Lecture 13

5.2. Fundamental Matrix Solution

In the previous section, we discussed a method for obtaining e^{At}. This method only worked for matrices with distinct eigenvalues. In this section, we deal with a more general framework for finding solutions to $\dot{\underline{x}} = A\underline{x}$ which works for distinct as well as repeated eigenvalues.

A fundamental matrix solution (FMS) of the system $\dot{\underline{x}} = A\underline{x}$ is a matrix $X(t)$ which satisfies the corresponding matrix differential equation $\dot{X} = AX$ and also has $|X(t_0)| \neq 0$ (ie a non-singular solution of $t = t_0$).

Lemma 5.3

If $X(t)$ is a fundamental matrix solution of the system $\dot{\underline{x}} = A\underline{x}$, then $|X(t)| \neq 0 \ \forall t$.

See proof in the note.

Theorem 5.2

Let $X(t)$ be any FMS of $\dot{\underline{x}} = A\underline{x}$. Then

$$e^{A(t-t_0)} = X(t)X^{-1}(t_0)$$

Proof:

i) $X(t)$ is a FMS of $\dot{\underline{x}} = A\underline{x}$, hence

$\dot{x} = Ax$

and

$$|X(t_0)| \neq 0$$

Thus $|X(t_0)|$ exists

ii) $e^{A(t-t_0)}$ is a FMS of $\dot{\underline{x}} = A\underline{x}$ since

$$\frac{d}{dt} e^{A(t-t_0)} = Ae^{(t-t_0)}$$

and

$$\left|e^{A(t-t_0)}\right| = \underline{I_n} = 1 \neq 0$$

iii) Recall that

$$X(t) = e^{A(t-t_0)} X_0, \text{ where } X_0 = X(t_0)$$

Since $|X(t_0)| \neq 0$, then $X^{-1}(t_0)$ exists,

hence $X(t) = e^{A(t-t_0)} X(t_0)$

Post multiplying by $X^{-1}(t_0)$ gives

$$X(t) \cdot X^{-1}(t_0) = e^{A(t-t_0)}.$$

Thus

$$e^{A(t-t_0)} = X(t) \cdot X^{-1}(t_0)$$

The problem has been transferred from finding $e^{A(t-t_0)}$ to finding $X(t)$, a FMS of $\dot{\underline{x}} = A\underline{x}$

5.2.1 If A has distinct eigenvalues

If A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with distinct eigenvectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$, then a FMS is given by

$$X(t) = \left(e^{\lambda_1 (t-t_0)} \underline{v_1}, e^{\lambda_2 (t-t_0)} \underline{v_2}, \ldots, e^{\lambda_n (t-t_0)} \underline{v_n} \right)$$

This is because

i. Distinct eigenvalues $\Rightarrow$ linearly independent eigenvectors
$\Rightarrow$ linearly independent columns at $t = t_0$
$\Rightarrow |X(t_0)| \neq 0$

ii, The matrix as given satisfies the matrix differential equation $\dot{X} = AX$.

$$\frac{d}{dt}\left(e^{\lambda_1 (t-t_0)} \underline{v_1} \; e^{\lambda_2 (t-t_0)} \underline{v_2} \ldots e^{\lambda_n (t-t_0)} \underline{v_n} \right) =$$

$$\left(\lambda_1 e^{\lambda_1 (t-t_0)} \underline{v_1} \; \lambda_2 e^{\lambda_2 (t-t_0)} \underline{v_2} \ldots \lambda_n e^{\lambda_n (t-t_0)} \underline{v_n} \right)$$

$$= A \left(e^{\lambda_1 (t-t_0)} \underline{v_1} \; e^{\lambda_2 (t-t_0)} \underline{v_2} \ldots e^{\lambda_n (t-t_0)} \underline{v_n} \right)$$

Since $Av_i = \lambda v_i$, $i = 1 \ldots, n$

$$= AX$$

Example:

Calculate e^{At} by first forming a FMS $X(t)$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. As a check, calculate

e^{At} from $M \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} M^{-1}$.

