GameStop: A study in the Black-Scholes Formula

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Abstract

This is a simple paragraph at the beginning of the document. A brief introduction to the main subject.

1 Introduction

2 What are shares, options, and short selling?

2.1 Shares

As defined in "Investopedia" [6], shares are "units of equity ownership in a corporation." Writing shares is an effective way to raise capital for a firm, as there is not legal mandate to be repaid to investors, and to not pay interest. Common shares usually offer a voting rights, giving shareholders more control over a business. The value of shares rises and falls with demand - meaning that owning a stock assumes the owner a level of risk. Many stocks also offer dividends to shareholders, which is a proportion of the businesses profits paid out directly.

2.2 Options

Again, as defined in [2], "the term option refers to a financial instrument based on the value of the underlying security, such as stocks." An options contract offers the holder the option, not the obligation (in contrast to a "future"), to buy or sell a share at a predetermined price. In themselves, they are a form of asset and as such have their own valuations and market value. A "call" option is when the writer of the option promises to sell the stock at a predetermined price, called the "strike" at a predetermined time, called the "strike date". This means that if you own a call option and the stock price increases above the strike, the writer will still have to honor the strike and so you can "exercise" the option and immediately sell the stock, where your profit is the difference between the strike and current price, less the cost of the option. However, if the price goes below the strike, you would be forced to let your option expire worthless, since by exercising the option you would pay above the market rate. In this way, the losses are limited but the profits are unlimited - you can only lose what you paid for the call options, but stand to gain an unlimited amount as the stock price increases.

On the other hand, if an investor expects a stock to decrease in value, they could write a call (also known as a short call). This means that if the stock has increase in value and you don't already own the security, you are obligated to purchase at the market rate, which could take any arbitrary value. Further to this, as the price increases over time, the short seller may be "margin called" - when to ensure the seller can pay, the broker requests that the seller puts more money into their account. If the short seller cannot afford this (and the further risk it entails), they can close their position - by buying the stocks they owe, the are protected from any further increases in price. However, by buying the stock off the open market they increase demand for the stock, increasing the price further, forcing other short sellers to close their positions, driving the price up further. This is what is known as a short squeeze.

There are also two types of options - a "european" and "american" derivative security. An european option can only be exercised at the strike date, while an american option can be exercised at any time up to the strike date. In this essay we shall only consider the simpler european option, although the american option is far more common in the financial markets.

3 What happened to GameStop?

3.1

4 Asset dynamics

4.1 A risk-free asset

A risk-free asset, as defined in The Black-Scholes Model [5] can be thought of as a money-market account with zero risk, for example a bank account with an interest rate. It is described by the deterministic function

$$dA(t) = rA(t)\mathrm{d}t \tag{1}$$

Where in 1, r represents the risk-free rate and A the value of the asset. We typically let $r > 0$ and $A(0) = 1$ for convenience. This can be rewritten as the ordinary differential equation $A'(t) = rA(t)$, which has the unique solution

$$A(t) = e^{rt} \tag{2}$$

4.2 Ito processes

According to Ito process [4], an Ito process is the stochastic (random) process $X = \{X_t, t >= 0\}$ which solves

$$X_t = X_0 + \int_0^t a(X_s, s)\mathrm{d}s + \int_0^t b(X_s, s)\mathrm{d}W_s \tag{3}$$

Here X_0 is the "scalar starting point", and a, b are stochastic processes defined for $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$. Then we call a the drift of the variable, and b is the diffusion, or more precisely in a financial context, volatity. 3 is commonly written as

$$X_t = a(X_t, t)\mathrm{d}t + b(X_t, t)\mathrm{d}W_t \tag{4}$$

or even

$$X_t = a_t\mathrm{d}t + b_t\mathrm{d}W_t \tag{5}$$

This converges at some time T to Brownian motion with instantaneous drift a_T and variance b_T^2. We let dW be normally distributed with mean zero and variance $\mathrm{d}t$. Then we can further simplify 3 as

$$\mathrm{d}X_t = a_t\mathrm{d}t + b_t\sqrt{\mathrm{d}t}\xi \tag{6}$$

where

$$\xi \sim \mathcal{N}(\mu, \sigma^2) \tag{7}$$

As defined in [4], Ito's lemma is

Lemma 4.1 (Ito's Lemma) *Suppose $f : R \to R$ is twice continuously differentiable and $\mathrm{d}X = a_t\mathrm{d}t + b_t\mathrm{d}W$. Then $f(X)$ is the Ito process,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)a_s\,\mathrm{d}s + \int_0^t f'(X_s)b_s\,\mathrm{d}W + \frac{1}{2}\int_0^t f''(X_s)b_s^2\,\mathrm{d}s \tag{8}$$

for $t \geq 0$

We shall also require Ito's existence/uniqueness theorem, as defined in [3]:

