Andrea Pascucci
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Financial Mathematics

Theory and Problems for Multi-period Models

Springer

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Theory and Problems for Multi-period Models

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Preface

Financial mathematics has recently undergone a considerable development, due mainly to new financial instruments that have been introduced in order to limit the risk in financial operations. The study of problems related to such instruments requires mathematical techniques that occasionally may be rather sophisticated and are to a great extent related to Probability.

Consequently, the financial institutions now offer job opportunities not only to economists, but also to experts in scientific-technical disciplines, in particular in mathematics. With the Bologna Accords the so-called 3+2+3 (bachelor-master-doctor) curriculum has been introduced in various countries with the intention that students may enter the job market already at the bachelor level. It thus turns out to be appropriate to have a financial mathematics course already at the bachelor level. Most mathematical techniques in use in financial mathematics are related to continuous time models and require therefore notions from stochastic analysis that are in general not familiar not only to economists but neither to mathematicians at the bachelor level. It is thus desirable to be able to transmit to bachelor students the basic notions and methodologies in use in financial mathematics without the technicalities from stochastic analysis that are inherent in continuous time models. This can be achieved by using discrete time (multi-period) models instead. On one hand they generalize to a dynamic context the one-period models that are still in wide use by economists, on the other hand they can also be seen as possible approximations to continuous time models. Multi period models have however also a genuine interest in their own and this also in view of possible practical applications.

The present volume is intended as a possible textbook for a course as described above and is the result of the teaching experience of the authors in the area of financial mathematics. For multi-period models there do not exist many textbooks (one of the best known is [18]) and so one of the purposes of the present volume is to fill in this gap. Although conceived mainly for a bachelor-level course in mathematics, the volume should also be appropriate for quantitative finance courses for economics students.

Evidently, we could not take into account in this book all possible topics in financial mathematics and so we have confined ourselves to those that one may consider as basic ones. The structure of the book originates from the idea of teaching by examples and counterexamples. It has been expanded beyond the examples to become a complete textbook that includes also the necessary theory. Consequently, and differently from other textbooks, this one includes many examples and solved problems. In this context we want to mention also [21] that contains examples for the specific binomial model and [19] that includes problems both from discrete as well as continuous time models.

The majority of the solution methods for multi-period models is based on recursive algorithms, for which the computational complexity increases considerably with the number of periods. In practice one has therefore to use computer programs to implement the algorithms. For problems in classrooms and at exam sessions it is therefore appropriate to limit oneself to situations where calculation can be performed "by hand". For this reason, in the examples and problems suggested in the book we consider small numbers of periods and numerical data that may not correspond to realistic situations but allow for easier calculations.

The book is divided into four chapters, in which we treat the following topics:

- pricing and hedging of European derivatives;
- portfolio optimization (dynamic programming and "martingale method");
- pricing, optimal exercise and hedging of American derivatives;
- multi-period models for the term structure of interest rates.

Each of the four chapters consists of two parts: a theoretical section and a problem section. In the latter section we describe in detail the solution for many possible problems.

Bologna/Padova, November 2011

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1

Pricing and hedging

In this chapter we introduce some of the basic ideas of the modern mathematical finance that will be used throughout the book. These are the concepts of risky and non-risky securities, primary assets and derivatives, including options, self-financing investment strategies and their portfolios. We also introduce the notion of arbitrage, the concept of equivalent martingale measure (EMM) and of a complete market. The issues covered in this chapter are basic problems of modern mathematical finance, i.e. the pricing and hedging of derivatives. To define the price of a derivative, which is not already traded on the market, we shall use one of the most common criteria namely the no-arbitrage principle. According to this principle, in a market in equilibrium, prices of various securities are such that it is not possible to make a profit without risk by investing in the market by a self-financing strategy.

We then describe two typical examples of discrete-time market models with a single risky asset: the binomial model that is a complete market model and the trinomial model that is an example of an incomplete market model. This last model can be completed by adding a second risky asset and this will constitute another example of complete market. These three market models will be the basis for all exercises discussed in the book. We also remark that, for a finite time horizon, these models can be naturally defined on a finite probability space Ω, as we assume throughout this book.

In a complete market any derivative can be replicated by a portfolio resulting from a self-financing investment strategy: furthermore, the prices of the various securities are uniquely determined by imposing the condition of absence of arbitrage opportunities. We shall also briefly discuss possible approaches to pricing and hedging in an incomplete market.

At last, using an alternative representation of the price of one of the basic options, that is the European Call option, we shall mention the so-called change of numeraire technique that is very useful in solving various problems in mathematical finance and in this book will be applied in Chapter 4.

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The exercises will focus on the evaluation and hedging of various types of derivatives, especially options. The last two exercises are an example of application of the techniques mentioned for the pricing and hedging in an incomplete market. For their solution, we apply a dynamic optimization methodology that will be studied in detail in Chapter 2.

For this chapter we have relied primarily on [17]. Since these are very basic topics, they are treated in almost all introductory books on mathematical finance: among those listed in the bibliography we mention here [3], [7], [11], [16], [18], [20], [21].

1.1 Primary securities and strategies

1.1.1 Discrete time markets

Consider a probability space $(\Omega, \mathcal{F}, P)$ where Ω has a finite number of elements and where we assume that $\mathcal{F}$ is the power set of Ω with $P(\{\omega\}) > 0$ for any $\omega \in \Omega$. We let $t_0, t_1, \dots, t_N \in \mathbb{R}$ with

$$t_0 < t_1 < \cdots < t_N$$

represent the trading dates: to fix the ideas, $t_0 = 0$ denotes today's date and $t_N = T$ the expiry date of a derivative.

A discrete time market model consists of a non-risky asset (bond) B and a certain number of risky assets (stocks) $S^1, \dots, S^d$, with $d \in \mathbb{N}$. The bond has the following deterministic dynamics: if B_n denotes the value of the bond at time t_n, we have

$$\begin{cases} B_0 = 1, \\ B_n = B_{n-1}(1 + r_n), \qquad n = 1, \dots, N, \end{cases} \tag{1.1}$$

where $r_n > -1$ denotes the risk-free rate in the n-th period $[t_{n-1}, t_n]$. Occasionally we will refer to this asset as the *bank account*.

The risky securities have the following stochastic dynamics: if S_n^i denotes the price at time t_n of the i-th asset, then we have

$$\begin{cases} S_0^i \in \mathbb{R}_+, \\ S_n^i = S_{n-1}^i \left(1 + \mu_n^i\right), \qquad n = 1, \dots, N, \end{cases} \tag{1.2}$$

where μ_n^i is a real random variable that represents the rate of return of the i-th asset in the n-th period $[t_{n-1}, t_n]$.

We set

$$\mu_n = (\mu_n^1, \dots, \mu_n^d)$$

and suppose that the process μ_n is adapted to a generic filtration $(\mathcal{F}_n)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Since in the market models considered in this book, based on

(1.2), the sequence μ_n will be the only source of randomness, we assume $\mathcal{F}_n$ generated by μ_n, that is

$$\mathcal{F}_n = \mathcal{F}_n^\mu := \sigma\{\mu_k \mid k \le n\}, \quad n = 1, \cdots, N. \tag{1.3}$$

Finally, since (1.2) establishes a bijective correspondence between the processes μ_n and S_n, the filtration $(\mathcal{F}_n)$ coincides with the filtration generated by S, that is $\mathcal{F}_n = \mathcal{F}_n^S$ for any n. In more general situations, however, we will typically have $\mathcal{F}_n^S \subset \mathcal{F}_n$ and occasionally, as later in the definition of derivatives, also in this book we will use the notation $\mathcal{F}_n^S$ when we want to emphasize that the filtration considered is the one generated by the underlying asset prices S. However in this book we will always have $\mathcal{F}_n = \mathcal{F}_n^\mu = \mathcal{F}_n^S$ and this σ-algebra represents the market information available at time t_n. We also assume that μ_n *is independent on* $\mathcal{F}_{n-1}$ *for any* $n = 1, \cdots, N$.

Notice that as elements ω of the basic probability space we can take the various realizations of the sequence μ_n. Therefore if μ_n assumes a finite number of values, as in the binomial (cf. Section 1.4.1) and trinomial (cf. Section 1.4.2) market models, then the set Ω contains a finite number of elements as we had supposed.

1.1.2 Self-financing and predictable portfolios

Definition 1.1. *A portfolio (or strategy) is a stochastic process in* $\mathbb{R}^{d+1}$

$$(\alpha, \beta) = (\alpha_n^1, \dots, \alpha_n^d, \beta_n)_{n=1,\dots,N}.$$

In the previous definition α_n^i (resp., β_n) represents the amount of asset S^i (resp., of bond) kept in the portfolio during the n-th period, that is from t_{n-1} to t_n. Therefore we denote the *value of the portfolio* (α, β) in the n-th period by

$$V_n^{(\alpha,\beta)} = \sum_{i=1}^d \alpha_n^i S_n^i + \beta_n B_n, \qquad n = 1, \dots, N, \tag{1.4}$$

and we also set

$$V_0^{(\alpha,\beta)} = \sum_{i=1}^d \alpha_1^i S_0^i + \beta_1 B_0.$$

The value $V^{(\alpha,\beta)}$ of the portfolio is a real stochastic process. Note that we allow α_n^i and β_n to assume negative values: in other words, short-selling of shares or borrowing money from the bank are allowed strategies. In the following, when the strategy (α, β) is fixed, we will often write V instead of $V^{(\alpha,\beta)}$.

Notation 1.2. *We use the vector notation for the price process*

$$S = (S^1, \dots, S^d).$$

If $\alpha = (\alpha^1, \dots, \alpha^d)$*, we denote by*

$$\alpha S = \sum_{i=1}^{d} \alpha^i S^i$$

the scalar product in $\mathbb{R}^d$*. In particular,* (1.4) *reads as*

$$V_n = \alpha_n S_n + \beta_n B_n.$$

Definition 1.3. *A strategy* (α, β) *is self-financing if it satisfies*

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1} \tag{1.5}$$

for any $n = 1, \dots, N$*.*

For a self-financing portfolio we have the equality

$$\alpha_{n-1} S_{n-1} + \beta_{n-1} B_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1}$$

which is interpreted as follows: *at time* t_{n-1}*, when the capital*

$$V_{n-1} = \alpha_{n-1} S_{n-1} + \beta_{n-1} B_{n-1}$$

is available, we build the strategy for the n*-th period* $[t_{n-1}, t_n]$ *with the new quantities* α_n, β_n *so as not to change the overall value of the portfolio.* We emphasize that (α_n, β_n) *denotes the composition of the portfolio that is built at time* t_{n-1}*.*

In the following we consider only investment strategies determined on the basis of the information available on the market (not knowing the future). Since, for a self-financing strategy, (α_n, β_n) indicates the composition of the portfolio that is built at time t_{n-1}, it is natural to assume that the process (α, β) is *predictable.*

Definition 1.4. *A strategy* (α, β) *is predictable if* (α_n, β_n) *is* $\mathcal{F}_{n-1}$*-measurable for any* $n = 1, \dots, N$*.*

Notation 1.5. *We denote by* $\mathcal{A}$ *the family of self-financing and predictable strategies.*

Since the self-financing condition (1.5) establishes a link between the processes α and β, we can identify a strategy $\mathcal{A}$ by using the pair (α, β) or, equivalently, by the pair (V_0, α) where $V_0 \in \mathbb{R}$ is the starting value of the strategy and α is a d-dimensional predictable process. More precisely, we have:

Proposition 1.6. *The value of the self-financing strategy* (α, β) *is determined by the initial value* V_0 *and recursively by the relation*

$$V_n = V_{n-1}(1 + r_n) + \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \left(\mu_n^i - r_n\right) \tag{1.6}$$

for $n = 1, \dots, N$*.*

Proof. By condition (1.5), the change in value of a self-financing portfolio in the period $[t_{n-1}, t_n]$ is equal to

$$\begin{aligned} V_n - V_{n-1} &= \alpha_n \left(S_n - S_{n-1}\right) + \beta_n \left(B_n - B_{n-1}\right) \\ &= \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \mu_n^i + \beta_n B_{n-1} r_n = \end{aligned} \tag{1.7}$$

(because, again by (1.5), we have $\beta_n B_{n-1} = V_{n-1} - \alpha_n S_{n-1}$)

$$= \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \left(\mu_n^i - r_n\right) + r_n V_{n-1}$$

and this implies (1.6). □

Corollary 1.7. *For any $V_0 \in \mathbb{R}$ and α a predictable process, there exists a unique predictable process β such that $(\alpha, \beta) \in \mathcal{A}$ and $V_0^{(\alpha,\beta)} = V_0$.*

Proof. Let $V_0 \in \mathbb{R}$ and α be a predictable process. We define the process

$$\beta_n = \frac{V_{n-1} - \alpha_n S_{n-1}}{B_{n-1}}, \qquad n = 1, \dots, N,$$

where (V_n) is defined recursively by (1.6). Clearly, by construction (β_n) is predictable and the strategy (α, β) is self-financing (cf. (1.5)). □

Remark 1.8. *Let $(\alpha, \beta) \in \mathcal{A}$. From (1.7), summing over n, we get*

$$V_n = V_0 + g_n^{(\alpha,\beta)}, \qquad n = 1, \dots, N, \tag{1.8}$$

where

$$\begin{aligned} g_n^{(\alpha,\beta)} &= \sum_{k=1}^{n} \left(\alpha_k \left(S_k - S_{k-1}\right) + \beta_k \left(B_k - B_{k-1}\right)\right) \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{d} \alpha_k^i S_{k-1}^i \mu_k^i + \beta_k B_{k-1} r_k \right) \end{aligned} \tag{1.9}$$

defines the process of the yield of the strategy. □

1.1.3 Relative portfolio

Sometimes it is convenient to express a portfolio in relative terms, indicating the proportions of the total value invested in the individual securities. So if $V_{n-1} \neq 0$ we denote by

$$\pi_n^i = \frac{\alpha_n^i S_{n-1}^i}{V_{n-1}}, \qquad i = 1, \dots, d, \tag{1.10}$$

and

$$\pi_n^0 = \frac{\beta_n B_{n-1}}{V_{n-1}} = 1 - \sum_{i=1}^{d} \pi_n^i, \tag{1.11}$$

the proportions invested in the n-th period $[t_{n-1}, t_n]$ for $n = 1, \dots, N$. Moreover, by convention, if $V_{n-1} = 0$, we set $\pi_n^i = 0$ for $i = 0, \dots, d$. Note that π_n^i is not a percentage since it does not necessarily belong to the interval $[0, 1]$.

We express now the self-financing condition in relative terms.

Proposition 1.9. *The value of a self-financing strategy (α, β) is determined by the initial value $V_0 \in \mathbb{R}$ and by the processes $\pi^1, \dots, \pi^d$ through the recursive relation*

$$V_n = V_{n-1}\left(1 + \pi_n \mu_n + \pi_n^0 r_n\right) \tag{1.12}$$

which is equivalent to

$$V_n = V_{n-1}\left(1 + r_n + \sum_{i=1}^{d} \pi_n^i \left(\mu_n^i - r_n\right)\right) \tag{1.13}$$

and to

$$\frac{V_n - V_{n-1}}{V_{n-1}} = \sum_{i=1}^{d} \pi_n^i \frac{S_n^i - S_{n-1}^i}{S_{n-1}^i} + \pi_n^0 \frac{B_n - B_{n-1}}{B_{n-1}}. \tag{1.14}$$

The latter condition expresses the fact that the relative yields of a self-financing portfolio are linear combinations of the returns of the assets composing the portfolio, with the weights given by the relative portfolio.

Proof. Identity (1.12) follows directly from the first equation in (1.7). Formula (1.13) is obtained by plugging the second equality of (1.11) into (1.12); finally, (1.14) follows from (1.12) by replacing μ_n and r_n with their expressions resulting from (1.1) and (1.2). □

Remark 1.10. *For a given $V_0 \in \mathbb{R}$ and $\pi^1, \dots, \pi^d$, predictable processes, we easily derive the corresponding strategy $(\alpha, \beta) \in \mathcal{A}$ by the formulas*

$$\alpha_n^i = \frac{\pi_n^i V_{n-1}}{S_{n-1}^i}, \qquad \beta_n = \frac{V_{n-1}}{B_{n-1}}\left(1 - \sum_{i=1}^{d} \pi_n^i\right) \tag{1.15}$$

where $V = (V_n)$ is defined by V_0 and $\pi^1, \dots, \pi^d$ through the recursive relation (1.13). □

1.1.4 Discounted market

The *discounted price* of the i-th asset is defined by

$$\widetilde{S}_n^i = \frac{S_n^i}{B_n}, \qquad n = 1, \dots, N,$$

and the *discounted value* of the strategy (α, β) is

$$\widetilde{V}_n = \alpha_n \widetilde{S}_n + \beta_n.$$

Note that the discounting of the prices is equivalent to using the bond B as unit of measure with respect to which one expresses the prices of all the assets in the market. In general it is possible to take as unit of measure any asset whose price is strictly positive: such an asset is called a *numeraire.*

We observe explicitly that, by the assumption $B_0 = 1$, we have $\widetilde{V}_0 = V_0$. The self-financing condition (1.5) is equivalent to

$$\widetilde{V}_{n-1} = \alpha_n \widetilde{S}_{n-1} + \beta_n, \qquad n = 1, \dots, N, \tag{1.16}$$

or

$$\widetilde{V}_n = \widetilde{V}_{n-1} + \alpha_n \left(\widetilde{S}_n - \widetilde{S}_{n-1} \right) \tag{1.17}$$

that also yields

$$\frac{\widetilde{V}_n - \widetilde{V}_{n-1}}{\widetilde{V}_{n-1}} = \sum_{i=1}^{d} \pi_n^i \frac{\widetilde{S}_n^i - \widetilde{S}_{n-1}^i}{\widetilde{S}_{n-1}^i}.$$

The latter formula is analogous to (1.14). Proposition 1.6 extends now as follows:

Proposition 1.11. *The discounted value of a self-financing strategy* (α, β) *is determined by the initial value* V_0 *and recursively by the relation* (1.17) *for* $n = 1, \dots, N$.

Moreover the following formula, analogous to (1.8), holds true:

$$\widetilde{V}_n = V_0 + G_n^{(\alpha)} \tag{1.18}$$

where

$$G_n^{(\alpha)} = \sum_{k=1}^{n} \alpha_k \left(\widetilde{S}_k - \widetilde{S}_{k-1} \right) \tag{1.19}$$

defines the process of the *discounted yield* of the strategy.

1.2 Arbitrage and martingale measures

An arbitrage is a financial investment that produces a sure gain without risk and at zero cost. In real markets arbitrage opportunities exist even though they generally have a short life because they are exploited by investors to instantly restore market balance.

In the mathematical modeling of financial markets the principle of absence of arbitrage opportunities asserts that the asset prices must be such as not to allow a sure gain without risk. In other words a mathematical market model is considered acceptable if it does not admit the existence of arbitrage opportunities.

In this section we introduce the formal notion of arbitrage strategy and characterize the properties of absence of arbitrage opportunities in terms of the existence of an appropriate probability measure, called equivalent martingale measure.

Definition 1.12. *An arbitrage is a self-financing strategy* $(\alpha, \beta) \in \mathcal{A}$ *verifying the following conditions:*

i) $V_0^{(\alpha,\beta)} = 0$;
ii) $V_N^{(\alpha,\beta)} \geq 0$;
iii) $P\left(V_N^{(\alpha,\beta)} > 0\right) > 0$.

We say that a market model is arbitrage-free if the family $\mathcal{A}$*, of self-financing predictable strategies, does not contain arbitrage strategies.*

We recall the notation $\widetilde{S}$ for the process of discounted prices and we give the following fundamental:

Definition 1.13. *An equivalent martingale measure (EMM) with numeraire* B *is a probability measure* Q *on* $(\Omega, \mathcal{F})$ *such that:*

i) Q *is equivalent*[1] *to* P;
ii) for any $n = 1, \dots, N$ *we have*

$$\widetilde{S}_{n-1} = E^Q\left[\widetilde{S}_n \mid \mathcal{F}_{n-1}\right], \tag{1.20}$$

that is $\widetilde{S}$ *is a* Q*-martingale.*

The following classical result holds.

Theorem 1.14 (First fundamental theorem of asset pricing). *A discrete-time market is arbitrage-free if and only if there exists at least one EMM.*

An EMM is sometimes called *risk neutral measure* because (1.20) can be economically interpreted as a risk neutral valuation formula. With respect to an EMM, not only are the processes of the discounted prices of each primitive security martingales, but so are also the discounted values of any self-financing predictable strategy.

Proposition 1.15. *Let* Q *be an EMM and* (α, β) *a strategy in* $\mathcal{A}$ *with value* V*. Then we have*

$$\widetilde{V}_{n-1} = E^Q\left[\widetilde{V}_n \mid \mathcal{F}_{n-1}\right], \qquad n = 1, \dots, N, \tag{1.21}$$

and in particular

$$V_0 = E^Q\left[\widetilde{V}_n\right], \qquad n = 1, \dots, N. \tag{1.22}$$

[1]Two probability measures are equivalent if they assign zero probability to the same events.

Proof. From the self-financing condition (1.17), by considering the expectation conditioned on $\mathcal{F}_n$, we get

$$E^Q\left[\widetilde{V}_n \mid \mathcal{F}_{n-1}\right] = \widetilde{V}_{n-1} + E^Q\left[\alpha_n(\widetilde{S}_n - \widetilde{S}_{n-1}) \mid \mathcal{F}_{n-1}\right] =$$

(since α is predictable)

$$= \widetilde{V}_{n-1} + \alpha_n E^Q\left[\widetilde{S}_n - \widetilde{S}_{n-1} \mid \mathcal{F}_{n-1}\right] = \widetilde{V}_{n-1},$$

by (1.20). □

We directly deduce the following remarkable version of the non-arbitrage principle:

Corollary 1.16. *In a arbitrage-free market, if two strategies (α, β), $(\alpha', \beta') \in \mathcal{A}$ have the same final value, $V_N^{(\alpha,\beta)} = V_N^{(\alpha',\beta')}$ a.s., then they are also such that*

$$V_n^{(\alpha,\beta)} = V_n^{(\alpha',\beta')} \quad a.s., \qquad n = 0, \dots, N.$$

Proof. Since the market is arbitrage-free, by Theorem 1.14 there exists an EMM Q. Then by Proposition 1.15 we have

$$\widetilde{V}_n^{(\alpha,\beta)} = E^Q\left[\widetilde{V}_N^{(\alpha,\beta)} \mid \mathcal{F}_n\right] = E^Q\left[\widetilde{V}_N^{(\alpha',\beta')} \mid \mathcal{F}_n\right] = \widetilde{V}_n^{(\alpha',\beta')} \quad \text{a.s.}$$ □

1.3 Pricing and hedging

1.3.1 Derivative securities

We consider a discrete time market model with process $S_n = (S_n^1, \cdots, S_n^d)$ of primary risky asset prices and we set an expiry date t_N which we denote simply by N.

Definition 1.17. *A European derivative with underlying assets $S = (S^1, \cdots, S^d)$ is a random variable X defined on the probability space $(\Omega, \mathcal{F}, P)$ and which is measurable with respect to the σ-algebra $\mathcal{F}_N^S := \sigma(\{S_n \mid n \le N\})$. We say that X is the* payoff *(or the* claim*) of the derivative.*

Remark 1.18. *Although in this book we assume that $\mathcal{F}_n = \mathcal{F}_n^S$ for any n (see (1.3) and the ensuing comments), in the preceding definition of derivative we showed explicitly the filtration $\left(\mathcal{F}_n^S\right)$ to highlight the fact that the value of a derivative depends on the underlying securities $S = (S^1, \cdots, S^d)$.* □

A typical example of derivatives are the European Call options. They are contracts that give the holder the right but not the obligation to buy a unit of the underlying asset, at a certain time t_N (the expiration date) for a certain price K (the strike price). The payoff of a Call option is therefore of the form

$$X = (S_N - K)^+.$$

In this case, the payoff depends only on the value of the underlying at maturity. More generally there are options, of European style, that depend on the whole trajectory of the underlying asset: an example is given by the so-called Asian options (average price) with payoff

$$X = \left(\frac{1}{N} \sum_{n=1}^{N} S_n - K \right)^+ .$$

Besides the European options, there are the American options (in discrete time also called Bermudan options) that can be exercised at any moment before expiration. The entire Chapter 3 will be devoted to the study of American options. Several other types of options will discussed in the exercises at the end of this chapter. Typically a derivative allows the holder to transfer to the counterparty the risk associated with the underlying assets. In the case of a Call option, in fact, the holder transfers the risk that is associated with the increase of the underlying price. Similarly, in the case of a Put option, whose payoff is given by

$$X = (K - S_N)^+$$

the holder transfers to the counterparty the risk of a lowering of the price of the underlying asset.

Several derivatives are traded on the market and therefore they have already a quoted price. Often, however, a derivative is designed to engage in a specific transaction on an asset or it is customized to the desires of the buyer on an ad hoc basis and therefore it does not have a market value. In this case, the problem of establishing the fair price arises both for the buyer/holder and for the seller/writer. This leads to the problem of the *evaluation or pricing* of derivatives.

On the other side, the seller/writer has also the problem of hedging the risk that he/she took over and this leads to the *hedging* problem.

1.3.2 Arbitrage pricing

One of the basic problems of the classical theory of arbitrage pricing is to establish conditions for the existence of a strategy $(\alpha, \beta) \in \mathcal{A}$ that takes at maturity the same value of a derivative X, i.e.

$$V_N^{(\alpha,\beta)} = X \qquad \text{a.s.}$$

If such a strategy exists, we say that X is *replicable* and (α, β) is a *replicating strategy* for X.

By the no-arbitrage principle in the form of Corollary 1.16, if in a free-arbitrage market two investments have the same final value then they must have the same value at any previous time. Consequently, the fair (or rational) price of a replicable derivative X must match the value $V_n^{(\alpha,\beta)}$ of a self-financing strategy replicating X.

This fact may also be justified in intuitive terms; indeed let us denote by H_n the price at time n of the derivative X. If $V_n^{(\alpha,\beta)} < H_n$, we could sell short the derivative at the price H_n and then invest the amount $V_n^{(\alpha,\beta)}$ in the replicating portfolio for X, to make sure to be able to repay the claim at maturity: indeed by the replication condition we have $V_N^{(\alpha,\beta)} = X$. Thus, by investing the remaining capital $H_n - V_n^{(\alpha,\beta)}$ in the non-risky asset, we would have a sure gain without risk. A similar argument can be used in the case $V_n^{(\alpha,\beta)} > H_n$, ending up in any case with an arbitrage. We formalize the previous ideas by introducing the families of super and sub-replicating portfolios for the derivative X:

$$\mathcal{A}_X^+ = \{(\alpha,\beta) \in \mathcal{A} \mid V_N^{(\alpha,\beta)} \geq X\}, \qquad \mathcal{A}_X^- = \{(\alpha,\beta) \in \mathcal{A} \mid V_N^{(\alpha,\beta)} \leq X\}.$$

Next we consider an arbitrage-free market and, for a fixed EMM Q which exists by Theorem 1.14, we set

$$H_n^Q := E^Q\left[X\frac{B_n}{B_N} \mid \mathcal{F}_n\right]. \tag{1.23}$$

Since H^Q is the expected value, with respect to a risk neutral measure, of the discounted payoff, we say that H^Q is the *Q-risk neutral price* of X: clearly H^Q depends on the selected EMM Q. The following results show that the risk-neutral price does not give rise to arbitrage opportunities (see in particular Proposition 1.22 below).

Lemma 1.19. *For every EMM Q with numeraire B, we have*

$$\sup_{(\alpha,\beta)\in\mathcal{A}_X^-} \widetilde{V}_n^{(\alpha,\beta)} \leq E^Q\left[\frac{X}{B_N} \mid \mathcal{F}_n\right] \leq \inf_{(\alpha,\beta)\in\mathcal{A}_X^+} \widetilde{V}_n^{(\alpha,\beta)},$$

for $n = 0,\dots,N$.

Proof. If $(\alpha,\beta) \in \mathcal{A}_X^-$ then, by Proposition 1.15, we have

$$\widetilde{V}_n^{(\alpha,\beta)} = E^Q\left[\widetilde{V}_N^{(\alpha,\beta)} \mid \mathcal{F}_n\right] \leq E^Q\left[\frac{X}{B_N} \mid \mathcal{F}_n\right],$$

and an analogous estimate holds for $(\alpha,\beta) \in \mathcal{A}_X^+$. □

The following theorem lays the foundations of arbitrage pricing.

Theorem 1.20. *Let X be a replicable derivative in an arbitrage-free market. Then for every replicating strategy $(\alpha, \beta) \in \mathcal{A}$ and for every EMM Q with numeraire B, we have*

$$E^Q \left[\frac{X}{B_N} \mid \mathcal{F}_n \right] = \frac{V_n^{(\alpha,\beta)}}{B_n}, \qquad n = 0, \dots, N. \tag{1.24}$$

The process $H := V^{(\alpha,\beta)}$ is called arbitrage price of X.

Proof. If $(\alpha, \beta), (\alpha', \beta') \in \mathcal{A}$ replicate X then they have the same terminal value and, by Corollary 1.16, they have the same value at all preceding times. Moreover, if $(\alpha, \beta) \in \mathcal{A}$ replicates X, then $(\alpha, \beta) \in \mathcal{A}_X^- \cap \mathcal{A}_X^+$ and by Lemma 1.19 we have

$$E^Q \left[\frac{X}{B_N} \mid \mathcal{F}_n \right] = \widetilde{V}_n^{(\alpha,\beta)},$$

for every EMM Q with numeraire B. □

We emphasize the fact that the arbitrage price coincides with the risk-neutral price in (1.23) which, if X is replicable, is thus independent of Q. We point out that the arbitrage price is defined in terms of a risk neutral measure Q, while the physical measure P does not intervene.

Consider now the case of a derivative X that is not replicable. Since there is no replicating strategy for X, definition (1.24) of arbitrage price loses consistency. On the other hand, if the market is arbitrage-free then there exists (though it is not necessarily unique) an EMM Q and formula (1.23) defines *in a non-unique way* (because of the Q-dependency) the risk-neutral price of X. The following remarkable result[2] states that if X is not replicable then formula (1.23) cannot provide an univocal definition of price.

Theorem 1.21. *In an arbitrage-free market, a derivative X is replicable if and only if $E^Q \left[\frac{X}{B_N} \right]$ has the same value for each martingale measure Q.*

The second important fact is that by taking (H_n^Q) in (1.23) as the price of X, whether or not replicable, then we do not create arbitrage opportunities. More specifically, we have:

Proposition 1.22. *For any EMM Q, the market model consisting of the bond B, the risky assets $(S^1, \dots, S^d)$ and H^Q in* (1.23) *is arbitrage-free.*

Proof. Since $\widetilde{H}^Q$ is a Q-martingale, then Q is an EMM for the market (B, S, H^Q) and the thesis follows from Theorem 1.14. □

[2] For the proof, which is based on a separation theorem of convex sets in finite dimension, see for example [18].

Remark 1.23. *An alternative way of proving Proposition 1.22 is to note that if we build a self-financing portfolio that includes the derivative, then the discounted value of this portfolio is a Q-martingale (see Proposition 1.15) and therefore it cannot be an arbitrage according to Definition 1.12.* □

1.3.3 Hedging

The hedging problem consists in determining a *replicating (hedging) strategy.* From the foregoing it is clear that not every derivative is replicable. A market in which each derivative is replicable is called a **complete market**. The completeness of the market is generally considered to be an unrealistic hypothesis, but it is very useful in the theoretical developments.

By Theorem 1.20, in a complete market the arbitrage price of any derivative is uniquely defined.

Theorem 1.24 (Second fundamental theorem of asset pricing). *An arbitrage-free market is complete if and only there exists one and only one EMM (with numeraire B).*

1.3.4 Put-Call parity

For the pricing of Call and Put options there exists a very useful relationship. For this purpose, given a generic underlying price process S_n and a martingale measure Q, let

$$H_n^{\mathrm{Call}} := E^Q\left[\frac{B_n}{B_N}(S_N - K)^+ \mid \mathcal{F}_n\right], \quad H_n^{\mathrm{Put}} := E^Q\left[\frac{B_n}{B_N}(K - S_N)^+ \mid \mathcal{F}_n\right]$$

denote the price in t_n of a Call and Put option respectively with expiration t_N and strike K. Noticing that

$$(K - S)^+ = (S - K)^+ + K - S$$

and recalling the definition of a martingale measure (see (1.20)) one immediately has:

Proposition 1.25.

$$H_n^{\mathrm{Put}} = H_n^{\mathrm{Call}} + \frac{B_n}{B_N}K - S_n. \tag{1.25}$$

Besides its usefulness for pricing, formula (1.25) also shows that a Put option can be replicated by a portfolio that does not change over time (buy and hold portfolio) and consists in buying one unit of the corresponding Call as well as of a contract guaranteeing K monetary units at maturity (zero coupon t_N-bond with face value K, see Chapter 4) while short selling one unit of the underlying asset.

1.4 Market models

Formulas (1.1) and (1.2) define a general discrete-time market, but to solve a specific problem such as the pricing or hedging, we need to define the model in more detail, particularly to define the sequence of random variables μ_n representing the yields of the risky assets. We will discuss two basic models: the first one provides an example of a complete market, the second one is an example of incomplete market model.

1.4.1 Binomial model

The simplest example is provided by the binomial market model. Assume that there exists a bond B with dynamics (1.1) where the short rate $r_n = r$ is constant, that is

$$B_n = (1+r)^n, \qquad n = 0, \dots, N. \tag{1.26}$$

We also assume that there is only one risky asset S with dynamics (1.2): more precisely, we have

$$S_n = S_{n-1}(1+\mu_n), \qquad n = 1, \dots, N$$

where μ_n are i.i.d. random variables such that

$$1+\mu_n = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p, \end{cases}$$

where $p \in]0,1[$ and $0 < d < u$. In other words, the distribution of μ_n is a linear combination of Dirac deltas $p\delta_{u-1} + (1-p)\delta_{d-1}$. We remark that

$$P(S_n = u^k d^{n-k} S_0) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \le k \le n \le N.$$

Figure 1.1 shows a binomial tree with three periods.

Martingale measure and risk neutral price

Theorem 1.26. *In the binomial model the condition*

$$d < 1+r < u, \tag{1.27}$$

is equivalent to the existence and uniqueness of the martingale measure Q. Under this condition, if

$$q = \frac{1+r-d}{u-d}, \tag{1.28}$$

then the EMM Q is defined by

$$Q(1+\mu_n = u) = 1 - Q(1+\mu_n = d) = q, \tag{1.29}$$

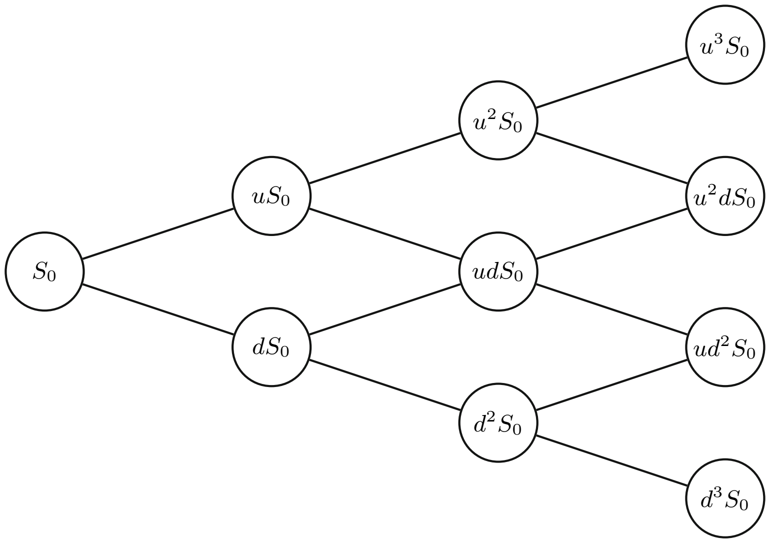

Fig. 1.1. Binomial tree with three periods

and the random variables $\mu_1, \dots, \mu_N$ *are* Q*-independent. Moreover we have*

$$Q(S_n = u^k d^{n-k} S_0) = \binom{n}{k} q^k (1-q)^{n-k}, \qquad 0 \le k \le n \le N. \tag{1.30}$$

Proof. By Definition 1.13, Q is a martingale measure if and only if

$$\widetilde{S}_{n-1} = E^Q \left[\widetilde{S}_n \mid \mathcal{F}_{n-1} \right], \tag{1.31}$$

or equivalently

$$S_{n-1}(1+r) = E^Q \left[S_{n-1} (1+\mu_n) \mid \mathcal{F}_{n-1} \right] = S_{n-1} E^Q \left[(1+\mu_n) \mid \mathcal{F}_{n-1} \right].$$

Then we have

$$\begin{aligned} r = E^Q \left[\mu_n \mid \mathcal{F}_{n-1} \right] =& (u-1) Q \left(\mu_n = u-1 \mid \mathcal{F}_{n-1} \right) \\ &+ (d-1) \left(1 - Q \left(\mu_n = u-1 \mid \mathcal{F}_{n-1} \right) \right), \end{aligned}$$

and

$$Q \left(\mu_n = u-1 \mid \mathcal{F}_{n-1} \right) = \frac{1+r-d}{u-d} = q. \tag{1.32}$$

Condition (1.27) is equivalent to the fact that q belongs to the interval $]0,1[$ and therefore Q, defined by (1.29), is a probability measure equivalent to P. Moreover, since the conditional probability in (1.32) is a real constant, we have that the random variables $\mu_1, \dots, \mu_N$ are independent in the measure Q as well and this proves (1.30). □

By the fundamental theorems of asset pricing, under condition (1.27) the binomial market is arbitrage-free and complete. Consequently, by Theorem 1.20, the arbitrage price of a derivative X is uniquely given by

$$H_n = \frac{1}{(1+r)^{N-n}} E^Q \left[X \mid \mathcal{F}_n \right], \tag{1.33}$$

and, when $X = F(S_N)$,

$$H_n = \frac{1}{(1+r)^{N-n}} \sum_{k=0}^{N-n} \binom{N-n}{k} q^k (1-q)^{N-n-k} F(u^k d^{N-n-k} S_n), \tag{1.34}$$

with q as in (1.28).