Solution

$$e^{A(t-t_0)} = X(t) \cdot X^{-1}(t_0)$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 0 = 0$$
$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$
$$\Rightarrow \lambda_1 = 1 \quad , \quad \lambda_2 = 3$$

$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvector corresponding to the eigenvalue $\lambda_1 = 1$

while the eigenvector corresponding to $\lambda_2 = 3$ is

$$\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} e^{\lambda_1(t-t_0)} \underline{v}_1 & e^{\lambda_2(t-t_0)} \underline{v}_2 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} e^{(t-t_0)}\begin{pmatrix}1\\0\end{pmatrix} & e^{3(t-t_0)}\begin{pmatrix}1\\1\end{pmatrix} \end{pmatrix}$$

$$X(t_0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X^{-1}(t_0) = 1\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$X^{-1}(t_0) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$e^{A(t-t_0)} = X(t) \cdot X^{-1}(t_0)$$

$$= \begin{pmatrix} e^{(t-t_0)}\begin{pmatrix}1\\0\end{pmatrix} & e^{3(t-t_0)}\begin{pmatrix}1\\1\end{pmatrix} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-t_0)} & e^{3(t-t_0)} \\ 0 & e^{3(t-t_0)} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-t_0)} & -e^{(t-t_0)} + e^{3(t-t_0)} \\ 0 & e^{3(t-t_0)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-t_0)} & e^{3(t-t_0)} - e^{(t-t_0)} \\ 0 & e^{3(t-t_0)} \end{pmatrix}$$

e^{At} (Assume $t_0 = 0$)

$$e^{At} = \begin{pmatrix} e^{t} & e^{3t} - e^{t} \\ 0 & e^{3t} \end{pmatrix}$$

Try:

Verify $e^{At} = \begin{pmatrix} e^{t} & e^{3t} - e^{t} \\ 0 & e^{3t} \end{pmatrix}$ using

$$e^{At} = M e^{\Lambda t} M^{-1}.$$

5.2.2 A has repeated eigenvalues

If A has repeated eigenvalues we need another technique to form a FMS. We need to be able to produce n linearly independent columns to ensure that $|X(t_0)| \neq 0$.

1. $e^{-\lambda I_n t} = e^{-\lambda t} I_n$ $e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

$= e^{-\lambda t}$

$\lambda_1 = \lambda_2 = \lambda$

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$= e^{\lambda t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ii, If we express

$$e^{A(t-t_0)} = e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)}$$

Let $X(t) = e^{A(t-t_0)} - e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)}$

$$X(t_0) = I_n - I_n = 0 \text{ matrix}$$

Also

$$\frac{d}{dt} X(t) = Ae^{A(t-t_0)} - (A-\lambda I_n)e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)}$$

$$- \lambda e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)}$$

$$= A\left(e^{A(t-t_0)} - e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)}\right)$$

$$= AX(t)$$

$$e^{A(t-t_0)} = e^{(A-\lambda I_n)(t-t_0)} \cdot e^{\lambda(t-t_0)} \quad \forall t.$$

iii, $e^{A(t-t_0)}$ and $e^{A(t-t_0)} \underline{v}$ for some constant vector $\underline{v}$ are always solution to the matrix and vector differential equations. The problem is to find $e^{A(t-t_0)} \underline{v}$ by using the above expression.

$$e^{A(t-t_0)} \underline{v} = e^{(A-\lambda I_n)(t-t_0)} e^{\lambda(t-t_0)} \cdot \underline{v} \quad \forall t$$

The expansion of $e^{(A-\lambda I_n)(t-t_0)}$ is finite provided

$$(A-\lambda I_n)^m = 0$$

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \dots$$

$$e^{(A-\lambda I_n)} = I + A - \lambda I_n + 1/2!\,(A-\lambda I_n)^2 + \dots$$

$$\frac{1}{m-1!}(A-\lambda I_n)^{m-1} + \cancel{\frac{1}{m!}(A-\lambda I_n)^m} \to 0$$

for some integers m (ie the series terminates after m terms).