Theorem 4.2 (Ito's Existence And Uniqueness Theorem) *If $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are uniformally Lipschitz, then the stochastic differential equation 3 has "strong solutions". This means that for any standard Brownian motion $\{W_t\}_{t\geq 0}$, any admissable filtration $\mathbb{F} = \{\mathcal{F}_t\}$ and any initial value $x \in \mathbb{R}$ there exists a unique proccess $X_t = X_t^x$ which solves 3.*

4.3 A risky asset

A risk asset, as defined in The Black-Scholes Model [5], can be thought of as a stock, is represented as the Ito process

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t) \tag{9}$$

with $S(0)$ a given starting price of the stock, $\mu \in \mathbb{R}$ is again the drift and $\sigma > 0$ is the volatility of the stock price S. Using 4.2 we can prove existence and uniqueness of 9:

First we rewrite 9 in integral form:

$$S(t) = S(0) + \mu\int_0^t S(u)\,\mathrm{d}u + \sigma\int_0^t S(u)\,\mathrm{d}W(u). \tag{10}$$

Then we rewrite our coefficients in the form of 3:

$$\mu S(t) = a(t, S(t)), \qquad a(t,x) = \mu x \tag{11}$$
$$\sigma S(t) = b(t, S(t), \qquad b(t,x) = \sigma x \tag{12}$$

then we check Lipschitz continuity:

$$|a(t,x) - a(t,y)| = |\mu(x-y)| \leq |\mu||x-y|, \tag{13}$$
$$|b(t,x) - b(t,y)| = |\sigma(x-y)| \leq \sigma|x-y| \qquad \text{Since } \sigma > 0. \tag{14}$$

We then have that the unique solution to 9 is of the form

$$S(t) = S(0)exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\}. \tag{15}$$

This can be easily substituted into 10 show that this solves 9.

4.4 Considering the model parameters

Now we have a soluition to a risky asset, we can consider $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) = \quad S(t)\mathbb{E}(exp\{\mu - \frac{1}{2}\sigma t + \sigma W(t)\}) \tag{16}$$
$$= \quad S(0)exp\{\mu t - \frac{1}{2}\sigma^2 t\}\mathbb{E}(exp\{\sigma W(t)\} \tag{17}$$
$$= \quad S(0)exp\{\mu t\}). \tag{18}$$

Where at 17 we use the fact that

$$\mathbb{E}(exp\{X\}) = exp\{\frac{1}{2}Var(X)\}$$

18 then shows that if $\mu = 0$ we know that the expectation of S(t) is constant in time - it is not "drifting" in any direction, which justifies the naming of μ as drift. We can then rearrange for μ:

$$\mu = \frac{1}{t}\ln\frac{\mathbb{E}(S(t))}{S(0)} \tag{19}$$

which is the logarithmic return of the expected price. The variance of the return is

$$Var(\mu t - \frac{\sigma^2}{2}t + \sigma W(t)) = \quad Var(\sigma W(t)) \tag{20}$$
$$= \quad \sigma^2 t \qquad \text{since Var(W(t)) = t} \tag{21}$$

as defined in [5]. Clearly we want to try to calculate μ and σ to apply to our model. 19 could suggest taking past values of the stock value and taking an average to find a value of μ, but according to [5] the accuracy of this is poor. A more effective approximation of the volatility is, for example, the process

$$\ln S(t) = \ln S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)) \tag{22}$$

which is an Ito process with the constant characteristics $a(t) = (\mu - \frac{1}{2}\sigma^2)$ and $b(t) = \sigma$. By [1], we have that

$$(\mathrm{d}S(t))^2 = b^2(t)\mathrm{d}t \tag{23}$$

If we then partition $[0, t]$ given by $0 = t_1 < \cdots < t_n = t$ with small mesh max width $[t_{k+1} - t_k]$, we have

$$(\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 \mathrm{d}t \tag{24}$$

then

$$\sum_k (\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 t \tag{25}$$

$$\implies \sigma = \sqrt{\frac{1}{2}\sum_k (\ln \frac{S(t_{k+1})}{S(t_k)})^2} \tag{26}$$

which is a much better estimate of the volatility coefficient, and is described as the sample volatility.