Remark 1.27. *It is possible to construct a binomial model with more than one risky asset as in Problem 1.40: however the model is arbitrage-free only if all the risky assets can be expressed as derivatives of a single asset.*

In general, in a binomial model with two assets, arbitrage is possible. In fact, for the model

$$\begin{cases} B_n = B_{n-1}(1+r), \\ S_n^i = S_{n-1}^i (1+\mu_n^i), \qquad i = 1,2 \end{cases}$$

the evolution of a self-financing portfolio is

$$V_n = V_{n-1} \left(1 + r_n + \sum_{i=1}^{d} \pi_n^i \left(\mu_n^i - r \right) \right).$$

Thus we can make an arbitrage if

$$V_n = V_{n-1} \left(1 + r_n + C \right)$$

with $C \geq 0$ and $P(C > 0) > 0$, because then V_n increases faster than B_n without risk. This can be achieved by solving (if possible) the system

$$\begin{cases} \pi_n^1 \left(u^1 - 1 - r \right) + \pi_n^2 \left(u^2 - 1 - r \right) = C^1 \\ \pi_n^1 \left(d^1 - 1 - r \right) + \pi_n^2 \left(d^2 - 1 - r \right) = C^2, \end{cases}$$

with

$$C^1, C^2 \geq 0 \quad \textit{and} \quad C^1 + C^2 > 0. \tag{1.35}$$

In the case of Problem 1.40, we would have

$$\begin{cases} 2\pi_n^1 + \pi_n^2 = C^1 \\ -\frac{2}{3}\pi_n^1 - \frac{1}{3}\pi_n^2 = C^2, \end{cases}$$

that has no solution if C^1, C^2 verify (1.35), being the coefficient matrix degenerate. □

Construction of an hedging strategy

In the binomial model we can build explicitly a hedging strategy (α, β) for a derivative X with maturity N. We set $V_n = \alpha_n S_n + \beta_n B_n$. If S_{N-1} denotes the price of the risky asset at time $N-1$, there are two possible final values of S:

$$S_N = \begin{cases} uS_{N-1}, \\ dS_{N-1}. \end{cases}$$

So the replication condition $V_N = X$ is equivalent to the system of equations

$$\begin{cases} \alpha_N u S_{N-1} + \beta_N B_N = X^u, \\ \alpha_N d S_{N-1} + \beta_N B_N = X^d, \end{cases} \tag{1.36}$$

where X^u and X^d represent the payoff in case of increase and decrease of the underlying asset respectively, given the information at time $N-1$. The linear system (1.36) has solution

$$\bar{\alpha}_N = \frac{X^u - X^d}{(u-d)S_{N-1}}, \qquad \bar{\beta}_N = \frac{uX^d - dX^u}{(1+r)^N(u-d)}, \tag{1.37}$$

and provides the strategy to be used at the time $N-1$ which ensures replication at the final time. By the self-financing condition, we have

$$H_{N-1} := V_{N-1} = \bar{\alpha}_N S_{N-1} + \bar{\beta}_N B_{N-1}$$

and this determines the arbitrage price of X at time $N-1$. A direct computation shows that this result is consistent with the risk neutral pricing formula (1.33): we have in fact

$$\bar{\alpha}_N S_{N-1} + \bar{\beta}_N B_{N-1} = \frac{qX^u + (1-q)X^d}{1+r} = \frac{1}{1+r} E^Q \left[X \mid \mathcal{F}_{N-1} \right].$$

The argument above can be used to determine, proceeding backwards, the entire hedging strategy down to the initial time. More precisely, if S_{n-1} (that, to fix ideas, we assume to be observable) denotes the price of the risky asset at time $n-1$, there are two possibilities:

$$S_n = \begin{cases} uS_{n-1}, \\ dS_{n-1}. \end{cases}$$

If V_n^u and V_n^d denote the values of the replicating strategy at time n in case of increase or decrease of the underlying respectively, we obtain the system

$$\begin{cases} \alpha_n u S_{n-1} + \beta_n B_n = V_n^u, \\ \alpha_n d S_{n-1} + \beta_n B_n = V_n^d, \end{cases} \tag{1.38}$$

with solution

$$\bar{\alpha}_n = \frac{V_n^u - V_n^d}{S_{n-1}(u-d)}, \qquad \bar{\beta}_n = \frac{uV_n^d - dV_n^u}{(1+r)^n(u-d)}, \tag{1.39}$$

that gives the hedging strategy at time $n-1$. By the self-financing condition we have

$$H_{n-1} := V_{n-1} = \bar{\alpha}_n S_{n-1} + \bar{\beta}_n B_{n-1} \tag{1.40}$$

and this determines the arbitrage price of X at time $n-1$. Equivalently we have

$$H_{n-1} = \bar{\alpha}_n S_{n-1} + \bar{\beta}_n B_{n-1} = \frac{qH_n^u + (1-q)H_n^d}{1+r} = \frac{1}{1+r} E^Q \left[H_n \mid \mathcal{F}_{n-1} \right]. \tag{1.41}$$

1.4.2 Trinomial model

In the trinomial model we assume that there is a bond B with dynamics (1.1) with $r_n \equiv r$ and one or more risky securities whose dynamics are driven by a stochastic process $(h_n)_{n=1,\dots,N}$ whose components are i.i.d. random variables such that

$$h_n = \begin{cases} 1 & \text{with probability } p_1, \\ 2 & \text{with probability } p_2, \\ 3 & \text{with probability } p_3 = 1 - p_1 - p_2, \end{cases}$$

where $p_1, p_2 > 0$ and $p_1 + p_2 < 1$. Below we consider the case where there is only one risky asset S^1 (this model is called *standard trinomial market*) and where there are two risky assets S^1 and S^2 (this model is called *completed trinomial market*) with $S_0^1, S_0^2 > 0$ and

$$S_n^i = S_{n-1}^i (1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2, \tag{1.42}$$

where

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3, \end{cases}$$

and $0 < d_i < m_i < u_i$. Figure 1.2 represents a two-period trinomial tree for generic values of u, m, d.

In the standard trinomial market, S^1 usually denotes the underlying of a derivative: as we shall see, the standard trinomial model is the simplest example of incomplete market. The completed trinomial market is a complete model that can be used for the pricing and hedging of an exotic derivative that is not negotiated in the market: a hedging strategy is built using the assets S^1 and S^2 which typically represent the underlying asset and a plain vanilla option on S^1, for example, a European Call option that is usually traded on the market.

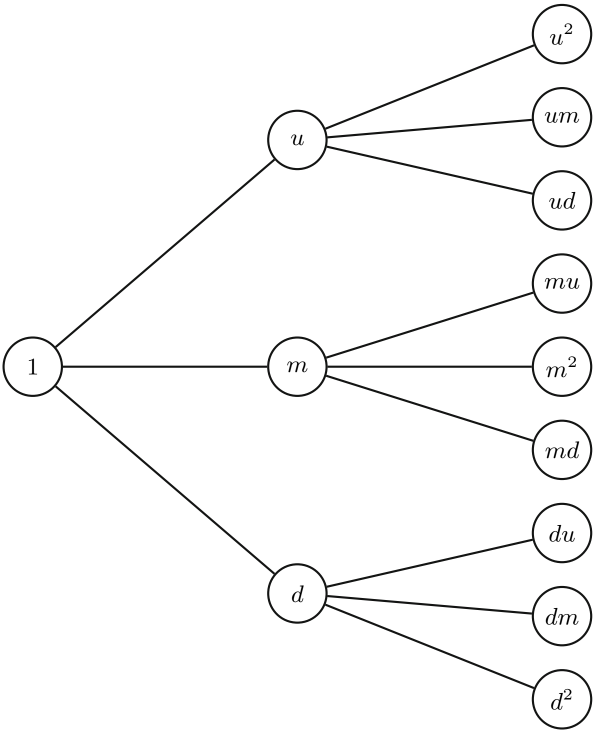

Fig. 1.2. Two-period trinomial tree with initial price $S_0 = 1$

Consider first the standard trinomial market. To determine a martingale measure Q, we proceed as in the case of the binomial model by imposing the martingale condition (1.31) that in this case becomes

$$S^1_{n-1}(1+r) = E^Q \left[S^1_{n-1} \left(1 + \mu(h_n)\right) \mid \mathcal{F}_{n-1} \right], \tag{1.43}$$

where $\mu(h) = \mu^1(h)$. Then, using the notation

$$q^n_j = Q(h_n = j \mid \mathcal{F}_{n-1}), \qquad j = 1, 2, 3, \quad n = 1, \dots, N,$$

we obtain the following system of equations

$$\begin{cases} u_1 q^n_1 + m_1 q^n_2 + d_1 q^n_3 = 1 + r, \\ q^n_1 + q^n_2 + q^n_3 = 1. \end{cases} \tag{1.44}$$

System (1.44) does not admit a unique solution and so in general there is more than one martingale measure; therefore (see Theorem 1.24) the market is incomplete. Furthermore, the random variables h_n are in general not independent with respect to a generic martingale measure. We note that the incompleteness of the market can be directly deduced by observing that the replication condition at maturity $V_N = X$, for a derivative with payoff X, translates into a linear system of three equations in two unknowns

$$\begin{cases} \alpha_N u S_{N-1} + \beta_N B_N = X^u, \\ \alpha_N m S_{N-1} + \beta_N B_N = X^m, \\ \alpha_N d S_{N-1} + \beta_N B_N = X^d, \end{cases}$$

which generally has no solution unless one of the equations is linearly dependent on the other two. This fact may also be interpreted by saying that in the market there is too much randomness with respect to the possibility of building a replicating portfolio.

Consider now the completed trinomial market: imposing the condition (1.43) for $S = S^i$ with $i = 1, 2$, we get the linear system

$$\begin{cases} u_1 q_1^n + m_1 q_2^n + d_1 q_3^n = 1 + r, \\ u_2 q_1^n + m_2 q_2^n + d_2 q_3^n = 1 + r, \\ q_1^n + q_2^n + q_3^n = 1. \end{cases} \tag{1.45}$$

Under suitable assumptions on the model parameters (these assumptions are equivalent to the absence of arbitrage opportunities), the system (1.45) has solution

$$\begin{aligned} q_1^n &= \frac{m_1(1+r-d_2) - d_1(1+r-m_2) - (1+r)(m_2-d_2)}{m_1(u_2-d_2) - u_1(m_2-d_2) - d_1(u_2-m_2)}, \\ q_2^n &= \frac{u_1(d_2-1-r) - d_1(u_2-1-r) + (1+r)(u_2-d_2)}{m_1(u_2-d_2) - u_1(m_2-d_2) - d_1(u_2-m_2)}, \\ q_3^n &= \frac{u_1(1+r-m_2) - m_1(1+r-u_2) - (1+r)(u_2-m_2)}{m_1(u_2-d_2) - u_1(m_2-d_2) - d_1(u_2-m_2)}, \end{aligned} \tag{1.46}$$

and the ratios in (1.46) are positive numbers belonging to $]0,1[$, so they define a probability measure Q equivalent to P. In this case the martingale measure Q is uniquely determined and moreover, since q_1^n, q_2^n and q_3^n are real constants (not random) and independent on n, the random variables h_n are i.i.d. with respect to Q. In this case the market is arbitrage-free and complete.

The hedging strategy of a derivative with price process H_n is determined similarly to the binomial case: assuming that the prices at time $n-1$ are known, to build the hedging strategy $(\alpha_n^1, \alpha_n^2, \beta_n)$ for the n-th period (from $n-1$ to n), we solve the linear system

$$\begin{cases} \alpha_n^1 u_1 S_{n-1}^1 + \alpha_n^2 u_2 S_{n-1}^2 + \beta_n (1+r)^n = H_n^u, \\ \alpha_n^1 m_1 S_{n-1}^1 + \alpha_n^2 m_2 S_{n-1}^2 + \beta_n (1+r)^n = H_n^m, \\ \alpha_n^1 d_1 S_{n-1}^1 + \alpha_n^2 d_2 S_{n-1}^2 + \beta_n (1+r)^n = H_n^d, \end{cases} \tag{1.47}$$

where H_n^u, H_n^m and H_n^d denote the prices of the derivative at time n in the three scenarios respectively. The solution of system (1.47) is given by:

$$\alpha_n^1 = \frac{d_2\left(H_n^m - H_n^u\right) + H_n^u m_2 - H_n^m u_2 + H_n^d\left(-m_2 + u_2\right)}{S_{n-1}^1\left(d_2\left(m_1 - u_1\right) + m_2 u_1 - m_1 u_2 + d_1\left(u_2 - m_2\right)\right)},$$
$$\alpha_n^2 = \frac{d_1\left(H_n^m - H_n^u\right) + H_n^u m_1 - H_n^m u_1 + H_n^d\left(u_1 - m_1\right)}{S_{n-1}^2\left(-m_2 u_1 + d_2\left(u_1 - m_1\right) + d_1\left(m_2 - u_2\right) + m_1 u_2\right)},$$
$$\beta_n = \frac{d_2\left(H_n^u m_1 - H_n^m u_1\right) + d_1\left(-H_n^u m_2 + H_n^m u_2\right) + H_n^d\left(m_2 u_1 - m_1 u_2\right)}{(1+r)^n\left(d_2\left(m_1 - u_1\right) + m_2 u_1 - m_1 u_2 + d_1\left(-m_2 + u_2\right)\right)}.$$

Remark 1.28. *We saw that, for a trinomial market model to be complete, it is necessary to have the opportunity to invest in two different risky securities. On the other hand, in the trinomial model we have three possible states of nature in every period: both prices may simultaneously go up, move to an intermediate level, or fall. In general we have that, for a market with m states of nature to be complete, there must exist at least $m-1$ tradable risky assets.* □

Remark 1.29. *Note that the coefficient matrix in (1.45) is given by*

$$\begin{pmatrix} u_1 & m_1 & d_1 \\ u_2 & m_2 & d_2 \\ 1 & 1 & 1 \end{pmatrix} \tag{1.48}$$

and this matrix induces a linear map $L : \Sigma^3 \to \mathbb{R}^3$ (Σ^3 is the simplex $\Sigma^3 = \{(q_1, q_2, q_3) \mid q_i \geq 0, \sum_{i=1}^3 q_i = 1\} \subset \mathbb{R}^3$) that associates $(1+r, 1+r, 1)$ to (q_1^n, q_2^n, q_3^n). The fact that the system (1.45) admits a unique solution is equivalent to stating that L is injective.

On the other hand, considering $\alpha_n^i S_{n-1}^i$, $i = 1, 2$ and $\beta_n(1+r)^n$ as unknowns in (1.47) (indeed S_{n-1}^i and $\beta_n(1+r)^n$ are observable at time n) the coefficient matrix in (1.47) is

$$\begin{pmatrix} u_1 & u_2 & 1 \\ m_2 & m_2 & 1 \\ d_1 & d_2 & 1 \end{pmatrix} \tag{1.49}$$

and it also induces a linear map $L^* : \mathbb{R}^3 \to \mathbb{R}^3$ that associates (H_n^u, H_n^m, H_n^d) to $(\alpha_n^1 S_{n-1}^1, \alpha_n^2 S_{n-1}^2, \beta_n(1+r)^n)$. The fact that the system (1.47) has solution for any value of (H_n^u, H_n^m, H_n^d) is equivalent to the fact that L^* is surjective.

Since the matrix (1.49) is the transpose of (1.48), we have that L^* is the adjoint operator of L. Then the fact that L^* is surjective if L is injective and viceversa, follows by a well-known mathematical result of linear algebra. This has been illustrated here for the pair of systems (1.45),(1.47), but could also be applied to the pair formed on one hand by the system satisfied by q and $1-q$ in the binomial model and on the other hand by system (1.38). More generally, a similar result holds true in any complete market model. These facts explain the main ideas underlying the second fundamental theorem of asset pricing (Theorem 1.24). □

1.5 On pricing and hedging in incomplete markets

In Section 1.4.2 we saw that an incomplete model, the trinomial model, can be completed. Often, the completion is not possible or not desirable. In this section we mention possible procedures for the pricing and hedging in an incomplete market.

By Theorem 1.14 in an arbitrage-free market there exists at least one EMM and by Theorem 1.20 the arbitrage price of a replicable derivative is given by the expected value of the discounted payoff with respect to a martingale measure.

It follows that in a complete market the no-arbitrage principle is sufficient to uniquely define the arbitrage price. This is not the case in an incomplete market, where many possible equivalent martingale measures exist. We recall that each specific choice of an EMM defines as in (1.23) a price for a derivative which is consistent with the no-arbitrage principle. On the other hand, by Theorem 1.24 it is not possible to find a replicating strategy for all the derivatives.

1. **Pricing.** In an incomplete market there exist different prices for a derivative which are consistent with the no-arbitrage principle. One way to define uniquely a price then consists in resorting to market data to determine, among all the possible ones, an EMM Q such that the theoretical prices given by (1.23) depart as little as possible from those actually observed in the market. This leads to an *inverse problem* namely the so-called problem of **calibration** to which some of the problems in Chapter 4 are dedicated. Alternatively, we may impose additional requirements such as to take into account the preference-structure of the agents in the market.

2. **Hedging.** As it is not possible to perfectly replicate any derivative, we must give up the perfect replication and introduce criteria to choose the best among imperfect hedging strategies. Before describing two of these criteria, we mention the so-called criterion of
   2.a. **Super-hedging.** We look for a self-financing strategy such that
   $$V_N^{(\alpha,\beta)} \geq X\,, \quad \text{a.s.}$$
   The main flaw of this approach is that, in general, it requires a large initial capital V_0.

   The following two hedging criteria are used most frequently:
   2.b. **Quadratic risk minimization.** We aim at determining a self-financing strategy that minimizes
   $$E_{S_0,V_0}\left[\left(X - V_N^{(\alpha,\beta)}\right)^2\right].$$
   The initial wealth V_0 can be given or the minimization procedure may also involve V_0. It is a symmetrical criterion that has the advantage

of being mathematically quite tractable, but has the flaw that it penalizes equally deviations in excess as in defect.

An asymmetric criterion is the following:

2.c. **"Shortfall" risk minimization.** We look for a strategy that minimizes

$$E_{S_0,V_0}\left[\left(X - V_N^{(\alpha,\beta)}\right)^+\right].$$

Also in this case the minimization can involve V_0 as well. It is a criterion that penalizes only deviations in defect *(downside-type risk)*. On the other hand, its asymmetrical nature makes it more difficult to be dealt with mathematically.

We refer to the Problems 1.47 and 1.48 for examples of hedging in an incomplete market following the lines described above.

Remark 1.30. *The above criteria can also be applied to a complete market where the investor does not possess sufficient initial capital to achieve perfect replication, i.e. if* $V_0 < E^Q\left[\frac{X}{B_N}\right]$. □

Remark 1.31. *When minimizing in 2.b. and 2.c. also with respect to* V_0*, the resulting minimal value is often considered as a possible price of the derivative. This price may indeed be considered also in accordance with the no-arbitrage principle in the sense that it provides the minimal initial capital from which we can obtain the best (imperfect) replication according to the criterion that is being adopted.* □

1.6 Change of numeraire

We briefly illustrate the technique of the change of numeraire starting from the pricing formula of a European Call option in the binomial model (see (1.34)) and then describing it in a more general setting. As we shall see in Chapter 4, this technique has significant practical applications.

1.6.1 A particular case

To introduce the technique, we apply the valuation formula (1.34) to the case of a European Call option with $X = F(S_N) = (S_N - K)^+$. By setting

$$a_n := \inf\left\{k \mid u^k d^{N-n-k} S_n > K\right\},$$

we have

$$H_n = \frac{1}{(1+r)^{N-n}} \sum_{k=a_n}^{N-n} \binom{N-n}{k} q^k (1-q)^{N-n-k} S_n u^k d^{N-n-k} +$$

$$
\begin{aligned}
&-\frac{K}{(1+r)^{N-n}}\sum_{k=a_n}^{N-n}\binom{N-n}{k}q^k(1-q)^{N-n-k}\\
=&S_n\sum_{k=a_n}^{N-n}\binom{N-n}{k}\left(\frac{qu}{1+r}\right)^k\left(\frac{(1-q)d}{1+r}\right)^{N-n-k}+\\
&-\frac{K}{(1+r)^{N-n}}\sum_{k=a_n}^{N-n}\binom{N-n}{k}q^k(1-q)^{N-n-k}\\
=&S_n\sum_{k=a_n}^{N-n}\binom{N-n}{k}\bar{q}^k(1-\bar{q})^{N-n-k}+\\
&-\frac{K}{(1+r)^{N-n}}\sum_{k=a_n}^{N-n}\binom{N-n}{k}q^k(1-q)^{N-n-k}\\
=&S_n\bar{Q}\left(S_N>K\mid\mathcal{F}_n\right)-\frac{K}{(1+r)^{N-n}}Q\left(S_N>K\mid\mathcal{F}_n\right)
\end{aligned}
\tag{1.50}
$$

where $\bar{q}:=\frac{qu}{1+r}\in(0,1)$ and $\bar{Q}$ is the measure induced by $\bar{q}$. In other words, the price of a European Call option in a binomial model can be computed by determining the probabilities, conditioned on $\mathcal{F}_n$, of the event $\{S_N>K\}$ (namely, that the option at maturity is "in the money") in the two measures $\bar{Q}$ and Q.

It turns out that not only is Q an EMM (with numeraire B), but so is also $\bar{Q}$ although with respect to a different numeraire. To verify this, we first introduce the general notion of *numeraire*. Considering then a general discrete time market (cf. Section 1.1.1), assume that the price process S^1 is positive and use it as a reference price, in units of which the prices of all other assets are measured, that is, take it as a numeraire. Hence we set

$$
\bar{B}_n=\frac{B_n}{S_n^1},\qquad \bar{S}_n^i=\frac{S_n^i}{S_n^1},\qquad i=1,\dots,d.
$$

If the market is arbitrage-free and complete, there exists a unique martingale measure $\bar{Q}$ with numeraire S^1, which is a measure such that:

i) $\bar{Q}$ is equivalent to P;
ii) the price processes $\bar{B}$ and $\bar{S}$ are $\bar{Q}$-martingales.

Returning to the binomial market model we now show that, while Q is a martingale measure with numeraire B, $\bar{Q}$ has S as numeraire. We can show it in an elementary way noticing that it is sufficient to verify that B_n and S_n (the only securities in the market under consideration), when expressed in units of S_n, are $\bar{Q}-$martingales. Now it is clear that $\frac{S_n}{S_n}\equiv 1$ is a martingale. It is then sufficient to show that

$$
E^{\bar{Q}}\left[\frac{B_{n+1}}{S_{n+1}}\mid\mathcal{F}_n\right]=E^{\bar{Q}}\left[\frac{B_{n+1}}{S_{n+1}}\mid S_n\right]=\frac{B_n}{S_n}
\tag{1.51}
$$

and this is equivalent to require that

$$\bar{q}\,\frac{1+r}{S_n u} + (1-\bar{q})\,\frac{1+r}{S_n d} = \frac{1}{S_n}. \tag{1.52}$$

Now we easily see that, according to $\bar{q} = \frac{qu}{1+r}$ and to the fact that $qu + (1-q)d = 1+r$ implies $\left(1 - \frac{qu}{1+r}\right)\frac{1+r}{d} = 1-q$, formula (1.52) follows immediately from the identity

$$\frac{q}{S_n} + \frac{1-q}{S_n} = \frac{1}{S_n}.$$

1.6.2 General case

This section is intended to make more explicit the relationship between martingale measures related to different numeraires and we give the expression of the Radon-Nikodym derivative of one measure with respect to another.

In the following let Y denote the price of a *traded* asset: this can be one of the primitive securities $S^1, \dots, S^d$ or the value of a self-financing and predictable strategy. From the mathematical point of view, the fact that the asset Y is *negotiated (or quoted) in the market corresponds to the fact that the process* $\widetilde{Y} = \left(\frac{Y_n}{B_n}\right)$ *is a Q-martingale* (cf. Proposition 1.15).

Theorem 1.32. *In an arbitrage-free market model, let Q be an EMM with numeraire B and $(Y_n)_{n \le N}$ a positive process such that $\widetilde{Y}$ is a Q-martingale (Y represents the price of a traded asset to be taken as the new numeraire). Then the measure Q^Y defined by*

$$\frac{dQ^Y}{dQ} = \frac{Y_N}{Y_0}\left(\frac{B_N}{B_0}\right)^{-1}, \tag{1.53}$$

is such that

$$B_n E^Q\left[\frac{X}{B_N} \mid \mathcal{F}_n\right] = Y_n E^{Q^Y}\left[\frac{X}{Y_N} \mid \mathcal{F}_n\right], \qquad n \le N, \tag{1.54}$$

for every integrable[3] *random variable X. Consequently Q^Y is an EMM with numeraire Y.*

Remark 1.33. *We can rewrite* (1.54) *in the form*

$$E^Q\left[D(n,N)X \mid \mathcal{F}_n\right] = E^{Q^Y}\left[D^Y(n,N)X \mid \mathcal{F}_n\right], \qquad n \le N, \tag{1.55}$$

where

$$D^Y(n,N) = \frac{Y_n}{Y_N}, \qquad n \le N,$$

[3] Here it is automatically true, since by assumption Ω is finite.

denotes the discount factor from N to n with respect to the numeraire Y (if $Y = B$, we simply write $D(n,N)$ instead of $D^B(n,N)$). Notice that the left (resp. right) hand side of formula (1.55) *represents the arbitrage price at time n of a European derivative with payoff X and maturity N, expressed in terms of conditional expectation of the discounted payoff w.r.t. the numeraire B (resp. Y) in the corresponding martingale measure Q (resp. Q^Y).* □

Proof. In (1.53), $Z := \frac{dQ^Y}{dQ}$ denotes the Radon-Nikodym derivative of Q^Y with respect to Q: this means that we have

$$E^{Q^Y}[X] = E^Q[XZ]$$

for every integrable random variable X.

From (1.53) we infer the following formula:

$$E^{Q^Y}[X \mid \mathcal{F}_n] = E^Q\left[X\frac{B_n}{B_N}\left(\frac{Y_n}{Y_N}\right)^{-1} \mid \mathcal{F}_n\right], \qquad n \le N. \tag{1.56}$$

In fact, by the Bayes formula[4] we have

$$E^{Q^Y}[X \mid \mathcal{F}_n] = \frac{E^Q[XZ \mid \mathcal{F}_n]}{E^Q[Z \mid \mathcal{F}_n]} = \frac{E^Q\left[X\frac{Y_N}{B_N} \mid \mathcal{F}_n\right]}{E^Q\left[\frac{Y_N}{B_N} \mid \mathcal{F}_n\right]}$$

which gives (1.56), because by assumption $\widetilde{Y}$ is a Q-martingale and therefore we have

$$E^Q\left[\frac{Y_N}{B_N} \mid \mathcal{F}_n\right] = \frac{Y_n}{B_n}.$$

Now (1.54) is a simple consequence of (1.56): indeed

$$B_n E^Q\left[\frac{X}{B_N} \mid \mathcal{F}_n\right] = E^Q\left[\frac{B_n}{B_N}\left(\frac{Y_n}{Y_N}\right)^{-1}\frac{Y_n X}{Y_N} \mid \mathcal{F}_n\right] =$$

(by (1.56))

$$= Y_n E^{Q^Y}\left[\frac{X}{Y_N} \mid \mathcal{F}_n\right].$$

By (1.54) we have that Q^Y is an EMM with numeraire Y: indeed, by definition of EMM we have

$$S_n = B_n E^Q\left[\frac{S_N}{B_N} \mid \mathcal{F}_n\right] =$$

[4] See, for instance, Theorem A.113 in [17].

(by (1.54) with $X = S_N$)

$$= Y_n E^{Q^Y}\left[\frac{S_N}{Y_N} \mid \mathcal{F}_n\right],$$

for any $n \le N$, and a similar relationship holds for B. □

Corollary 1.34. *Under the assumptions of Theorem 1.32, for any $A \in \mathcal{F}_n$ we have*

$$Q^Y(A) = E^Q\left[\frac{Y_n}{Y_0}\left(\frac{B_n}{B_0}\right)^{-1} \mathbb{1}_A\right]. \tag{1.57}$$

Proof. We have

$$Q^Y(A) = E^{Q^Y}[\mathbb{1}_A] =$$

(by (1.53))

$$= E^Q\left[\mathbb{1}_A \frac{Y_N}{Y_0}\left(\frac{B_N}{B_0}\right)^{-1}\right] =$$

(using that $A \in \mathcal{F}_n$)

$$= E^Q\left[\mathbb{1}_A E^Q\left[\frac{Y_N}{Y_0}\left(\frac{B_N}{B_0}\right)^{-1} \mid \mathcal{F}_n\right]\right]$$

and the thesis follows from the fact that $\frac{Y}{B}$ is a Q-martingale. □

Example 1.35. Consider the binomial model of Section 1.4.1, where the EMM with numeraire B is defined by

$$q := Q(1 + \mu_n = u) = 1 - Q(1 + \mu_n = d) = \frac{1 + r - d}{u - d}.$$

To determine the EMM $\bar{Q}$ with numeraire S, we may proceed analogously to the proof of Theorem 1.26. Alternatively we can directly use the Corollary 1.34: since $\{1 + \mu_1 = u\} \in \mathcal{F}_1$, by (1.57) we have

$$\begin{aligned}\bar{q} := \bar{Q}(1 + \mu_1 = u) &= \int_{\{1+\mu_1=u\}} \frac{S_1^1}{B_1}\left(\frac{S_0^1}{B_0}\right)^{-1} dQ \\ &= \int_{\{1+\mu_1=u\}} \frac{1 + \mu_1}{1 + r} dQ = \frac{uq}{1 + r}.\end{aligned} \tag{1.58}$$

In general, if we consider the elementary event

$$A = \{\mu_1 = \bar{\mu}_1, \dots, \mu_N = \bar{\mu}_N\} \in \mathcal{F}_N,$$

where $1 + \bar{\mu}_n \in \{u, d\}$ for $n = 1, \dots, N$, we have

$$\begin{aligned}\bar{Q}(A) &= \int_A \frac{S_N^1}{B_N} \left(\frac{S_0^1}{B_0}\right)^{-1} dQ = \frac{(1+\bar{\mu}_1)\cdots(1+\bar{\mu}_N)}{(1+r)^N} Q(A) \\ &= \frac{(uq)^k (d(1-q))^{N-k}}{(1+r)^N},\end{aligned}$$

where k denotes the number of $\bar{\mu}_n$ taking the value $u - 1$. □

1.7 Solved problems

Problem 1.36. Consider a binomial market model where, using the notations of Section 1.4.1, $S_0 = 1$, $u = 2$, $d = 1/2$, $r = 0$ and $N = 2$. For a "look-back Call" option with payoff

$$X = (S_N - m_N)^+ = S_N - m_N, \qquad \text{where} \quad m_N := \min_{n \le N} S_n,$$

determine:

i) the initial price H_0 and the prices at time $n = 1$ in the two scenarios $S_1 = 1$ and $S_1 = 1/2$;
ii) the proportion π_1 to be invested in the risky asset at time $n = 1$, in the two scenarios $S_1 = 1$ and $S_1 = 1/2$, to hedge the option. Determine also the proportion π_0 to be invested in the risky asset at time $n = 0$, to hedge the option.

Solution of Problem 1.36
i) Figure 1.3 shows the binomial tree of the underlying asset prices and the values of m_N and the payoff.

According to the pricing formula (1.33), to directly determine the price in $n = 0$ we have to compute:

$$\begin{aligned}H_0 = &\frac{1}{(1+r)^2}\Big[q^2 \left(u^2 - \min\{1, u, u^2\}\right)^+ + q(1-q)\left(ud - \min\{1, u, ud\}\right)^+ \\ &+ q(1-q)\left(du - \min\{1, d, du\}\right)^+ + (1-q)^2 \left(d^2 - \min\{1, d, d^2\}\right)^+\Big].\end{aligned}$$

In our case, recalling (1.28), the martingale measure is defined in terms of

$$q = \frac{1 + r - d}{u - d} = \frac{1}{3},$$

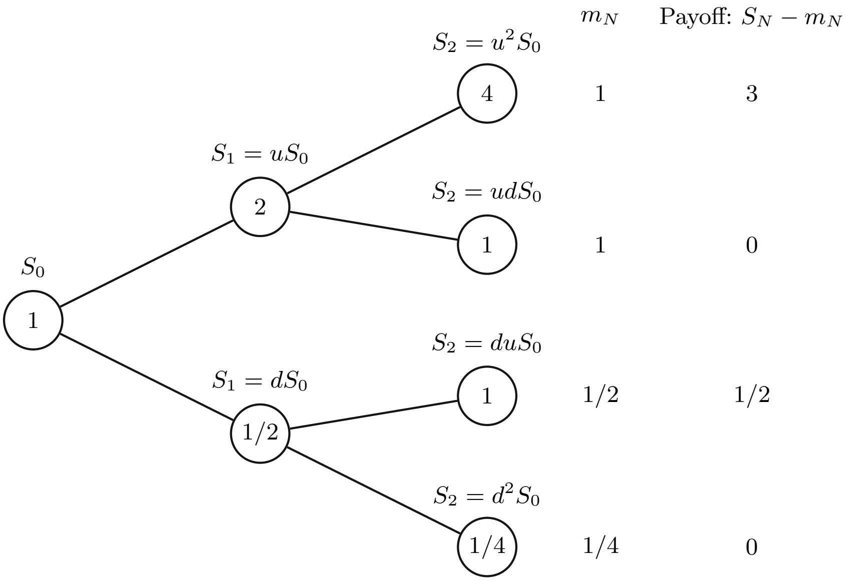

Fig. 1.3. Two-period binomial tree: values of $m_N := \min_{n \le N} S_n$ and of the payoff of a look-back option

and then

$$H_0 = 3q^2 + 0 \cdot q(1-q) + \frac{1}{2} q(1-q) + 0 \cdot (1-q)^2 = \frac{4}{9}. \tag{1.59}$$

In $n = 1$ we have two scenarios, $1 + \mu_1 = u$ and $1 + \mu_1 = d$, and we denote the corresponding prices of the derivative by H_1^u and H_1^d respectively:

$$H_1^u = \frac{1}{1+r} \left[q \left(u^2 - \min\{1, u, u^2\} \right)^+ + (1-q) \left(ud - \min\{1, u, ud\} \right)^+ \right],$$
$$H_1^d = \frac{1}{1+r} \left[q \left(du - \min\{1, d, du\} \right)^+ + (1-q) \left(d^2 - \min\{1, d, d^2\} \right)^+ \right].$$

Thus we have

$$H_1^u = 3q = 1, \qquad H_1^d = \frac{q}{2} = \frac{1}{6}. \tag{1.60}$$

According to (1.41), a simple check shows that by (1.59) we have

$$\frac{4}{9} = H_0 = \frac{1}{1+r} \left(q H_1^u + (1-q) H_1^d \right) = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{6}.$$

ii) To determine the proportion to be invested in $n = 1$ in the scenario $S_1 = S_1^u := 2$, first we determine the quantities α_2^u, β_2^u that denote the units of risky and riskless asset respectively, of the hedging strategy in the second period $[t_1, t_2]$. To this end, with obvious notations, we impose the replication condition (1.38):

$$\begin{cases} \alpha_2 u S_1 + \beta_2 B_2 = u S_1 - m_2^u, \\ \alpha_2 d S_1 + \beta_2 B_2 = d S_1 - m_2^d, \end{cases} \tag{1.61}$$

equivalent to

$$\begin{cases} 4\alpha_2 + \beta_2 = 3, \\ \alpha_2 + \beta_2 = 0, \end{cases}$$

that has solution

$$\alpha_2^u = 1, \qquad \beta_2^u = -1.$$

We may verify that, by the self-financing condition, we have

$$H_1^u := V_1^u = \alpha_2^u S_1^u + \beta_2^u = 1,$$

in agreement with (1.60). By definition (1.10), the proportion of risky asset is equal to

$$\pi_2^u = \frac{\alpha_2^u S_1^u}{H_1^u} = 2;$$

in other words, starting from a wealth $V_1^u = 1$, the strategy consists of borrowing a unit of bond to buy one unit of risky asset whose value is $S_1^u = 2$.

