Skoltech

Skolkovo Institute of Science and Technology

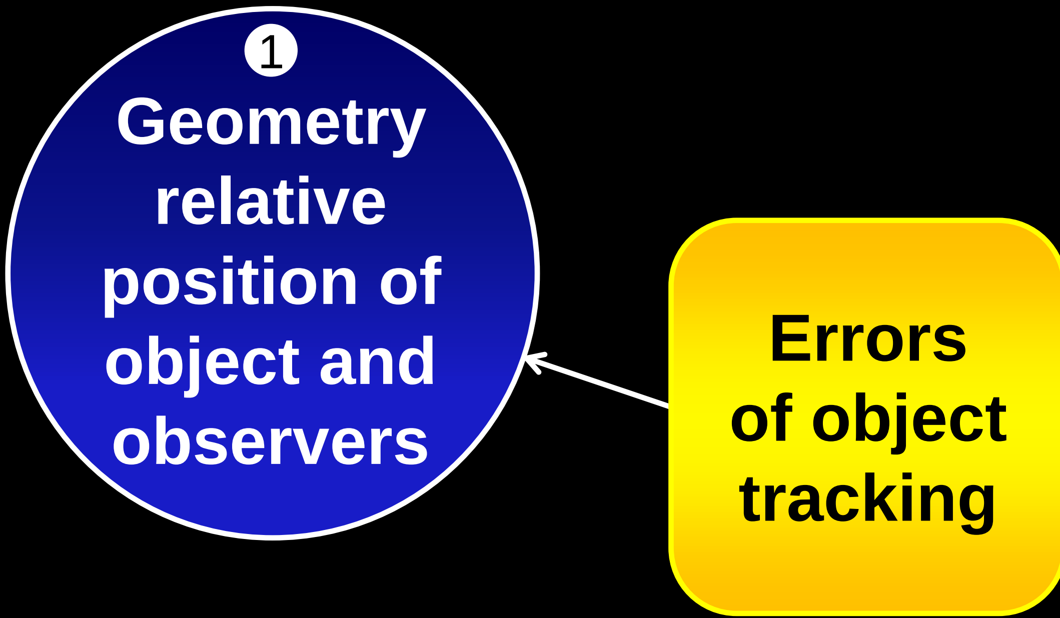
"Space Data Processing:
Making Sense of Experimental Data"

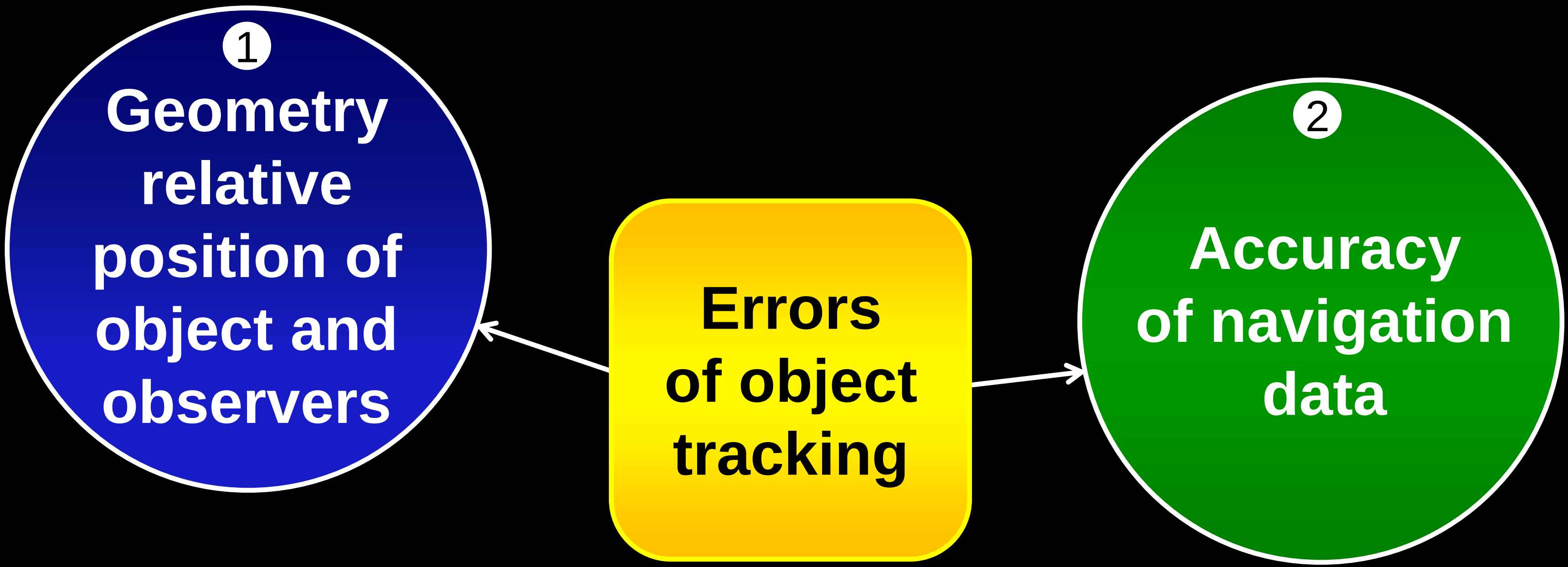
Topic 5
"Model construction at state space under uncertainty"
Part I. Noise statistics identification

