GameStop: A study in the Black-Scholes Formula

Luke Stanislaus 2009701

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Contents

Abstract

This essay is about the Black-Scholes formula, and how it could be used to explain the short squeeze of GameStop in early 2021. We begin with explaining shares, options and short selling, and move on to deriving the Black-Scholes formula, and finally apply it to GameStop.

1 What are shares, options, and short selling?

1.1 Shares

As defined in [**shares**], shares are "units of equity ownership in a corporation." Writing shares can be an effective way to raise capital for a firm, as there is no legal mandate to be repaid to investors, and they do not need to pay interest. However, common shares usually offer voting rights, giving shareholders control over a business. There can be some cases of businesses where being private means it can struggle to functions effectively. For example, SpaceX, owned by Elon Musk, has remained a private company, as shareholders can tend to look very closely at short term profits, taking less considerating into the long term of a company, and a space launch company can struggle to create consistent income, and in this case a small group of investors who are fully onboard with the company vision will allow the company to lose money in the short term in order to gain longer term profits. This was also the plan with Tesla, as the company struggled to scale up production its new car and had to spend lots of money in research.

The value of shares rises and falls with demand - meaning that owning a stock assumes the owner a certain level of risk. Many stocks also offer dividends to shareholders, which is a proportion of the businesses profits paid out directly and allows low growth, highly profitable business to offer value to shareholders.

1.2 Options

Again, as defined in [**options**], "the term option refers to a financial instrument based on the value of the underlying security, such as stocks." An options contract offers the holder the option, not the obligation (in contrast to a "future"), to buy or sell a share at a predetermined price. In themselves, they are a form of asset and as such have their own valuations and market value. A "call" option is when the writer of the option promises to sell the stock at a predetermined price, (called the "strike") at a predetermined time, (called the "exercise date"). This means that if you own a call option and the stock price increases above the strike price, the writer will still have to honor the strike and so you can "exercise" the option and immediately sell the stock, where your profit is the difference between the strike and current price, less the cost of the option. However, if the price goes below the strike, you would be forced to let your option expire worthless, since by exercising the option you would pay above the market rate. In this way, the losses are limited but the profits are unlimited - you can only lose what you paid for the call options, but stand to gain an unlimited amount as the stock price increases.

On the other hand, if an investor expects a stock to decrease in value, they could write a call (also known as a short call). This means that if the stock has increase in value and you don't already own the security, you are obligated to purchase at the market rate, which could take any arbitrary value. Further to this, as the price increases over time, the short seller may be "margin called" - when to ensure the seller can pay, the broker requests that the seller puts more money into their account. If the short seller cannot afford this (and the further risk it entails), they can close their position - by buying the stocks they owe, they are protected from any further increases in price. However, by buying the stock off the open market they increase demand for the stock, increasing the price further, forcing other short sellers to close their positions, driving the price up further. This is what is known as a short squeeze. There isn't really a symmetrical "long squeeze" - the lowest a stock can be valued is zero, so is limited to the value of the stocks sold long, hence a short squeeze is a unique phenomenon in this respect.

There are also two types of options - a "european" and "american" derivative security. An european option can only be exercised at the exercise date, while an american option can be exercised at any time up to the exercise date. In this essay we shall only consider the simpler european option, although the american option is far more common in the financial markets.

2 Short squeezes and GameStop

A short squeeze is relatively uncommon occurance in financial markets, but they do happen, normally due to substancial short interest in the stop, i.e. a large number of the shares sold short, an a small available float - the number of shares available to purchase on the market. This means, as the short sellers (traders writing call options) are forced to close their positions, they are fighting over a small number of stocks, inflating the price to extraordinary values.

An extreme of an extremely artificial short squeeze was Volkswagen in 2008, where Porsche began to buy large amounts of Volkswagen stock off the market, as they were looking to purchase the business. This meant that only 6% of the stock was available on the market, while 13% of the stock was sold short. This caused a scramble for the

