2} y = B x^{-3} \qquad \text{———} \quad (6)$$

In equation (6), we can perform the change of variables

$$y = \frac{u}{x} = u x^{-1}$$

$$\begin{aligned} y' &= u(-x^{-2}) + u' x^{-1} \\ &= u \cdot \left(\frac{-1}{x^2}\right) + \frac{1}{x} u' \\ &= \frac{u'}{x} - \frac{u}{x^2} \end{aligned}$$

Substituting this in equation (6) gives

$$\frac{u}{x}\left(\frac{u'}{x}-\frac{u}{x^2}\right)+\frac{A}{x^2}\left(\frac{u}{x}\right)=\frac{B}{x^3}$$

$$\frac{uu'}{x^2}-\frac{u^2}{x^3}+\frac{Au}{x^3}=\frac{B}{x^3} \quad \text{———} \quad (7)$$

Multiply through by x^3

$$xuu' - u^2 + Au = B \quad \text{———} \quad (8)$$

$$xuu' = u^2 - Au + B$$

$$xu\frac{du}{dx} = u^2 - Au + B$$

$$u\,du = \frac{1}{x}\left(u^2 - Au + B\right)dx$$

$$\frac{u}{u^2 - Au + B}\,du = \frac{1}{x}\,dx \quad \text{———} \quad (9)$$

which is now seperable

General ODEs of order n

Basic properties:

The most general scalar nth order linear ODE is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} +$$

$$a_0(x)\,y = b(x) \quad \text{———} \quad (10).$$

$$a_2\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)$$

where the a_i's and b are functions of x.

Equation (10) is said to be homogeneous if $b(x)$ is zero; otherwise, it is said to be a non-homogeneous equation.

For example:
$2y'' + 3y' - 5y = 0$ is a homogeneous, linear, second order differential equation, where as
$x^3y''' + 6y' + 10y = e^x$ is a non-homogeneous, linear third order differential equation.

Introduce the differential operator (or the D operator)

$$D = \frac{d}{dx}$$

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2} \ldots D^ny = \frac{d^ny}{dx^n},$$

then equation 10 becomes

$$a_n(x)D^ny + a_{n-1}(x)D^{n-1}y + \ldots + a_1(x)Dy + a_0(x)y = b(x) \quad \text{———} \quad (11)$$

OR

$$\left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots a_1(x)D + a_0(x)\right)y = b(x) \quad \text{———} \quad (12)$$

If we let

$$L_n = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x)$$

then equation 12 becomes:

$$L_n y = b(x) \quad \text{———} \quad (13)$$

Note that to solve equation 10, since it is an nth order differential equation the general solution will also contain n arbitrary constants.

Also, to solve the non-homogeneous equation 13, we must first be able to solve the associated homogeneous equation

$$L_n y = 0 \quad \text{———} \quad (14)$$

Linear dependence / independence

A set of functions $y_1(x), y_2(x) \dots y_n(x)$ is said to be linearly dependent on an interval I if there exist constants $C_1, C_2, \dots, C_n$, such that

$$C_1 y_1(x) + C_2 y_2(x) \dots + C_n y_n(x) = 0 \quad \text{—— (15)}$$

for every x in the interval. If the set of functions is not linearly dependent, then it is linearly independent.

For example:

Given $y_1 = x$, $y_2 = 5x$, $y_3 = 1$, $y_4 = \sin x$

then $C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4 = 0$ if

$C_1 = -5$, $C_2 = 1$, $C_3, C_4 = 0$

$-5x + 5x + 0 + 0 = 0$

Since $C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4 = 0$ even though all the C's are not equal then the set y_1, y_2, y_3 & y_4 are linearly dependent.

A set of functions is linearly dependent if there exist another set of constants not all zero that satisfies equation (15).

If the only solution to equation (15) is $C_1 = C_2 = \dots = C_n = 0$, then the set of functions is linearly independent.

The nth order linear homogeneous differential equation in (14) always have n linearly independent solutions.

If $y_1(x), y_2(x), \ldots y_n(x)$ represent these solution, then the general solution of equation (14) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \quad \text{--- (16)}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants

Any set of n linearly independent solution of (14) is called fundamental set of solutions.

The general solution to equation (10) then becomes

$$y = y_c + y_p$$

where $y_c = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ is called complementary solution and y_p is called particular solution.

Note that y_p has no arbitrary constant in it.