Skolkovo Institute of Science and Technology

“Space Data Processing:
Making Sense of Experimental Data”

Topic 3
"Optimal approximation at state space”

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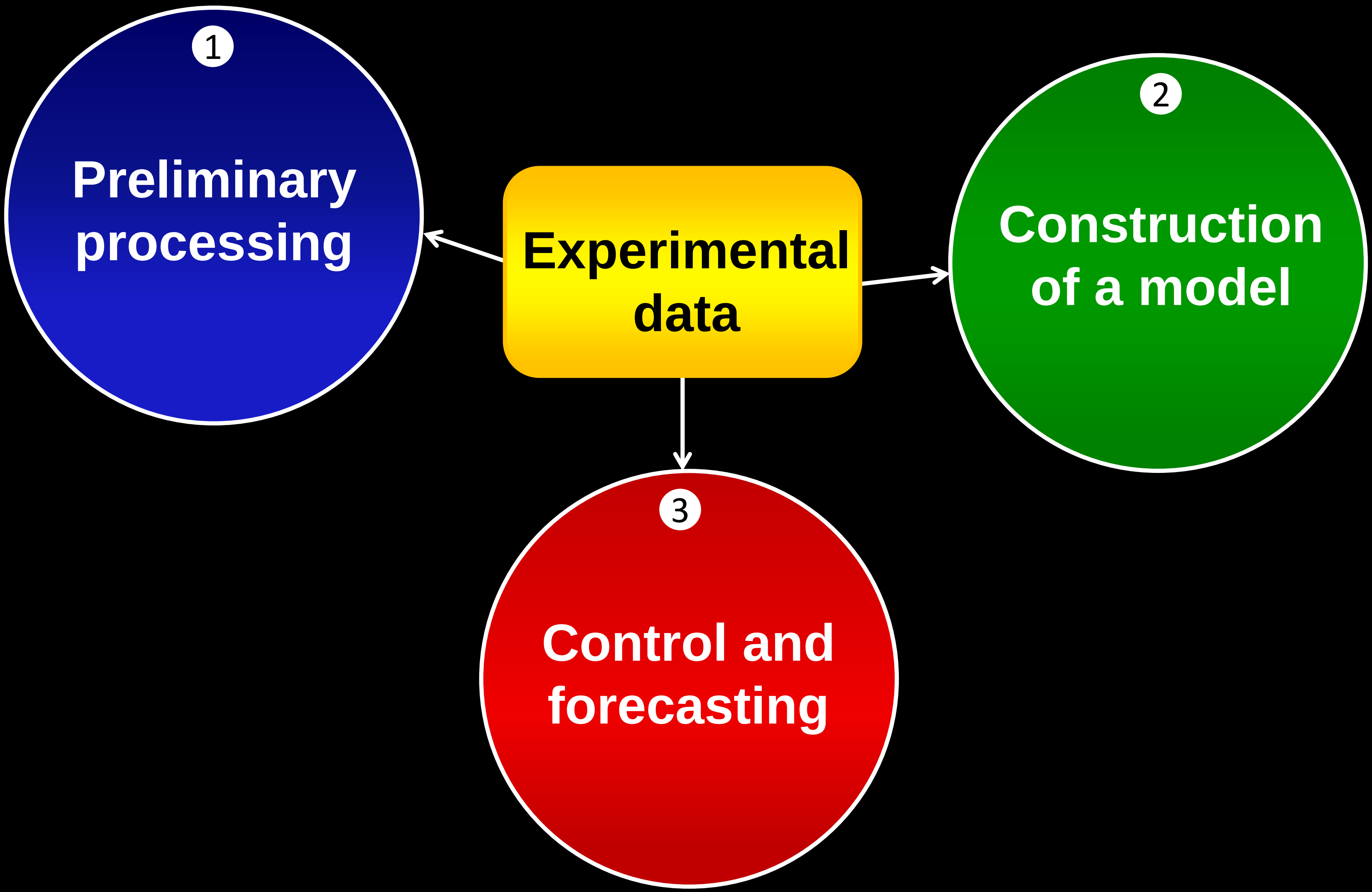
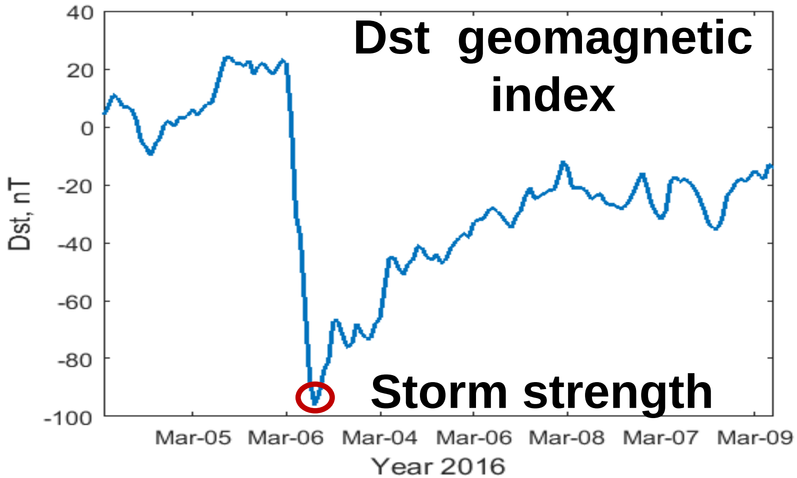
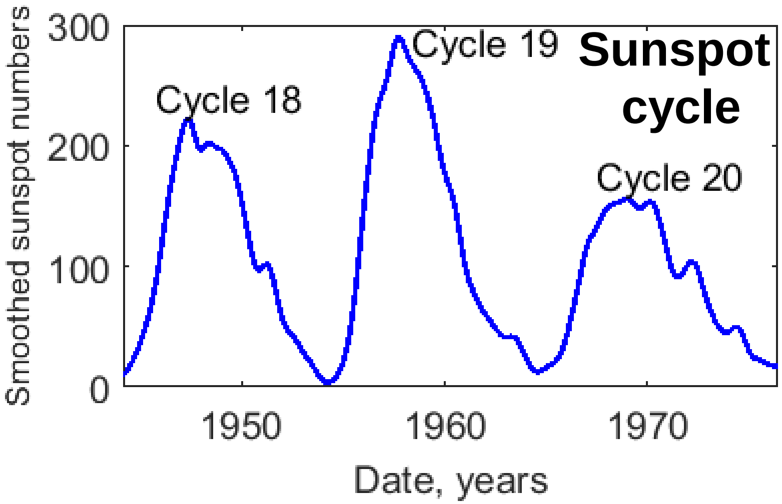
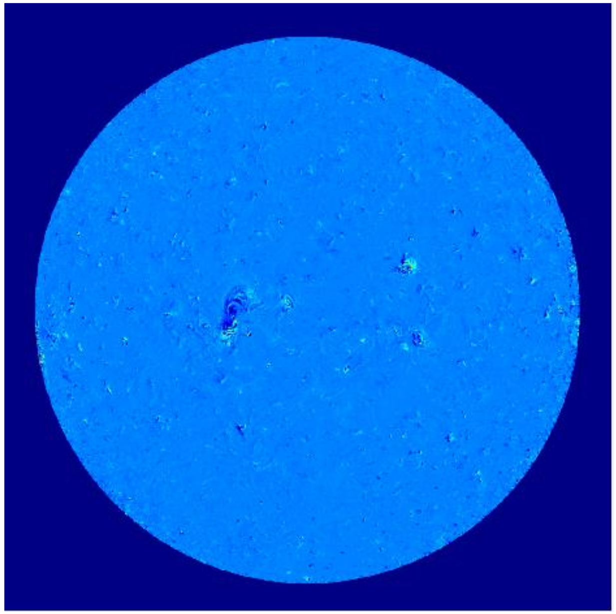
Traditional approach to estimation and forecasting



Creation of a mathematical model of insufficiently studied processes is quite problematic







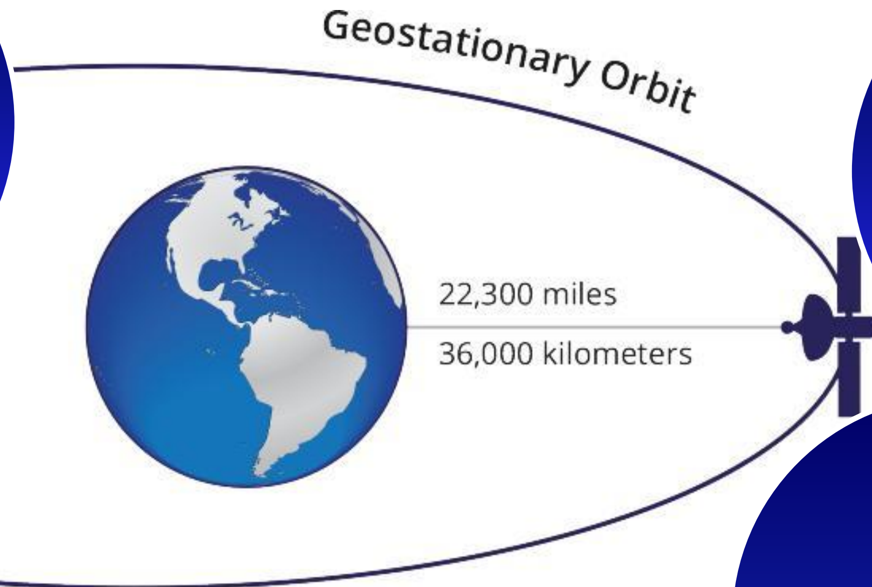
Extreme ultraviolet coronal wave December 7, 2007

Forces that affect motion of a geostationary satellite

Motion of controlled objects is usually described by a mathematical model

1

Disturbance due to nonspherical gravity field of Earth



2

Gravitational disturbances of Moon and Sun

3

Light pressure from Sun

4

Other disturbances

However it is problematic to describe random deviations from the orbit

Application area of quasi-optimal methods

Mathematical model requires any prior assumptions about a process

These assumptions may significantly distort estimation output

Quasi-optimal methods do not need any prior assumptions that may distort a process

