How many ordered triples $(x,y,z)$ of positive integers satisfy $\text{lcm}(x,y) = 72, \text{lcm}(x,z) = 600$ and $\text{lcm}(y,z)=900$?

$\textbf{(A)}\ 15\qquad\textbf{(B)}\ 16\qquad\textbf{(C)}\ 24\qquad\textbf{(D)}\ 27\qquad\textbf{(E)}\ 64$

We prime factorize $72,600,$ and $900$. The prime factorizations are

$2^3\times 3^2$

, $2^3\times 3\times 5^2$

 and $2^2\times 3^2\times 5^2$, respectively.

Let $x=2^a\times 3^b\times 5^c$,

$y=2^d\times 3^e\times 5^f$ and

$z=2^g\times 3^h\times 5^i$. We know that

\[\max(a,d)=3\]

\[\max(b,e)=2\]

\[\max(a,g)=3\]

\[\max(b,h)=1\]

\[\max(c,i)=2\]

\[\max(d,g)=2\]

\[\max(e,h)=2\]

From here we can see a few things. Note that since the max of d and g is 2, a must equal 3. Because the max of b and h is 1, e must equal 2. From here, we only care about

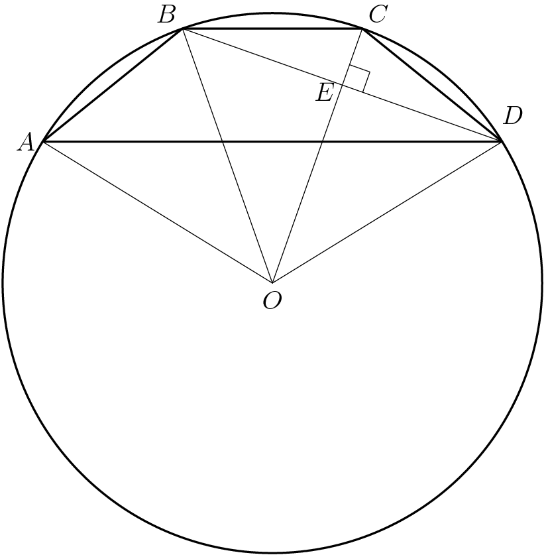
\[\max(b,h)=1\]

\[\max(d,g)=2\]

Running a casework on both of them, we have the first has 3, and second 5, so 3 \* 5 = 15.

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length $200$. What is the length of the fourth side?

$\textbf{(A) }200\qquad \textbf{(B) }200\sqrt{2}\qquad\textbf{(C) }200\sqrt{3}\qquad\textbf{(D) }300\sqrt{2}\qquad\textbf{(E) } 500$



Let us divide everything by 200 and multiply that later, so the side lengths are all 1 and the radius √2.

Let BE = ED = x, so CE = √(12-x2) and OE = √(√22-x2). We also know CE + OE = √2.

We can then find x, then use Ptolemy’s theorem to solve for AD = 500.

The number $5^{867}$ is between $2^{2013}$ and $2^{2014}$. How many pairs of integers $(m,n)$ are there such that $1\leq m\leq 2012$ and\[5^n<2^m<2^{m+2}<5^{n+1}?\]$\textbf{(A) }278\qquad \textbf{(B) }279\qquad \textbf{(C) }280\qquad \textbf{(D) }281\qquad \textbf{(E) }282\qquad$

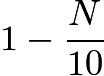
Between each consecutive power of 5, there are either 2 or 3 powers of two. This is because 22 = 4, and if a power of 2 is say 1 greater than a power of 5, then if we multiply that power of 2 by 22 it will still be less than the next power of 5, therefore there are 3 in this interval.

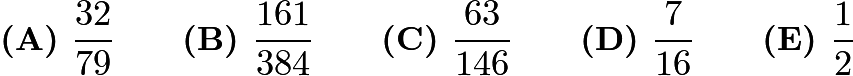
We know that up to 5867 there are 2013 powers of 2, so let x be the number of intervals with 2 powers of 2, and y with 3 powers of 2.

x + y = 867

2x + 3y = 2013.

Solve and y = 279

In a small pond there are eleven lily pads in a row labeled $0$ through $10$. A frog is sitting on pad $1$. When the frog is on pad $N$, $0<N<10$, it will jump to pad $N-1$ with probability $\frac{N}{10}$ and to pad $N+1$ with probability . Each jump is independent of the previous jumps. If the frog reaches pad $0$ it will be eaten by a patiently waiting snake. If the frog reaches pad $10$ it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?



The probability is ½ at Lili pad 5. If we let Pk be the probability the frog will escape at pad k, then we obtain the following equations.

