Classification of first order ODEs   Lecture 5

1. Separable equations assume the form:

$$f(y)\, y' + g(x) = 0$$

where $f(y)$ and $g(x)$ are arbitrary functions.

In order to integrate this equation, first we rewrite it as

$$f(y) \frac{dy}{dx} + g(x) = 0$$

$$f(y)\, dy + g(x)\, dx = 0$$

and then integrate to obtain

$$\int f(y)\, dy + \int g(x)\, dx = c,$$

where c is an arbitrary constant.

A first order ordinary differential equation of this form is said to be separable or have separable variables.

Exercise: Which of the following is separable?

a. $y' = xy^2 e^{3x+4y}$

b. $y' = y + \sin x$

Example: Solve the following differential equations:

1. $x\,dx - y^2\,dy = 0$

2. $(1+x)\,dy - y\,dx = 0$

3. $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2} \Rightarrow \ln|y| + y^2 = \sin x + c$

4. $(e^y+1)^2 e^{-y} dx + (e^x+1)^3 e^{-x} dy = 0$
$\Rightarrow (e^x+1)^{-2} + 2(e^y+1)^{-1} = c$

Solutions

1. $x\,dx - y^2 dy = 0 \Rightarrow \int g(x)dx + \int f(y)dy = c$

$$\int x\,dx - \int y^2 dy = c$$
$$\frac{x^2}{2} - \frac{y^3}{3} = c$$

Solving for y;

$$\frac{y^3}{3} = \frac{x^2}{2} - c$$
$$\frac{y^3}{3} = \frac{x^2 - 2c}{2}$$
$$2y^3 = 3x^2 - 6c$$
$$y^3 = \frac{3x^2}{2} - 3c$$
$$y^3 = \frac{3x^2}{2} + c_1 \quad (c_1 = -3c)$$
$$\therefore y = \left(\frac{3x^2}{2} + c_1\right)^{1/3}.$$

2. $(1+x)dy - y\,dx = 0$

To separate the variables, divide both sides by $y(1+x)$

$$\frac{1+x}{y(1+x)}dy - \frac{y}{y(1+x)}dx = 0$$

$$\frac{1}{y}dy - \frac{1}{1+x}dx = 0 \Rightarrow f(y)dy + g(x)dx = 0$$

Integrating:

$$\int \frac{1}{y}dy - \int \frac{1}{1+x}dx = c$$

$$\ln|y| - \ln|1+x| = c$$

$$\ln\left|\frac{y}{1+x}\right| = c \quad \text{(using laws of logarithms)}$$

$$\left|\frac{y}{1+x}\right| = e^c \quad \text{(taking exponential of both sides)}$$

$$|y| = |1+x|e^c$$

$$y = \pm e^c|1+x|$$

$$y = c_1(1+x) \quad \text{where } c_1 = \pm e^c$$

Some equations do not present themselves readily in separable form. We discuss a few examples but define homogeneous equation first.

Homogeneous Equations

Given the first order ODE

$$y' = f(x, y) \quad \text{—— (1)}$$

Equation 1 is homogeneous if

$$f(tx, ty) = f(x,y)$$

for any real number t

For example: $y' = \frac{y+x}{x} = f(x,y)^*$

$$f(tx, ty) = \frac{ty + tx}{tx} = \frac{t(y+x)}{t(x)}$$

$$= \frac{y+x}{x} = f(x,y)$$

hence $y' = \frac{y+x}{x}$ is homogeneous of order 1

$y' = \frac{x^2+y}{x^3}$ is not homogenous because

$$f(x,y) = \frac{x^2+y}{x^3} \qquad , (tx)^2$$

$$f(tx, ty) = \frac{t^2x^2 + ty}{t^3x^3} = \frac{\cancel{t}(tx^2+y)}{\cancel{t}(t^2x^3)}$$

$$= \frac{tx^2+ty}{t^2x^2} \neq f(x,y)$$

Equations reducible to separable form (Homogeneous equations)

