Integrating rational functions

What is a rational function? A rational function is a function that can be expressed as a ratio of two functions.

ie $R(x) = \dfrac{P(x)}{Q(x)}$

If $P(x) = 3x+2$, while $Q(x) = 2x^2+x-3$.

then $R(x) = \dfrac{3x+2}{2x^2+x-3}$

To integrate a rational function, we use partial fractions to split the function into easily integrable parts.

Review of partial fractions

$$\frac{3x+2}{2x^2+x-3} = \frac{3x+2}{(2x+3)(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}$$

$$\frac{3x+2}{(2x+3)(x-1)} = \frac{A(x-1) + B(2x+3)}{(2x+3)\ (x-1)}$$

$$3x+2 = A(x-1) + B(2x+3)$$

$x-1=0$
$x=1$

$2x+3=0$
$2x=-3$
$x=-3/2$

Let $x=1$

$$3(1)+2 = A(\cancel{1-1})^{0} + B(2(1)+3)$$

$$3+2 = A(0) + B(2+3)$$

$$5 = 5B$$

$$B = 5/5, \quad \therefore \boxed{B=1}$$

Let $x = -3/2$

$$3(-3/2) + 2 = A\left(\frac{-3}{2} - 1\right) + B\left(2(-3/2) + 3\right)$$

$$\frac{-9}{2} + 2 = A\left(\frac{-5}{2}\right) + B(-3+3)$$

$$\frac{-9+4}{2} = A\left(\frac{-5}{2}\right) + B(0)$$

$$\frac{-5}{2} = \frac{-5}{2}A$$

Multiply both sides by $-2/5$

$$\frac{-5}{2} \times -\frac{2}{5} = A\left(\frac{-5}{2}\right) \times -\frac{2}{5}$$

$$\boxed{1 = A}$$

$$\therefore \frac{3x+2}{(2x+3)(x-1)} = \frac{1}{2x+3} + \frac{1}{x-1}$$

numerator

Denominator

The method of partial fractions consist of the following:

1. Find a common denominator
2. Set the numerator equal
3. Set the coefficients of the different powers equal.
4. Solve the system of equations obtained from the unknown coefficients.

Note: To solve partial fractions, the degree of the numerator must be lesser than the degree of the denominator. If this is not the case, the long division is required before the solution.

Example: Find the integral of $\int \frac{x+6}{(3x-2)(x+1)} dx$

Solution:

$$\frac{x+6}{(3x-2)(x+1)} = \frac{A}{(3x-2)} + \frac{B}{(x+1)}$$

$$\frac{x+6}{(3x-2)(x+1} = \frac{A(x+1) + B(3x-2)}{(3x-2)(x+1)}$$

$$x+6 = A(x+1) + B(3x-2)$$

$$x+6 = Ax + A + 3Bx - 2B$$

$$x+6 = (A+3B)x + A - 2B$$

$$1 = A+3B$$

$$6 = A - 2B$$

$$-5 = 0 + 5B$$

$$B = -1 \quad \text{and} \quad A = 4$$

if $B = -1$

using

$$1 = A + 3B$$

$$1 = A + 3(-1)$$

$$1 = A - 3$$

$$A = 4$$

OR

$$x+6 = A(x+1) + B(3x-2)$$

$x+1=0$

$x=-1$

$$-1+6 = A(-1+1) + B(3(-1)-2)$$

$$5 = A(0) + B(-5)$$

$$5 = -5B$$

$$\therefore B = -1$$

$3x-2=0$

$3x = 2$

$x = 2/3$

$$2/3 + 6 = A(2/3 + 1) + B(3(2/3) - 2)$$

$$\frac{2+18}{3} = A\left(\frac{2+3}{3}\right) + B\left(\frac{6}{3} - 2\right)$$

$$\frac{20}{3} = \frac{5}{3}A + B(2-2)$$

$$\frac{20}{\cancel{3}} \times 3/5 = A\left(\frac{\cancel{5}}{\cancel{3}}\right) \times \cancel{3}/\cancel{5} \quad \therefore A = 20/5 = 4$$

$$\int \frac{x+6}{(3x-2)(x+1)}\,dx = \int \left(\frac{4}{(3x-2} - \frac{1}{x+1} \right) dx$$

$$= 4\int \frac{1}{3x-2}\,dx - \int \frac{1}{x+1}\,dx$$

$$4\int \frac{1}{3x-2}\,dx =$$

Let $u = 3x-2$

$$\frac{du}{dx} = 3$$

$$dx = \frac{du}{3}$$

$$4\int \frac{1}{3x-2}\,dx = 4\int \frac{1}{u} \cdot \frac{du}{3}$$

$$= \frac{4}{3}\int \frac{1}{u}\,du$$

$$= \frac{4}{3}\ln|u|$$

and $u = 3x-2$

$$= \frac{4}{3}\ln|3x-2|$$

$$\therefore \int \frac{x+6}{(3x-2)(x+1)}\,dx = \frac{4}{3}\ln|3x-2| - \ln|x+1| + C$$

$$= \frac{4}{3}\ln|3x-2| - \ln|x+1| + C$$

Remark: for a polynomial whose denominator does not factor out completely, for example, if

$$\frac{5}{(x^2+1)(x+4)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{x+4}$$

