GameStop: A study in the Black-Scholes Formula

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Abstract

This is a simple paragraph at the beginning of the document. A brief introduction to the main subject.

1 Introduction

2 What are shares, options, and short selling?

2.1 Shares

As defined in "Investopedia" [6], shares are "units of equity ownership in a corporation." Writing shares is an effective way to raise capital for a firm, as there is not legal mandate to be repaid to investors, and to not pay interest. Common shares usually offer a voting rights, giving shareholders more control over a business. The value of shares rises and falls with demand - meaning that owning a stock assumes the owner a level of risk. Many stocks also offer dividends to shareholders, which is a proportion of the businesses profits paid out directly.

2.2 Options

Again, as defined in [2], "the term option refers to a financial instrument based on the value of the underlying security, such as stocks." An options contract offers the holder the option, not the obligation (in contrast to a "future"), to buy or sell a share at a predetermined price. In themselves, they are a form of asset and as such have their own valuations and market value. A "call" option is when the writer of the option promises to sell the stock at a predetermined price, called the "strike" at a predetermined time, called the "strike date". This means that if you own a call option and the stock price increases above the strike, the writer will still have to honor the strike and so you can "exercise" the option and immediately sell the stock, where your profit is the difference between the strike and current price, less the cost of the option. However, if the price goes below the strike, you would be forced to let your option expire worthless, since by exercising the option you would pay above the market rate. In this way, the losses are limited but the profits are unlimited - you can only lose what you paid for the call options, but stand to gain an unlimited amount as the stock price increases.

On the other hand, if an investor expects a stock to decrease in value, they could write a call (also known as a short call). This means that if the stock has increase in value and you don't already own the security, you are obligated to purchase at the market rate, which could take any arbitrary value. Further to this, as the price increases over time, the short seller may be "margin called" - when to ensure the seller can pay, the broker requests that the seller puts more money into their account. If the short seller cannot afford this (and the further risk it entails), they can close their position - by buying the stocks they owe, the are protected from any further increases in price. However, by buying the stock off the open market they increase demand for the stock, increasing the price further, forcing other short sellers to close their positions, driving the price up further. This is what is known as a short squeeze.

3 What happened to GameStop?

4 Asset dynamics

4.1 A risk-free asset

A risk-free asset, as defined in The Black-Scholes Model [5] can be thought of as a money-market account with zero risk, for example a bank account with an interest rate. It is described by the deterministic function

$$dA(t) = rA(t)\mathrm{d}t \tag{1}$$

Where in 1, r represents the risk-free rate and A the value of the asset. We typically let $r > 0$ and $A(0) = 1$ for convenience. This can be rewritten as the ordinary differential equation $A'(t) = rA(t)$, which has the unique solution

$$A(t) = e^{rt} \tag{2}$$

4.2 Ito processes

According to Ito process [4], an Ito process is the stochastic (random) process $X = \{X_t, t >= 0\}$ which solves

$$X_t = X_0 + \int_0^t a(X_s, s)\mathrm{d}s + \int_0^t b(X_s, s)\mathrm{d}W_s \tag{3}$$

Here X_0 is the "scalar starting point", and a, b are stochastic processes defined for $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$. Then we call a the drift of the variable, and b is the diffusion, or more precisely in a financial context, volatity. 3 is commonly written as

$$X_t = a(X_t, t)\mathrm{d}t + b(X_t, t)\mathrm{d}W_t \tag{4}$$

or even

$$X_t = a_t\mathrm{d}t + b_t\mathrm{d}W_t \tag{5}$$

This converges at some time T to Brownian motion with instantaneous drift a_T and variance b_T^2. We let dW be normally distributed with mean zero and variance dt. Then we can further simplify 3 as

$$\mathrm{d}X_t = a_t \mathrm{d}t + b_t \sqrt{\mathrm{d}t}\xi \tag{6}$$

where

$$\xi \sim \mathcal{N}(\mu, \sigma^2) \tag{7}$$

As defined in [4], Ito's lemma is

Lemma 4.1 (Ito's Lemma) *Suppose $f : R \to R$ is twice continuously differentiable and $\mathrm{d}X = a_t \mathrm{d}t + b_t \mathrm{d}W$. Then $f(X)$ is the Ito process,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) a_s \,\mathrm{d}s + \int_0^t f'(X_s) b_s \,\mathrm{d}W + \frac{1}{2}\int_0^t f''(X_s) b_s^2 \,\mathrm{d}s \tag{8}$$

for $t \geq 0$

We shall also require Ito's existence/uniqueness theorem, as defined in [3]:

