Project report for Numerical Methods for PDEs

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MAT: 250759

Let $\Omega = (0,\ 1)^2$ and consider the problem

$$-\Delta u = f \quad in\ \ \Omega$$
$$u = g \quad on\ \ \partial\Omega$$

Let u_h be the continuous piecewise linear finite element approximation on a given triangulation, for $h = 2^{-k}$ and $k \in \{2,\ 3,\ 4,\ 5,\ 6\}$.

(a) Consider $f(x, y) = -4$ and $g(x, y) = \left(x - \frac{3}{4}\right)^2 + (y+1)^2$. Observe that the solution to the problem is $u = g$, since

$$\frac{\partial g}{\partial x} = 2\left(x - \frac{3}{4}\right) \Longrightarrow \frac{\partial^2 g}{\partial x^2} = 2$$
$$\frac{\partial g}{\partial y} = 2\,(y+1) \Longrightarrow \frac{\partial^2 g}{\partial y^2} = 2$$
$$\Longrightarrow -\Delta g = -\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) = -4 = f$$

- Let's consider the first triangulation 1 of the domain.

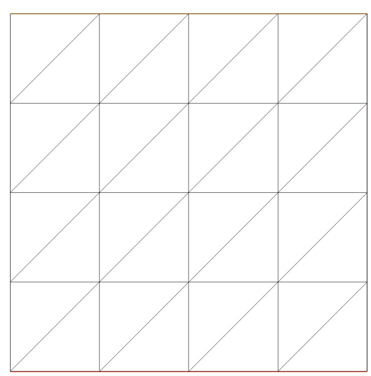

FIGURE 1. Plot of the first type of triangulation, obtained considering $h = 2^{-2}$: the implementation is in the code **first_mesh.edp**.

Over this triangulation we obtained:

- For k = 2,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 8.88178e - 16$$
$$\|\prod_h^1 u - u_h\|_{L^2} = 0.0218502$$
$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.204124$$

- For k = 3,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 8.88178e-16$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.00546255$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.102062$$

- For k = 4,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 4.88498e-15$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.00136564$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.051031$$

- For k = 5,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 2.22045e-15$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.000341409$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0255155$$

- For k = 6,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 2.66454e-15$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 8.53523e-05$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0127578$$

Moreover, we report the following convergence rates:

- L^∞ convergence: -0.263034
- L^2 convergence: 2
- L^2 “gradient” convergence: 1

In 2 we show the solution u_h, for $h = \frac{1}{2^4}$.

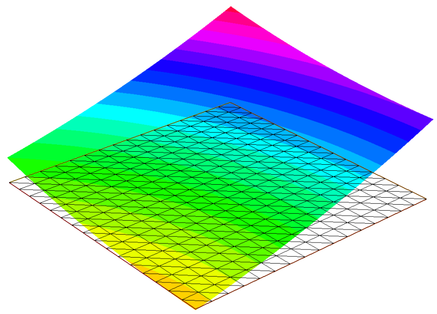

FIGURE 2. Plot of the solution u_h, obtained considering $h = 2^{-4}$ and as mesh the first mesh 1.

- Let's now consider the so–called "criss–cross" triangulation 3:

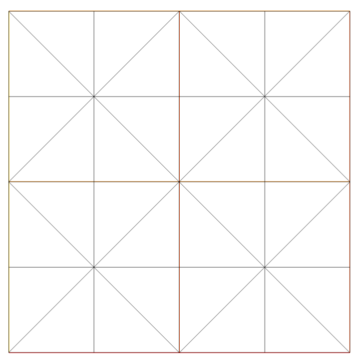

FIGURE 3. Plot of the second type of triangulation, obtained considering $h = 2^{-2}$: : the implementation is in the code **second_mesh.edp**.

– For k = 2,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.0208333$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.0156828$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.186339$$

– For k = 3,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.00520833$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.00392069$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0931695$$

– For k = 4,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.00130208$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.000980173$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0465847$$

– For k = 5,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.000325521$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.000245043$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0232924$$

– For k = 6,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 8.13802e - 05$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 6.12608e - 05$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0116462$$

Moreover, we report the following convergence rates:

– L^∞ convergence: 2
– L^2 convergence: 2
– L^2 "gradient" convergence: 1

As we can observe, we generally obtain the expected convergence rates for both meshes, but the L^∞ convergence rate over the first mesh 1 is unexpected: oddly enough, it seems to grow as h shrinks.