Example: Evaluate $\int \frac{6x+2}{x^2-2x-3}\,dx$

$$\frac{6x+2}{x^2+2x-3} = \frac{6x+2}{(x-1)(x+3)}$$

$$\frac{6x+2}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$$

$$\frac{6x+2}{(x-1)(x+3)} = \frac{A(x+3) + B(x-1)}{(x-1)(x+3)}$$

$$6x+2 = A(x+3) + B(x-1)$$
$$6x+2 = Ax + 3A + Bx - B$$
$$6x+2 = (A+B)x + 3A - B$$
$$6 = A + B \quad \times 3$$
$$2 = 3A - B \quad \times 1$$

Solving Simultaneously

$$18 = 3A + 3B$$
$$2 = 3A - B$$
$$16 = 0 + 4B$$

$B = 4$ and $A = 2$

Since $B = 4$

Using $6 = A + B$

$$6 = A + 4$$
$$6 - 4 = A$$
$$A = 2$$

$$\int \frac{6x+2}{x^2+2x-3}\,dx = \int \frac{2}{x-1} + \frac{4}{x+3}\,dx$$

$$= \int \frac{2}{x-1}\,dx + \int \frac{4}{x+3}\,dx$$

$$= 2\int \frac{1}{x-1}\,dx + 4\int \frac{1}{x+3}\,dx$$

$$= 2\ln|x-1| + 4\ln|x+3| + c$$

Considering a situation where the numerator is not of lower degree than the denominator.

Example: Solve $\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx$

In the above example, the degree of the numerator is 4 and the degree of the denominator is 3. Therefore partial fraction fractions can't be done on this rational expression.

To fix this up, we'll need to do long division on this to get it into a form that we can deal with.

Here is the workings for that

$$\begin{array}{r|l} & x-2 \\ \hline x^3-3x^2 & x^4-5x^3+6x^2-18 \\ & -(x^4-3x^3) \\ \hline & \quad -2x^3+6x^2-18 \\ & \quad -(-2x^3+6x^2) \\ \hline & \qquad\qquad -18 \end{array}$$

So, from the long division, we see that,

$$\frac{x^4-5x^3+6x^2-18}{x^3-3x^2} = x-2-\frac{18}{x^3-3x^2}$$

and the integral becomes

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx = \int x-2-\frac{18}{x^3-3x^2}\,dx$$

$$= \int x-2\,dx - \int \frac{18}{x^3-3x^2}\,dx$$

$$\int x-2\,dx = \frac{x^2}{2}-2x$$

$$\int \frac{18}{x^3-3x^2}\,dx = \int \frac{18}{x^2(x-3)}\,dx$$

$$\frac{18}{x^2(x-3)} = \frac{Ax+B}{x^2}+\frac{C}{x-3}$$

Setting the numerators equal gives us

$$18 = (Ax+B)(x-3) + Cx^2$$
$$18 = Ax^2 - 3Ax + Bx - 3B + Cx^2$$
$$18 = (A+C)x^2 \; (-3A+B)x - 3B$$
$$18 = -3B \quad \therefore \quad B = -6$$
$$A + C = 0 \quad \text{—} \; (i)$$
$$-3A + B = 0 \quad \text{—} \; (ii), \quad B \text{ is Known}$$

from $\ddot{u}$ (ii), $-3A + B = 0$

$$-3A + B = 0$$
$$-3A + (-6) = 0$$
$$-3A - 6 = 0$$
$$-3A = 6$$
$$A = -2$$

Since A is known, from (i)

$$A + C = 0$$
$$-2 + C = 0$$
$$C = 2$$

$\therefore A = -2, \; B = -6$
$C = 2$

$$\therefore \int \frac{x^4 - 5x^3 - 6x^2 - 18}{x^3 - 3x^2}\,dx = \int x - 2\,dx - \left(\int -\frac{2}{x} - \frac{6}{x^2} - \frac{2}{x-3}\,dx\right)$$
$$= \int x - 2\,dx + 2\int \frac{1}{x}\,dx - 6\int \frac{1}{x^2}\,dx - 2\int \frac{1}{x-3}\,dx$$
$$= x^2 - 2x + 2\ln|x| - \frac{6}{x} - 2\ln|x-3| + C$$

Example: Determine $\int \frac{3x^2 + 18x + 3}{3x^2 + 5x - 2}\,dx$

In the above example, the degree of the numerator and denominator are the same, therefore, a long division is required.

$$\int \frac{3x^2+18x+3}{3x^2+5x-2}\,dx$$

$$\begin{array}{r|l} & \quad 1 \\ 3x^2+5x-2 & 3x^2+18x+3 \\ & -\ 3x^2+5x-2 \\ \hline & \qquad 13x+5 \end{array}$$

