

GameStop: A study in the Black-Scholes Formula

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Abstract

This essay is about the Black-Scholes formula, and how it could be used to explain the short squeeze of GameStop in early 2021. We begin with explaining shares, options and short selling, and move on to deriving the Black-Scholes formula, and finally apply it to GameStop.

1 What are shares, options, and short selling?

1.1 Shares

As defined in "Investopedia" [9], shares are "units of equity ownership in a corporation." Writing shares is an effective way to raise capital for a firm, as there is not legal mandate to be repaid to investors, and to not pay interest. Common shares usually offer a voting rights, giving shareholders more control over a business. The value of shares rises and falls with demand - meaning that owning a stock assumes the owner a level of risk. Many stocks also offer dividends to shareholders, which is a proportion of the businesses profits paid out directly.

1.2 Options

Again, as defined in [3], "the term option refers to a financial instrument based on the value of the underlying security, such as stocks." An options contract offers the holder the option, not the obligation (in contrast to a "future"), to buy or sell a share at a predetermined price. In themselves, they are a form of asset and as such have their own valuations and market value. A "call" option is when the writer of the option promises to sell the stock at a predetermined price, called the "strike" at a predetermined time, called the "strike date". This means that if you own a call option and the stock price increases above the strike, the writer will still have to honor the strike and so you can "exercise" the option and immediately sell the stock, where your profit is the difference between the strike and current price, less the cost of the option. However, if the price goes below the strike, you would be forced to let your option expire worthless, since by exercising the option you would pay above the market rate. In this way, the losses are limited but the profits are unlimited - you can only lose what you paid for the call options, but stand to gain an unlimited amount as the stock price increases.

On the other hand, if an investor expects a stock to decrease in value, they could write a call (also known as a short call). This means that if the stock has increase in value and you don't already own the security, you are obligated to purchase at the market rate, which could take any arbitrary value. Further to this, as the price increases over time, the short seller may be "margin called" - when to ensure the seller can pay, the broker requests that the seller puts more money into their account. If the short seller cannot afford this (and the further risk it entails), they can close their position - by buying the stocks they owe, the are protected from any further increases in price. However, by buying the stock off the open market they increase demand for the stock, increasing the price further, forcing other short sellers to close their positions, driving the price up further. This is what is known as a short squeeze.

There are also two types of options - a "european" and "american" derivative security. An european option can only be exercised at the strike date, while an american option can be exercised at any time up to the strike date. In this essay we shall only consider the simpler european option, although the american option is far more common in the financial markets.

2 Short squeezes and GameStop

A short squeeze is relatively uncommon occurrence in financial markets, but they do happen, normally due to substantial short interest in the stock, i.e. a large number of the shares sold short, and a small available float - the number of shares available to purchase on the market. This means, as the short sellers (traders writing call options) are forced to close their positions, they are fighting over a small number of stocks, inflating the price to extraordinary values.

For example, in 2008, Porsche began to buy large amounts of Volkswagen stock off the market. This meant that only 6% of the stock was available on the market, while 13% of the stock was sold short. This caused a scramble for the final few shares, and as

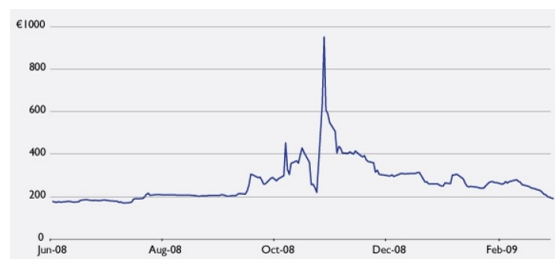


Figure 1: The stock price of Volkswagen during the short squeeze of 2008

short sellers were squeezed out by the high price and were forced to close their position

by buying more of the stock, the price went higher and higher, and Volkswagen briefly became the highest value company in the world. The squeeze only ended when Porsche announced it would sell around 5% of its stake in order to make life easier for the hedge funds - it is estimated that the squeeze cost short sellers around £30 billion, according to [8], with massive profits going over to Porsche.