If there exist an integer m such that

$(A-\lambda I_n)^m \underline{v} = 0$ and $(A-\lambda I)^{m-1} \underline{v} \neq 0$ then

$$e^{A(t-t_0)}\underline{v} = \left[I_n + (A-\lambda I_n)(t-t_0) + \frac{1}{2!}(A-\lambda I_n)^2(t-t_0)^2 + \dots + \frac{1}{(m-1)!}(A-\lambda I_n)^{m-1}(t-t_0)^{m-1} \right]$$

$$\cdot \; e^{\lambda(t-t_0)} \underline{v}$$

$$= e^{\lambda(t-t_0)} \left[\underline{v} + (A-\lambda I_n)\underline{v} + \frac{1}{2!}(A-\lambda I_n)^2 \underline{v} + \dots + \frac{1}{(m-1)!}(A-\lambda I_n)^{m-1}\underline{v} \right]$$

$$= e^{\lambda(t-t_0)} \left[\underline{v} + (A-\lambda I_n)(t-t_0)\underline{v} + \frac{1}{2!}(A-\lambda I_n)^2(t-t_0)^2 \underline{v} + \dots + \frac{1}{(m-1)!}(A-\lambda I_n)^{m-1}(t-t_0)^{m-1}\underline{v} \right];$$

then we have a procedure for forming $e^{A(t-t_0)}\underline{v}$ which is another linearly independent solution of the vector differential equation $\dot{\underline{x}} = A\underline{x}$ (i.e. another FMS).

Example:
Find three linearly independent solutions of
$\dot{\underline{x}} = A\underline{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

You may assume that $t_0 = 0$.

Solution.
Here since A is a upper triangular matrix, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 1$ twice

$$|A - \lambda I| = 0 \quad \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$1-\lambda \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 \\ 0 & 2-\lambda \end{bmatrix} + 0 \begin{bmatrix} 0 & 1-\lambda \\ 0 & 0 \end{bmatrix}$$

$$1-\lambda\,[(1-\lambda)(2-\lambda) - 0] - 1[0] + 0[0]$$

$$1-\lambda[2-\lambda-2\lambda+\lambda^2] \quad \Big| \quad 1-\lambda\,[(\lambda-2)(\lambda-1)] = 0$$
$$1-\lambda[\lambda^2 - 3\lambda + 2] = 0 \quad \Big| \quad (\lambda-1)(\lambda-2)(\lambda-1) = 0$$

ie $\lambda_1 = 2$ and $\lambda_2 = 1$ twice.

For $\lambda_1 = 2$, the eigenvectors are $\underline{V}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

How?

$-x_1 + x_2 + 0x_3 = 0$ and $x_3 = 1 \Rightarrow$ since
$0x_1 - x_2 + 0x_3 = 0$ x_3 is arbitrary
$0x_1 + 0x_2 + x_3 = 0$

$\therefore \; x_1 \div x_2 = 0$
$x_1 = x_2 = 0$

For $\lambda_2 = 1$, the eigenvectors are $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Let

$$(A - \lambda I_n)^2 \underline{V_3} = 0$$

$$\Rightarrow (A - \lambda I_3)^2 \underline{v} = 0$$

So that $(A - \lambda I_3)\underline{v} \neq 0$

then

$$e^{A(t-t_0)} \underline{V_3} = e^{\lambda(t-t_0)} \left[\underline{V_3} + (A - I_3)(t-t_0)\underline{v} + \cancel{\frac{1}{2!}(A - \lambda I_3)^2 (t-t_0)^2}^{\,0} \right] \underline{v}$$

$$= e^{\lambda(t-t_0)} \left(\underline{V_3} + (A - \lambda I_3)(t-t_0)\underline{V_3} \right)$$

To solve for $\underline{v}$, find $(A - \lambda I_3)^2 \underline{v} = 0$, where $\lambda = 1$

$$(A - I_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A - I_3)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A - I_3)^2 \underline{V_3} = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \underline{V_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ $\underline{v}_3 + (A - \lambda I_n)(t - t_0)\underline{v}_3$, gives

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (t - t_0)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (t - t_0)$$

Substituting back into

$$e^{A(t-t_0)}\underline{v} = e^{\lambda(t-t_0)}\left[\underline{v}_3 + (A - \lambda I_n)(t - t_0)\underline{v}_3\right]$$

Since $\lambda = 1$, we have

$$\Rightarrow e^{A(t-t_0)}\underline{v} = e^{(t-t_0)}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(t - t_0)\right]$$