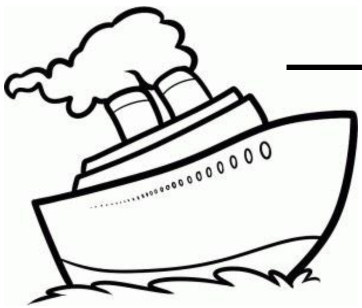
Thus they can extract hidden regularities for long-term forecasting of complicated processes

From Gauss to Kalman

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$



V_i

Unintentional maneuver can be described by random acceleration a_i

ship pitching or undercurrents x

Classical least – square method provides estimations of constant parameters

Development

Kalman filter provides estimations of variable parameters x_i, V_i

A. Legendre, 1806
J. Gauss, 1809

R. Kalman, 1960

State equation

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

State vector

It contains full information about the state of system at time i

State equation

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

State vector

It contains full information about the state of system at time i

State equation

$$X_i = \Phi X_{i-1} + G a_{i-1}$$

$$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$$

Transition matrix

$$G = \begin{vmatrix} T^2/2 \\ T \end{vmatrix}$$

Input matrix

Measurement equation

Measurements of coordinate x_i with error η_i → $z_i = x_i + \eta_i$

Measurement equation → $z_i = HX_i + \eta_i$

$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$

$\mathrm{H} = |1 \quad 0|$ → Observation matrix

State space model

State equation

$$X_i = \Phi_{i,i-1} X_{i-1} + w_i$$

Stochastic model

X_i
State vector

$\Phi_{i,i-1}$
Transition matrix, that relates states X_i and X_{i-1}

w_i
State noise describing model errors with covariance matrix Q_i

Measurement equation

$$z_i = H_i X_i + \eta_i$$

z_i
Measurements

H_i
Observation matrix, that relates state X_i with measurements z_i

η_i
Measurement noise with the covariance matrix R_i

State space model

State equation → $X_i = \Phi_{i,i-1} X_{i-1} + w_i$

Measurement equation → $z_i = H_i X_i + \eta_i$

w_i
Noise intrinsic to the process itself that should not be filtered

State space model separates noises in contrast to linear regression

η_i
Measurement noise that should be filtered

Kalman filter estimate from Least-Squares method

Let's minimize functional J

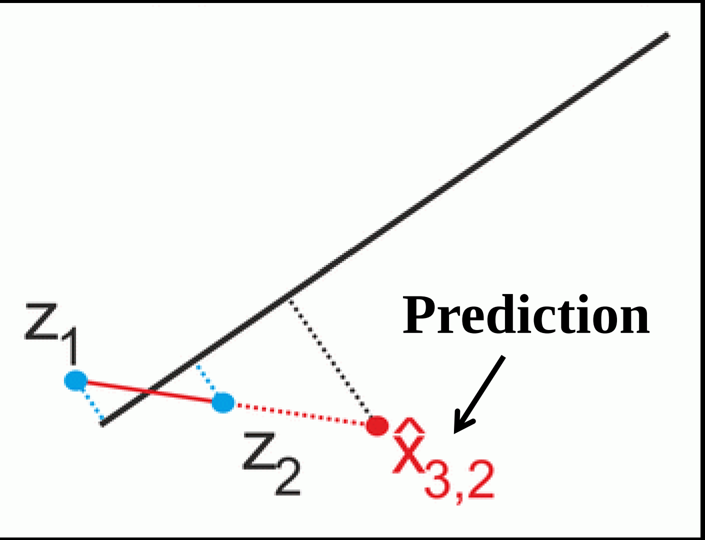
$$J = \sum_{i=1}^{N} (z_i - H_i X_i)^T R_i^{-1} (z_i - H_i X_i)$$

w_i, η_i Gaussian noises → Best estimate → Variance of estimation error is minimal

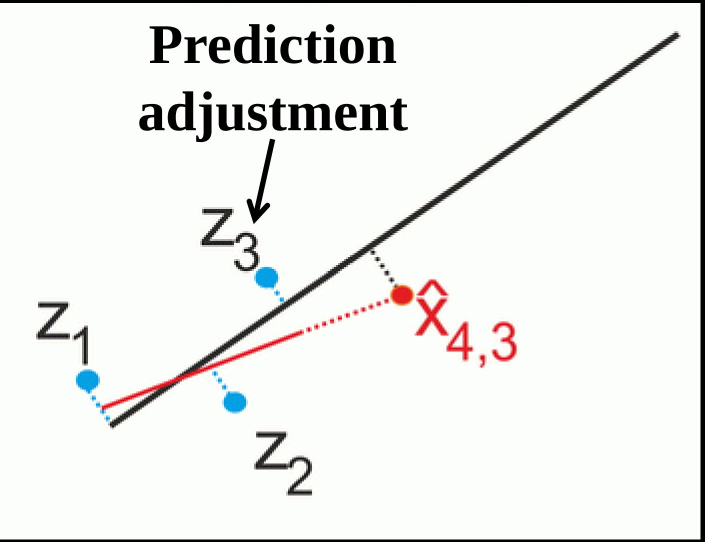
w_i, η_i Non - Gaussian noises → Best estimate for linear filters

Recurrent algorithm of Kalman filter

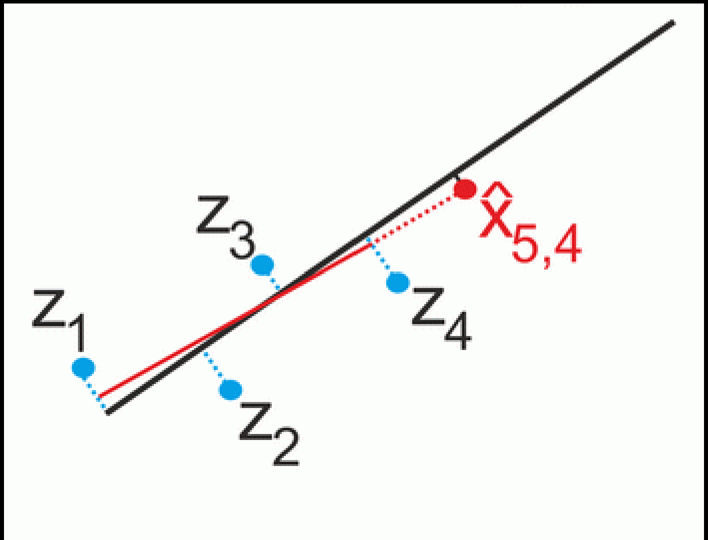
2 measurements



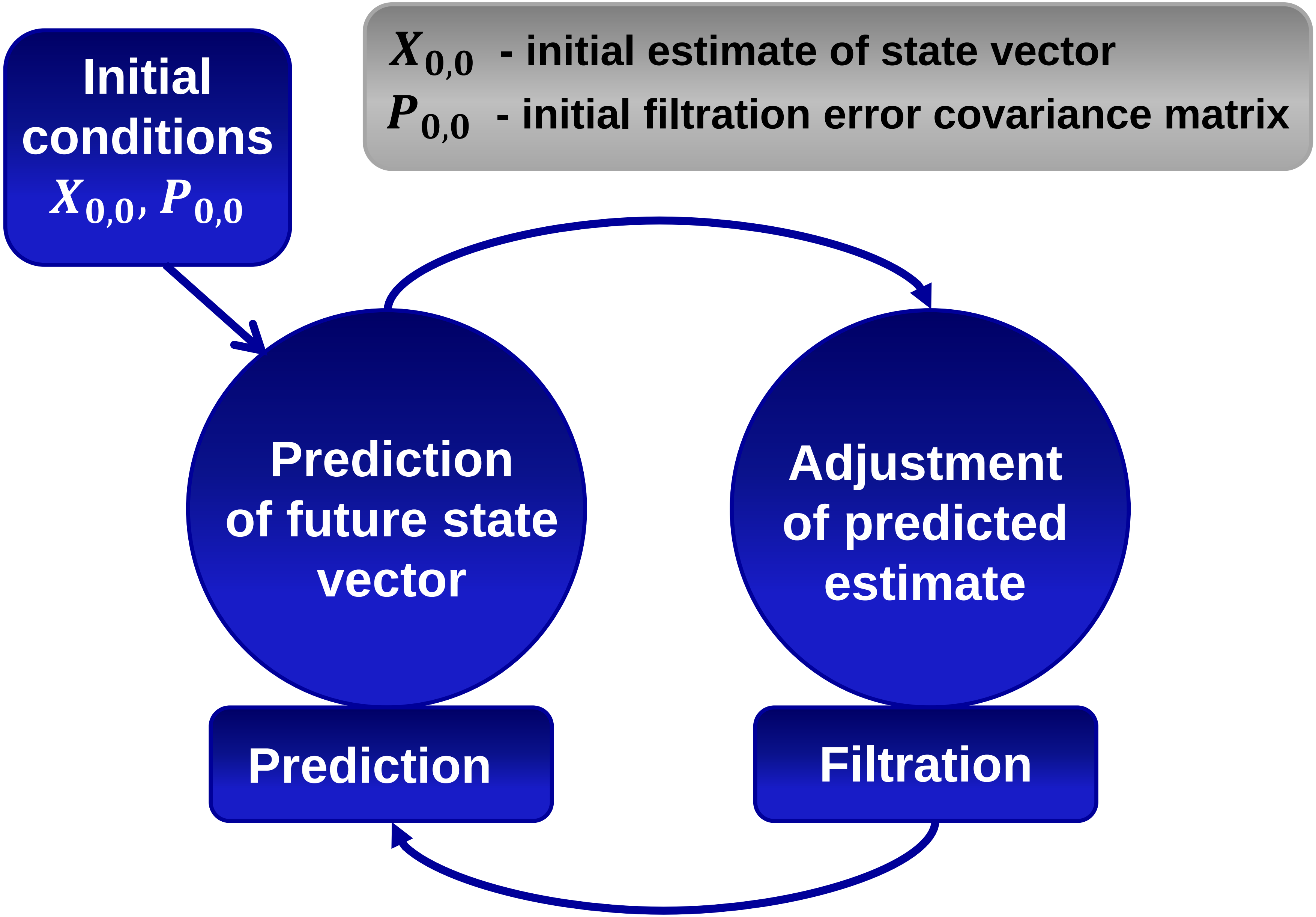
3 measurements



4 measurements



Recurrent algorithm of Kalman filter



Recurrent algorithm of Kalman filter

❶ Prediction (extrapolation)