The case of GameStop is similar - there was an extreme amount of short interest in the share, so that at one point more than %100 of the stock was sold short. With COVID

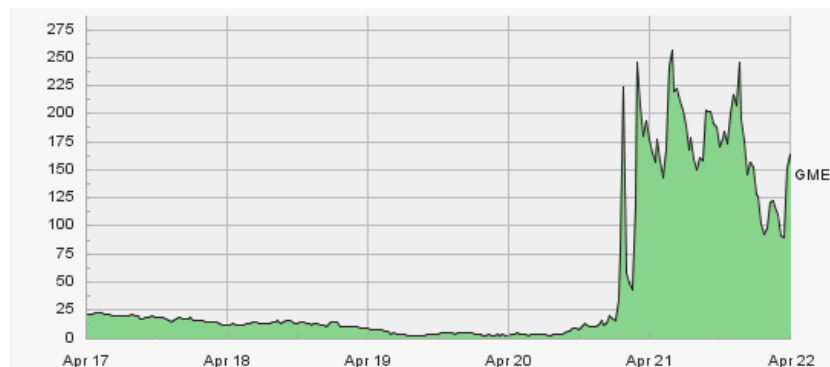


Figure 2: The stock price of GameStop over the past 5 years.

and the death of the high street, a brick and mortar store like GameStop was considered an extremely safe short sell - and as you can see from 2 until early 2021, the price of GameStop was dropping extremely consistently. This meant that it had an extremely high short interest, as you can see in 3. This was noticed by a messageboard on the internet, called "r/wallstreetbets" [7]. The independant investors saw this as greed from

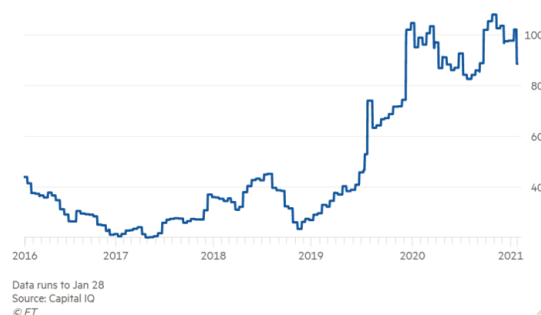


Figure 3: The short interest in GameStop over the past 5 years.

the large hedge funds making billions of safe money from the demise of a high street store, and when Ryan Cohen, who had just created the hugely sucessful "Chewy", the largest online pet retailer, joined GameStop's board as chairman. Investors on r/wallstreetbets saw this as a very positive sign, as they imagined Ryan could turn GameStop into

”Amazon, but for computer games”. Hence they attempted to create a short squeeze in GameStop, hoping to profit from the greed of the hedge funds. They began purchasing the stock, both increasing the price and reducing the available float, so that as short sellers attempted to close their positions due to rising prices, they found themselves fighting over a smaller and smaller number of shares, similar to the Volkswagen case. This caused a significant short squeeze in the price, profiting the independant investors who had invested early on, as seen in 4.

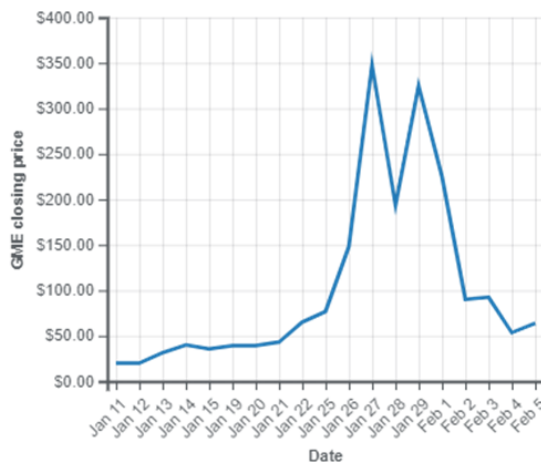


Figure 4: The stock price of GameStop in early 2021, during the squeeze.

3 Asset dynamics

However, the squeeze was short lived - as the stock increased tenfold, independant investors began to cash out on their profits and sell their stock, the pressure on the short sellers became reduced and they were able to exit their positions. This is in contrast to the Volkswagen case, where the shares were all owned by one party, who completely refused to sell its shares, pushing the share price much higher.