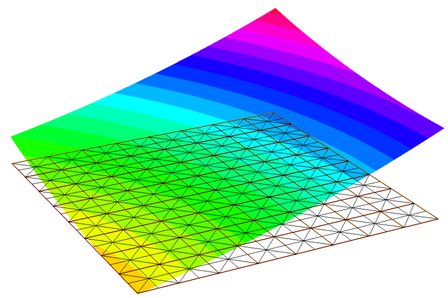

FIGURE 4. Plot of the solution u_h, obtained considering $h = 2^{-4}$ and as mesh the second mesh 3.

(b) Consider now $f(x,y) = 0$ and $g(x,y) = \log\left[\left(x-\frac{3}{4}\right)^2 + (y+1)^2\right]$. As before, the solution is $u = g$, since

$$\frac{\partial g}{\partial x} = \frac{2\left(x-\frac{3}{4}\right)}{\left(x-\frac{3}{4}\right)^2 + (y+1)^2} \Longrightarrow \frac{\partial^2 g}{\partial x^2} = \frac{2}{\left(x-\frac{3}{4}\right)^2 + (y+1)^2} - \frac{4\left(x-\frac{3}{4}\right)^2}{\left[\left(x-\frac{3}{4}\right)^2 + (y+1)^2\right]^2}$$

$$\frac{\partial g}{\partial y} = \frac{2(y+1)}{\left(x-\frac{3}{4}\right)^2 + (y+1)^2} \Longrightarrow \frac{\partial^2 g}{\partial x^2} = \frac{2}{\left(x-\frac{3}{4}\right)^2 + (y+1)^2} - \frac{4(y+1)^2}{\left[\left(x-\frac{3}{4}\right)^2 + (y+1)^2\right]^2}$$

$$\Longrightarrow -\Delta g = -\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) = 0 = f$$

- Let's consider the first triangulation 1 of the domain. Over this triangulation we obtained:
  - For k = 2,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.000761101$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.00443971$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.108998$$

  - For k = 3,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.000203629$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.0011188$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0548127$$

– For k = 4,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 5.22383e - 05$$
$$\|\prod_h^1 u - u_h\|_{L^2} = 0.000280258$$
$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0274469$$

– For k = 5,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 1.32342e - 05$$
$$\|\prod_h^1 u - u_h\|_{L^2} = 7.00995e - 05$$
$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0137286$$

– For k = 6,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 3.31464e - 06$$
$$\|\prod_h^1 u - u_h\|_{L^2} = 1.7527e - 05$$
$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.00686493$$

Moreover, we report the following convergence rates:
– L^∞ convergence: 1.99734
– L^2 convergence: 1.99982
– L^2 “gradient” convergence: 0.999865

In 5 we show the solution u_h, for $h = \frac{1}{2^4}$.

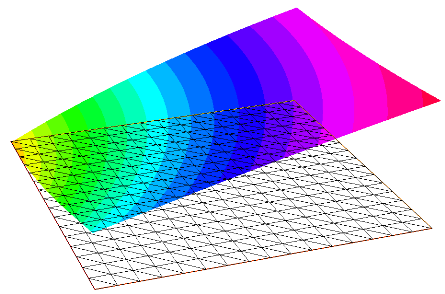

FIGURE 5. Plot of the solution u_h, obtained considering $h = 2^{-4}$ and as mesh the first mesh 1.

- Now, consider the triangulation 3 of the domain. Over this triangulation we obtained:
  - For k = 2,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 0.00052081$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.0043543$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.109006$$

  - For k = 3,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 6.71978e - 05$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.00109619$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0548151$$

  - For k = 4,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 5.78189e - 06$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 0.000274483$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0274473$$

– For k = 5,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 4.17754e-07$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 6.86472e-05$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.0137286$$

– For k = 6,

$$\|\prod_h^1 u - u_h\|_{L^\infty} = 2.7944e-08$$

$$\|\prod_h^1 u - u_h\|_{L^2} = 1.71634e-05$$

$$\|\nabla\left(\prod_h^1 u - u_h\right)\|_{L^2} = 0.00686493$$

Moreover, we report the following convergence rates:
- L^∞ convergence: 3.90204
- L^2 convergence: 1.99986
- L^2 "gradient" convergence: 0.999869

In 6 we show the solution u_h, for $h = \frac{1}{2^4}$. As expected, we get the usual convergence rates but also a greater rate for the L^∞ norm in the second experiment.

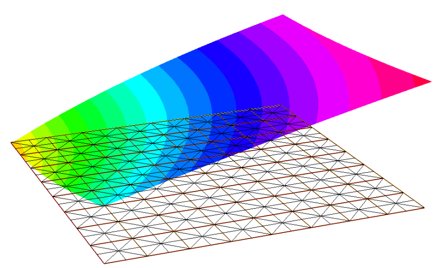

FIGURE 6. Plot of the solution u_h, obtained considering $h = 2^{-4}$ and as mesh the second mesh 3.