Consider the ODE:

$$y' = \frac{M(x,y)}{N(x,y)} \qquad \text{——————} \quad 2$$

Where M and N are arbitrary functions of the

Same degree, say n. That is

$$M(x,y) = x^n f(y/x) \ \& \ N(x,y) = x^n g(y/x)$$

Then equation (2) becomes

$$y' = \frac{f(y/x)}{g(y/x)} = h(y/x) \quad \text{———} \quad (3)$$

This equation is known as a homogeneous first order ODE. clearly as it stands, this equation is not seperable by changing the dependent variable to

$$v = \frac{y}{x}$$

$$y = vx$$

Using product rule, on $y = vx$,

$$y' = v'x + v \quad \text{———} \quad (4a)$$

and keeping the independent variable as is we obtain:

$$y' = h(y/x) = h(v) \quad \text{———} \quad (4b)$$

where $v = y/x$

$$h(v) = v'x + v$$

$$v'x = h(v) - v$$

$$v' = \frac{h(v) - v}{x}$$

$$\frac{dv}{dx} = \frac{h(v) - v}{x}$$

$$h(v) - v \, dx = x \, dv$$

$$\frac{1}{h(v) - v} \, dv = \frac{1}{x} \, dx$$

integrating, we have

$$\int \frac{1}{h(v)-v}\,dv = \int \frac{1}{x}\,dx$$

$$\int \frac{1}{h(v)-v}\,dv = \ln|x| + c$$

This is now a separable equation

Example: Consider the ODE

$$y' = \frac{y+x}{x} = f(x,y)$$ Homogeneous in this state as previously verified

Introduce the change of variable:

let $v = y/x$

$$y' = \frac{y/x + x/x}{x/x}$$

$$y' = \frac{y/x + 1}{1} \Rightarrow y/x + 1$$

$$\Rightarrow v + 1$$

Since from equation 4a, $y' = v'x + v$

then equating we obtain

$$v'x + v = v + 1$$

$$v'x = v + 1 - v$$

$$v'x = 1$$

$$v' = \frac{1}{x}$$ → This is now a separable equation

$$\frac{dv}{dx} = \frac{1}{x}$$

$$dv = \frac{1}{x}\,dx$$

$$\int dv = \int \frac{1}{x}\,dx$$

$$v = \ln|x| + \ln|c|$$

$v = \ln|xc|$ but $v = y/x$

$\frac{y}{x} = \ln|xc|$

$\therefore y = x \ln|xc|$

Another example

$$y' = \frac{2y^4 + x^4}{x y^3} = f(x,y)$$

$$f(tx, ty) = \frac{2(ty)^4 + (tx)^4}{tx\,(ty)^3}$$

Always check if it is homogeneous and order of t the homogeneity before solving

$$= \frac{2t^4y^4 + t^4x^4}{tx\ t^3y^3}$$

$$= \frac{2t^4y^4 + t^4x^4}{t^4 x\ y^3}$$

$$= \frac{\cancel{t^4}(2y^4 + x^4)}{\cancel{t^4}(xy^3)}$$

$$= \frac{2y^4 + x^4}{xy^3} \Rightarrow f(x,y)$$

The equation is homogeneous with t of order 4.

$$y' = \frac{2(y^4/x^4) + x^4/x^4}{xy^3/x^4}$$

$$= \frac{2(y^4/x^4) + 1}{y^3/x^3} \Rightarrow \frac{2(y/x)^4 + 1}{(y/x)^3}$$

$$y' = \frac{2v^4 + 1}{v^3}$$

(Recall $v = y/x$ and $y' = v'x + v$)

$$v'x + v = \frac{2v^4 + 1}{v^3}$$

$$v'x = \frac{2v^4+1}{v^3} - v \implies \frac{2v^4+1-v^4}{v^3}$$

$$\frac{dv}{dx}x = \frac{v^4+1}{v^3} \implies \frac{x\,dv}{x} = \frac{v^4+1}{v^3}\frac{dx}{x}$$

$$\frac{v^3}{v^4+1}dv = \frac{1}{x}dx$$ $\implies$ This is now a separable equation

