APPM2021A/ APPM2022A

Lecture 1

Mathematical methods and modelling

Brief introduction:

What is mathematical modelling?

Mathematical modelling is the process of describing real world problem in mathematical terms usually in the form of an equation.

The phases of mathematical modelling process are:

1. Identification of the problem
2. Formulation of the model.
3. Analysis of the model
4. Interpretation of the solution
5. Validation of the prediction with experimental data.

Review of integration techniques

1. $\int ax^2 + bx + c \, dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + k$

2. $\int e^{3x+2} \, dx \Rightarrow \int e^u \cdot \frac{du}{3} = \int \frac{e^u}{3} \, du$

Let $u = 3x + 2$

$\frac{du}{dx} = 3$

$du = 3dx$

$= \frac{1}{3} e^u + c$

$= \frac{1}{3} e^{3x+2} + c$

1. Differential of a function

Function of one variable

$y = f(x)$, $\frac{dy}{dx} = f'(x)$

$\boxed{dy = f'(x) \, dx}$

$y = f$

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

It is safe to assume

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad *$$

Let $\Delta y = \dfrac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \Delta x$

$$\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x$$

$y = f(x)$

$\dfrac{dy}{dx} = f'(x)$

Assume $\Delta y = dy$ & $\Delta x = dx$

$$dy = \frac{dy}{dx} \cdot dx$$

$$\boxed{dy = f'(x)\, dx}$$ → For a function of one variable, dy is called the differential of $f(x)$

For example, find the differential of $y = \frac{1}{2}x^2$

Differentiate $\dfrac{dy}{dx} = \cancel{2} \cdot \dfrac{1}{\cancel{2}} x$

$$\frac{dy}{dx} = x$$

The differential ⟹ $dy = x\,dx$

Let $f(x) = \frac{1}{2}x^2$, $f'(x) = x$

$$dy = f'(x)\,dx = x\,dx$$

$y = \frac{1}{2}x^2$, Let $f(x) = \frac{1}{2}x^2$

$dy = f'(x)\,dx$ $f'(x) = x$

$$\boxed{dy = x\,dx}$$

Function of two variables

Let $u = u(x_1, x_2)$

$\Delta u = u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1, x_2)$

OR:

$\Delta u = u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1 + \Delta x_1, x_2) +$
$u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)$

$u(x_1 + \Delta x_1, x_2) - u(x_1, x_2) \rightarrow$ x_2 is constant and x_1 is changing

$u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1 + \Delta x_1, x_2)$
$\downarrow$
x_1 is constant and x_2 is changing

$\rightarrow u(x_1 + \Delta x_1, x_2) - u(x_1, x_2) = \frac{\partial u}{\partial x_1} \cdot \Delta x_1$

$\rightarrow u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1 + \Delta x_1, x_2) = \frac{\partial u}{x_2} \cdot \Delta x_2$

$$\Delta u = \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \cdot \Delta x_2$$

$\Delta u = du$, $\Delta x_1 = dx_1$ & $\Delta x_2 = dx_2$

$$\boxed{\therefore du = \frac{\partial u}{\partial x_1} \cdot dx_1 + \frac{\partial u}{\partial x_2} \cdot dx_2}$$ — Differential for function of two variables

Example: Find the differential of $u = \sqrt{x_1^2 + 2x_2^2}$

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

$$u = (x_1^2 + 2x_2^2)^{1/2}$$

$$\frac{\partial u}{\partial x_1} = \frac{1}{\cancel{2}}(x_1^2 + 2x_2^2)^{-1/2} \cdot \cancel{2}x_1$$

$$= (x_1^2 + 2x_2^2)^{-1/2} \cdot x_1$$

$$= \frac{x_1}{\sqrt{x_1^2 + 2x_2^2}}$$

$$\frac{\partial u}{\partial x_2} = \frac{1}{\cancel{2}}(x_1^2 + 2x_2^2)^{-1/2} \cdot \cancel{4}^{2}x_2$$

$$= (x_1^2 + 2x_2^2)^{-1/2} \cdot 2x_2$$

$$= \frac{2x_2}{\sqrt{x_1^2 + 2x_2^2}}$$

$$du = \frac{x_1}{\sqrt{x_1^2 + 2x_2^2}} dx_1 + \frac{2x_2}{\sqrt{x_1^2 + 2x^2}} dx_2$$

Function of Several variables

Consider the function $u = u(x_1, x_2, \cdots, x_n)$ then the differential is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n$$

Try: If $u = \dfrac{x_1^5 x_2^6}{x_1 + x_2^2}$ find the differential of u

Solution: Here, you use quotient rule.

Below are the steps:

$$\frac{\partial u}{\partial x_1} = \frac{(x_1 + x_2^2) \cdot 5x_1^4 x_2^6 - x_1^5 x_2^6 \cdot (1)}{(x_1 + x_2^2)^2}$$

$$= \frac{5x_1^5 x_2^6 + 5x_1^4 x_2^8 - x_1^5 x_2^6}{(x_1 + x_2^2)^2}$$

$$= \frac{x_1^4 x_2^6 (5x_1 + 5x^2 - x_1)}{(x_1 + x_2^2)^2}$$

$$= \frac{x_1^4 x_2^6 (4x_1 + 5x_2^2)}{(x_1 + x_2^2)^2}$$

$$u = \frac{x_1^5 x_2^6}{x_1 + x_2^2}$$

$$\frac{\partial u}{\partial x_2} = \frac{(6x_1^5 x_2^5)(x_1 + x_2^2) - x_1^5 x_2^6 \cdot 2x_2}{(x_1 + x_2^2)^2}$$

$$\frac{\partial u}{\partial x_2} = \frac{6x_1^6 x_2^5 + 6x_1^5 x_2^7 - 2x_1^5 x_2^8}{(x_1 + x_2^2)^2}$$

$$= \frac{x_1^5 x_2^5 \left[6x_1 + 6x_2^2 - 2x_2^2\right]}{(x_1 + x_2^2)^2}$$

$$= \frac{x_1^5 x_2^5 \left[6x_1 + 4x_2^2\right]}{(x_1 + x_2^2)^2}$$

$$\therefore du = \frac{x_1^4 x_2^6 (4x_1 + 5x_2^2)}{(x_1 + x_2^2)} dx_1 + \frac{x_1^5 x_2^5 (6x_1 + 4x_2^2)}{(x_1 + x_2^2)^2} dx_2$$