P1 = 9/10 P2

P2 = 1/5 P1 + 4/5 P3

P3 = 3/10 P2 + 7/10 P4

P4 = 2/5 P3 + 3/5 P5

We can then plug in P5  and solve P1 = 63/146

Let $a$, $b$, and $c$ be positive integers with $a\ge$ $b\ge$ $c$ such that $a^2-b^2-c^2+ab=2011$ and $a^2+3b^2+3c^2-3ab-2ac-2bc=-1997$.

What is $a$?

$\textbf{(A)}\ 249\qquad\textbf{(B)}\ 250\qquad\textbf{(C)}\ 251\qquad\textbf{(D)}\ 252\qquad\textbf{(E)}\ 253$

Adding the two equations, we get

2a2 + 2b2+ 2c2 – 2ab -2ac -2bc = 14

This can be factored into

(a-b)2 + (a-c)2 + (b-c)2 = 14

They are all integers, so note 14 = 9 + 4 + 1, or 32  + 22 + 12

We know that a-c is the largest, so a-c = 3.

We then do casework on a-b = 1 or 2, and solve for a in either case, and find a = 253.

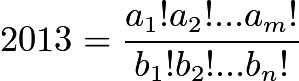
In base $10$, the number $2013$ ends in the digit $3$. In base $9$, on the other hand, the same number is written as $(2676)_{9}$ and ends in the digit $6$. For how many positive integers $b$ does the base-$b$-representation of $2013$ end in the digit $3$?

$\textbf{(A)}\ 6\qquad\textbf{(B)}\ 9\qquad\textbf{(C)}\ 13\qquad\textbf{(D)}\ 16\qquad\textbf{(E)}\ 18$

We are essentially looking for numbers such that 2013modb = 3, in other words factors of 2010.

2010 has 16 factors, but we cannot use 1 2 or 3 because their base representations cannot contain the digit 3, so we have a total of 13.

The number $2013$ is expressed in the form 

,

where $a_1 \ge a_2 \ge \cdots \ge a_m$ and $b_1 \ge b_2 \ge \cdots \ge b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

2013 = 61 \* 11 \* 3. Because of this, a1 = 61 since we need a factor of 61 at the top and it also is the smallest possible.

The denominator needs to cancel every prime other than 11 and 3 that is less than 61, and the next prime is 59. Therefore, b1 = 59. So the answer is 2.

What is the hundreds digit of $2011^{2011}$?

$\textbf{(A)}\ 1 \qquad\textbf{(B)}\ 4 \qquad\textbf{(C)}\ 5 \qquad\textbf{(D)}\ 6 \qquad\textbf{(E)}\ 9$

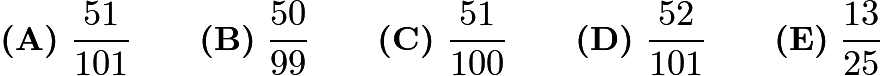
Note that this is equivalent to (2000 + 10 + 1)2011. We do not care about the 2000, however, since we are looking for mod 1000.

So we have (10 + 1)2011. By the polynomial theorem, we know that the first couple thousand terms have powers of ten that are ultimately all 0mod1000, so we do not care about those. The first term we care about is 2011C2 \* 102, and everything after that.

So

2011c2 \* 102 + 2011 \* 10 + 1. The hundreds digit of which is 1.

A lattice point in an $xy$-coordinate system is any point $(x, y)$ where both $x$ and $y$ are integers. The graph of $y = mx +2$ passes through no lattice point with $0 < x \le 100$ for all $m$ such that $1/2 < m < a$. What is the maximum possible value of $a$?



We know that the denominator of m must be greater than 100, otherwise some value of x will be able to cancel it thus resulting in an integer y.

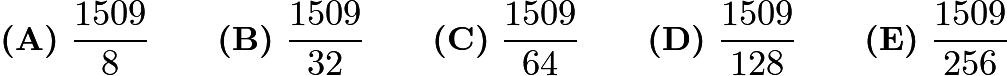
Knowing that the MAA sorts the answer choices in increasing value, we know that since 50/99 will give an integer y for x = 99, no value above 50/99 will work, effectively eliminating C, D, and E.

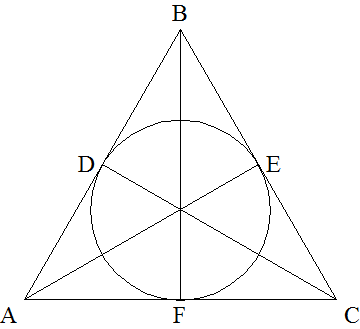
Now we consider A and B. Writing these two in common denominators, we find they become

5049/9999 and 5050/9999.