In the scenario $S_1 = S_1^d := \frac{1}{2}$, system (1.61) is equivalent to

$$\begin{cases} \alpha_2 + \beta_2 = \frac{1}{2}, \\ \frac{\alpha_2}{4} + \beta_2 = 0, \end{cases}$$

with solution

$$\alpha_2^d = \frac{2}{3}, \qquad \beta_2^d = -\frac{1}{6}.$$

We may verify that

$$H_1^d := V_1^d = \alpha_2^d S_1^d + \beta_2^d = \frac{1}{6},$$

in accordance with (1.60). Therefore

$$\pi_2^d = \frac{\alpha_2^d S_1^d}{H_1^d} = 2;$$

in other words, starting from an initial wealth $V_1^d = \frac{1}{2}$, the strategy consists of borrowing $\frac{1}{6}$ of the riskless asset to buy $\frac{2}{3}$ of risky asset whose value is $S_1^d = \frac{1}{2}$.

In $n = 0$, finally we have to solve

$$\begin{cases} \alpha_1 u S_0 + \beta_1 B_1 = H_1^u, \\ \alpha_1 d S_0 + \beta_1 B_1 = H_1^d, \end{cases}$$

equivalent to

$$\begin{cases} 2\alpha_1 + \beta_1 = 1, \\ \frac{\alpha_1}{2} + \beta_1 = \frac{1}{6}, \end{cases}$$

with solution

$$\alpha_1 = \frac{5}{9}, \qquad \beta_1 = -\frac{1}{9}.$$

We may verify that

$$H_0 := V_0 = \alpha_1 S_0 + \beta_1 = \frac{4}{9}$$

in accordance with formula (1.59). Thus

$$\pi_1 = \frac{\alpha_1 S_0}{H_0} = \frac{5}{4};$$

in other words, the amount $\frac{1}{9}$, obtained by selling the bond, is added to the initial wealth $V_0 = \frac{4}{9}$, received from the sale of the option, and then it is invested to buy $\frac{5}{9}$ units of risky asset at the unit price of $S_0 = 1$. □

Problem 1.37. In a binomial market model with parameters $S_0 = 1$, $u = 2$, $d = 1/2$, $r = 0$ and $N = 2$, consider an Asian Put option with "floating strike" and payoff

$$X = (M - S_N)^+, \qquad \text{where} \quad M := \frac{S_0 + S_1 + S_2}{3};$$

i) determine the price process, denoted here by $(A_n)_{n=0,1}$, and the hedging strategy for $n = 0, 1$;
ii) consider a European Put option with strike K, payoff $H_2 = (K - S_2)^+$ and price H. Show that there exists a unique positive value of K such that $H_0 = A_0$: for that particular K, compare the Asian and European hedging strategies for the first period.

Solution of Problem 1.37

i) We represent in Figure 1.4 the binomial tree of the underlying prices, the values of the average M and the payoff. The martingale measure is given in terms of $q = \frac{1+r-d}{u-d} = \frac{1}{3}$.

By the valuation formula (1.33), the derivative prices are then

$$\begin{aligned} A_0 =& \frac{1}{(1+r)^2} \left(q^2 X^{uu} + q(1-q) \left(X^{ud} + X^{du} \right) + (1-q)^2 X^{dd} \right) = \frac{2}{9}, \\ A_1^u =& \frac{1}{1+r} \left(q X^{uu} + (1-q) X^{ud} \right) = \frac{2}{9}, \\ A_1^d =& \frac{1}{1+r} \left(q X^{du} + (1-q) X^{dd} \right) = \frac{2}{9}. \end{aligned}$$

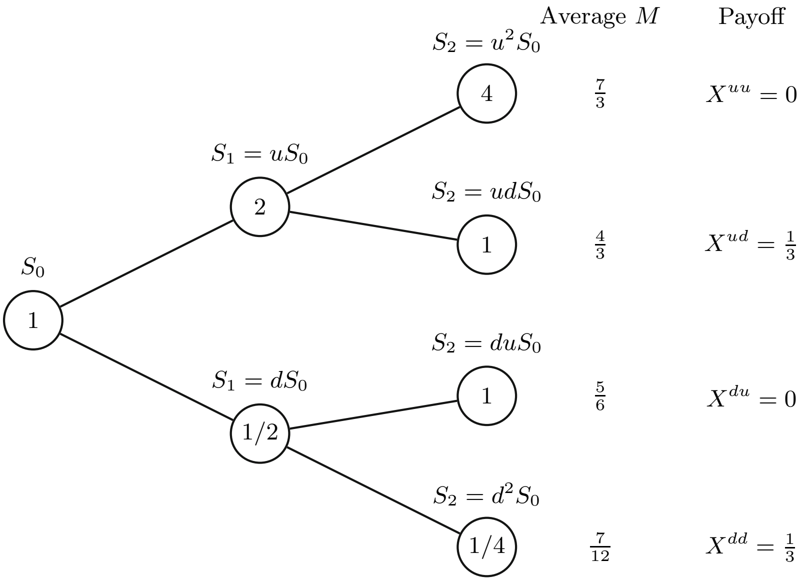

Fig. 1.4. Two-period binomial tree: value of the arithmetic average $M := \frac{S_0+S_1+S_2}{3}$ and the payoff of an Asian Put option with floating strike

For the hedging strategy in the first period, consider the system

$$\begin{cases} \alpha_1 u S_0 + \beta_1 B_1 = A_1^u, \\ \alpha_1 d S_0 + \beta_1 B_1 = A_1^d, \end{cases}$$

which is equivalent to

$$\begin{cases} 2\alpha + \beta = \frac{2}{9}, \\ \frac{\alpha}{2} + \beta = \frac{2}{9}, \end{cases}$$

and has solution

$$\alpha_1 = 0, \qquad \beta_1 = \frac{2}{9}.$$

In the second period, for the scenario $S_1 = 2$ we have to solve the system

$$\begin{cases} \alpha_2 2u + \beta_2 B_2 = X^{uu}, \\ \alpha_2 2d + \beta_2 B_2 = X^{ud}, \end{cases}$$

which is equivalent to

$$\begin{cases} 4\alpha + \beta = 0, \\ \alpha + \beta = \frac{1}{3}, \end{cases}$$

and has solution

$$\alpha_2^u = -\frac{1}{9}, \qquad \beta_2^u = \frac{4}{9}.$$

Analogously, for the scenario $S_1 = \frac{1}{2}$ we find

$$\alpha_2^d = -\frac{4}{9}, \qquad \beta_2^u = \frac{4}{9}.$$

The hedging strategy for the Put option requires thus a short position in the risky asset which is greater (in units of the asset) when the prices decrease. Finally, a simple verification shows that

$$A_0 = \alpha_1 S_0 + \beta_1 B_0 = \frac{2}{9}, \quad \text{and, for } S_1 = uS_0, \quad A_1^u = \alpha_2^u S_1 + \beta_2^u B_1 = \frac{2}{9}.$$

An analogous result holds for A_1^d.

ii) We have

$$H_0(K) = \frac{q^2(K-4)^+ + 2q(1-q)(K-1)^+ + (1-q)^2\,(K-1/4)^+}{(1+r)^2},$$

and therefore $K \mapsto H_0(K)$ is a continuous and monotone increasing function. Moreover

$$0 = H_0\,(1/4) < A_0 < H_0(1) = \frac{1}{3},$$

and therefore there is only one $K \in]1/4, 1[$ such that $H_0(K) = A_0 = \frac{2}{9}$. In particular, for $K \in]1/4, 1[$, we have

$$H_0(K) = \frac{4}{9}\,(K - 1/4)$$

and, by imposing the condition $H_0(K) = A_0$, we get $K = \frac{3}{4}$.

Now we set $K = \frac{3}{4}$. Before determining the hedging strategy at the initial time, we notice that a direct computation shows that $H_1^u = 0$ and $H_1^d = \frac{1}{3}$. Thus, by solving the system

$$\begin{cases} 2\alpha + \beta = 0, \\ \frac{\alpha}{2} + \beta = \frac{1}{3}, \end{cases}$$

we obtain the initial strategy

$$\alpha_1^{\text{Put}} = -\frac{2}{9}, \qquad \beta_1^{\text{Put}} = \frac{4}{9}.$$

So, unlike the Asian Put, the hedging strategy of the European Put requires taking a short position on the risky asset already at the beginning and to invest more on the bond. □

Problem 1.38. Consider a binomial market model where

$$S_n = S_{n-1}(1+\mu_n), \qquad \mu_n \text{ i.i.d. and } 1+\mu_n \in \{u, d\},\ S_0 = 1,$$

and a Call option with payoff $(S_N - K)^+$. Choose the following numerical data: $N = 3$, $u = 2$, $d = \frac{1}{2}$, $K = 1$ and $r = 0$,

i) recalling the pricing formula (1.50) for a Call option

$$H_0 = S_0 \bar{Q}(S_N > K) - \frac{K}{(1+r)^N} Q(S_N > K),$$

determine the initial price H_0 of the option;

ii) check the result by a direct computation using the risk-neutral valuation formula

$$H_0 = \frac{1}{(1+r)^N} E^Q \left[(S_N - K)^+ \right].$$

Solution of Problem 1.38

i) First we determine the probabilities of elementary events in the two martingale measures. We have

$$\begin{aligned} Q(1+\mu_n = u) &= \frac{1+r-d}{u-d} = \frac{1}{3} =: q, \\ Q(1+\mu_n = d) &= \frac{u-1-r}{u-d} = \frac{2}{3} = 1-q, \end{aligned}$$

and, recalling (1.58),

$$\begin{aligned} \bar{Q}(1+\mu_n = u) &= \frac{uq}{1+r} = \frac{2}{3} =: \bar{q}, \\ \bar{Q}(1+\mu_n = d) &= \frac{(1-q)d}{u-d} = \frac{1}{3} = 1-\bar{q}. \end{aligned}$$

Then we have

$$\begin{aligned} H_0 &= S_0 \bar{Q}(S_3 > 1) - Q(S_3 > 1) \\ &= \Big(\bar{Q}(S_3 = 8) + \bar{Q}(S_3 = 2) \Big) - \Big(Q(S_3 = 8) + Q(S_3 = 2) \Big) \\ &= \bar{q}^3 + 3\bar{q}^2(1-\bar{q}) - \left(q^3 + 3q^2(1-q) \right) = \frac{13}{27}. \end{aligned}$$

Here we used the fact that the random variables μ_n are independent not only under the measure P, but also under the martingale measures Q and $\bar{Q}$. In fact, proceeding as in the proof of Theorem 1.26, we have

$$\bar{Q}(1+\mu_n = u \mid \mathcal{F}_{n-1}) = \bar{q} = \bar{Q}(1+\mu_n = u).$$

ii) Using the risk-neutral valuation formula we have

$$\begin{aligned} H_0 = E^Q \left[(S_3 - 1)^+ \right] &= q^3(u^3 - 1)^+ + 3q^2(1-q)(u^2 d - 1)^+ \\ &\qquad + 3q(1-q)^2(ud^2 - 1)^+ + (1-q)^3(d^3 - 1)^+ \\ &= \frac{13}{27}. \end{aligned}$$

□

Problem 1.39. In a binomial market model with $S_0 = 1, u = 2, d = \frac{1}{2}, r = 0$ consider a European Call option with maturity $N = 3$ and strike $K = 1$. Denote by Q the unique equivalent martingale measure:

i) determine the initial price H_0 of the option;
ii) consider the variant (*contingent premium* or *pay later option*), for which the holder pays a fraction $\alpha \in (0, 1)$ of the price, call it V, at maturity in $n = N = 3$ and this only if he/she exercises the option, whereas he/she pays the remaining fraction $(1 - \alpha)$ in $n = 0$. The value at maturity is therefore
$$H_3^{\mathrm{CP}} = \begin{cases} (S_3 - 1) - \alpha V & \text{if } S_3 > 1, \\ 0 & \text{otherwise.} \end{cases}$$
Determine V such that the arbitrage price H_0^{CP} of this variant in $n = 0$ is $H_0^{\mathrm{CP}} = (1 - \alpha) V$. Give an intuitive explanation for how H_0^{CP} compares with H_0;
iii) choosing $\alpha = \frac{1}{2}$, determine the hedging strategy (α_3, β_3) for the CP variant in $n = 2$ in the two nodes corresponding to the scenarios where the price has increased both times and where it has decreased both times.

Solution of Problem 1.46

i) Noticing that for the given data we have $q = \frac{1}{3}$, we can compute

$$\begin{aligned} H_0 &= E^Q\left[(S_3 - 1)^+\right] \\ &= \sum_{k=0}^{3} \binom{3}{k} q^k (1 - q)^{3-k} \left(2^k \left(\frac{1}{2}\right)^{3-k} - 1\right)^+ \\ &= 3 \frac{1}{3^2} \frac{2}{3} \left(\frac{4}{2} - 1\right)^+ + \frac{1}{3^3} \left(2^3 - 1\right)^+ = \frac{13}{27}. \end{aligned}$$

ii) Notice first that, defining

$$a := \min\{k \le 3 \mid S_0 u^k d^{3-k} > 1\}$$

we have

$$Q\left(S_3 > 1\right) = \sum_{k=a}^{3} \binom{3}{k} q^k (1 - q)^{3-k} = \frac{7}{27}.$$

The following has now to hold

$$\begin{aligned} (1 - \alpha) V &= E^Q\left[(S_3 - 1 - \alpha V) \mathbb{1}_{\{S_3 > 1\}}\right] \\ &= E^Q\left[(S_3 - 1)^+\right] - \alpha V E^Q\left[\mathbb{1}_{\{S_3 > 1\}}\right] \\ &= H_0 - \alpha V Q\left(S_3 > 1\right) = \frac{13}{27} - \alpha V \frac{7}{27}. \end{aligned}$$

It follows that

$$V\left(1 - \alpha\left(1 - \frac{7}{27}\right)\right) = \frac{13}{27}$$

which leads to

$$V = \frac{13}{27}\left(1 - \frac{20}{27}\alpha\right)^{-1} > \frac{13}{27} = H_0.$$

Notice that, in the case of $\alpha = 1$, the above inequality becomes $V = \frac{13}{7} > \frac{13}{27}$; in fact, since the CP-variant is more convenient to the holder, he/she has to pay more than in the case of an ordinary Call. Furthermore, comparing H_0^{CP} with H_0 we obtain

$$H_0^{\mathrm{CP}} = (1-\alpha)V = (1-\alpha)\frac{13}{27}\left(1 - \frac{20}{27}\alpha\right)^{-1} = \frac{1-\alpha}{1 - \frac{20}{27}\alpha} H_0 \leq H_0$$

with equality on the right hand side if and only if $\alpha = 0$. The intuition here is that, since for the CP-variant a fraction α of the price is paid at expiration, the cost at the beginning is less than for an ordinary Call.
iii) Noticing that, for $\alpha = \frac{1}{2}$, we have $V = \frac{13}{17}$, the system to be satisfied by the strategy in the first node is

$$\begin{cases} \alpha_3 u^3 + \beta_3 = \left(u^3 - 1 - \frac{13}{34}\right) \\ \alpha_3 u^2 d + \beta_3 = \left(u^2 d - 1 - \frac{13}{34}\right) \end{cases}$$

namely

$$\begin{cases} 8\alpha_3 + \beta_3 = \frac{225}{34} \\ 2\alpha_3 + \beta_3 = \frac{21}{34} \end{cases}$$

that has as solution $\alpha_3 = 1, \beta_3 = -\frac{47}{34}$. It means that the writer of the option has to go short of $\frac{47}{34}$ units of the money account (riskless asset) which, in addition to his/her current wealth given by

$$E^Q\left[H_3^{\mathrm{CP}} \mid \mathcal{F}_2^{uu}\right] = \frac{1}{3}\frac{225}{35} + \frac{2}{3}\frac{21}{34} = \frac{267}{102}$$

leads to an amount of $\frac{267+141}{102} = 4$ exactly the amount to buy the requested unit of the risky asset of which the current price is $S_2 = 4$.

Analogously, in the second node we have

$$\begin{cases} \frac{1}{2}\alpha_3 + \beta_3 = 0 \\ \frac{1}{8}\alpha_3 + \beta_3 = 0 \end{cases}$$

leading to $\alpha_3 = 0, \beta_3 = 0$ which is justified by the fact that, for the given scenario, the holder will not exercise the option and so no capital is needed on the side of the writer. □

Problem 1.40. Consider a binomial market model with two risky assets (in addition to a riskless one) whose dynamics is given by

$$S_n^i = S_{n-1}^i(1 + \mu^i(h_n)), \qquad n = 1, \ldots, N, \quad i = 1, 2,$$

with h_n i.i.d. random variables with values in $\{-1, 1\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ d_i & \text{if } h = -1. \end{cases}$$

Choosing $u_1 = 3$, $u_2 = 2$, $d_1 = \frac{1}{3}$, $d_2 = \frac{2}{3}$ and $r = 0$:

i) show that the model is arbitrage free and complete;
ii) consider a two-period evolution, i.e. $N = 2$, starting from $S_0^1 = S_0^2 = 1$, and determine the arbitrage price process of the exchange option with payoff
$$H_2 = \left(S_2^2 - S_2^1\right)^+;$$
iii) finally determine all possible hedging strategies $(\alpha_1^1, \alpha_1^2, \beta_1)$ for the first period, showing that it is generally enough to invest on any two of the three assets available S^1, S^2 and B to replicate the payoff.

Solution of Problem 1.40

i) To show that the model is arbitrage free and complete, it is enough to prove the existence and uniqueness of the martingale measure. By Theorem 1.26, the discounted price $\widetilde{S}^i$ is a martingale in the measure defined by

$$Q(h = 1) = \frac{1 + r - d_i}{u_i - d_i}, \quad i = 1, 2.$$

With the given numerical data, we have

$$\frac{1 + r - d_1}{u_1 - d_1} = \frac{1 - \frac{1}{3}}{3 - \frac{1}{3}} = \frac{1}{4}, \qquad \text{and} \qquad \frac{1 + r - d_2}{u_2 - d_2} = \frac{1 - \frac{2}{3}}{2 - \frac{2}{3}} = \frac{1}{4},$$

and therefore the martingale measure is uniquely determined by $q := Q(h = 1) = \frac{1}{4}$.

ii) Since it will be useful for the third point below, we compute by a backward procedure the arbitrage prices in the first period, in the two possible scenarios $h = -1, 1$ (cases of increase and decrease of the underlying assets): we denote the arbitrage prices by H_1^u and H_1^d respectively. Then we have

$$\begin{aligned} H_1^u &= \frac{1}{1+r} E^Q \left[H_2 \mid h_1 = 1 \right] \\ &= \frac{1}{1+r} E^Q \left[H_2 \mid S_1^1 = u_1 S_0^1,\ S_1^2 = u_2 S_0^2 \right] = \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}, \\ H_1^d &= \frac{1}{1+r} E^Q \left[H_2 \mid h_1 = -1 \right] \\ &= \frac{1}{1+r} E^Q \left[H_2 \mid S_1^1 = d_1 S_0^1,\ S_1^2 = d_2 S_0^2 \right] = \frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

At the initial time, we have

$$H_0 = \frac{1}{1+r} \left(q H_1^u + (1-q) H_1^d \right) = \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{5}{16}.$$

iii) To determine the hedging strategy in the first period, we impose the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1,$$

equivalent to the following system of two equations in the three unknowns $\alpha_1^1, \alpha_1^2, \beta_1$:

$$\begin{cases} \alpha_1^1 u^1 + \alpha_1^2 u^2 + \beta_1 = H^u, \\ \alpha_1^1 d^1 + \alpha_1^2 d^2 + \beta_1 = H^d. \end{cases}$$

Substituting the data we obtain the system

$$\begin{cases} 3\alpha_1^1 + 2\alpha_1^2 + \beta_1 = \frac{1}{4}, \\ \frac{1}{3}\alpha_1^1 + \frac{2}{3}\alpha_1^2 + \beta_1 = \frac{1}{3}, \end{cases}$$

with solution

$$\alpha_1^2 = -\frac{1}{16} - 2\alpha_1^1, \qquad \beta_1 = \frac{3}{8} + \alpha_1^1, \tag{1.62}$$

with arbitrary α_1^1. It is clear that by choosing appropriately α_1^1, we can form different hedging portfolios that involve only two of the three available assets: precisely, from (1.62) by choosing $\alpha_1^1 = 0$ we get a strategy on the assets S^2 and B; by setting $\alpha_1^1 = -\frac{1}{32}$ we get a strategy on the assets S^1 and B; eventually, setting $\alpha_1^1 = -\frac{3}{8}$ we get a strategy on the assets S^1 and S^2. □

Problem 1.41. Consider a market with two risky assets (in addition to a riskless one) where prices follow the trinomial model (1.42) with

$$S_n^i = S_{n-1}^i (1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2,$$

where h_n are i.i.d. with values in $\{1, 2, 3\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3. \end{cases}$$

Choosing $u_1 = 2$, $u_2 = \frac{8}{3}$, $m_1 = 1$, $m_2 = \frac{8}{9}$, $d_1 = \frac{1}{2}$, $d_2 = \frac{1}{3}$ and $r = 0$ it turns out that the unique martingale measure Q is defined by

$$Q(h_n = 1) = q_1 = \frac{1}{6}, \quad Q(h_n = 2) = q_2 = \frac{1}{2}, \quad Q(h_n = 3) = q_3 = \frac{1}{3}.$$

Consider a two-period evolution, i.e. $N = 2$, starting from $S_0^1 = S_0^2 = 1$ and an exchange option with payoff

$$H_2 = \left(S_2^2 - S_2^1\right)^+.$$

Determine:

i) the initial price H_0 of the option;
ii) the hedging strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the first period.

Solution of Problem 1.41

i) Figure 1.5 depicts the trinomial tree of the asset prices and the payoff of the option. Since they shall be needed in the second point below, we compute the arbitrage prices at time $n = 1$ and in the three scenarios $h = 1, 2, 3$ which we shall denote by the superscript u, m and d respectively. We have

$$\begin{aligned} H_1^u =& \frac{1}{1+r} E^Q \left[\left(S_2^2 - S_2^1 \right)^+ \mid h_1 = 1 \right] \\ =& \left(q_1 \left((u_2)^2 - (u_1)^2 \right)^+ + q_2 \left(u_2 m_2 - u_1 m_1 \right)^+ + q_3 \left(u_2 d_2 - u_1 d_1 \right)^+ \right) = \frac{19}{27}. \end{aligned}$$

Analogously, we have

$$\begin{aligned} H_1^m =& \frac{1}{1+r} E^Q \left[\left(S_2^2 - S_2^1 \right)^+ \mid h_1 = 2 \right] = \frac{5}{81}, \\ H_1^d =& \frac{1}{1+r} E^Q \left[\left(S_2^2 - S_2^1 \right)^+ \mid h_1 = 3 \right] = 0. \end{aligned}$$

Then the initial arbitrage price is equal to

$$H_0 = \frac{1}{1+r} E^Q [H_1] = q_1 H_1^u + q_2 H_1^m + q_3 H_1^d = \frac{4}{27}.$$

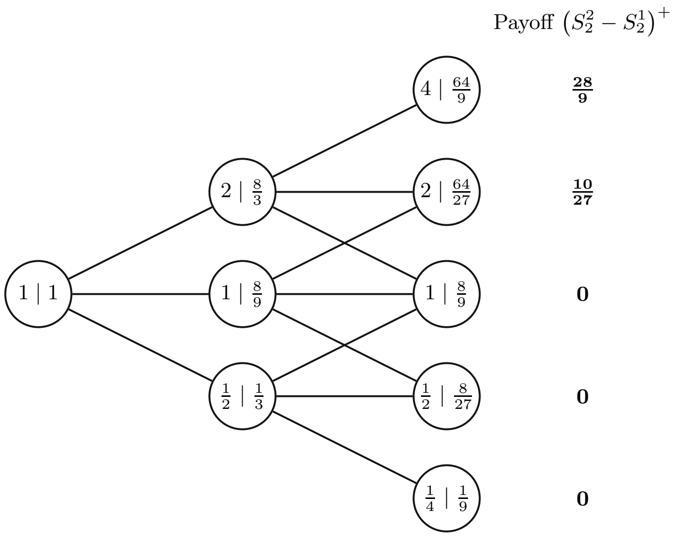

Fig. 1.5. Two-period trinomial tree: prices of the assets S^1, S^2 and payoff of an exchange option

A direct verification shows that the same result is obtained using the risk-neutral valuation formula (1.23).

ii) To determine the initial hedging strategy, we impose the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1$$

which amounts to

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases} \tag{1.63}$$

and provides the system

$$\begin{cases} 2\alpha_1^1 + \frac{8}{3}\alpha_1^2 + \beta_1 = \frac{19}{27}, \\ \alpha_1^1 + \frac{8}{9}\alpha_1^2 + \beta_1 = \frac{5}{81}, \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \beta_1 = 0. \end{cases}$$

Thus we get

$$\alpha_1^1 = -\frac{20}{27}, \qquad \alpha_1^2 = \frac{7}{9}, \qquad \beta_1 = \frac{1}{9}.$$

Finally, we check that

$$V_0 = -\frac{20}{27} S_0^1 + \frac{7}{9} S_0^2 + \frac{1}{9} B_0 = \frac{4}{27} = H_0. \qquad \Box$$

Problem 1.42. Consider a market with two risky assets (in addition to a riskless one) where prices follow a trinomial model (1.42) for which $S_0^1 = S_0^2 = 1$ and

$$S_n^i = S_{n-1}^i (1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2,$$

with h_n i.i.d. with values in $\{1, 2, 3\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3. \end{cases}$$

Choosing $u_1 = \frac{7}{3}$, $u_2 = \frac{22}{9}$, $m_1 = m_2 = 1$, $d_1 = \frac{1}{2}$, $d_2 = \frac{1}{3}$ and $r = \frac{1}{2}$, it turns out that the unique martingale measure is defined by

$$Q(h_n = 1) = q_1 = \frac{1}{2}, \quad Q(h_n = 2) = q_2 = \frac{1}{6}, \quad Q(h_n = 3) = q_3 = \frac{1}{3}.$$

Consider a "Backward Put" with underlying S^2, whose payoff is of the form

$$H_N = \left(M_N - S_N^2\right)^+ = M_N - S_N^2 \qquad \text{with} \quad M_N = \max\{S_1^2, \cdots, S_N^2\}.$$

In the case $N = 2$, determine:

i) the initial price H_0 of the option;
ii) the hedging strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the initial period.

Solution of Problem 1.42

i) The derivative concerns only the asset S^2. For simplicity we thus use the notation $u = u_2$, $m = m_2$ and $d = d_2$. Notice that for this case the trinomial tree is not "recombining", namely we have e.g. that $ud \neq m^2$. Therefore we have 9 possible scenarios (Fig. 1.2):

$$(u,u),\ (u,m),\ (u,d),\ (m,u),\ (m,m),\ (m,d),\ (d,u),\ (d,m),\ (d,d).$$

We have

$$\begin{aligned}H_0 = \frac{1}{(1+r)^2}\Big(& q_1^2\left(\max\{u,u^2\} - u\right) + q_1 q_2 \left(\max\{u, um\} - um\right)\\ & + q_1 q_3 \left(\max\{u, ud\} - ud\right) + q_2 q_1 \left(\max\{m, mu\} - mu\right)\\ & + q_2^2\left(\max\{m,m^2\} - m^2\right) + q_2 q_3 \left(\max\{m, md\} - md\right)\\ & + q_3 q_1 \left(\max\{d, du\} - du\right) + q_3 q_2 \left(\max\{d, dm\} - dm\right)\\ & + q_3^2\left(\max\{d,d^2\} - d^2\right)\Big) = \frac{4}{27}.\end{aligned}$$

ii) For the hedging strategy the following condition has to be satisfied

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta(1+r) = H_1$$

in all three possible scenarios, which we characterize with the superscripts u, m and d respectively. Consider thus the following linear system, corresponding to (1.47) with $n = 1$:

$$\begin{cases}\alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u,\\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m,\\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d.\end{cases}$$

We need to compute first the three values taken by $H_1 = \frac{1}{1+r} E^Q \left[H_2 \mid S_1^2\right]$:

$$H_1 = \begin{cases}H_1^u = \frac{1}{1+r} E^Q \left[M_2 - S_2 \mid S_1^2 = u\right] = \frac{88}{243},\\ H_1^m = \frac{1}{1+r} E^Q \left[M_2 - S_2 \mid S_1^2 = m\right] = \frac{4}{27},\\ H_1^d = \frac{1}{1+r} E^Q \left[M_2 - S_2 \mid S_1^2 = d\right] = \frac{4}{81}.\end{cases}$$

We then have

$$\begin{cases}\frac{7}{3}\alpha_1^1 + \frac{22}{9}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{88}{243},\\ \alpha_1^1 + \alpha_1^2 + \frac{3}{2}\beta_1 = \frac{4}{27},\\ \frac{\alpha_1^1}{2} + \frac{\alpha_1^2}{3} + \frac{3}{2}\beta_1 = \frac{4}{81},\end{cases}$$

from which

$$\alpha_1^1 = 0, \qquad \alpha_1^2 = \frac{4}{27}, \qquad \beta_1 = 0.$$

The amount $H_0 = \frac{4}{27}$ received from the sale of the option is thus entirely invested in the second asset; in other words, we buy $\frac{4}{27}$ units which, at the unit price of $S_0^2 = 1$, correspond to the initial wealth

$$V_0 = \alpha_1^1 S_0^1 + \alpha_1^2 S_0^2 + \beta = \frac{4}{27} = H_0. \qquad \square$$

Problem 1.43. Consider a market with two risky assets (in addition to a riskless one) where prices follow the trinomial model (1.42) where $S_0^1 = S_0^2 = 1$ and

$$S_n^i = S_{n-1}^i (1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2,$$

with h_n i.i.d. with values in $\{1, 2, 3\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3. \end{cases}$$

Choosing $u_1 = 2$, $m_1 = 1$, $d_1 = \frac{1}{2}$, $u_2 = \frac{7}{3}$, $m_2 = \frac{7}{9}$, $d_2 = \frac{1}{3}$ and $r = \frac{1}{4}$ we have that the martingale measure is defined by

$$Q(h_n = 1) = q_1 = \frac{3}{8}, \quad Q(h_n = 2) = q_2 = \frac{3}{8}, \quad Q(h_n = 3) = q_3 = \frac{1}{4}.$$

Consider a two-period evolution ($N = 2$) and an option of the type "collar" with underlying S^1, whose payoff is given by

$$H_2 = \min\{\max\{S_2^1, K_1\}, K_2\} \qquad \text{with} \quad K_1 = 1, \ K_2 = 2.$$

Determine:

i) the initial price H_0 of the option;
ii) the hedging strategy $(\alpha^1, \alpha^2, \beta)$.

Solution of Problem 1.43

i) In Figure 1.6 we represent the tree of prices of S^1 and the values of the payoff of the option. As we saw in Problem 1.42, to determine the hedging strategy as requested in the second point, we have to compute the arbitrage prices of the derivative at time $n = 1$. So, instead of applying directly the pricing formula (1.23), we first compute the prices in the first period by the risk-neutral formula $H_1 = \frac{1}{1+r} E^Q \left[H_2 \mid S_1^2 \right]$. We have

$$H_1^u := \frac{1}{1+r} E^Q \left[H_2 \mid S_1^1 = u_1 \right] = \frac{1}{1 + \frac{1}{4}} (2q_1 + 2q_2 + q_3) = \frac{7}{5}.$$

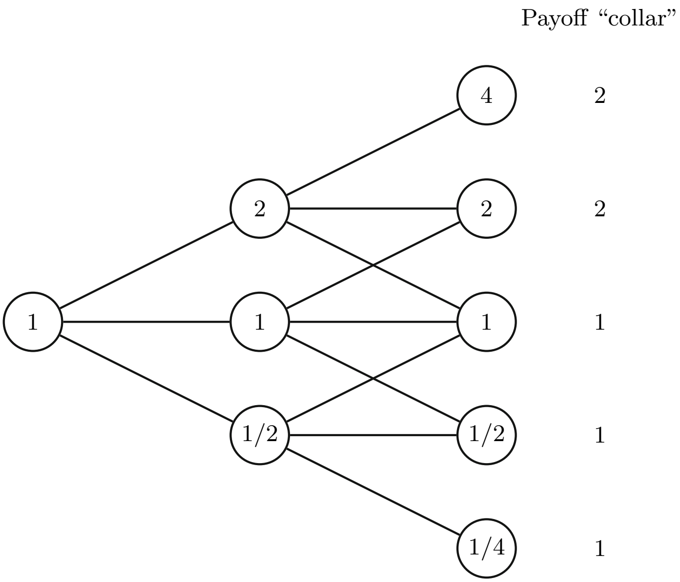

Fig. 1.6. Two-period trinomial tree: price of the asset S^1 and payoff of a "collar" option

Analogously we have

$$H_1^m = \frac{11}{10}, \qquad H_1^d = \frac{4}{5}.$$

Now we can compute the initial price of the option

$$H_0 = \frac{1}{1+r} E^Q\left[H_1\right] = \frac{1}{1+\frac{1}{4}}\left(\frac{3}{8}H_1^u + \frac{3}{8}H_1^m + \frac{1}{4}H_1^d\right) = \frac{91}{100}.$$

ii) As regards the first period, the replication condition

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases}$$

yields the system

$$\begin{cases} 2\alpha_1^1 + \frac{7}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{7}{5}, \\ \alpha_1^1 + \frac{7}{9}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{11}{10}, \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{4}{5}, \end{cases}$$

with solution

$$\alpha_1^1 = 1, \qquad \alpha_1^2 = -\frac{9}{20}, \qquad \beta_1 = \frac{9}{25}.$$

We verify that

$$V_0 = S_0^1 - \frac{9}{20}S_0^2 + \frac{9}{25}B_0 = \frac{91}{100} = H_0.$$

Concerning the second period, there are three scenarios: when $S_1^1 = u_1$, we have to solve the system

$$\begin{cases} \alpha_2^1 u_1 S_1^1 + \alpha_2^2 u_2 S_1^2 + \beta_2(1+r)^2 = H_2^{uu}, \\ \alpha_2^1 m_1 S_1^1 + \alpha_2^2 m_2 S_1^2 + \beta_2(1+r)^2 = H_2^{um}, \\ \alpha_2^1 d_1 S_1^1 + \alpha_2^2 d_2 S_1^2 + \beta_2(1+r)^2 = H_2^{ud}, \end{cases}$$

equivalent to

$$\begin{cases} 4\alpha_2^1 + \frac{49}{9}\alpha_2^2 + \frac{25}{16}\beta_2 = 2, \\ 2\alpha_2^1 + \frac{49}{27}\alpha_2^2 + \frac{25}{16}\beta_2 = 2, \\ \alpha_2^1 + \frac{7}{9}\alpha_2^2 + \frac{25}{16}\beta_2 = 1; \end{cases}$$

and then

$$\alpha_2^1 = \frac{7}{3}, \qquad \alpha_2^2 = -\frac{9}{7}, \qquad \beta_2 = -\frac{16}{75}.$$

A simple computation shows that, if $S_1^1 = u_1 = 2$ then

$$V_1^u = \frac{7}{3}S_1^1 - \frac{9}{7}S_1^2 - \frac{16}{75}B_1 = \frac{7}{5} = H_1^u.$$

Similarly, in the scenario $S_1^1 = m_1$ we get

$$\alpha_2^1 = -\frac{4}{3}, \qquad \alpha_2^2 = \frac{27}{14}, \qquad \beta_2 = \frac{56}{75},$$

and finally, in the scenario $S_1^1 = d_1$, we have

$$\alpha_2^1 = 0, \qquad \alpha_2^2 = 0, \qquad \beta_2 = \frac{16}{25}.$$ □

Problem 1.44. Consider a market with two risky assets (in addition to a riskless one) where prices follow the trinomial model (1.42) for which

$$S_n^i = S_{n-1}^i(1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2,$$

with h_n i.i.d. with values in $\{1, 2, 3\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3. \end{cases}$$

Choosing $u_1 = \frac{7}{3}$, $u_2 = \frac{22}{9}$, $m_1 = m_2 = 1$, $d_1 = \frac{1}{2}$, $d_2 = \frac{1}{3}$ and $r = \frac{1}{2}$, we have that the unique equivalent martingale measure Q is defined by

$$Q\,(h_n = 1) = q_1 = \frac{1}{2}, \quad Q\,(h_n = 2) = q_2 = \frac{1}{6}, \quad Q\,(h_n = 3) = q_3 = \frac{1}{3}.$$

Consider a two-period evolution, i.e. $N = 2$, starting from $S_0^1 = S_0^2 = 1$ and an option of the type "forward start" with underlying S^2, whose the payoff is

$$H_2 = \left(S_2^2 - S_1^2\right)^+.$$

Determine:

i) the initial arbitrage price H_0 of the option;
ii) the hedging strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the initial period, that is from $n = 0$ to $n = 1$.

