Skolkovo Institute of Science and Technology

“Space Data Processing:
Making Sense of Experimental Data”

Topic 5
"Model construction at state space under uncertainty”
II. Extended Kalman filter for navigation and tracking

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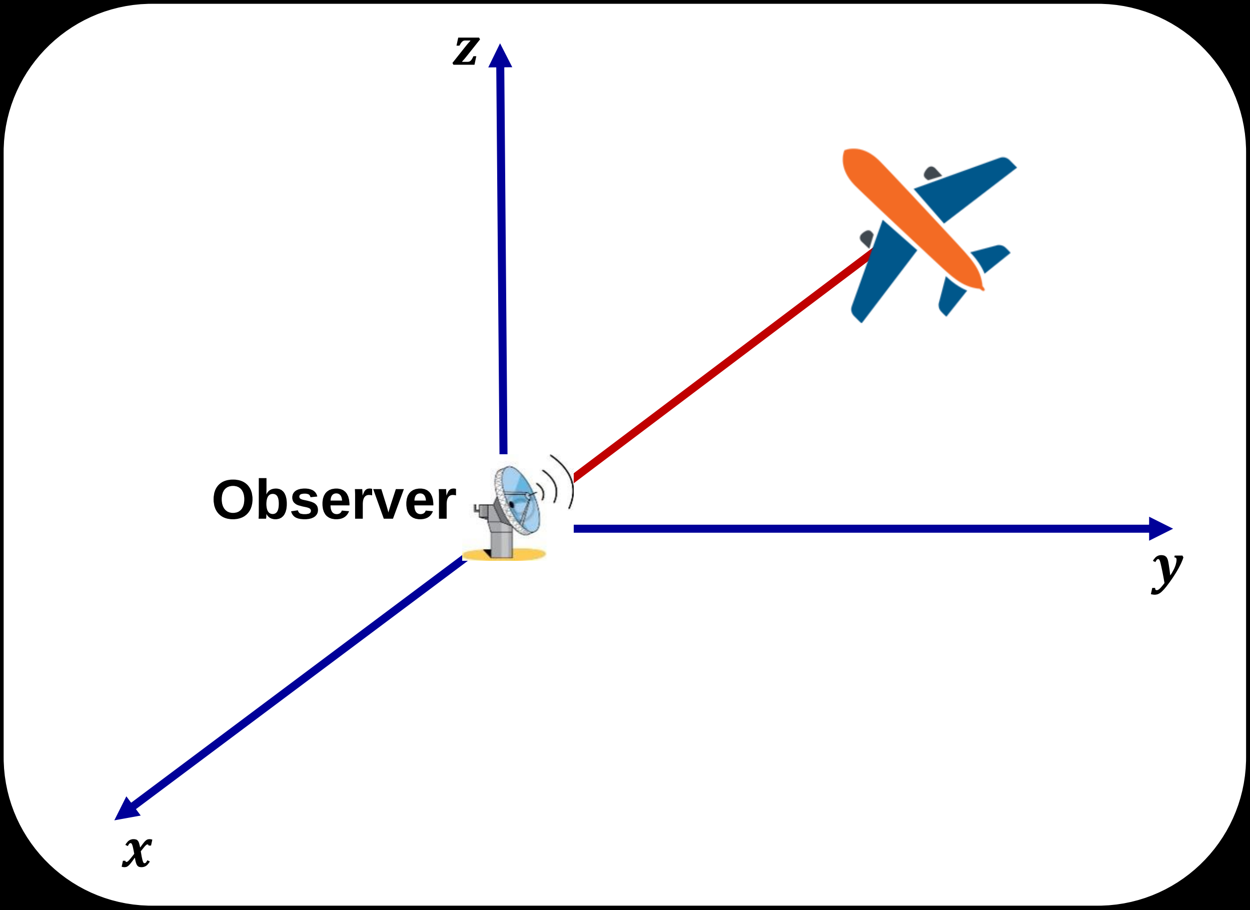
State of a moving object is characterized by state vector in Cartesian coordinate system