Prediction of state vector at time i using $i-1$ measurements

$$X_{i,i-1} = \Phi_{i,i-1} X_{i-1,i-1}$$

Prediction error covariance matrix

$$P_{i,i-1} = \Phi_{i,i-1} P_{i-1,i-1} \Phi_{i,i-1}^{T} + Q_i$$

$$P_{i,i-1} = E\left[\left(X_i - X_{i,i-1}\right)\left(X_i - X_{i,i-1}\right)^{T}\right]$$

$X_{i,i-1}$ → First subscript i denotes time on which the prediction is made → Second subscript $i-1$ represents the number of measurements to get $X_{i,i-1}$

Recurrent algorithm of Kalman filter

❷ Filtration
Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$X_{i,i} = X_{i,i-1} + K_i(\underbrace{z_i - HX_{i,i-1}}_{\text{Residual}})$$

Filter gain, weight of residual

$$K_i = P_{i,i-1}H_i^T\left(H_iP_{i,i-1}H_i^T + R_i\right)^{-1}$$

Filtration error covariance matrix

$$P_{i,i} = (I - K_iH_i)P_{i,i-1}$$

$$P_{i,i} = E\left[(X_i - X_{i,i})\left(X_i - X_{i,i}\right)^T\right]$$

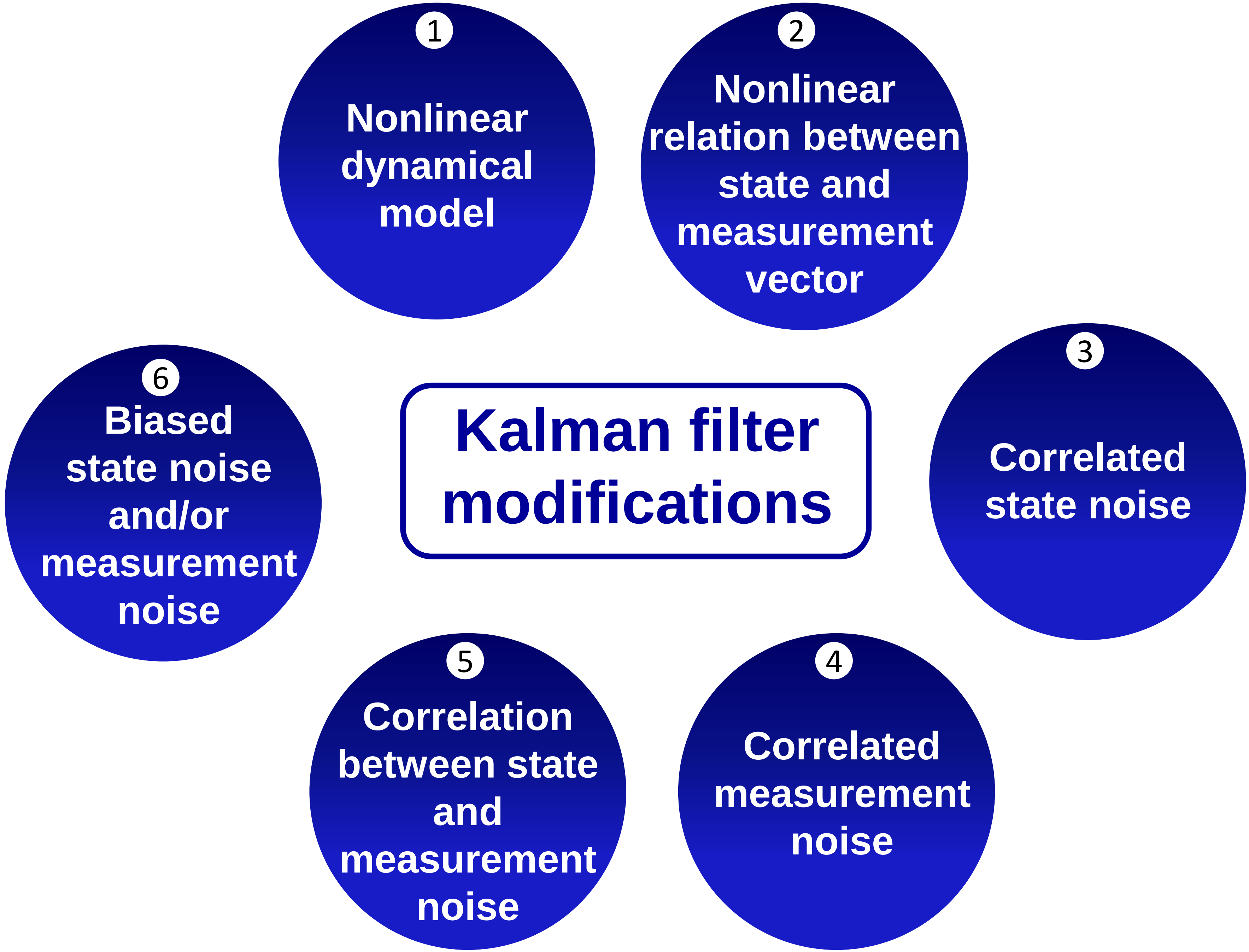
Classical Least-Squares method (LSM) is particular case of Kalman filter

Dynamical model is deterministic. Covariance matrix of state noise w

$$Q = 0$$

The Kalman filter solution is equivalent to that of LSM

However recurrent form of Kalman filter solution has great advantage for implementation

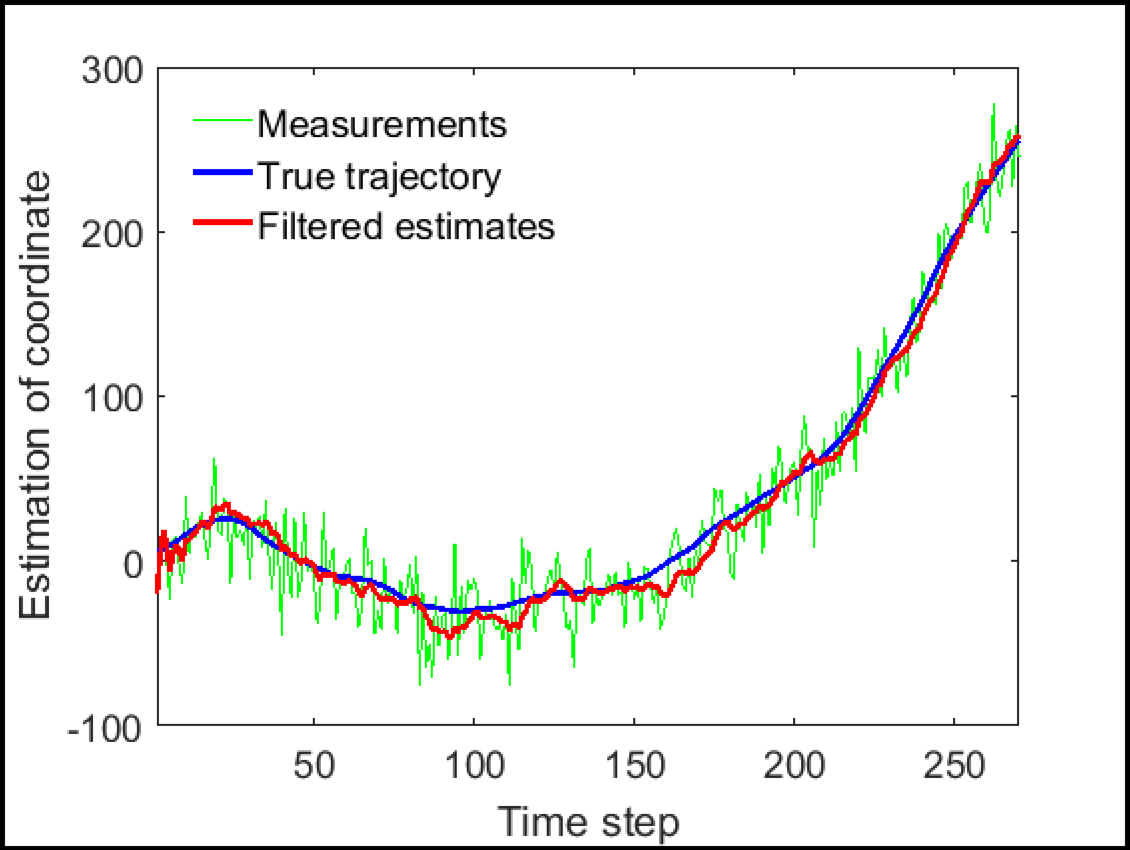

Kalman filter modifications
1
Nonlinear dynamical model
2
Nonlinear relation between state and measurement vector
3
Correlated state noise
4
Correlated measurement noise
5
Correlation between state and measurement noise
6
Biased state noise and/or measurement noise