Solution of Problem 1.44

i) The derivative concerns only the asset S^2. For simplicity we thus use the notation $u = u_2$, $m = m_2$ and $d = d_2$. We have

$$\begin{aligned}
H_0 =& \frac{1}{(1+r)^2} E^Q\left[\left(S_2^2 - S_1^2\right)^+\right] \\
=& \frac{1}{(1+r)^2}\bigg(q_1^2\left(u^2 - u\right)^+ + q_1q_2\left(um - u\right)^+ + q_1q_3\left(ud - u\right)^+ \\
&+ q_2q_1\left(mu - m\right)^+ + q_2^2\left(m^2 - m\right)^+ + q_2q_3\left(md - m\right)^+ \\
&+ q_3q_1\left(du - d\right)^+ + q_3q_2\left(dm - d\right)^+ + q_3^2\left(d^2 - d\right)^+\bigg) = \frac{13}{27}.
\end{aligned}$$

ii) We compute the prices in $n = 1$ in the three possible scenarios:

$$\begin{aligned}
H_1^u &= \frac{1}{1+r}\left(q_1\left(u^2 - u\right)^+ + q_2\left(um - u\right)^+ + q_3\left(ud - u\right)^+\right) = \frac{286}{243}, \\
H_1^m &= \frac{1}{1+r}\left(q_1\left(mu - m\right)^+ + q_2\left(m^2 - m\right)^+ + q_3\left(md - m\right)^+\right) = \frac{13}{27}, \\
H_1^d &= \frac{1}{1+r}\left(q_1\left(du - d\right)^+ + q_2\left(dm - d\right)^+ + q_3\left(d^2 - d\right)^+\right) = \frac{13}{81}.
\end{aligned}$$

For the hedging strategy we impose the replication condition (1.63) that is

$$\begin{cases}
\frac{7}{3}\alpha_1^1 + \frac{22}{9}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{286}{243}, \\
\alpha_1^1 + \alpha_1^2 + \frac{3}{2}\beta_1 = \frac{13}{27}, \\
\frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{13}{81},
\end{cases}$$

from which

$$\alpha_1^1 = 0, \qquad \alpha_1^2 = \frac{13}{27}, \qquad \beta_1 = 0.$$

In the second period, in case of increase $h_1 = 1$, the hedging strategy is the solution of

$$\begin{cases}
\alpha_2^1 u_1 S_1^1 + \alpha_2^2 u_2 S_1^2 + \beta_2(1+r)^2 = H_2^{uu}, \\
\alpha_2^1 m_1 S_1^1 + \alpha_2^2 m_2 S_1^2 + \beta_2(1+r)^2 = H_2^{um}, \\
\alpha_2^1 d_1 S_1^1 + \alpha_2^2 d_2 S_1^2 + \beta_2(1+r)^2 = H_2^{ud},
\end{cases}$$

equivalent to

$$\begin{cases} \frac{49}{9}\alpha_2^1 + \frac{484}{81}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{286}{81}, \\ \frac{7}{3}\alpha_2^1 + \frac{22}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{7}{6}\alpha_2^1 + \frac{22}{27}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \end{cases}$$

from which

$$\alpha_2^1 = \frac{1144}{189}, \qquad \alpha_2^2 = -\frac{13}{3}, \qquad \beta_2 = -\frac{1144}{729}.$$

In case $h_1 = 2$, we have

$$\begin{cases} \frac{7}{3}\alpha_2^1 + \frac{22}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{13}{9}, \\ \alpha_2^1 + \alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{1}{2}\alpha_2^1 + \frac{1}{3}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \end{cases}$$

from which

$$\alpha_2^1 = \frac{52}{9}, \qquad \alpha_2^2 = -\frac{13}{3}, \qquad \beta_2 = -\frac{52}{81}.$$

Finally, in the scenario $h_1 = 3$, we have

$$\begin{cases} \frac{7}{6}\alpha_2^1 + \frac{22}{27}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{13}{27}, \\ \frac{1}{2}\alpha_2^1 + \frac{1}{3}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{1}{4}\alpha_2^1 + \frac{1}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \end{cases}$$

from which

$$\alpha_2^1 = \frac{104}{27}, \qquad \alpha_2^2 = -\frac{13}{3}, \qquad \beta_2 = -\frac{52}{243}.$$ □

Problem 1.45. In a trinomial market with the same numerical data of Problem 1.44, consider a Put and a Call option with strike $K = 1$ and underlying asset S^2. Determine:

i) the initial arbitrage prices H_0^{Call} and H_0^{Put};
ii) the hedging strategy of the Put option involving the underlying risky and riskless assets (recall that one may also hedge the Put as mentioned in Section 1.3.4).

Solution of Problem 1.45
i) Recall that

$$q_1 = \frac{1}{2}, \quad q_2 = \frac{1}{6}, \quad q_3 = \frac{1}{3}.$$

Then we have

$$\begin{aligned} H_0^{\text{Call}} =& \frac{1}{(1+r)^2} E^Q\left[\left(S_2^2 - 1\right)^+\right] \\ =& \frac{1}{(1+r)^2}\Big(q_1^2\left(u^2-1\right)^+ + q_1 q_2 (um-1)^+ + q_1 q_3 (ud-1)^+ \\ &+ q_2 q_1 (mu-1)^+ + q_2^2\left(m^2-1\right)^+ + q_2 q_3 (md-1)^+ \\ &+ q_3 q_1 (du-1)^+ + q_3 q_2 (dm-1)^+ + q_3^2\left(d^2-1\right)^+\Big) = \frac{481}{729}. \end{aligned}$$

The price of the corresponding Put option can be obtained:

a) using the Put-Call parity formula (see Section 1.3.4):

$$H_0^{\text{Put}} = H_0^{\text{Call}} + \frac{K}{(1+r)^2} - S_0^2 = \frac{76}{729};$$

b) through a direct computation

$$\begin{aligned}
H_0^{\text{Put}} =& \frac{1}{(1+r)^2} E^Q\left[\left(1 - S_2^2\right)^+\right] \\
=& \frac{1}{(1+r)^2}\Big(q_1^2\left(1-u^2\right)^+ + q_1 q_2 \left(1-um\right)^+ + q_1 q_3 \left(1-ud\right)^+ \\
& + q_2 q_1 \left(1-mu\right)^+ + q_2^2 \left(1-m^2\right)^+ + q_2 q_3 \left(1-md\right)^+ \\
& + q_3 q_1 \left(1-du\right)^+ + q_3 q_2 \left(1-dm\right)^+ + q_3^2 \left(1-d^2\right)^+\Big) = \frac{76}{729}.
\end{aligned}$$

ii) From now on, for simplicity we set $H^{\text{Put}} = H$. To determine the hedging strategy, we first compute the prices in $n = 1$:

$$\begin{aligned}
H_1^u &= \frac{1}{1+r}\left(q_1\left(1-u^2\right)^+ + q_2\left(1-um\right)^+ + q_3\left(1-ud\right)^+\right) = \frac{10}{243}, \\
H_1^m &= \frac{1}{1+r}\left(q_1\left(1-mu\right)^+ + q_2\left(1-m^2\right)^+ + q_3\left(1-md\right)^+\right) = \frac{4}{27}, \\
H_1^d &= \frac{1}{1+r}\left(q_1\left(1-du\right)^+ + q_2\left(1-dm\right)^+ + q_3\left(1-d^2\right)^+\right) = \frac{1}{3}.
\end{aligned}$$

For the initial hedging strategy, we impose the replication condition (1.63) which is equivalent to

$$\begin{cases}
\frac{7}{3}\alpha_1^1 + \frac{22}{9}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{10}{243}, \\
\alpha_1^1 + \alpha_1^2 + \frac{3}{2}\beta_1 = \frac{4}{27}, \\
\frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{1}{3},
\end{cases}$$

from which

$$\alpha_1^1 = \frac{286}{243}, \qquad \alpha_1^2 = -\frac{94}{81}, \qquad \beta_1 = \frac{64}{729}.$$

To determine the strategy in $n = 1$, we have to consider the three scenarios $h = 1, 2, 3$. In case of increase $h_1 = 1$, corresponding to $S_1^i = u_i$ for $i = 1, 2$, the hedging strategy is the solution of

$$\begin{cases}
\alpha_2^1 u_1 S_1^1 + \alpha_2^2 u_2 S_1^2 + \beta_2(1+r)^2 = H_2^{uu}, \\
\alpha_2^1 m_1 S_1^1 + \alpha_2^2 m_2 S_1^2 + \beta_2(1+r)^2 = H_2^{um}, \\
\alpha_2^1 d_1 S_1^1 + \alpha_2^2 d_2 S_1^2 + \beta_2(1+r)^2 = H_2^{ud},
\end{cases}$$

equivalent to

$$\begin{cases} \frac{49}{9}\alpha_2^1 + \frac{484}{81}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{7}{3}\alpha_2^1 + \frac{22}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{7}{6}\alpha_2^1 + \frac{22}{27}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{5}{27}, \end{cases}$$

from which

$$\alpha_2^1 = \frac{130}{189}, \qquad \alpha_2^2 = -\frac{20}{33}, \qquad \beta_2 = -\frac{40}{729}.$$

If $h_1 = 2$, we have

$$\begin{cases} \frac{7}{3}\alpha_2^1 + \frac{22}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \alpha_2^1 + \alpha_2^2 + \frac{9}{4}\beta_2 = 0, \\ \frac{1}{2}\alpha_2^1 + \frac{1}{3}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{2}{3}, \end{cases}$$

from which

$$\alpha_2^1 = \frac{52}{9}, \qquad \alpha_2^2 = -\frac{16}{3}, \qquad \beta_2 = -\frac{16}{81}.$$

Lastly, in the scenario $h_1 = 3$ we have

$$\begin{cases} \frac{7}{6}\alpha_2^1 + \frac{22}{27}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{5}{27}, \\ \frac{1}{2}\alpha_2^1 + \frac{1}{3}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{2}{3}, \\ \frac{1}{4}\alpha_2^1 + \frac{1}{9}\alpha_2^2 + \frac{9}{4}\beta_2 = \frac{8}{9}, \end{cases}$$

from which

$$\alpha_2^1 = 0, \qquad \alpha_2^2 = -1, \qquad \beta_2 = \frac{4}{9}.$$

□

Problem 1.46. Consider a market with two risky assets (in addition to a riskless one) where prices follow the trinomial model (1.42) for which

$$S_n^i = S_{n-1}^i(1 + \mu^i(h_n)), \qquad n = 1, \dots, N, \quad i = 1, 2,$$

with $S_0^1 = S_0^2 = 1$ and with h_n i.i.d. with values in $\{1, 2, 3\}$ and

$$1 + \mu^i(h) = \begin{cases} u_i & \text{if } h = 1, \\ m_i & \text{if } h = 2, \\ d_i & \text{if } h = 3. \end{cases}$$

Choosing $u_1 = \frac{7}{3}$, $u_2 = \frac{22}{9}$, $m_1 = m_2 = 1$, $d_1 = \frac{1}{2}$, $d_2 = \frac{1}{3}$ and $r = \frac{1}{2}$, we have that there exists a unique equivalent martingale measure Q for which, defining $q_i := Q(h_n = i)$, $i = 1, 2, 3,$ it holds that

$$q_1 = \frac{1}{2}, \quad q_2 = \frac{1}{6}, \quad q_3 = \frac{1}{3}.$$

Defining analogously $p_i := P(h_n = i)$ with P the "physical measure", suppose that $p_1 = p_2 = p_3$.

Consider a Call option with underlying S^1, maturity $N = 3$ and strike $K = 2$.

i) Determine the initial price H_0 of the option considered as a European option.

ii) Consider the variant (*contingent premium* or *pay later option*), for which the holder pays a fraction $\alpha \in (0,1)$ of the price, call it V, at maturity in $n = N = 3$ and this only if he/she exercises the option, whereas he/she pays the remaining fraction $1-\alpha$ in $n=0$. The value at maturity is therefore

$$H_3^{\mathrm{CP}} = \begin{cases} (S_3^1 - 2) - \alpha V & \text{if } S_3^1 > 2, \\ 0 & \text{otherwise.} \end{cases}$$

Determine V such that the arbitrage price H_0^{CP} of this variant in $n=0$ is $H_0^{\mathrm{CP}} = (1-\alpha)\, V$.
(Hint: $H_0^{\mathrm{CP}} = \frac{1}{(1+r)^3} E^Q \left[\left((S_3^1 - 2) - \alpha V\right) \mathbb{1}_{\{S_3^1 > 2\}}\right]$).

iii) Choosing $\alpha = \frac{1}{2}$, identify the linear system of equations to be solved in order to determine the hedging strategy $(\alpha_3^1, \alpha_3^2, \beta_3)$ for the CP variant in $n=2$ in the node corresponding to the scenario for which $(h_1, h_2) = (1,2)$.

Solution of Problem 1.46

i) We have

$$\begin{aligned} H_0 &= \frac{1}{(1+r)^3} E^Q \left[(S_3^1 - 2)^+\right] \\ &= \left(\frac{2}{3}\right)^3 \left(q_1^3 \left(\frac{343}{27} - 2\right) + 3q_1^2 q_3 \left(\frac{49}{18} - 2\right)\right. \\ &\quad \left. + 3q_1^2 q_2 \left(\frac{49}{9} - 2\right) + 3q_1 q_2^2 \left(\frac{7}{3} - 2\right)\right) \\ &= \left(\frac{2}{3}\right)^3 \frac{289 + 39 + 93 + 3}{8 \cdot 27} = \frac{424}{243}. \end{aligned}$$

ii) Indicating by ν_n^i the random number of events for which $h_k = i$ for $k = 1, \cdots, n$ and $i = 1,2,3$, notice first that the event

$$\{S_3^1 > 2\} = \left\{S_0^1 u_1^{\nu_3^1} m_1^{\nu_3^2} d_1^{3-\nu_3^1-\nu_3^2} > 2\right\}$$

is equivalent to the union of the following disjoint events

$$\{\nu_3^1 = 3, \nu_3^2 = 0\} \cup \{\nu_3^1 = 2, \nu_3^2 = 0\} \cup \{\nu_3^1 = 2, \nu_3^2 = 1\} \cup \{\nu_3^1 = 1, \nu_3^2 = 2\}$$

of which each one, except for the first, may occur in three different ways. It follows that

$$Q\left(S_3^1 > 2\right) = q_1^3 + 3q_1^2 q_3 + 3q_1^2 q_2 + 3q_1 q_2^2 = \frac{13}{24}.$$

By the hint, the following has now to hold

$$
\begin{aligned}
(1-\alpha)V &= \frac{1}{(1+r)^3}E^Q\left[(S_3^1-2-\alpha V)\mathbb{1}_{\{S_3^1>2\}}\right] \\
&= \frac{1}{(1+r)^3}E^Q\left[(S_3^1-2)^+\right] - \frac{1}{(1+r)^3}\alpha V E^Q\left[\mathbb{1}_{\{S_3^1>2\}}\right] \\
&= H_0 - \frac{1}{(1+r)^3}\alpha V Q\left(S_3^1>2\right) \\
&= \frac{424}{243} - \alpha V\left(\frac{2}{3}\right)^3\frac{13}{24} = \frac{424}{243} - \alpha V\frac{13}{81}.
\end{aligned}
$$

It follows that

$$
V\left(1-\alpha\left(1-\frac{13}{81}\right)\right) = \frac{424}{243}
$$

which leads to

$$
V = \frac{424}{243}\left(1-\frac{68}{81}\alpha\right)^{-1} = \frac{424}{243}\frac{81}{81-68\alpha}.
$$

iii) The system that the strategy has to satisfy in the indicated node is

$$
\begin{cases}
\alpha_3^1 u_1^2 m_1 + \alpha_3^2 u_2^2 m_2 + \beta_3(1+r)^3 = u_1^2 m_1 - 2 - \alpha V \\
\alpha_3^1 u_1 m_1^2 + \alpha_3^2 u_2 m_2^2 + \beta_3(1+r)^3 = u_1 m_1^2 - 2 - \alpha V \\
\alpha_3^1 u_1 m_1 d_1 + \alpha_3^2 u_2 m_2 d_2 + \beta_3(1+r)^3 = 0
\end{cases}
$$

where the 0 on the right of the last equation derives from the fact that in that case we have $S_3^1 < 2$. For the specific values this system now becomes

$$
\begin{cases}
\frac{49}{9}\alpha_3^1 + \frac{484}{81}\alpha_3^2 + \frac{27}{8}\beta_3 = \frac{2185}{846} \\
\frac{7}{3}\alpha_3^1 + \frac{22}{9}\alpha_3^2 + \frac{27}{8}\beta_3 = -\frac{149}{282} \\
\frac{7}{6}\alpha_3^1 + \frac{22}{27}\alpha_3^2 + \frac{27}{8}\beta_3 = 0.
\end{cases}
$$

□

Problem 1.47 (Hedging in incomplete markets with constraints on the strategy). In a standard trinomial market model with two periods, the dynamics of the price of the risk asset is given by

$$
S_n = S_{n-1}(1+\mu_n), \qquad n = 1, 2
$$

with $S_0 = 1$ and μ_n i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and such that

$$
P\left(\mu_n = -1/2\right) = P\left(\mu_n = 0\right) = P\left(\mu_n = 1\right) = \frac{1}{3}, \qquad n = 1, 2.
$$

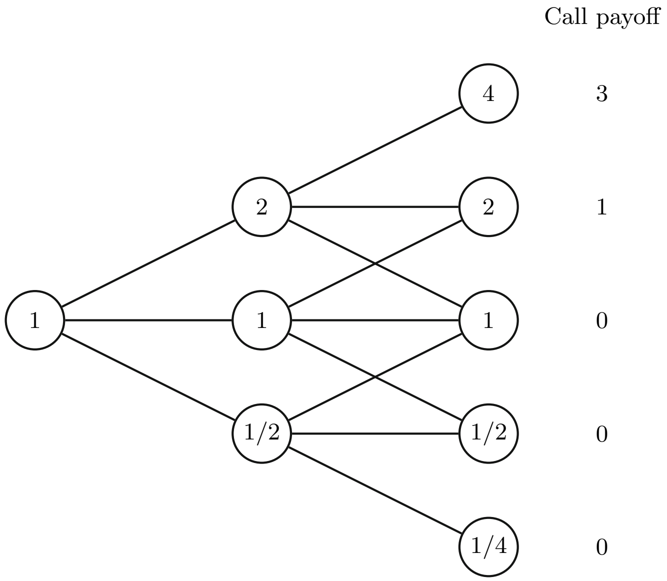

Fig. 1.7. Two-period trinomial tree: price of the underlying asset and payoff of a European Call option with unitary strike

Assume that the risk-free rate is zero, $r = 0$. Recalling that the market is incomplete, consider the problem of hedging by the shortfall risk criterion a European Call option with payoff

$$\varphi(S_2) = (S_2 - 1)^+ .$$

More precisely, using the algorithm of Dynamic Programming[5] (DP), determine the self-financing strategy with non-negative[6] value V that minimizes the following criterion of risk

$$E^P [\mathcal{U}(V_2, S_2)] ,$$

where

$$\mathcal{U}(V, S) = (\varphi(S) - V)^+$$

is the "shortfall" risk function.

Solution of Problem 1.47

The trinomial tree of prices of the underlying asset is displayed in Figure 1.7. Since $r = 0$, by (1.17) the dynamics of the value V of a self-financing strategy

[5] For an overview of the Dynamic Programming algorithm, we refer to Section 2.3.

[6] V is non-negative if $V_n \geq 0$ for any n.

(α, β) is given by

$$V_n = V_{n-1} + \alpha_n S_{n-1}\mu_n = V_{n-1} + \begin{cases} \alpha_n S_{n-1}, \\ 0, \\ -\frac{\alpha_n S_{n-1}}{2}. \end{cases} \tag{1.64}$$

Then a necessary and sufficient condition for $V_n \geq 0$, for any n, is that $V_0 \geq 0$ and

$$-\frac{V_{n-1}}{S_{n-1}} \leq \alpha_n \leq \frac{2V_{n-1}}{S_{n-1}}, \qquad n = 1, 2.$$

In a N-period model, the DP algorithm consists of two steps:

1) compute

$$R_{N-1}(V, S) := \min_{\alpha \in \left[-\frac{V}{S}, \frac{2V}{S}\right]} E^P \left[\mathcal{U}\left(V + S\alpha\mu_N, S\left(1 + \mu_N\right)\right)\right]$$

for S varying among the possible values of S_{N-1}. We recall that we consider only predictable strategies and we denote the corresponding minimum point by $\alpha_N = \alpha_N(V)$ for V varying among the possible values of V_{N-1};

2) for $n \in \{N-1, N-2, \dots, 1\}$, we compute

$$R_{n-1}(V, S) := \min_{\alpha \in \left[-\frac{V}{S}, \frac{2V}{S}\right]} E^P \left[R_n\left(V + S\alpha\mu_n, S\left(1 + \mu_n\right)\right)\right]$$

for S varying among the possible values of S_{n-1}. We denote the corresponding minimum point by $\alpha_n = \alpha_n(V)$ for V varying among the possible values of V_{n-1}.

In our case, the first step of the DP algorithm consists of computing $R_1(V, S)$ for $S \in \left\{2, 1, \frac{1}{2}\right\}$. We have

$$\begin{aligned} R_1(V, 2) &= \min_{\alpha \in [-V/2, V]} E^P \left[\mathcal{U}\left(V + 2\alpha\mu_2, 2(1 + \mu_2)\right)\right] \\ &= \min_{\alpha \in [-V/2, V]} E^P \left[\left(\left(2(1 + \mu_2) - 1\right)^+ - \left(V + 2\alpha\mu_2\right)\right)^+\right] \\ &= \min_{\alpha \in [-V/2, V]} \frac{1}{3}\left((3 - V - 2\alpha)^+ + (1 - V)^+\right) = \frac{4}{3}(1 - V)^+, \end{aligned}$$

the minimum being attained at

$$\alpha_2 = V. \tag{1.65}$$

Furthermore, we have

$$\begin{aligned} R_1(V, 1) &= \min_{\alpha \in [-V, 2V]} E^P \left[\mathcal{U}\left(V + \alpha\mu_2, 1 + \mu_2\right)\right] \\ &= \min_{\alpha \in [-V, 2V]} E^P \left[\left(\mu_2^+ - \left(V + \alpha\mu_2\right)\right)^+\right] \\ &= \min_{\alpha \in [-V, 2V]} \frac{1}{3}(1 - V - \alpha)^+ = \frac{1}{3}(1 - 3V)^+, \end{aligned}$$

the minimum being attained at

$$\alpha_2 = 2V. \tag{1.66}$$

Finally, we have

$$\begin{aligned} R_1\left(V, \frac{1}{2}\right) &= \min_{\alpha \in [-2V, 4V]} E^P\left[\mathcal{U}\left(V + \frac{\alpha\mu_2}{2}, \frac{1+\mu_2}{2}\right)\right] \\ &= \min_{\alpha \in [-2V, 4V]} E^P\left[\left(\underbrace{\left(\frac{1+\mu_2}{2} - 1\right)^+}_{=0} - \underbrace{\left(V + \frac{\alpha\mu_2}{2}\right)}_{\geq 0}\right)^+\right] = 0, \end{aligned}$$

and the minimum is attained at any point

$$\alpha_2 \in [-2V, 4V]. \tag{1.67}$$

The second step consists of computing the risk at the initial time:

$$\begin{aligned} R_0\left(V, 1\right) &= \min_{\alpha \in [-V, 2V]} E^P\left[R_1\left(V + \alpha\mu_1, 1 + \mu_1\right)\right] \\ &= \frac{1}{3} \min_{\alpha \in [-V, 2V]} \left(R_1\left(V, 1\right) + R_1\left(V + \alpha, 2\right)\right) \\ &= \frac{1}{3} \min_{\alpha \in [-V, 2V]} \left(\frac{1}{3}\left(1 - 3V\right)^+ + \frac{4}{3}\left(1 - \left(V + \alpha\right)\right)^+\right) \\ &= \frac{5}{9}\left(1 - 3V\right)^+, \end{aligned}$$

and the minimum is attained at

$$\alpha_1 = 2V. \tag{1.68}$$

From the expression of $R_0\left(V, 1\right)$ we deduce that an initial wealth $V \geq \frac{1}{3}$ is sufficient to eliminate the shortfall risk or, in more explicit terms, *to super-replicate the payoff.*

Next we find the shortfall strategy, that is the strategy that minimizes the shortfall risk: denoting the initial wealth by V_0, from (1.68) we get $\alpha_1 = 2V_0$. Consequently, by (1.64) we have

$$V_1 = V_0 + \begin{cases} 2V_0, & \text{if } \mu_1 = 1, \\ 0, & \text{if } \mu_1 = 0, \\ -V_0, & \text{if } \mu_1 = -\frac{1}{2}. \end{cases}$$

and by (1.65)-(1.66)-(1.67)

$$\alpha_2 = \begin{cases} 3V_0, & \text{if } S_1 = 2, \\ 2V_0, & \text{if } S_1 = 1, \\ 0, & \text{if } S_1 = \frac{1}{2}. \end{cases}$$

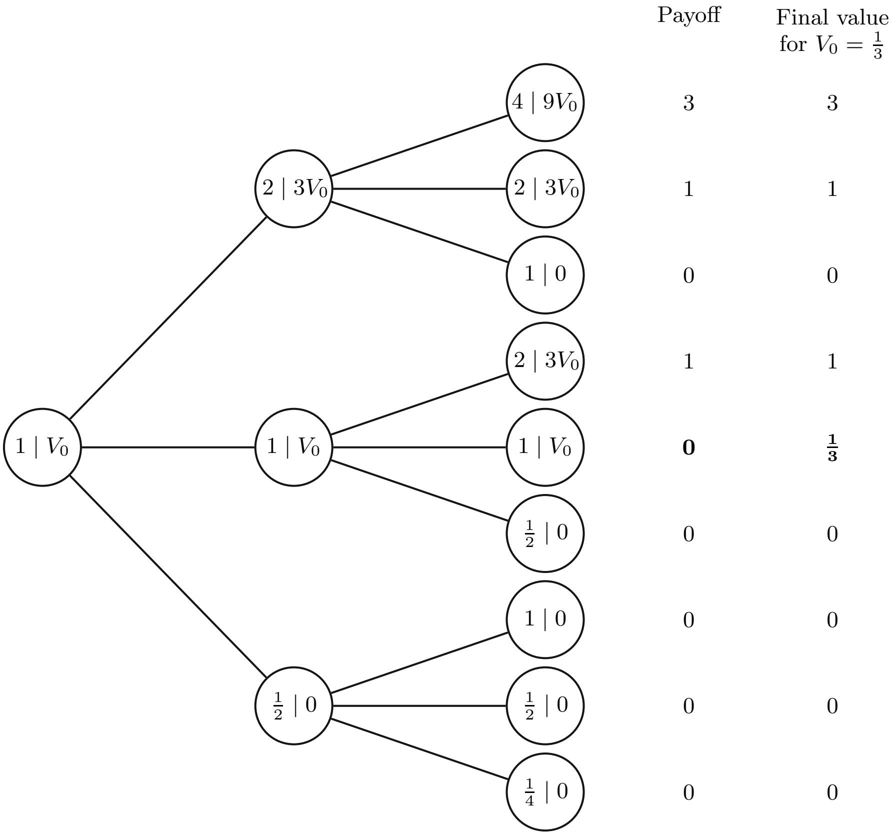

Fig. 1.8. Two-period trinomial tree: prices of the underlying asset (inside the circle, on the left) and values of the shortfall strategy (inside the circle, on the right)

Then the final value V_2 is easy to compute, again by using (1.64). The trinomial tree with asset prices and values of the shortfall strategy, represented inside the circles, is shown in Figure 1.8. In the columns on the right, the values of the European Call and the final values of the shortfall strategy with initial value $V_0 = \frac{1}{3}$ are shown. Notice that the strategy replicates the payoff in all cases except for the trajectory $S_0 = S_1 = S_2 = 1$ at which it super-replicates: the final value of the shortfall strategy $V_2 = \frac{1}{3}$ is strictly greater than the payoff of the Call option that is null. $\square$

Problem 1.48 (Hedging in a complete market with insufficient initial capital). In a binomial market model, consider a Put option with payoff $X = (K - S_N)^+$ and assume the following numerical data (with notations as in Section 1.4.1): $S_0 = 1$, $u = 2$, $d = 1/2$, $r = 0$, $K = 1$ and $N = 2$.

i) Verify that the initial price of the option is $H_0 = \frac{1}{3}$ and compute the option price at time $n = 1$ in the two scenarios $S_1 = 2$ and $S_1 = \frac{1}{2}$. Determine furthermore the hedging strategy (π_1, π_2) in periods 0 and 1 (expressed in terms of the proportion invested in the risky asset);

ii) assuming an initial capital $V_0 < \frac{1}{3}$, perfect hedging is not possible. Then by the Dynamic Programming algorithm[7], determine the self-financing strategy (π_1, π_2) which minimizes the quadratic risk criterion

$$E^P\left[\left(V_2 - (K - S_2)^+\right)^2\right],$$

where $p := P(1 + \mu_n = u) = \frac{1}{2}$;

iii) proceeding as in ii), determine the self-financing strategy that minimizes the shortfall risk criterion

$$E^P\left[\left((K - S_2)^+ - V_2\right)^+\right],$$

with the constraint $V_n \geq 0$, $n = 1, 2$.

Solution of Problem 1.48

i) We have

$$q = \frac{1 + r - d}{u - d} = \frac{1}{3}$$

and

$$\begin{aligned} H_0 &= q^2\left(1 - u^2\right)^+ + 2q(1 - q)(1 - ud)^+ + (1 - q)^2\left(1 - d^2\right)^+ \\ &= (1 - q)^2\left(1 - d^2\right)^+ = \frac{1}{3}. \end{aligned}$$

Moreover

$$\begin{aligned} H_1^u &:= E^Q[X \mid S_1 = u] = q\left(1 - u^2\right)^+ + (1 - q)(1 - ud)^+ = 0, \\ H_1^d &:= E^Q[X \mid S_1 = d] = q(1 - ud)^+ + (1 - q)\left(1 - d^2\right)^+ = \frac{1}{2}. \end{aligned}$$

By a simple calculation, we also get

$$H_0 = E^Q[H_1] = qH_1^u + (1 - q)H_1^d = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$

To determine the hedging strategy we use formula (1.37) for α, while for β we use the definition of value of the portfolio:

$$\alpha_1 = \frac{H_1^u - H_1^d}{S_0(u - d)} = -\frac{1}{3}, \qquad \beta_1 = H_0 - \alpha_1 S_0 = \frac{2}{3}.$$

In the second period we have

$$\alpha_2^u = \beta_2^u = 0,$$

[7] For an overview of the Dynamic Programming algorithm, we refer to Section 2.3.

in the scenario $S_1 = u$, and

$$\alpha_2^d = \frac{H_2^{du} - H_2^{dd}}{d\,(u-d)} = -1, \qquad \beta_2^d = H_1^d - \alpha_2^d d = 1,$$

in the scenario $S_1 = d$. In the end, by the definition (1.10) of relative portfolio (here we adopt the notation $\pi_n = \pi_n^1$), we have

$$\pi_1 = \frac{\alpha_1 S_0}{H_0} = -1,$$

and

$$\pi_2^u = 0, \qquad \pi_2^d = \frac{\alpha_2^d d}{H_1^d} = -1.$$

ii) According to (1.12), since by assumption $r = 0$, the dynamics of the value V of a self-financing strategy (π_1, π_2) is given by

$$V_n = V_{n-1}\left(1 + \pi_n \mu_n\right) = \begin{cases} V_{n-1}\left(1 + \pi_n(u-1)\right) = V_{n-1}(1+\pi_n), \\ V_{n-1}\left(1 + \pi_n(d-1)\right) = V_{n-1}\left(1 - \frac{\pi_n}{2}\right), \end{cases} \tag{1.69}$$

for $n = 1, 2$.

As explained in the solution of Problem 1.47, the first step of the DP algorithm consists of calculating

$$R_1\left(V, S_1\right) := \min_{\pi_2 \in \mathbb{R}} E^P\left[\left(V\left(1 + \pi_2 \mu_2\right) - \left(K - S_1\left(1 + \mu_2\right)\right)^+\right)^2\right]$$

in the two states $S_1 = u$ and $S_1 = d$. In the scenario $S_1 = u = 2$, we have

$$\begin{aligned} R_1\left(V, 2\right) &= \frac{1}{2} \min_{\pi_2 \in \mathbb{R}} \left(V^2(1+\pi_2)^2 + V^2\left(1 - \frac{\pi_2}{2}\right)^2\right) \\ &= \frac{V^2}{8} \min_{\pi_2 \in \mathbb{R}} \left(5\pi_2^2 + 4\pi_2 + 8\right), \end{aligned}$$

and the minimum point can be easily determined at $\pi_2^u = -\frac{2}{5}$. Then we have

$$R_1\left(V, 2\right) = \frac{9V^2}{10}. \tag{1.70}$$

On the other hand, in the scenario $S_1 = d = \frac{1}{2}$, we have

$$\begin{aligned} R_1\left(V, \frac{1}{2}\right) &= \frac{1}{2} \min_{\pi_2 \in \mathbb{R}} \left(V^2(1+\pi_2)^2 + \left(\frac{3}{4} - V\left(1 - \frac{\pi_2}{2}\right)\right)^2\right) \\ &= \frac{1}{32} \min_{\pi_2 \in \mathbb{R}} \left(4V^2(5\pi_2^2 + 4\pi_2 + 8) + 12V(\pi_2 - 2) + 9\right). \end{aligned}$$

By imposing the condition

$$0 = \partial_{\pi_2} \left(4V^2(5\pi_2^2 + 4\pi_2 + 8) + 12V(\pi_2 - 2)\right) = 8V^2(5\pi_2 + 2) + 12V,$$

we find the minimum point $\pi_2^d = -\frac{4V+3}{10V}$ and therefore we have

$$R_1\left(V, \frac{1}{2}\right) = \frac{9}{40}(2V-1)^2. \tag{1.71}$$

In the second (and final) step of the DP algorithm, we calculate

$$R_0(V, S_0) := \min_{\pi_1 \in \mathbb{R}} E^P\left[R_1\big(V(1+\pi_1\mu_1), S_0(1+\mu_1)\big)\right].$$

We have

$$R_0(V,1) = \frac{1}{2}\min_{\pi_1 \in \mathbb{R}} \left(R_1(V(1+\pi_1), 2) + R_1\left(V\left(1 - \frac{\pi_1}{2}\right), \frac{1}{2}\right)\right) =$$

(using expressions (1.70) and (1.71))

$$= \frac{1}{2}\min_{\pi_1 \in \mathbb{R}} \left(\frac{9}{10}V^2(1+\pi_1)^2 + \frac{9}{40}\left(2V\left(1-\frac{\pi_1}{2}\right) - 1\right)^2\right).$$

The first derivative (with respect to π_1) of the function we have to minimize is equal to

$$\frac{9V}{40}(V(5\pi_1 + 2) + 1),$$

so that the minimum is attained at $\pi_1 = -\frac{2V+1}{5V}$ and we have

$$R_0(V,1) = \frac{9}{100}(3V-1)^2.$$

Summing up, starting from an initial wealth V, the strategy that minimizes the quadratic risk is given by

$$\pi_1 = -\frac{2V+1}{5V}, \qquad \pi_2^u = -\frac{2}{5}, \qquad \pi_2^d = -\frac{4V+3}{10V}. \tag{1.72}$$

By (1.69) we calculate the value of the strategy, represented in Figure 1.9: notice that if the initial capital is equal to $V = H_0 = \frac{1}{3}$ then we have perfect replication. On the contrary, if $V < \frac{1}{3}$ then the strategy has a final value that is less than the payoff in all scenarios. Note also that, since there are no constraints on π, the values in (1.72) can be arbitrarily large.

iii) Preliminarily, let us recall that $V_0 \geq 0$ by assumption and that the dynamics of a replicating portfolio is given by (1.69): then the constraint $V_n \geq 0$ is equivalent to

$$1 + \pi_n \geq 0 \qquad \text{and} \qquad 1 - \frac{\pi_n}{2} \geq 0,$$

or, more simply, $\pi_n \in [-1, 2]$ for $n = 1, 2$.