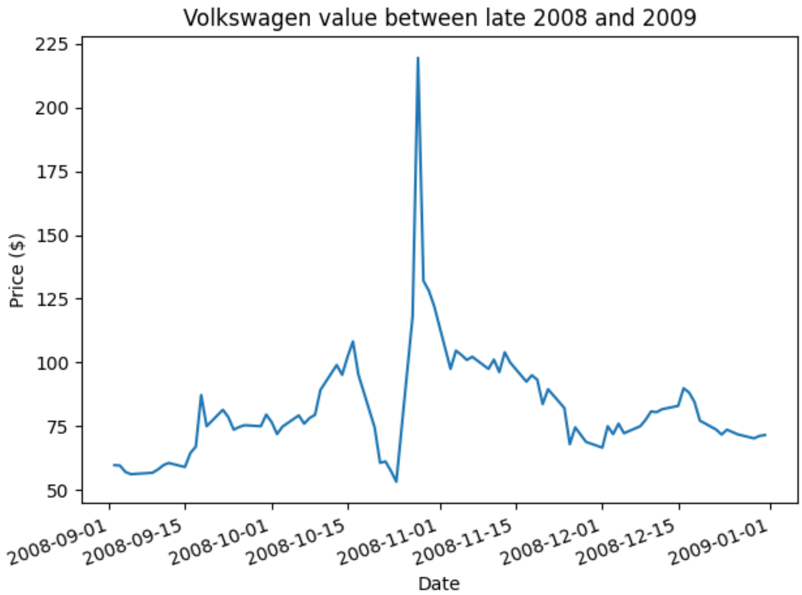

Figure 1: The stock price of Volkswagen during the short squeeze of 2008

final few shares, and as short sellers were squeezed out by the high price and were forced to close their position by buying more of the stock, the price went higher and higher, and Volkswagen briefly became the highest value company in the world, reaching a peak value of around \$1000, up from around \$50 just before. The squeeze only ended when Porsche announced it would sell around 5% of its stake in order to make life easier for the hedge funds - it is estimated that the squeeze cost short sellers around £30 billion, according to [**volkswagen**], with massive profits going over to Porsche, in the midst of a financial crisis!

The case of GameStop is similar - there was an extreme amount of short interest in the share, so that at one point more than 100% of the stock was sold short. With COVID and the death of the high street, a brick and mortar store like GameStop was considered an extremely safe short sell - and as you can see from 2 until mid 2020, the price of GameStop was dropping extremely consistently. As we shall study, a low volatilty history will mean that short sellers can become highly leveraged against a stock, as low stock volatility implies low option volatilty - it is very unlikely that the short sellers is forced to close their position by their broker due to lack of funds - the worst possible outcome for the short seller, as there is a chance that a jump is short term the underlying stock drops below its strike price before the option expiry. This meant that it had an

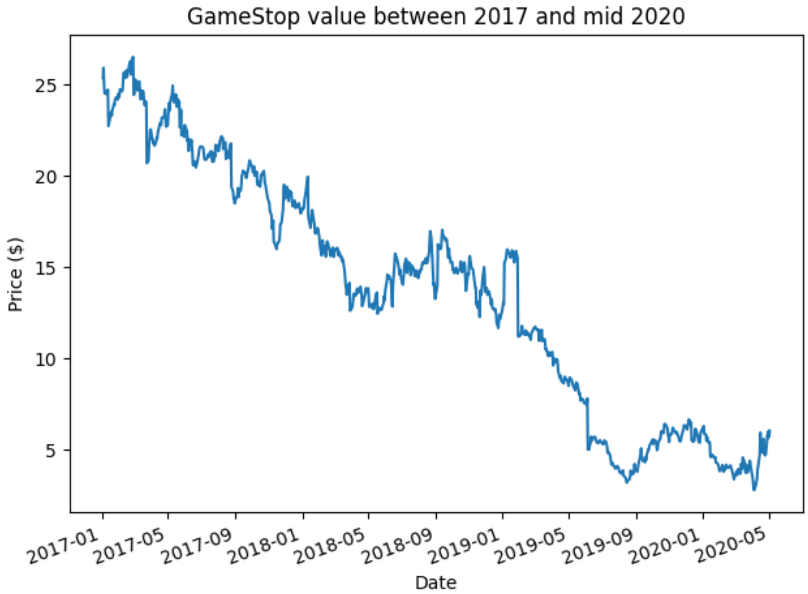

Figure 2: GameStop daily closing price, data from [**nasdaq**]