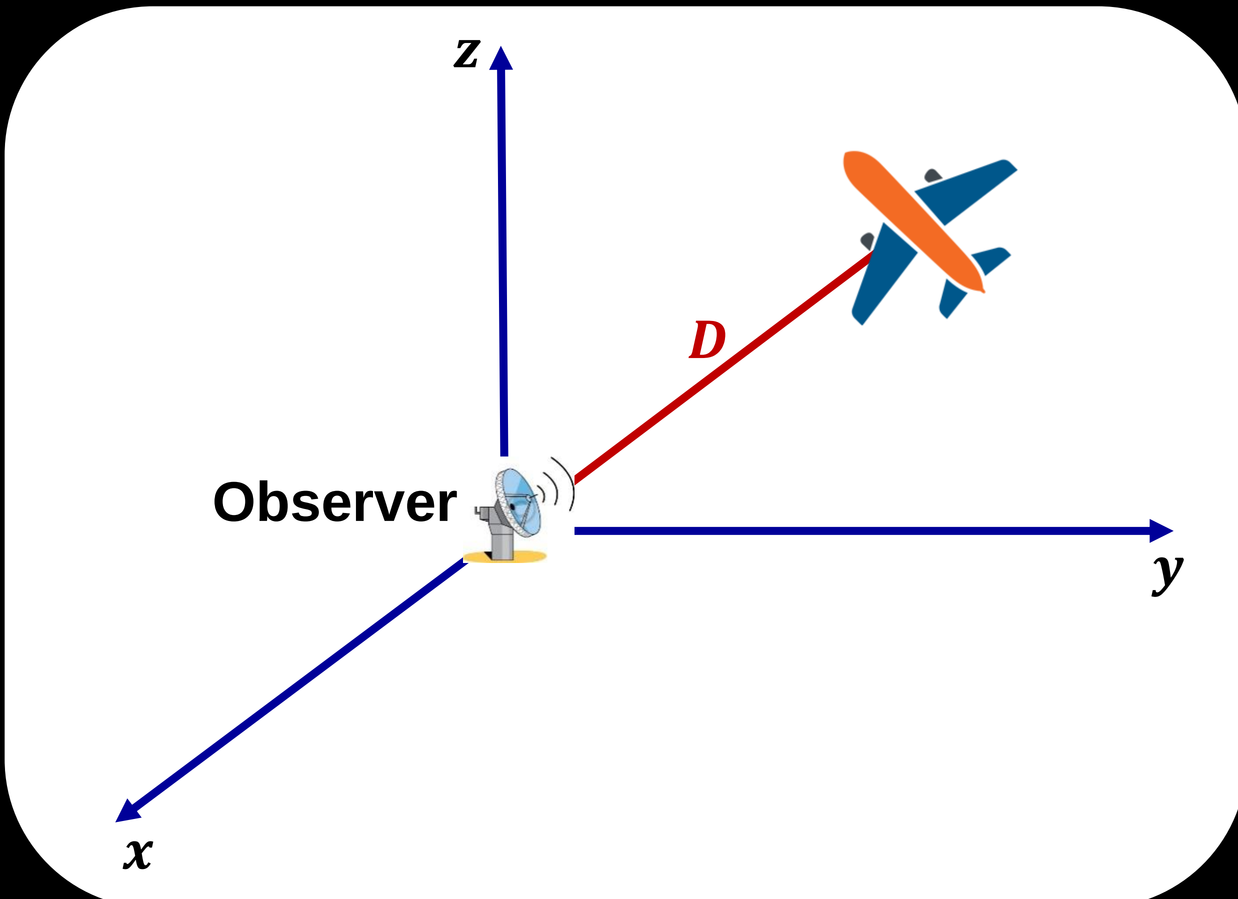


State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{vmatrix}$$

D

1. Estimation of coordinates using measurements of distance D, azimuth β, and angle of elevation ε

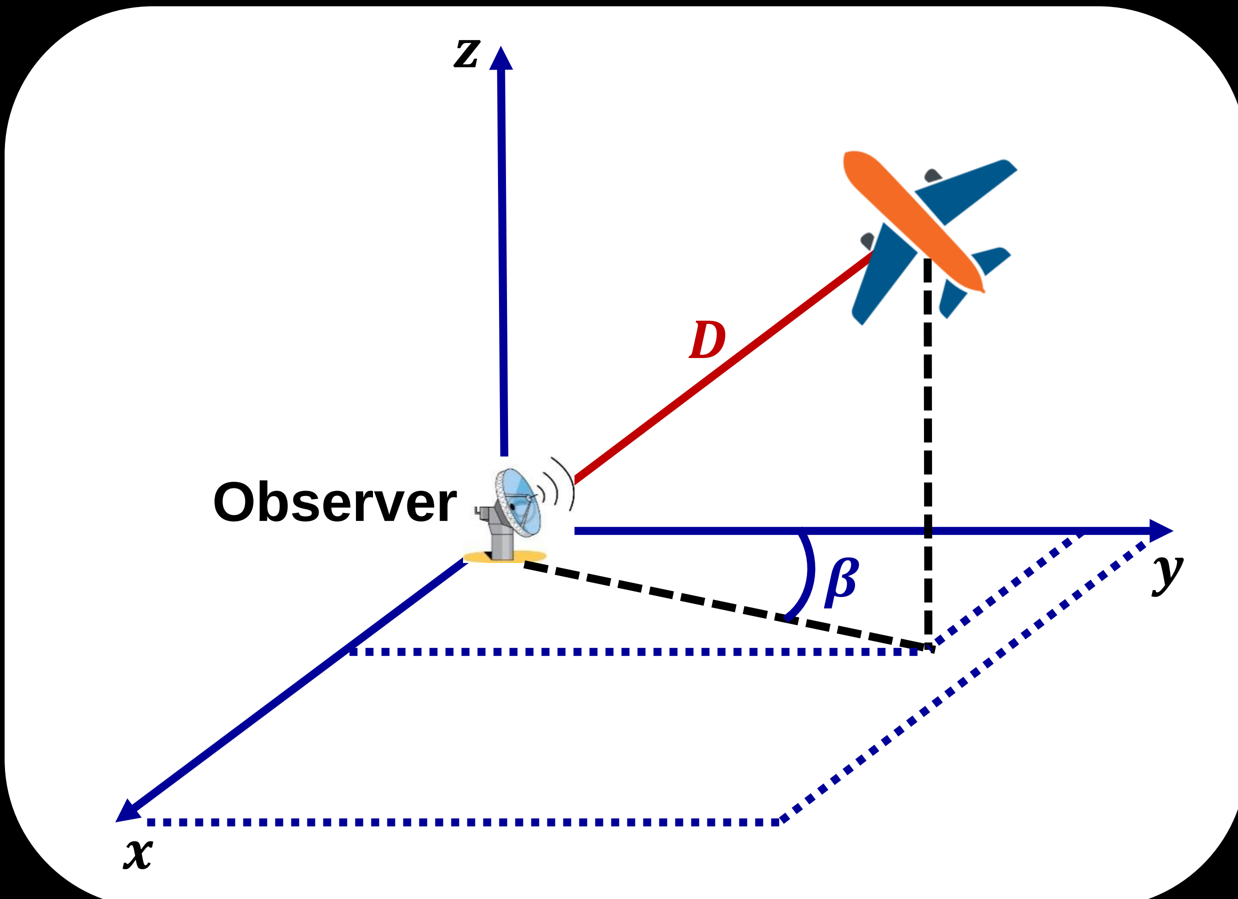


State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{vmatrix}$$

Range D → **Distance from an observer to a moving object** → $D_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$