3.1 A risk-free asset

A risk-free asset, as defined in The Black-Scholes Model [6] can be thought of as a money-market account with zero risk, for example a bank account with an interest rate. It is described by the deterministic function

$$dA(t) = rA(t)dt \quad (1)$$

Where in 1, r represents the risk-free rate and A the value of the asset. We typically let $r > 0$ and $A(0) = 1$ for convenience. This can be rewritten as the ordinary differential equation $A'(t) = rA(t)$, which has the unique solution

$$A(t) = e^{rt} \quad (2)$$

3.2 Ito processes

According to Ito process [5], an Ito process is the stochastic (random) process $X = \{X_t, t \geq 0\}$ which solves

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s \quad (3)$$

Here X_0 is the "scalar starting point", and a, b are stochastic processes defined for $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$. Then we call a the drift of the variable, and b is the diffusion, or more precisely in a financial context, volatility. 3 is commonly written as

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t \quad (4)$$

or even

$$X_t = a_t dt + b_t dW_t \quad (5)$$

This converges at some time T to Brownian motion with instantaneous drift a_T and variance b_T^2 . We let dW be normally distributed with mean zero and variance dt . Then we can further simplify 3 as

$$dX_t = a_t dt + b_t \sqrt{dt} \xi \quad (6)$$

where

$$\xi \sim \mathcal{N}(\mu, \sigma^2) \quad (7)$$

As defined in [5], Ito's lemma is

Lemma 3.1 (Ito's Lemma) *Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is twice continuously differentiable and $dX = a_t dt + b_t dW$. Then $f(X)$ is the Ito process,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) a_s ds + \int_0^t f'(X_s) b_s dW + \frac{1}{2} \int_0^t f''(X_s) b_s^2 ds \quad (8)$$

for $t \geq 0$

We shall also require Ito's existence/uniqueness theorem, as defined in [4]:

Theorem 3.2 (Ito's Existence And Uniqueness Theorem) *If $\mu : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}_+$ are uniformly Lipschitz, then the stochastic differential equation 3 has "strong solutions". This means that for any standard Brownian motion $\{W_t\}_{t \geq 0}$, any admissible filtration $\mathbb{F} = \{\mathcal{F}_t\}$ and any initial value $x \in \mathbb{R}$ there exists a unique process $X_t = X_t^x$ which solves 3.*

3.3 A risky asset

A risk asset, as defined in The Black-Scholes Model [6], can be thought of as a stock, is represented as the Ito process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (9)$$

with $S(0)$ a given starting price of the stock, $\mu \in \mathbb{R}$ is again the drift and $\sigma > 0$ is the volatility of the stock price S . Using 3.2 we can prove existence and uniqueness of 9:

First we rewrite 9 in integral form:

$$S(t) = S(0) + \mu \int_0^t S(u) du + \sigma \int_0^t S(u) dW(u). \quad (10)$$

Then we rewrite our coefficients in the form of 3:

$$\mu S(t) = a(t, S(t)), \quad a(t, x) = \mu x \quad (11)$$

$$\sigma S(t) = b(t, S(t)), \quad b(t, x) = \sigma x \quad (12)$$

then we check Lipschitz continuity:

$$|a(t, x) - a(t, y)| = |\mu(x - y)| \leq |\mu||x - y|, \quad (13)$$

$$|b(t, x) - b(t, y)| = |\sigma(x - y)| \leq \sigma|x - y| \quad \text{Since } \sigma > 0. \quad (14)$$

We then have that the unique solution to 9 is of the form

$$S(t) = S(0)\exp\left\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\right\}. \quad (15)$$

This can be easily substituted into 10 show that this solves 9.