extremely high short interest, as you can see in **??**. This was noticed by a messageboard on the internet, called "r/wallstreetbets" [**wsb**]. The independant investors saw this as greed from the large hedge funds making billions of safe money preying on the demise of a beloved high street store. and when Ryan Cohen, who had just created the hugely sucessful "Chewy", the largest online pet retailer, joined GameStop's board as chairman. Investors on *r/wallstreetbets* saw this as a very positive sign, as they invisaged Ryan turning GameStop into Amazon, but for video games. Hence they organised a concerted attempt to create a short squeeze in GameStop, hoping to profit from the eccessive greed of the hedge funds. They began purchasing the stock, both increasing the price and reducing the available float, so that as short sellers attempted to close their positions due to rising prices, they found themselves fighting over a smaller and smaller number of shares, similar to the Volkswagen case. This caused a signfiicant short squeeze in the price, profiting the internet investors who had invested early on, as seen in **??**. In fact, I myself was a regular visitor to these internet boards, and in early 2020 I myself ended up investing 10 stocks back when it was worth just $40 - the period of massive growth afterwards was an extremely exciting time for me, as the world learned about what a short squeeze is. However, the squeeze was short lived - as the stock increased tenfold, independant investors began to cash out on their profits and sell their stock, the pressure on the short sellers became reduced and they were able to exit their positions. This is in contrast to the Volkswagen case, where the shares were all owned by one party, who completely refused to sell its shares, pushing the share price much higher. Meanwhile the messageboards were desperately sharing the message of "diamond hands", the concept that shareholders should hold onto their stocks, minimizing the available

Figure 3: The short interest in GameStop over the past 5 years.

float and exacerbaing the squeeze.

3 Asset dynamics

In our attempt to study the Black-Scholes equation, set out our underlying variables and the different variables we need to find in order to study a stock and its options.

3.1 A risk-free asset

First we shall define a risk-free asset, which we shall need to define in order to calculate the cost of value of an option at time t. This is because we can't do as described in 1.2, where we could simply buy another option from the market. Since this is what we are trying to calculate, this would make no sense! Therefore we must take out a loan for the stock, where in the case of A risk-free asset, as defined in the Black-Scholes Model [**blackscholes**] can be thought of as a money-market account with zero risk, for example a bank account with an interest rate. It is described by the deterministic function

$$dA(t) = rA(t)\mathrm{d}t \tag{1}$$

Where in (**??**), r represents the risk-free rate and A the value of the asset. We typically let $r > 0$ and $A(0) = 1$ for convenience. This can be rewritten as the ordinary differential equation $A'(t) = rA(t)$, which has the unique solution

$$A(t) = e^{rt} \tag{2}$$

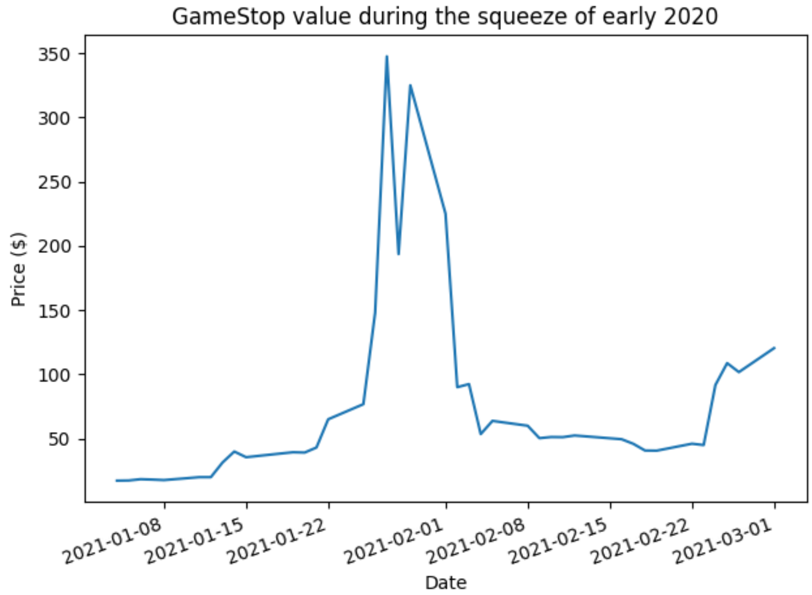

Figure 4: The stock price of GameStop during the squeeze.