$$\int \frac{v^3}{v^4+1}dv = \int \frac{1}{x}dx$$

Use your substitution for this part

$\ln|c| = c$

if c=1 then $\ln|1| = 0$

which is a constant

$$\frac{1}{4}\ln|v^4+1| = \ln|x| + \ln|c|$$

$$\frac{1}{4}\ln|v^4+1| = \ln|xc|$$ - Applying logarithm laws

$$\ln|v^4+1| = 4\ln|xc|$$

$$\ln|v^4+1| = \ln|xc|^4$$ - By laws of log

Taking exponent on both sides

$$v^4+1 = (xc)^4$$

$$v^4+1 = x^4c^4$$

$$v^4+1 = x^4c_1$$ where $C_1 = c^4$

$$v^4 = x^4c_1 - 1$$ (Remember $v = y/x$)

$$(y/x)^4 = c_1x^4 - 1$$

$$\frac{y^4}{x^4} = c_1x^4 - 1$$

Multiplying all through by x^4

Remember $C_1 = C^4$

$\therefore C_2 = C_1^4$

$$y^4 = C_1^4 x^8 - x^4$$
$$y^4 = C_2 x^8 - x^4 \qquad (C_2 = C_1^4)$$
$$y = (C_2 x^8 - x^4)^{1/4}$$

In the above example, we have been to show a situation where our homogeneous equation can be transformed into a variable separable equation.

2. $y' = \dfrac{ax + by + c}{dx + ey + f}$

Another scenario where your ODE can be transformed to homogeneous form and further reduced to variable separable

where a, b, c and d, e, f are constants satisfying $\frac{a}{d} \neq \frac{b}{e} \Rightarrow ae \neq bd$

$$\Rightarrow ae - bd \neq 0$$

then make the change

$$y' = \frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f} = \frac{Y}{X}$$

$Y = ax + by + c$ and $X = dx + ey + f$

$dY = adx + bdy$ and $dX = ddx + edy$

$$\frac{dY}{dX} = \frac{a dx + b dy}{d dx + e dy}$$

$$= \frac{a + b\,{}^{dy}/_{dx}}{d + e\,{}^{dy}/_{dx}}$$ Dividing by dx

$$= \frac{a + b(Y/X)}{d + e(Y/X)}$$

This equation is now homogeneous and can be solved using the techniques discussed above. $Y \mp y$ and $X \mp x$

What would be done if $ae - bd = 0$?

If $\frac{a}{e} = \frac{b}{e}$, then we use the substitution

$$Y = x + y$$
$$\frac{dY}{dx} = dx + dy$$
$$\frac{dY}{dx} = 1 + dy/dx$$

Remember $y' = \frac{ax + by + c}{dx + ey + f}$

$$= \frac{a(x+y) + c}{d(x+y) + f}$$

$ax + by + c$
$ax + ay + c$
$a(x+y) + c$

$\Rightarrow$ Since $\frac{a}{d} = \frac{b}{e}$

$$y' = \frac{aY + c}{dY + f} = \frac{dy}{dx}$$

$$\frac{dY}{dx} = 1 + \frac{dy}{dx} \Longrightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dx} - 1}$$

$$\boxed{\frac{dY}{dx} - 1 = \frac{aY + c}{dY + f}}$$

The above is now a variable separable equation.

Example: Consider the ODE

$$\frac{dy}{dx} = \frac{x + y + 3}{2x + 2y + 1} = \frac{ax + by + c}{dx + ey + f}$$

$a=1, \quad b=1, \quad c=3, \quad d=2, \quad e=2$ and $f=1$.

$$ae-bd = 1(2) - 1(2)$$
$$= 2-2 = 0$$

$$\frac{dy}{dx} = \frac{(x+y)+3}{2(x+y)+1} = \frac{Y+3}{2Y+1}$$

$$\frac{dY}{dx} - 1 = \frac{Y+3}{2Y+1}$$

$$\frac{dY}{dx} = \frac{Y+3}{2Y+1} + 1 \Rightarrow \frac{Y+3+2Y+1}{2Y+1}$$
$$= \frac{3Y+4}{2Y+1}$$

