Finite Difference Schemes for the McKendrick Von Foerster Equation

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Abstract

This is my Abstract

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Introduction

Chapter 2

Finite Difference Equations

As we will see, the equations that are derived in chapter 3 do not have well formed analytics solutions. In fact, generally, most partial differential equations do not have explicit analytic solutions. Consequently we must find accurate numerical approximations and methods for solving these problems.

In this chapter we introduce the required mathematics for understanding the numerical methods used in chapter 4. We introduce the idea of a finite difference equation, and the finite difference method for numerically solving partial differential equations. As well as the required knowledge to understand the stability of the methods described.

2.1 Relation to Differential Calculus

As Gorge Boole writes in Boole and Moulton (1880), *"Differential Calculus is occupied about the limits to which such ratios approach as the increments are indefinitely diminished." However, "The Calculus of Finite Differences may be strictly defined as the science which is occupied about the ratios of the simultaneous increments of quantities mutually dependent."* In simple terms the calculus of finite differences is only concerned about ratios of infinitesimals, in line with the ideas of Newtonian calculus.

To understand this, consider the simple *ordinary* differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y. \tag{2.1}$$

In differential calculus we can represent the left hand of Equation 2.1 by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(y) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}, \tag{2.2}$$

but what instead if we represented it with respect to some infinitesimal Δx and in terms of some ratio Δ, prefixed to any function of x, which increments the value of that function by

Δx. This would give that

$$\Delta y = y(x + \Delta x) - y(x) \tag{2.3}$$

and then introduce the the quotient

$$\frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x} \tag{2.4}$$

which we can see is the foundation of the operator $\frac{\mathrm{d}}{\mathrm{d}x}$. Thus, if we can say that $\frac{\mathrm{d}}{\mathrm{d}x}$ is the fundamental operator in differential calculus; then $\frac{\Delta}{\Delta x}$ is the fundamental operator in finite difference calculus. From undergraduate know that

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y_{x+\Delta x} - y_x}{\Delta x} &= \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} \\ &= \frac{\mathrm{d}y}{\mathrm{d}x}. \end{aligned} \tag{2.5}$$

However $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not a true fraction, $\mathrm{d}y$ and $\mathrm{d}x$ do not have any intrinsic value where as Δy and Δx have exact value. As Boole and Moulton (1880) notes in the opening remarks: *"In consequence of the fundamental difference above noted between the Differential Calculus and the Calculus of Finite Differences, the term Finite ceases to be necessary as a mark of distinction. The former is a calculus of limits, not of differences."*

2.2 Difference Quotients

In section we begin to construct the frameworks needed to numerically model and solve partial differential equations in the discrete space of finite difference equations. We begin by introducing a deeper model of differences using a directional differences, and then using this to construct approximations to partial derivatives of any order.

2.2.1 Difference Notation

In this section we assume that our fractional $\Delta x > 0$ is fixed throughout.

In section 2.1 we introduced the definition of a difference as

$$\Delta u = u(x + \Delta x) - u(x). \tag{2.6}$$

Following the notation of Milne-Thomson (1933) we define this as the first forward difference of u, and generalize this to a function $u : \mathbb{R}^n \to \mathbb{R}$.

Definition 2.1 (First Forward Difference)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ then the forward difference of f, is defined by

$$\Delta[f](x) = f(x_1, ..., x_i + \Delta x, ...) - f(x_1, ..., x_i, ...) \tag{2.7}$$

Remark 2.1 (Properties of the Difference Operator)*.* Just as with the differential operator we have that it is linear and satisfies the Leibniz rule. The proof of this is left to the reader, but follows exactly the same method as for derivatives.

Following this principle it is easy to construct the n^{th} forward difference $(\Delta^n[f](x))$ by simply considering $\Delta[\Delta^{n-1}[f]](x)$.

The forward difference is just one way of taking a difference. We can define a difference to be either *forwards*, *backwards*, or *central*. A backwards difference can be defined by simply considering a fractional of $-\Delta x$ and is denoted by $\nabla[f](x)$, and can be useful in different situations.

