Finite Difference Schemes for the McKendrick-von Foerster Equation

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University of York

Author

Luke Sheard

Supervisor

Dr. Gustav Delius

Abstract

While the mechanics of marine ecosystems have been researched for years, recent efforts by Datta et al. (2010) and Hartvig et al. (2011) derive fully non-linear equations to describe their mechanics. Unfortunately these equations are difficult to solve and require numerical methods to solve and methods with a high order of approximation are yet to be developed thoroughly.

This project investigates the derivation of the equations used to model marine ecosystems, in particular the Jump-Growth Equation and various forms of the McKendrick-von Foerster Equation. It shows that particular forms of the McKendrick-von Foerster Equation are approximations to the Jump-Growth Equation and construct the coefficients necessary to approximate the Jump-Growth Equation. Using the second order approximation we show that, for particular parameters, this approximation satisfies a power law steady state.

This project considers numerical methods for solving these equations including the history of finite differences and the construction of accurate approximations. It contains a full discussion of the errors generated by using finite difference approximations and how to resolve the issues that arise from using them. This is helpful when it considers applying these methods to the McKendrick-von Foerster Equation. It shows that while the simple forms of the McKendrick-von Foerster Equation are easy to solve numerically, more work is needed before accurate methods can be used on the non-linear forms of the McKendrick-von Foerster Equation.

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Chapter 1

Introduction

The study of partial differential equations has been of interest due to their close relation to modelling real world ecosystems. In particular a class of partial differential equations called Transport Equations relate real world convection-diffusion systems to analytic equations and are studied extremely widely. In this project we study the details around one particular Transport Equation: the McKendrick-von Foerster Equation.

$$\frac{\partial u(t,w)}{\partial t} = -\frac{\partial}{\partial w}\left(g(w)\cdot u(t,w)\right) - \mu(w)\cdot u(t,w) \tag{1.1}$$

Project Outline

This project investigates the current methods used in research around Equation 1.1 and similar equations, focussing on trying to derive an higher order finite difference equation for the Jump-Growth Equation

$$\begin{aligned}\frac{\partial \phi(w)}{\partial t} = \int \big(& -k(w',w)\phi(w)\phi(w') \\ & - k(w,w')\phi(w')\phi(w) \\ & + k(w-Kw',w')\phi(w-Kw')\phi(w')\big)\,\mathrm{d}w'.\end{aligned} \tag{1.2}$$

In Chapter 2 we discuss the historical background and derivation of the McKendrick-von Foerster Equation. We study similarities between the equation Transport Equations which will serve as the basis for our approach towards constructing finite difference approximations for the equations in Chapter 5. Lastly we give a short overview of the Jump-Growth Equation derived by Datta et al. (2010). We show that the McKendrick-von Foerster Equation with diffusion is an approximation of the Jump-Growth Equation and focus on the simpler equation

given by this approximation.

Chapter 3 constructs the model equations that will be used in Chapter 5 by considering the derivation of the coefficients in the Jump-Growth Equation. In the second half of the chapter we consider the steady states of these equations which will be useful when analysing the solutions to the numerical approximations as we can consider how close to the steady states the solutions are.

Before performing the final calculations and discussion of approximating the Jump-Growth Equation and the McKendrick-von Foerster Equation in Chapter 5, we introduce the mathematical framework of finite difference equations in Chapter 4. We discuss the history of difference equations and their application to the simple model problem of the Advection Equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = du. \tag{1.3}$$

Finally we discuss how to handle the errors generated by approximating the equations derived in Chapter 3 by finite difference equations and how we aim to minimise this error so that the approximations converge correctly.

Chapter 2

Size Spectrum Theory

In this chapter we consider the models in McKendrick (1925) and Foerster (1959) who describe a system indexed by age. We consider the advances in Silvert and Platt (1978) who describe new equations indexed by weight and derive the standard McKendrick-von Foerster Equation for weight indexed systems.

This work serves as the basis for Datta et al. (2010), who construct a stochastic model for the dynamics of population which they call the "Deterministic Jump-Growth Equation". We consider the similarities between this equation and the McKendrick-von Foerster Equation.

2.1 Age Indexed Models

McKendrick (1925) first posed the idea of modelling biological processes for medicinal science by a single characteristic. In a process if individuals meet, they transfer information within the system and if we consider these individuals as particles in a system moving according to a dimension indexed by this single characteristic then their movement becomes a study in kinetics.

Foerster (1959) extended the principle equations derived in McKendrick (1925). Trucco (1965) gives a rigorous discussion of the advancements made in Foerster (1959) and further considers the steady state solutions of what he calls "The Von Foerster Equation".

2.1.1 The Von Foerster Equation

The Von Foerster Equation was a major extension to the work of McKendrick (1925). It defines the standard in age indexed population models and reads

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -m(a)n, \tag{2.1}$$

for a mortality function dependent on age $m(a)$. We derive Equation 2.1 following the method discussed in Trucco (1965) based upon the discussion in Foerster (1959). Suppose that $n(t, a)$ represents the density of individuals at time t in the age category $[a, a + \Delta a)$, then

$$\begin{aligned} \frac{\partial}{\partial t}\left(n(a,t)\Delta a\right) \quad = \quad & + \text{rate of entry of } a \\ & - \text{rate of departure at } (a + \Delta a) \\ & - \text{deaths in } [a, a + \Delta a). \end{aligned} \tag{2.2}$$

We can express Equation 2.2 in mathematical terms for some flux, $J(t, a)$, which describes the rate of movement of individuals in $[a, a + \Delta a)$ as

$$\frac{\partial n}{\partial t} = \frac{J(t,a) - J(t, a + \Delta a)}{\Delta a} - M \cdot n(a, t), \tag{2.3}$$

for some per capita mortality rate $M([a, a + \Delta a))$. We take the flux, J, to define the movements of individuals within the system. As individuals become older the flux can be assumed to be proportional to the density of individuals with some velocity $v(t, a)$. If the ageing corresponds to the passing of time then

$$v(t, a) = \frac{\partial a}{\partial t} = 1$$

and so $J(t, a) = n(t, a)$. Substituting this into Equation 2.3 it is clear that in the limit as $\Delta a \to 0$, Equation 2.3 tends to Equation 2.1.

2.2 The Transport Equation

After considering Equation 2.1 we consider a more general form of equation. Transport Equations, or convection-diffusion equations, take the general form

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) - \nabla \cdot (\vec{v} u) + R. \tag{2.4}$$

They describe particles undergoing diffusion and convection. In one dimension Equation 2.4

reduces to

$$\frac{\partial u}{\partial t} = -vu_x + R + (Du_x)_x. \tag{2.5}$$

Comparing the coefficient definitions from Stocker (2011) to the ideas from the derivation of Equation 2.1 we take u as the quantity of interest (population or population density), then we can interpret each term in Equation 2.5.

The first term $-vu_x$ describes the convection (movement due to mass) in the system. v describes the velocity particles in the system are moving at. This is analogous to the growth of individuals in the population, the rate at which they grow and move through the weight range.

The second term R describes the creation or destruction of u. In Equation 2.1 this becomes the death (naturally or predatorily) of the population.

For the diffusion term $(Du_x)_x$, imagine that c is the concentration of a chemical. When concentration is low somewhere compared to the surrounding areas (e.g. a local minimum of concentration), the substance will diffuse in from the surroundings, so the concentration will increase. The reverse case applies, also. This is analogous to phenomena that are exhibited in approximations the Jump-Growth Equation which we talk about in the next section.