$$\frac{dY}{dx} = \frac{3Y+4}{2Y+1} \Rightarrow 3Y+4\,dx = 2Y+1\,dY$$

$$\frac{2Y+1}{3Y+4}\,dY = dx$$

$\hookrightarrow$ Use long division first before integrating

$$\begin{array}{r|l} & \frac{2}{3} \\ \hline 3Y+4 & 2Y+1 \\ & 2Y+8/3 \\ \hline & 0 \quad -5/3 \end{array}$$

$$\frac{2Y+1}{3Y+4} = \frac{2}{3} - \frac{5}{3(3Y+4)}$$

$$\int\left(\frac{2}{3} - \frac{5}{3(3Y+4)}\right) dY = \int dx$$

$$\frac{2}{3}Y - \frac{5}{3}\ln|3Y+4| \cdot \frac{1}{3} = x + c$$

$$\frac{2}{3}Y - \frac{5}{9}\ln|3Y+4| = x+c$$

$$\frac{2}{3}(x+y) - \frac{5}{9}\ln|3x+3y+4| = x+c$$ Since $Y = x+y$

Exact Equations

A first order ODE can be written in the form

$$y' = \frac{-M(x,y)}{N(x,y)} \quad \text{——— (1)}$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

$$dy = \frac{-M(x,y)}{N(x,y)}\,dx$$

$$N(x,y)\,dy = -M(x,y)\,dx$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{——— (2)}$$

Equation (2) is said to be exact if it can be expressed as

$$M(x,y)\,dx + N(x,y)\,dy = du \quad \text{——— (3)}$$

where $u(x,y) = c$

But the RHS of equation (2) is zero.

Therefore, comparing equations (2) and (3) gives;

$$du = 0 \quad \text{——— (4)}$$

Integrating equation (4), gives

$$u(x,y) = c$$

Recall in differentials, the differential of u is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dx = 0 \qquad (5)$$

Differential of functions in two variables x & y

Comparing equations (5) and (2) gives

$$\boxed{M(x,y)\,dx} + \boxed{N(x,y)\,dy} = du = \boxed{\frac{\partial u}{\partial x}dx} + \boxed{\frac{\partial u}{\partial y}dy} \qquad (6)$$

Hence, equation (1) is exact if and only if there is a function $u(x,y)$ such that

$$M(x,y) = \frac{\partial u}{\partial x} \text{ and } N(x,y) = \frac{\partial u}{\partial y} \qquad (7)$$

If $u(x,y)$ is twice differentiable,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(N(x,y)\right) = \frac{\partial}{\partial y}\left(M(x,y)\right)$$

$$N_x = M_y \qquad (8)$$

Equation (8) provides a pratical criteria for equation (1) to be exact

Example: check that the equation

$$(2x - 5y)\,dx + (3y^2 - 5x)\,dy = 0$$

is exact and find its solution.

Solution:

$$(2x-5y)\,dx + (3y^2-5x)\,dy = 0$$

$$\underset{M(x,y)\,dx}{\downarrow} \qquad \underset{N(x,y)\,dy}{\downarrow}$$

$M = 2x-5y$ and $N = 3y^2-5x$

$M_y = \frac{\partial}{\partial y}(2x-5y)$, $N_x = \frac{\partial}{\partial x}(3y^2-5x)$

$= -5$, $= -5$

Since $M_y = N_x = -5$, then the equation is exact.

To solve, we have

$$M(x,y) = \frac{\partial u}{\partial x} = 2x-5y$$

$$u(x,y) = x^2-5xy+g(y)$$ upon integrating $\frac{\partial u}{\partial x}$

where the arbitrary constant $g(y)$ is the "constant" of integration

Hence $\boxed{u(x,y) = x^2-5xy+g(y)}$

$$N(x,y) = \frac{\partial u}{\partial y} = 3y^2-5x$$

$$\frac{\partial}{\partial y}\left(x^2-5xy+g(y)\right) = 3y^2-5x$$

$$-5x+g'(y) = 3y^2-5x$$

$$g'(y) = 3y^2-5x+5x$$
$$= 3y^2$$

integrating wrt y

$$\int g'(y)\,dy = \int 3y^2\,dy$$

$g(y) = y^3$

Since $u = x^2 - 5xy + g(y)$

then $u(x,y) = x^2 - 5xy + y^3 = C$