3.2 Ito processes

According to Ito process [**itoprocess**], an Ito process is the stochastic (random) process $X = \{X_t, t >= 0\}$ which solves

$$X_t = X_0 + \int_0^t a(X_s, s)\mathrm{d}s + \int_0^t b(X_s, s)\mathrm{d}W_s \tag{3}$$

Here X_0 is the "scalar starting point", and a, b are stochastic processes defined for $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$. Then we call a the drift of the variable, and b is the diffusion, or more precisely in a financial context, volatity. (**??**) is commonly written as

$$X_t = a(X_t, t)\mathrm{d}t + b(X_t, t)\mathrm{d}W_t \tag{4}$$

or even

$$X_t = a_t\mathrm{d}t + b_t\mathrm{d}W_t \tag{5}$$

This converges at some time T to Brownian motion with instantaneous drift a_T and variance b_T^2. We let dW be normally distributed with mean zero and variance dt. Then we can further simplify (**??**) as

$$\mathrm{d}X_t = a_t\mathrm{d}t + b_t\sqrt{\mathrm{d}t}\xi \tag{6}$$

where

$$\xi \sim \mathcal{N}(\mu, \sigma^2) \tag{7}$$

As defined in [**itoprocess**], Ito's lemma is

Lemma 3.1 (Ito's Lemma). *Suppose $f : R \to R$ is twice continuously differentiable and $\mathrm{d}X = a_t \mathrm{d}t + b_t \mathrm{d}W$. Then $f(X)$ is the Ito process,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) a_s \,\mathrm{d}s + \int_0^t f'(X_s) b_s \,\mathrm{d}W + \frac{1}{2}\int_0^t f''(X_s) b_s^2 \,\mathrm{d}s \tag{8}$$

for $t \geq 0$

Proof. Let G be a continuous a differentiable function of variables x, t. Then by Taylor's Theorem we find

$$\Delta G = \frac{\delta G}{\delta x}\Delta x + \frac{\delta G}{\delta t}\Delta t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}\Delta x^2 + \frac{1}{2}\frac{\delta^2 G}{\delta t^2}\Delta t^2 + \dots \tag{9}$$

which in the limit of $\Delta x, \Delta t \to 0$, (**??**) approaches

$$\mathrm{d}G = \frac{\delta G}{\delta x}\mathrm{d}x + \frac{\delta G}{\delta t}\mathrm{d}t \tag{10}$$

Now we consider (**??**) to cover functions following Ito processes as in (**??**), i.e.

$$\mathrm{d}x = a(x,t)\mathrm{d}t + b(x,t)\mathrm{d}z \tag{11}$$

and G is a function of x of time t. We can rewrite (**??**) as discrete variables:

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t} \qquad \text{where } \epsilon \sim \mathcal{N}(0,1) \tag{12}$$

using the definiton of a Wiener process, as defined in [**optionsderivatives**]. Then we have $\mathrm{E}(\epsilon^2) = 1$ and $\mathrm{E}(\epsilon^2 \Delta t) = \Delta t$ and $\mathrm{Var}(\epsilon^2 \Delta t) = 2\Delta t^2$. Since the variance is too small to have a stochastic component, we can treat Δx^2 as nonstochastic (or non-random) and drop the normal distribution, so we have

$$\Delta x^2 = b^2 \Delta x.$$

then we are done: using (**??**)

$$\begin{aligned}
\Delta G = & \quad \frac{\delta G}{\delta x}\mathrm{d}x + \frac{\delta G}{\delta t}\mathrm{d}t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}\mathrm{d}x^2 + \frac{1}{2}\frac{\delta^2 G}{\delta t^2}\mathrm{d}t^2 + \dots \approx & (13)\\
= & \quad \frac{\delta G}{\delta x}\mathrm{d}x + \frac{\delta G}{\delta t}\mathrm{d}t + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2\mathrm{d}x & (14)\\
= & \quad \left(\frac{\delta G}{\delta x}a + \frac{\delta G}{\delta t} + \frac{1}{2}\frac{\delta^2 G}{\delta x^2}b^2\right)\mathrm{d}t + \frac{\delta G}{\delta x}b\mathrm{d}z & (15)
\end{aligned}$$

This completes our proof of Ito's lemma, based on the proof of [**optionsderivatives**]. □

We shall also require Ito's existence/uniqueness theorem, as defined in [**SDE**]:

Theorem 3.2 (Ito's Existence And Uniqueness Theorem). *If $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are uniformally Lipschitz, then the stochastic differential equation (**??**) has "strong solutions". This means that for any standard Brownian motion $\{W_t\}_{t\geq 0}$, any admissable filtration $\mathbb{F} = \{\mathcal{F}_t\}$ and any initial value $x \in \mathbb{R}$ there exists a unique proccess $X_t = X_t^x$ which solves (**??**).*