Assume $t_0 = 0$

$$e^{At}\underline{v} = e^t\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= e^t\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} te^t \\ e^t \\ 0 \end{pmatrix}$$

$$x_1(t) = e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_2(t) = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_3(t) = e^t\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$$

Remark:

If one found v such that

$$(A - \lambda I_n)^2 \underline{v} = 0 \quad \text{and} \quad (A - \lambda I_n) \underline{v} \neq 0$$

and still need further linearly independent solutions, one can solve for v such that

$$(A - \lambda I_n)^3 \underline{v} = 0 \quad \text{and} \quad (A - \lambda I_n)^2 \underline{v} \neq 0$$

5.3 Homogeneous, variable coefficients, linear system

We extend the idea of a FMS from the constant A case to the time dependent $A(t)$ case with the following theorem:

Theorem 5.3

If $A(t)$ is an $n \times n$ matrix whose elements are continuous functions of time ie

$$t \in [t_0, t_f]$$

then there is a unique solution $\underline{x}(t)$ of $\dot{\underline{x}} = A(t)\underline{x}$ which is defined on $[t_0, t_f]$ and takes the value x_0 at t_0.

Proof:

Let $\phi_i(t - t_0)$ be a solution to the system $\dot{\underline{x}} = A(t)\underline{x}$ with $x(t_0) = e_i$ (the standard basis vector). Define the matrix

$$\Phi(t, t_0) = [\phi_1 \; \phi_2 \cdots \phi_n]$$

$\Phi(t, t_0)$ is called the transition matrix of the system $\dot{\underline{x}} = A(t)\underline{x}$

Note that $\phi(t,t_0)$ will satisfy $\dot{\phi} = A\phi$ and $\phi(t_0,t_0) = I_n$, hence $|\phi(t_0,t_0)| \neq 0$.

If $\dot{\underline{x}} = A(t)\underline{x}$ is a system with $\underline{x}(t_0) = \underline{x}_0$, then the system has a solution of

$$\boxed{\underline{x}(t) = \phi(t,t_0)\,\underline{x}_0 .}$$

A special case for which the transition matrix $\phi(t,t_0)$ can be found is if

$$A(t)\left(\int_{t_0}^{t} A(\tau)\,d\tau\right) = \left(\int_{t_0}^{t} A(\tau)\,d\tau\right)\cdot A(t) \ \forall t$$

then the system $\dot{\underline{x}} = A(t)\,\underline{x}$ has

$$\phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)\,d\tau}$$

as its transition matrix

If A is constant

$$\phi(t,t_0) = e^{A(t-t_0)}$$

$x(t_0) = x_0$

Example: Solve $\dot{\underline{x}} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\underline{x}$, $\underline{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Solution

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad A(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

$$\int_{t_0}^{t} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} d\tau = \left.\begin{pmatrix} \tau & \frac{\tau^2}{2} \\ c & \tau \end{pmatrix}\right|_{t_0}^{t}$$

$$= \begin{pmatrix} t - t_0 & \frac{(t^2 - t_0^2)}{2} \\ 0 & t - t_0 \end{pmatrix}$$

$$A(t) \cdot \int_{t_0}^{t} A(\tau)\, d\tau = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t - t_0 & (t^2 - t_0^2)/2 \\ 0 & t - t_0 \end{pmatrix}$$

$$= \begin{pmatrix} t - t_0 & \frac{t^2 - t_0^2}{2} + t(t - t_0) \\ 0 & (t - t_0) \end{pmatrix}$$

$$\int_{t_0}^{t} A(\tau)\, d\tau \cdot A(t) = \begin{pmatrix} t - t_0 & \frac{(t^2 - t_0^2)}{2} \\ 0 & t - t_0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} t - t_0 & t - t_0 + \frac{t^2 - t_0^2}{2} \\ 0 & t - t_0 \end{pmatrix}$$

$$\phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)\, d\tau}$$

$$= e^{\begin{pmatrix} t - t_0 & \frac{(t^2 - t_0^2)}{2} \\ 0 & t - t_0 \end{pmatrix}}$$