Definition 2.2 (First Central Difference)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ then the forward difference of f, is defined by

$$\delta[f](x) = f(x_1, ..., x_i + \Delta x/2, ...) - f(x_1, ..., x_i - \Delta x/2, ...) \tag{2.8}$$

2.2.2 Relation to Derivatives

Now that we have the definitions of differences we can study their relation to derivatives in more detail. Consider the first forward difference of some function $u : \mathbb{R} \to \mathbb{R}$. Then from section 2.1 we know that

$$u'(x) = \lim_{\Delta x \to 0} \frac{\Delta[u](x)}{\mathrm{d}x}. \tag{2.9}$$

Hence we can see that

$$\frac{\Delta[u](x)}{\mathrm{d}x} - u'(x) = O(\Delta x) \to 0 \tag{2.10}$$

as $\Delta x \to 0$ and this is derived in Hildebrand (1987) using Taylor's Theorem. Now we ask the question: "How, in general, do we approximate the n^{th} derivative of u by a difference?".

Theorem 2.1. *Suppose that $u \in \mathcal{C}^n([a, b])$. Then for any fractional Δx there exists a finite difference equation to represent $u^{(k)}(x)$ for all $k \leq n$ and all $x \in [a, b]$.*

Proof. The explicit details of this proof are left for the reader, but can be found in Fornberg (1988). A proof for the first derivative goes as follows.

Suppose that $u \in \mathcal{C}^1([a, b])$ then we have that

$$u(x + \Delta x) = u(x) + \Delta x \frac{\mathrm{d}u}{\mathrm{d}x} + E_1 \tag{2.11}$$

for some error E_1 which is at most $O(\Delta x^2)$. Rearranging we have that

$$\frac{u(x+\Delta x)-u(x)}{\Delta} = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{E_1}{\Delta x}. \tag{2.12}$$

We label $E_1/\Delta x$ as T_1 which is defined as *truncation error* (See subsection 2.3.1). It is clear from this that the left hand side is an approximation to $\frac{\mathrm{d}u}{\mathrm{d}x}$. □

2.3 Transforming Differential Equations to Difference Equations

Suppose now that we are given a partial differential equation of the form

$$L(u) = \frac{\partial u}{\partial t} - \mathcal{L}u \tag{2.13}$$

for some linear differential operator $\mathcal{L}$ which is a function $\mathcal{L}(x, u, x_x, ..., u_{x^n})$ for some $n \in \mathbb{N}$, that is that $L : \mathcal{C}^m \to \mathcal{C}$.

Example 2.2. The simplest example of this is the Advection Equation where L takes the form

$$L(u) = \frac{\partial u}{\partial t} - \upsilon\frac{\partial u}{\partial x}. \tag{2.14}$$

for some velocity constant υ. However this advection equation can be extended further to add a source term to yield

$$L(u) = \frac{\partial u}{\partial t} - \left(\upsilon\frac{\partial u}{\partial x} - \gamma u\right). \tag{2.15}$$

We consider a mesh-grid which is a partition in each variable (t, x) with a constant h in the x and a constant increment k in the t direction (Figure 2.1). This allows us to construct a general difference equation $D : \mathcal{C}^m \to \mathbb{R}$ which can approximate each point in the mesh grid.

Example 2.3. Consider 2.2 and consider approximating the derivatives at any point (t, x) by $u(t, x)$, then we can write a difference equation as

$$D(u) = u - (\upsilon u - \gamma u) \tag{2.16}$$

or, at any point in the grid $(t, x) = (n \cdot k, i \cdot h) \leftrightarrow (n, i)$ we can represent the value of $u(t, x)$ as u_i^n and so a general finite difference takes the form of these indexed values. In our example this would be illustrated as

$$D(u) = u_i^n - (\upsilon u_i^n - \gamma u_i^n) = (1 - \upsilon + \gamma)u_i^n \tag{2.17}$$

2.3.1 Truncation Error

Suppose that we have some difference equation $D : \mathcal{C}^m \to \mathcal{C}^n$ which is supposed to be an approximation to L. Denote ϕ be the exact solution to $D(u) = 0$, so $D(\phi) = 0$, and let Φ be the exact solution to $L(u) = 0$.

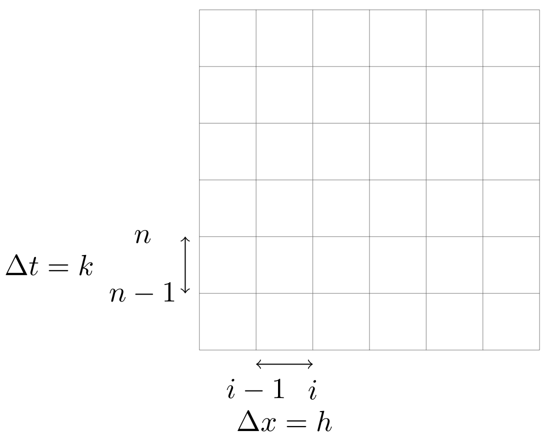

Figure 2.1: Mesh Grid example

Chapter 3

Fundamental Equations

3.1 Fundamental Equations

3.2 Feeding Kernels

The work of Datta et al. (2010) and the Jump-Growth Equation relies on the principle of a feeding kernel, the preferred weight a predator would like to prey on.

