Logistic Regression

University of the Witwatersrand

2022

Review Question

- I roll two regular dice and add the values on the upward faces up. Find the probability that I get a seven.

Lesson Plan

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- Idea of multivariate regression

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- Mathematical formalisation

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- $\frac{6}{36} = \frac{1}{6}$

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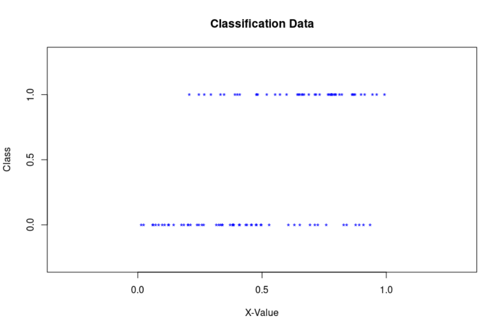
- The idea of logistic regression is that we'll set up an indicator random variable. 1 if in the group 0 if not (two class classification problem).
- Then we try to do regression on it.

A problem

- 

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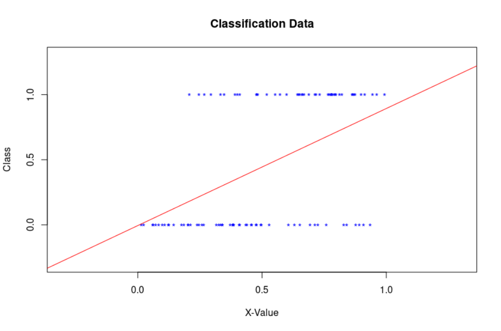

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- Could use a lot of functions. In machine learning we sometimes do. For logistic regression we use a sigmoid function

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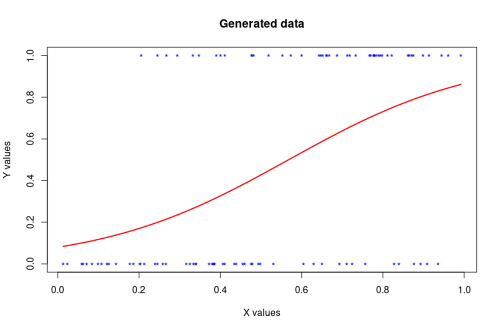

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$$
\begin{aligned}
p(X) &= \frac{e^{\beta^T X}}{1 + e^{\beta^T X}} \\
p(X) &= \frac{Z}{1 + Z} \\
p(X)(1 + Z) &= Z \\
p(X) + Zp(X) &= Z
\end{aligned}
$$

Sigmoid Function

- $$\begin{aligned} p(X) + Zp(X) &= Z \\ p(X) &= Z(1 - p(X)) \\ Z &= \frac{p(X)}{1 - p(X)} \\ \frac{p(X)}{1 - p(X)} &= e^{\beta^T X} \\ \ln \frac{p(X)}{1 - p(X)} &= \beta^T X \end{aligned}$$

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- That is to say that the "log odds" is modeled as a linear function of the X variables.

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- Due to similarities with Linear Regression we can do inference on β. We won't here but your favourite software package will have tests.

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- $$\begin{aligned} l(\beta) &= \sum_i y_i \ln p(x_i) + (1 - y_i) \ln[1 - p(x_i)] \\ &= \sum_i y_i \ln \frac{p(x_i)}{1 - p(x_i)} + \ln[1 - p(x_i)] \\ &= \sum_i y_i(X^T\beta) - \ln[1 + e^{X^T\beta}] \end{aligned}$$

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- Numerics