3.3 A risky asset

A risk asset, as defined in The Black-Scholes Model [**blackscholes**], can be thought of as a stock, is represented as the Ito process

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t) \tag{16}$$

with $S(0)$ a given starting price of the stock, $\mu \in \mathbb{R}$ is again the drift and $\sigma > 0$ is the volatility of the stock price S. Using (**??**) we can prove existence and uniqueness of (**??**):

First we rewrite (**??**) in integral form:

$$S(t) = S(0) + \mu \int_0^t S(u)\,\mathrm{d}u + \sigma \int_0^t S(u)\,\mathrm{d}W(u). \tag{17}$$

Then we rewrite our coefficients in the form of (**??**):

$$\mu S(t) = a(t, S(t)), \qquad a(t,x) = \mu x \tag{18}$$

$$\sigma S(t) = b(t, S(t), \qquad b(t,x) = \sigma x \tag{19}$$

then we check Lipschitz continuity:

$$|a(t,x) - a(t,y)| = |\mu(x-y)| \leq |\mu||x-y|, \tag{20}$$

$$|b(t,x) - b(t,y)| = |\sigma(x-y)| \leq \sigma|x-y| \qquad \text{Since } \sigma > 0. \tag{21}$$

We then have that the unique solution to (**??**) is of the form

$$S(t) = S(0)exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\}. \tag{22}$$

This can be easily substituted into (**??**) show that this solves (**??**).

3.4 Considering the model parameters

Now we have a soluition to a risky asset, we can consider $\mathbb{E}(S(t))$:

$$\mathbb{E}(S(t)) = \qquad S(t)\mathbb{E}(exp\{\mu - \frac{1}{2}\sigma t + \sigma W(t)\}) \tag{23}$$

$$= \qquad S(0)exp\{\mu t - \frac{1}{2}\sigma^2 t\}\mathbb{E}(exp\{\sigma W(t)\} \tag{24}$$

$$= \qquad S(0)exp\{\mu t\}). \tag{25}$$

Where at (**??**) we use the fact that

$$\mathbb{E}(exp\{X\}) = exp\{\frac{1}{2}Var(X)\}$$

(**??**) then shows that if $\mu = 0$ we know that the expectation of S(t) is constant in time - it is not "drifting" in any direction, which justifies the naming of μ as drift. We can then rearrange for μ:

$$\mu = \frac{1}{t} \ln \frac{\mathbb{E}(S(t))}{S(0)} \tag{26}$$

which is the logarithmic return of the expected price. The variance of the return is

$$Var(\mu t - \frac{\sigma^2}{2} t + \sigma W(t)) = \quad Var(\sigma W(t)) \tag{27}$$

$$= \quad \sigma^2 t \qquad \text{since Var(W(t)) = t} \tag{28}$$

as defined in [**blackscholes**]. Clearly we want to try to calculate μ and σ to apply to our model. (**??**) could suggest taking past values of the stock value and taking an average to find a value of μ, but according to [**blackscholes**] the accuracy of this is poor. A more effective approximation of the volatility is, for example, the process

$$\ln S(t) = \ln S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)) \tag{29}$$

which is an Ito process with the constant characteristics $a(t) = (\mu - \frac{1}{2}\sigma^2)$ and $b(t) = \sigma$. By [**quadtraticvariation**], we have that

$$(\mathrm{d}S(t))^2 = b^2(t)\mathrm{d}t \tag{30}$$

If we then partition $[0, t]$ given by $0 = t_1 < \cdots < t_n = t$ with small mesh max width $[t_{k+1} - t_k]$, we have

$$(\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 \mathrm{d}t \tag{31}$$

then

$$\sum_k (\ln S(t_{k+1}) - \ln S(t_k))^2 \approx \sigma^2 t \tag{32}$$

$$\implies \sigma = \sqrt{\frac{1}{2} \sum_k (\ln \frac{S(t_{k+1})}{S(t_k)})^2} \tag{33}$$

which is a much better estimate of the volatility coefficient, and is described as the sample volatility.