3.2.1 Prey Size Selection

Under the assumption that larger fish eat smaller fish (Figure 3.1) we assume that feeding take places locally on a preferred weight smaller, $\beta = 100$ than the predators own. The preference is then normally distributed around this weight

As described in Benoıt and Rochet (2004) we assume that the feeding rate takes the form

$$k(w, w') = w^{\alpha} S\left(\frac{w}{w'}\right) \tag{3.1}$$

for some α, and further we take S to be a local Gaussian of the the form

$$S\left(\mathrm{e}^{z}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(z-\beta)^2}{2\sigma^2}\right) \tag{3.2}$$

for some β, σ to be determined experimentally.

Chapter 4

Method

This chapter outlines the method used to construct a finite difference scheme for the deterministic Jump Growth Equation, Equation 3.8. It begins with a discussion about the steady state of the McKendrick-von Foerster Equation and how this helps to solve the Jump Growth Equation. Following this we apply the mathematics in chapter 2 and Rosinger (2008) to construct a finite difference scheme that is stable and consistent, and thus convergent.

4.1 Steady State Solutions

In marine ecosystems it's been found that the abundance of organisms within weight classes is roughly constant (Sheldon et al. (1972)) if those weight classes are distributed logarithmically. Further, averaged over time, this abundance changes rather little (Datta et al. (2011)) suggesting that the system is near steady state. Benoıt and Rochet (2004) found that the McKendrick-von Foerster Equation has a power law steady steady state of the form $\phi(w) \propto w^{\gamma}$. We combine this reasearch to now show that the the McKendrick-von Foerster Equation with diffusion (Equation 3.8) can be transformed using the logarithmic change of variables and appropriate change of function to have a constant solution if the work of Benoıt and Rochet (2004) is correct.

It first helps to transform to a dimensionless variable $x = \log w$, with $\varphi(x)\mathrm{d}x = \phi(w)\mathrm{d}w$ which we will show makes our equations behave much more nicely under translation. Following the method found in section A.1 we can change variables from our weight indexed equation

$$u_t = -(g \cdot u)_w - \mu \cdot u + \frac{1}{2}(D \cdot u)_{xx} \tag{4.1}$$

to a dimensionless equation for $\rho(x) = \mathrm{e}^{(1-\gamma)\log w}\, u(w)$,

$$\mathrm{e}^{\gamma x}\, \rho_t = -(\bar{g}\rho)_x - \bar{\mu}\, \mathrm{e}^{\gamma x}\, \rho + \frac{\mathrm{e}^{-x}}{2}\left((\bar{D}\rho)_{xx} - (\bar{D}\rho)_x\right), \tag{4.2}$$

for the coefficient functions $\bar{g}, \bar{\mu}, \bar{D}$ found in section A.1. Through this transform and the knowledge from section 4.1 we find that at steady state $\rho(x) \propto 1$ and so if $u(w) = u_0 w^{\gamma}$ then

$\rho(x) = u_0$ which is much easier to run tests against since for a initial condition which consists of a small perturbation $\epsilon(x)$ we can much more easily see if the system returns to a steady state.

4.2 Coefficient Observations

We first observe that under the logarithmic weight index that $\bar{S}(x)$ is a gaussian curve which we will say takes the form defined in Equation 3.12. Thus under the coefficient functions $\bar{S}(x - x')$ takes the form

$$\bar{S}(x - x') = C \exp\left(-\frac{(x - x' - \beta)^2}{2\sigma^2}\right) = C \exp\left(-\frac{(x' - (x - \beta))^2}{2\sigma^2}\right), \tag{4.3}$$

where $C = (2\sigma^2\pi)^{-1}$, and again this takes the form of a Gaussian centered at $x - \beta$. Further if $Gau_{\beta,\sigma}(x)$ is a Gaussian then by completing the square it is easy to see that

$$\mathrm{e}^{\gamma x}\, Gau_{\beta,\sigma}(x) \propto Gau_{\lambda,\eta}(x). \tag{4.4}$$