Expanding the series using e^{At} gives

$$e^{At} = I_n + e^{\begin{pmatrix} t-t_0 & \frac{(t-t_0)^2}{2} \\ 0 & t-t_0 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t-t_0 & \frac{(t-t_0)^2}{2} \\ 0 & t-t_0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} (t-t_0)^2 & \frac{(t-t_0)^4}{2} \\ 0 & (t-t_0)^2 \end{pmatrix}$$

$$+ \frac{1}{3!}\begin{pmatrix} (t-t_0)^3 & \frac{(t-t_0)^6}{2} \\ 0 & (t-t_0)^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+(t-t_0)+\frac{1}{2!}(t-t_0)^2+\frac{1}{3!}(t-t_0)^3 & 0+\frac{(t-t_0)^2}{2}+\frac{1}{2!}\frac{(t-t_0)^4}{2}+\frac{1}{3!}\frac{(t-t_0)^6}{2} \\ 0 & 1+(t-t_0)+\frac{1}{2!}(t-t_0)^2+\frac{1}{3!}(t-t_0)^3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(t-t_0)} & \frac{(t-t_0)^2}{2}e^{(t-t_0)} \\ 0 & e^{(t-t_0)} \end{pmatrix}$$

$$\therefore \phi(t, t_0) = \begin{pmatrix} e^{(t-t_0)} & \frac{(t-t_0)^2}{2}e^{(t-t_0)} \\ 0 & e^{(t-t_0} \end{pmatrix}$$

$$\underline{x}(t) = \phi(t-t_0)\,x_0 \quad \text{and} \quad x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x(0)\,x_0 = \begin{pmatrix} e^t & \frac{t^2}{2}e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & t^2e^t \\ & 2e^t \end{pmatrix} = e^t\begin{pmatrix} 1 & +t^2 \\ & 2 \end{pmatrix}$$

5.4 Properties of transition matrix

1. ϕ is always invertible

   ie

   $$|\phi(t,t_0)| \neq 0 \quad \text{for} \quad t_0 \leq t < \infty$$

2. $\phi(t,t_0) = \phi(t,t_1) \cdot \phi(t_1,t_0)$

3. $\phi^{-1}(t,t_0) = \phi(t_0,t)$

5.5 General non-homogeneous linear systems

Theorem 5.6

If $\phi(t,t_0)$ is the transition matrix for the system $\dot{\underline{x}} = A\underline{x}$, then the unique solution of

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{f}(t) \quad \text{with} \quad x(t_0) = x_0$$

where $\underline{f}(t)$ is continuous on $[t_0, t_f]$, is

$$\underline{x}(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,\tau) f(\tau)\, d\tau$$

Thus, once the transition matrix $\phi(t,t_0)$ is known, one can calculate the complete solution.

If A is a constant matrix, then the solution to the system $\dot{\underline{x}} = A\underline{x}$ is

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x_0} + \int_{t_0}^{t} e^{A(t-\tau)} f(\tau)\, d\tau$$

Proof of theorem 5.6

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{f}(t), \qquad x(t_0) = x_0$$

the solution to the non-homogeneous system is

$$\underline{x}(t) = \phi(t,t_0)\,\underline{x}_0 + \int_{t_0}^{t} \phi(t,\tau)\, f(\tau)\, d\tau$$

$$\underline{x}(t_0) = \phi(t_0,t_0)\,\underline{x}_0 + \cancelto{0}{\int_{t_0}^{t_0} \phi(t_0,\tau)\, f(\tau)\, d\tau}.$$