4 Options

5 The Black-Scholes Model

Starting from the model derived and fully defined in **??**, we may consider final and boundary conditions of a European Call, as defined in 1.2. We define this as $C(t, z)$, with exercise price K and expiry date T from above. Now for the final condition $t = T$ we appeal to the definition of a call as in (**??**):

$$C(T, z) = \max(z - K, 0) \tag{34}$$

Now for our boundary conditions again we consider u for $z = 0$ and $z \to \infty$. Simiarly to (**??**) we have

$$C(t, 0) = 0 \qquad \text{for } t \in (0, T) \tag{35}$$

Then for $z \to \infty$, again we can ignore out max. However, in this case, we must purchase the stock and hold on to it until the exercise date, as we have a european option. This money spent on the stock could otherwise be put in a money market account, so the profit from exercising the option must account for this. From (**??**), we get that

$$C(t, z) = z - Ke^{-r(T-t)} \qquad \text{as } z \to \infty \tag{36}$$

We then finally get the following, the Black-Scholes European Call option IBVP:

$$u_t(t, z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t, z) + rzu_z(t, z) - ru(t, z) = 0 \qquad \text{for } (t, z) \in (0, T) \times \mathbb{R}_+ \tag{37}$$

$$\text{with initial condition:} \qquad u(0, z) = \max(z - K, 0),\ z \in \mathbb{R}_+ \tag{38}$$

$$\text{and boundary conditions:} \qquad u(t, 0) = 0,\ u(t, z) = z - Ke^{-r(T-t)} \tag{39}$$

$$\text{as } z \to \infty,\ t \in [0, T] \tag{40}$$

6 Modelling GameStop

We begin by using (**??**) a calculate a value of σ. Using data from [**nyse**] between 2018 - 2020, in the attached excel book we calculate $\sigma = 1.007310782$. We also set $r = 1.004$, approximately the average bank rate between 2018 - 2020. (**??**) shows how returns from options are far more consistent and safe with a low a low volatility, like in the GameStop case. In the above case we have used the solution from [**wikipedia**], so the upper to sigma is $x > 0$ so:

$$x = \ln\frac{z}{K} + (r - \frac{1}{2}\sigma^2)\tau > 0 \qquad \text{with } \tau,\ x \text{ as defined in [\textbf{wikipedia}]} \tag{41}$$

$$\implies |\sigma| < \sqrt{\frac{2}{\tau}\ln\frac{z}{K} + 2r} \approx 1.435 \tag{42}$$

Modifying values of K, the strike price, we see in (**??**) the increased risk of buying options with a more expensive strike price, which makes sense - if the strike of an option

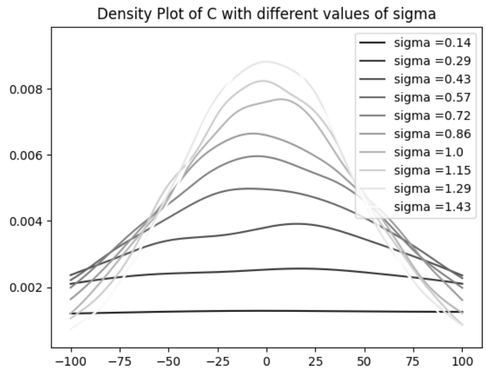

Figure 5

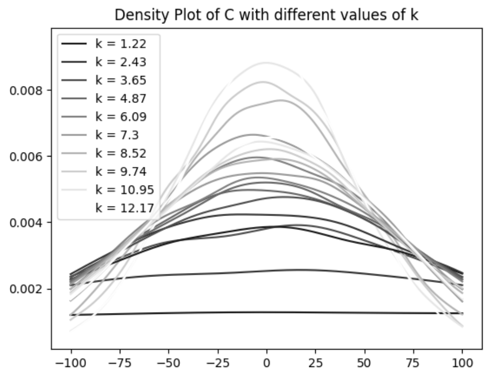

Figure 6

is near the current price, it is more likely that the stock drops below this and so expires worthless. Again here we have a mathematical limit on the value of k:

$$\ln \frac{z}{K} + (r - \frac{1}{2}\sigma^2)\tau > 0 \tag{43}$$

$$\implies |k| < ze^{\frac{1}{2}\sigma^2 - r} \approx 12.171166822082068 \tag{44}$$

We finish with an analysis for different values of τ, where we notice that the time to expiry has no effect on the distribution of u, the value of the option.

$$\ln \frac{z}{K} + (r - \frac{1}{2}\sigma^2)\tau > 0 \tag{45}$$

$$\implies |\tau| > \frac{\ln \frac{K}{z}}{r - \frac{1}{2}\sigma^2} \approx -0.0510 \quad \text{although note that time is defined to be nonegative.} \tag{46}$$