If, for a simple test, we take ρ to be at the steady state and thus constant $\rho(x) = \upsilon_0$ and consider the growth coefficient $\bar{g}(x)$ then

$$\bar{g}(x) = \upsilon_0 K \int \mathrm{e}^{(\gamma+1)x'}\, \bar{S}(x - x')\, \mathrm{d}x'. \tag{4.5}$$

Considering the inner function we just said that this will take the form of a Gaussian curve for various x, and this can be numerically validated using a simple MATLAB script (See Figure 4.1). However as can be seen on the the red curve and partially on the yellow curve for this range of x we see that much of the Gaussian curve is cut off and thus, if this was to be integrated would be missing much of the area.

Figure 4.1: Numerical validation that the integrand for the growth coeffient takes a Gaussian form if ρ is at steady state.

Appendix A

Proof of Steady State

A.1 Change of Variables

Starting with the McKendrick-von Foerster Equation with diffusion (Equation 3.8) we first need to change to a dimensionless variable $x = \log w$, so that $\varphi(x)\partial x = u(w)\partial w$. In a first step this gives that

$$\mathrm{e}^{-x}\varphi_t = -(g(w)\,\mathrm{e}^{-x}\varphi)_w - \mathrm{e}^{-x}\mu(w)\varphi + \frac{1}{2}(D\,\mathrm{e}^{-x}\varphi)_{ww} \tag{A.1}$$

We then transform the derivatives from w to x using the chain rule

$$\frac{\partial}{\partial w} = \frac{\partial x}{\partial w}\frac{\partial}{\partial x} = \mathrm{e}^{-x}\frac{\partial}{\partial x}. \tag{A.2}$$

Which shows that

$$\begin{aligned}
\mathrm{e}^{-x}\varphi_t &= -\,\mathrm{e}^{-x}\frac{\partial}{\partial x}(\hat{g}\varphi) - \mathrm{e}^{-x}\hat{\mu}(x)\varphi + \frac{\mathrm{e}^{-x}}{2}\frac{\partial}{\partial x}\left(\mathrm{e}^{-x}\frac{\partial}{\partial x}\left(\hat{D}\varphi\right)\right) \\
\varphi_t &= -\frac{\partial}{\partial x}(\hat{g}\varphi) - \hat{\mu}(x)\varphi + \frac{1}{2}\frac{\partial}{\partial x}\left(\mathrm{e}^{-x}\frac{\partial}{\partial x}\left(\hat{D}\varphi\right)\right) \\
&= -\,(\hat{g}\varphi)_x - \hat{\mu}(x)\varphi + \frac{\mathrm{e}^{-x}}{2}\left(\left(\hat{D}\varphi\right)_{xx} - \left(\hat{D}\varphi\right)_x\right).
\end{aligned} \tag{A.3}$$

Now we need to concentrate on the coefficient functions for growth, death and diffusion. First let us consider the growth function, $g(w)$ which from subsection 3.2.3 takes the form

$$g(w) = \int Kw'k(w,w')u(w')\,\mathrm{d}w' = \int Kw'w^{\alpha}S\left(\frac{w}{w'}\right)u(w')\,\mathrm{d}w' \tag{A.4}$$

then following the change of variables in Equation A.3 we take $\hat{g}(x) = \mathrm{e}^{-x}g(\mathrm{e}^{x})$, and since $\varphi(x)\partial x = u(w)\partial w$ we can change from u to φ in the integral very simply and thus

$$\hat{g}(x) = \mathrm{e}^{-x}g(\mathrm{e}^{x}) = \int K\,\mathrm{e}^{x'}\,\mathrm{e}^{(\alpha-1)x}\,\bar{S}(x-x')\varphi(x')\,\mathrm{d}x', \tag{A.5}$$

where $\bar{S}(z) = S(\mathrm{e}^{z})$. Similarly for the diffusion coefficient

$$D(w) = \int (Kw')^2k(w,w')u(w')\,\mathrm{d}w' = \int (Kw')^2w^{\alpha}S\left(\frac{w}{w'}\right)u(w')\,\mathrm{d}w' \tag{A.6}$$

we change variables so that

$$\hat{D}(x) = \mathrm{e}^{-x}\, D(\mathrm{e}^{x}) = \int K^2\, \mathrm{e}^{2x'+(\alpha-1)x}\, \bar{S}(x-x')\varphi(x')\, \mathrm{d}x'. \tag{A.7}$$