Confident now that no number in between those two can reduce to a fraction with denominator less than or equal to 100, we simply pick the larger of the two values with is B.

Let $T_1$ be a triangle with sides $2011, 2012,$ and $2013$. For $n \ge 1$, if $T_n = \triangle ABC$ and $D, E,$ and $F$ are the points of tangency of the incircle of $\triangle ABC$ to the sides $AB, BC$ and $AC,$ respectively, then $T_{n+1}$ is a triangle with side lengths $AD, BE,$ and $CF,$ if it exists. What is the perimeter of the last triangle in the sequence $( T_n )$?





We know that AD = AF, BD = BE, and CE= CF. If AD = x, BD = y, and CE = z, and let x + z = m be the middle value, which is 2012 for the first triangle.

We have that

x + y = m – 1

x + z = m

y + z = m + 1

And once we solve we have that

x = m/2 – 1

y = m/2

z = m/2 + 1

This is true for all triangles, so we simply keep dividing until we can no longer satisfy the triangle inequality, which is when the perimeter is D

How many four-digit integers $abcd$, with $a \neq 0$, have the property that the three two-digit integers $ab<bc<cd$ form an increasing arithmetic sequence? One such number is $4692$, where $a=4$, $b=6$, $c=9$, and $d=2$.

$\textbf{(A)}\ 9\qquad\textbf{(B)}\ 15\qquad\textbf{(C)}\ 16\qquad\textbf{(D)}\ 17\qquad\textbf{(E)}\ 20$

Note that the numbers are 10a + b, 10b + c, and 10c + d and that a <= b <= c.

So the arithmetic sequence is 10c + d – 10b +c = 10b + c – 10a + b, rearranged as

10(c -2b + a) = 2c -b -d

Because the left hand side is a multiple of 10, the right hand side must be either 0 or 10 (they are all digits and it cannot be -10 since c >= b >= a).

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$P(1) = P(3) = P(5) = P(7) = a$, and  
$P(2) = P(4) = P(6) = P(8) = -a$.

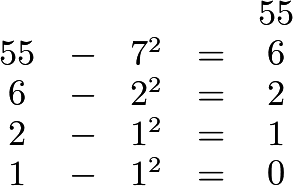
What is the smallest possible value of $a$?

Because P(1) = P(3) = P(5) = P(7) = a, then R(x) can be defined such that R(x) = P(x) – a, and has roots when P(x) = a; when x = 1, 3, 5, 7.

Therefore, (x-1)(x-3)(x-5)(x-7) can be factored out of R(x), so a new function can be defined as

Q(x)(x-1)(x-3)(x-5)(x-7) = R(x) = P(x) – a

Jim starts with a positive integer $n$ and creates a sequence of numbers. Each successive number is obtained by subtracting the largest possible integer square less than or equal to the current number until zero is reached. For example, if Jim starts with $n = 55$, then his sequence contains $5$numbers:



Let $N$ be the smallest number for which Jim’s sequence has $8$ numbers. What is the units digit of $N$?

We use the Greedy Algorithm, which is the method of taking the most optimal choice whenever possible.

We want the smallest possible value of N, so we simply find the smallest possible value of each term at each point.

So we start with

0

Which is from 1

Which is from 2

Which is from 3

Which is from 7

Which is from 23

Which is from 167

Which is from 7223

So 7223 is our 8th term, and 3 is our units digit.

Seven students count from 1 to 1000 as follows:

•Alice says all the numbers, except she skips the middle number in each consecutive group of three numbers. That is, Alice says 1, 3, 4, 6, 7, 9, . . ., 997, 999, 1000.

•Barbara says all of the numbers that Alice doesn't say, except she also skips the middle number in each consecutive group of three numbers.

•Candice says all of the numbers that neither Alice nor Barbara says, except she also skips the middle number in each consecutive group of three numbers.

•Debbie, Eliza, and Fatima say all of the numbers that none of the students with the first names beginning before theirs in the alphabet say, except each also skips the middle number in each of her consecutive groups of three numbers.

•Finally, George says the only number that no one else says.

What number does George say?

So we look at this with modular arithmetic. Alice skips all numbers of 2 mod 3,

Barbara will then skip the second of all the numbers Alice skips, which is all numbers 5 mod 9

Candice will skip all numbers 14 mod 27

Debbie will skip all numbers 41 mod 81

Eliza then skips 122 mod 243

Fatima skips 365 mod 729

So Fatima skips only 365, therefore George says the last number, which is 365.x