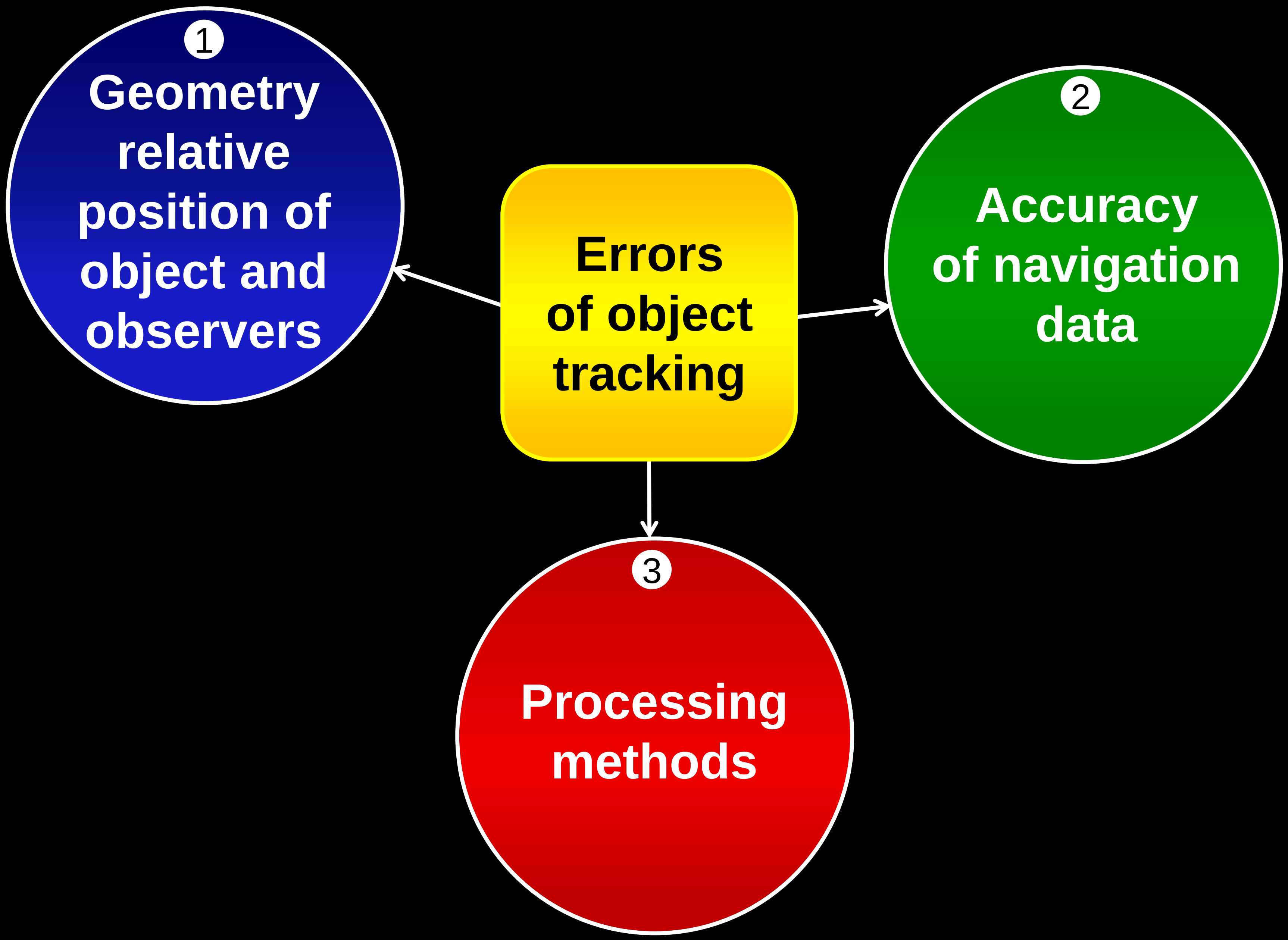
Tatiana Podladchikova   Rupert Gerzer

Term 4, March 28 – May 27, 2016

t.podladchikova@skoltech.ru


1
Geometry relative position of object and observers
Errors of object tracking


1
Geometry relative position of object and observers
Errors of object tracking
2
Accuracy of navigation data

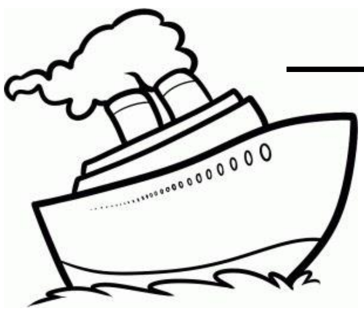

Errors of object tracking
1
Geometry relative position of object and observers
2
Accuracy of navigation data
3
Processing methods

Process noise should not be filtered

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$



V_i

Unintentional maneuver can be described by random acceleration a_i

x

ship pitching or undercurrents

a_i

Noise intrinsic to the process itself that should not be filtered

Stochastic model

State equation → $X_i = \Phi_{i,i-1} X_{i-1} + Ga_i$

Measurement equation → $z_i = H_i X_i + \eta_i$

Ga_i
Noise intrinsic to the process itself that should not be filtered

State space model separates noises in contrast to linear regression

η_i
Measurement noise that should be filtered

Noise statistics identification

State equation → $X_i = \Phi_{i,i-1}X_{i-1} + Ga_i$

Measurement equation → $z_i = H_iX_i + \eta_i$

Gq
Bias of state noise a_i
$q = E[a_i]$

$Q = GG^T\sigma_a^2$
Covariance matrix of state noise w_i
$\sigma_a^2 = \mathrm{E}\left[(a_i - q)^2\right]$

$R = \sigma_\eta^2$
Covariance matrix of measurement noise η_i
