Chapter 1

Mathematics of Discretization

The major benefit of studying equations which hold the form of a Transport equation is the large body of research of their solutions and behavior. We begin this chapter by considering the most basic transport equation, the Advection Equation which will allow the introduction of finite difference techniques.

We then follow this through with a discussion of the approach used by **?** on the food web framework that they have developed using semi-implicit methods.

1.1 Advection Equation

The Advection Equation is a simple hyperbolic partial differential equation, in one dimension we take

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = du, \tag{1.1}$$

for some transfer velocity v and destruction rate d on the domain $\mathbb{R}^+ \times [a, b]$ for any $a, b, \in \mathbb{R}$. The linear advection has been studied extensively and thus we can gain some insight in to potential issues that might occur in our more complicated equations such as **??**.

1.1.1 Analytic Solution

As with ordinary differential equations, partial differential equations can be solved with the "method of characteristics" (or method of lines). Say we consider Equation 1.1 on the domain $\mathbb{R}^+ \times \mathbb{R}^+$, as to line up with the McKendrick-von Foerster Equation. Then we define some boundary value and initial value problem (BVP and IVP):

$$
\begin{aligned}
u_t + v\,u_x &= +d\,u \\
u(0,x) &= I(x) \\
u(t,0) &= B(x)
\end{aligned}
\tag{1.2}
$$

Using the method of characteristics we consider a point in the domain $\{(t,x) : t,x > 0\}$ and solve the equation along some characteristic line $(t(s),x(s)) = y(s)$ stemming from an $y_0 \in \Gamma = \{(0,x) : x \in \mathbb{R}^+\} \cup \{(t,0) : t \in \mathbb{R}^+\}$. The full details of the method are left to the reader however we find that

$$
u(t,x) = \begin{cases} I(x-vt)\,\mathrm{e}^{-d\,t} & x - vt > 0 \\ B(x-vt)\,\mathrm{e}^{-d\,t} & x - vt \leq 0 \end{cases}
\tag{1.3}
$$

along any characteristic $x = vt$. We note that the McKendrick-von Foerster will take a form similar to this if the growth and death terms are constants, but the problem will become more complicated if the coefficients are functions of t, x or non-linear. However this problem does allow us to introduce the idea of domain of dependence.

Definition 1.1 (Analytic Domain of dependence)**.** Given any BVP and/or IVP on $U \subseteq \mathbb{R}^n$ the with solution φ, the domain of dependence for for any $x \in U$ is the set $V \subseteq U$ that $\varphi(x)$ depends on to be calculated.

Figure 1.1 shows the characteristic lines for, the blue lines represent characteristics which depend on the time axis boundary condition while the red arrows depend on the spatial axis initial condition. Looking at the analytic solution to Equation 1.2 we see that the initial condition is carried along the characteristic.

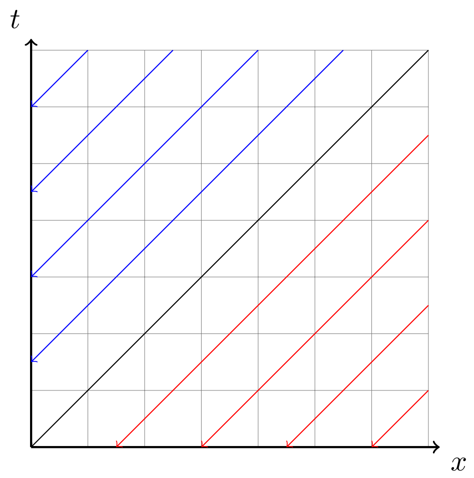

Figure 1.1: The characteristic lines for the solution of Equation 1.3 with constant coefficients. There is no overlap between the lines which means that the solution can be smooth if the boundary conditions along Γ are smooth. The domain of dependence for traverses the direction of each arrowhead line.