2.3 Weight Indexed Models

2.3.1 McKendrick-von Foerster Equation

Silvert and Platt (1978) introduced a more general construction of the McKendrick-von Foerster Equation, notwithstanding a change from age indexed populations to sized based populations, in a model which allowed growth and mortality to be functions of body mass. Their changes are widely used in mathematical biology and a full derivation can be found in Silvert and Platt (1978). However a simple argument is that the flux described in Equation 2.3 is changed for a flux that depends on the growth of individuals

$$J(t, w) = g(t, w)n(t, w) \tag{2.6}$$

and $m(a)$ becomes a mortality function in weight rather than age $\mu(w)$. Thus the McKendrick-von Foerster Equation reads

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial w}(g \cdot n) = -\mu \cdot n. \tag{2.7}$$

Considering this equation in the context of Section 2.2 we can see Equation 2.7 is a Transport Equation with no diffusion term. While a Transport Equation is defined on the region of $\mathbb{R}^+ \times \mathbb{R}$, simply requiring an initial condition, the Von Foerster Equation is only defined on $\mathbb{R}^+ \times \mathbb{R}^+$, thus requiring a left boundary condition along the line $(t, 0)$. Generally this boundary condition is defined as the births across the population which could simply be defined with

$$u(t, 0) = 0. \tag{2.8}$$

This would be an uninteresting problem since the system would only include growth and death, but no birth. Thus instead of Equation 2.8 we introduce a birth rate $b(w)$ and integrate over the population to give a boundary condition

$$u(t, 0) = \int_0^\infty b(w)u(t, w)\,\mathrm{d}t. \tag{2.9}$$

Numerical methods cannot handle the domain $[0, \infty]$ and thus we need to introduce a right hand boundary condition at some weight W. This will be dealt with in due course.

2.3.2 Jump-Growth Equation

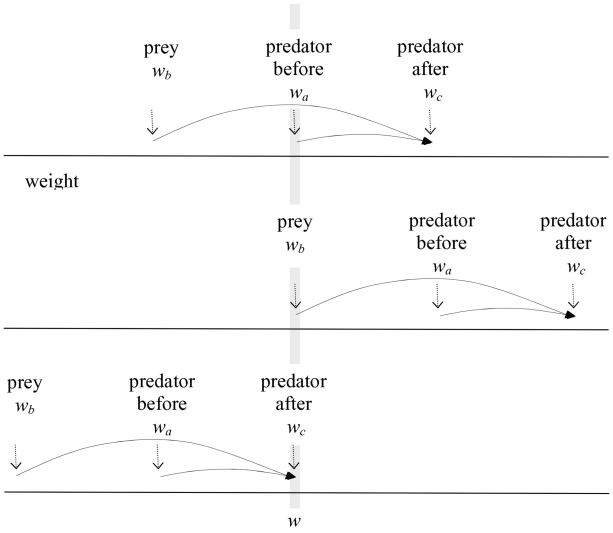
While the mathematical framework presented in Silvert and Platt (1978), Datta et al. (2010) presents a different method of describing weight indexed population under the assumption that predation is a Markov process. The predation event extends the ideas of Silvert and Platt (1980) where predation is modelled by coupling the death of an individual in one weight interval $[w, w + \Delta w)$ with the growth of an individual in a different weight interval $[w', w' + \Delta w)$. This is illustrated in Figure 2.1 where we see there are three types of predation to yield 'new' individuals.

Resolving the macroscopic behaviour of these predation events yields the deterministic equation

$$\frac{\partial \varphi_i}{\partial t} = \sum_j \left(-k_{ij}\varphi_i\varphi_j - k_{ji}\varphi_j\varphi_i + k_{mj}\varphi_m\varphi_j\right), \tag{2.10}$$

where m is the index satisfying the weight bracket $w_m \leq w_i - Kw_j < w_{m+1}$ and k_{xy} is the rate of predation events, indexed predator before prey. Datta et al. (2010) explains that these three terms correspond to the three types of predation event seen in Figure 2.1: losses from bracket i

Figure 2.1: The death of an individual with weight w_b is attributed to the growth of another individual with weight w_a who dies to yield a new individual with weight $w_c = w_a + Kw_b$ for some predation efficiency K. Taken from from Datta et al. (2010).



(negative terms) occur because individuals in this bracket eat prey and become heavier and because these individuals are themselves eaten. Gains into weight bracket i (the positive term) occur through smaller predators growing into this bracket by eating prey.

We can create an analytic equation for use in calculations by taking the continuum limit of Equation 2.10 and using a continuous feeding rate function $k(w, w')$ to replace the rate constants k_{xy}. Thus Equation 2.10 becomes

$$\begin{aligned}\frac{\partial \phi(w)}{\partial t} = \int (& - k(w', w)\phi(w)\phi(w') \\ & - k(w, w')\phi(w')\phi(w) \\ & + k(w - Kw', w')\phi(w - Kw')\phi(w')) \, \mathrm{d}w'. \end{aligned} \tag{2.11}$$

Datta et al. (2010) defines Equation 2.11 as the "Deterministic Jump-Growth Equation" and again the three terms represent the methods of transferring weight within the system through predation corresponding to the methods in Figure 2.1: Predation on prey to become larger, being predated and being fed the correct weight to increase.

2.3.3 McKendrick-von Foerster Equation with Diffusion

The McKendrick-von Foerster Equation can be shown to be an approximation to the Jump-Growth Equation, but only to first order. A second order approximation can be yielded by

including a diffusion term. If we consider the third term in the Jump-Growth Equation as a Taylor Series, then

$$\begin{aligned} k(w - Kw', w')\phi(w - Kw') =& k(w, w')\varphi(w) \\ &+ (-Kw')\frac{\partial}{\partial w}\left(k(w, w')\varphi(w')\varphi(w)\right) \\ &+ \frac{1}{2}(-Kw')^2\frac{\partial^2}{\partial w^2}\left(k(w, w')\varphi(w)\right) + \ldots \end{aligned} \tag{2.12}$$

Once this is substituted back into Equation 2.11

$$\begin{aligned} \frac{\partial\phi(w)}{\partial t} =& \int -k(w', w)\phi(w)\phi(w')\,\mathrm{d}w' \\ &- \frac{\partial}{\partial w}\int Kw'k(w, w')\varphi(w)\varphi(w')\,\mathrm{d}w' \\ &+ \frac{1}{2}\frac{\partial^2}{\partial w^2}\int (Kw')^2k(w, w')\varphi(w)\varphi(w')\,\mathrm{d}w' + \ldots \end{aligned} \tag{2.13}$$

which can be seen as an approximation to the McKendrick-von Foerster Equation, with a diffusion term added. Thus for an appropriate choice of coefficients we can restrict our attention to equations in a Transport Equation form

$$\frac{\partial u}{\partial t} = -(g \cdot u)_w - \mu \cdot u + \frac{1}{2}(Du)_{xx}. \tag{2.14}$$

2.4 Summary

In this chapter we have introduced the mathematical framework that will be used to model populations and shown that the macroscopic stochastic Jump-Growth equation is comparable to the Transport Equation form of the McKendrick-von Foerster Equation. In Chapter 3 we introduce a numerical construction of the coefficients giving rise to the true models that will be solved in Chapter 5.

Chapter 3

Model Equations

In this chapter we build on Chapter 2 to construct a fully derived equation to model in Chapter 5. We begin by considering numerical interpretations of "feeding kernels" and prey selection, then construct the various predation coefficients for Equation 2.13.

3.1 Feeding Kernels

Central to the models we are studying is the idea of a feeding kernel which is the rate at which a predator eats prey of a different size. While there are many interpretations of feeding kernels, we use an interpretation centralised around the idea that 'big fish eat smaller fish'. We assume that the likelihood a predator of weight w feeds upon prey with weight w' is based upon both these weights.

Based upon this key idea Benoıt and Rochet (2004) showed that the feeding kernel for an individual takes the form

$$k(w, w') = sw^{\alpha} S\left(\frac{w}{w'}\right) \tag{3.1}$$

for some parameters s, α to be determined and for the size selection function s. This states the rate at which a predator of weight w feeds is a product of the volumetric search rate sw^{α} and the feeding preference function S. In this thesis we make the assumption that $s = 1$ for convenience.