1. Estimation of coordinates using measurements of distance D, azimuth β, and angle of elevation ε

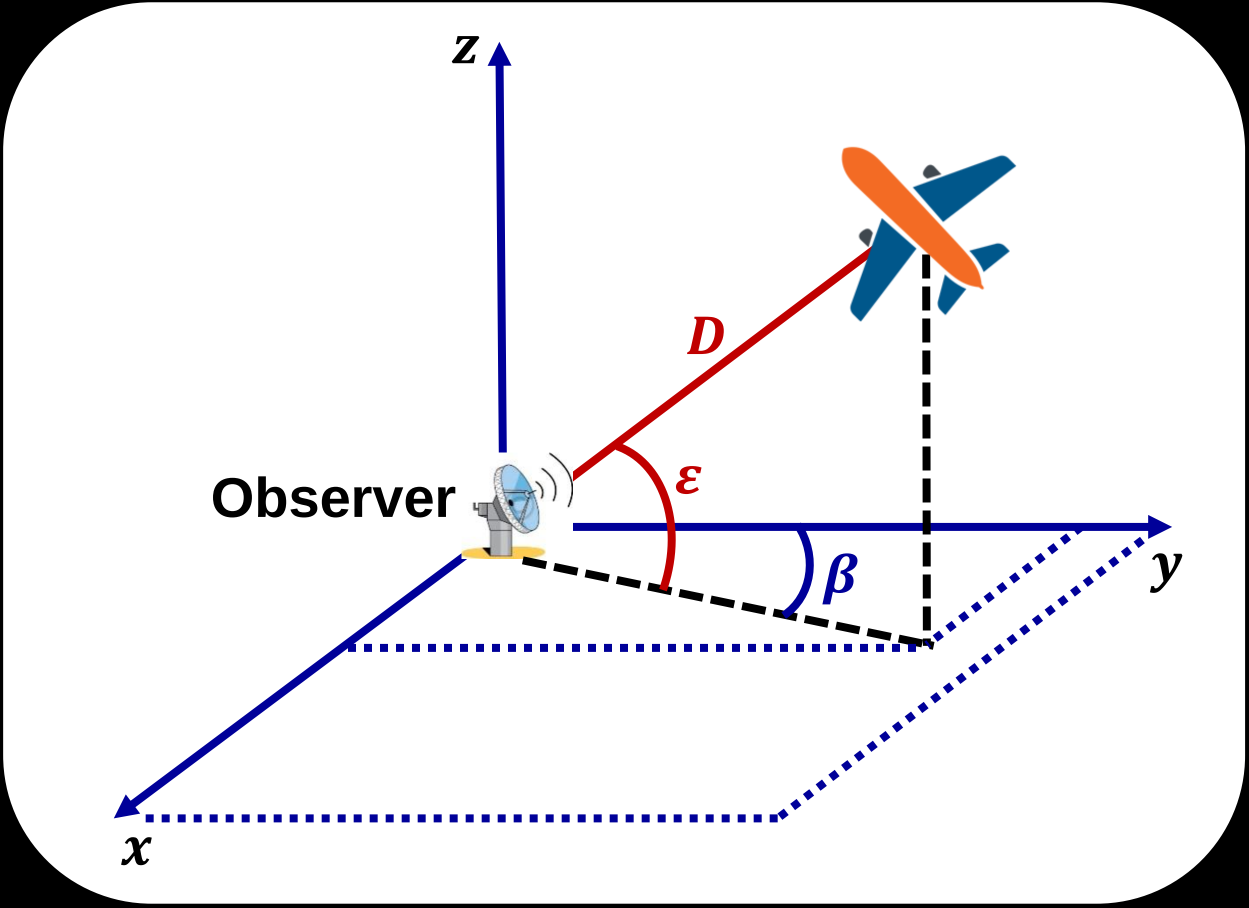


State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{vmatrix}$$

Azimuth β → Angle between direction of North and projection line in horizontal plane → $\beta_i = arctg\left(\frac{x_i}{y_i}\right)$

ε **1. Estimation of coordinates using measurements of distance D, azimuth β, and angle of elevation ε**