$\mathrm{E}[\eta_i] = 0, \sigma_\eta^2 = \mathrm{E}[\eta_i^2]$

How to identify q, σ_a^2, and σ_η^2 using measurements?

Characteristics of residuals of optimal filter

Filtration → $X_{i,i} = X_{i,i-1} + K_i(z_i - HX_{i,i-1})$

Residual

$\nu_i = z_i - HX_{i,i-1}$ → **Mismatch between measurement and prediction**

Characteristics of residuals of optimal filter

Filtration → $X_{i,i} = X_{i,i-1} + K_i(z_i - HX_{i,i-1})$

Residual

$\nu_i = z_i - HX_{i,i-1}$ → Mismatch between measurement and prediction

1 Mathematical expectation of ν_i → $E[\nu_i] = 0$

Characteristics of residuals of optimal filter

Filtration

$$X_{i,i} = X_{i,i-1} + K_i(z_i - HX_{i,i-1})$$

Residual

$$\nu_i = z_i - HX_{i,i-1}$$

Mismatch between measurement and prediction

1 Mathematical expectation of v_i

$$E[\nu_i] = 0$$

2 Covariance of v_i

$$E\left[\nu_i \nu_i^T\right] = HP_{i,i-1}H^T + R$$

Characteristics of residuals of optimal filter

Filtration → $X_{i,i} = X_{i,i-1} + K_i(z_i - HX_{i,i-1})$

Residual

$\nu_i = z_i - HX_{i,i-1}$ → Mismatch between measurement and prediction

1 Mathematical expectation of ν_i → $E[\nu_i] = 0$

2 Covariance of ν_i → $E\left[\nu_i \nu_i^T\right] = HP_{i,i-1}H^T + R$

3 Correlation moment of ν_i → $E\left[\nu_i \nu_j^T\right] = 0, i \neq j$

Consistent identification methods

1 R. Mehra (1970), On the identification of variances and adaptive Kalman filtering in IEEE Transactions on Automatic Control, vol. 15, no. 2, pp. 175-184