3.1.1 Prey Selection

As discussed the prey selection will peak around some fixed ratio w/w' for predator / prey, which we call β. Assuming there is some spread for which individuals will vary from this ratio, taken as σ, we can say the prey selection function S will take the form

$$S(z) = \kappa \exp\left(-\frac{(z-\beta)^2}{2\sigma^2}\right) \tag{3.2}$$

for some κ that we discuss later.

3.1.2 Coefficient Functions

When we consider the Transport Equation form of the McKendrick-von Foerster Equation we can extract coefficients. Taking Equation 2.13 in the form of Equation 2.14 it reads

$$\begin{aligned}\frac{\partial u}{\partial t} = &-\left(\left(\int (Kw')k(w,w')u(w')\,\mathrm{d}w'\right)\cdot u\right)_w \\ &-\left(\int k(w',w)u(w')\,\mathrm{d}w'\right)\cdot u \\ &+\frac{1}{2}\left(\left(\int (Kw')^2 k(w,w')u(w')\,\mathrm{d}w'\right)\cdot u\right)_{xx}.\end{aligned} \tag{3.3}$$

Consider the growth term in Equation 3.3,

$$g(w) = \int (Kw')k(w,w')u(w'), \tag{3.4}$$

we can think of this as the expected value of the mass eaten by individuals of weight w. Thus the prey selection function can be thought of as a probability distribution and so we take

$$\kappa = \frac{1}{\sigma\sqrt{2\pi}}.$$

If we take Equation 3.4 and let $v = g$, and define

$$R = \mu(w) = \int k(w',w)u(w')\,\mathrm{d}w', \tag{3.5}$$

$$D(w) = \int (Kw')^2 k(w,w')w(w')\,\mathrm{d}w' \tag{3.6}$$

we have Equation 2.13 in the form of Equation 2.5.

3.2 Steady State Solutions

In marine ecosystems it is been found that the abundance of organisms within weight classes is roughly constant (Sheldon et al. (1972)) if these weight classes are distributed logarithmically. Averaged over time, Datta et al. (2011) observes that this abundance changes little, suggesting the system is near steady state.

Benoıt and Rochet (2004) found the McKendrick-von Foerster Equation has a power law steady state of the form $\phi(w) \propto w^{\gamma}$, for $\gamma \in \mathbb{R}^{-}$. We note that in the real world such a power law steady state must break down since it predicts an infinite number of individuals with weight $w = 0$.

Combining this research to show that Equation 3.3 can be transformed using the logarithmic change of variables and appropriate change of function to have a constant solution as Benoıt and Rochet (2004) discuss.

3.2.1 Logarithmic Scale

Sheldon and Parsons (1967), who first introduced the idea of a size spectrum to organise counts of particles in the ocean, stipulates the fundamental conjecture that the total mass within logarithmically spaced size groups was constant over the size range from bacteria to whales.

Interpreting this conjecture biologically Silvert and Platt (1980) poses that the total mass of prey is the same as the mass of their predators, but can be written in terms of the number spectrum, where the number of individuals with size m per volume is described by a density function $N(w) \propto w^{-\gamma}$. This was confirmed later by Benoıt and Rochet (2004).

Since these conjectures assume a logarithmic weight scale, we perform a change of variables from Equation 3.3 using the dimensionless variable $x = \log w$ assuming that $\phi(x)\mathrm{d}x = u(w)\mathrm{d}w$. Following Equation 2.14 we have

$$u_t = -(g \cdot u)_w - \mu \cdot u + \frac{1}{2}(D \cdot u)_{ww}. \tag{3.7}$$

After changing variables using a logarithmic weight scale we gain a new equation for ϕ reading

$$\phi_t = -(\hat{g} \cdot \phi)_x - \hat{\mu} \cdot \phi + \frac{\mathrm{e}^{-x}}{2}\left(\left(\hat{D} \cdot \phi\right)_{xx} - \left(\hat{D} \cdot \phi\right)_x\right). \tag{3.8}$$

However, if we are to maintain a constant steady state as stipulated by Sheldon and Parsons (1967), we must perform a simple change of variables. The steady state $u(w) \propto w^{\gamma}$ reduces to $\phi(x) \propto \mathrm{e}^{x(\gamma+1)}$, which is clearly not constant. If we reduce to a constant by taking $\mathrm{e}^{\gamma x}\,\rho(x) = \phi(x)$ we have our constant, but ρ is governed by the more complicated equation

$$\mathrm{e}^{\gamma x}\,\rho_t = -(\bar{g}\cdot\rho)_x - \bar{\mu}\,\mathrm{e}^{\gamma x}\,\rho + \frac{\mathrm{e}^{-x}}{2}\left(\left(\bar{D}\cdot\rho\right)_{xx} - \left(\bar{D}\cdot\rho\right)_{x}\right), \tag{3.9}$$

for transformed coefficients $\overline{g}, \overline{\mu}, \overline{D}$.

3.2.2 Constant Steady State

Following the change of variables $\mathrm{e}^{\gamma x}\,\rho(x)\mathrm{d}x = u(w)\mathrm{d}w$ we find that the coefficients of Equation 3.9 read

$$\overline{g}(x) = \int K\,\mathrm{e}^{x'+(\alpha+\gamma-1)x}\,S(\mathrm{e}^{x-x'})\,\mathrm{e}^{\gamma x'}\,\rho(x')\,\mathrm{d}x' \tag{3.10}$$

$$\overline{D}(x) = \int K^2\,\mathrm{e}^{2x'+(\alpha+\gamma-1)x}\,S(\mathrm{e}^{x-x'})\,\mathrm{e}^{\gamma x'}\,\rho(x')\,\mathrm{d}x' \tag{3.11}$$

$$\overline{\mu}(x) = \int \mathrm{e}^{\alpha x'}\,S(\mathrm{e}^{x'-x})\,\mathrm{e}^{\gamma x'}\,\rho(x')\,\mathrm{d}x'. \tag{3.12}$$

Taking the same assumption as Datta et al. (2011) that $\alpha+\gamma-1=0$ and substituting $\rho(x) = C$ as the steady state, we can find an equation for α which will guarantee a constant steady state. Cancelling some factors and taking the convention that $r = x - w'$ the steady state condition for the McKendrick-von Foerster Equation with diffusion reads

$$0 = \int S(\mathrm{e}^{r})\left(-1 + \alpha K\,\mathrm{e}^{(\alpha+1)r} + + \alpha(\alpha+1)\frac{K}{2}\,\mathrm{e}^{(\alpha+2)r}\right)\,\mathrm{d}r. \tag{3.13}$$

If we do not include the diffusion term, we can ignore the K^2 terms in Equation 3.13.

3.3 Summary

In this chapter we have outlined the realised equations that will be used in Chapter 5. Next we will discuss some of the methods that we can use to approach these equations analytically and numerically.

Chapter 4

Mathematics of Discretization

The major benefit of studying equations which hold the form of a Transport equation is the large body of research of their solutions and behaviour. We begin this chapter by considering the most basic Transport Equation, the Advection Equation, which will allow the introduction of finite difference techniques.

Following this we discuss the theory of errors for finite difference equations and how these determine the convergence and stability of the solutions when difference approximations are used in the equations.

4.1 Advection Equation

The Advection Equation is a simple hyperbolic partial differential equation. In one dimension we take

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = du, \tag{4.1}$$

for some transfer velocity v and destruction rate d on the domain $\mathbb{R}^+ \times R$ for some $R \subseteq \mathbb{R}$. The linear advection has been studied extensively and we can gain some insight into potential issues that might occur in more complicated equations such as Equation 3.8.

