

## **Laboratory work 7**

### **Development of forward-backward Kalman filter in conditions of correlated state noise**

Performance -Thursday, April 21, 2016  
Due to submit a performance report – Tuesday, April 26, 2016

The objective of this laboratory work is to develop optimal Kalman filter that takes into account correlated random acceleration to track a moving object. This problem is typical for many practical control and forecasting problems. This will bring about a deeper understanding of main difficulties of practical Kalman filter implementation and skills to overcome these difficulties to get optimal assimilation output. Additional important outcome of this exercise is experience in developing algorithms to improve Kalman filter estimates.

This laboratory work is performed in the class by students as in teams of 2 on April 21, 2016 and the team will submit one document reporting about the performance till Tuesday, April 26, 2016. Within your group, you may discuss all issues openly, and discuss and debate until you reach a consensus.

This laboratory work consists of two parts:

- I. Development of optimal Kalman filter in conditions of correlated state noise
- II. Development of optimal smoothing to increase the estimation accuracy

***Here is the recommended procedure for part I:***

***Development of optimal Kalman filter in conditions of correlated state noise***

1. Generate a true trajectory  $X_i$  of an object motion disturbed by a correlated in time random acceleration with variance  $\sigma_a^2$ . Let's assume that this random acceleration is first-order Gauss-Markov process. It means that

$$a_i = e^{-\lambda T} a_{i-1} + \zeta_i \quad (1)$$

$\zeta_i$  - uncorrelated random noise with variance  $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$ ;

$\sigma_a^2$  – variance of correlated noise  $\sigma_a^2 = 0.2^2$ ;

$T = 1$  – time interval between measurements.

$\lambda$  – value that is inverse to correlation interval.

For example,

(a) if  $\lambda = 1000$ , then  $a_i = \sigma_a \xi_i$  and is uncorrelated noise  
(substitute 1000 to equation (1) for  $\lambda$ )

(b) if  $\lambda = 0.1$ , then  $a_i$  is correlated noise on interval over 10 steps. It means that inside every 10 steps correlation is significant.

Then true trajectory is generated using random acceleration  $a_i$  obtained according to equation (1). Use  $\lambda = 0.1$  that means that  $a_i$  is correlated noise on interval over 10 steps.

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$

Size of trajectory is  $N = 200$  points.

Initial conditions:  $x_1 = 5; V_1 = 1;$

2. Generate measurements  $z_i$  of the coordinate  $x_i$

$$z_i = x_i + \eta_i$$

$\eta_i$  –normally distributed uncorrelated random noise with zero mathematical expectation and variance  $\sigma_\eta^2 = 20^2$ .

3. Preparations to develop optimal Kalman filter algorithm that takes into account correlated acceleration.

- (a) In case of correlated random acceleration the motion of object is described by following equations

$$\begin{aligned}x_i &= x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2} \\V_i &= V_{i-1} + a_{i-1}T \\a_i &= e^{-\lambda T}a_{i-1} + \zeta_i\end{aligned}$$

Here  $x_i$  – coordinate,  $V_i$  – velocity,  $a_i$  – correlated random acceleration.

Here  $e^{-\lambda T}a_{i-1}$  depends on previous value of random acceleration  $a_{i-1}$

$\zeta_i$  - uncorrelated random noise with variance  $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$ ;

$\lambda$  – value that is inverse to correlation interval;

$T$  – time interval between measurements.

- (b) To apply Kalman filter let's present the system at state space.

State vector in this case is presented as

$$X_i = \begin{bmatrix} x_i \\ V_i \\ a_i \end{bmatrix}$$

It means that state vector  $X_i$  is extended by inclusion of correlated random acceleration  $a_i$ . Therefore, Kalman filter algorithm will provide estimates not only of coordinate  $x_i$  and velocity  $V_i$ , but also acceleration  $a_i$ . If random acceleration is correlated, then it is characterized by a certain dynamics that should be estimated in contrast to white noise.

**State equation** is given by

$$X_i = \Phi X_{i-1} + G\zeta_i \quad (2)$$

Determine transition matrix  $\Phi$  and input matrix  $G$  for the motion given in (a)

$G\zeta_i$  - state noise;

$\zeta_i$  - uncorrelated noise with variance  $\sigma_\zeta^2 = \sigma_a^2(1 - e^{-2\lambda T})$ ;

**Measurement equation** is given by

$$z_i = HX_i + \eta_i \quad (3)$$

Determine the observation matrix  $H$  if only coordinate  $x_i$  is measured.

Check that determined matrices  $\Phi, G$ , and  $H$  satisfy state-space model (2,3)

4. Initial conditions needed for Kalman filter algorithm

Initial filtered estimate

$$X_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Initial filtration error covariance matrix

$$P_{0,0} = \begin{vmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{vmatrix}$$

5. Obtain estimates of state vector  $X_i = \begin{bmatrix} x_i \\ V_i \\ a_i \end{bmatrix}$  by Kalman filter over  $M = 500$  runs and compare true estimation error with errors of estimation  $P_{i,i}$  provided by Kalman filter algorithm.
- (a) error of filtered estimates of coordinate  $x_i$ ;
  - (b) error of filtered estimates of velocity  $V_i$ ;
  - (c) error of filtered estimates of acceleration  $a_i$ ;

Compare also errors of filtered estimates with errors of extrapolated estimates of  $x_i, V_i, a_i$

***Here is the recommended procedure for part II:***

***Development of optimal smoothing to increase the estimation accuracy***

1. Reminder

Smoothing procedure is performed in backward in time and is applied to forward Kalman filter estimates. Smoothing takes into account both current and future measurements and therefore provides improved estimation compared to Kalman filter.

2. Develop backward smoothing algorithm to get improved estimates of state vector  $X_i$

*Hint*

The recurrent algorithm of smoothing is presented on charts Topic\_3\_Optimal approximation at state space.pdf, page 52.

3. Make  $M = 500$  runs of smoothing and compare true estimation error with errors of smoothing  $P_{i,N}$  provided by smoothing algorithm.
- (d) error of smoothed estimates of coordinate  $x_i$ ;
  - (e) error of smoothed estimates of velocity  $V_i$ ;
  - (f) error of smoothed estimates of acceleration  $a_i$ ;
4. Compare smoothing errors of estimation with filtration errors of estimation.

***Performance report***

1. Performance report should contain all the items listed
2. The code should be commented. It should include:
- Title of the laboratory work, for example  
% Converting a physical distance to a grid distance using least-square method
  - The names of a team, indication of Skoltech, and date, for example,  
%Tatiana Podladchikova, Skoltech, 2016  
Main procedures also should be commented, for example  
%13-month running mean  
...here comes the code
3. If your report includes a plot, then it should contain: title, title of x axis, title of y axis, legend of lines on plot.