Difficult to implement in real -time

2 Anderson, W. N. et al (1969), Consistent estimates of the parameters of a linear system, Ann. Math. Stat., 40(3), 2064–2075.

All components of state vector should be measured

Stochastic model

State equation → $X_i = \Phi_{i,i-1} X_{i-1} + G a_i$

Measurement equation → $z_i = H_i X_i + \eta_i$

$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$ → **State vector**

$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$ → **Transition matrix**

$G = \begin{vmatrix} T^2/2 \\ T \end{vmatrix}$ → **Input matrix**

$\mathrm{H} = \begin{vmatrix} 1 & 0 \end{vmatrix}$ → **Observation matrix**

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Let's create pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Measurements

$$z_i$$

Let's create pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^*$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Let's create pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^*$$

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-1} \\ \dfrac{z_{i-1} - z_{i-2}}{T} \end{vmatrix}$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Measurements

$$z_i$$

Let's create pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^*$$

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-1} \\ \dfrac{z_{i-1} - z_{i-2}}{T} \end{vmatrix}$$

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^* = \begin{vmatrix} z_i - 2z_{i-1} + z_{i-2} \\ \dfrac{z_i - 2z_{i-1} + z_{i-2}}{T} \end{vmatrix}$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Measurements

$$z_i$$

Let's create pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^*$$

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-1} \\ \dfrac{z_{i-1} - z_{i-2}}{T} \end{vmatrix}$$

$$Z_i^* - \Phi_{i,i-1} Z_{i-1}^* = \begin{vmatrix} z_i - 2z_{i-1} + z_{i-2} \\ \dfrac{z_i - 2z_{i-1} + z_{i-2}}{T} \end{vmatrix}$$

Let's consider

$$\nu_i = z_i - 2z_{i-1} + z_{i-2}$$

Noise statistics identification

Let's consider $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$z_i = x_i + \eta_i$

$$\nu_i = z_i - 2z_{i-1} + z_{i-2} = x_i + \eta_i - 2(x_{i-1} + \eta_{i-1}) + (x_{i-2} + \eta_{i-2})$$

Noise statistics identification

Let's consider $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$z_i = x_i + \eta_i$

$$\nu_i = z_i - 2z_{i-1} + z_{i-2} = x_i + \eta_i - 2(x_{i-1} + \eta_{i-1}) + (x_{i-2} + \eta_{i-2})$$

$$\nu_i = (x_i - x_{i-1}) - (x_{i-1} - x_{i-2}) + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Noise statistics identification

Let's consider → $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$z_i = x_i + \eta_i$

$$\nu_i = z_i - 2z_{i-1} + z_{i-2} = x_i + \eta_i - 2(x_{i-1} + \eta_{i-1}) + (x_{i-2} + \eta_{i-2})$$

$$\nu_i = (x_i - x_{i-1}) - (x_{i-1} - x_{i-2}) + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$\nu_i = V_{i-1}T + \frac{a_{i-1}T^2}{2} - V_{i-2}T - \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Noise statistics identification

Let's consider $\rightarrow$ $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$z_i = x_i + \eta_i$

$$\nu_i = z_i - 2z_{i-1} + z_{i-2} = x_i + \eta_i - 2(x_{i-1} + \eta_{i-1}) + (x_{i-2} + \eta_{i-2})$$

$$\nu_i = (x_i - x_{i-1}) - (x_{i-1} - x_{i-2}) + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

$x_i = x_{i-1} + V_{i-1}T + \dfrac{a_{i-1}T^2}{2}$

$$\nu_i = V_{i-1}T + \frac{a_{i-1}T^2}{2} - V_{i-2}T - \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