State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{vmatrix}$$

Angle of elevation ε → Angle between the horizontal plane and direction of an object →

$$\varepsilon_i = arcsin\left(\frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}}\right)$$

1. Estimation of coordinates using measurements of distance D, azimuth β, and angle of elevation ε

Measurement equation → $z_i = h(X_i) + \eta_i$

Measurement vector z_i

$$z_i = \begin{vmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{vmatrix}$$

1. Estimation of coordinates using measurements of distance D, azimuth β, and angle of elevation ε

Measurement equation → $z_i = h(X_i) + \eta_i$

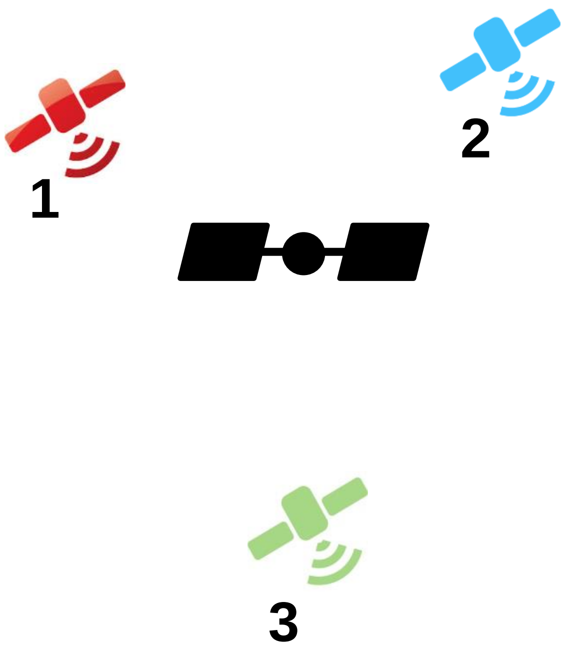
Measurement vector z_i

$$z_i = \begin{vmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{vmatrix}$$

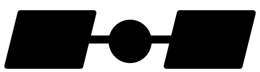
Nonlinear function $h(X_i)$

$$h(X_i) = \begin{vmatrix} \sqrt{x_i^2 + y_i^2 + z_i^2} \\ arctg\left(\frac{x_i}{y_i}\right) \\ arcsin\left(\frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}}\right) \end{vmatrix}$$

2. Estimation of coordinates using measurements only of distance D



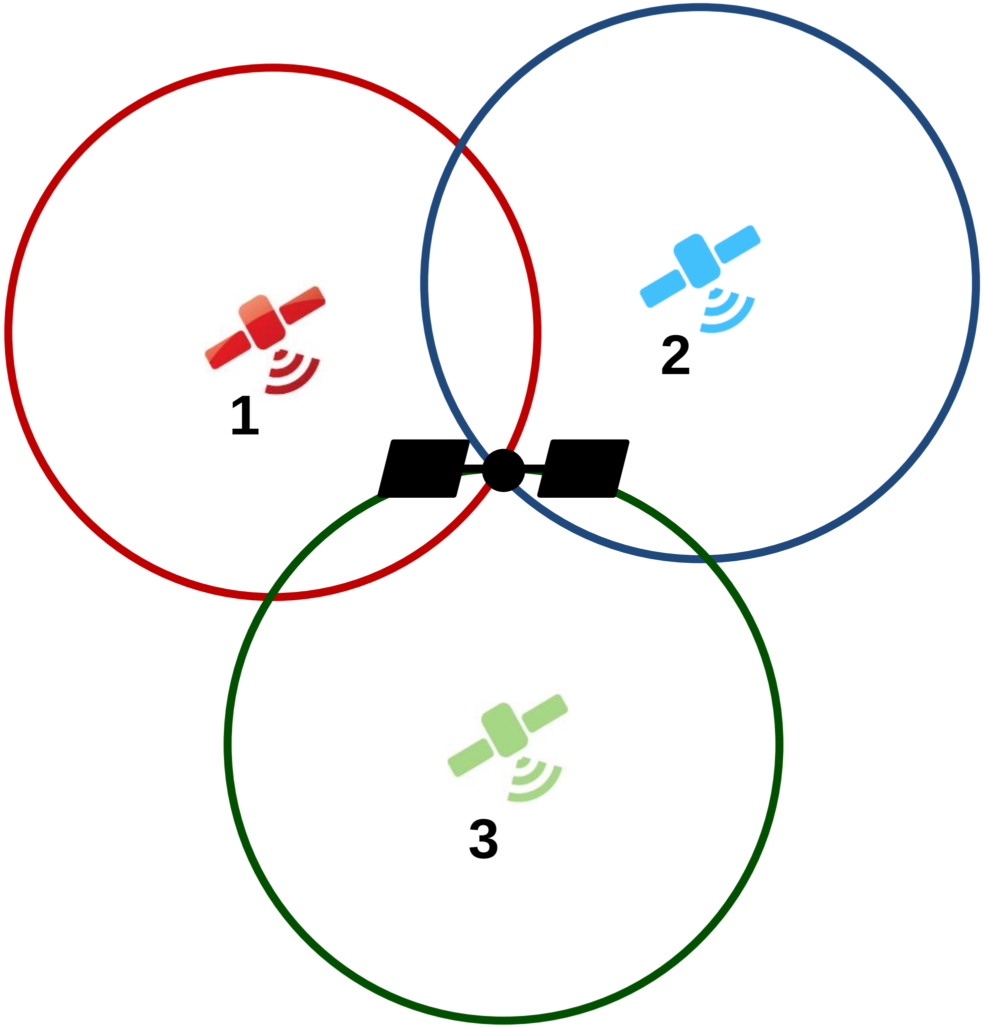
Three navigation stations measure distance D to a moving object