3.4 Considering the model parameters

Now we have a solution to a risky asset, we can consider $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) = S(t)\mathbb{E}(\exp\{\mu - \frac{1}{2}\sigma^2 t + \sigma W(t)\}) \quad (16)$$

$$= S(0)\exp\{\mu t - \frac{1}{2}\sigma^2 t\}\mathbb{E}(\exp\{\sigma W(t)\}) \quad (17)$$

$$= S(0)\exp\{\mu t\}. \quad (18)$$

Where at 17 we use the fact that

$$\mathbb{E}(\exp\{X\}) = \exp\{\frac{1}{2}\text{Var}(X)\}$$

18 then shows that if $\mu = 0$ we know that the expectation of $S(t)$ is constant in time - it is not "drifting" in any direction, which justifies the naming of μ as drift. We can then rearrange for μ :

$$\mu = \frac{1}{t} \ln \frac{\mathbb{E}(S(t))}{S(0)} \quad (19)$$

which is the logarithmic return of the expected price. The variance of the return is

$$\text{Var}(\mu t - \frac{\sigma^2}{2}t + \sigma W(t)) = \text{Var}(\sigma W(t)) \quad (20)$$

$$= \sigma^2 t \quad \text{since } \text{Var}(W(t)) = t \quad (21)$$

as defined in [6]. Clearly we want to try to calculate μ and σ to apply to our model. 19 could suggest taking past values of the stock value and taking an average to find a value of μ , but according to [6] the accuracy of this is poor. A more effective approximation of the volatility is, for example, the process

$$\ln S(t) = \ln S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \quad (22)$$

which is an Ito process with the constant characteristics $a(t) = (\mu - \frac{1}{2}\sigma^2)$ and $b(t) = \sigma$. By [2], we have that

$$(dS(t))^2 = b^2(t)dt \quad (23)$$

If we then partition $[0, t]$ given by $0 = t_1 < \dots < t_n = t$ with small mesh max width $[t_{k+1} - t_k]$, we have

$$(\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 dt \quad (24)$$

then

$$\sum_k (\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 t \quad (25)$$

$$\implies \sigma = \sqrt{\frac{1}{2} \sum_k \left(\ln \frac{S(t_{k+1})}{S(t_k)} \right)^2} \quad (26)$$

which is a much better estimate of the volatility coefficient, and is described as the sample volatility.

4 Options

4.1 Assumptions

As in [6], before we can begin to derive the Black-Scholes Model, we must state some assumptions.

4.1.1 Existence of replicating strategy

This assumption says that any option investment can be perfectly replciated by holdings in the stock and a money market account directly. A key part of investing is the substantial "leverage" employed in an option, as for a small (as a percentage of the share value) cost you can purchase a large number of shares and sell them for a profit, without having to risk investing such a large amount of money directly into the stock. Our assumption, however, says that a "replicating strategy" always exists. More rigorously, we say that there exists a pair of processes (x, y) which satisfy

$$H(t) = x(t)S(t) + y(t)A(t) \quad (27)$$

where $H(t)$ is the option value at time T , an Ito proces. $S(t)$ and $A(t)$ are the risky and riskless assets as defined in Asset Dynamics 3. As described in [6], this assumption "captures the idea that changes in the values and holdings of assets are sole drivers of changes of wealth".

Additionally we assume that the process $H(t)$ is of the form

$$H(t) = u(t, S(t)). \quad (28)$$

This deterministic function $u(t, z)$ is not dependant on the history of the stock price. It is assmued to have continuous first derivative wrt $x \in [0, T]$ and continuous first and second derivatives in $z \in \mathbb{R}$.

4.1.2 Assumption 2

There again exists a replicating strategy (x, y) which satisfies

$$dH(t) = x(t)dS(t) + y(t)dA(t) \quad (29)$$

where H , S , x , y and A are as defined in 4.1.1.

4.1.3 Assumption 3

There exists a probability Q

Applying 3.1 with $u(t, Z) = f(Z)$ we get

$$du(Z) = u_z(Z)\mu dt + u_z(Z)\sigma dW + \frac{1}{2}u_{zz}(Z)\sigma^2 dt \quad \text{and} \quad (30)$$

$$du(t) = u_t \quad \text{then} \quad (31)$$

$$dH = du = du(Z) + du(t) = (u_t + \mu S u_z + \frac{1}{2}\sigma^2 S^2 u_{zz})dt + + \sigma S u_z dW \quad (32)$$

as a stochastic differential. From 29, we have

$$dH = x(t)dS(t) + dx(t)S(t) \quad (33)$$

so

$$dH = (x\mu S + ryA)dt + x\sigma SdW \quad (34)$$

from 1 and 9.