Tracking moving object using Kalman filter
Stochastic model

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$



$$P_{0,0}^{-1} = \infty$$

$$\sigma_a^2 = 0.04$$

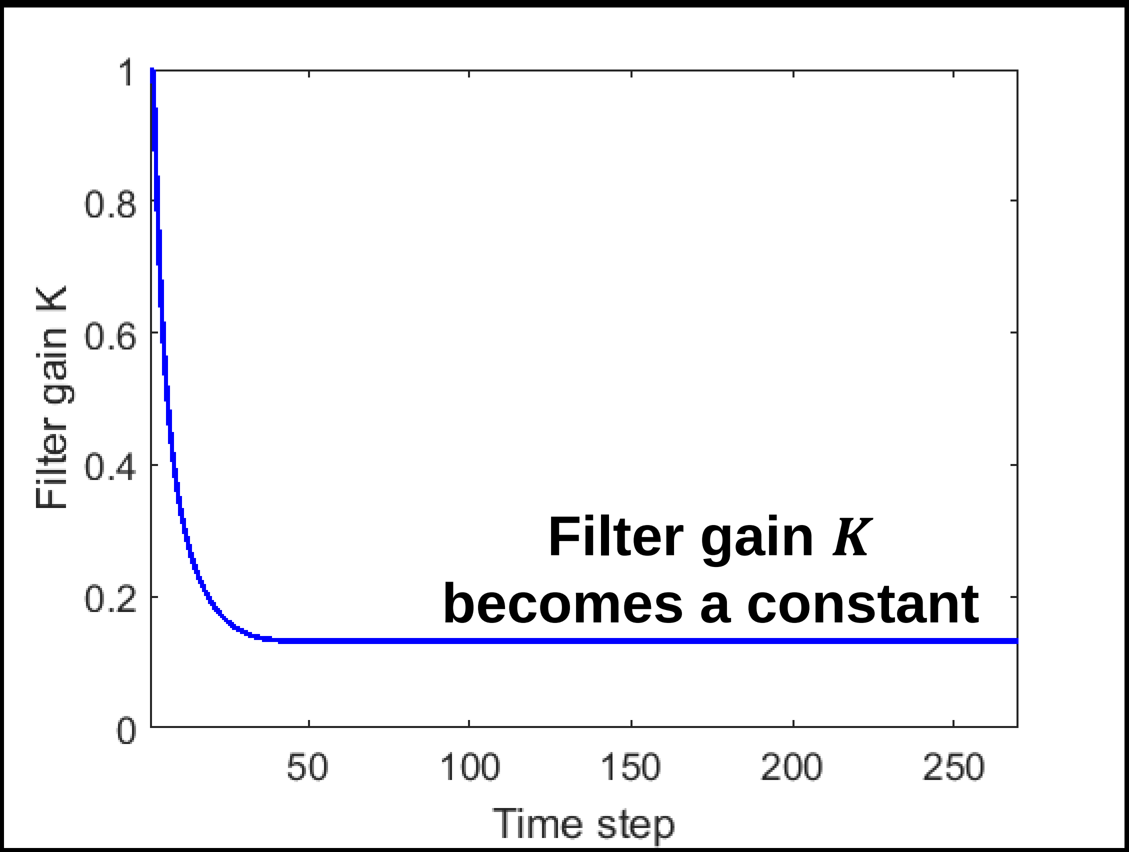
$$\sigma_\eta^2 = 400$$

Tracking moving object using Kalman filter
Stochastic model

Motion model

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$

$$V_i = V_{i-1} + a_{i-1}T$$



Kalman filter becomes stationary

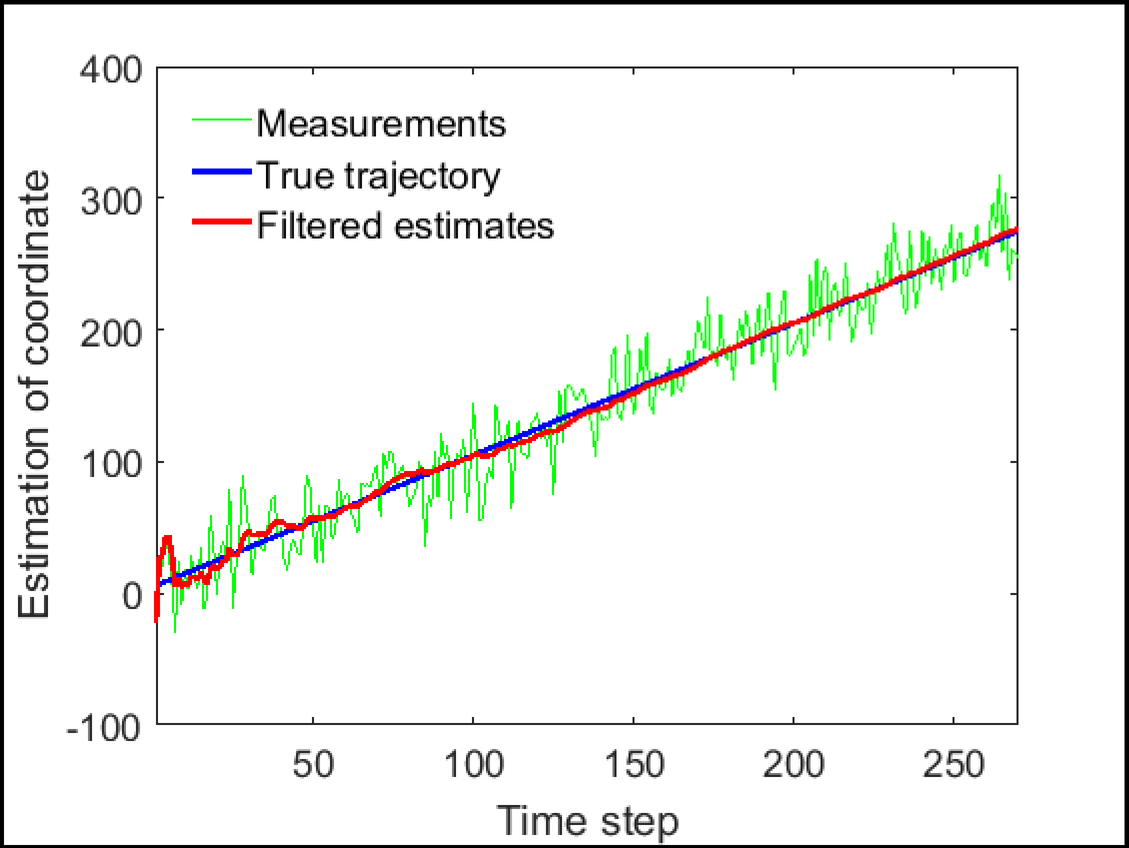
After that there is no increase of estimation accuracy

Measurements always adjust prediction

Tracking moving object using Kalman filter
Deterministic model

Uniform motion

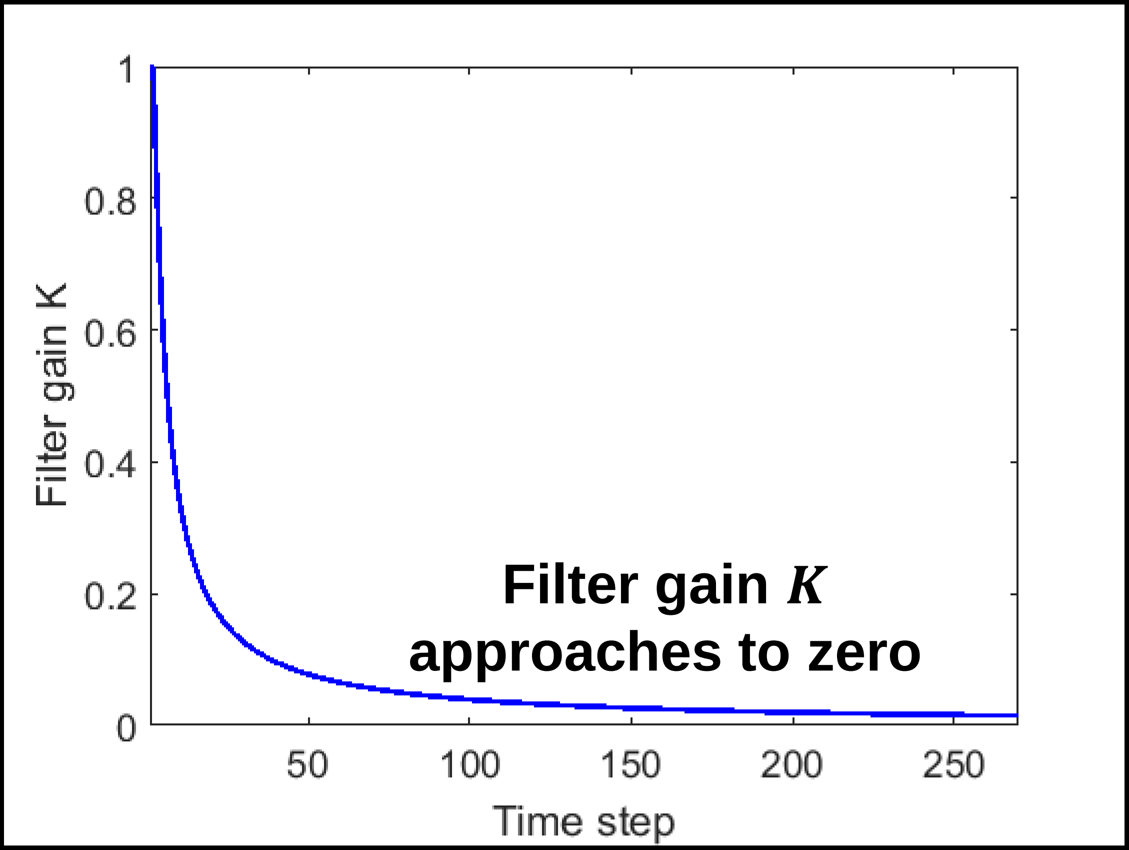
$$x_i = x_{i-1} + V_{i-1}T$$
$$V_i = V_{i-1}$$



$$P_{0,0}^{-1} = \infty$$
$$\sigma_a^2 = 0$$
$$\sigma_\eta^2 = 400$$

Tracking moving object using Kalman filter
Deterministic model

Uniform motion → $x_i = x_{i-1} + V_{i-1}T$
$V_i = V_{i-1}$



High estimation accuracy achieved

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Filter switches off from measurements