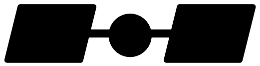
Moving object with unknown coordinates x, y, z

Navigation stations with known coordinates
(x_1, y_1, z_1); (x_2, y_2, z_2),
(x_3, y_3, z_3)

2. Estimation of coordinates using measurements only of distance D



The position of a satellite is at the intersecting points of circles

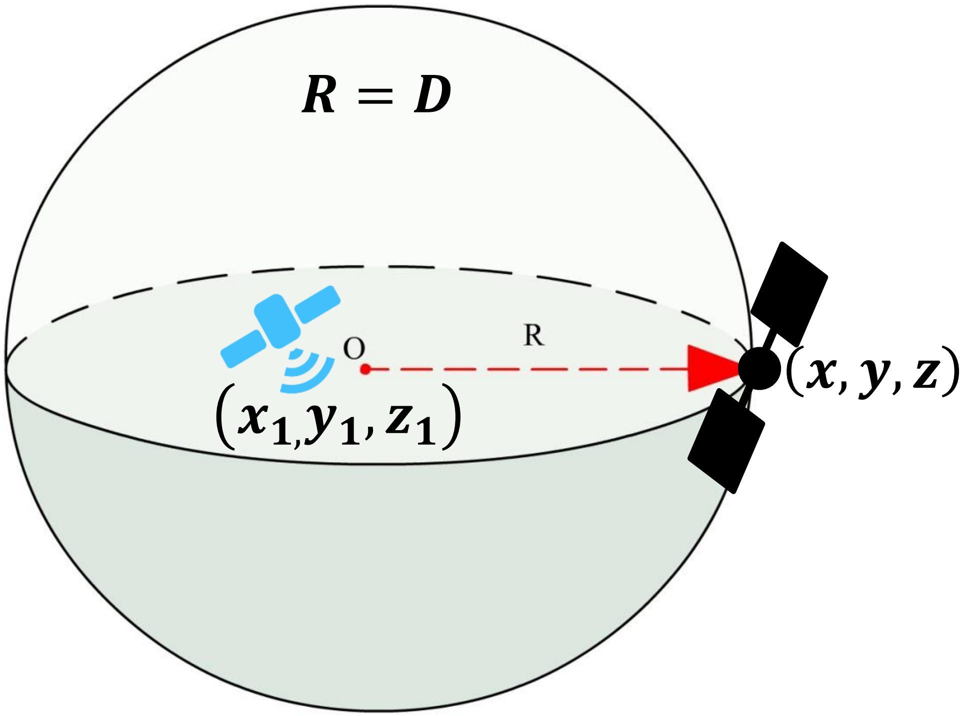


Moving object with unknown coordinates x, y, z

Navigation stations with known coordinates
(x_1, y_1, z_1); (x_2, y_2, z_2), (x_3, y_3, z_3)

2. Estimation of coordinates using measurements only of distance D

Equation of a 3-D sphere



Unknown coordinates x, y, z can be obtained by solving system of equations

$$D_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2}$$
$$D_2 = \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2}$$
$$D_3 = \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2}$$