4.1.1 Analytic Solution

As with ordinary differential equations, partial differential equations can be solved with the "method of characteristics" (or method of lines). Consider Equation 4.1 on the domain $\mathbb{R}^+ \times \mathbb{R}^+$ so that when the correct boundary conditions are applied, the problem aligns with the

McKendrick-von Foerster Equation. Then we define some boundary value and initial value problem (BVP and IVP):

$$\begin{aligned} u_t + v\,u_x &= +d\,u \\ u(0,x) &= I(x) \\ u(t,0) &= B(t) \end{aligned} \tag{4.2}$$

Using the method of characteristics we consider a point in the domain $\{(t,x) : t,x > 0\}$ and solve the equation along some characteristic line $(t(s),x(s)) = y(s)$ stemming from an $y_0 \in \Gamma = \{(0,x) : x \in \mathbb{R}^+\} \cup \{(t,0) : t \in \mathbb{R}^+\}$. The full details of the method are left to the reader, or can be found in Chapter 4 of Holmes (2006). The solution of Equation 4.2 is defined by

$$u(t,x) = \begin{cases} I(x-vt)\,\mathrm{e}^{-d\,t} & x - vt > 0 \\ B(x-vt)\,\mathrm{e}^{-d\,t} & x - vt \leq 0 \end{cases} \tag{4.3}$$

along the characteristics $x - vt = C$. We note the McKendrick-von Foerster Equation will take a form similar to this if the growth and death terms are constants, but the problem will become more complicated if the coefficients are functions of t,x or non-linear. However this problem does allow us to introduce the idea of domain of dependence.

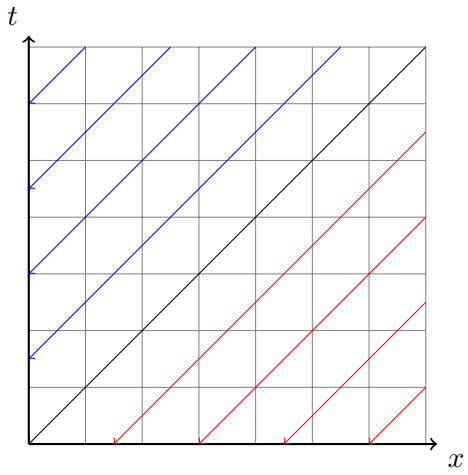


Figure 4.1: The characteristic lines for the solution of Equation 4.3 with constant coefficients. There is no overlap between the lines which means the solution can be smooth if the boundary conditions along Γ are smooth. The analytic domain of dependence traverses the direction of each arrowhead line.

Given any BVP and/or IVP on $U \subseteq \mathbb{R}^n$ with solution φ, the domain of dependence for any $x \in U$ is

the set $V \subseteq U$ that $\varphi(x)$ depends on to be calculated. Figure 4.1 shows the characteristic lines for Equation 4.3; the blue lines represent characteristics which depend on the time axis boundary condition. The red arrows depend on the spatial axis initial condition. Looking at the analytic solution to Equation 4.2 we see that the initial condition is carried along the characteristic.

While it is not obvious how this definition will be useful now, it will become extremely important in the analysis of stability of the numerical methods that we discuss in the Subsection 5.1.1.

4.1.2 Numerical Solutions to the Advection Equation

When constructing a numerical method for solving the Advection Equation we can consider writing the equation using limits.

$$\lim_{\Delta t \to 0} \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + v \lim_{\Delta x \to 0} \frac{u(t, x+\Delta x) - u(t,x)}{\Delta t} = du(t,x). \tag{4.4}$$

If we ignore the limits and substituted Newtonian infinitesimals $\Delta t, \Delta x$ we find

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + v \frac{u(t, x+\Delta x) - u(t,x)}{\Delta x} = du(t,x) \tag{4.5}$$

which can be rearranged to solve for $u(t+\Delta t, x)$. This yields a explicit disctrize algorithm for calculating u at a point forward in time. This method of constructing algorithms for solving discrete steps of the solution forms the fundamental idea for finite difference equations. These equations approximate a problem by considering a discrete mesh on the domain and then solving the equation using suitable difference approximations.

4.2 Difference Equations

In general, most partial differential equations do not have explicit analytic solutions. Consequently we must find accurate numerical approximations and methods for solving these problems. In this section we introduce the required mathematics for understanding the numerical methods used to solve problems in partial differential equation theory.

Boole and Moulton (1880) writes "Differential Calculus is occupied about the limits to which such ratios approach as the increments are indefinitely diminished." However "The Calculus of Finite Differences may be strictly defined as the science which is occupied about the ratios of the simultaneous increments of quantities mutually dependent." In simple terms, the calculus of finite differences is only concerned about ratios of infinitesimals, in line with the ideas of

Newtonian calculus.

In Equation 4.4 we broke down the limits in the derivitives by simply ignoring the limit, in this section we reconstruct this equation using difference equations derived from Boole and Moulton (1880) and Hildebrand (1987).

Definition 4.1 (First Forward Difference)**.** For a function $u : U \to \mathbb{R}$ the first forwards difference of u is

$$\Delta_f[u](x) = u(x + \delta x) - u(x). \tag{4.6}$$

Definition 4.2 (First Backwards Difference)**.** For a function $u : U \to \mathbb{R}$ the first backwards difference of u is

$$\Delta_b[u](x) = u(x) - u(x - \delta x). \tag{4.7}$$

These two definitions form the basis for finite difference theory. We can take any difference $\Delta[u](x)$ as a combination of forward and backward differences. Just as with the differential operator, the difference operator is linear and satisfies the Leibniz rule. The proof of this is left to the reader, but follows exactly the same method as for derivatives. It is easy to construct the n^{th} difference ($\Delta^n[f](x)$) by simply considering $\Delta[\Delta^{n-1}[f]](x)$.

When we consider the partial derivative of $u : \mathbb{R} \to \mathbb{R}$ we have that

$$u'(x) = \lim_{\delta x \to 0} \frac{\Delta[u](x)}{\delta x} \tag{4.8}$$

which can be extended to higher dimensions.

4.2.1 Taylor's Theorem, Big and Little O Notation

When calculating an equation we can replace terms by their approximate sizes if the exact size is not required. In such cases we generally replace terms by their big (or little) O size. If $f, g : \mathbb{R} \to \mathbb{R}$, the informal assertions that $f(x)$ is big-O (or little-o) of $g(x)$ can be defined rigorously by comparing their sizes in limits around a point.

Definition 4.3 (Big O)**.** We define

$$f(x) = O(g(x)) \tag{4.9}$$

as $x \to a$ if and only if

$$\limsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty. \tag{4.10}$$

Definition 4.4 (Little O)**.** We define

$$f(x) = o(g(x)) \tag{4.11}$$

as $x \to a$ if and only if

$$\lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| = 0. \tag{4.12}$$

Intuitively calling $f(x)$ big-O of $g(x)$ states that $f(x)$ is "about as big" as $g(x)$, while if $f(x)$ is little-o of $g(x)$, then it is "much smaller" in size than $g(x)$. A very classical theorem in calculus is Taylor's theorem which is used to construct finite difference approximations.

Theorem 4.1 (Taylor)**.** If $u : \mathbb{R} \to \mathbb{R}$ is k times differentiable in a region around $x \in \mathbb{R}$ then $u(x + \delta x)$ can be written as

$$u(x + \delta x) = u(x) + (\delta x)\, u'(x) + \ldots + \frac{(\delta x)^k}{k!} u^{(k)}(x) + h_k(x + \delta x)(\delta x)^k \tag{4.13}$$

for some function $h_k(x)$ such that $\lim_{\delta x \to 0} h_k(x + \delta x) = 0$.