Alpha-beta filter – simplified case of Kalman filter

Object moves uniformly → $x_i = x_{i-1} + VT$

Measurements → $z_i = x_i + \eta_i$

Let's use these parameters in Kalman filter algorithm

$$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$$

$$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$$

$$\mathrm{H} = \begin{vmatrix} 1 & 0 \end{vmatrix}$$

$$Q = 0$$

$$P_{0,0}^{-1} = 0$$

Alpha-beta filter – simplified case of Kalman filter

Predicted estimate

$$x_{i,i-1} = x_{i-1,i-1} + V_{i-1,i-1}T$$

$$V_{i,i-1} = V_{i-1,i-1}$$

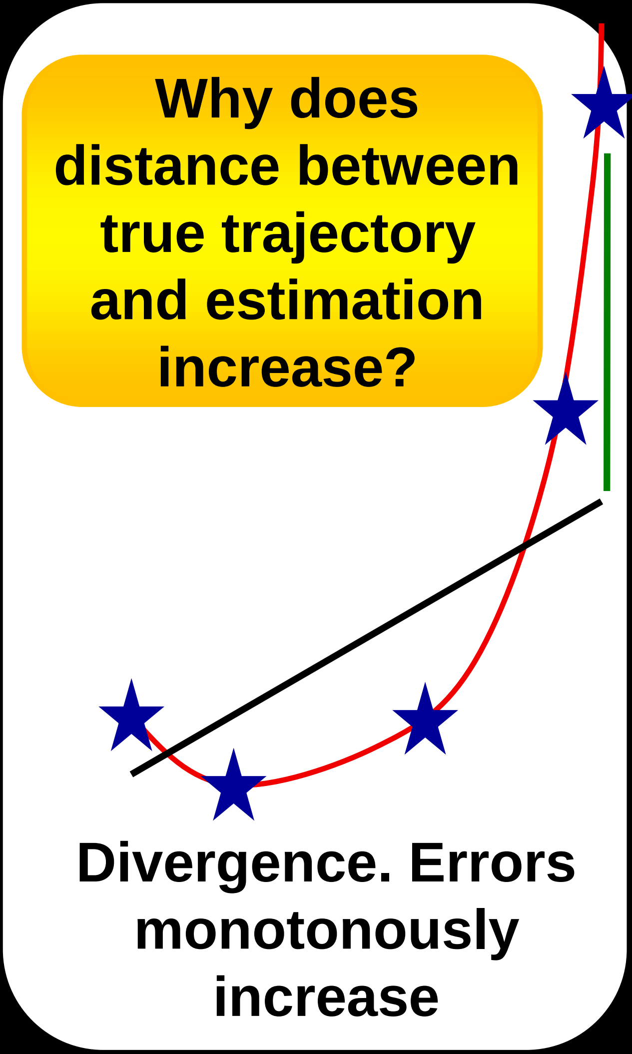
Filtered estimate

$$x_{i,i} = x_{i,i-1} + \alpha(z_i - x_{i,i-1})$$

$$V_{i,i} = V_{i,i-1} + \beta\,(z_i - x_{i,i-1})$$

$$\alpha = \frac{2(2i-1)}{i(i+1)}$$

$$\beta = \frac{6}{i(i+1)T}$$



Alpha-beta filter – simplified case of Kalman filter

Filtered estimate

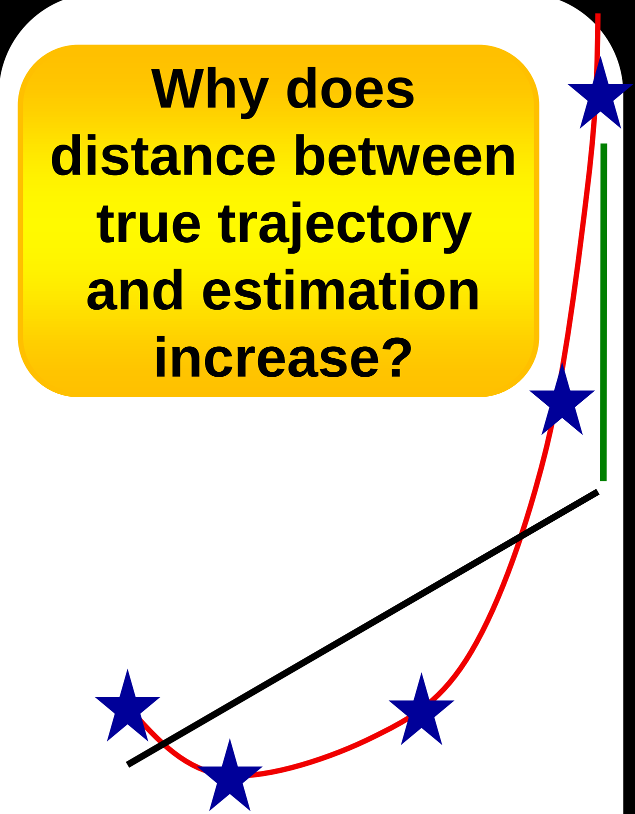
$$x_{i,i} = x_{i,i-1} + \alpha(z_i - x_{i,i-1})$$

$$V_{i,i} = V_{i,i-1} + \beta\,(z_i - x_{i,i-1})$$

$$\alpha = \frac{2(2i-1)}{i(i+1)} \qquad \beta = \frac{6}{i(i+1)T}$$

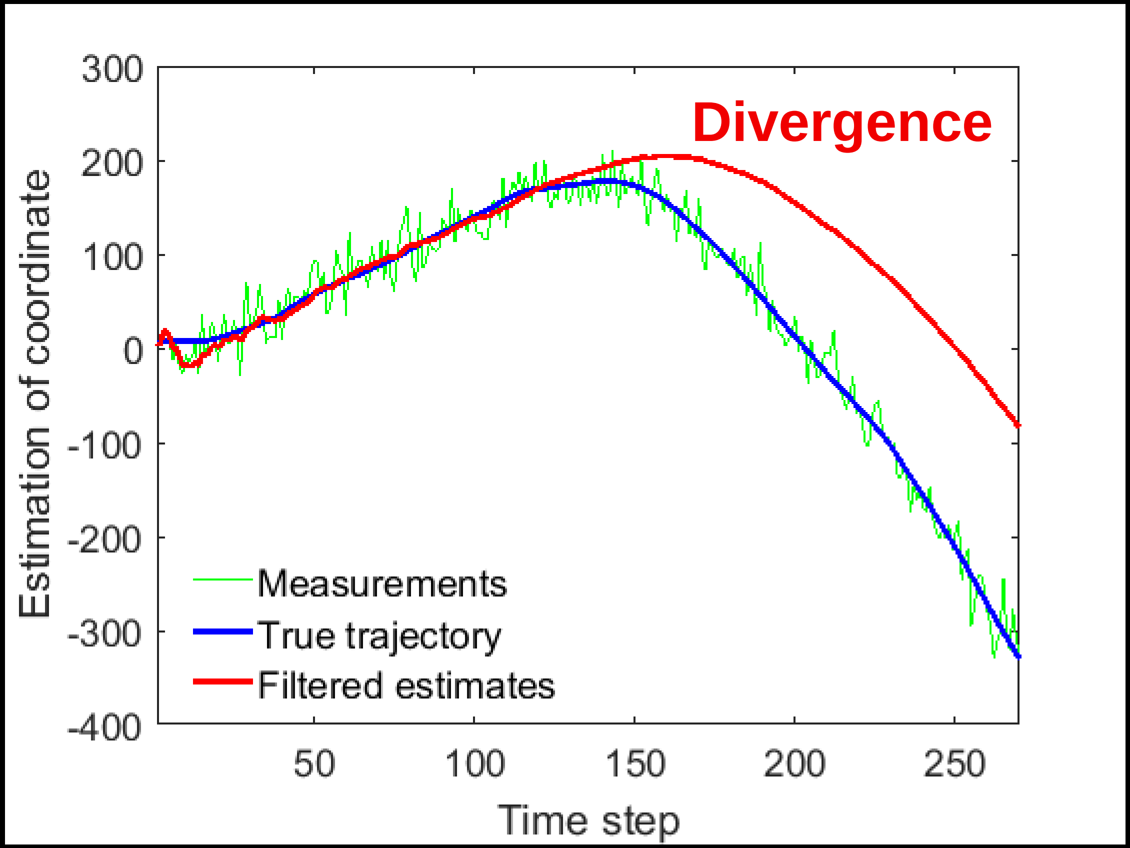
With increase of i coefficients $\alpha, \beta \to 0$

Filter switches off from measurements



Divergence. Errors monotonously increase

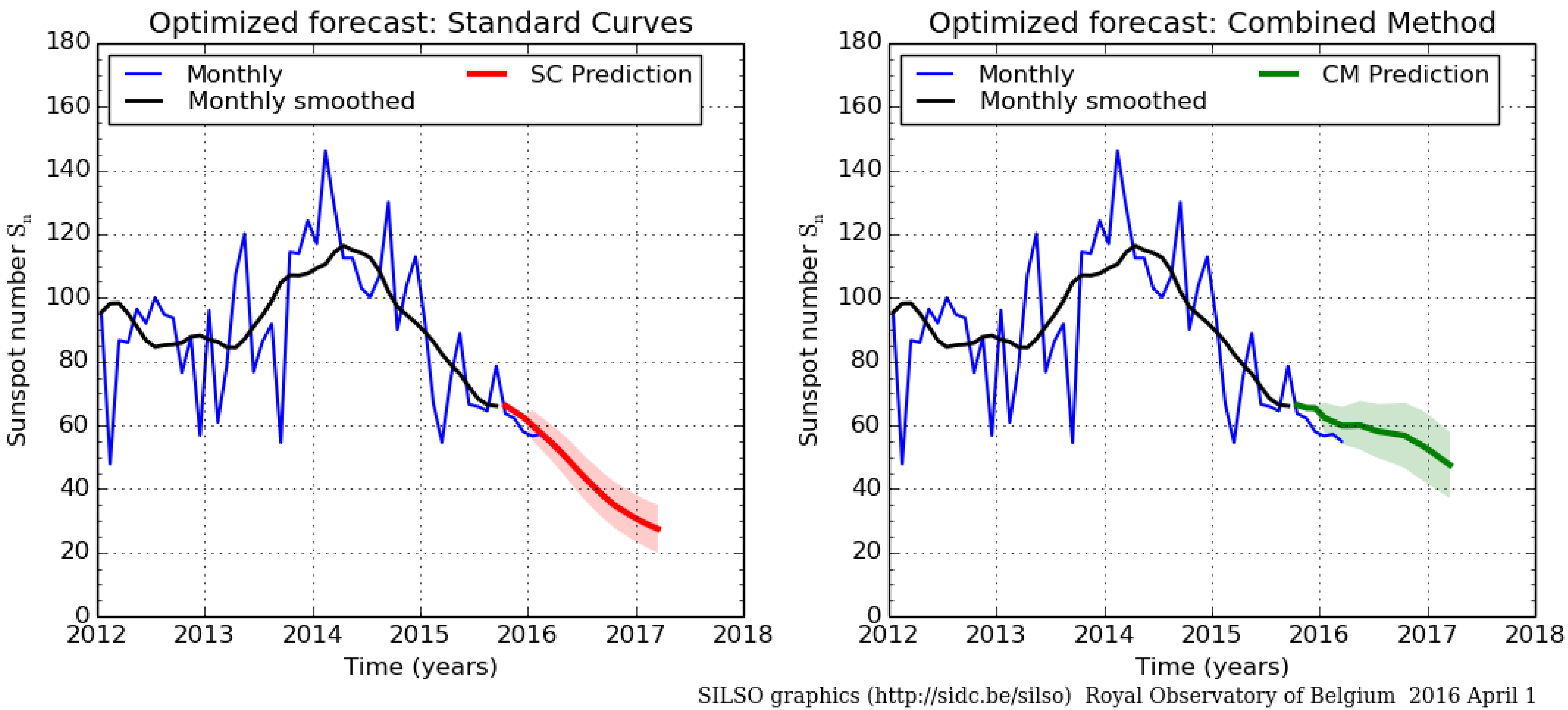
What happens if we use deterministic model, but in fact it is stochastic model?