2. Estimation of coordinates using measurements only of distance D

Measurement equation

$$z_i = h(X_i) + \eta_i$$

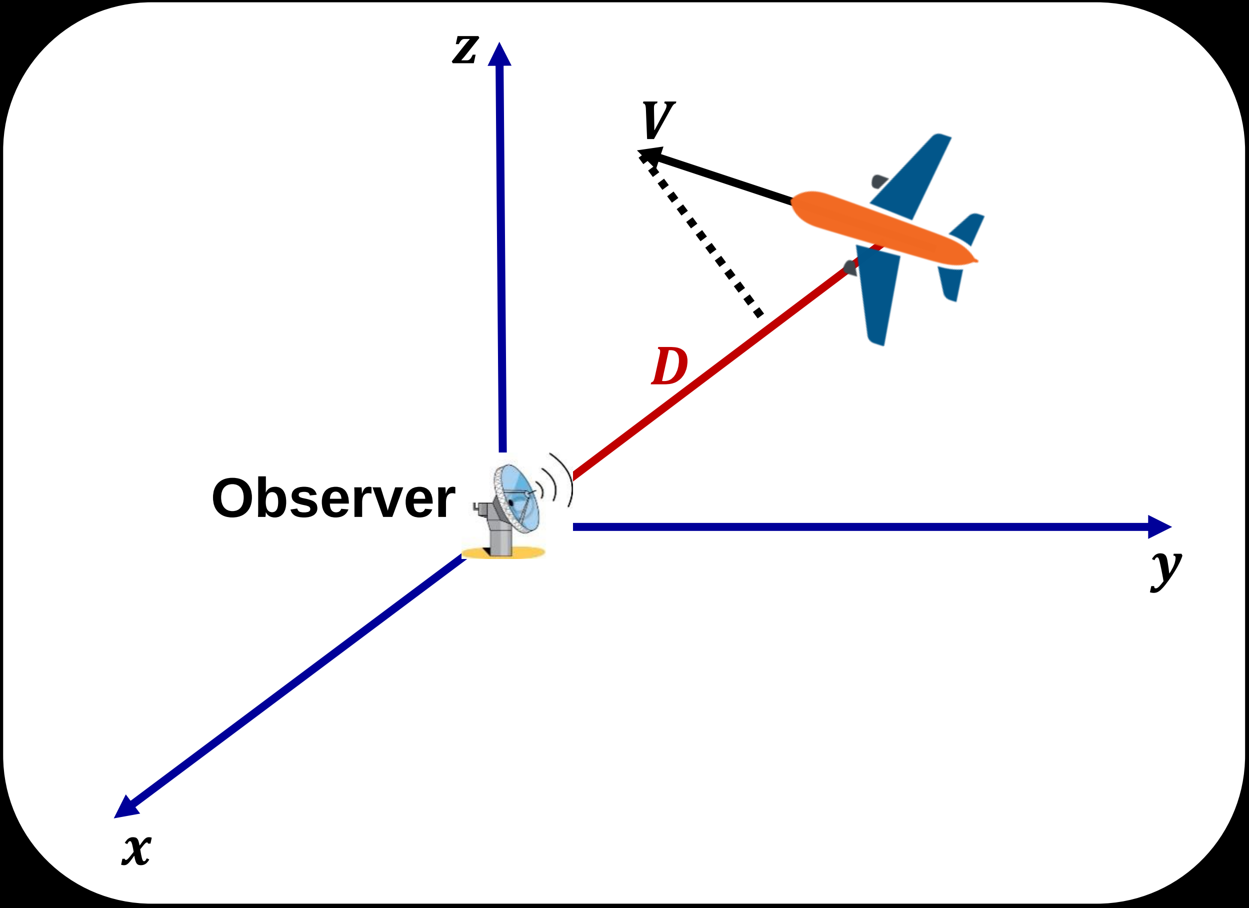
Measurement vector z_i

Nonlinear function $h(X_i)$

$$z_i = \begin{vmatrix} D_1^m \\ D_2^m \\ D_3^m \end{vmatrix}$$

$$h(X_i) = \begin{vmatrix} \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{vmatrix}$$

3. Estimation of velocity using measurements of Doppler velocity



State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{vmatrix}$$

Doppler velocity V_d → **Projection of V on vector D - radial component of V** → $$V_d = \frac{xV^x + yV^y + zV^z}{D}$$

3. Estimation of velocity using measurements of Doppler velocity

Measurement equation → $z_i = h(X_i) + \eta_i$

Measurement vector z_i

$$z_i = |V_d^m|$$

Nonlinear function $h(X_i)$

$$h(X_i) = \frac{xV^x + yV^y + zV^z}{\sqrt{x^2 + y^2 + z^2}}$$

4. Nonlinear model of a geostationary satellite orbit

System of differential equations in Celestial Reference System (CRS)

$$\begin{cases} \dot{r} = V \\ \dot{V} = -GM_o \dfrac{r}{|r|^3} + F(r, V, t) \end{cases}$$

$$r = \begin{vmatrix} x \\ y \\ z \end{vmatrix}, V = \begin{vmatrix} V^x \\ V^y \\ V^z \end{vmatrix}$$

Coordinates and components of velocity of a geostationary satellite in CRS

GM_o

Gravitational constant multiplied by the Earth's mass

$F(r, V, t)$

Disturbing acceleration related with:
-nonspherical gravity field of the Earth;
-gravitational disturbances of Moon and Sun;
-light pressure from Sun

Brumberg V.A. Essential relativistic celestial mechanics. – CRC Press, 1991

Extended Kalman filter

1 Nonlinear state space model →

- State equation → $X_{i+1} = f(X_i) + w_i$
- Measurement equation → $z_i = h(X_i) + \eta_i$

X_{i+1} → State vector

$f(X_i)$ → Transition function

$h(X_i)$ → Observation function

Extended Kalman filter

$\widehat{X}_{i,i}, \widehat{X}_{i+1,i}$ ⇒ **Filtered and predicted estimates at time i**

Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$ → **Filtered and predicted estimates at time i**

Let's produce Taylor series for
$f(X_i)$ and $h(X_i)$ around estimates $\hat{X}_{i,i}$ and $\hat{X}_{i+1,i}$

Extended Kalman filter

$\widehat{X}_{i,i}$, $\widehat{X}_{i+1,i}$ → **Filtered and predicted estimates at time i**

Let's produce Taylor series for
$f(X_i)$ and $h(X_i)$ around estimates $\widehat{X}_{i,i}$ and $\widehat{X}_{i+1,i}$

State equation →

$$f(X_i) \approx f\left(\widehat{X}_{i,i}\right) + \frac{df\left(\widehat{X}_{i,i}\right)}{dX_i}\left(X_i - \widehat{X}_{i,i}\right)$$