If Equation 4.13 is reformulated to give a polynomial estimation $P(x)$ for $u(x)$ then the approximation error for a Taylor polynomial $R_k(x) = u(x) - P_k(x)$ can be estimated as $o((\delta x)^k)$. Using this theorem we know that if u is a differentiable function then by taking $k = 1$

$$u(x + \delta x) = u(x) + \delta x u'(x) + o(\delta x) \tag{4.14}$$

and so we can take the difference equation

$$\frac{\Delta[u](x)}{\delta x} = u(x) + O(\delta x) \tag{4.15}$$

an approximation to a first derivative of u. Given a difference equation $d_k(x)$ that depends on $k = \delta x$ that represents the left hand side of Equation 4.15 that approximates an n^{th} derivative of u we define the order of the approximation to be the integer N that satisfies

$$d_k(x) - u^{(n)}(x) = O(k^N). \tag{4.16}$$

4.3 Finite Difference Equation Errors

Since finite difference equations have become a staple in numerical analysis there are many texts that supply the different coefficients required to approximate derivatives accurately. By combining n^{th} forward and backward differences we can construct higher order approximations to derivatives at higher orders and higher orders of accuracy. Fornberg (1988) provides a detailed derivation as well as a set of tables for almost all derivatives required for general partial differential equation problems up to high orders in both accuracy and order. We define a finite difference scheme as a particular choice of approximation for a collection of derivatives.

However, as with all numerical problems, issues arise when constructing approximations due to errors that occur with the generation of those approximations themselves. In this section we begin to outline the two main types of error that occur in approximations and introduce the idea of consistency and convergence.

4.3.1 Notation

Before we move forward to discuss the rigorous theory of finite difference equations, their errors and use to approximate partial differential equations, we introduce some general theory. As we saw in Chapter 2 we deal with equations with the form

$$L(u) = \frac{\partial u(t,x)}{\partial t} - \mathcal{L}(u(t,x)), \tag{4.17}$$

where $\mathcal{L}$ is an m times differential operator in x (with most cases satisfying $m < 3$). Given this we have that if u is in the set of real valued functions, C^m, which are one times differentiable in t and m times differentiable in x on some region in the plane $\Omega = \mathbb{R}^+ \times U$ for some $U \subset \mathbb{R}^+$ then $L : C^M \to C$. We aim to solve the partial differential equation $L(u) = 0$ for specified boundary conditions on $\partial\Omega$.

To do this we first discretise our domain into a mesh grid. In Figure 4.2 we discretise an area of the plane $[0,T] \times [0,X]$ into a grid spaced with $\delta x = k$ and $\delta t = h$. We index the time axis by n and the spatial axis by j, thus for any function $u : [0,T] \times [0,X] \to \mathbb{R}$ we write

$$u(t,x) = u(nh, jk) = u_j^n. \tag{4.18}$$

With this numerical realisation of the domain for a partial differential equation, and a choice of finite difference approximation, we construct a finite difference equation D_j^n which takes a C^m

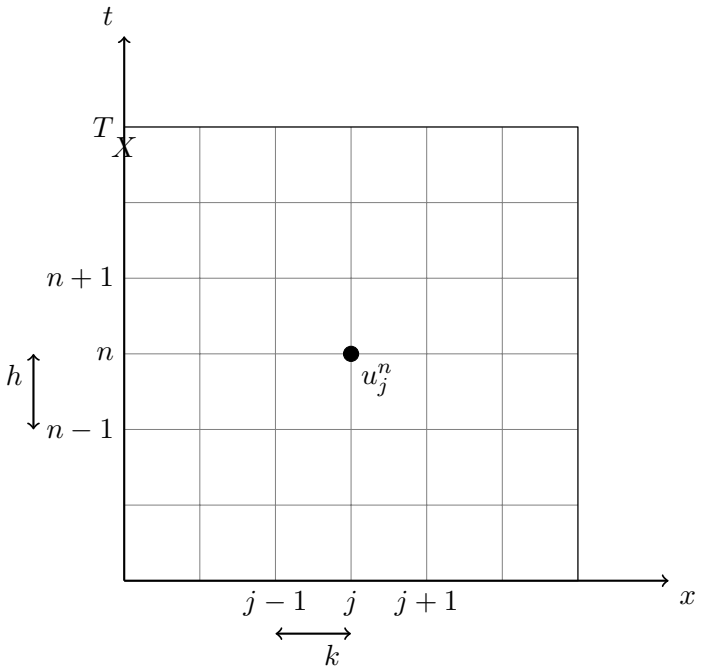


Figure 4.2: An example mesh-grid for the region $[0, T] \times [0, X]$ with an illustration of a grid on the central region of the domain.

function, u, and returns a combination of u_i^m for relative values of m and i to n and j.

Example 4.1. Consider the Advection Equation with a first order approximation for the time and spatial derivatives. Then our finite difference equation states that

$$D_j^n(u) = \frac{1}{h} u_j^{n+1} + \left(\frac{v}{k}\right) u_{j+1}^n + \left(\frac{-1}{h} - \frac{v}{k}\right) u_j^n. \tag{4.19}$$

4.3.2 Consistency

When considering how appropriate a choice of finite difference equation is to model a particular partial differential equation there are three issues that may arise.

The first of these we introduce is consistency, its broad definition is that a finite difference equation D is consistent with a partial differential equation L if the truncation error of D from L tends to 0 for smaller mesh grid approximations.

To introduce formally consistency consider a partial differential equation in the form of Equation 4.17 with a true solution Φ. Suppose we take any finite difference equation D_j^n to approximate Equation 4.17 and let ϕ be a non-zero solution to $D(u) = 0$.

Definition 4.5 (Truncation Error)**.** For any problem described above, the truncation error for a function $\psi : \Omega \to \mathbb{R}$ at any grid point (n, j) is given as

$$T_j^n(\psi) = D_j^n(\psi) - L_j^n(\psi) \tag{4.20}$$

where L_j^n represents $L(u)$ evaluated at $(t, x) = (nh, jk)$.

While we can consider the global truncation of a finite difference approximation by considering D acting upon arbitrary functions v, if $v = \Phi$ is the true solution of L and is known, then the local truncation error τ_j^n is defined as $T_j^n(\Phi)$. If $\tau_j^n = O(h^p)$, then D is said to be an approximation order p method. In reality most finite difference equations are of order p for some $p \in \mathbb{N}$. However this is not always the case and this is why we rely on consistency in a numerical calculation. T_j^n gives an estimate of the error of replacing $L_j^n(v)$ by $D_j^n(v)$. This gives arise to a formal definition of consistency.

Definition 4.6 (Consistency). A finite difference approximation D_j^n is consistent to a partial differential equation L described above. If v is any function, with a sufficient number of continuous derivatives such that $L_j^n(v)$ can be evaluated, D_j^n is considered consistent with L if

$$\lim_{k,h\to 0} T_j^n(v) = \lim_{k,h\to 0} D_j^n(v) - L_j^n(v) = 0. \tag{4.21}$$

If $v = \Phi$, then $T_j^n(v) = \tau_j^n$, so another definition of consistency is the limiting value of the local truncation error is 0. Suppose we review Equation 4.5 as an approximation to Equation 4.1, having engineered the scheme using Fornberg (1988) seen in Equation 4.19, we know that rearranging to gain an equation in the form $D(u) = 0$ the truncation error can be written as

$$T(v) = D(v) - L(v) = O(h) + vO(k) - dO(1) = O(h) + O(k) \tag{4.22}$$

and thus as $h, k \to 0$ then $T(v) \to 0$ and we have a consistent difference equation of order $p = 1$.

4.3.3 Stability

Numerical stability theory stems directly from analytical stability theory. Suppose that $L(u)$ is a partial differential equation problem with two analytic solutions Ψ_1 and Ψ_2. We can analyse how likely a perturbation in the initial condition is to affect the system as it tends to one of the two analytic solutions. This principle applies to numerical methods. The essential idea defining stability is that the numerical process should not cause any small perturbations introduced through rounding at any stage to grow and ultimately dominate the solution, similar to the study of whether a small perturbation introduced into an analytic solution will cause the solution to

tend to a dominant solution.

