Laboratory work 7
Development of forward-backward Kalman filter in conditions of correlated state noise

Performance -Thursday, April 21, 2016
Due to submit a performance report – Tuesday, April 26, 2016

The objective of this laboratory work is to develop optimal Kalman filter that takes into account correlated random acceleration to track a moving object. This problem is typical for many practical control and forecasting problems. This will bring about a deeper understanding of main difficulties of practical Kalman filter implementation and skills to overcome these difficulties to get optimal assimilation output. Additional important outcome of this exercise is experience in developing algorithms to improve Kalman filter estimates.

This laboratory work is performed in the class by students as in teams of 2 on April 21, 2016 and the team will submit one document reporting about the performance till Tuesday, April 26, 2016. Within your group, you may discuss all issues openly, and discuss and debate until you reach a consensus.

This laboratory work consists of two parts:

I. Development of optimal Kalman filter in conditions of correlated state noise
II. Development of optimal smoothing to increase the estimation accuracy

Here is the recommended procedure for part I:
Development of optimal Kalman filter in conditions of correlated state noise

1. Generate a true trajectory X_i of an object motion disturbed by a correlated in time random acceleration with variance σ_a^2 . Let's assume that this random acceleration is first-order Gauss-Markov process. It means that

$$a_i = e^{-\lambda T} a_{i-1} + \zeta_i \ (1)$$

ζ_i - uncorrelated random noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;
σ_a^2 – variance of correlated noise $\sigma_a^2 = 0.2^2$;
$T = 1$ – time interval between measurements.

λ – value that is inverse to correlation interval.
For example,
(a) if $\lambda = 1000$, then $a_i = \sigma_a \xi_i$ and is uncorrelated noise
(substitute 1000 to equation (1) for λ)
(b) if $\lambda = 0.1$, then a_i is correlated noise on interval over 10 steps. It means that inside every 10 steps correlation is significant.

Then true trajectory is generated using random acceleration a_i obtained according to equation (1). Use $\lambda = 0.1$ that means that a_i is correlated noise on interval over 10 steps.

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$

Size of trajectory is $N = 200$ points.

Initial conditions: $x_1 = 5; V_1 = 1;$

2. Generate measurements z_i of the coordinate x_i

$$z_i = x_i + \eta_i$$

η_i –normally distributed uncorrelated random noise with zero mathematical expectation and variance $\sigma_\eta^2 = 20^2$.

3. Preparations to develop optimal Kalman filter algorithm that takes into account correlated acceleration.

(a) In case of correlated random acceleration the motion of object is described by following equations

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$
$$a_i = e^{-\lambda T}a_{i-1} + \zeta_i$$

Here x_i – coordinate, V_i – velocity, a_i – correlated random acceleration.
Here $e^{-\lambda T}a_{i-1}$ depends on previous value of random acceleration a_{i-1}
ζ_i - uncorrelated random noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;
λ – value that is inverse to correlation interval;
T – time interval between measurements.

(b) To apply Kalman filter let's present the system at state space.
State vector in this case is presented as

$$X_i = \begin{vmatrix} x_i \\ V_i \\ a_i \end{vmatrix}$$

It means that state vector X_i is extended by inclusion of correlated random acceleration a_i. Therefore, Kalman filter algorithm will provide estimates not only of coordinate x_i and velocity V_i, but also acceleration a_i. If random acceleration is correlated, then it is characterized by a certain dynamics that should be estimated in contrast to white noise.

State equation is given by

$$X_i = \Phi X_{i-1} + G\zeta_i \ (2)$$

Determine transition matrix Φ and input matrix G for the motion given in (a)
$G\zeta_i$ - state noise;
ζ_i - uncorrelated noise with variance $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$;

Measurement equation is given by

$$z_i = \mathrm{H}X_i + \eta_i \ (3)$$

Determine the observation matrix H if only coordinate x_i is measured.

Check that determined matrices Φ, G, and H satisfy state-space model (2,3)

4. Initial conditions needed for Kalman filter algorithm
Initial filtered estimate

$$X_0 = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$$

Initial filtration error covariance matrix

$$P_{0,0} = \begin{vmatrix} 10000 & 0 \\ 0 & 10000 \end{vmatrix}$$

5. Obtain estimates of state vector $X_i = \begin{vmatrix} x_i \\ V_i \\ a_i \end{vmatrix}$ by Kalman filter over $M = 500$ runs and compare true estimation error with errors of estimation $P_{i.i}$ provided by Kalman filter algorithm.
   (a) error of filtered estimates of coordinate x_i;
   (b) error of filtered estimates of velocity V_i;
   (c) error of filtered estimates of acceleration a_i;

   Compare also errors of filtered estimates with errors of extrapolated estimates of x_i, V_i, a_i

Here is the recommended procedure for part II:
Development of optimal smoothing to increase the estimation accuracy

1. Reminder

Smoothing procedure is performed in backward in time and is applied to forward Kalman filter estimates. Smoothing takes into account both current and future measurements and therefore provides improved estimation compared to Kalman filter.

2. Develop backward smoothing algorithm to get improved estimates of state vector X_i

   Hint
   The recurrent algorithm of smoothing is presented on charts Topic_3_Optimal approximation at state space.pdf, page 52.

3. Make $M = 500$ runs of smoothing and compare true estimation error with errors of smoothing $P_{i.N}$ provided by smoothing algorithm.
   (d) error of smoothed estimates of coordinate x_i;
   (e) error of smoothed estimates of velocity V_i;
   (f) error of smoothed estimates of acceleration a_i;

4. Compare smoothing errors of estimation with filtration errors of estimation.

Performance report

1. Performance report should contain all the items listed

2. The code should be commented. It should include:
   - Title of the laboratory work, for example
     % Converting a physical distance to a grid distance using least-square method
   - The names of a team, indication of Skoltech, and date, for example,
     %Tatiana Podladchikova, Skoltech, 2016
     Main procedures also should be commented, for example
     %13-month running mean
     …here comes the code

3. If your report includes a plot, then it should contain: title, title of x axis, title of y axis, legend of lines on plot.