Filter gain K approaches to zero for deterministic model

Filter diverges as it switches off from measurements

Kalman filter to forecast sunspot number 12 months ahead



Extrapolation 12 steps ahead

$$X_{12,1} = \Phi_{12,1} X_{1,1}$$

$$\Phi_{12,1} = \Phi_{12,11} \Phi_{11,10} + \cdots \Phi_{3,2} \Phi_{2,1}$$

Non-observable system. Example 1

Uniform motion

$$x_i = x_{i-1} + V_{i-1}T$$

$$V_i = V_{i-1}$$

Measurements of only velocity V_i are available

$$z_i = V_i + \eta_i$$

Measurements of coordinate x_i are not available

Non-observable system. Example 1

Uniform motion

$$x_i = x_{i-1} + V_{i-1}T$$
$$V_i = V_{i-1}$$

Measurements of only velocity V_i are available
$$z_i = V_i + \eta_i$$

Measurements of coordinate x_i are not available

Let's present the system at state space

State equation: $X_i = \Phi X_{i-1}$

Measurement equation: $z_i = \mathrm{H} X_i + \eta_i$

$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$ — State vector

$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$ — Transition matrix

$H = \begin{vmatrix} 0 & 1 \end{vmatrix}$ — Observation matrix

Non-observable system. Example 1

Uniform motion

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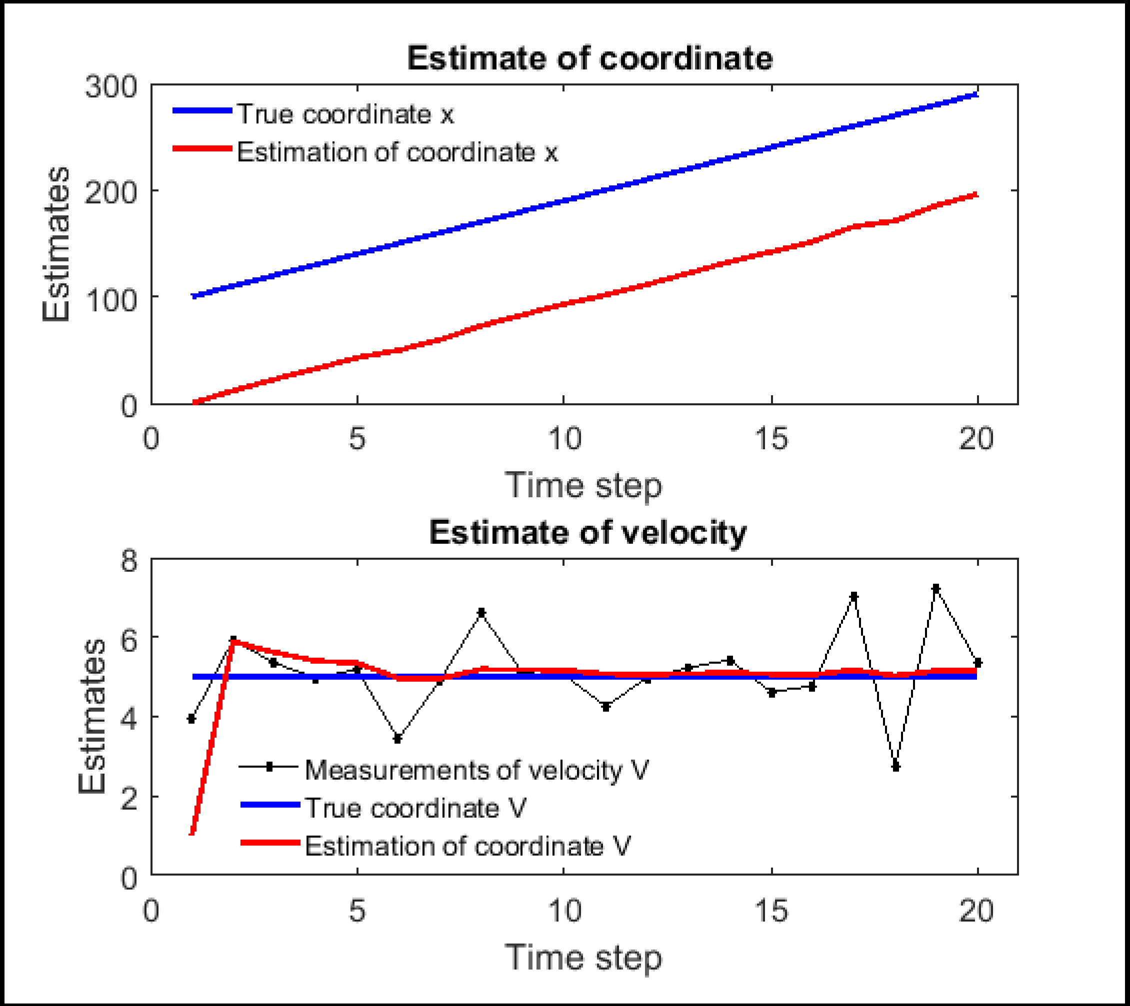
Is it possible to estimate coordinate x_i using Kalman filter?

$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$ — Transition matrix

$H = \begin{vmatrix} 0 & 1 \end{vmatrix}$ — Observation matrix

Non-observable system. Example 1



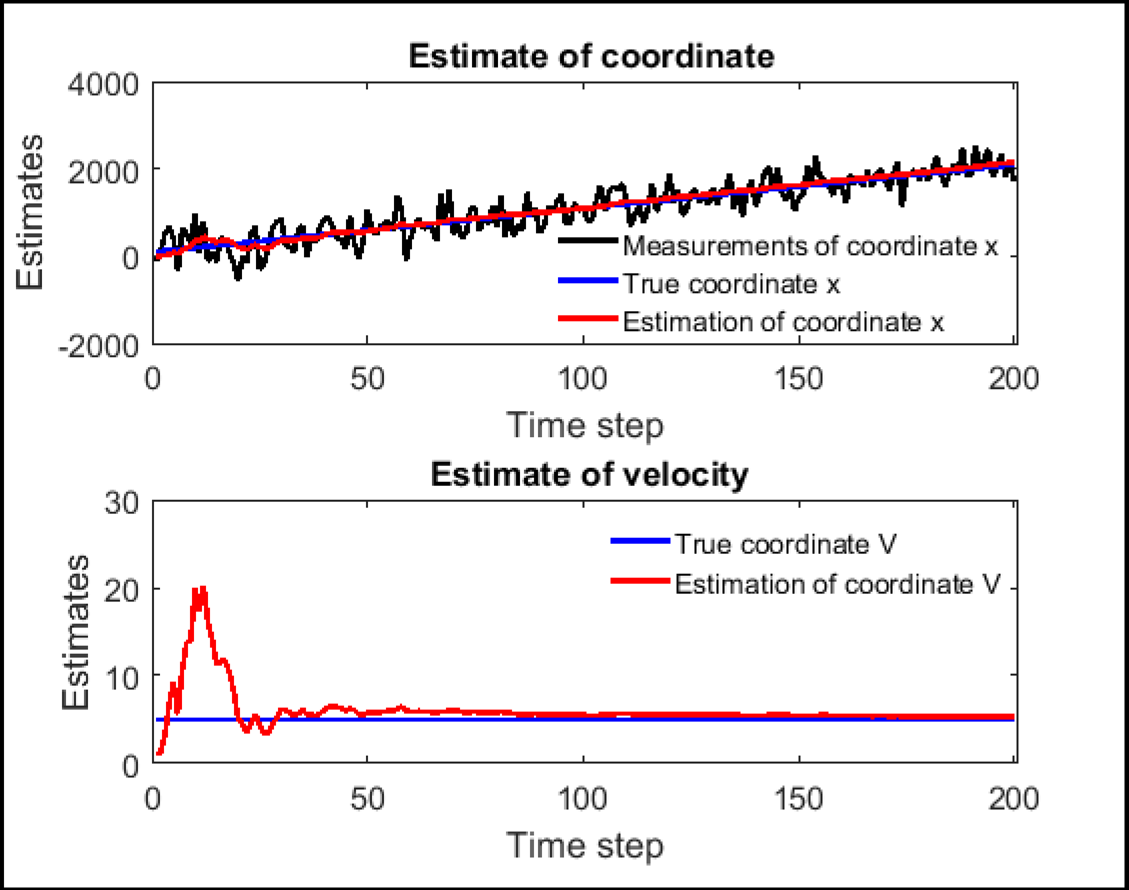
Coordinate x_i cannot be adjusted by measurements of V_i

Kalman filter minimizes the estimation error variance of velocity V, but not the coordinate x

The term "optimality" is applicable only for observable components

The initial error x_0 is kept during all the filtration interval

Non-observable system. Example 1



Measurements of only coordinate x_i are available

↓

System is observable

↓

Kalman filter provides estimation of full state vector X_i

Non-observable system. Example 2

$$X_i = \begin{vmatrix} x_i \\ y_i \end{vmatrix}$$

State vector

Measurements of sum
$x_i + y_i$
are available

Measurements of
x_i, and y_i
are not available

Non-observable system. Example 2

$X_i = \begin{vmatrix} x_i \\ y_i \end{vmatrix}$ → State vector

Measurements of sum $x_i + y_i$ are available

Measurements of x_i, and y_i are not available

The system at state space

State equation → $X_i = \Phi X_{i-1}$

Measurement equation → $z_i = \mathrm{H} X_i + \eta_i$

$\Phi = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ → Transition matrix

$H = \begin{vmatrix} 1 & 1 \end{vmatrix}$ → Observation matrix

Is it possible to estimate state vector X_i?