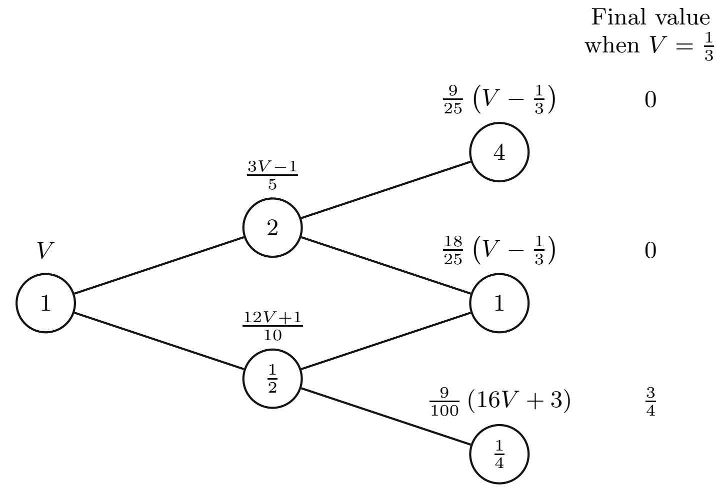

Fig. 1.9. Price of the underlying asset (inside the circles) and value of the strategy that minimizes the quadratic risk (over the circles)

We proceed now as in point ii): the first step consists of calculating

$$R_1(V,S_1) := \min_{\pi_2\in[-1,2]} E^P\left[\left((K-S_1(1+\mu_2))^+ - V(1+\pi_2\mu_2)\right)^+\right]$$

in the two states $S_1=u$ and $S_1=d$. In the scenario $S_1=u=2$, we have

$$R_1(V,2)=\frac{1}{2}\min_{\pi_2\in[-1,2]}\left((-V(1+\pi_2))^+ + \left(-V\left(1-\frac{\pi_2}{2}\right)\right)^+\right)=0,$$

for any $V\geq 0$, because

$$(-V(1+\pi_2))^+ + \left(-V\left(1-\frac{\pi_2}{2}\right)\right)^+ = 0$$

for any $V\geq 0$ and $\pi_2\in[-1,2]$.

In the scenario $S_1=d=\frac{1}{2}$, we have

$$\begin{aligned}
R_1\left(V,\frac{1}{2}\right) &= \frac{1}{2}\min_{\pi_2\in[-1,2]}\left((-V(1+\pi_2))^+ + \left(\frac{3}{4}-V\left(1-\frac{\pi_2}{2}\right)\right)^+\right)\\
&= \frac{1}{2}\min_{\pi_2\in[-1,2]}\left(\frac{3}{4}-V\left(1-\frac{\pi_2}{2}\right)\right)^+\\
&= \frac{3}{4}\left(\frac{1}{2}-V\right)^+,
\end{aligned}$$

and the minimum is attained at $\pi_2^d=-1$.

In the second step of the DP algorithm, we calculate

$$
\begin{aligned}
R_0(V, S_0) :=& \min_{\pi_1 \in [-1,2]} E^P \left[R_1\big(V(1+\pi_1\mu_1), S_0(1+\mu_1)\big) \right] \\
=& \frac{1}{2} \min_{\pi_1 \in [-1,2]} \left(R_1(V(1+\pi_1), 2) + R_1\left(V\left(1 - \frac{\pi_1}{2}\right), \frac{1}{2} \right) \right) \\
=& \frac{1}{2} \min_{\pi_1 \in [-1,2]} \frac{3}{4} \left(\frac{1}{2} - V\left(1 - \frac{\pi_1}{2}\right) \right)^+ \\
=& \frac{9}{16} \left(\frac{1}{3} - V \right)^+,
\end{aligned}
$$

and also in this case the minimum is attained at $\pi_1 = -1$.

Summing up, the strategy, with initial value V, that minimizes the shortfall risk is given by[8]

$$\pi_1 = -1, \qquad \pi_2^u = 0, \qquad \pi_2^d = -1;$$

the value of this strategy is shown in Figure 1.10. Again, if the initial capital is equal to $V = H_0 = \frac{1}{3}$ then we have perfect replication. However, unlike the quadratic case, the shortfall strategy starting from an initial value $V < \frac{1}{3}$, replicates the payoff in all cases except $S_2 = d^2$, because of the constraint $V_n \geq 0$. Thus the shortfall strategy concentrates the replication error in the scenario $S_2 = d^2$, while the quadratic strategy distributes the error among all different scenarios.

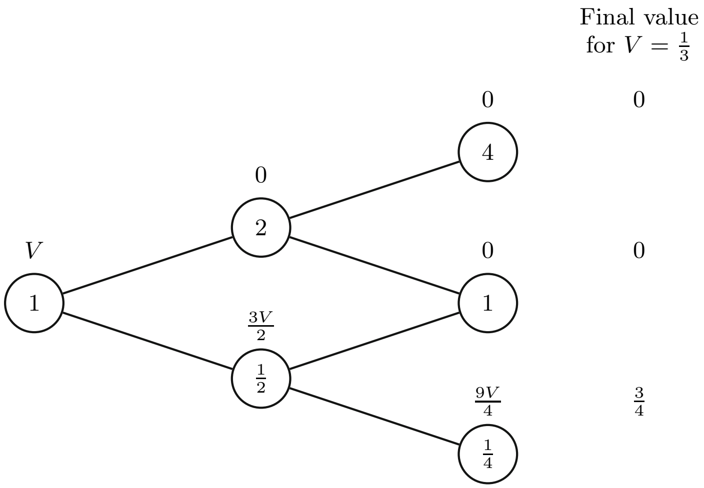

Fig. 1.10. Price of the underlying asset (inside the circles) and value of the strategy that minimizes the shortfall risk (above the circles)

[8] Recall that according to the definition given in Section 1.1.3, $\pi_2^u = 0$ because $V_1^u = 0$ regardless of its initial value V: in this regard, see also Figure 1.10.

Using the final values of V previously computed for the quadratic and shortfall strategies (see Figures 1.9 and 1.10 respectively), as well as the values of the underlying, we can easily compute the average error of replication $E^P[V_N - X]$: in the quadratic case it is equal to $\frac{27}{100}(3V-1)$ and in the shortfall case it is equal to $\frac{3}{16}(3V-1)$. $\square$

2

Portfolio optimization

This chapter concerns portfolio optimization, which is one of the historically first problems in financial mathematics. Indeed, one of the basic problems that a given subject (physical person or institution), possessing a certain amount of wealth, has to deal with is the following: how to invest this wealth in the financial market over a given period of time in order to be able to consume optimally (according to a given criterion) from this portfolio over time and at the end have a residual amount leading to benefit, also this one optimal according to a given criterion. The optimality criterion that we shall consider is the most common one, namely the *maximization of expected utility* of the various monetary amounts.

We shall start in Section 2.1 with a more detailed description of the problem distinguishing it, as it is usually being done, into the simpler sub-problems of *maximization of expected utility of terminal wealth* only and into that of *maximization of expected utility from consumption and of the residual final wealth*. In the examples and exercises we shall deal with both types of problems, whereby in the second one we shall exclusively consider intermediate consumption (at the end of the period of interest one consumes all the residual wealth).

For the solution of the problem we shall describe two main approaches: the so-called "martingale approach" and that of Dynamic Programming. The method of *Dynamic Programming (DP)* is a general method for the intertemporal optimization both in a deterministic as well as in a stochastic context. The *martingale method*, even if it can be used also outside the financial context, originates from the financial context itself being inspired by the hedging problem of a derivative. According to the *martingale method* a dynamic optimization problem decomposes into two *sub-problems*:

i) a "static" problem, in which we determine the optimal terminal value of a self-financing portfolio that can reached starting from a given initial wealth and following a self-financing strategy;

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ii) the problem of determining the optimal self-financing investment strategy seen as a strategy that replicates the optimal terminal value of the portfolio.

We point out that, while with DP we do not have to distinguish between complete and incomplete markets, this distinction has however to be made for the martingale method. As such, the completeness of the market plays in fact no essential role in the optimization of a portfolio and enters only at the methodological level of the martingale approach because the latter is inspired by the replication of a derivative.

For the description of the problem and the solution methods we have not followed any particular textbook. Among the many general references for the two methods described in this book, we limit ourselves to cite [7] for the martingale method and [2] for the Dynamic Programming. As general reference to portfolio optimization we cite [15].

Before coming to the Section 2.5 of "Solved problems" we have inserted a Section 2.4, in which we exemplify the two methods in the specific case of a log-utility function. In Section 2.5 of solved problems we then consider utility functions other than log-utility. The examples in the text as well as the problems concern each of the two mentioned sub-problems: the maximization of expected terminal utility only and maximization of expected utility from intermediate consumption. We consider complete markets, namely the binomial market (see Section 1.4.1) and the completed trinomial market (see Section 1.4.2) and, for the sole DP, also incomplete markets, namely the standard trinomial market (see Section 1.4.2).

2.1 Maximization of expected utility

2.1.1 Strategies with consumption

In the context of a discrete time market as introduced in Section 1.1, a consumption process is a *non-negative and adapted* stochastic process $C = (C_n)_{n=0,\dots,N}$, where C_n denotes the amount of wealth consumed at time t_n. A consumption strategy is a triple (α, β, C) where (α, β) is an investment strategy and C is a consumption process.

Definition 2.1. *A strategy with consumption* (α, β, C) *is self-financing if*

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1} + C_{n-1}, \qquad n = 1, \dots, N. \tag{2.1}$$

Furthermore, a self-financing strategy with consumption (α, β, C) *is admissible if*

$$V_N \geq C_N.$$

Note that, in particular, the terminal value V_N of an admissible strategy is non-negative.

Proposition 1.6 has the following simple extension:

Proposition 2.2. *The value of a self-financing strategy with consumption* (α, β, C) *is determined by its initial value* V_0 *and, recursively, by the relation*

$$V_n = (V_{n-1} - C_{n-1})(1 + r_n) + \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \left(\mu_n^i - r_n\right) \tag{2.2}$$

for $n = 1, \dots, N$.

Proof. By condition (2.1), the change in the portfolio value in the period $[t_{n-1}, t_n]$ is

$$\begin{aligned} V_n - V_{n-1} &= \alpha_n (S_n - S_{n-1}) + \beta_n (B_n - B_{n-1}) - C_{n-1} \\ &= \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \mu_n^i + \beta_n B_{n-1} r_n - C_{n-1} = \end{aligned} \tag{2.3}$$

(since, always by (2.1), we have $\beta_n B_{n-1} = V_{n-1} - \alpha_n S_{n-1} - C_{n-1}$)

$$= r_n V_{n-1} + \sum_{i=1}^{d} \alpha_n^i S_{n-1}^i \left(\mu_n^i - r_n\right) - C_{n-1}(1 + r_n)$$

from which (2.2) follows. □

Corollary 2.3. *Given* $V_0 \in \mathbb{R}$, *a predictable process* α *and a consumption process* C, *there exists a unique predictable process* β *such that* (α, β, C) *is a self-financing strategy with consumption and initial value* V_0.

Proof. Given $V_0 \in \mathbb{R}$ and the processes α and C, define the process

$$\beta_n = \frac{V_{n-1} - \alpha_n S_{n-1} - C_{n-1}}{B_{n-1}}, \qquad n = 1, \dots, N, \tag{2.4}$$

where (V_n) is defined recursively by (1.6). It is clear by construction that (β_n) is predictable and the strategy (α, β, C) is self-financing (see (2.1)). □

In the sequel it will be convenient to adopt the following notation that turns out to be well posed by Corollary 2.3.

Notation 2.4. *Having fixed* $V_0 \in \mathbb{R}$ *and a predictable process* α, *denote by* $V^{(\alpha)}$ *the value process of the self-financing strategy* (α, β) *with initial value* V_0.

Analogously, having fixed also a consumption process C, *denote by* $V^{(\alpha,C)}$ *the value process of the self-financing strategy with consumption* (α, β, C) *having initial value* V_0.

Remark 2.5. *From the proof of Corollary 2.3, it becomes evident that the process* β *in* (2.4) *depends on* V_0, *on the predictable process* α *and on the adapted process* $(C_0, \dots, C_{N-1})$ *but does not depend on* C_N.

Given then $V_0 \in \mathbb{R}$ and a predictable process α, the following definition is well posed: we say that a consumption process C is (V_0, α)-admissible if the self-financing strategy with consumption (α, β, C) and initial value V_0 (uniquely determined by Corollary 2.3) is admissible, i.e. if

$$C_N \le V_N^{(\alpha,C)}.$$

□

Remark 2.6. *Let (α, β, C) be a self-financing strategy with consumption. From (2.3), by summing over n, we obtain*

$$V_n = V_0 + g_n^{(\alpha,\beta)} - \sum_{k=0}^{n-1} C_k, \tag{2.5}$$

where $g^{(\alpha,\beta)}$ denotes the process of the returns of the strategy defined in (1.9), namely

$$g_n^{(\alpha,\beta)} = \sum_{k=1}^{n} \left(\sum_{i=1}^{d} \alpha_k^i S_{k-1}^i \mu_k^i + \beta_k B_{k-1} r_k \right), \qquad n = 1, \dots, N.$$

□

Remark 2.7. *The self-financing condition for the strategy with consumption (α, β, C), expressed in terms of discounted values becomes*

$$\widetilde{V}_n = \widetilde{V}_{n-1} - \widetilde{C}_{n-1} + \alpha_n \left(\widetilde{S}_n - \widetilde{S}_{n-1} \right). \tag{2.6}$$

Summing over n we obtain the analog of formula (1.18)

$$\widetilde{V}_n = \widetilde{V}_0 + G_n^{(\alpha)} - \sum_{k=0}^{n-1} \widetilde{C}_k, \tag{2.7}$$

where $G^{(\alpha,\beta)}$ denotes the process of the discounted returns of the strategy defined in in (1.19), namely

$$G_n^{(\alpha)} = \sum_{k=1}^{n} \alpha_k \left(\widetilde{S}_k - \widetilde{S}_{k-1} \right).$$

Recall that, with respect to a martingale measure, the process $G^{(\alpha)}$ is a zero-mean martingale (cf. Proposition 1.15). From (2.6) we obtain directly the following version with consumption of the risk neutral valuation formula:

$$V_{n-1} = E^Q \left[\frac{V_n}{1+r} \mid \mathcal{F}_{n-1} \right] + C_{n-1}, \qquad n = 1, \dots, N. \tag{2.8}$$

□

Finally, we express the self-financing condition in relative terms. Recall the notation

$$\pi_n^i = \frac{\alpha_n^i S_{n-1}^i}{V_{n-1}}, \qquad i = 1, \dots, d \tag{2.9}$$

for the strategy expressed in relative terms.

Proposition 2.8. *The value of a self-financing strategy with consumption (α, β, C) is determined by the initial value $V_0 \in \mathbb{R}$ and by the processes $\pi^1, \dots, \pi^d$ and C according to the following recursive relation*

$$V_n = (V_{n-1} - C_{n-1})(1 + r_n) + V_{n-1} \sum_{i=1}^{d} \pi_n^i \left(\mu_n^i - r_n\right). \tag{2.10}$$

Proof. Equation (2.10) follows directly from (2.2) on the basis of the notation (2.9). □

Remark 2.9. *Given $V_0 \in \mathbb{R}$, predictable processes $\pi^1, \dots, \pi^d$ and a consumption process C, we easily obtain the corresponding strategy with consumption (α, β, C) by means of the formulae*

$$\alpha_n^i = \frac{\pi_n^i V_{n-1}}{S_{n-1}^i}, \qquad \beta_n = \frac{V_{n-1} - C_{n-1}}{B_{n-1}} - \frac{V_{n-1}}{B_{n-1}} \sum_{i=1}^{d} \pi_n^i, \tag{2.11}$$

with (V_n) defined by $V_0, \pi^1, \dots, \pi^d, C$ on the basis of the recursive relation (2.10). □

2.1.2 Utility functions

In the sequel let I denote the real interval $]a, +\infty[$ where $a \leq 0$ is a fixed constant, possibly $a = -\infty$.

Definition 2.10. *A utility function is a function of class C^1*

$$u : I \longrightarrow \mathbb{R}$$

which is

(H1) *strictly increasing;*
(H2) *strictly concave.*

The domain of a utility function u is commonly extended by convention to all of $\mathbb{R}$ by putting[1]

$$u(v) = -\infty \quad \text{for } v \leq a.$$

[1] Since we are interested in the utility maximization problem, putting $u(v) = -\infty$ for $v \in \mathbb{R} \setminus I$ is equivalent to excluding values outside the interval I from being optimal.

Some classical examples of utility functions are the following:

- the logarithmic utility function
$$u(v) = \log v, \qquad v \in \mathbb{R}_+;$$
- the power utility function
$$u(v) = \frac{v^\gamma}{\gamma}, \qquad v \in \mathbb{R}_+,$$
where γ is a real parameter such that $\gamma < 1$, $\gamma \neq 0$;
- the exponential utility function
$$u(v) = -e^{-v}, \qquad v \in \mathbb{R}.$$

In what follows we shall also make the following technical assumption:

(H3) *in the case $a > -\infty$, it holds that $\lim_{v \to a^+} u'(v) = +\infty$; in the case $a = -\infty$, u is bounded from above.*

This condition is clearly satisfied by all the above-mentioned utility functions.

Recall now the Notation 2.4: having fixed $V_0 \in \mathbb{R}$, a predictable process α and a consumption process C, denote by $V^{(\alpha)}$ and $V^{(\alpha,C)}$ the value processes of the self-financing strategies (α, β) and (α, β, C) respectively having initial value V_0. Having assigned the utility functions $u, u_0, u_1, \dots, u_N$, defined on I, we are interested in the following classical problems of portfolio optimization:

- **Maximization of expected utility of terminal wealth.** Having fixed $V_0 \in \mathbb{R}_+$, the problem consists in determining, if it exists,
$$\max_{\alpha} E\left[u\left(V_N^{(\alpha)}\right)\right] \tag{2.12}$$
where the maximum is over the predictable processes α such that $V_N^{(\alpha)} \in I$;
- **Maximization of expected utility from intermediate consumption and terminal wealth.** Having fixed $V_0 \in \mathbb{R}_+$, the problem consists in determining, if it exists,
$$\max_{\alpha, C} E\left[\sum_{n=0}^{N} u_n\left(C_n\right) + u\left(V_N^{(\alpha,C)} - C_N\right)\right] \tag{2.13}$$
where the maximum is over the processes α that are predictable and of consumption C that are (V_0, α)-admissible such that $C_0, \dots, C_N \in I$ and $(V_N^{(\alpha,C)} - C_N) \in I$.

Remark 2.11. *A particular case of problem* (2.13) *is that in which $u \equiv 0$, namely where we consider the maximization of expected utility from the sole intermediate consumption. In this case, by the admissibility condition of the consumption strategy and the monotonicity property of the utility functions, the optimal strategy is such that $C_N = V_N^{(\alpha,C)}$, namely at the terminal time the entire wealth is being consumed.* □

In the following sections we prove that *in a market free of arbitrage the expected utility maximization problems* (2.12) *and* (2.13) *have a solution.*

2.1.3 Expected utility of terminal wealth

In the sequel we suppose that u is a utility function that satisfies properties **H1, H2** and **H3** and recall that we had supposed Ω to have finite cardinality. The following theorem holds:

Theorem 2.12. *For problem* (2.12) *there exists an optimal strategy if and only if the market is free of arbitrage.*

Proof. Let α be an optimal strategy with initial value v and assume that there exists an arbitrage strategy $\bar{\alpha}$ for which

$$V_0^{(\bar{\alpha})} = 0, \qquad V_N^{(\bar{\alpha})} \geq 0, \qquad P\left(V_N^{(\bar{\alpha})} > 0\right) > 0.$$

Then the strategy $\alpha + \bar{\alpha}$ is such that

$$V_0^{(\alpha+\bar{\alpha})} = v, \quad V_N^{(\alpha+\bar{\alpha})} = V_N^{(\alpha)} + V_N^{(\bar{\alpha})} \geq V_N^{(\alpha)}, \quad P\left(V_N^{(\alpha+\bar{\alpha})} > V_N^{(\alpha)}\right) > 0, \tag{2.14}$$

and this contradicts the optimality of α, since the function u is strictly increasing. This proves that, if there exists an optimal strategy, then the market is free of arbitrage.

Viceversa, denote by $\mathcal{V}_v$ the set of terminal values that are reachable by a self-financing strategy with initial value v:

$$\mathcal{V}_v = \left\{V_N^{(\alpha)} \mid \alpha \text{ predictable, } V_0^{(\alpha)} = v\right\}.$$

To prove the statement let us embed the problem in an Euclidean space: denote by M the cardinality of Ω and by $\omega_1, \dots, \omega_M$ the elementary events. If Y is a real valued random variable on Ω, put $Y(\omega_j) = Y_j$ and identify Y with the following vector in $\mathbb{R}^M$

$$(Y_1, \dots, Y_M). \tag{2.15}$$

Then we have

$$E^P[Y] = \sum_{j=1}^{M} Y_j P(\{\omega_j\}), \tag{2.16}$$

and the optimization problem (2.12) becomes equivalent to a maximization problem for the function

$$f(V) := \sum_{j=1}^{M} u(V_j) P(\{\omega_j\}) = E^P[u(V)], \qquad V \in \mathcal{V}_v \cap I^M.$$

We observe explicitly that $\mathcal{V}_v \cap I^M \neq \emptyset$ for all $v > 0$: in fact, the strategy with initial value $v > 0$, which consists in keeping the entire wealth in the riskless asset has terminal value[2] $v(1+r)^N > a$.

[2]Recall that by our assumptions $a \leq 0$ and $1 + r > 0$.

Observe also that, by (1.18)-(1.19), we have

$$\mathcal{V}_v = \left\{ B_N v + B_N \sum_{n=1}^{N} \alpha_n \left(\widetilde{S}_n - \widetilde{S}_{n-1} \right) \mid \alpha \text{ predictable} \right\}.$$

In particular, $\mathcal{V}_v$ is an affine subspace of $\mathbb{R}^M$ and is thus a *closed* set.

By the assumption of absence of arbitrage and by the first fundamental theorem of asset pricing there exists a martingale measure Q. With respect to Q, each $V \in \mathcal{V}_v$ satisfies the following condition

$$v = E^Q \left[B_N^{-1} V \right] = B_N^{-1} \sum_{j=1}^{M} V_j Q(\{\omega_j\}). \tag{2.17}$$

Now use the property **H3** of u. We first prove the statement assuming that the domain I of the utility function is bounded from below, namely $I =]a, +\infty[$ with $a > -\infty$. Let (V^n) be a sequence in $\mathcal{V}_v \cap I^M$ such that

$$\lim_{n \to \infty} E\left[u(V^n)\right] = \sup_{V \in \mathcal{V}_v \cap I^M} E\left[u(V)\right]. \tag{2.18}$$

Since $V_j^n > a$, from (2.17) it follows that the components V_j^n are uniformly bounded in j and n. Then, by possibly passing to a subsequence, (V^n) converges to a limit $\hat{V}$ and this limit belongs to $\mathcal{V}_v$, since $\mathcal{V}_v$ is closed: in particular, there exists $\hat{\alpha}$ predictable such that $\hat{V} = V_N^{(\hat{\alpha})}$ and $V_0^{(\hat{\alpha})} = v$.

We conclude the proof by showing that $\hat{V} \in I^M$ namely that

$$\hat{V}_j > a, \qquad j = 1, \dots, M. \tag{2.19}$$

Assume by contradiction that $F := \{\hat{V} = a\} \neq \emptyset$. Consider the strategy α, with initial value v, which consists in keeping the entire wealth in the nonrisky asset. Then by the concavity of u, for each $\varepsilon \in]0, 1[$ we have

$$\begin{aligned}
&E\left[u\left(\varepsilon V_N^{(\alpha)} + (1-\varepsilon) V_N^{(\hat{\alpha})} \right) - u\left(V_N^{(\hat{\alpha})} \right) \right] \\
&\geq \varepsilon E\left[u'\left(\varepsilon V_N^{(\alpha)} + (1-\varepsilon) V_N^{(\hat{\alpha})} \right) \left(V_N^{(\alpha)} - V_N^{(\hat{\alpha})} \right) \right] \\
&= \varepsilon \Bigg(E\left[\mathbb{1}_F u'\left(\varepsilon V_N^{(\alpha)} + (1-\varepsilon) a \right) \left(v(1+r)^N - a \right) \right] \\
&\quad + \varepsilon E\left[\mathbb{1}_{\Omega \setminus F} u'\left(\varepsilon V_N^{(\alpha)} + (1-\varepsilon) V_N^{(\hat{\alpha})} \right) \left(v(1+r)^N - V_N^{(\hat{\alpha})} \right) \right] \Bigg) \\
&=: \varepsilon \left(I_1(\varepsilon) + I_2(\varepsilon) \right).
\end{aligned}$$

At this point, to arrive at a contradiction, it suffices to show that there exists ε for which the above expression is positive. In fact

$$u'\left(\varepsilon V_N^{(\alpha)} + (1-\varepsilon) a \right) = u'\left(\varepsilon v(1+r)^N + (1-\varepsilon) a \right) \longrightarrow +\infty$$

for $\varepsilon \to 0^+$, on the basis of the assumption **H3**. Therefore also

$$I_1(\varepsilon) + I_2(\varepsilon) \longrightarrow +\infty$$

for $\varepsilon \to 0^+$, being $I_2(\varepsilon)$ bounded as a function of ε. Thus

$$\varepsilon\,(I_1(\varepsilon) + I_2(\varepsilon)) > 0$$

if ε is sufficiently small: this contradicts the optimality of $\hat{V}$ and concludes the proof in the case of $a > -\infty$.

We prove now the statement assuming that u is bounded from above and $I = \mathbb{R}$, namely $a = -\infty$. Observe first that, by the concavity of u, we have

$$\lim_{v \to -\infty} u(v) = -\infty. \tag{2.20}$$

As in the previous case, consider a sequence (V^n) in $\mathcal{V}_v$ that verifies (2.18): the statement consists in proving that (V^n) admits a converging subsequence. Proceed now by contradiction and assume that (V^n) does not admit any converging subsequence and is consequently unbounded. Using the equation (2.17) it is easy to prove that there exist two sequences of indexes (k_n) and (j_n) such that

$$\lim_{n \to \infty} V_{j_n}^{k_n} = -\infty.$$

Then, by the assumption of upper boundedness of u and by (2.20) it follows that

$$\lim_{n \to \infty} E\left[u(V^n)\right] = -\infty$$

and this contradicts (2.18). Thus (V^n) converges, possibly along a subsequence, in $\mathcal{V}_v$ to the terminal value of an optimal strategy. □

Remark 2.13. *The optimal terminal value is unique as a consequence of the strict concavity of the utility function u.* □

Remark 2.14. *In Section 2.1.2 we had assumed that the domain I of a utility function is not bounded from above. In fact, when we remove this assumption, the existence of an optimal strategy for problem* (2.12) *is not guaranteed, not even in the case when the market is free of arbitrage: to this effect see the example in Remark 2.58. The unboundedness from above of I has also been used implicitly in the proof of Theorem 2.12, in the argument by contradiction based on* (2.14)*.* □

Corollary 2.15. *In an arbitrage-free market let $\bar{V} = V_N^{(\bar{\alpha})}$ be the optimal terminal value for problem* (2.12)*. Then the measure Q defined by*

$$Q(\{\omega\}) = \frac{u'(\bar{V}(\omega))B_N}{E^P\left[B_N u'(\bar{V})\right]} P(\{\omega\}), \qquad \omega \in \Omega, \tag{2.21}$$

is a martingale measure.

Proof. For simplicity let us consider only the one-period case [3] $N = 1$. Putting

$$f(\alpha) = E\left[u(V_1^{\alpha})\right], \qquad \alpha \in \mathbb{R}^d,$$

we have, by (1.18)-(1.19),

$$\max_{\alpha \in \mathbb{R}^d} f(\alpha) = f(\bar{\alpha}) = E\left[u\left(B_1\left(v + \bar{\alpha}\left(\widetilde{S}_1 - \widetilde{S}_0\right)\right)\right)\right].$$

Consequently

$$\begin{aligned} 0 = \partial_{\alpha^i} f(\bar{\alpha}) &= E\left[u'\left(B_1\left(v + \bar{\alpha}\left(\widetilde{S}_1 - \widetilde{S}_0\right)\right)\right) B_1\left(\widetilde{S}_1^i - \widetilde{S}_0^i\right)\right] \\ &= E^Q\left[\widetilde{S}_1^i - \widetilde{S}_0^i\right] E^P\left[u'(\bar{V})B_1\right], \qquad i = 1, \dots, d, \end{aligned} \tag{2.22}$$

where Q is the measure defined by

$$\frac{dQ}{dP}(\omega) = \frac{Q(\{\omega\})}{P(\{\omega\})} = \frac{u'(\bar{V}(\omega))B_1}{E^P\left[u'(\bar{V})B_1\right]}, \qquad \omega \in \Omega.$$

By the assumption **H1** we have $u' > 0$ and so the measures P and Q are equivalent. Furthermore Q is a martingale measure because (2.22) is equivalent to

$$\widetilde{S}_0^i = E^Q\left[\widetilde{S}_1^i\right], \qquad i = 1, \dots, d.$$

□

2.1.4 Expected utility from intermediate consumption and terminal wealth

The following result is analogous to Theorem 2.12.

Theorem 2.16. *For problem* (2.13) *there exists an optimal strategy if and only if the market is free of arbitrage.*

Proof. The proof is analogous to that of Theorem 2.12 and so we sketch it only. We first show that, if there exists an optimal strategy, then the market is free of arbitrage. Let (α, C) be an optimal strategy with initial value v and suppose by contradiction that there exists an arbitrage strategy $\bar{\alpha}$ for which

$$V_0^{(\bar{\alpha})} = 0, \qquad V_N^{(\bar{\alpha})} \geq 0, \qquad P\left(V_N^{(\bar{\alpha})} > 0\right) > 0.$$

Then the strategy $(\alpha + \bar{\alpha}, C)$ is such that

$$V_0^{(\alpha+\bar{\alpha},C)} = v, \quad V_N^{(\alpha+\bar{\alpha},C)} \geq V_N^{(\alpha,C)},$$

and so C is $(v, \alpha + \bar{\alpha})$-admissible; furthermore, since the function u is increasing, the fact that $P\left(V_N^{(\alpha+\bar{\alpha},C)} > V_N^{(\alpha,C)}\right) > 0$ contradicts the optimality of (α, C).

[3] For a full proof see for example Proposition 2.7.2 in [7].

Viceversa, having fixed $v > 0$, put

$$\mathcal{W}_v = \Big\{ (V, C) \mid C \text{ consumption process, } C_N \le V = V_N^{(\alpha,C)} \\ \text{with } \alpha \text{ predictable, } V_0^{(\alpha,C)} = v \Big\}. \tag{2.23}$$

Embed the problem in an Euclidean space as in the proof of Theorem 2.12 and recall that the utility function u is defined on the interval $I =]a, +\infty[$. The optimization problem (2.13) becomes equivalent to the problem of determining the maximum of the function

$$f(V, C) := \sum_{j=1}^{M} \left(\sum_{k=0}^{N} u_k(C_{k,j}) + u(V_j - C_{N,j}) \right) P(\{\omega_j\})$$

on the set

$$\mathcal{W}_{v,a} = \mathcal{W}_v \cap \{(V, C) \mid \ C_{k,j} > a, \ V_j - C_{N,j} > a, \ j = 1, \dots, M\}.$$

Recall that, by assumption, $a \le 0$ and note that $\mathcal{W}_{v,a} = \mathcal{W}_v$ if $a < 0$.

On the basis of (2.7) we have the representation

$$\mathcal{W}_v = \Big\{ (V, C) \mid \ C \text{ adapted and non-negative, } C_N \le V, \\ V = B_N \left(v + \sum_{k=1}^{N} \alpha_k \left(\widetilde{S}_k - \widetilde{S}_{k-1} \right) - \sum_{h=0}^{N-1} \widetilde{C}_h \right), \ \alpha \text{ predictable} \Big\}.$$

Consequently, $\mathcal{W}_v$ is a *closed* set. Furthermore, by the assumption of the existence of a martingale measure Q, we have the "budget condition"

$$E^Q \left[B_N^{-1} V + \sum_{k=0}^{N-1} B_k^{-1} C_k \right] = v, \tag{2.24}$$

for each $(V, C) \in \mathcal{W}_v$. From (2.24) it follows easily that $\mathcal{W}_v$ is bounded and thus a *compact* set. This is sufficient to conclude the proof in the case $a < 0$: in fact, we have already noticed that in this case $\mathcal{W}_{v,a} = \mathcal{W}_v$ and so the continuous function f has a maximum on this domain.

In the case $a = 0$ the existence of an optimal strategy is proved by proceeding as in the proof of Theorem 2.12: consider a sequence $(V^n, C^n) \in \mathcal{W}_{v,a}$ such that

$$\lim_{n \to \infty} f(V^n, C^n) = \sup_{\mathcal{W}_{v,a}} f.$$

By compactness, and passing possibly to a subsequence, there exists

$$\lim_{n \to \infty} (V^n, C^n) =: (\hat{V}, \hat{C}) \in \mathcal{W}_v.$$

Finally, to show that indeed $(\hat{V}, \hat{C}) \in \mathcal{W}_{v,a}$ we may use an argument similar to that in the proof of Theorem 2.12, based on the assumption $\lim_{v \to 0^+} u'(v) = +\infty$. $\square$

2.2 "Martingale" method

The martingale method is used for the solution of dynamic stochastic optimization problems, but it has its origins in the problem of hedging of derivatives.

In Chapter 1 we have seen that, given a European derivative X, namely a random variable that represents the payoff of a derivative, the hedging problem consists in determining an initial value $V_0 = v$ and a self-financing strategy α such that $V_N^{(\alpha)} = X$ almost surely in P and thus almost surely also in any martingale measure Q. On the other hand, the discounted value of any self-financing and predictable strategy is a martingale with respect to any martingale measure. From the replication equation we thus obtain the condition

$$v = E^Q\left[\widetilde{V}_N^{(\alpha)}\right] = E^Q\left[B_N^{-1}X\right] \tag{2.25}$$

for each martingale measure Q. Recall also that, by Theorem 1.21, a derivative X is replicable if and only if $E^Q\left[B_N^{-1}X\right]$ takes the same value for each martingale measure Q.

The problem of determining a hedging strategy may be interpreted as a "martingale representation problem" in the following sense. Fix for a moment a martingale measure Q and define the martingale

$$\widetilde{M}_n := E^Q\left[B_N^{-1}X \mid \mathcal{F}_n\right], \qquad n = 0, \dots, N.$$

If we determine a strategy α such that

$$\widetilde{M}_n = \widetilde{V}_n^{(\alpha)} = v + \sum_{k=1}^{n} \alpha_k \left(\widetilde{S}_k - \widetilde{S}_{k-1}\right), \qquad n = 0, \dots, N, \tag{2.26}$$

where the second equation follows from (1.18)-(1.19) then, starting from the initial value v and following the strategy α, we have $\widetilde{V}_N^{(\alpha)} = \widetilde{X}$ and thus also $V_N^{(\alpha)} = X$. Consequently, determining a strategy α is equivalent to finding a representation for the martingale $\widetilde{M}$ in the form of (2.26).

2.2.1 Complete market: terminal wealth

Consider the problem of maximization of expected utility of terminal wealth

$$\max_{\alpha} E\left[u\left(V_N^{(\alpha)}\right)\right], \tag{2.27}$$

starting from an initial wealth v.

The martingale method consists of three steps:

(P1) recalling the Notation 2.4, determine the set of terminal values that can be reached by a self financing and predictable strategy

$$\mathcal{V}_v = \left\{V \mid V = V_N^{(\alpha)},\ \alpha \text{ predictable},\ V_0^{(\alpha)} = v\right\};$$

(P2) determine the optimal terminal reachable value $\bar{V}_N$ namely the one that realizes the maximum in (2.27);

(P3) determine a self-financing strategy $\bar{\alpha}$ such that $V_N^{(\bar{\alpha})} = \bar{V}_N$.

The first problem is generally solved by using the martingale condition that established a link between the expected terminal value and the initial value (cf. the characterization (2.28) below). The second step consists in a maximization problem that can be solved by using the standard tools for constrained optimization such as the Lagrange multiplier theorem (see the proof of Theorem 2.18). Since the last step corresponds to a standard hedging problem (where the payoff to be replicated is the optimal terminal value $\bar{V}_N$), in what follows we shall mainly deal with problems **P1** and **P2**.

Notice that the martingale method decomposes the original *dynamic* portfolio optimization problem into a *static* problem (the determination of the optimal terminal value) and a hedging problem (corresponding to a “martingale representation problem”).