From the long division, we see that $\frac{3x^2+18x+3}{3x^2+5x-2} = 1 + \frac{13x+5}{3x^2+5x+2}$

$$\therefore \int \frac{3x^2+18x+3}{3x^2+5x-2}\,dx = \int 1 + \frac{13x+5}{3x^2+5x+2}\,dx$$

$$\frac{13x+5}{3x^2+5x+2} = \frac{A}{3x-1} + \frac{B}{x+2}$$

$$13x+5 = A(x+2) + B(3x-1)$$
$$13x+5 = Ax + 2A + 3Bx - B$$
$$13x+5 = (A+3B)x + 2A - B$$

$$13 = A + 3B \quad \times 2 \quad \text{— i,}$$
$$5 = 2A - B \quad \times 1 \quad \text{— ii,}$$

Solving Simultaneously

$$26 = 2A + 6B$$
$$5 = 2A - B$$
$$\overline{21 = 7B}$$
$$B = \frac{21}{7} = 3 \quad \therefore \quad B = 3$$

use $B = 3$ in i, or ii,

using i, $13 = A + 3B$

$$13 = A + 3(3)$$
$$13 - 9 = A \quad \therefore \quad A = 4$$

$$\therefore \quad \frac{13x-5}{(3x-1)(x+2)} = \frac{4}{3x-1} - \frac{5}{(x+2)}$$

$$\int \frac{3x^2+18x+3}{3x^2+5x+2}\,dx = \int \left(1 + \frac{4}{3x-1} + \frac{3}{x+2}\right) dx$$

$$= \int 1\,dx + 4\int \frac{1}{3x-1}\,dx + 3\int \frac{1}{x+2}\,dx$$

$$= x + \frac{4\ln|3x-1|}{3} + 3\ln|x+2| + C$$

Study Section 1.4.5 in the notes to familiarise yourself with the table of integrals given.

Scalar nth order ordinary differential equations (ODEs)

Preliminaries

Given a function $y = e^{0.1x^2}$

$$y' = 0.2x\underbrace{\boxed{e^{0.1x^2}}}_{}$$ with y written above the boxed term:

$y' = 0.2xy$ – Differential equation

$$y' - 0.2xy = 0$$

Differential equation: An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.

In the above, y is the dependent variable and x is the independent variable.

Notations

Prime symbol

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{d^2y}{dx^2}$$

$$y''' = \frac{d^3y}{dx^3}$$

$$\vdots$$

$$y^{(n)} = \frac{d^ny}{dx^n}$$

dot symbol

$$\dot{y} = \frac{dy}{dt}$$

$$\ddot{y} = \frac{d^2y}{dt^2}$$

Ordinary differential equation:- Is an equation that contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

Let x be the independent variable, and y be the dependent variable.

Also, let $y', y'' \dots y^{(n)}$ be the derivatives of y with respect to x, then any equation which involves at least one of these derivatives is called an ordinary differential equation.

If the number of equations is one, then it is called a SCALAR ODE; otherwise it is a SYSTEM OF ODES

Example of ODES

1. $y' - 3y = e^x$ ⟶ First order

2. $y'' + a(x)y' - b(x)y = c(x)$ ⟶ Second order

3. $x'(t) + y'(t) = 2x(t) + y(t)$ ⟶ First order

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

Remark:

Relation 3 is an example of an ODE that contains more than one dependent variable $x(t)$ and $y(t)$ and one independent variable (t).

Partial differential equation: Is an equation that contains partial derivatives of one or more dependent variable with respect to two or more independent variable.

Examples of PDEs

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ dependent variable: u
independent variable: x & y

2. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$ dependent variable: u
Independent variable x & t

Note:

For the purpose of this course, we will not deal with integration of PDEs.

ORDER: The order of a differential equation is the order of the highest derivative in the equation.

Example:

1. $$\underbrace{\frac{d^2y}{dx^2}}_{\text{Second order}} + x^2\underbrace{\left(\frac{dy}{dx}\right)^3}_{\text{First order}} = -4 \Rightarrow \text{Is of order } 2$$

2. An nth order ODE in one dependent variable has the general form

$$F(x, y, y', y'' \ldots\ldots y^{(n)}) = 0$$

Class work:

Determine the order, dependent and independent variable in the following

1. $y''' - 5xy' = e^{x} + 1$ is of order 3
dependent is y
independent is x

2. $t\dfrac{d^2y}{dt^2} + t^2\dfrac{dy}{dt} - \sin t\sqrt{y} = t^2 - t + 1$

3. $s^2\dfrac{d^2t}{ds^2} + st\dfrac{dt}{ds} = s$

4. $5\left(\dfrac{d^4b}{dp^4}\right) + 7\left(\dfrac{db}{dp}\right)^{10} + b^7 - b^5 = p$