Then we equate the right hand side of 30 and 34:

$$u_t + \mu S u_z + \frac{1}{2} \sigma^2 S^2 u_{zz} = x\mu S + ryA \quad \text{and} \quad (35)$$

$$\sigma S u_z = x\sigma S. \quad (36)$$

Now we can start to solve these equations: 36 gives us

$$x(t) = u_z(t, S(t)) \quad (37)$$

which we can use to eliminate x from 35:

$$u_t + \frac{1}{2} \sigma^2 S^2 u_{zz} = ryA \quad (38)$$

which we then may use to solve for $y(t)$:

$$y(t) = \frac{1}{rA(t)} (u_t(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) u_{zz}(t, S(t))). \quad (39)$$

where I have reintroduced the arguments for all the functions.

Finally, we use 4.1.1 and our expressions for x and y to give us

$$u(t, S(t)) = u_z(t, S(t))S(t) + \frac{1}{r} \left(u_t(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) u_{zz}(t, S(t)) \right) \quad (40)$$

$$\text{since } H(t) = u(t, S(t)) \quad (41)$$

We now simply rearrange for $u_t(t, z)$:

$$u_t(t, z) = -\frac{1}{2} \sigma^2 z^2 u_{zz}(t, z) - rz u_z(t, z) + ru(t, z) \quad \text{for } 0 < t < T, z \in \mathbb{R}. \quad (42)$$

We also have the boundary condition that $H(T)$ is equal to the option payoff at the strike date, so in the case of a call option we can immediately buy the stock to lock in our profits, so at $t = 0$ we have

$$u(0, z) = \max(z - K, 0) \quad \text{for } z \in \mathbb{R} \quad (43)$$

where K is the strike price of the option. We use the max since the option has no value if the strike is above the market value. We can then consider when $z = 0$ and $z \rightarrow \infty$, as in [1]. When $z = 0$ we notice from 9 that $dS = 0$ so $z = 0$ is constant and

$$u(t, 0) = 0. \quad (44)$$

As $z \rightarrow \infty$, it becomes more likely we exercise the option, and the strike price becomes less relevant to u , so the value of $u(t, Z)$ is equivalent to z . Therefore we have the boundary condition

$$u(t, z) = z \quad \text{as } z \rightarrow \infty \quad (45)$$

We have finally reached the general Black-Scholes initial value boundary problem

$$u(t, z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t, z) + rz u_z(t, z) - ru(t, z) = 0 \quad \text{for } (t, z) \in (0, T) \times \mathbb{R}_+ \quad (46)$$

$$\text{with initial condition:} \quad u(0, z) = \max(z - K, 0), \quad z \in \mathbb{R}_+ \quad (47)$$

$$\text{and boundary conditions:} \quad u(t, 0) = 0, \quad u(t, z) = z \text{ as } z \rightarrow \infty, \quad t \in [0, T] \quad (48)$$

Where we remember that:

$$u(t, z) - \text{price of the option} \quad (49)$$

$$z - \text{price of the underlying stock} \quad (50)$$

$$K - \text{strike price of the option} \quad (51)$$

$$r - \text{annualized risk-free interest rate, continuously compounded} \quad (52)$$

$$t - \text{time, generally in years, with now as } t = 0 \text{ and expiry } t = T \quad (53)$$

$$\sigma - \text{the volatility of the underlying stock} \quad (54)$$

5 The Black-Scholes Model

Now we have derived and fully defined our model, we may consider final and boundary conditions of a European Call, as defined in 1.2. We define this as $C(t, z)$, with exercise price K and expiry date T from above. Now for the final condition $t = T$ we appeal to the definition of a call as in 43:

$$C(T, z) = \max(z - K, 0) \quad (55)$$

Now for our boundary conditions again we consider u for $z = 0$ and $z \rightarrow \infty$. Similarly to 44 we have

$$C(t, 0) = 0 \quad \text{for } t \in (0, T) \quad (56)$$

Then for $z \rightarrow \infty$

6 Modelling GameStop

7 Extensions to the Black-Scholes Model

8 Conclusion

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