$V_{i-1}T = V_{i-2}T + a_{i-2}T^2$

$$\nu_i = \frac{a_{i-1}T^2}{2} + \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

ν_i depends only on noises a_i, η_i

Noise statistics identification

Let's consider $\Rightarrow$ $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$$\nu_i = \frac{a_{i-1}T^2}{2} + \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Noise statistics identification

Let's consider → $$\nu_i = z_i - 2z_{i-1} + z_{i-2}$$

$$\nu_i = \frac{a_{i-1}T^2}{2} + \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Mathematical expectation of ν_i → $$E[\nu_i] = qT^2$$

$$E[\nu_i] \approx \frac{1}{N-2}\sum_{i=3}^{N} \nu_i$$

Noise statistics identification

Let's consider → $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$$\nu_i = \frac{a_{i-1}T^2}{2} + \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Mathematical expectation of ν_i → $E[\nu_i] = qT^2$

$$E[\nu_i] \approx \frac{1}{N-2}\sum_{i=3}^{N} \nu_i$$

Bias q → $q = \dfrac{E[\nu_i]}{T^2}$

Noise statistics identification

Let's consider → $\nu_i = z_i - 2z_{i-1} + z_{i-2}$

$$\nu_i = \frac{a_{i-1}T^2}{2} + \frac{a_{i-2}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Mathematical expectation of ν_i → $E[\nu_i] = qT^2$

$$E[\nu_i] \approx \frac{1}{N-2}\sum_{i=3}^{N} \nu_i$$

Bias q → $q = \frac{E[\nu_i]}{T^2}$

Variance of ν_i → $E\left[\left(\nu_i - qT^2\right)^2\right] = \frac{1}{2}\sigma_a^2 T^4 + 6\sigma_\eta^2$

$$E\left[\left(\nu_i - qT^2\right)^2\right] \approx \frac{1}{N-2}\sum_{i=3}^{N}\left(\nu_i - qT^2\right)^2$$

σ_a^2 - variance of a_i

σ_η^2 - variance of η_i

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-2} Z_{i-2}^*$$

$$\Phi_{i,i-2} = \Phi_{i,i-1} \Phi_{i-1,i-2}$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-2} Z_{i-2}^*$$

$$\Phi_{i,i-2} = \Phi_{i,i-1} \Phi_{i-1,i-2}$$

$$Z_i^* - \Phi_{i,i-2} Z_{i-1}^* = \begin{vmatrix} z_i \\ \frac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & 2T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-2} \\ \frac{z_{i-2} - z_{i-3}}{T} \end{vmatrix}$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-2} Z_{i-2}^*$$

$$\Phi_{i,i-2} = \Phi_{i,i-1} \Phi_{i-1,i-2}$$

$$Z_i^* - \Phi_{i,i-2} Z_{i-1}^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & 2T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-2} \\ \dfrac{z_{i-2} - z_{i-3}}{T} \end{vmatrix}$$

$$Z_i^* - \Phi_{i,i-2} Z_{i-1}^* = \begin{vmatrix} z_i - 3z_{i-2} + 2z_{i-3} \\ \dfrac{z_i - z_{i-1} - z_{i-2} + z_{i-3}}{T} \end{vmatrix}$$

Noise statistics identification

State vector

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

Pseudo-measurement vector

$$Z_i^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix}$$

Measurements

$$z_i$$

Let's consider difference

$$Z_i^* - \Phi_{i,i-2} Z_{i-2}^*$$

$$\Phi_{i,i-2} = \Phi_{i,i-1} \Phi_{i-1,i-2}$$

$$Z_i^* - \Phi_{i,i-2} Z_{i-1}^* = \begin{vmatrix} z_i \\ \dfrac{z_i - z_{i-1}}{T} \end{vmatrix} - \begin{vmatrix} 1 & 2T \\ 0 & 1 \end{vmatrix} \begin{vmatrix} z_{i-2} \\ \dfrac{z_{i-2} - z_{i-3}}{T} \end{vmatrix}$$

$$Z_i^* - \Phi_{i,i-2} Z_{i-1}^* = \begin{vmatrix} z_i - 3z_{i-2} + 2z_{i-3} \\ \dfrac{z_i - z_{i-1} - z_{i-2} + z_{i-3}}{T} \end{vmatrix}$$