In the case when the market is free of arbitrage and complete, there exists a unique martingale measure Q. For what concerns problem **P1**, we then have the characterization

$$\mathcal{V}_v = \left\{V \mid E^Q\left[B_N^{-1}V\right] = v\right\}, \tag{2.28}$$

for the set of terminal values that are reachable by means of a self-financing strategy with initial value v.

Before attacking problem **P2** let us make the following:

Remark 2.17. *Let u be a utility function. Since*

$$u' : I \longrightarrow \mathbb{R}_+$$

is a continuous and strictly decreasing function, the inverse function

$$\mathcal{I} := (u')^{-1} \tag{2.29}$$

is a continuous and strictly decreasing function as well. For example:

- *for the logarithmic utility we have*

$$u(v) = \log v, \quad u'(v) = \frac{1}{v}, \quad \mathcal{I}(w) = \frac{1}{w}, \qquad v, w \in \mathbb{R}_+;$$

- *for the power utility we have*

$$u(v) = \frac{v^\gamma}{\gamma}, \quad u'(v) = v^{\gamma-1}, \quad \mathcal{I}(w) = w^{\frac{1}{\gamma-1}}, \qquad v, w \in \mathbb{R}_+;$$

- *for the exponential utility we have*

$$u(v) = -e^{-v}, \quad u'(v) = e^{-v}, \quad \mathcal{I}(w) = -\log w, \qquad v \in \mathbb{R},\ w \in \mathbb{R}_+. \ \square$$

The problem **P2** is solved according to the following:

Theorem 2.18. *In a market that is complete and free of arbitrage consider the problem of maximization of the expected utility of terminal wealth* (2.27) *starting from an initial capital* $v \in \mathbb{R}_+$*. Under the condition*

$$u'(I) = \mathbb{R}_+, \tag{2.30}$$

the optimal terminal value is given by

$$\bar{V}_N = \mathcal{I}\left(\lambda \widetilde{L}\right) \tag{2.31}$$

where $\widetilde{L} = B_N^{-1} L$ *with* $L = \frac{dQ}{dP}$*, being* Q *the martingale measure, and* $\lambda \in \mathbb{R}$ *is determined by the equation*

$$E^P\left[\mathcal{I}\left(\lambda \widetilde{L}\right) \widetilde{L}\right] = v. \tag{2.32}$$

Equation (2.32) *is called the "budget equation".*

Proof. Embedding the problem in an Euclidean space as in the proof of Theorem 2.12, the expected utility maximization problem of terminal wealth becomes equivalent to a standard constrained optimization problem in $\mathbb{R}^M$. In fact, adopting the notations (2.15)-(2.16) and putting, for simplicity, $P_i = P(\{\omega_i\})$ and $Q_i = Q(\{\omega_i\})$ for $i = 1, \dots, M$, we may reformulate the problem in terms of maximization of the function

$$f(V) := \sum_{i=1}^{M} u(V_i) P_i = E^P[u(V)]$$

subject to the constraint $V \in \mathcal{V}_v \cap I^M$ which, by (2.28), is expressed by

$$g(V) := \sum_{i=1}^{M} B_N^{-1} V_i Q_i - v = E^Q\left[B_N^{-1} V\right] - v = 0 \quad \text{and} \quad V_i > a, \quad i = 1, \dots, M.$$

By Theorem 2.12 and thanks to the assumption **H3** we know that there exists a unique $\bar{V}_N \in \mathcal{V}_v$, solution of the optimization problem, such that $\bar{V}_N > a$ (cf. (2.19)). To determine this optimal value we use the Lagrange multiplier theorem and introduce the Lagrangian for the function f over the constraint $\mathcal{V}_v = \{g = 0\}$:

$$\mathcal{L}(V, \lambda) = f(V) - \lambda g(V).$$

Putting the gradient equal to zero we obtain the system of equations

$$\partial_{V_i} \mathcal{L}(V, \lambda) = u'(V_i) P_i - B_N^{-1} \lambda Q_i = 0, \qquad i = 1, \dots, M, \tag{2.33}$$

$$\partial_\lambda \mathcal{L}(V, \lambda) = \sum_{i=1}^{M} B_N^{-1} V_i Q_i - v = 0. \tag{2.34}$$

By the assumption (2.30), the function u' is bijective from I into $\mathbb{R}_+$: thus (2.33) has the unique solution

$$\left(\bar{V}_N\right)_i = \mathcal{I}\left(B_N^{-1}\lambda \frac{Q_i}{P_i}\right), \qquad i = 1, \dots, M$$

and this equation is equivalent to (2.31). Notice that, by construction, $\bar{V}_N \in \mathcal{V}_v$ and $\bar{V}_N > a$.

Inserting (2.31) into (2.34) in order to determine λ, we obtain

$$h(\lambda) := \sum_{i=1}^{M} B_N^{-1}\mathcal{I}\left(B_N^{-1}\lambda \frac{Q_i}{P_i}\right) Q_i = v \tag{2.35}$$

which is equivalent to (2.32). Based on Remark 2.17, the function h defined in (2.35) is continuous and strictly decreasing. Consequently, for each $v \in \mathbb{R}_+$, there exists a unique λ solution of (2.35). $\square$

Since the Radon-Nikodym derivatives of Q with respect to P play an important role in the martingale method, in the next two examples we shall derive their expressions in the case of the binomial and the completed trinomial model. The examples/problems on the application of the martingale method itself follow in Section 2.4 for the case of a logarithmic utility and in Section 2.5 also for other utility functions.

Example 2.19 (Binomial model). Consider a binomial market model with N periods where the asset price's up and down movements are characterized by the parameters u and d respectively, the riskless interest rate is r and the probability for an up-move of the price is p. We may identify the elementary events ω of the probability space with the N-tuples of the form

$$\omega = (0, 1, 0, 0, 1, 0, \dots).$$

The risky asset's price then satisfies

$$S_n = u^{\nu_n} d^{n-\nu_n} S_0, \tag{2.36}$$

where ν_n is the random variable that “counts the number of price increases” in n periods (or, equivalently, the number of 1 among the first n elements of the N-tuple ω). Notice that

$$E\left[\nu_N\right] = pN.$$

Recalling Theorem 1.26, the martingale measure is defined as

$$Q(\{\omega\}) = q^{\nu_N(\omega)}(1-q)^{N-\nu_N(\omega)}, \qquad \omega \in \Omega,$$

where

$$q = \frac{1+r-d}{u-d}.$$

Furthermore, the random variables $\mu_1, \dots, \mu_N$ are Q-independent. Consequently, the Radon-Nikodym derivative of Q with respect to P is

$$L(\omega) = \left(\frac{q}{p}\right)^{\nu_N(\omega)} \left(\frac{1-q}{1-p}\right)^{N-\nu_N(\omega)}, \qquad \omega \in \Omega. \tag{2.37}$$

For later use, in particular in Theorem 2.24, we derive the expression of the process

$$L_n = E^P \left[L \mid \mathcal{F}_n\right] = E^P \left[\frac{dQ}{dP} \mid \mathcal{F}_n\right].$$

On the basis of the definition of conditional expectation, for each $\omega \in \Omega$ such that $\nu_n(\omega) = k$, we have

$$\begin{aligned} E\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right](\omega) &= E\left[\frac{dQ}{dP} \mid \nu_n = k\right] \\ &= \frac{1}{P(\nu_n = k)} \int_{\{\nu_n = k\}} \left(\frac{dQ}{dP}\right) dP = \frac{Q(\nu_n = k)}{P(\nu_n = k)}, \end{aligned}$$

namely

$$L_n = \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n}. \tag{2.38}$$

An alternative, more elementary way to prove (2.38) and which we shall use also later, is the following: for each $n < N$, we have

$$\begin{aligned} L &= \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N} \\ &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \left(\frac{q}{p}\right)^{\nu_N-\nu_n} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_N-\nu_n)}. \end{aligned}$$

Therefore, being ν_n an $\mathcal{F}_n$-measurable random variable and since $\nu_N - \nu_n$ has the same distribution as ν_{N-n} and is Q-independent from $\mathcal{F}_n$, we have

$$\begin{aligned} L_n = E\left[L \mid \mathcal{F}_n\right] &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \\ &\cdot E\left[\left(\frac{q}{p}\right)^{\nu_N-\nu_n} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_N-\nu_n)} \mid \mathcal{F}_n\right] \\ &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \cdot E\left[\left(\frac{q}{p}\right)^{\nu_{N-n}} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_{N-n})}\right] \\ &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \end{aligned}$$

being

$$E\left[\left(\frac{q}{p}\right)^{\nu_{N-n}}\left(\frac{1-q}{1-p}\right)^{N-n-(\nu_{N-n})}\right]$$
$$=\sum_{k=0}^{N-n}\binom{N-n}{k}p^k(1-p)^{N-n-k}\left(\frac{q}{p}\right)^k\left(\frac{1-q}{1-p}\right)^{N-n-k}=1. \qquad \square$$

Example 2.20 (Completed trinomial model). Consider a completed trinomial market model (cf. Paragraph 1.4.2) with N periods and parameters u_i, m_i, d_i for the two risky asset prices S^i, $i = 1, 2$, with riskless interest rate r and probabilities p_1, p_2, p_3. We may identify the elementary events ω with the N-tuples of the form

$$\omega = (1, 1, 2, 3, 1, 3, \dots).$$

The cardinality of the probability space Ω is 3^N and we have

$$P(\{\omega\}) = p_1^{\nu_N^1(\omega)} p_2^{\nu_N^2(\omega)} p_3^{N-\nu_N^1(\omega)-\nu_N^2(\omega)}, \qquad \omega \in \Omega,$$

where ν_N^i, $i = 1, 2$, are the random variables that count the movements corresponding to u and m respectively in N periods (or, equivalently, the number of 1 and 2 in the N-tuple ω). Furthermore,

$$E\left[\nu_N^i\right] = p_i N \qquad i = 1, 2.$$

We have the representation

$$S_n^i = u_i^{\nu_n^1} m_i^{\nu_n^2} d_i^{n-\nu_n^1-\nu_n^2} S_0^i, \qquad i = 1, 2, \tag{2.39}$$

for the prices of the risky assets. The martingale measure is defined by

$$Q(\{\omega\}) = q_1^{\nu_N^1(\omega)} q_2^{\nu_N^2(\omega)} q_3^{N-\nu_N^1(\omega)-\nu_N^2(\omega)}, \qquad \omega \in \Omega,$$

with q_1, q_2, q_3 in (1.46). Consequently, the Radon-Nikodym derivative of Q with respect to P is given by

$$L(\omega) = \left(\frac{q_1}{p_1}\right)^{\nu_N^1(\omega)}\left(\frac{q_2}{p_2}\right)^{\nu_N^2(\omega)}\left(\frac{q_3}{p_3}\right)^{N-\nu_N^1(\omega)-\nu_N^2(\omega)}, \qquad \omega \in \Omega. \tag{2.40}$$

Proceeding as in the binomial case, we obtain for each $\omega \in \Omega$ such that $\nu_n^1(\omega) = k_1$ and $\nu_n^2(\omega) = k_2$:

$$L_n = E\left[L \mid \mathcal{F}_n\right](\omega) = E\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right](\omega)$$
$$= E\left[\frac{dQ}{dP} \mid \nu_n^1 = k_1,\ \nu_n^2 = k_2\right] = \frac{Q(\nu_n^1 = k_1,\ \nu_n^2 = k_2)}{P(\nu_n^1 = k_1,\ \nu_n^2 = k_2)},$$

namely

$$L_n = \left(\frac{q_1}{p_1}\right)^{\nu_n^1}\left(\frac{q_2}{p_2}\right)^{\nu_n^2}\left(\frac{q_3}{p_3}\right)^{n-\nu_n^1-\nu_n^2}. \tag{2.41}$$

$\square$

2.2.2 Incomplete market: terminal wealth

In the case when the market is free of arbitrage and incomplete, the set of martingale measures is infinite and therefore the problem **P1** is in general more delicate. Observe first of all that, on the basis of Theorem 1.21, the set of terminal values that can be obtained starting from an initial wealth v has the following characterization:

$$\mathcal{V}_v = \left\{ V \mid E^Q \left[B_N^{-1} V \right] = v \ \text{ for each martingale measure} Q \right\}.$$

Secondly, the family of martingale measures (identified with the vectors of $\mathbb{R}^M$ as in the proof of Theorem 2.18) is the intersection of an affine space of $\mathbb{R}^M$ with the set of strictly positive probability measures[4]

$$\mathbb{R}_+^M = \{Q = (Q_1, \dots, Q_M) \mid Q_j > 0, \ j = 1, \dots, M\}.$$

In particular, there exist measures $Q^{(1)}, \dots, Q^{(r)} \in \overline{\mathbb{R}_+^M}$ such that every martingale measure Q can be expressed as a linear combination of the form

$$Q = a_1 Q^{(1)} + \dots a_r Q^{(r)}$$

in which the sum of the weights a_i is one. Consequently we have

$$\mathcal{V}_v = \left\{ V \mid E^{Q^{(j)}} \left[B_N^{-1} V \right] = v \text{ for } j = 1, \dots, r \right\}. \tag{2.42}$$

Once the "extremal" measures $Q^{(1)}, \dots, Q^{(r)}$ have been identified, the following result, which generalizes Theorem 2.18, solves the problem **P2** of determining the optimal terminal reachable values $\bar{V}$ when starting from an initial capital $v \in I$.

Theorem 2.21. *Under the condition*

$$u'(I) = \mathbb{R}_+, \tag{2.43}$$

the optimal terminal value is

$$\bar{V}_N = \mathcal{I} \left(\sum_{j=1}^{r} \lambda_j \widetilde{L}^{(j)} \right) \tag{2.44}$$

[4] For example, in the one-period case, assuming for simplicity $r = 0$, we have that $Q \in \mathbb{R}_+^M$ is a martingale measure if

$$\sum_{j=1}^{M} Q_j = 1 \quad \text{and} \quad E^Q \left[S_1^i \right] = \sum_{j=1}^{M} (S_1^i)_j Q_j = S_0^i, \qquad i = 1, \dots, d.$$

where $\widetilde{L}^{(j)} = B_N^{-1} L^{(j)}$ *with* $L^{(j)} = \frac{dQ^{(j)}}{dP}$ *and* $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ *are determined from the system of budget equations*

$$E^P \left[\mathcal{I}\left(\sum_{k=1}^{r} \lambda_k \widetilde{L}^{(k)} \right) \widetilde{L}^{(j)} \right] = v, \qquad j = 1, \dots, r. \tag{2.45}$$

Proof. The proof is analogous to that of Theorem 2.18 and consists in reformulating the problem in terms of maximization of the function

$$f(V) := \sum_{i=1}^{M} u(V_i) P_i = E^P \left[u(V) \right]$$

subject to the constraint $V \in \mathcal{V}_v$ which, by (2.42), can be expressed as

$$g^{(j)}(V) := \sum_{i=1}^{M} B_N^{-1} V_i Q_i^{(j)} - v = E^{Q^{(j)}} \left[B_N^{-1} V \right] - v = 0, \qquad j = 1, \dots, r.$$

In this case we consider the Lagrangian

$$\mathcal{L}(V, \lambda) = f(V) - \sum_{k=1}^{r} \lambda_k g^{(k)}(V)$$

and, setting the gradient equal to zero, we obtain the system of equations

$$\partial_{V_i} \mathcal{L}(V, \lambda) = u'(V_i) P_i - B_N^{-1} \sum_{k=1}^{r} \lambda_k Q_i^{(k)} = 0, \qquad i = 1, \dots, M, \tag{2.46}$$

$$\partial_{\lambda_j} \mathcal{L}(V, \lambda) = \sum_{i=1}^{M} B_N^{-1} V_i Q_i^{(j)} - v = 0, \qquad j = 1, \dots, r. \tag{2.47}$$

By the assumption (2.43) the equation (2.46) is equivalent to

$$(\bar{V}_N)_i = \mathcal{I}\left(B_N^{-1} \sum_{k=1}^{r} \lambda_k \frac{Q_i^{(k)}}{P_i} \right), \qquad i = 1, \dots, M$$

i.e. to (2.44).

Inserting these expressions into (2.47) we obtain the system of equations

$$\sum_{i=1}^{M} B_N^{-1} \mathcal{I}\left(B_N^{-1} \sum_{k=1}^{r} \lambda_k \frac{Q_i^{(k)}}{P_i} \right) Q_i^{(j)} = v, \qquad j = 1, \dots, r$$

which is equivalent to (2.45). □

Remark 2.22. *We have seen that the solution of the problem of maximization of expected utility of terminal wealth by the martingale method requires the determination of the extremal martingale measures. We point out that this leads to considerable practical difficulties especially if N is large. Limiting ourselves to the case of an incomplete market given by a trinomial model, notice that the martingale measures, and thus also the extremal ones, are defined (cf. Section 1.4.2 and Example 2.20) on a probability space Ω where the elementary events ω can be identified with the N-tuples of the form*

$$\omega = (1, 1, 2, 3, 1, 3, \cdots)$$

and the cardinality of Ω is equal to 3^N.

Considering now the case of $N = 2$, let $\omega = (\omega^1, \omega^2)$ where $\omega^i \in \{1, 2, 3\}$ for $i = 1, 2$, and write for simplicity $Q(\omega)$ instead of $Q(\{\omega\})$. We have $Q(\omega) = Q(\omega^1, \omega^2) = Q^1(\omega^1)Q(\omega^2 \mid \omega^1)$, where the marginal $Q^1(\omega^1)$ and the conditional probabilities $Q(\omega^2 \mid \omega^1)$ satisfy (1.44) as consequence of (1.43). *Even if, as mentioned after (1.44), $Q(\omega^2 \mid \omega^1)$ may in general depend on $\mathcal{F}_1$, namely on ω^1, the form of (1.44) allows us to consider for simplicity the subclass of martingale measures in which $Q(\omega^2 \mid \omega^1) \equiv Q^2(\omega^2)$ for a marginal Q^2 that satisfies (1.44) and is independent of ω^1. Always on the basis of (1.44), when Q and thus the marginals Q^i vary, the triples $(q_1^i, q_2^i, q_3^i) = (Q^i(1), Q^i(2), Q^i(3))$ form, for $i = 1, 2$, a same set of possible values given by a segment of $\mathbb{R}$ and which thus admits two extremal values (which are always martingale measures but not necessarily equivalent to P), call them $(Q^{e,0}(1), Q^{e,0}(2), Q^{e,0}(3))$ and $(Q^{e,1}(1), Q^{e,1}(2), Q^{e,1}(3))$ respectively. We may thus write*

$$\begin{aligned} Q^1(\omega^1) &= \gamma_1 Q^{e,0}(\omega^1) + (1 - \gamma^1)Q^{e,1}(\omega^1) \quad \forall \omega^1 \in \{1, 2, 3\} \\ Q^2(\omega^2) &= \gamma_2 Q^{e,0}(\omega^2) + (1 - \gamma^2)Q^{e,1}(\omega^2) \quad \forall \omega^2 \in \{1, 2, 3\} \end{aligned}$$

where $\gamma_1, \gamma_2 \in (0, 1)$ and where, on the basis of the above assumption that $Q(\omega^2 \mid \omega^1) \equiv Q^2(\omega^2)$, independently of ω^1 and thus of $\mathcal{F}_1$, also γ_2 is chosen independent of ω^1 and thus of $\mathcal{F}_1$. It thus follows that

$$\begin{aligned} Q(\omega) &= \left(\gamma_1 Q^{e,0}(\omega^1) + (1 - \gamma^1)Q^{e,1}(\omega^1)\right)\left(\gamma_2 Q^{e,0}(\omega^2) + (1 - \gamma^2)Q^{e,1}(\omega^2)\right) \\ &= \gamma_1\gamma_2 Q^{e,0}(\omega^1)Q^{e,0}(\omega^2) + \gamma_1(1 - \gamma_2)\gamma_2 Q^{e,0}(\omega^1)Q^{e,1}(\omega^2) \\ &\quad + (1 - \gamma_1)\gamma_2 Q^{e,1}(\omega^1)Q^{e,0}(\omega^2) + (1 - \gamma_1)(1 - \gamma_2)Q^{e,1}(\omega^1)Q^{e,1}(\omega^2) \end{aligned}$$

namely $Q(\omega)$ appears, already with the simplification that $Q(\omega^2 \mid \omega^1) \equiv Q^2(\omega^2)$ is independent of ω^1, as a convex combination of four extremal martingale measures, namely

$$\begin{matrix} \bar{Q}^1(\omega) = Q^{e,0}(\omega^1)Q^{e,0}(\omega^2)\,, & \bar{Q}^2(\omega) = Q^{e,0}(\omega^1)Q^{e,1}(\omega^2) \\ \bar{Q}^3(\omega) = Q^{e,1}(\omega^1)Q^{e,0}(\omega^2)\,, & \bar{Q}^4(\omega) = Q^{e,1}(\omega^1)Q^{e,1}(\omega^2) \end{matrix}.$$

Generalizing, for a generic value of N we would then have 2^N extremal martingale measures.

For this reason, in the problems with the martingale method we shall limit ourselves, both for the case of terminal wealth and intermediate consumption, to a complete market context. □

2.2.3 Complete market: intermediate consumption

In this section we consider the problem of maximization of expected utility from intermediate consumption, namely

$$\max_{\alpha,C} E\left[\sum_{n=0}^{N} u_n\left(C_n\right)\right] \tag{2.48}$$

which corresponds to problem (2.13) with $u = 0$. In (2.48), $u_0, \dots, u_N$ are utility functions defined on the interval I. In what follows we consider only the case $I = \mathbb{R}_+$, namely $a = 0$.

Since in (2.48) the maximum is over the set of admissible strategies with consumption, recalling Remark 2.11, we have that the martingale method consists of the following three steps:

(P1) determine, recalling Notation 2.4, the set of “reachable” consumption processes

$$\mathcal{C}_v = \left\{C \text{ consumption proc.} \mid C_N = V_N^{(\alpha,C)} \text{with } \alpha \text{ predictable, } V_0^{(\alpha,C)} = v\right\};$$

(P2) determine the optimal reachable consumption process $\bar{C}$ which leads to the maximum in (2.48);

(P3) determine the self-financing strategy with consumption that corresponds to the optimal reachable consumption.

Step P1. In an arbitrage free and complete market the martingale measure Q exists and is unique: by analogy to (2.28) we thus have the following characterization of the family $\mathcal{C}_v$.

Lemma 2.23. *We have*

$$\mathrm{C}_v = \left\{C \text{ consumption process} \mid E^Q\left[\sum_{n=0}^{N} B_n^{-1} C_n\right] = v\right\}. \tag{2.49}$$

Proof. If $C \in \mathrm{C}_v$ then by (2.7) we have

$$v + G_N^{(\alpha)} = B_N^{-1} V_N^{(\alpha,C)} + \sum_{n=0}^{N-1} B_n^{-1} C_n = \sum_{n=0}^{N} B_n^{-1} C_n$$

and, since $G_N^{(\alpha)}$ has zero expectation under Q, it holds that

$$E^Q\left[\sum_{n=0}^{N} B_n^{-1} C_n\right] = v. \tag{2.50}$$

Viceversa let C be a consumption process that satisfies (2.50). Since by assumption the market is complete, for each $n = 1, \dots, N$ there exists a self-financing and predictable strategy without consumption $(\alpha^{(n)}, \beta^{(n)})$ that replicates the payoff C_n at time t_n, namely such that

$$V_n^{(\alpha^{(n)}, \beta^{(n)})} = C_n. \tag{2.51}$$

We modify this strategy by putting

$$\alpha_k^{(n)} = 0, \qquad \beta_k^{(n)} = 0, \qquad \text{for } k > n,$$

and consider the predictable processes

$$\alpha = \alpha^{(1)} + \cdots + \alpha^{(N)}, \qquad \beta = \beta^{(1)} + \cdots + \beta^{(N)}.$$

Then the strategy with consumption (α, β, C) is self-financing since, denoting by V its value, for $n = 1, \dots, N$ we have

$$V_{n-1} = \sum_{k=n-1}^{N} \left(\alpha_{n-1}^{(k)} S_{n-1} + \beta_{n-1}^{(k)} B_{n-1} \right) =$$

(by the self-financing property, over the n-th period, of the strategies without consumption $\left(\alpha^{(k)}, \beta^{(k)}\right)$ for $k = n, \dots, N$)

$$= \alpha_{n-1}^{(n-1)} S_{n-1} + \beta_{n-1}^{(n-1)} B_{n-1} + \sum_{k=n}^{N} \left(\alpha_n^{(k)} S_{n-1} + \beta_n^{(k)} B_{n-1} \right) =$$

(by the replication condition (2.51))

$$= C_{n-1} + \alpha_n S_{n-1} + \beta_n B_{n-1}.$$

Finally, by construction we obviously have

$$V_N = \alpha_N^{(N)} S_N + \beta_N^{(N)} B_N = C_N. \tag{2.52}$$

To conclude we show that $V_0 = v$: since the strategy with consumption (α, β, C) is self-financing by construction, by (2.7) we have

$$V_0 = E^Q \left[B_N^{-1} V_N + \sum_{n=0}^{N-1} B_n^{-1} C_n \right] =$$

(by (2.52) and the assumption (2.50))

$$= E^Q \left[\sum_{n=0}^{N} B_n^{-1} C_n \right] = v. \qquad \square$$

Step P2. The problem of determining the optimal consumption process is solved by the following result, analogous to Theorem 2.18.

Theorem 2.24. *In an arbitrage free and complete market consider the problem of maximization of expected utility from intermediate consumption* (2.48) *starting from an initial capital* $v \in \mathbb{R}_+$. *Under the condition*

$$u_n'(\mathbb{R}_+) = \mathbb{R}_+, \qquad n = 0, \dots, N, \tag{2.53}$$

the optimal consumption process is given by

$$\bar{C}_n = \mathcal{I}_n\left(\lambda \widetilde{L}_n\right), \qquad n = 0, \dots, N, \tag{2.54}$$

where $\mathcal{I}_n = (u_n')^{-1}$ *and* $\widetilde{L}_n = B_n^{-1} L_n$ *with* $L_n = E^P\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right]$, *being* Q *the martingale measure. Furthermore,* $\lambda \in \mathbb{R}$ *is determined by the budget equation*

$$E^P\left[\sum_{n=0}^{N} \widetilde{L}_n \mathcal{I}_n\left(\lambda \widetilde{L}_n\right)\right] = v. \tag{2.55}$$

Proof. The proof is analogous to that of Theorem 2.18: the problem is equivalent to a standard constrained optimization problem in an Euclidean space for the function

$$f(C) := \sum_{i=1}^{M} \sum_{n=0}^{N} u_n(C_{n,i}) P_i = E^P\left[\sum_{n=0}^{N} u_n(C_n)\right]$$

subject to the constraint $C \in \mathrm{C}_v$, expressed in terms of Lemma 2.23. In the previous equation we use, as is common, the notation $C_{n,i} = C_n(\omega_i)$ for $i = 1, \dots, m$.

We make the preliminary observation that, being the process C adapted, equation (2.50) is equivalent to

$$\begin{aligned} v &= E^Q\left[\sum_{n=0}^{N} B_n^{-1} C_n\right] = E^P\left[\sum_{n=0}^{N} B_n^{-1} C_n L\right] \\ &= E^P\left[\sum_{n=0}^{N} E^P\left[B_n^{-1} C_n L \mid \mathcal{F}_n\right]\right] = E^P\left[\sum_{n=0}^{N} C_n \widetilde{L}_n\right]. \end{aligned}$$

The need of introducing the adapted process $\left(\widetilde{L}_n\right)$ for the probability measure change becomes evident from formula (2.54), since by definition every consumption process is adapted.

By Theorem 2.16 there exists an optimal consumption process $\bar{C} \in \mathrm{C}_v$ such that $\bar{C}_n > 0$ for each n. It follows that, putting

$$g(C) = \sum_{i=1}^{M} \sum_{n=0}^{N} C_{n,i} \widetilde{L}_{n,i} P_i - v = E^Q\left[\sum_{n=0}^{N} B_n^{-1} C_n\right] - v,$$

the optimal consumption can be determined by using the Lagrange multiplier theorem, thereby setting equal to zero the gradient of the Lagrangian

$$\mathcal{L}(C, \lambda) = f(C) - \lambda g(C).$$

We thus obtain the system of equations

$$\partial_{C_{n,i}} \mathcal{L}(C, \lambda) = u_n'(C_{n,i}) P_i - \lambda \widetilde{L}_{n,i} P_i = 0, \tag{2.56}$$

for $i = 1, \dots, M, \ n = 0, \dots, N$ and

$$\partial_\lambda \mathcal{L}(C, \lambda) = \sum_{i=1}^{M} \sum_{n=0}^{N} C_{n,i} \widetilde{L}_{n,i} P_i - v = 0. \tag{2.57}$$

By the assumption (2.53), the function u_n' is bijective and thus (2.56) has a unique solution

$$\bar{C}_{n,i} = \mathcal{I}_n \left(\lambda \widetilde{L}_{n,i} \right),$$

equivalent to (2.54).

Inserting the expression for $\bar{C}_{n,i}$ into (2.57) for the purpose of determining λ, we obtain

$$h(\lambda) := \sum_{i=1}^{M} \sum_{n=0}^{N} \mathcal{I}_n \left(\lambda \widetilde{L}_{n,i} \right) \widetilde{L}_{n,i} P_i = v \tag{2.58}$$

which is equivalent to (2.55). On the basis of Remark 2.17, the function h is continuous and strictly decreasing so that for each $v \in \mathbb{R}_+$ there exists a unique λ solution of (2.58). □

Step P3. Once problem **P2** has been solved and thus the optimal reachable consumption process $\bar{C}$ for the criterion (2.48) has been determined, the third and last step consists in determining the self-financing strategy with consumption corresponding to $\bar{C}$. For this purpose we have seen in the proof of Lemma 2.23 that the optimal strategy α can be expressed as the sum

$$\alpha = \alpha^{(1)} + \cdots + \alpha^{(N)}$$

where $\alpha^{(k)}$ is the self-financing strategy (without consumption), defined over the period $[0, k]$ which replicates $\bar{C}_k$. This α can be determined solving the N replication problems relative to the payoffs $C_1, \dots, C_N$.

Nevertheless, from a practical point of view it is preferable to use the following recursive algorithm:

- having fixed $v > 0$, determine (α_N, β_N) imposing the replication condition

$$\alpha_N S_N + \beta_N B_N = \bar{V}_N = \bar{C}_N \tag{2.59}$$

  which leads to a system of linear equations;

- at the generic step n, assuming to have computed $\bar{V}_n$, we can determine (α_n, β_n) imposing that

$$\alpha_n S_n + \beta_n B_n = \bar{V}_n; \tag{2.60}$$

- in order to determine $\bar{V}_n$, which represents the value in n of a self-financing portfolio corresponding to the consumption given by $\bar{C}_n$, we recall (2.8) that leads to the recursive relation

$$\bar{V}_n = \frac{1}{1+r} E^Q\{\bar{V}_{n+1} \mid \mathcal{F}_n\} + \bar{C}_n. \tag{2.61}$$

  Depending on the utility function, this latter recursion may lead to explicit formulae (see (2.63) below).

Example 2.25. In the case of a logarithmic utility, in which (see (2.103) below)

$$\bar{C}_n = \frac{v(1+r)^n}{N+1} \left(\frac{p}{q}\right)^{\nu_n} \left(\frac{1-p}{1-q}\right)^{n-\nu_n} \tag{2.62}$$

we have

$$\bar{V}_n = (N+1-n)\,\bar{C}_n. \tag{2.63}$$

We show (2.63) by backwards induction on n. For $n = N$ equation (2.63) is true by the replication condition $\bar{V}_N = \bar{C}_N$. Assuming (2.63) true for $n+1$, from (2.61) and taking into account (2.61) we have

$$\begin{aligned}
\bar{V}_n &= \bar{C}_n + \frac{v(1+r)^n}{N+1}(N-n) \\
&\quad \cdot \left(q \left(\frac{p}{q}\right)^{\nu_n+1} \left(\frac{1-p}{1-q}\right)^{n-\nu_n} + (1-q) \left(\frac{p}{q}\right)^{\nu_n} \left(\frac{1-p}{1-q}\right)^{n+1-\nu_n} \right) \\
&= \bar{C}_n + (N-n)\,\frac{v(1+r)^n}{N+1} \left(\frac{p}{q}\right)^{\nu_n} \left(\frac{1-p}{1-q}\right)^{n-\nu_n} (p + (1-p)) \\
&= (N-n+1)\bar{C}_n.
\end{aligned}$$

Taking into account (2.59) and (2.60), the systems of linear equations to be solved in each period $n \le N$ in order to determine (α_n, β_n) are all analogous to one another. □

Remark 2.26. *The difference between the problems of intermediate consumption in a complete and an incomplete market with r extremal martingale measures consists essentially in the fact that, in an incomplete market,*

$$\bar{C}_n = \mathcal{I}_n \left(\sum_{k=1}^{r} \lambda_k \widetilde{L}_n^{(k)} \right)$$

where, in order to determine the Lagrange multipliers λ_k, instead of having a single budget equation namely (2.55), we have r of them, more precisely (see (2.45)*),*

$$E^P \left[\sum_{n=0}^{N} \mathcal{I}_n \left(\sum_{k=1}^{r} \lambda_k \widetilde{L}_n^{(k)} \right) \widetilde{L}_n^{(j)} \right] = v \quad j = 1, \dots, r.$$

$\square$

2.2.4 Complete market: intermediate consumption and terminal wealth

In this section we mention synthetically the problem of maximization of expected utility from intermediate consumption and terminal wealth

$$\max_{\alpha, C} E \left[\sum_{n=0}^{N} u_n (C_n) + u \left(V_N^{(\alpha,C)} - C_N \right) \right] \tag{2.64}$$

corresponding to problem (2.13). In (2.64), $u, u_0, \dots, u_N$ are utility functions defined on $\mathbb{R}_+$.

The martingale method consists of the following three steps:

(P1) determine the set of reachable terminal values and consumption processes

$$\mathcal{W}_v = \Big\{ (V, C) \mid C \text{ consumption process, } C_N \le V = V_N^{(\alpha,C)} \\ \text{with } \alpha \text{ predictable, } V_0^{(\alpha,C)} = v \Big\};$$

(P2) determine the optimal reachable terminal value and consumption process $(\bar{V}, \bar{C})$ that achieve the maximum in (2.64);

(P3) determine the self-financing strategy with consumption corresponding to $(\bar{V}, \bar{C})$.

Step P1. The following result can be proved like Lemma 2.23.

Lemma 2.27. *We have*

$$\mathcal{W}_v = \Bigg\{ (V, C) \mid C \text{ consumption process s.t. } C_N \le V \text{ and} \\ E^Q \left[B_N^{-1} V + \sum_{n=0}^{N-1} B_n^{-1} C_n \right] = v \Bigg\}. \tag{2.65}$$

Step P2. The problem of determining the optimal consumption process is solved by the following result, analogous to Theorem 2.24.