The death term provides an easier change of variables with

$$\hat{\mu}(x) = \mu(\mathrm{e}^{x}) = \int \mathrm{e}^{\alpha x'}\, \bar{S}(x'-x)\varphi(x')\, \mathrm{d}x' - \delta. \tag{A.8}$$

If we now assume that $\mathrm{e}^{\gamma x}\,\rho(x) = \varphi(x)$, so that if the steady state of $\varphi(x) \propto \mathrm{e}^{\gamma x}$ then the steady state of $\rho(x) \propto 1$, then we have that Equation A.3 implies

$$\begin{aligned}
\mathrm{e}^{\gamma x}\,\rho_t &= -(\hat{g}\,\mathrm{e}^{\gamma x}\,\rho)_x - \hat{\mu}\,\mathrm{e}^{\gamma x}\,\rho + \frac{\mathrm{e}^{-x}}{2}\Big((\hat{D}\,\mathrm{e}^{\gamma x}\,\rho)_{xx} - (\hat{D}\,\mathrm{e}^{\gamma x}\,\rho)_x\Big)\\
&= -(\bar{g}\rho)_x - \bar{\mu}\,\mathrm{e}^{\gamma x}\,\rho + \frac{\mathrm{e}^{-x}}{2}\big((\bar{D}\rho)_{xx} - (\bar{D}\rho)_x\big)
\end{aligned} \tag{A.9}$$

with the coefficient equations

$$\bar{g}(x) = \int K\,\mathrm{e}^{(\gamma+1)x'+(\gamma+\alpha-1)x}\,\bar{S}(x-x')\rho(x')\,\mathrm{d}x' \tag{A.10}$$

$$\bar{D}(x) = \int K^2\,\mathrm{e}^{(\gamma+2)x'+(\gamma+\alpha-1)x}\,\bar{S}(x-x')\rho(x')\,\mathrm{d}x' \tag{A.11}$$

$$\bar{\mu}(x) = \int \mathrm{e}^{(\alpha+\gamma)x'}\,\bar{S}(x'-x)\rho(x')\,\mathrm{d}x' - \delta \tag{A.12}$$

If we then follow the example of Datta et al. (2011) and set $\alpha+\gamma-1=0$, then these are reduced to

$$\begin{aligned}
\bar{g}(x) &= \int K\,\mathrm{e}^{(\gamma+1)x'}\,\bar{S}(x-x')\rho(x')\,\mathrm{d}x'\\
&= \int K\,\mathrm{e}^{(\gamma+1)(x-y)}\,\bar{S}(y)\rho(x-y)\,\mathrm{d}y
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
\bar{D}(x) &= \int K^2\,\mathrm{e}^{(\gamma+2)x'}\,\bar{S}(x-x')\rho(x')\,\mathrm{d}x'\\
&= \int K^2\,\mathrm{e}^{(\gamma+2)(x-y)}\,\bar{S}(y)\rho(x-y)\,\mathrm{d}y
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
\bar{\mu}(x) &= \int \mathrm{e}^{x'}\,\bar{S}(x'-x)\rho(x')\,\mathrm{d}x' - \delta\\
&= -\int \mathrm{e}^{x+y}\,\bar{S}(y)\rho(x+y)\,\mathrm{d}y - \delta
\end{aligned} \tag{A.15}$$

A.2 Steady State Condition

At steady state we have that $\varphi(x) = \varphi_0\,\mathrm{e}^{\gamma x}$, so $\rho(x) = \upsilon_0 = \varphi_0$, thus we have that, after cancelling υ_0

$$0 = \mathrm{e}^{-\gamma x}\,\bar{g}_x + \bar{\mu} - \frac{\mathrm{e}^{-(\gamma+1)x}}{2}\big(\bar{D}_{xx} - \bar{D}_x\big) \tag{A.16}$$

which, after substituting in the coefficient functions and cancelling some overall factors gives a steady state condition for the equation.

$$\int \bar{S}(y)\, \mathrm{e}^{x+y} \left(-1 + (\gamma+1)K\, \mathrm{e}^{-(\gamma+2)y} - (\gamma+2)(\gamma+3)\frac{K^2}{2}\, \mathrm{e}^{-(\gamma+3)y} \right)\, \mathrm{d}y - \eta \tag{A.17}$$

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