Let's consider

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

Noise statistics identification

Let's consider → $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

Noise statistics identification

Let's consider → $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

Noise statistics identification

Let's consider $\rightarrow$ $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2} \rightarrow x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-2}T + a_{i-2}T^2 + \frac{a_{i-1}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$z_i = x_i + \eta_i$$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-2}T + a_{i-2}T^2 + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$z_i = x_i + \eta_i$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-2}T + a_{i-2}T^2 + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_i - x_{i-2} = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$z_i = x_i + \eta_i$$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3} = x_i + \eta_i - 3(x_{i-2} + \eta_{i-2}) + 2(x_{i-3} + \eta_{i-3})$$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + V_{i-2}T + \frac{a_{i-2}T^2}{2} + V_{i-2}T + a_{i-2}T^2 + \frac{a_{i-1}T^2}{2}$$

$$x_i = x_{i-2} + 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_i - x_{i-2} = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_{i-2} - x_{i-3} = V_{i-3}T + \frac{a_{i-3}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i - x_{i-2} = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_{i-2} - x_{i-3} = V_{i-3}T + \frac{a_{i-3}T^2}{2}$$

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i - x_{i-2} = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_{i-2} - x_{i-3} = V_{i-3}T + \frac{a_{i-3}T^2}{2}$$

$$\rho_i = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} - 2V_{i-3}T - a_{i-3}T^2 + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

Noise statistics identification

Let's consider → $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = (x_i - x_{i-2}) - 2(x_{i-2} - x_{i-3}) + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$x_i - x_{i-2} = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2}$$

$$x_{i-2} - x_{i-3} = V_{i-3}T + \frac{a_{i-3}T^2}{2}$$

$$\rho_i = 2V_{i-2}T + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} - 2V_{i-3}T - a_{i-3}T^2 + \eta_i - 3\eta_{i-2} + 2\eta_{i-3}$$

$$2V_{i-2}T = 2V_{i-3}T + 2a_{i-3}T^2$$

$$\rho_i = a_{i-3}T^2 + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

ρ_i depends only on noises a_i, η_i

Noise statistics identification

Let's consider $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = a_{i-3}T^2 + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Noise statistics identification

Let's consider → $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = a_{i-3}T^2 + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Mathematical expectation of ρ_i → $E[\rho_i] = 3qT^2$

Noise statistics identification

Let's consider $\rightarrow$ $\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$

$$\rho_i = a_{i-3}T^2 + \frac{3a_{i-2}T^2}{2} + \frac{a_{i-1}T^2}{2} + \eta_i - 2\eta_{i-1} + \eta_{i-2}$$

Mathematical expectation of ρ_i $\rightarrow$ $E[\rho_i] = 3qT^2$

Variance of ρ_i $\rightarrow$ $E\left[\left(\rho_i - 3qT^2\right)^2\right] = \frac{7}{2}\sigma_a^2 T^4 + 14\sigma_\eta^2$

$$E\left[\left(\rho_i - 3qT^2\right)^2\right] \approx \frac{1}{N-2}\sum_{i=3}^{N}\left(\rho_i - 3qT^2\right)^2$$

σ_a^2 - variance of a_i

σ_η^2 - variance of η_i

Summary: Noise statistics identification

Bias q → $q = \frac{E[\nu_i]}{T^2}$

$$E[\nu_i] \approx \frac{1}{N-2}\sum_{i=3}^{N}\nu_i$$

σ_a^2 and σ_η^2 are determined from the solution of system of equations

$$E\left[\left(\nu_i - qT^2\right)^2\right] = \frac{1}{2}\sigma_a^2 T^4 + 6\sigma_\eta^2$$

$$E\left[\left(\rho_i - qT^2\right)^2\right] = \frac{7}{2}\sigma_a^2 T^4 + 14\sigma_\eta^2$$

$$\nu_i = z_i - 2z_{i-1} + z_{i-2}$$

$$\rho_i = z_i - 3z_{i-2} + 2z_{i-3}$$

Podladchikova et al. (2014), Noise statistics identification for Kalman filtering of the electron radiation belt observations: 1. Model errors 2. Filtration and smoothing, J. Geophys. Res. Space Physics, 119