Theorem 2.28. *In an arbitrage free and complete market consider the problem of maximization of expected utility from intermediate consumption and terminal wealth* (2.64) *starting from an initial capital* $v \in \mathbb{R}_+$*. Under the condition*

$$u'(\mathbb{R}_+) = u_n'(\mathbb{R}_+) = \mathbb{R}_+, \qquad n = 0, \dots, N, \tag{2.66}$$

the optimal consumption process is given by

$$\bar{C}_n = \mathcal{I}_n\left(\lambda \widetilde{L}_n\right), \qquad n = 0, \dots, N, \tag{2.67}$$

where $\mathcal{I}_n = (u_n')^{-1}$ *and* $\widetilde{L}_n = B_n^{-1} L_n$ *with* $L_n = E^P\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right]$*, being* Q *the martingale measure, and the optimal terminal value is*

$$\bar{V}_N = \mathcal{I}_N(\lambda \widetilde{L}_N) + \mathcal{I}(\lambda \widetilde{L}_N), \tag{2.68}$$

where $\mathcal{I} = (u')^{-1}$*. Furthermore,* $\lambda \in \mathbb{R}$ *is determined from the budget equation*

$$E^P\left[\widetilde{L}_N \mathcal{I}(\lambda \widetilde{L}_N) + \sum_{n=0}^{N} \widetilde{L}_n \mathcal{I}_n\left(\lambda \widetilde{L}_n\right)\right] = v. \tag{2.69}$$

Proof. Differently from the proof of Theorem 2.24, the budget equation there, namely (2.50), is here given by

$$\begin{aligned} v &= E^Q\left[B_N^{-1} V_N + \sum_{n=0}^{N-1} B_n^{-1} C_n\right] = E^P\left[B_N^{-1} V_N L + \sum_{n=0}^{N-1} B_n^{-1} C_n L\right] \\ &= E^P\left[B_N^{-1} V_N L + \sum_{n=0}^{N-1} E^P\left[B_n^{-1} C_n L \mid \mathcal{F}_n\right]\right] = E^P\left[\widetilde{L}_N V_N + \sum_{n=0}^{N-1} \widetilde{L}_n C_n\right]. \end{aligned}$$

Here too though the optimization problem is equivalent to a standard constrained optimization problem in a Euclidean space obtained by considering the elementary events. Assuming the number of elementary events is M, the Lagrangian is given by

$$\begin{aligned} &\sum_{i=1}^{M}\left(\sum_{n=0}^{N} u_n(C_{n,i}) + u(V_{N,i} - C_{N,i})\right) P_i \\ &- \lambda \sum_{i=1}^{M}\left(\widetilde{L}_{N,i} V_{N,i} + \sum_{n=0}^{N-1} \widetilde{L}_{n,i} C_{n,i}\right) P_i - \lambda v. \end{aligned}$$

Differentiating with respect to $V_{N,i}, C_{N,i}$ and $C_{n,i}$ we obtain

$$\begin{cases} u'(V_{N,i} - C_{N,i}) - \lambda \widetilde{L}_{N,i} = 0 \\ u_N'(C_{N,i}) - u'(V_{N,i} - C_{N,i}) = 0 \\ u_n'(C_{n,i}) - \lambda \widetilde{L}_{n,i} = 0, \qquad \text{for } n < N. \end{cases} \tag{2.70}$$

Summing the first two equations we obtain

$$u'_N(C_{N,i}) - \lambda \widetilde{L}_{N,i} = 0$$

i.e. also for $n = N$ a relation as for $n < N$ (third equation above) and thus

$$\bar{C}_n = \mathcal{I}_n(\lambda \widetilde{L}_n) \quad \text{for } n = 0, \cdots, N. \tag{2.71}$$

Furthermore, from the first of the equations (2.70) we obtain

$$V_{N,i} - C_{N,i} = \mathcal{I}(\lambda \widetilde{L}_{N,i})$$

from which, using also (2.71),

$$\bar{V}_N = \mathcal{I}_N(\lambda \widetilde{L}_N) + \mathcal{I}(\lambda \widetilde{L}_N).$$

The budget condition can be rewritten as

$$E^P \left[\widetilde{L}_N \mathcal{I}(\lambda \widetilde{L}_N) + \sum_{n=0}^{N} \widetilde{L}_n \mathcal{I}_n \left(\lambda \widetilde{L}_n \right) \right] = v. \qquad \Box$$

Step P3. Finally, the self-financing strategy with consumption corresponding to $(\bar{V}, \bar{C})$ is easily determined by modifying appropriately the algorithm of Step **P3** in Section 2.2.3.

2.3 Dynamic Programming Method

2.3.1 Recursive algorithm

On a finite probability space $(\Omega, \mathcal{F}, P)$ consider a stochastic process $(V_n)_{n=0,\dots,N}$ (to fix ideas, we may think of V as the value process of a portfolio) the evolution of which depends on the choice of a "control process" (typically, an investment strategy and/or a consumption process). More precisely, suppose that the following recursive relation holds

$$V_k = G_k(V_{k-1}, \mu_k; \eta_{k-1}(V_{k-1})) \tag{2.72}$$

for $k = 1, \dots, N$, where:

- $\mu_1, \dots, \mu_N$ are d-dimensional *independent* random variables (typically, they represent the risk factors that drive the dynamics of the asset prices in a discrete time market);
- $\eta_0, \dots, \eta_N$ are generic functions

  $$\eta_k : \mathbb{R} \longrightarrow \mathbb{R}^\ell, \qquad k = 0, \dots, N,$$

  with $\ell \in \mathbb{N}$, called control functions or, more simply, *controls*;

- $G_1, \dots, G_N$ are generic functions

$$G_k : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^\ell \longrightarrow \mathbb{R}, \qquad k = 1, \dots, N.$$

Example 2.29. In a discrete time market of the type (1.1)-(1.2), consider a self-financing strategy and denote by $\pi^1, \dots, \pi^d$ the ratios of wealth invested in the risky assets, as defined in (2.9). Under the assumption that the strategy is a function of the value of the portfolio, namely that

$$\alpha_k = \alpha_k(V_{k-1}), \qquad k \geq 1,$$

then, by Proposition 2.8, the value of the strategy of pure investment without consumption satisfies the recursive relation (2.72) where

$$G_k(v, \mu_k; \eta_{k-1}) = v\left(1 + r_k + \sum_{i=1}^{d} \eta_{k-1}^i \left(\mu_k^i - r_k\right)\right) \tag{2.73}$$

and

$$\eta_k = \begin{cases} \left(\pi_{k+1}^1, \dots, \pi_{k+1}^d\right) & \text{for } k = 0, \dots, N-1 \\ 0 & \text{for } k = N. \end{cases} \tag{2.74}$$

□

Example 2.30. Consider a self-financing strategy with consumption and suppose that the processes of investment and consumption are functions of the portfolio value, namely that

$$\alpha_k = \alpha_k(V_{k-1}), \qquad C_k = C_k(V_k), \qquad k \geq 1.$$

Then, by Proposition 2.8, the value of the strategy satisfies the recursive relation (2.72) with

$$G_k(v, \mu_k; \eta_{k-1}) = v\left(1 + r_k + \sum_{i=1}^{d} \eta_{k-1}^i \left(\mu_k^i - r_k\right)\right) - (1 + r_k)\eta_{k-1}^{d+1}$$

where η is the $(d+1)$-dimensional process, the components of which are the ratios invested in the risky assets as well as the consumption:

$$\eta_k = \begin{cases} \left(\pi_{k+1}^1, \dots, \pi_{k+1}^d, C_k\right) & \text{for } k = 0, \dots, N-1, \\ (0, \dots, 0, C_N) & \text{for } k = N. \end{cases}$$

□

Notation 2.31. *Having fixed $v \in \mathbb{R}_+$ and $n \in \{0, 1, \dots, N-1\}$, denote by*

$$(V_k^{n,v})_{k=n,\dots,N}$$

the process defined by $V_n^{n,v} = v$ *and, recursively, by* (2.72) *for* $k > n$. *Furthermore, put*

$$U^{n,v}(\eta_n, \dots, \eta_N) = E\left[\sum_{k=n}^{N} u_k\left(V_k^{n,v}, \eta_k\left(V_k^{n,v}\right)\right)\right], \tag{2.75}$$

where $u_0, \dots, u_N$ *are given functions*

$$u_n : \mathbb{R} \times \mathbb{R}^\ell \longrightarrow \mathbb{R}, \qquad n = 0, \dots, N.$$

We are interested in the optimization problem that consists in determining the supremum of $U^{0,v}(\eta_0, \dots, \eta_N)$ over the controls $\eta_0, \dots, \eta_N$, namely

$$\sup_{\eta_0, \dots, \eta_N} U^{0,v}(\eta_0, \dots, \eta_N). \tag{2.76}$$

We are also interested in determining, whenever they exist, the optimal controls that achieve this supremum.

The method of Dynamic Programming (in the sequel DP) to solve the optimization problem (2.76) is based on the idea that *if a control is optimal over an entire sequence of periods, then it has to be optimal over each single period.* More precisely, the method of DP is based on the following result, of which the proof is postponed to the Section 2.3.2:

Theorem 2.32. *For each* $n = 0, \dots, N$ *we have*

$$\sup_{\eta_n, \dots, \eta_N} U^{n,v}(\eta_n, \dots, \eta_N) = W_n(v) \tag{2.77}$$

where W_n *is defined recursively by*

$$\begin{cases} W_N(v) = \sup\limits_{\xi \in \mathbb{R}^\ell} u_N(v, \xi), & \textit{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup\limits_{\xi \in \mathbb{R}^\ell} \left(u_{n-1}(v, \xi) + E\left[W_n\Big(G_n\left(v, \mu_n; \xi\right)\Big)\right]\right). \end{cases} \tag{2.78}$$

We point out that (2.78) leads to a recursive algorithm in which, at every step, we perform a standard optimization of a function of real variables. In particular, under suitable assumptions that guarantee that the supremum in (2.77) is attained (namely is actually a maximum), the algorithm allows also to determine the optimal controls $\bar{\eta}_0, \dots, \bar{\eta}_N$. In fact, they result from the points where the functions to be maximized in (2.78) attain their suprema: more precisely, suppose that for each n there exists $\bar{\xi}_n \in \mathbb{R}^\ell$ which maximizes the function

$$\xi \mapsto u_{n-1}(v, \xi) + E\left[W_n\Big(G_n\left(v, \mu_n; \xi\right)\Big)\right],$$

and notice that $\bar{\xi}_n$ depends implicitly on v; then the function $\bar{\eta}_{n-1}$ defined by $\bar{\eta}_{n-1}(v) = \bar{\xi}_n$ is an optimal control.

Summing up, the method of DP leads to a *deterministic algorithm* in which at every step we determine (by backwards recursion), the optimal value and the optimal controls by means of a standard scalar maximization procedure.

Example 2.33 (Maximization of expected utility of terminal wealth). The value of a self-financing strategy is defined recursively by

$$V_k = G_k(V_{k-1}, \mu_k; \pi_k) = V_{k-1}\left(1 + r_k + \sum_{i=1}^{d} \pi_k^i \left(\mu_k^i - r_k\right)\right). \tag{2.79}$$

We have

$$U^{n,v}(\pi_{n+1}, \dots, \pi_N) = E\left[u\left(V_N^{n,v}\right)\right],$$

and, by Theorem 2.32,

$$\sup_{\pi_{n+1},\dots,\pi_N} E\left[u\left(V_N^{n,v}\right)\right] = W_n(v) \tag{2.80}$$

where

$$\begin{cases} W_N(v) = u(v), & \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup\limits_{\bar{\pi}_n \in \mathbb{R}^d} E\left[W_n\left(G_n(v, \mu_n; \bar{\pi}_n)\right)\right]. \end{cases} \tag{2.81}$$

□

Notation 2.34. *In* (2.81) *we use the overbar symbol $\bar{\pi}$ to distinguish the vectors in $\mathbb{R}^d$ from the functions, denoted simply by π in the optimization problem* (2.80)*. We shall keep this distinction in the first examples but, as the context becomes clearer, we shall omit the overbar in order not to overburden the notation.*

Example 2.35 (Maximization of expected utility from intermediate consumption and terminal wealth). The value of a self-financing strategy with consumption is defined recursively by

$$\begin{aligned} V_k &= G_k(V_{k-1}, \mu_k; \pi_k, C_{k-1}) \\ &= (V_{k-1} - C_{k-1})(1 + r_k) + V_{k-1}\sum_{i=1}^{d} \pi_k^i \left(\mu_k^i - r_k\right). \end{aligned}$$

In this case we have

$$U^{n,v}(\pi_{n+1}, \dots, \pi_N, C_n, \dots, C_N) = E\left[\sum_{k=n}^{N} u_k(C_k) + u(V_N^{n,v} - C_N)\right].$$

On the basis of Theorem 2.32 we have

$$\sup_{\substack{\pi_{n+1},\dots,\pi_N \\ C_n,\dots,C_N}} E\left[\sum_{k=n}^{N} u_k(C_k) + u(V_N^{n,v} - C_N)\right] = W_n(v)$$

where

$$\begin{cases} W_N(v) = \sup_{\bar{C}_N \le v} \left(u_N(\bar{C}_N) + u(v - \bar{C}_N) \right), & \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup_{\substack{\bar{\pi}_n \in \mathbb{R}^d, \\ \bar{C}_{n-1} \in \mathbb{R}_+}} \left(u_{n-1}(\bar{C}_{n-1}) + E\left[W_n\left(G_n\left(v, \mu_n; \bar{\pi}_n, \bar{C}_{n-1} \right) \right) \right] \right). \end{cases} \tag{2.82}$$

□

2.3.2 Proof of Theorem 2.32

Lemma 2.36. *Let X, Y be independent real random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a measurable and bounded function. Putting*

$$f(x) = E\left[g(x, Y)\right], \qquad x \in \mathbb{R},$$

we have

$$E\left[f(X)\right] = E\left[g(X, Y)\right].$$

Proof. Denote by P^X and P^Y the law of X and Y respectively. Then

$$\begin{aligned} E\left[f(X)\right] &= \int f(x) P^X(dx) = \int E\left[g(x, Y)\right] P^X(dx) \\ &= \int \int g(x, y) P^Y(dy) P^X(dx) = \end{aligned}$$

(by the independence assumption)

$$= \int \int g(x, y) P^{(X,Y)}(dx dy) = E\left[g(X, Y)\right].$$

□

Proof (of Theorem 2.32). Let us prove the statement by induction. For $n = N$ we have

$$\begin{aligned} W_N(v) &= \sup_{\xi \in \mathbb{R}^\ell} u_N(v, \xi) = \sup_{\eta_N} u_N(v, \eta_N(v)) \\ &= \sup_{\eta_N} E\left[u_N(v, \eta_N(v))\right] = \sup_{\eta_N} U^{N,v}(\eta_N). \end{aligned}$$

Assuming that (2.77) holds for n, we prove the statement for $n - 1$:

$$\begin{aligned} W_{n-1}(v) &= \sup_{\xi \in \mathbb{R}^\ell} E\left[u_{n-1}(v, \xi) + W_n\Big(G_n\left(v, \mu_n; \xi\right) \Big) \right] \\ &= \sup_{\eta_{n-1}} E\left[u_{n-1}\big(v, \eta_{n-1}(v)\big) + W_n\Big(G_n\big(v, \mu_n; \eta_{n-1}(v)\big) \Big) \right] = \end{aligned}$$

(by the inductive hypothesis)

$$= \sup_{\eta_{n-1}} E\left[u_{n-1}(v, \eta_{n-1}(v)) + \sup_{\eta_n,\dots,\eta_N} U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n,\dots,\eta_N)\right] =$$

(see the footnote[5])

$$= \sup_{\eta_{n-1},\dots,\eta_N} E\left[u_{n-1}(v, \eta_{n-1}(v)) + U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n,\dots,\eta_N)\right] =$$

(by (2.75) and Lemma 2.36, given the independence of the random variables $\mu_1, \dots, \mu_N$)

$$= \sup_{\eta_{n-1},\dots,\eta_N} E\Bigg[u_{n-1}(v, \eta_{n-1}(v)) + \sum_{k=n}^{N} u_k\left(V_k^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}, \eta_k\left(V_k^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}\right)\right)\Bigg] =$$

(observing that $V_k^{n,G_n(v,\mu_n;\eta_{n-1}(v))} = V_k^{n-1,v}$ for $k = n, \dots, N$)

$$= \sup_{\eta_{n-1},\dots,\eta_N} E\left[\sum_{k=n-1}^{N} u_k\left(V_k^{n-1,v}, \eta_k\left(V_k^{n-1,v}\right)\right)\right]$$
$$= \sup_{\eta_{n-1},\dots,\eta_N} U^{n-1,v}(\eta_{n-1},\dots,\eta_N)$$

and this concludes the proof. □

[5] The inequality "$\geq$" is obvious. To show the inverse inequality, fix $\varepsilon > 0$ and consider functions $\eta_n^\varepsilon, \dots, \eta_N^\varepsilon$ such that

$$\sup_{\eta_n,\dots,\eta_N} U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n,\dots,\eta_N) \leq U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n^\varepsilon,\dots,\eta_N^\varepsilon) + \varepsilon.$$

Then, in the mean, we obtain

$$E\left[\sup_{\eta_n,\dots,\eta_N} U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n,\dots,\eta_N)\right]$$
$$\leq E\left[U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n^\varepsilon,\dots,\eta_N^\varepsilon)\right] + \varepsilon$$
$$\leq \sup_{\eta_n,\dots,\eta_N} E\left[U^{n,G_n\left(v,\mu_n;\eta_{n-1}(v)\right)}(\eta_n,\dots,\eta_N)\right] + \varepsilon,$$

from which the thesis follows, given the arbitrariness of ε.

2.4 Logarithmic utility: examples

2.4.1 Terminal utility in the binomial model: MG method

We use the martingale method to solve the problem of maximization of expected utility of terminal wealth in the case of the logarithmic utility and in a binomial model. Recalling that for a logarithmic utility $\mathcal{I}(w) = \frac{1}{w}$, by Theorem 2.18 and by the expression (2.37) of the Radon-Nikodym derivative of Q with respect to P, we have

$$\bar{V}_N = (\lambda \widetilde{L})^{-1} = \frac{(1+r)^N}{\lambda} \left(\frac{p}{q}\right)^{\nu_N} \left(\frac{1-p}{1-q}\right)^{N-\nu_N}$$

where λ is determined by (2.32):

$$v = E\left[\frac{\widetilde{L}}{\lambda \widetilde{L}}\right] = \lambda^{-1}.$$

Therefore

$$\bar{V}_N = \frac{v}{\widetilde{L}} = v(1+r)^N \left(\frac{p}{q}\right)^{\nu_N} \left(\frac{1-p}{1-q}\right)^{N-\nu_N},$$

and the optimal value of expected utility is

$$\begin{aligned} E\left[\log \bar{V}_N\right] &= \log v + N\log(1+r) + E\left[\nu_N\right]\log\frac{p}{q} + (N - E\left[\nu_N\right])\log\frac{1-p}{1-q} \\ &= \log v + N\log(1+r) + Np\log\frac{p}{q} + N(1-p)\log\frac{1-p}{1-q}. \end{aligned} \tag{2.83}$$

The last step consists in determining the optimal strategy as a hedging strategy for the derivative $\bar{V}_N$. We proceed backwards as in Section 1.4.1 and impose the replication condition for the last period

$$\alpha_N S_N + \beta_N B_N = \bar{V}_N; \tag{2.84}$$

assuming that $S_{N-1} = S_0 u^k d^{N-1-k}$ namely that $\nu_{N-1} = k$ for $k < N$. Equation (2.84) is equivalent to the following system of equations in the unknowns α_N, β_N:

$$\begin{cases} \alpha_N u S_{N-1} + \beta_N B_N = v(1+r)^N \left(\frac{p}{q}\right)^{k+1} \left(\frac{1-p}{1-q}\right)^{N-k-1}, \\ \alpha_N d S_{N-1} + \beta_N B_N = v(1+r)^N \left(\frac{p}{q}\right)^{k} \left(\frac{1-p}{1-q}\right)^{N-k}. \end{cases}$$

We have

$$\alpha_N = \frac{v(1+r)^N \left(\frac{p}{q}\right)^k \left(\frac{p-1}{q-1}\right)^{N-k} (p-q)}{q(1-p)S_{N-1}(u-d)},$$

$$\beta_N = \frac{v \left(\frac{p}{q}\right)^k \left(\frac{p-1}{q-1}\right)^{N-k} ((p-1)qu - dp(q-1))}{q(p-1)(u-d)}.$$

Notice also that (cf. Example 2.19)

$$\begin{aligned}\alpha_N &= \frac{v(1+r)^N}{(u-d)S_{N-1}} \left(\frac{p}{q} - \frac{1-p}{1-q}\right) \left(\frac{p}{q}\right)^{\nu_{N-1}} \left(\frac{p-1}{q-1}\right)^{N-1-\nu_{N-1}} \\ &= \frac{v(1+r)^N}{(u-d)S_{N-1}} \left(\frac{p}{q} - \frac{1-p}{1-q}\right) L_{N-1}^{-1}.\end{aligned}$$

In general, to compute the strategy for the n-th period, we have first to determine $\bar{V}_n$: to this end notice that, putting

$$L_n := E\left[L \mid \mathcal{F}_n\right] = \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n},$$

for each $n < N$, we have

$$\begin{aligned}L &= \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N} \\ &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \left(\frac{q}{p}\right)^{\nu_N-\nu_n} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_N-\nu_n)} \\ &= L_n \left(\frac{q}{p}\right)^{\nu_N-\nu_n} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_N-\nu_n)}.\end{aligned}$$

Therefore, being ν_n an $\mathcal{F}_n$-measurable random variable and since $\nu_N - \nu_n$ has the same distribution as ν_{N-n} and is Q-independent of $\mathcal{F}_n$, we have

$$\begin{aligned}\bar{V}_n &= \frac{1}{(1+r)^{N-n}} E^Q\left[\bar{V}_N \mid \mathcal{F}_n\right] \\ &= v(1+r)^n E^Q\left[\frac{1}{L} \mid \mathcal{F}_n\right] \\ &= \frac{v(1+r)^n}{L_n} E^Q\left[L_{N-n}^{-1}\right] = \frac{v(1+r)^n}{L_n}.\end{aligned} \tag{2.85}$$

Notice that $\bar{V}_n$ has an expression similar to $\bar{V}_N$ so that the calculations to determine the optimal strategy α_n, β_n are formally analogous to the previous ones: in fact, the replication condition

$$\alpha_n S_n + \beta_n B_n = \bar{V}_n,$$

is equivalent to the system

$$\begin{cases} \alpha_n u S_{n-1} + \beta_n B_n = v(1+r)^n \left(\frac{p}{q}\right)^{\nu_{n-1}+1} \left(\frac{1-p}{1-q}\right)^{N-1-\nu_{n-1}}, \\ \alpha_n d S_{n-1} + \beta_n B_n = v(1+r)^n \left(\frac{p}{q}\right)^{\nu_{n-1}} \left(\frac{1-p}{1-q}\right)^{N-\nu_{n-1}}. \end{cases}$$

In particular we have

$$\alpha_n = \frac{v(1+r)^n}{(u-d)S_{n-1}} \left(\frac{p}{q} - \frac{1-p}{1-q}\right) L_{n-1}^{-1}. \tag{2.86}$$

Notice, finally, that the ratio invested in the risky asset is constant, independently of the period and of the state of the system: in fact from (2.85) and (2.86) we have

$$\pi_n = \frac{\alpha_n S_{n-1}}{\bar{V}_{n-1}} = \frac{(1+r)(p-q)}{(u-d)q(1-q)}, \qquad n = 1, \dots, N. \tag{2.87}$$

Remark 2.37. *The fact that the optimal strategy consists in investing in the risky asset the same ratio of wealth in every period does not mean that the strategy, expressed in units of the assets kept in the portfolio, remains constant. In fact, in order to keep the invested ratio constant, with each change in the price of the underlying we have to change the number of units of the risky asset in the portfolio.* □

2.4.2 Terminal utility in the binomial model: DP method

Consider the problem of maximization of expected utility from terminal wealth for a logarithmic utility function in a binomial model of N periods (cf. Paragraph 1.4.1), where the asset price's up and down movements are characterized by the parameters u and d respectively, the riskless interest rate is r and the probability for an up-move of the price is p.

We use the Dynamic Programming method. Following Example 2.33, the dynamics of the portfolio value are given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = \begin{cases} V_{n-1}\left(1 + r + \pi_n(u-1-r)\right) & \text{if } \mu_n = u-1, \\ V_{n-1}\left(1 + r + \pi_n(d-1-r)\right) & \text{if } \mu_n = d-1, \end{cases}$$

where π (which denotes the ratio of the risky asset held in the portfolio) forms the control process. Observe that, starting from $V_{n-1} > 0$, we have that $V_n > 0$ if and only if

$$\begin{cases} 1 + r + \pi_n(u-1-r) > 0 \\ 1 + r + \pi_n(d-1-r) > 0 \end{cases}$$

or, recalling that the condition $d < 1 + r < u$ for the absence of arbitrage holds, if equivalently

$$\pi_n \in D =]a, b[\qquad \text{where} \qquad a = -\frac{1+r}{u-1-r}, \qquad b = \frac{1+r}{1+r-d}. \tag{2.88}$$

On the basis of the DP algorithm (2.81) we have, for $v > 0$,

$$\begin{aligned} W_N(v) &= \log v, \\ W_{N-1}(v) &= \max_{\bar{\pi}_N \in D} E\left[\log G_N(v, \mu_N; \bar{\pi}_N)\right] = \log v + \max_D f, \end{aligned}$$

where

$$f(\pi) = p \log(1 + r + \pi(u - 1 - r)) + (1-p) \log\left(1 + r + \pi(d - 1 - r)\right).$$

We have

$$f'(\pi) = p \frac{u-1-r}{1+r+\pi(u-1-r)} + (1-p) \frac{d-1-r}{1+r+\pi(d-1-r)}$$

and this derivative vanishes at the point

$$\bar{\pi} = \frac{(1+r)(pu + (1-p)d - 1 - r)}{(u-1-r)(1+r-d)}. \tag{2.89}$$

A simple calculation shows that $\bar{\pi} \in D =]a, b[$ with a, b as in (2.88), for any choice of the parameters $p \in]0, 1[$ and $d < 1 + r < u$. Observing that

$$\lim_{\pi \to a^+} f(\pi) = \lim_{\pi \to b^-} f(\pi) = -\infty,$$

we have that $\bar{\pi}$ is the global maximizer of f and determines the optimal strategy $\pi_N^{\max}(v) \equiv \bar{\pi}$, $v \in \mathbb{R}_+$. Furthermore,

$$\begin{aligned} \max_D f &= f\left(\bar{\pi}\right) \\ &= p \log\left(\frac{p(u-d)}{1+r-d}\right) + (1-p) \log\left(\frac{(1-p)(u-d)}{u-1-r}\right) + \log(1+r). \end{aligned} \tag{2.90}$$

At the next step we have

$$\begin{aligned} W_{N-2}(v) &= \max_{\bar{\pi}_{N-1} \in D} E\left[\log G_{N-1}(v, \mu_{N-1}; \bar{\pi}_{N-1})\right] + \max_D f \\ &= \log v + 2f\left(\bar{\pi}\right), \end{aligned}$$

and an analogous formula holds at the generic step n, namely

$$W_{N-n}(v) = \log v + nf\left(\bar{\pi}\right).$$

Summing up, the optimal value of expected utility starting from an initial wealth $v > 0$, is equal to

$$W_0(v) = \log v + N f(\bar{\pi}),$$

with $f(\bar{\pi})$ as in (2.90), and the corresponding optimal strategy is constant and equal to

$$\pi_n^{\max}(v) = \bar{\pi}, \qquad v \in \mathbb{R}_+, \; n = 1, \dots, N,$$

with $\bar{\pi}$ as defined in (2.89).

Remark 2.38. *Recall the expression of the martingale measure in the binomial model, namely*

$$q = Q(1 + \mu_n = u) = \frac{1 + r - d}{u - d}.$$

A simple calculation shows that for the optimal strategy we have the following expression, equivalent to (2.89)*,*

$$\bar{\pi} = \frac{(1+r)(p-q)}{(u-d)q(1-q)}$$

which coincides with the one obtained with the martingale method (see (2.87)). Furthermore, we also have

$$W_n(v) = \log v + (N - n)\left(p \log \frac{p}{q} + (1-p)\log\frac{1-p}{1-q} + \log(1+r)\right)$$

which corresponds to (2.83). □

Example 2.39. Consider the following numerical values for the parameters: $N = 2$, $r = 0$, $u = 2$, $d = \frac{1}{2}$ and $p = \frac{4}{9}$. The dynamics for the portfolio value are then given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = V_{n-1}(1 + \pi_n \mu_n) = \begin{cases} V_{n-1}(1 + \pi_n), \\ V_{n-1}(1 - \frac{\pi_n}{2}), \end{cases} \tag{2.91}$$

where, as usual, π denotes the ratio of the risky asset held in the portfolio.

On the basis of the DP algorithm, for $v > 0$ we have

$$\begin{aligned}
W_2(v) &= \log v, \\
W_1(v) &= \max_{\bar{\pi}_2 \in]-1,2[} E\left[\log G_2(v, \mu_2; \bar{\pi}_2)\right] \\
&= \log v + \max_{\bar{\pi}_2 \in]-1,2[} \left[\frac{4}{9}\log(1 + \bar{\pi}_2) + \frac{5}{9}\log\left(1 - \frac{\bar{\pi}_2}{2}\right)\right] = \log v + M, \\
W_0(v) &= \max_{\bar{\pi}_1 \in]-1,2[} E\left[W_1\left(G_1(v, \mu_1; \bar{\pi}_1)\right)\right] \\
&= \log v + M + \max_{\bar{\pi}_1 \in]-1,2[} \left[\frac{4}{9}\log(1 + \bar{\pi}_1) + \frac{5}{9}\log\left(1 - \frac{\bar{\pi}_1}{2}\right)\right] \\
&= \log v + 2M,
\end{aligned}$$

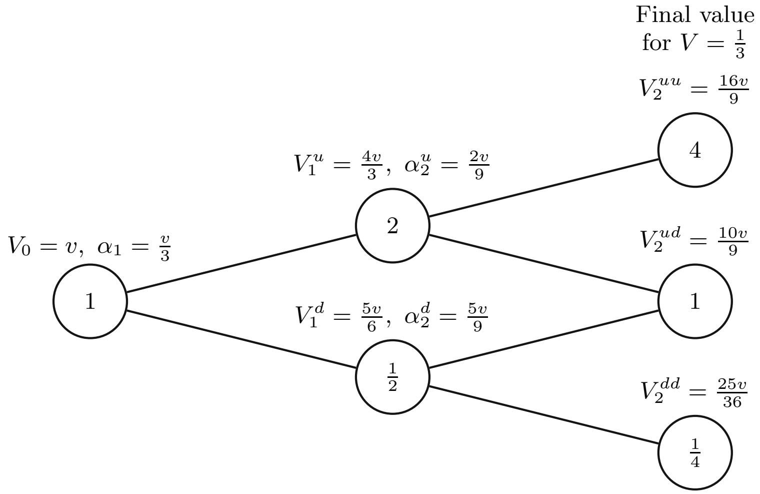

Fig. 2.1. Price of the underlying (inside the circles); optimal value and strategy for logarithmic terminal utility (above the circles)

where

$$M = \max_{\pi \in]-1,2[} \left[\frac{4}{9} \log(1+\pi) + \frac{5}{9} \log\left(1 - \frac{\pi}{2}\right) \right] = \frac{1}{3} \log 2 - \log 3 + \frac{5}{9} \log 5,$$

the maximum being attained in $\bar{\pi} = \frac{1}{3}$.

Summing up, the optimal value of expected utility starting from an initial wealth $v > 0$, is equal to

$$W_0(v) = \log v + 2 \left(\frac{1}{3} \log 2 - \log 3 + \frac{5}{9} \log 5 \right),$$

and the corresponding optimal strategy is constant and equal to

$$\pi_1^{\max}(v) = \pi_2^{\max}(v) = \frac{1}{3}, \qquad v \in \mathbb{R}_+.$$

Using (2.91), in Figure 2.1 we represent the value of the optimal strategy on a binomial tree: we represent also the optimal strategy α computed with the formulae (2.11) from which we can see that, despite of the fact that π is constant, α is not. □

2.4.3 Terminal utility in the completed trinomial model: MG method

We use the martingale method to solve the problem of maximization of expected utility of terminal wealth in the case of the logarithmic utility and in a completed trinomial model. As in the binomial case, by Theorem 2.18 and the expression (2.40) of the Radon-Nikodym derivative of Q with respect to P, we have

$$\begin{aligned}\bar{V}_N(\omega) &= \frac{1}{\lambda \widetilde{L}(\omega)} \\ &= v(1+r)^N \left(\frac{p_1}{q_1}\right)^{\nu_N^1(\omega)} \left(\frac{p_2}{q_2}\right)^{\nu_N^2(\omega)} \left(\frac{p_3}{q_3}\right)^{N-\nu_N^1(\omega)-\nu_N^2(\omega)}, \quad \omega \in \Omega.\end{aligned}$$

The optimal value of the terminal expected utility is

$$\begin{aligned}E\left[\log \bar{V}_N\right] =& \log v + N \log(1+r) + E\left[\nu_N^1\right] \log \frac{p_1}{q_1} + E\left[\nu_N^2\right] \log \frac{p_2}{q_2} \\ &+ \left(N - E\left[\nu_N^1 + \nu_N^2\right]\right) \log \frac{p_3}{q_3} \\ =& \log v + N \left(\log(1+r) + p_1 \log \frac{p_1}{q_1} + p_2 \log \frac{p_2}{q_2} + p_3 \log \frac{p_3}{q_3}\right).\end{aligned} \tag{2.92}$$

The last step consists in determining the optimal strategy as a hedging strategy for the derivative $\bar{V}_N$. We proceed backwards as in the Paragraph 1.4.2 and impose the replication condition for the last period

$$\alpha_N^1 S_N^1 + \alpha_N^2 S_N^2 + \beta_N B_N = \bar{V}_N. \tag{2.93}$$

Assuming that $S_{N-1}^i = S_0 u_i^{n_1} m_i^{n_2} d^{N-n_1-n_2}$ namely that we are in the case $\nu_{N-1}^j = n_j$ with $n_1 + n_2 < N$, the equation (2.93) becomes equivalent to the following system of equations in the unknowns $\alpha_N^1, \alpha_N^2, \beta_N$

$$\begin{cases} \alpha_N^1 u_1 S_{N-1}^1 + \alpha_N^2 u_2 S_{N-1}^2 + \beta_N B_N = \\ \qquad = v(1+r)^N \left(\frac{p_1}{q_1}\right)^{n_1+1} \left(\frac{p_2}{q_2}\right)^{n_2} \left(\frac{p_3}{q_3}\right)^{N-1-n_1-n_2}, \\ \alpha_N^1 m_1 S_{N-1}^1 + \alpha_N^2 m_2 S_{N-1}^2 + \beta_N B_N = \\ \qquad = v(1+r)^N \left(\frac{p_1}{q_1}\right)^{n_1} \left(\frac{p_2}{q_2}\right)^{n_2+1} \left(\frac{p_3}{q_3}\right)^{N-1-n_1-n_2}, \\ \alpha_N^1 d_1 S_{N-1}^1 + \alpha_N^2 d_2 S_{N-1}^2 + \beta_N B_N = \\ \qquad = v(1+r)^N \left(\frac{p_1}{q_1}\right)^{n_1} \left(\frac{p_2}{q_2}\right)^{n_2} \left(\frac{p_3}{q_3}\right)^{N-n_1-n_2}, \end{cases}$$

from which it is possible to obtain the optimal strategy for the last period. Form the risk neutral valuation formula we have

$$\bar{V}_{N-1} = \frac{1}{1+r} E^Q \left[\bar{V}_N \mid \mathcal{F}_{N-1}\right],$$

from which we obtain the optimal value in $N-1$ as

$$\bar{V}_{N-1} = v(1+r)^{N-1} \left(\frac{p_1}{q_1}\right)^{n_1} \left(\frac{p_2}{q_2}\right)^{n_2} \left(\frac{p_3}{q_3}\right)^{N-1-n_1-n_2}.$$

Observe that, since $\bar{V}_{N-1}$ has an expression similar to $\bar{V}_N$, the calculations in the succeeding steps are formally analogous and, proceeding backwards, it is possible to determine the entire optimal strategy.

2.4.4 Terminal utility in the completed trinomial model: DP method

We use the general notations of Paragraph 1.4.2 and we recall Example 2.33: the dynamics of the portfolio value are given by

$$\begin{aligned} V_n &= G_n(V_{n-1}, \mu_n; \pi_n) \\ &= \begin{cases} V_{n-1}\left(1 + r + \pi_n^1 (u_1 - 1 - r) + \pi_n^2 (u_2 - 1 - r)\right), \\ V_{n-1}\left(1 + r + \pi_n^1 (m_1 - 1 - r) + \pi_n^2 (m_2 - 1 - r)\right), \\ V_{n-1}\left(1 + r + \pi_n^1 (d_1 - 1 - r) + \pi_n^2 (d_2 - 1 - r)\right), \end{cases} \end{aligned} \tag{2.94}$$

where $\pi = (\pi^1, \pi^2)$ is the vector of the ratios of the risky assets held in the portfolio and which constitutes the control process.

Observe that

$$E\left[\log G_n(v, \mu_n; \pi_n^1, \pi_n^2)\right] = \log v + f(\pi_n^1, \pi_n^2),$$

where f is the function

$$\begin{aligned} f(\pi^1, \pi^2) = &p_1 \log\left(1 + r + \pi^1 (u_1 - 1 - r) + \pi^2 (u_2 - 1 - r)\right) \\ &+ p_2 \log\left(1 + r + \pi^1 (m_1 - 1 - r) + \pi^2 (m_2 - 1 - r)\right) \\ &+ p_3 \log\left(1 + r + \pi^1 (d_1 - 1 - r) + \pi^2 (d_2 - 1 - r)\right), \end{aligned} \tag{2.95}$$

defined on the set D of the values of (π^1, π^2) such that the arguments of the logarithmic functions in (2.95) are positive.

Assume that the function f admits a global maximum on D (this has to be verified case by case, see Example 2.40) and put

$$M = \max_D f.$$

On the basis of the DP algorithm we have

$$\begin{aligned} W_N(v) &= \log v, \\ W_{N-1}(v) &= \max_{\bar{\pi}_N \in D} E\left[\log G_N(v, \mu_N; \bar{\pi}_N)\right] = \log v + M, \end{aligned}$$

and at the generic step n it holds that $W_{N-n}(v) = \log v + nM$. Consequently, the optimal value of the expected utility, starting from an initial capital v is equal to

$$W_0(v) = \log v + NM,$$

and the corresponding optimal strategy is constant and equal to

$$\pi_n^{\max}(v) \equiv \left(\bar{\pi}^1, \bar{\pi}^2\right), \qquad v \in \mathbb{R}_+, \ n = 1, \dots, N,$$

where $\left(\bar{\pi}^1, \bar{\pi}^2\right)$ is a maximizer of the function f in (2.95).