To apply Kalman filter we need to analyze observability of a system

The observability is determined only by pair of matrices Φ and H.

$$rank[H^T\ \Phi^T H^T\ (\Phi^T)^2 H^T\ \dots\ (\Phi^T)^{n-1} H^T] = n$$

For stationary system

n – dimension of state vector

To apply Kalman filter we need to analyze observability of a system

The observability is determined only by pair of matrices Φ and H.

$$rank[H^T \; \Phi^T H^T \; (\Phi^T)^2 H^T \; \dots \; (\Phi^T)^{n-1} H^T] = n$$

For stationary system

n – dimension of state vector

$$rank\left[H^T \; \Phi^T H^T \; (\Phi^T)^2 H^T \; \dots \; (\Phi^T)^{n-1} H^T\right] = q < n$$

Partial observability

$\frac{q}{n}$ → Observability degree

To apply Kalman filter we need to analyze observability of a system

$$rank[H^T\ \Phi^T H^T\ (\Phi^T)^2 H^T \ldots\ (\Phi^T)^{n-1} H^T] = n$$

Analysis of system observability for example 1

$\Phi = \begin{vmatrix} 1 & T \\ 0 & 1 \end{vmatrix}$ → Transition matrix

$H = |0 \quad 1|$ → Observation matrix

$X_i = \begin{vmatrix} x_i \\ V_i \end{vmatrix}$ → Dimension of state vector $n = 2$

$$rank[H^T\ \Phi^T H^T] = rank\left[\begin{vmatrix} 0 \\ 1 \end{vmatrix}\ \begin{vmatrix} 1 & 0 \\ T & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ 1 \end{vmatrix}\right] = rank\left[\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}\right] = 1$$

System is partly observable → Only one component is observable

To apply Kalman filter we need to analyze observability of a system

Observability Gramian W
for non-stationary system

$$W = \sum_{i=1}^{n} \Phi_{i,n}^T H_i^T H_i \Phi_{i,n} \succ 0$$

Positive-definite matrix

$\Phi_{i,n}$ is inverse matrix to transition matrix $\Phi_{n,i}$

$$\Phi_{n,i} = \Phi_{n,n-1} \cdot \Phi_{n-1,n-2} \cdots \cdot \Phi_{i+1,i}$$

Ill-conditioned problem
Example 1: scalar form

x ⇨ **Unknown variable**

Ill-conditioned problem
Example 1: scalar form

x → Unknown variable

We know → $c = 0.001x$ → $x = 1000c$

Ill-conditioned problem
Example 1: scalar form

x → Unknown variable

We know → $c = 0.001x$ → $x = 1000c$

Measurements of c → $z = c + \eta$ → $z = 0.001x + \eta$

Ill-conditioned problem
Example 1: scalar form

x → Unknown variable

We know → $c = 0.001x$ → $x = 1000c$

Measurements of c → $z = c + \eta$ → $z = 0.001x + \eta$

Estimation of x → $\hat{x} = 1000z$

Ill-conditioned problem
Example 1: scalar form

x → Unknown variable

We know → $c = 0.001x$ → $x = 1000c$

Measurements of c → $z = c + \eta$ → $z = 0.001x + \eta$

Estimation of x → $\hat{x} = 1000z$

Estimation error → $x - \hat{x} = 1000(z - c)$ → $x - \hat{x} = 1000\eta$

Ill-conditioned problem
Example 1: scalar form

x → Unknown variable

We know → $c = 0.001x$ → $x = 1000c$

Measurements of c → $z = c + \eta$ → $z = 0.001x + \eta$

Estimation of x → $\hat{x} = 1000z$

Estimation error → $x - \hat{x} = 1000(z - c)$ → $x - \hat{x} = 1000\eta$

Estimation error is very high

Ill-conditioned problem
Example 2: matrix form

X → **Unknown vector**

Ill-conditioned problem
Example 2: matrix form

X → Unknown vector

We know → $C = AX$ → $X = A^{-1}C$

Ill-conditioned problem
Example 2: matrix form

X → Unknown vector

We know → $C = AX$ → $X = A^{-1}C$

Measurements of C → $Z = C + \mathrm{E}$

E → Vector of measurement noise

Ill-conditioned problem
Example 2: matrix form

X → Unknown vector

We know → $C = AX$ → $X = A^{-1}C$

Measurements of C → $Z = C + \mathrm{E}$

E → Vector of measurement noise

Estimation of X → $\widehat{X} = A^{-1}Z$

Ill-conditioned problem
Example 2: matrix form

X → Unknown vector

We know → $C = AX$ → $X = A^{-1}C$

Measurements of C → $Z = C + \mathrm{E}$

E → Vector of measurement noise

Estimation of X → $\widehat{X} = A^{-1}Z$

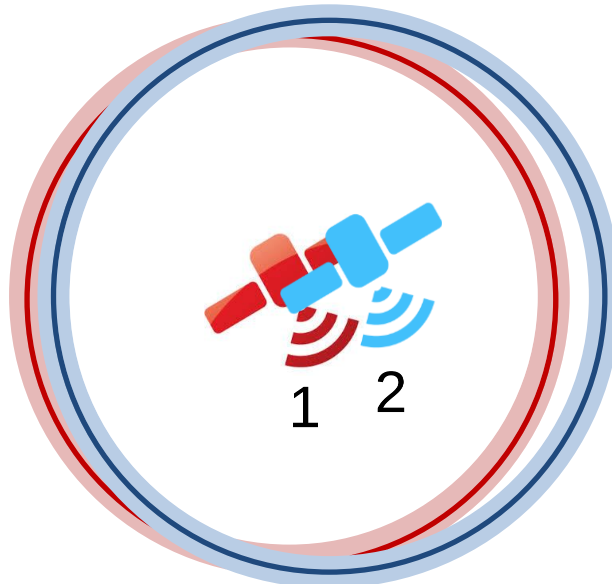
Estimation error → $X - \hat{X} = A^{-1}(Z - C)$ → $X - \hat{X} = A^{-1}E$

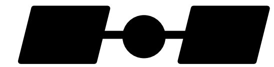
If matrix A is ill-conditioned, than it is close to singular

Ill-conditioned problem

1

Validity of applying a technique

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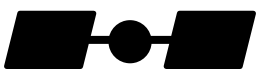
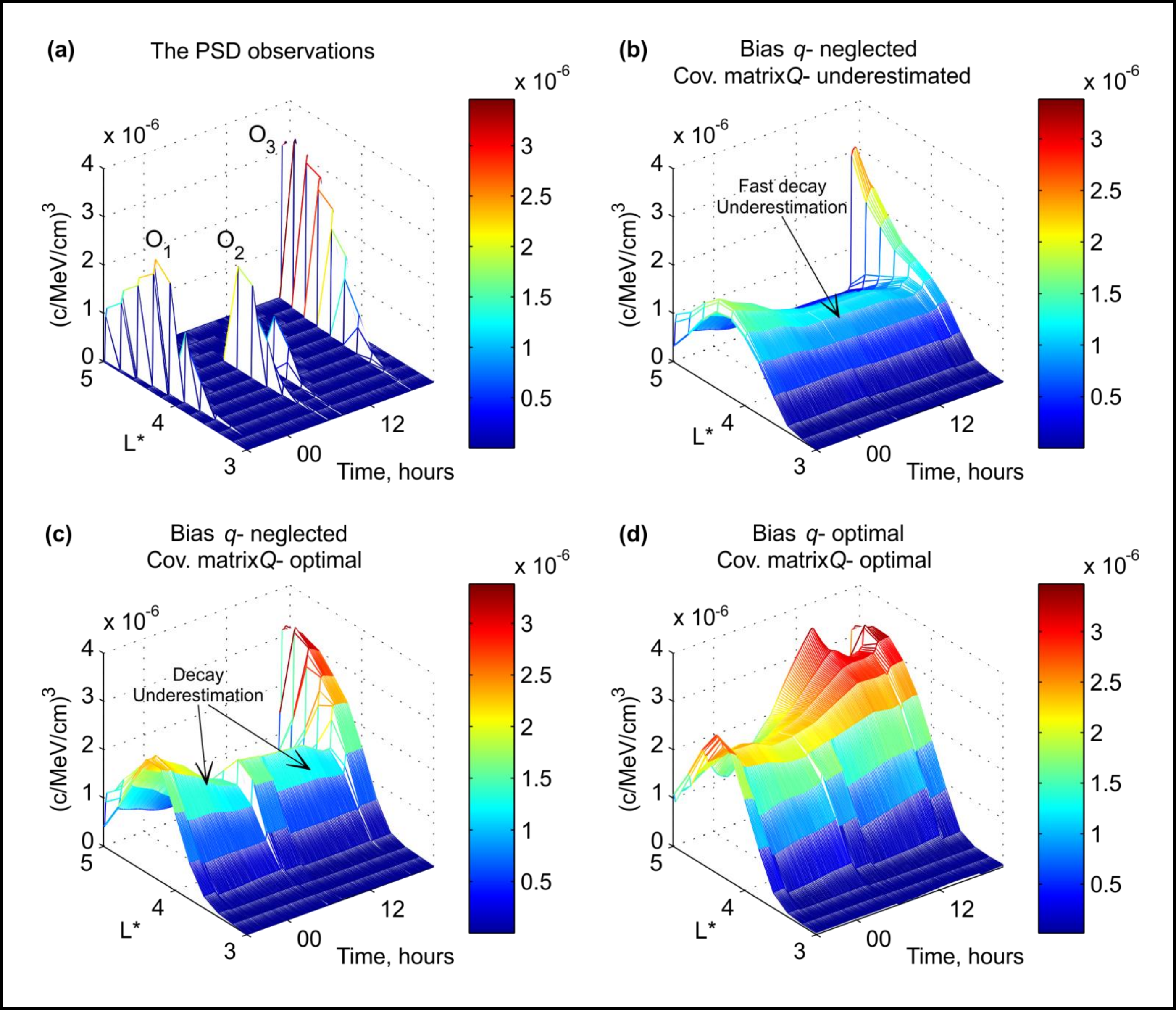

Man-made satellite


Navigation satellite

Il-conditioned problem

Satellite position is undefined!