For mathematical analysis a definition of stability is given with relation to the growth of the exact solution to the finite difference equation. Therefore, if rounding errors of perturbations to the solution are introduced at any stage in time, these will be bounded if the exact solution is bounded.

Definition 4.7 (Numerical Stability). A numerical method is stable if, for a fixed spatial domain $[0, X]$, for every $T > 0$ there exists a constant $c(T) > 0$ such that

$$||\boldsymbol{\phi}^n||_k < c(T) \tag{4.23}$$

for all $n = 0, 1, .., T/h$ and $h \to 0$, $k \to 0$.

The norm $||\cdot||_k$ is a grid dependent Euclidean Norm for the space, for example

$$||\mathbf{u}||_k = \left(k \sum |u_j|^2\right)^{1/2},$$

acting on the vector $\boldsymbol{\phi}^n = (\phi_j^n : j = 0, .., X/k) \in \mathbb{R}^{(X/k)+1}$.

Eigenvalue Method

If a finite difference approximation can be written in matrix form $u^{n+1} = S^{n+1}u^0$, where u^n represents a vector of the values of u_j^n for all j and S^{n+1} is an invertible matrix which consolidates the finite difference approximation. Then we can determine the stability of that approximation by considering the eigenvalues of the matrix S. If all the eigenvalues, λ_S, of S satisfy $|\lambda_S| \leq 1$, then we have that

$$|S^n z| < \infty \tag{4.24}$$

as $n \to \infty$ for all vectors z. Thus, if we consider the eigenvalues for this matrix, we can determine the stability of the system.

Example 4.2. Suppose that we consider the approximation for the Advection Equation seen in Equation 4.19, this can be written with S in the tridiagonal form

$$\begin{pmatrix} a & b & & & \\ c & a & b & & \\ & c & \ddots & \ddots & \\ & & \ddots & a & b \\ & & & c & a \end{pmatrix} \tag{4.25}$$

and thus has eigenvalues $\lambda_j = a + 2\sqrt{bc}\cos\left(\frac{j\pi}{J}\right)$ for $j = 1, ..., J-1$ if there are J points in the mesh grid along the x axis. Considering the worst case for this eigenvalue spectrum will yield a relation between h and k that must be satisfied for the method to be stable.

This way of analysing the stability of a scheme is not easily generalised since it involves finding the eigenvalues of the corresponding S-matrix. Therefore we look at a different way of determining stability - the Fourier method or von Neumann method.

Fourier Method

The Fourier method is one of the most common methods for analyzing the stability of a finite difference equation. If the spatial differential operator $\mathcal{L}$, in L, is linear then in the limit as $h, k \to 0$ the numerical error for a finite difference equation also satisfies L since

$$L(\phi - \Phi) = (\phi - \Phi)_t - \mathcal{L}(\phi - \Phi) = (\phi_t - \mathcal{L}(\phi)) - (\Phi_t - \mathcal{L}(\Phi)).$$

We assume that the scheme admits a solution in the form

$$v_j^n = \lambda^n(\omega)\, \mathrm{e}^{ij\omega\Delta x} \tag{4.26}$$

and define

$$G(\omega) = \frac{\lambda^{n+1}(\omega)}{\lambda^n(\omega)} \tag{4.27}$$

to give the amplification factor which governs the growth of the solution at each time step. The von Neumann stability condition is given by

$$|G(w)| \leq 1 \tag{4.28}$$

for all $0 \leq \omega \Delta x \leq \pi$. If a finite difference equation is stable then it is said to be conditionally stable if there is dependence on $G(\omega)$ for the stability condition to hold, otherwise the scheme is unconditionally stable. Whilst the Fourier method is the most widely used but can break down for non-linear methods since it can be more difficult to solve for $G(\omega)$.

4.4 Convergence Theory

In Subsection 4.3.2 we defined the Truncation Error of a finite difference equation as the approximation error for a partial differential equation. In this section we deal with the numerical error.

Definition 4.8 (Numerical Error). Given a BVP/IVP problem with true solution $\Phi(t, x)$ and a finite difference equation $D(u)$ with true solution ϕ the numerical error of ϕ is defined as

$$e_j^n(u) = \phi_j^n - \Phi_j^n. \tag{4.29}$$

Broadly speaking, we define convergence of a problem in a simple way. Does our numerical solution accurately model the real solution? However the numerical error and true solution, and thus convergence, of a general problem is quite difficult to calculate since it relies on the knowledge of the solution. Thus more theory is needed to deal with this issue. Before we move onto this theory we give the rigorous definition of convergence.

Definition 4.9 (Convergence). Given a BVP/IVP problem with true solution $\Phi(t, x)$ and a finite difference equation $D(u)$, with $\Delta t = h$ and $\Delta x = k$, the solution D, ϕ, is said to converge to Φ if

$$\lim_{k \to 0} \left(\max_{n=0,1,\ldots,T/k} ||\boldsymbol{\phi}_j - \boldsymbol{\Phi}_j||_k \right) = 0 \tag{4.30}$$

for every initial condition and for every $T > 0$.

The most major result in the theory of linear finite difference equations is the Lax-Richtmyer Equivalence theorem.

Theorem 4.2 (Lax-Richtmyer). For a uniformly solvable linear finite difference scheme which is consistent with a well-posed linear evolution problem, stability is a necessary and sufficient condition for its convergence.

In plain terms, the theorem states that **Consistency + Stability implies convergence**. This is an extremely useful result which allows numerical mathematicians to ignore issues with finding

the true solution of a partial differential equation to calculate the convergence of finite difference equations.

The issue that arises from this theorem is that it only holds for linear problems. Thus in our research on Equation 3.8 with non-linear coefficients, the theorem will break down. Luckily Theorem 4.2 has been extended by the work of Rosinger (2008) who classes a more general set of partial differential equations using nonlinear semigroup theory. This allows us to continue with the problem for non-linear equations assuming that we can treat our coefficients in finite difference approximations for the McKendrick-von Foerster Equation with diffusion as if they were linear for stability and consistency calculations.

Chapter 5

Method

In this chapter we review the current methods for numerically solving partial-integro differential equations. We begin by considering the simplest model for the McKendrick-von Foerster Equation with constant coefficients and a simple first order approximation to solve it.

To further this we consider the work of Hartvig et al. (2011) who discuss a stable first order method for the McKendrick-von Foerster Equation when integral coefficients are used, but works equally well for constant coefficients. Finally, we discuss the boundary conditions and add the diffusion term to the problem.

5.1 First Order Upwind / Downwind Scheme

We first consider the McKendrick-von Foerster Equation with constant coefficients

$$L(u) = \frac{\partial u(t,w)}{\partial t} + g\frac{\partial u(t,w)}{\partial w} + \mu \cdot u(t,w) \tag{5.1}$$

on the discretised domain $[0,T] \times [0,W]$. By simply replacing the time derivative with a forward difference and the spatial derivative with a backwards difference, we yield a first order approximation

$$D_j^n(u) = \frac{u_j^{n+1} - u^n}{h} + g\frac{u_j^n - u_{j-1}^n}{k} + \mu u_j^n. \tag{5.2}$$

5.1.1 Courant Condition

At this point one might question the choice of the downwind scheme (backwards difference) as a choice for the spatial derivative. Ultimately this is for stability of the overall scheme. As discussed in Chapter 4 solutions to partial differential equations have an analytic domain of dependence. In Figure 4.1 we saw the domain of dependence for the Advection Equation; due to the underlying structure the McKendrick-von Foerster Equation with constant coefficients inherits it shares the same domain of dependence.