Theorem 4.2 (Ito's Existence And Uniqueness Theorem) *If $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are uniformally Lipschitz, then the stochastic differential equation 3 has "strong solutions". This means that for any standard Brownian motion $\{W_t\}_{t\geq 0}$, any admissable filtration $\mathbb{F} = \{\mathcal{F}_t\}$ and any initial value $x \in \mathbb{R}$ there exists a unique proccess $X_t = X_t^x$ which solves 3.*

4.3 A risky asset

A risk asset, as defined in The Black-Scholes Model [5], can be thought of as a stock, is represented as the Ito process

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t) \tag{9}$$

with $S(0)$ a given starting price of the stock, $\mu \in \mathbb{R}$ is again the drift and $\sigma > 0$ is the volatility of the stock price S. Using 4.2 we can prove existence and uniqueness of 9:

First we rewrite 9 in integral form:

$$S(t) = S(0) + \mu \int_0^t S(u)\,\mathrm{d}u + \sigma \int_0^t S(u)\,\mathrm{d}W(u). \tag{10}$$

Then we rewrite our coefficients in the form of 3:

$$\mu S(t) = a(t, S(t)), \qquad a(t,x) = \mu x \tag{11}$$

$$\sigma S(t) = b(t, S(t), \qquad b(t,x) = \sigma x \tag{12}$$

then we check Lipschitz continuity:

$$|a(t,x) - a(t,y)| = |\mu(x-y)| \leq |\mu||x-y|, \tag{13}$$
$$|b(t,x) - b(t,y)| = |\sigma(x-y)| \leq \sigma|x-y| \qquad \text{Since } \sigma > 0. \tag{14}$$

We then have that the unique solution to 9 is of the form

$$S(t) = S(0)exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\}. \tag{15}$$

This can be easily substituted into 10 show that this solves 9.

4.4 Considering the model parameters

Now we have a soluition to a risky asset, we can consider $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) = S(t)\mathbb{E}(exp\{\mu - \frac{1}{2}\sigma t + \sigma W(t)\}) \tag{16}$$
$$= S(0)exp\{\mu t - \frac{1}{2}\sigma^2 t\}\mathbb{E}(exp\{\sigma W(t)\} \tag{17}$$
$$= S(0)exp\{\mu t\}). \tag{18}$$

Where at 17 we use the fact that

$$\mathbb{E}(exp\{X\}) = exp\{\frac{1}{2}Var(X)\}$$

18 then shows that if $\mu = 0$ we know that the expectation of S(t) is constant in time - it is not "drifting" in any direction, which justifies the naming of μ as drift. We can then rearrange for μ:

$$\mu = \frac{1}{t}\ln\frac{\mathbb{E}(S(t))}{S(0)} \tag{19}$$

which is the logarithmic return of the expected price. The variance of the return is

$$Var(\mu t - \frac{\sigma^2}{2}t + \sigma W(t)) = Var(\sigma W(t)) \tag{20}$$
$$= \sigma^2 t \qquad \text{since Var(W(t)) = t} \tag{21}$$

as defined in [5]. Clearly we want to try to calculate μ and σ to apply to our model. 19 could suggest taking past values of the stock value and taking an average to find a value of μ, but according to [5] the accuracy of this is poor. A more effective approximation of the volatility is, for example, the process

$$\ln S(t) = \ln S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)) \tag{22}$$

which is an Ito process with the constant characteristics $a(t) = (\mu - \frac{1}{2}\sigma^2)$ and $b(t) = \sigma$. By [1], we have that

$$(\mathrm{d}S(t))^2 = b^2(t)\mathrm{d}t \tag{23}$$

If we then partition $[0, t]$ given by $0 = t_1 < \cdots < t_n = t$ with small mesh max width $[t_{k+1} = t_k]$, we have

$$(\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 \mathrm{d}t \tag{24}$$

then

$$\sum_k (\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 t \tag{25}$$

$$\implies \sigma = \sqrt{\frac{1}{2}\sum_k (\ln \frac{S(t_{k+1})}{S(t_k)})^2} \tag{26}$$

which is a much better estimate of the volatility coefficient, and is described as the sample volatility.

5 Options

6 The Black-Scholes Model

7 Modelling GameStop

8 Extensions to the Black-Scholes Model

9 Conclusion

References

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