$$\underline{x}(t_0) = \underbrace{\phi(t_0,t_0)}_{I_n}\,\underline{x}_0$$

$$I_n\,\underline{x}_0 = \underline{x}_0$$

$$\underline{x}(t_0) = \underline{x}_0$$

$$\underline{x}(t) = x_c + x_p$$

$$\underline{x}_c(t) = \phi(t,t_0)\,\underline{x}_0$$

To find x_p, we use variation of parameters and assume a solution of the form

$$x_p = \phi(t,t_0)\,V(t)$$
$$= \phi(v)$$

$A(t) = A$

$V(t) = v$

$\underline{f}(t) = \underline{f}$

$\dot{\phi} = A\phi$

$$\text{Since} \quad \dot{x}_p = A x_p + \underline{f}$$

$$\dot{x}_p = \dot{\phi} v + \phi \dot{v}$$

$$\dot{\phi} v + \phi \dot{v} = A\phi v + \underline{f}$$

$$\cancel{A\phi v} + \phi\dot{v} = \cancel{A\phi v} + \underline{f}$$

$$\phi\dot{v} = \underline{f}$$

Premultiply by ϕ^{-1}

$$\phi^{-1}\phi\,\dot{v} = \phi^{-1} f$$

$$\dot{v} = \Phi^{-1}\underline{f}$$

$$\dot{v}(t) = \Phi^{-1}\underline{f}$$

$f(t)$
$\phi^{-1}(t, t_0)$
$\phi^{-1}(\tau, t_0)$
$f(\tau)$

$$\int_{t_0}^{t} \dot{v}(\tau)\,d\tau = \int_{t_0}^{t} \Phi^{-1}\underline{f}\,d\tau$$

$$v(t) - v(t_0) = \int_{t_0}^{t} \phi^{-1}(\tau, t_0)\, f(\tau)\,d\tau$$

take $v(t_0) = 0$

$$v(t) = \int_{t_0}^{t} \phi^{-1}(\tau, t_0)\, f(\tau)\,d\tau.$$

$$x_p = \phi(t, t_0)\, v(t)$$

$$x_p = \phi(t, t_0)\int_{t_0}^{t} \phi^{-1}(\tau, t_0)\, f(\tau)\,d\tau.$$

$$x(t) = x_c + x_p$$

$$= \phi(t, t_0) + \phi(t, t_0)\int_{t_0}^{t} \phi^{-1}(\tau, t_0)\, f(\tau)\,d\tau.$$

Example: Solve

$$\dot{x}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x(t) + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t}}_{\underline{f}(t)}$$

$$x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution:

For the homogeneous part:

$$\dot{\underline{x}}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \underline{x}(t)$$

The solution is

$$x_c(t) = e^t \begin{pmatrix} t^2+1 \\ 2 \end{pmatrix}$$

$$x_p = \phi(t,t_0) \int_{t_0}^{t} \phi^{-1}(\tau,t_0)\, f(\tau)\, d\tau .$$

but

$$\phi(t,t_0) = \begin{pmatrix} e^{t-t_0} & \frac{(t-t_0)^2}{2} e^{(t-t_0)} \\ 0 & e^{t-t_0} \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t_0 = 0$$

$$\phi(t,0) = \begin{pmatrix} e^t & \frac{t^2}{2} e^t \\ 0 & e^t \end{pmatrix}$$

$$\phi^{-1} = \frac{1}{|\phi|} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$$

$$\phi^{-1}(t,0) = \frac{1}{e^{2t}} \begin{pmatrix} e^t & -\frac{t^2}{2} e^t \\ 0 & e^t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -\frac{t^2}{2} e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

Since $f(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$

then $\Phi^{-1}(t,0) f(t) = \begin{pmatrix} e^{-t} & -\frac{t^2}{2} e^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$

$$= \begin{pmatrix} 1 - t^2 \\ 2 \end{pmatrix}$$

$$\int_0^t \begin{pmatrix} 1 - \tau^2 \\ 2 \end{pmatrix} d\tau = \begin{pmatrix} \tau - \tau^3/3 \\ 2\tau \end{pmatrix} \Bigg|_0^t$$

$$= \begin{pmatrix} t - t^3/3 \\ 2t \end{pmatrix}$$

$$\Phi(t,t_0) \int_{t_0}^t \phi(t,\tau) f(\tau)\, d\tau$$

$$= \begin{pmatrix} e^t & t^2/2\, e^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} t - t^3/3 \\ 2t \end{pmatrix}$$

$$= \begin{pmatrix} te^t - \frac{t^3}{3} e^t + t^3 e^t \\ 2te^t \end{pmatrix}$$

$$= \begin{pmatrix} te^t - \frac{2}{3} t^3 e^t \\ 2te^t \end{pmatrix}$$

$$x(t) = x_c + x_p$$

$$x(t) = \begin{pmatrix} t^2+1 \\ 2 \end{pmatrix} e^t + \begin{pmatrix} t - \frac{2}{3}t^3 \\ 2t \end{pmatrix} e^t .$$