Courant et al. (1928) researched the idea of a Courant Number. In simple terms, the numerical domain of dependence (the region on which the numerical solution depends for information about its solution) must contain the analytic domain of dependence.

Formally Courant et al. (1928) stated that if the Courant Number

$$C = \frac{\upsilon \delta t}{\delta x} > C_{\text{max}}, \tag{5.3}$$

where υ is the largest magnitude of velocity that information travels in the system and C_{max} is a constant that depends on the approximations used (normally 1), then the finite difference approximation is unstable.

5.1.2 Stability of the Scheme

If we are to to solve the problem $L(u) = 0$ then we can rearrange D_j^n to solve for u_j^{n+1}, which gives a recursive formula for $\mathbf{u}^n$:

$$u_j^{n+1} = \left(1 + h \cdot \mu - \frac{gh}{k}\right) u_j^n + \frac{gh}{k} u_{j-1}^n. \tag{5.4}$$

If we let $\lambda = ghk^{-1}$ then the matrix problem form of Equation 5.4 is given by $\mathbf{u}^{n+1} = A\mathbf{u}^n$ where

$$A = \begin{pmatrix} 1 + h \cdot \mu - \lambda & 0 & & & \\ \lambda & 1 + h \cdot \mu - \lambda & 0 & & \\ & \lambda & \ddots & \ddots & \\ & & \ddots & 1 + h \cdot \mu - \lambda & 0 \\ & & & \lambda & 1 + h \cdot \mu - \lambda \end{pmatrix}. \tag{5.5}$$

Using the eigenvalue method we find the stability condition for this method is given by $k \leq g\mu^{-1}$.

5.1.3 Solution

To numerically solve the problems described in this chapter we will create an iPython notebook for each problem and produce a contour plot for each solution.

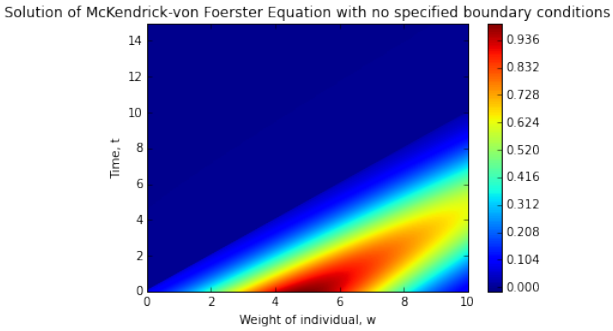


Figure 5.1: Numerical solution of the Advection Equation with left boundary conditions unspecified. As expected we see that the initial condition decays along characteristic lines of the form $w - gt = C$ for $C \in \mathbb{R}$.

Using Equation 5.4 as a model for iterating through each time step and using a Gaussian distribution as the initial condition we produce Figure 5.1. This shows a decreasing solution through time. However we note that above the line $w = gt$ the solution is zero, since there has been no specified left boundary condition. This is because the matrix A in Equation 5.5 does not appropriately deal with the derivatives in the boundary. On the left boundary for $w = 0$ we find that

$$u_0^{n+1} = (1 + h \cdot \mu - \lambda)\, u_0^n, \tag{5.6}$$

which is clearly not an accurate approximation for $L(u)$. While plotting the numerical solution for this would, without further inspection, show that the left boundary is decreasing exponentially, which can be seen in Figure 5.2.

To solve this issue and throughout the problems that follow, we imposed periodic boundary conditions on the domain so that

$$u(t, W_{\text{Low}}) = u(t, W_{\text{High}}) \tag{5.7}$$

for our choice of spatial domain $[W_{\text{Low}}, W_{\text{High}}]$. Biologically we can interpret this in terms of birth and death. On the boundary $w = 0$ the approximation considers the oldest individuals and the growth rate g to determine the number of individuals who are born and have a weight less than the lower boundary of the domain. By taking $W_{\text{low}} = k$ then we have that $u_{-1}^n = 0$ which is

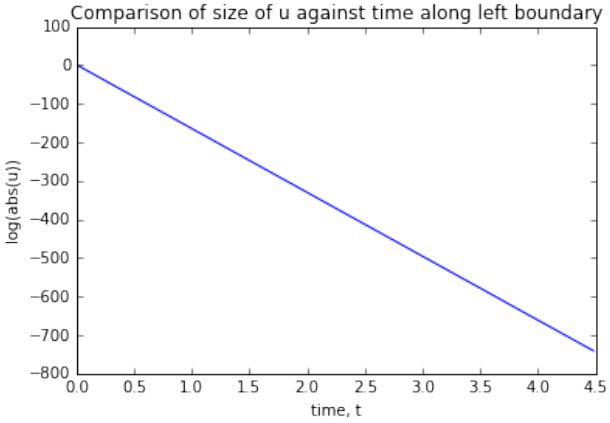


Figure 5.2: Logarithmic size of the solution to Figure 5.1 along the left boundary $w = 0$.

an appropriate birth weight. When we add this condition, the matrix A is rewritten to include the extra term in the top right and

$$A = \begin{pmatrix} 1 + h \cdot \mu - \lambda & 0 & & & & \lambda \\ \lambda & 1 + h \cdot \mu - \lambda & 0 & & & \\ & \lambda & \ddots & \ddots & & \\ & & \ddots & 1 + h \cdot \mu - \lambda & 0 & \\ & & & \lambda & 1 + h \cdot \mu - \lambda & \end{pmatrix}. \tag{5.8}$$

A simulation confirms that the boundary conditions works. We see the solution decaying further over time and continuing to move to the right through the spatial domain for weight.

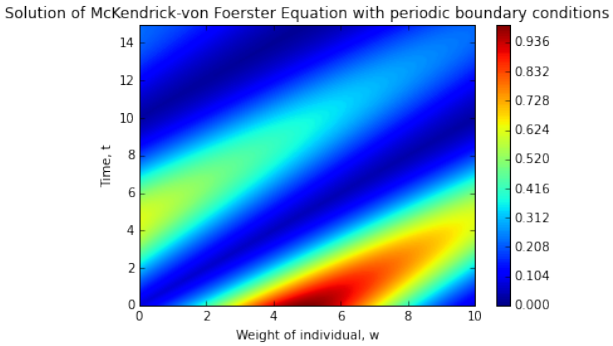
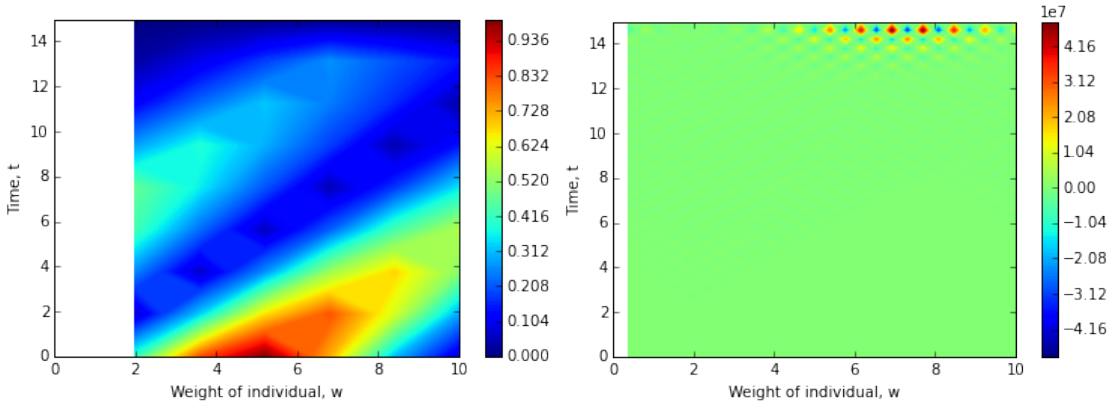


Figure 5.3: The numerical solution of the McKendrick-von Foerster Equation with periodic boundary conditions. The solution continues decaying from the left boundary $w = 0$ when the solution carried along any characteristic meets the right boundary.