Measurement equation →

$$h(X_{i+1}) \approx h\left(\widehat{X}_{i+1,i}\right) + \frac{dh\left(\widehat{X}_{i+1.i}\right)}{dX_i}\left(X_{i+1} - \widehat{X}_{i+1,i}\right)$$

Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$ → Filtered and predicted estimates at time i

Let's produce Taylor series for
$f(X_i)$ and $h(X_i)$ around estimates $\hat{X}_{i,i}$ and $\hat{X}_{i+1,i}$

State equation →

$$f(X_i) \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i}(X_i - \hat{X}_{i,i})$$

Measurement equation →

$$h(X_{i+1}) \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1.i})}{dX_i}(X_{i+1} - \hat{X}_{i+1,i})$$

Let's substitute these expressions for
$f(X_i)$ and $h(X_{i+1})$ in state space model (1)

Extended Kalman filter

State equation

$$X_{i+1} \approx f\left(\hat{X}_{i,i}\right) + \frac{df\left(\hat{X}_{i,i}\right)}{dX_i}\left(X_i - \hat{X}_{i,i}\right) + w_i$$

Measurement equation

$$z_{i+1} \approx h\left(\hat{X}_{i+1,i}\right) + \frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}}\left(X_{i+1} - \hat{X}_{i+1,i}\right) + \eta_i$$

Extended Kalman filter

State equation

$$X_{i+1} \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i}(X_i - \hat{X}_{i,i}) + w_i$$

Measurement equation

$$z_{i+1} \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}}(X_{i+1} - \hat{X}_{i+1,i}) + \eta_i$$

State equation

$$X_{i+1} \approx \frac{df(\hat{X}_{i,i})}{dX_i}X_i + w_i + f(\hat{X}_{i,i}) - \frac{df(\hat{X}_{i,i})}{dX_i}\hat{X}_{i,i}$$

Measurement equation

$$z_{i+1} \approx \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}}X_{i+1} + \eta_i + h(\hat{X}_{i+1,i}) - \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}}\hat{X}_{i+1,i}$$

Unknown terms

Known terms

Extended Kalman filter

State equation

$$X_{i+1} \approx \frac{df\left(\hat{X}_{i,i}\right)}{dX_i} X_i + w_i + u_i$$

Measurement equation

$$z_{i+1} \approx \frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}} X_{i+1} + \eta_{i+1} + y_{i+1}$$

Known values

$$u_i = f\left(\hat{X}_{i,i}\right) - \frac{df\left(\hat{X}_{i,i}\right)}{dX_i} \hat{X}_{i,i}$$

$$y_{i+1} = h\left(\hat{X}_{i+1,i}\right) - \frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}} \hat{X}_{i+1,i}$$

Recurrent algorithm of Kalman filter

❶ Prediction (extrapolation)

Prediction of state vector at time i

$$\widehat{X}_{i+1,i} = \frac{df\left(\widehat{X}_{i,i}\right)}{dX_i}\widehat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df\left(\widehat{X}_{i,i}\right)}{dX_i}P_{i,i}\left(\frac{df\left(\widehat{X}_{i,i}\right)}{dX_i}\right)^T + Q_i$$

Recurrent algorithm of Kalman filter

❶ Prediction (extrapolation)

Prediction of state vector at time i

$$\hat{X}_{i+1,i} = \frac{df\left(\hat{X}_{i,i}\right)}{dX_i}\hat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df\left(\hat{X}_{i,i}\right)}{dX_i} P_{i,i} \left(\frac{df\left(\hat{X}_{i,i}\right)}{dX_i}\right)^T + Q_i$$

More accurate prediction from state equation

$$\hat{X}_{i+1,i} = f\left(\hat{X}_{i,i}\right)$$

Recurrent algorithm of Kalman filter

❷ Filtration
Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\widehat{X}_{i+1,i+1} = \widehat{X}_{i+1,i} + K_{i+1}\left(z_{i+1} - h(\widehat{X}_{i+1,i})\right)$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i}\left(\frac{dh(\widehat{X}_{i+1,i})}{dX_{i+1}}\right)^{T}\left[\left(\frac{dh(\widehat{X}_{i+1,i})}{dX_{i+1}}\right)P_{i,i}\left(\frac{dh(\widehat{X}_{i+1,i})}{dX_{i+1}}\right)^{T} + R_i\right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[I - K_{i+1}\left(\frac{dh(\widehat{X}_{i+1,i})}{dX_{i+1}}\right)\right]P_{i+1,i}$$

