APPM 4350/5350: Fourier Series and Boundary Value Problems
Homework #2 Monday, September 11, 2023
Due: Monday 3PM, September 25, 2023

In what follows, and in the future, all 'arbitrary' functions are assumed to be piecewise smooth unless otherwise specified.

1. (20 points) Consider:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L, t > 0, \kappa > 0 \text{ constant}$$

with BCs: $T(x = 0, t) = T(x = L, t) = 0$.

Solve for $T(x, t)$ with the following initial values:

a) $T(x, 0) = T_0 \sin \frac{\pi x}{L}$,

b) $T(x, 0) = T_0 \sin \frac{\pi x}{L} \cos \frac{\pi x}{L}$

where T_0 constant.

2. (25 points) Suppose:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad 0 < x < L, t > 0, \kappa > 0 \text{ constant}$$

with boundary conditions (BCs); $T(x = 0, t) = 0$, $\frac{\partial T}{\partial x}(x = L, t) = 0$ and initial condition (IC): $T(x, 0) = f(x)$. Find $T(x, t)$ and the the equilibrium, temperature. Hint: direct integration can be used to show that the eigenfunctions are orthogonal.

3. (25 points) Consider

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \alpha T, \quad 0 < x < L, t > 0, \kappa > 0, \alpha > 0 \text{ constants}$$

with BCs: $\frac{\partial T}{\partial x}(x = 0, t) = \frac{\partial T}{\partial x}(x = L, t) = 0$ and IC: $T(x, 0) = f(x)$. Find the the equilibrium temperature and $T(x, t)$. Find the long time asymptotic limit (i.e. the limit as $t \to \infty$) of T and compare it to the equilibrium temperature.

4. (20 points) 2.3.10 a,c

5. (20 points) 2.5.1 b

6. (20 points) Solve Laplace's equation

$$\nabla^2 u = 0$$

for $u = u(r, \theta)$ for the bounded solution inside the semicircle $r < R$ and above the real axis where $\frac{\partial u}{\partial \theta} = 0$ on the real axis and $\frac{\partial u}{\partial r}(r = R, \theta) = g(\theta)$ on the semicircle.

Hint: The boundary condition implies that separated solutions $u_s(r, \theta) = F(r)G(\theta), G(\theta)$ satisfies $\frac{\partial G}{\partial \theta} = 0$, when $\theta = 0$ and $\theta = \pi$.

7. (25 points) Solve Laplace's equation
$$\nabla^2 \phi = 0$$
inside a circular annulus $R_1 < r < R_2$ with boundary conditions: $\phi(R_1, \theta) = 0$, $\dfrac{\partial \phi}{\partial r}(R_2, \theta) = f(\theta)$.

8. (10 points) Use the maximum principle to show that the solution of $\nabla^2 \phi = f(x, y)$ in a domain D with $\phi = g(x, y)$ specified on a curve C bounding the region D is unique.

9. (15 points) Consider
$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad 0 < x < L$$
with $\phi(x = 0, t) = \phi(x = L, t) = 0$ and initial conditions:

   a) $\phi(x, 0) = 0, \frac{\partial \phi}{\partial t}(x, 0) = 0$. Find the solution $\phi(x, t)$.

   b) Suppose we perturb the initial conditions a little bit; namely suppose we have the initial conditions $\phi(x, 0) = \frac{1}{N} \sin \frac{N\pi x}{L}, \frac{\partial \phi}{\partial t}(x, 0) = 0, N$ a large positive integer. Find the solution and explain why the equation is *not* well-posed.

10. (20 points) Solve for the bounded solution of Laplace's equation
$$\nabla^2 u = 0$$
in a strip: $0 < x < L,\ 0 < y < \infty$ with boundary conditions $u(x = 0, y) = u(x = L, y) = 0$ and $u(x, 0) = f(x)$.

11. XC (20 points) In this problem you will solve for the exterior ideal flow around a circle using the velocity potential: $\boldsymbol{u} = \boldsymbol{\nabla} \phi$ where $\boldsymbol{u}$ is the velocity of the flow: $\boldsymbol{u} = \phi_x \hat{\boldsymbol{i}} + \phi_y \hat{\boldsymbol{j}} = \phi_r \hat{\boldsymbol{r}} + \frac{1}{r} \phi_\theta \hat{\boldsymbol{\theta}}$ where $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}$ are unit vectors in the x, y directions respectively and $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ are unit vectors in the radial and circular directions respectively.

    The problem is posed as follows. Find the solution $\phi = \phi(r, \theta)$ where ϕ satisfies
$$\nabla^2 \phi = 0, \quad \frac{\partial \phi}{\partial r}(r = R, \theta) = 0, \quad \phi \to U_0 x = U_0 r \cos \theta \ \text{ as } \ r \to \infty$$
    Then determine ϕ with prescribed circulation
$$\Gamma = \oint_C \boldsymbol{u} \cdot \hat{\boldsymbol{t}} ds$$
    where the integral is over the simple closed path C, $\boldsymbol{u}$ is the velocity, $\hat{\boldsymbol{t}}$ is the unit tangent and ds is the differential arc length. Find all coefficients in the solution ϕ in terms of U_0 and Γ.

    Hint: The conditions of periodicity only apply to the velocity: $\dfrac{\partial \phi}{\partial r}, \dfrac{\partial \phi}{\partial \theta}$. This problem was discussed in class using the stream function ψ where $\boldsymbol{u} = \psi_y \hat{\boldsymbol{i}} - \psi_x \hat{\boldsymbol{j}}$.