5.2 Diffusion Term and Implicit Methods

Issues begin to arise when we consider the diffusion term within the upwind/downwind approximations.Figure 5.4 illustrates what happens for two different values of k when we add the diffusion term using a second order downwind approximation.



(a) Solution $k = 2$ (b) Solution $k = 0.4$

Figure 5.4: Comparison of the upwind/downwind scheme.

Even though the diffusion term should increase the analytic stability of the solution, we find that it increases the instability of the numerical approximations. To solve this we take a new approach, using implicit approximations instead of the explicit approximations.

Instead of finding an equation for u_j^{n+1} in terms of $u_i^n, i = ..., j-1, j, j+1, ...$ we aim to find an equation in the form $C \cdot \mathbf{u}^{n+1} = c \cdot \mathbf{u}^n$. If we begin to consider the McKendrick-von Foerster Equation with non-constant coefficients so that

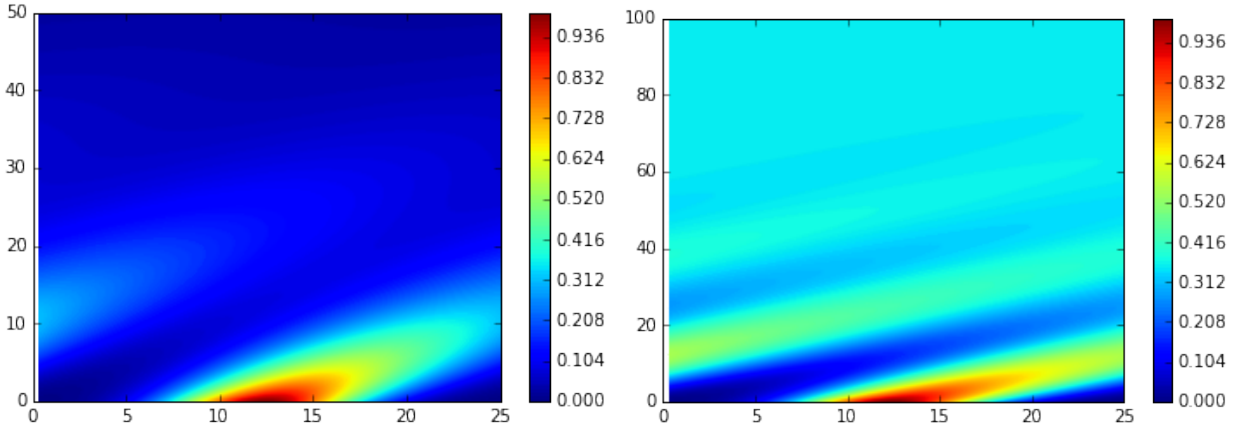
$$L(u) = u_t + (g(w) \cdot u)_w + \mu(w) \cdot u. \tag{5.9}$$

Using the approximation from Hartvig et al. (2011), which relies upon the semi-implicit derivative for products in Press (1992), we can construct a first order implicit approximation

$$D_j^n(u) = \frac{u_j^n - u_j^{n-1}}{h} + \frac{g_j^{n-1}u_j^n - g_{j-1}^{n-1}u_{j-1}^n}{k} + \mu_j^{n-1}u_j^n. \tag{5.10}$$

This method is stable under the assumption that g, μ are non-linear. However we note that the approximation for the derivative in weight is only first order. To look for a higher order method we consider the average of the forwards and backwards derivatives and add in the diffusion term to yield a finite difference equation

$$D_j^n(u) = \frac{u_j^n - u_j^{n-1}}{h} + \frac{g_{j+1}^{n-1}u_{j+1}^n - g_{j-1}^{n-1}u_{j-1}^n}{2k} + \mu_j^{n-1}u_j^n - \frac{D_{j+1}^{n-1}u_{j+1}^n - 2D_j^{n-1}u_j^n + D_{j-1}^{n-1}u_{j-1}^n}{k^2}. \quad (5.11)$$



(a) $\mu = 0.05$ (b) $\mu = 0$

Figure 5.5: Comparison between values of μ for the McKendrick-von Foerster Equation with diffusion and periodic boundary conditions.

While the solution shown in Figure 5.5 shows that the scheme is stable, it still diffuses to a stationary distribution of $u_{\text{stab}}(t, x) = 0$. However we note that the inclusion of the $\mu \cdot u$ term will kill the population over time, much like the solution of the McKendrick-von Foerster Equation without diffusion. Comparing this to the solution of the equation with $\mu = 0$ we see that the solution tends to a constant over time, closer to the solution that we aim to see in Chapter 3.

Chapter 6

Discussion

In this project we have discussed the historical context of the McKendrick-von Foerster Equation and the Jump-Growth Equation and the numerical methods which are used when approaching similar problems. However we have not tried to apply these techniques to the McKendrick-von Foerster Equation with integral coefficients.

6.1 Integral Coefficients

If we consider modelling the integral coefficients using the approximation in Equation 5.11 derived from the work of Hartvig et al. (2011), applying the same method used for the earlier results in Figure 5.5 with the implicit approximations from Equation 5.11, we can attempt to solve the equation numerically.

Unfortunately, as Figure 6.1 shows, the solution does not behave as expected using the same choice of parameters as Figure 2 Datta et al. (2011). While we do see the same decay as Figure 5.5, the growth and diffusion terms do not have as much effect on the equation and thus the solution does not shift through the domain as much as the constant coefficient solution.

6.2 Summary

While this project has not made direct advances on numerical methods for the various forms of the McKendrick-von Foerster Equation and its approximations to the Jump-Growth Equation, we have outlined a detailed framework for developing new methods from the existing methods and the work of Hartvig et al. (2011).

Problems in partial differential equation theory do have numerical methods, but the study of

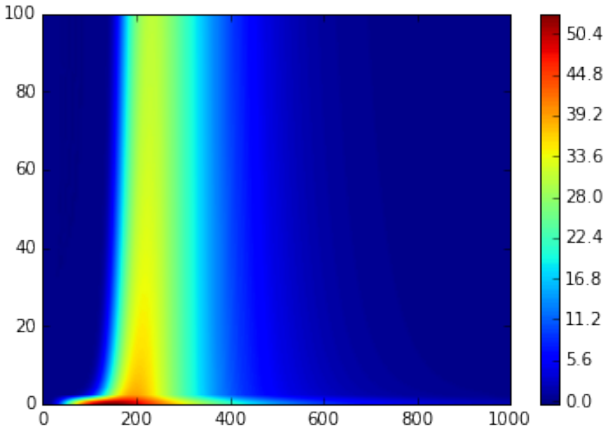


Figure 6.1: Numerical solution to the McKendrick-von Foerster Equation with integral coefficients. The solution diffuses slower than the McKendrick-von Foerster Equation with constant coefficients but still decays over time, though at a slower rate than the constant coefficient problem.

the non-linear equations in this project is still in its infancy because methods are not developed thoroughly. Much of the research and time spent constructing our attempt at solving the McKendrick-von Foerster Equation with diffusion and non-linear coefficients has come from studying the problem with constants instead of functions and experimentation with approximations which are stable for those methods.

Figure 5.4 and Figure 5.5 show that, while the diffusion term can greatly improve the stability of the solution to the equations, we will need to study the death term $\mu(w) \cdot u(t, w)$ if we are to avoid this dominating the derivative and making the finite difference equation converge to a $u(t, x) = 0$ steady state solution.

Going forward, research will need to focus on dealing with the approximation error and stability in the integral coefficients, since it is clear from Figure 6.1 that they are the root cause of the difficulty when studying the non-linear equations. As we saw in Chapter 4, there is a large body of knowledge already developed around the approximation error and stability of finite difference equations and the same principles developed for approximating differentials can be applied to calculating integrals.

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