Laboratory work 11
Motion model is in Cartesian coordinate system

$$x_i = x_{i-1} + V_{i-1}^x T + \frac{a_{i-1}^x T}{2}$$

$$V_i^x = V_{i-1}^x + a_{i-1}^x T$$

$$y_i = y_{i-1} + V_{i-1}^y T + \frac{a_{i-1}^y T}{2}$$

$$V_i^y = V_{i-1}^y + a_{i-1}^y T$$

x_i, y_i → **Cartesian coordinates**

V_i^x, V_i^y → **Components of velocity V_i**

a_i^x, a_i^y → **Random acceleration**

State-space model, state equation

State equation

$$X_i = \Phi_{i,i-1} X_{i-1} + G a_{i-1}$$

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \end{vmatrix}$$

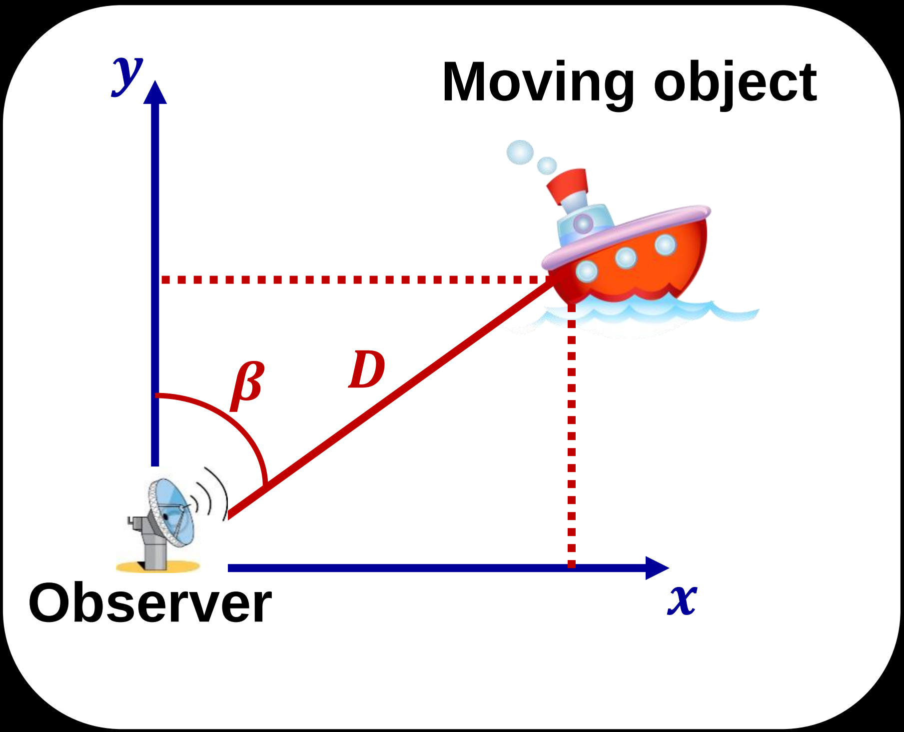
Transition matrix

$$\Phi_{i,i-1} = \begin{vmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Input matrix

$$G = \begin{vmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{vmatrix}$$

State-space model, measurement equation



$$D = \sqrt{x^2 + y^2}$$

$$\beta = arctg\left(\frac{x}{y}\right)$$

$$x = D sin\beta$$
$$y = D cos\beta$$

Measurement vector → $z_i = \begin{vmatrix} D_i^m \\ \beta_i^m \end{vmatrix}$

D_i^m → Measurements of range D

β_i^m → Measurements of azimuth β

State-space model, measurement equation

Measurement equation → $z_i = h(X_i) + \eta_i$

$$\eta_i = \begin{vmatrix} \eta_i^D \\ \eta_i^\beta \end{vmatrix}$$

Measurement vector z_i

$$z_i = \begin{vmatrix} D_i^m \\ \beta_i^m \end{vmatrix}$$

Nonlinear function $h(X_i)$

$$h(X_i) = \begin{vmatrix} \sqrt{x_i^2 + y_i^2} \\ arctg\left(\frac{x_i}{y_i}\right) \end{vmatrix}$$

Recurrent algorithm of Kalman filter

❶ Prediction (extrapolation)

Prediction of state vector at time $i+1$ using i measurements

$$\hat{X}_{i+1,i} = \Phi_{i+1,i}\hat{X}_{i,i}$$

Prediction error covariance matrix

$$P_{i+1,i} = \Phi_{i+1,i}P_{i,i}\Phi^{T}_{i+1,i} + Q_i$$

$$P_{i+1,i} = E\left[(X_{i+1} - X_{i+1,i})\left(X_{i+1} - X_{i+1,i}\right)^{T}\right]$$

$X_{i+1,i}$ → First subscript $i+1$ denotes time on which the prediction is made → Second subscript i represents the number of measurements to get $X_{i+1,i}$

Recurrent algorithm of Kalman filter

❷ Filtration

Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\hat{X}_{i+1,i+1} = \hat{X}_{i+1,i} + K_{i+1}\left(z_{i+1} - h\left(\hat{X}_{i+1,i}\right)\right)$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i}\left(\frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}}\right)^T\left[\left(\frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}}\right)P_{i,i}\left(\frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}}\right)^T + R_i\right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[I - K_{i+1}\left(\frac{dh\left(\hat{X}_{i+1,i}\right)}{dX_{i+1}}\right)\right]P_{i+1,i}$$

Recurrent algorithm of Kalman filter

Nonlinear function $h(X_i)$

$$h(X_i) = \begin{vmatrix} \sqrt{x_i^2 + y_i^2} \\ arctg\left(\frac{x}{y}\right) \end{vmatrix}$$

Derivative with respect to X_{i+1} at point $\widehat{X}_{i+1,i}$

$$\frac{dh(\widehat{X}_{i+1,i})}{dX_{i+1}} = \begin{vmatrix} \dfrac{x_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 & \dfrac{y_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 \\ \dfrac{y_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 & -\dfrac{x_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 \end{vmatrix}$$