Kalman filter needs noise statistics identification



Smoothing with fixed interval

Smoothing is performed in backward in time

$$X_{i,N} = X_{i,i} + A_i(X_{i+1,N} - \Phi_{i+1,\mathrm{i}} X_{i,i})$$

$$i = N-1, N-2, \cdots 1$$

Coefficient $A_i = P_{i,i}\Phi_{i+1,i}P_{i+1,i}^{-1}$

Smoothing error covariance matrix

$$P_{i,N} = P_{i,i} + A_i\left(P_{i+1,N} - P_{i+1,i}\right)A_i^T$$

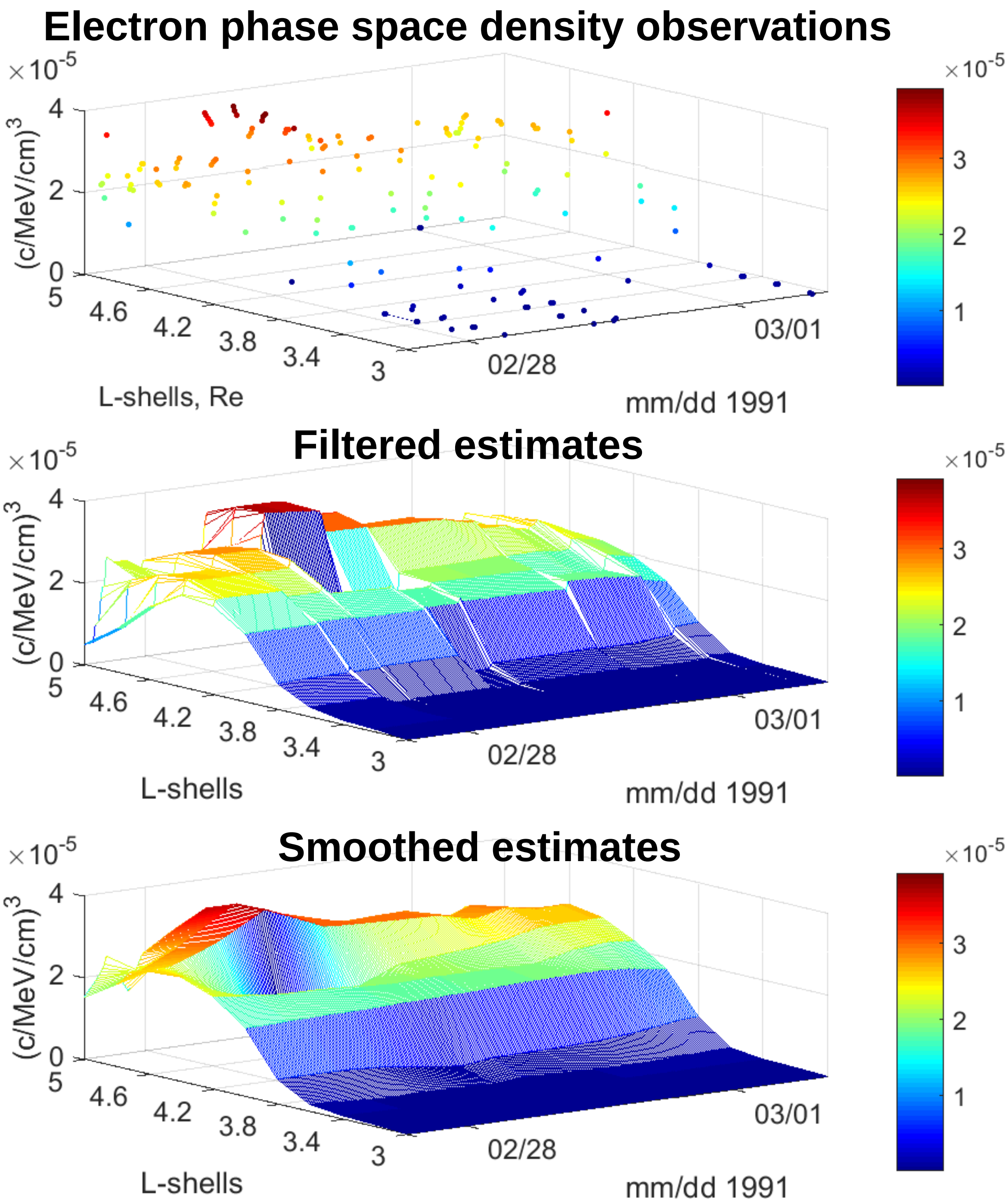
$X_{i,i}$ - filtered estimate, $X_{N,N}$ - initial estimate

$P_{i,i}$ - filtration error covariance matrix

$P_{i+1,i}$ - prediction error covariance matrix

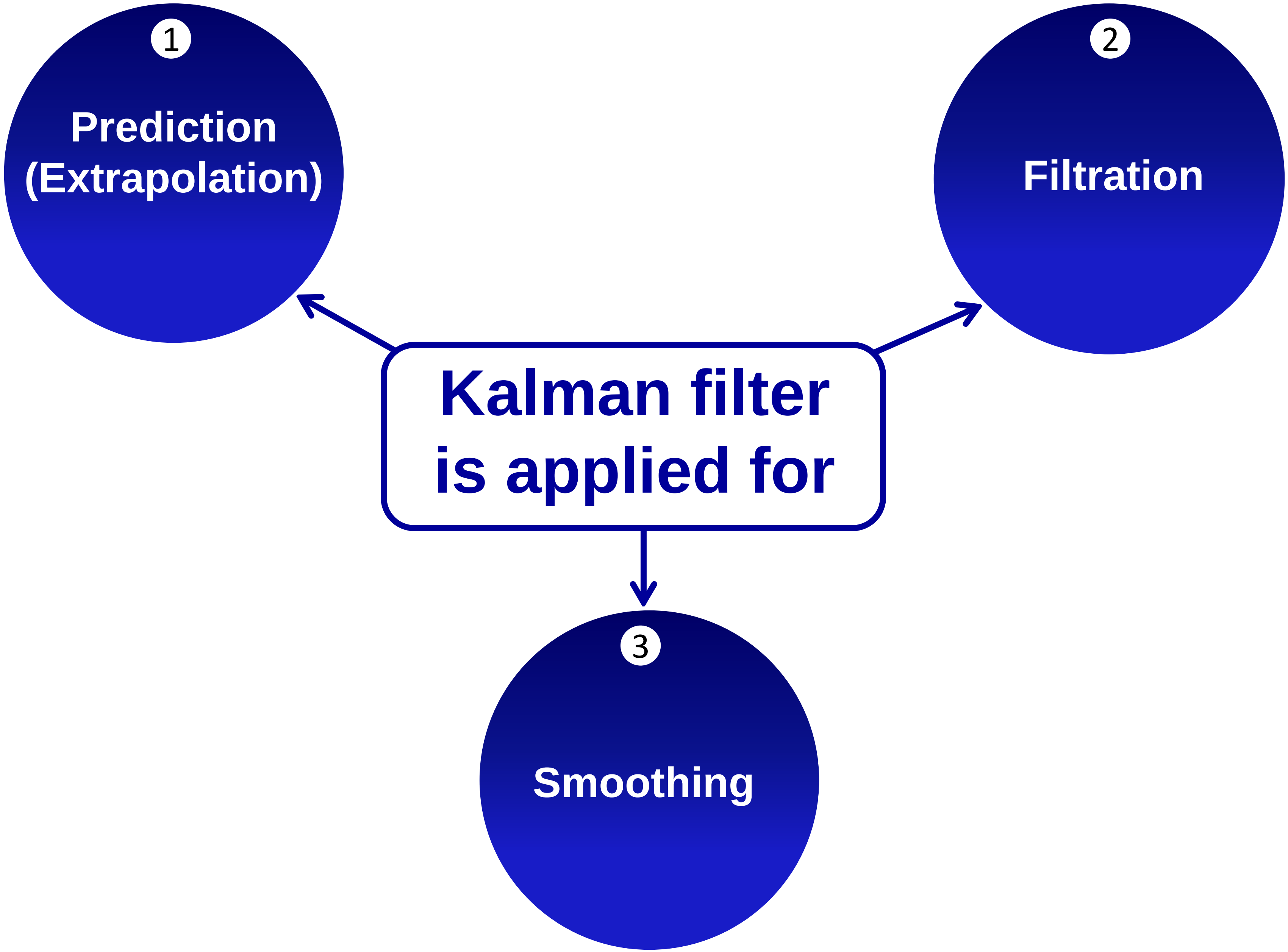
Smoothing takes into account both current and future measurements and therefore provides improved estimation

Reconstructing the dynamics of relativistic electrons in Earth’s radiation belts



Smoothing takes into account both current and future measurements and therefore provides improved estimation

Podladchikova et al. (2014), Noise statistics identification for Kalman filtering of the electron radiation belt observations: 2. Filtration and smoothing, J. Geophys. Res. Space Physics, 119


1
Prediction
(Extrapolation)
2
Filtration
Kalman filter
is applied for
3
Smoothing

Equivalence of exponential smoothing and stationary Kalman filter

Random walk model →

$$x_i = x_{i-1} + w_i$$
$$z_i = x_i + \eta_i$$

This is state space model with following parameters

$X_i = |x_i|$ → **State vector**

$\Phi = 1$ → **Transition matrix**

$H = 1$ → **Observation matrix**

Stationary Kalman filter →

$$x_{i,i} = x_{i-1,i-1} + K(z_i - x_{i-1,i-1})$$

Filter gain K becomes a constant

Exponential smoothing →

$$x_i = x_{i-1} + \alpha(z_i - x_{i-1})$$

Optimal α → $\alpha = K$

Conclusions

Kalman filter is effective tool for estimation and forecasting

However it requires good hands for tuning