5 Options

5.1 Assumptions

As in [5], before we can begin to derive the Black-Scholes Model, we must state some assumptions.

5.1.1 Existence of replicating strategy

This assumption says that any option investment can be perfectly replciated by holdings in the stock and a money market account directly. A key part of investing is the substantial "leverage" employed in an option, as for a small (as a percentage of the share value) cost you can purchase a large number of shares and sell them for a profit, without having to risk investing such a large amount of money directly into the stock. Our assumption, however, says that a "replicating strategy" always exists. More rigorously, we say that there exists a pair of processes (x, y) which satisfy

$$H(t) = x(t)S(t) + y(t)A(t) \tag{27}$$

where $H(t)$ is the option value at time T, an Ito proces.$S(t)$ and $A(t)$ are the risky and riskless assets as defined in Asset Dynamics 4. As described in [5], this assumption "captures the idea that changes in the values and holdings of assets are sole drivers of changes of wealth".

Additionally we assume that the process $H(t)$ is of the form

$$H(t) = u(t, S(t)). \tag{28}$$

This deterministic function $u(t, z)$ is not dependant on the history of the stock price. It is assmued to have continuous first derivative wrt $x \in [0, T]$ and continuous first and second derivatives in $z \in \mathbb{R}$.

5.1.2 Assumption 2

There again exists a replicating strategy (x, y) which satisfies

$$\mathrm{d}H(t) = x(t)\mathrm{d}S(t) + y(t)\mathrm{d}A(t) \tag{29}$$

where H, S, x, y and A are as defined in 5.1.1.

5.1.3 Assumption 3

There exists a probability Q

Applying 4.1, we find that $H(t)$ is the Ito process with stochastic differential

$$\mathrm{d}H = (u_t + \mu S u_z + \frac{1}{2}\sigma^2 S^2 u_{zz})\mathrm{d}t + \sigma S u_z \mathrm{d}W \tag{30}$$

From 29, we have

$$\mathrm{d}H = x(t)\mathrm{d}S(t) + \mathrm{d}x(t)S(t) \tag{31}$$

so

$$\mathrm{d}H = (x\mu S + ryA)\mathrm{d}t + x\sigma S\mathrm{d}W \tag{32}$$

from 1 and 9.

Then we equate the right hand side of 30 and 32:

$$u_t + \mu S u_z + \frac{1}{2}\sigma^2 S^2 u_{zz} = x\mu S + ryA \quad \text{and} \tag{33}$$

$$\sigma S u_z = x\sigma S. \tag{34}$$

Now we can start to solve these equations: 34 gives us

$$x(t) = u_z(t, S(t)) \tag{35}$$

which we can use to eliminate x from 33:

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{zz} = ryA \tag{36}$$

which we then may use to solve for $y(t)$:

$$y(t) = \frac{1}{rA(t)}(u_t(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)u_{zz}(t, S(t))). \tag{37}$$

where I have reintroduced the arguements for all the functions.

Finally, we use 5.1.1 and our expressions for x and y to give us

$$u(t, S(t)) = u_z(t, S(t))S(t) + \frac{1}{r}\left(u_t(t, S(t) + \frac{1}{2}\sigma^2 S^2(t)u_{zz}(t, S(t))\right) \tag{38}$$

$$\text{since } H(t) = u(t, S(t)) \tag{39}$$

We now simply rearrange for $u_t(t, z)$:

$$u_t(t, z) = -\frac{1}{2}\sigma^2 x^2 u_{zz}(t, z) - rzu_z(t, z) + ru(t, z) \qquad \text{for } 0 < t < T,\ z \in \mathbb{R}. \tag{40}$$

We also have the boundary condition that $H(T)$ is equal to the option payoff at the strike date, in other words

$$u(T, z) = \max(z - K, 0) \qquad \text{for } z \in \mathbb{R} \tag{41}$$

where K is the strike price of the option. We use the max since there is no point exericsing the option if the strike is above the market value.

6 The Black-Scholes Model

7 Modelling GameStop

8 Extensions to the Black-Scholes Model

9 Conclusion

References

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