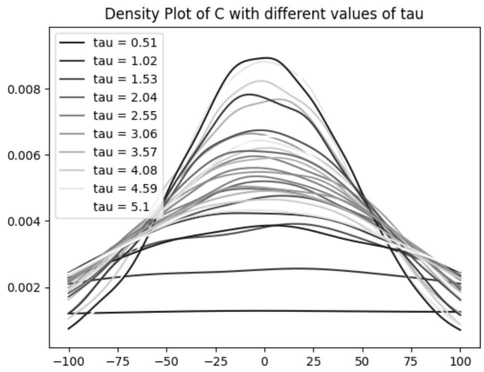

Figure 7

7 Conclusion

In conclusion, it is apparent that GameStop was an extremely safe investment for these hedge funds - with low volatility and and slow decline, these head funds could gain millions in low risk money of the period of many years, which is how it ended up gaining such a massive short interest which allowed the squeeze to occur so easily. However, this is a perfect example of where the Black-Scholes formula becomes extremely ineffective - when there is a sudden change in volatility, the Black-Scholes will become a less effective model as it essentialy expects a stock to perform similarly as in the past, so if there is a high volatility (induced, for example, by people on the internet) alongside high short interest, due to the way the markets are set up billions of dollars can change hands in a very short amount of time!

A Risk-free value of an option

A key fact to note is that even with a european option, all parties involved can close their positions at any time.

Option holders

A call options holder has the option to either have the option worth nothing, i.e. losing the premium paid , or could lock in the value of the option at time t as equal to difference between the strike price and the market price. At time t the investor can decide to lock in their profits risk-free, by taking a loan in stocks equal to the size of the option and immediately selling the loaned stock. We then give the money back to the person who loaned to us, so we pay the risk-free rate. Then at expiry, exercise the option and use these stocks to pay back the loan. Then the value is the difference between the current price at time t and the strike K plus the interest paid i on the loan of the underlying security z. Considering a call option we have

$$C(z,t) = z - (K + i)$$

and then we calculate interest using (**??**) with negative r, so over time period $T - t$ where T is expiry, we have risk-free interest paid equal to $z - ze^{-r(T-t)}$ and putting this together we finish with

$$C(z,t) = \max\left(0, z - (K + z - ze^{-r(T-t)})\right) = \max\left(0, ze^{-r(T-t)} - k\right)$$

B Derivation of Black-Scholes Formula

B.1 Assumptions

As in [**blackscholes**], before we can begin to derive the Black-Scholes Model, we must state some assumptions. Note that since the mathematics applied here quickly approaches a very high level, I shall brush over some more specific steps while trying to offer a satisfying proof of our equation.

B.1.1 Existence of replicating strategy

This assumption says that any option investment can be perfectly replciated by holdings in the stock and a money market account directly. A key part of investing is the substantial "leverage" employed in an option, as for a small (as a percentage of the share value) cost you can purchase a large number of shares and sell them for a profit, without having to risk investing such a large amount of money directly into the stock. Our assumption, however, says that a "replicating strategy" always exists. More rigorously, we say that there exists a pair of processes (x, y) which satisfy

$$H(t) = x(t)S(t) + y(t)A(t) \tag{47}$$

where $H(t)$ is the option value at time T, an Ito proces.$S(t)$ and $A(t)$ are the risky and riskless assets as defined in Asset Dynamics **??**. As described in [**blackscholes**], this assumption "captures the idea that changes in the values and holdings of assets are sole drivers of changes of wealth".

Additionally we assume that the process $H(t)$ is of the form

$$H(t) = u(t, S(t)). \tag{48}$$

This deterministic function $u(t, z)$ is not dependant on the history of the stock price. It is assmued to have continuous first derivative wrt $x \in [0, T]$ and continuous first and second derivatives in $z \in \mathbb{R}$.

B.1.2 Assumption 2

There again exists a replicating strategy (x, y) which satisfies

$$\mathrm{d}H(t) = x(t)\mathrm{d}S(t) + y(t)\mathrm{d}A(t) \tag{49}$$

where H, S, x, y and A are as defined in (**??**).

B.1.3 Assumption 3

There exists a probability Q

Applying (**??**) with $u(t,Z) = f(Z)$ we get

$$\mathrm{d}u(Z) = u_z(Z)\mu\mathrm{d}t + u_z(Z)\sigma\mathrm{d}W + \frac{1}{2}u_{zz}(Z)\sigma^2\mathrm{d}t \qquad \text{and} \tag{50}$$

$$\mathrm{d}u(t) = u_t \qquad \text{then} \tag{51}$$

$$\mathrm{d}H = \mathrm{d}u = \mathrm{d}u(Z) + \mathrm{d}u(t) = (u_t + \mu S u_z + \frac{1}{2}\sigma^2 S^2 u_{zz})\mathrm{d}t + +\sigma S u_z \mathrm{d}W \tag{52}$$

as a stochastic differential. From (**??**), we have

$$\mathrm{d}H = x(t)\mathrm{d}S(t) + \mathrm{d}x(t)S(t) \tag{53}$$

so

$$\mathrm{d}H = (x\mu S + ryA)\mathrm{d}t + x\sigma S\mathrm{d}W \tag{54}$$

from (**??**) and (**??**).

Then we equate the right hand side of (**??**) and (**??**):

$$u_t + \mu S u_z + \frac{1}{2}\sigma^2 S^2 u_{zz} = \qquad x\mu S + ryA \qquad \text{and} \tag{55}$$

$$\sigma S u_z = \qquad x\sigma S. \tag{56}$$

Now we can start to solve these equations: (**??**) gives us

$$x(t) = u_z(t, S(t)) \tag{57}$$

which we can use to eliminate x from (**??**):

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{zz} = ryA \tag{58}$$

which we then may use to solve for $y(t)$:

$$y(t) = \frac{1}{rA(t)}(u_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t) u_{zz}(t, S(t))). \tag{59}$$

where I have reintroduced the arguements for all the functions.

Finally, we use (**??**) and our expressions for x and y to give us

$$u(t,S(t)) = u_z(t,S(t))S(t) + \frac{1}{r}\left(u_t(t,S(t) + \frac{1}{2}\sigma^2 S^2(t) u_{zz}(t,S(t))\right) \tag{60}$$

$$\text{since } H(t) = u(t, S(t)) \tag{61}$$

We now simply rearrange for $u_t(t,z)$:

$$u_t(t,z) = -\frac{1}{2}\sigma^2 z^2 u_{zz}(t,z) - rzu_z(t,z) + ru(t,z) \qquad \text{for } 0 < t < T,\ z \in \mathbb{R}. \tag{62}$$

We also have the boundary condition that $H(T)$ is equal to the option payoff at the exercise date, so in the case of a call option we can immediately buy the stock to lock in our profits, so a $t = 0$ we have

$$u(0, z) = \max(z - K, 0) \qquad \text{for } z \in \mathbb{R} \tag{63}$$

where K is the strike price of the option. We use the max since the option has no value if the strike is above the market value. We can then consider when $z = 0$ and $z \to \infty$, as in [**scholesapplication**]. When $z = 0$ we notice from (**??**) that $\mathrm{d}S = 0$ so $z = 0$ is constant and

$$u(t, 0) = 0. \tag{64}$$

As $z \to \infty$, it becomes more likely we exercise the option, and the strike price becomes less relevant to u, so the value of $u(t, Z)$ is equivalent to z. Therefore we have the boundary condition

$$u(t, z) = z \qquad \text{as } z \to \infty \tag{65}$$

We have finally reached the general Black-Scholes inital value boundary problem

$$u_t(t, z) + \frac{1}{2}\sigma^2 z^2 u_{zz}(t, z) + rzu_z(t, z) - ru(t, z) = 0 \qquad \text{for } (t, z) \in (0, T) \times \mathbb{R}_+ \tag{66}$$

$$\text{with initial condition:} \qquad u(0, z) = \max(z - K, 0),\ z \in \mathbb{R}_+ \tag{67}$$

$$\text{and boundary conditions:} \qquad u(t, 0) = 0,\ u(t, z) = z \text{ as } S \to \infty,\ t \in [0, T] \tag{68}$$

Where we remember that:

$$
\begin{aligned}
u(t, z)- &\quad \text{price of the option} && (69)\\
z- &\quad \text{price of the underlying stock} && (70)\\
K- &\quad \text{strike price of the option} && (71)\\
r- &\quad \text{annualized risk-free interest rate, continouusly compounded} && (72)\\
t- &\quad \text{time, generally in years, with now as } t = 0 \text{ and expiry } t = T && (73)\\
\sigma- &\quad \text{the volatility of the underlying stock} && (74)
\end{aligned}
$$