Example 2.40. Consider the following numerical values for the parameters:

$$u_1 = 2, \ m_1 = 1, \ d_1 = \frac{1}{2}, \ u_2 = \frac{8}{3}, \ m_2 = \frac{8}{9}, \ d_2 = \frac{1}{3}, \ r = 0,$$

and $p_1 = p_2 = \frac{1}{3}$. By (2.94) the dynamics of the portfolio value are then given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = \begin{cases} V_{n-1}\left(1 + \pi_n^1 + \frac{5\pi_n^2}{3}\right) \\ V_{n-1}\left(1 - \frac{\pi_n^2}{9}\right) \\ V_{n-1}\left(1 - \frac{\pi_n^1}{2} - \frac{2\pi_n^2}{3}\right) \end{cases}$$

and we have

$$E\left[\log G_n(v, \mu_n; \pi_n)\right] = \log v + f(\pi_n),$$

where f is the function in (2.95) which in this case is

$$\begin{aligned} f(\pi^1, \pi^2) = \frac{1}{3}\log\left(1 + \pi^1 + \frac{5\pi^2}{3}\right) + \frac{1}{3}\log\left(1 - \frac{\pi^2}{9}\right) \\ + \frac{1}{3}\log\left(1 - \frac{\pi^1}{2} - \frac{2\pi^2}{3}\right). \end{aligned} \tag{2.96}$$

Observe that the domain D of f is bounded. In fact, imposing that the arguments of the logarithms in (2.96) are positive, we obtain the conditions $\pi^2 < 9$ and

$$\pi^1 + \frac{5\pi^2}{3} > -1, \qquad -\pi^1 - \frac{4\pi^2}{3} > -2. \tag{2.97}$$

Adding the two inequalities in (2.97) we obtain $\pi^2 > -9$ and from the boundedness of π^2 it follows easily that also π^1 is bounded.

To determine the maximum of f, compute its partial derivatives:

$$\begin{aligned} \partial_{\pi^1} f(\pi^1, \pi^2) &= \frac{1}{3\left(1 + \pi^1 + \frac{5\pi^2}{3}\right)} - \frac{1}{6\left(1 - \frac{\pi^1}{2} - \frac{2\pi^2}{3}\right)}, \\ \partial_{\pi^2} f(\pi^1, \pi^2) &= \frac{5}{9\left(1 + \pi^1 + \frac{5\pi^2}{3}\right)} - \frac{1}{27\left(1 - \frac{\pi^2}{9}\right)} - \frac{2}{9\left(1 - \frac{\pi^1}{2} - \frac{2\pi^2}{3}\right)}. \end{aligned}$$

The gradient of f vanishes in

$$\bar{\pi}^1 = -4, \qquad \bar{\pi}^2 = 3 \tag{2.98}$$

and these values belong to D.

Since D is bounded, this critical point is also a global maximizer of f and we have

$$M = \max f = f(-4,3) = \frac{1}{3}\log\frac{4}{3}.$$

In conclusion, the optimal value of the expected utility starting from an initial capital $v > 0$, is equal to

$$W_0(v) = \log v + NM,$$

which coincides with the optimal value obtained by the martingale method in (2.92). With the data of this example we have in fact

$$\log(1+r) + p_1 \log\frac{p_1}{q_1} + p_2 \log\frac{p_2}{q_2} + p_3 \log\frac{p_3}{q_3} = \frac{1}{3}\log\frac{4}{3}.$$

The corresponding optimal (constant) strategy is given by (2.98). □

2.4.5 Terminal utility in the standard trinomial model: DP method

Consider the problem of maximization of expected utility from terminal wealth for a logarithmic utility function in a standard trinomial model of N periods (cf. Paragraph 1.4.2), with parameters u, m, d and short rate r. For this case of an incomplete market we use only the DP method.

Following the Example 2.33, the dynamics of the portfolio value are given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = \begin{cases} V_{n-1}\left(1 + r + \pi_n(u - 1 - r)\right) & \text{if } \mu_n = u - 1, \\ V_{n-1}\left(1 + r + \pi_n(m - 1 - r)\right) & \text{if } \mu_n = m - 1, \\ V_{n-1}\left(1 + r + \pi_n(d - 1 - r)\right) & \text{if } \mu_n = d - 1, \end{cases}$$

where π denotes the ratio of the risky asset held in the portfolio and constitutes the control process. Starting from $V_{n-1} > 0$ we have $V_n > 0$ if and only if

$$\begin{cases} 1 + r + \pi_n(u - 1 - r) > 0 \\ 1 + r + \pi_n(m - 1 - r) > 0 \\ 1 + r + \pi_n(d - 1 - r) > 0 \end{cases}$$

or, assuming $d < 1 + r < u$, if equivalently

$$\pi_n \in D =]a, b[\qquad \text{where} \qquad a = -\frac{1+r}{u-1-r}, \quad b = \frac{1+r}{1+r-d}. \tag{2.99}$$

On the basis of the DP algorithm we have

$$\begin{aligned} W_N(v) &= \log v, \\ W_{N-1}(v) &= \max_{\bar{\pi}_N \in D} E\left[\log G_N(v, \mu_N; \bar{\pi}_N)\right] = \log v + \max_D f, \end{aligned}$$

where

$$f(\pi) = p_1 \log(1 + r + (u - 1 - r)\pi) + p_2 \log(1 + r + (m - 1 - r)\pi) \\ + (1 - p_1 - p_2) \log (1 + r + (d - 1 - r)\pi) .$$

In the next step we have

$$W_{N-2}(v) = \max_{\bar{\pi}_{N-1} \in D} E [\log G_{N-1}(v, \mu_{N-1}; \bar{\pi}_{N-1})] + \max_D f = \log v + 2 \max_D f,$$

and at the generic step n

$$W_{N-n}(v) = \log v + n \max_D f.$$

Summing up, the optimal value of expected utility starting from an initial wealth $v > 0$, is equal to

$$W_0(v) = \log v + N \max_D f$$

and the corresponding optimal strategy is constant and given by the maximizer of the function f.

Example 2.41. Consider the following numerical values for the parameters:

$$u = 2, \quad m = \frac{5}{4}, \quad d = \frac{1}{2}, \quad r = 0, \quad p_1 = p_2 = \frac{1}{3}.$$

The dynamics of the portfolio value are then given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = V_{n-1} (1 + \pi_n \mu_n) = \begin{cases} V_{n-1} (1 + \pi_n) \\ V_{n-1} \left(1 + \frac{\pi_n}{4}\right) \\ V_{n-1} \left(1 - \frac{\pi_n}{2}\right) \end{cases} \tag{2.100}$$

where π denotes the ratio of the risky asset held in the portfolio.

Observe that we have

$$E [\log G_n(v, \mu_n; \pi_n)] = \log v + f(\pi_n),$$

where f is the function

$$f(\pi) = \frac{1}{3} \log(1 + \pi) + \frac{1}{3} \log \left(1 + \frac{\pi}{4}\right) + \frac{1}{3} \log \left(1 - \frac{\pi}{2}\right), \tag{2.101}$$

defined for $\pi \in D :=]-1, 2[$ and of which the graph is represented in Figure 2.2. Furthermore put $M = \max_{]-1,2[} f$ and observe that, since

$$f'(\pi) = -\frac{1}{6 \left(1 - \frac{\pi}{2}\right)} + \frac{1}{12 \left(1 + \frac{\pi}{4}\right)} + \frac{1}{3(1 + \pi)},$$

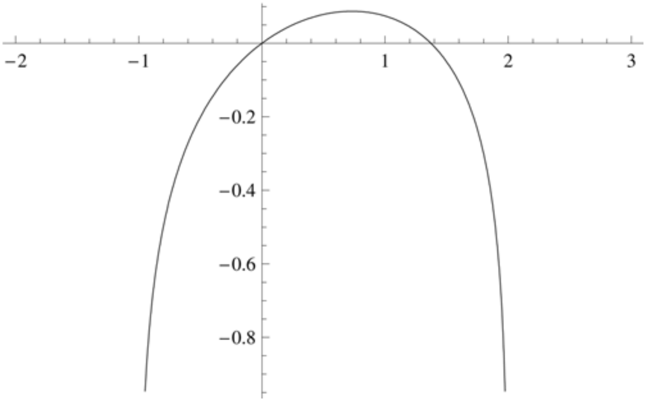

Fig. 2.2. Graph of the function f in (2.101)

then $f'(\pi) = 0$ for $\pi = -1 + \sqrt{3}$ and so

$$\begin{aligned} M &= f(-1+\sqrt{3}) \\ &= \frac{\log 3}{6} + \frac{1}{3}\log\left(1 + \frac{1}{2}\left(1-\sqrt{3}\right)\right) + \frac{1}{3}\log\left(1 - \frac{1}{4}\left(1-\sqrt{3}\right)\right). \end{aligned} \tag{2.102}$$

On the basis of the DP algorithm we have

$$\begin{aligned} W_N(v) &= \log v, \\ W_{N-1}(v) &= \max_{\bar{\pi}_N \in]-1,2[} E\left[\log G_N(v, \mu_N; \bar{\pi}_N)\right] = \log v + \max_{]-1,2[} f = \log v + M, \\ &\vdots \\ W_{N-n}(v) &= \max_{\bar{\pi}_{N-n+1} \in]-1,2[} E\left[\log G_n(v, \mu_n; \bar{\pi}_{N-n+1})\right] + (n-1)M \\ &= \log v + nM. \end{aligned}$$

Summing up, the optimal value of expected utility starting from an initial wealth $v > 0$, is equal to

$$W_0(v) = \log v + NM,$$

with M as in (2.102) and the corresponding optimal strategy is constant and equal to

$$\pi_n^{\max}(v) \equiv -1 + \sqrt{3}, \qquad v \in \mathbb{R}_+,\ n = 1, \dots, N.$$

Using (2.100), in Figure 2.3 we represent the optimal value and the optimal strategy on a trinomial tree. Furthermore, we also represent the optimal strategy α computed with the formulae (2.11). □

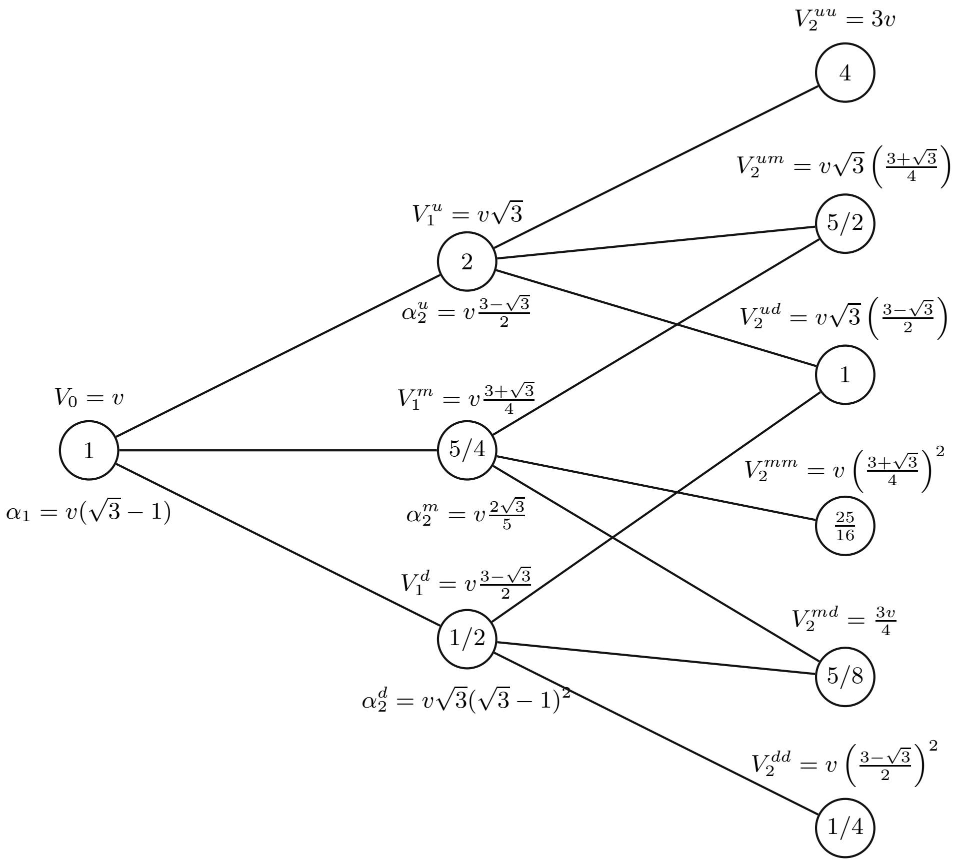

Fig. 2.3. Price of the underlying (inside the circles), optimal value (above the circles) and optimal strategy (below the circles) for logarithmic terminal utility

2.4.6 Intermediate consumption in the binomial model: MG method

We use the martingale method to solve the problem of maximization of expected utility from intermediate consumption in the case of a logarithmic utility in the binomial model. Recall that, in the case of logarithmic utility, $\mathcal{I}(w) = \frac{1}{w}$ and putting $L_n = E\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right]$ for the process of the Radon-Nikodym derivative of Q with respect to P as well as $\widetilde{L}_n = B_n^{-1} L_n$, by Theorem 2.24 we have

$$\bar{C}_n = \mathcal{I}_n\left(\lambda \widetilde{L}_n\right) = \frac{1}{\lambda \widetilde{L}_n}, \qquad n = 0, \dots, N,$$

where λ is determined by (2.55):

$$v = E^P\left[\sum_{n=0}^{N}\widetilde{L}_n\mathcal{I}_n\left(\lambda\widetilde{L}_n\right)\right] = E^P\left[\sum_{n=0}^{N}\frac{1}{\lambda}\right] = \frac{N+1}{\lambda}.$$

Therefore, recalling the expression of L_n in (2.38)

$$L_n = \left(\frac{q}{p}\right)^{\nu_n}\left(\frac{1-q}{1-p}\right)^{n-\nu_n},$$

we obtain the following expression for the optimal consumption:

$$\bar{C}_n(\omega) = \frac{v(1+r)^n}{N+1}\left(\frac{p}{q}\right)^{\nu_n(\omega)}\left(\frac{1-p}{1-q}\right)^{n-\nu_n(\omega)}, \qquad \omega\in\Omega. \tag{2.103}$$

We can now compute the optimal value of the expected utility:

$$\begin{aligned}
&E\left[\sum_{n=0}^{N}\log\bar{C}_n\right]\\
&=\sum_{n=0}^{N}\left(\log\left(\frac{v(1+r)^n}{N+1}\right)+E\left[\nu_n\right]\log\frac{p}{q}+\left(n-E\left[\nu_n\right]\right)\log\frac{1-p}{1-q}\right)\\
&=(N+1)\log\frac{v}{N+1}+\sum_{n=0}^{N}\left(\log(1+r)^n+np\log\frac{p}{q}+n\left(1-p\right)\log\frac{1-p}{1-q}\right)\\
&=(N+1)\left(\log\frac{v}{N+1}+\frac{N}{2}\left(\log(1+r)+p\log\frac{p}{q}+(1-p)\log\frac{1-p}{1-q}\right)\right).
\end{aligned}$$

The last step consists in determining the optimal investment strategy following the procedure of Step **P3** in Section 2.2.3: impose the replication condition for the last step

$$\alpha_N S_N + \beta_N B_N = \bar{C}_N,$$

which is equivalent to the system of equations

$$\begin{cases}\alpha_N u S_{N-1}+\beta_N B_N = \frac{v(1+r)^N}{N+1}\left(\frac{p}{q}\right)^{\nu_{N-1}+1}\left(\frac{1-p}{1-q}\right)^{N-1-\nu_{N-1}},\\ \alpha_N d S_{N-1}+\beta_N B_N = \frac{v(1+r)^N}{N+1}\left(\frac{p}{q}\right)^{\nu_{N-1}}\left(\frac{1-p}{1-q}\right)^{N-\nu_{N-1}},\end{cases}$$

and from which

$$\begin{aligned}
\alpha_N &= \frac{v(1+r)^N}{(N+1)(u-d)S_{N-1}}\left(\frac{p}{q}-\frac{1-p}{1-q}\right)\left(\frac{p}{q}\right)^{\nu_{N-1}}\left(\frac{p-1}{q-1}\right)^{N-1-\nu_{N-1}}\\
&= \frac{v(1+r)^N}{(N+1)(u-d)S_{N-1}}\left(\frac{p}{q}-\frac{1-p}{1-q}\right)L_{N-1}^{-1}.
\end{aligned}$$

In the generic period $[t_{n-1}, t_n]$ impose the replication condition (2.60) namely

$$\alpha_n S_n + \beta_n B_n = \bar{V}_n \tag{2.104}$$

for which we need the value of $\bar{V}_n$. Here too we find a relation equal to (2.63) namely

$$\bar{V}_n = (N + 1 - n)\,\bar{C}_n \tag{2.105}$$

which may be proved by backwards induction on n using the recursive relation (2.61). In fact, for $n = N$ relation (2.105) is true by the replication condition $\bar{V}_N = \bar{C}_N$. Assuming (2.105) true for $n+1$, from (2.61) and (2.103) we obtain

$$\begin{aligned}
\bar{V}_n &= \frac{1}{1+r} E^Q\left[(N-n)\bar{C}_{n+1} \mid \mathcal{F}_{N-1}\right] + \bar{C}_n \\
&= (N-n)\frac{v(1+r)^n}{N+1} \\
&\quad \cdot \left(q\left(\frac{p}{q}\right)^{\nu_n+1}\left(\frac{1-p}{1-q}\right)^{n-\nu_n} + (1-q)\left(\frac{p}{q}\right)^{\nu_n}\left(\frac{1-p}{1-q}\right)^{n+1-\nu_n}\right) + \bar{C}_n \\
&= (N-n)\frac{v(1+r)^n}{N+1}\left(\frac{p}{q}\right)^{\nu_n}\left(\frac{1-p}{1-q}\right)^{n-\nu_n}\left(q\,\frac{p}{q} + (1-q)\,\frac{1-p}{1-q}\right) + \bar{C}_n \\
&= (N+1-n) + \bar{C}_n.
\end{aligned}$$

Always by (2.103) the relation (2.105) becomes

$$\bar{V}_n = \frac{v(1+r)^n(N+1-n)}{N+1}\left(\frac{p}{q}\right)^{\nu_n}\left(\frac{1-p}{1-q}\right)^{n-\nu_n}. \tag{2.106}$$

Imposing then the replication condition (2.104), a simple computation shows that

$$\begin{aligned}
\alpha_n &= \frac{v(1+r)^n(N+1-n)}{(N+1)(u-d)S_{n-1}}\left(\frac{p}{q} - \frac{1-p}{1-q}\right)\left(\frac{p}{q}\right)^{\nu_{n-1}}\left(\frac{p-1}{q-1}\right)^{n-1-\nu_{n-1}} \\
&= \frac{v(1+r)^n(N+1-n)}{(N+1)(u-d)S_{n-1}}\left(\frac{p}{q} - \frac{1-p}{1-q}\right)L_{n-1}^{-1}.
\end{aligned} \tag{2.107}$$

Notice finally that the ratio invested in the risky asset is

$$\pi_n = \frac{\alpha_n S_{n-1}}{\bar{V}_{n-1}} = \frac{(N+1-n)(p-q)(1+r)}{(N+2-n)(1-q)q(u-d)}, \qquad n = 1, \ldots, N \tag{2.108}$$

and it depends thus on the period, but is independent of the state of the system.

Example 2.42. Consider the following numerical values for the parameters: $N = 2$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $r = 0$ and $p = \frac{4}{9}$. Then, on the basis of (2.103) the optimal consumption process is given by

$$\bar{C}_n = \frac{v 2^{3\nu_n - n} 5^{n-\nu_n}}{3^{n+1}}, \qquad n = 0, 1, 2,$$

and this is in accordance with what will be obtained in Example 2.43 by the DP method.

On the basis of (2.107) the optimal strategy is given by

$$\alpha_n = \frac{2^{3\nu_{n-1}-1} 5^{1-\nu_{n-1}} v}{27 S_{n-1}}, \qquad \beta_n = \frac{5^{1-\nu_{n-1}} 8^{\nu_{n-1}} v}{27}, \qquad n = 1, 2$$

and this corresponds to what will be shown in Figure 2.5 in the case of the DP method. □

2.4.7 Intermediate consumption in the binomial model: DP method

We use now the DP method to solve the problem of maximization of expected utility from intermediate consumption in the case of a logarithmic utility in a binomial model with N periods, where the asset price's up and down movements are characterized by the parameters u and d respectively, the riskless interest rate is r and the probability for an up-move of the price is p.

By (2.10) the dynamics of the portfolio value are given by

$$\begin{aligned} V_n &= G_n(V_{n-1}, \mu_n; \pi_n, C_{n-1}) \\ &= (V_{n-1} - C_{n-1})(1 + r) + V_{n-1}\pi_n(\mu_n - r) \\ &= (V_{n-1} - C_{n-1})(1 + r) + \begin{cases} V_{n-1}\pi_n(u - 1 - r) \\ V_{n-1}\pi_n(d - 1 - r), \end{cases} \end{aligned}$$

where π denotes the ratio of the risky asset held in the portfolio and C is the consumption process.

Contrary to what we did in the examples with terminal utility, here we do not impose on π and C to guarantee that $V_n \geq 0$. This would imply an additional constraint in the optimization problem and thus introduce some complications in the calculations. We shall rather limit ourselves to verify case by case whether $V_n \geq 0$ as in the Example 2.43 that follows.

Following Example 2.35, with the choice of the utility functions

$$u_n(C) = \log C \quad \text{per} \quad n = 0, \dots, N \quad \text{and} \quad u(C) \equiv 0,$$

on the basis of the DP algorithm (2.82) we have, for $v > 0$,

$$W_N(v) = \max_{\bar{C}_N \leq v} u_N(\bar{C}_N) = \log v,$$

and, for $n = N, \dots, 1$,

$$\begin{aligned} W_{n-1}(v) &= \max_{\bar{\pi}_n, \bar{C}_{n-1}} \left(\log \bar{C}_{n-1} + E\left[W_n \left(G_n \left(v, \mu_n; \bar{\pi}_n, \bar{C}_{n-1} \right) \right) \right] \right) \\ &= \max_{\bar{\pi}_n, \bar{C}_{n-1}} f_{n,v} \left(\bar{\pi}_n, \bar{C}_{n-1} \right), \end{aligned}$$

where

$$\begin{aligned} f_{n,v}(\pi, C) = \log C &+ pW_n\Big((v - C)(1 + r) + \pi v(u - 1 - r) \Big) \\ &+ (1 - p) W_n \Big((v - C)(1 + r) + \pi v \, (d - 1 - r) \Big). \end{aligned}$$

Assuming that $f_{n,v}$ admits a global maximizer in $\left(\bar{\pi}_{n,v}, \bar{C}_{n-1,v}\right)$, this determines the optimal strategy[6]:

$$\pi_n^{\max}(v) = \bar{\pi}_{n,v}, \qquad C_{n-1}^{\max}(v) = \bar{C}_{n-1,v}, \qquad v \in \mathbb{R}_+, \ n = 1, \dots, N.$$

Example 2.43. Consider the following numerical values for the parameters: $N = 2$, $S_0 = 1$, $u = 2$, $d = \frac{1}{2}$, $r = 0$ and $p = \frac{4}{9}$. The dynamics of the portfolio value are then

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n, C_{n-1}) = \begin{cases} V_{n-1}(1 + \pi_n) - C_{n-1}, \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) - C_{n-1}. \end{cases}$$

On the basis of the DP algorithm we have

$$\begin{aligned} W_2(v) &= \log v, \\ W_1(v) &= \max_{\bar{\pi}_2, \bar{C}_1} \left(\log \bar{C}_1 + E\left[\log G_2(v, \mu_2; \bar{\pi}_2, \bar{C}_1) \right] \right) \\ &= \max_{\bar{\pi}_2, \bar{C}_1} \left(\log \bar{C}_1 + \frac{4}{9} \log \left(v \left(1 + \bar{\pi}_2 \right) - \bar{C}_1 \right) + \frac{5}{9} \log \left(v \left(1 - \frac{\bar{\pi}_2}{2} \right) - \bar{C}_1 \right) \right) \\ &= 2 \log v + \frac{4}{9} \log \frac{8}{5} - \log \frac{24}{5}, \end{aligned} \tag{2.109}$$

the maximum being attained in

$$\bar{\pi}_{2,v} = \frac{1}{6}, \qquad \bar{C}_{1,v} = \frac{v}{2},$$

which is the only critical point of the function

$$f_{2,v}(\pi, C) = \log C + \frac{4}{9} \log \left(v \left(1 + \pi \right) - C \right) + \frac{5}{9} \log \left(v \left(1 - \frac{\pi}{2} \right) - C \right), \tag{2.110}$$

that is represented in Figure 2.4 in the case of $v = 1$. Notice that $\bar{C}_{1,v} > 0$ which is thus in the domain of the function $u(C) = \log C$. In Figure 2.4 we also represent the region of π and C in which the portfolio value is positive.

[6] It holds, furthermore, that $C_N^{\max}(v) = v$.

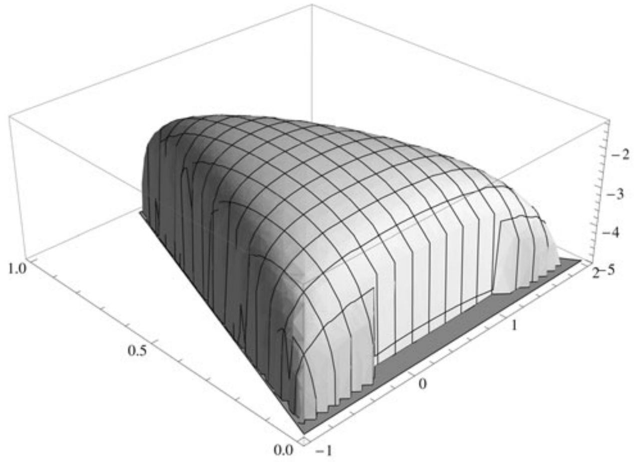

Fig. 2.4. Graph of the function $f_{2,v}$ in (2.110) with $v = 1$, for $\pi \in]-1, 2[$ and $C \in]0, 1[$

Compute now the optimal initial expected utility: recalling the expression of W_1 in (2.109), we have

$$
\begin{aligned}
W_0(v) &= \max_{\bar{\pi}_1, \bar{C}_0} \left(\log \bar{C}_0 + E\left[W_1\left(v\left(1 + \bar{\pi}_1 \mu_1\right) - \bar{C}_0 \right) \right] \right) \\
&= \max_{\bar{\pi}_1, \bar{C}_0} \Bigg(\log \bar{C}_0 + \frac{4}{9} W_1\left(v\left(1 + \bar{\pi}_1 (u-1)\right) - \bar{C}_0 \right) \\
&\qquad + \frac{5}{9} W_1\left(v\left(1 + \bar{\pi}_1 (d-1)\right) - \bar{C}_0 \right) \Bigg) \\
&= 3 \log v + \frac{2}{3} \log 5 + \log \frac{10}{729},
\end{aligned}
$$

the maximum being attained in

$$
\bar{\pi}_{1,v} = \frac{2}{9}, \qquad \bar{C}_{0,v} = \frac{v}{3},
$$

which results by setting equal to zero the gradient of the function to be maximized. Here too we have $\bar{C}_{0,v} > 0$ which is thus in the domain of the function $u(C) = \log C$. The optimal strategy is therefore given by

$$
\begin{aligned}
&\pi_1^{\max}(v) = \frac{2}{9}, \qquad \pi_2^{\max}(v) = \frac{1}{6}, \\
&C_0^{\max}(v) = \frac{v}{3}, \qquad C_1^{\max}(v) = \frac{v}{2}, \qquad C_2^{\max}(v) = v.
\end{aligned}
$$

Notice that the strategy $\pi^{\max}$ that was obtained here coincides, for the data of this Example, among which $N = 2, r = 0, q = \frac{1}{3}$, with the one obtained by the MG method in (2.108). With these specific data, (2.108) becomes in fact

$$\pi_n = \frac{3-n}{3(4-n)}$$

and therefore we have $\pi_1 = \frac{2}{9}$, $\pi_2 = \frac{1}{6}$. For what concerns the consumption, from the expression (2.105) that was obtained by the martingale method, we obtain

$$\bar{C}_n = \frac{\bar{V}_n}{N+1-n}$$

and therefore $\bar{C}_0 = \frac{\bar{V}_0}{3}$, $\bar{C}_1 = \frac{\bar{V}_1}{2}$, $\bar{C}_2 = \bar{V}_2$.

In Figure 2.5 we represent on a binomial tree the value of the optimal strategy as defined recursively by $V_0 = v$ and

$$V_n = V_{n-1} - C_{n-1}^{\max}(V_{n-1}) + \begin{cases} V_{n-1}\pi_n^{\max}(V_{n-1}) \\ -\frac{V_{n-1}\pi_n^{\max}(V_{n-1})}{2}, \end{cases}$$

where the last term represents the values in case of a price increase or decrease respectively. Furthermore, using formulae (2.11), we obtain for the optimal strategy α:

$$\alpha_n = \frac{\pi_n^{\max} V_{n-1}}{S_{n-1}}, \qquad n = 1, 2.$$

As we may verify directly by observing the binomial tree, the values V_n of the optimal strategy are positive for all n. □

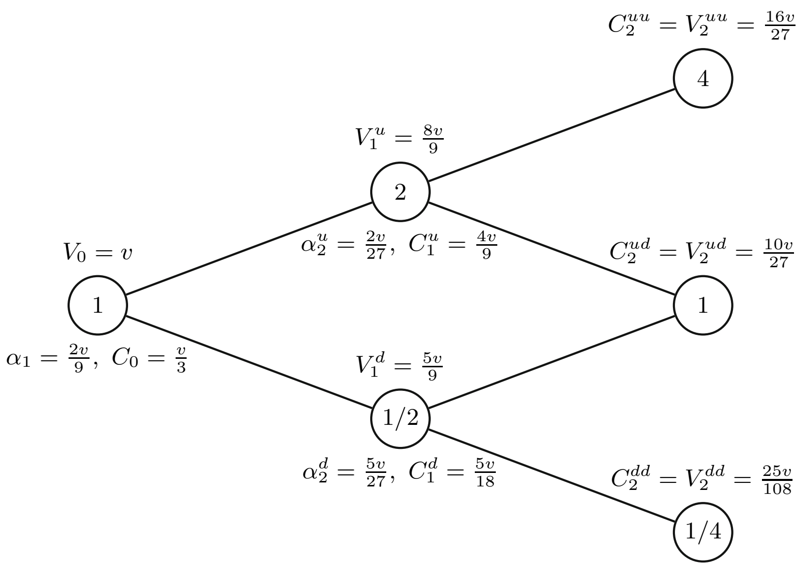

Fig. 2.5. Price of the underlying (inside the circles), optimal value (above the circles) and optimal strategy with consumption (below the circles) for logarithmic utility

2.4.8 Intermediate consumption in the completed trinomial model: MG method

As in the binomial case we have

$$\bar{C}_n = \mathcal{I}_n\left(B_n^{-1}\lambda L_n\right) = \frac{(1+r)^n}{\lambda L_n}, \qquad n = 0, \dots, N,$$

where λ is determined by (2.55), namely $\lambda = \frac{N+1}{v}$. Thus, recalling the expression of L_n in (2.41)

$$L_n = \left(\frac{q_1}{p_1}\right)^{\nu_n^1} \left(\frac{q_2}{p_2}\right)^{\nu_n^2} \left(\frac{q_3}{p_3}\right)^{n-\nu_n^1-\nu_n^2},$$

we obtain the following expression for the optimal consumption:

$$\bar{C}_n = \frac{v(1+r)^n}{N+1} \left(\frac{p_1}{q_1}\right)^{\nu_n^1} \left(\frac{p_2}{q_2}\right)^{\nu_n^2} \left(\frac{p_3}{q_3}\right)^{n-\nu_n^1-\nu_n^2}. \tag{2.111}$$

For what concerns the optimal investment strategy, following the procedure of Step **P3** in Section 2.2.3, from the replication condition in the last period

$$\alpha_N^1 S_N^1 + \alpha_N^2 S_N^2 + \beta_N B_N = \bar{C}_N,$$

we obtain the system of equations

$$\begin{cases} \alpha_N^1 u_1 S_{N-1}^1 + \alpha_N^2 u_2 S_{N-1}^2 + \beta_N B_N \\ \qquad = \frac{v(1+r)^N}{N+1} \left(\frac{p_1}{q_1}\right)^{\nu_{N-1}^1+1} \left(\frac{p_2}{q_2}\right)^{\nu_{N-1}^2} \left(\frac{p_3}{q_3}\right)^{N-1-\nu_{N-1}^1-\nu_{N-1}^2}, \\ \alpha_N^1 m_1 S_{N-1}^1 + \alpha_N^2 m_2 S_{N-1}^2 + \beta_N B_N \\ \qquad = \frac{v(1+r)^N}{N+1} \left(\frac{p_1}{q_1}\right)^{\nu_{N-1}^1} \left(\frac{p_2}{q_2}\right)^{\nu_{N-1}^2+1} \left(\frac{p_3}{q_3}\right)^{N-1-\nu_{N-1}^1-\nu_{N-1}^2}, \\ \alpha_N^1 d_1 S_{N-1}^1 + \alpha_N^2 d_2 S_{N-1}^2 + \beta_N B_N \\ \qquad = \frac{v(1+r)^N}{N+1} \left(\frac{p_1}{q_1}\right)^{\nu_{N-1}^1} \left(\frac{p_2}{q_2}\right)^{\nu_{N-1}^2} \left(\frac{p_3}{q_3}\right)^{N-\nu_{N-1}^1-\nu_{N-1}^2}. \end{cases} \tag{2.112}$$

Using the risk neutral valuation formula (2.8) and recalling that $\bar{V}_N = \bar{C}_N$, compute

$$\begin{aligned}
\bar{V}_{N-1} &= \frac{1}{1+r} E^Q \left[\bar{V}_N \mid \mathcal{F}_{N-1} \right] + \bar{C}_{N-1} \\
&= \frac{v(1+r)^{N-1}}{N+1} \left(q_1 \left(\frac{p_1}{q_1} \right)^{\nu^1_{N-1}+1} \left(\frac{p_2}{q_2} \right)^{\nu^2_{N-1}} \left(\frac{p_3}{q_3} \right)^{N-1-\nu^1_{N-1}-\nu^2_{N-1}} \right. \\
&+ q_2 \left(\frac{p_1}{q_1} \right)^{\nu^1_{N-1}} \left(\frac{p_2}{q_2} \right)^{\nu^2_{N-1}+1} \left(\frac{p_3}{q_3} \right)^{N-1-\nu^1_{N-1}-\nu^2_{N-1}} \\
&+ \left. q_3 \left(\frac{p_1}{q_1} \right)^{\nu^1_{N-1}} \left(\frac{p_2}{q_2} \right)^{\nu^2_{N-1}} \left(\frac{p_3}{q_3} \right)^{N-\nu^1_{N-1}-\nu^2_{N-1}} \right) + \bar{C}_{N-1} \\
&= 2\bar{C}_{N-1}.
\end{aligned}$$

In general we can show by induction that

$$\bar{V}_n = (N+1-n)\bar{C}_n = \frac{v(1+r)^n (N+1-n)}{N+1} \left(\frac{p_1}{q_1} \right)^{\nu^1_n} \left(\frac{p_2}{q_2} \right)^{\nu^2_n} \left(\frac{p_3}{q_3} \right)^{n-\nu^1_n-\nu^2_n}.$$

Therefore, to determine the optimal strategy in the succeeding steps, it suffices to solve a linear system that is formally analogous to (2.112).

2.4.9 Optimal consumption in the completed trinomial model: DP method

Recalling Example 2.35 the dynamics of the portfolio value are given by

$$\begin{aligned}
V_n &= (V_{n-1} - C_{n-1})(1+r) \\
&+ \begin{cases} V_{n-1} \left(\pi^1_n (u_1 - 1 - r) + \pi^2_n (u_2 - 1 - r) \right), \\ V_{n-1} \left(\pi^1_n (m_1 - 1 - r) + \pi^2_n (m_2 - 1 - r) \right), \\ V_{n-1} \left(\pi^1_n (d_1 - 1 - r) + \pi^2_n (d_2 - 1 - r) \right), \end{cases}
\end{aligned} \tag{2.113}$$

where $\pi = (\pi^1, \pi^2)$ is the vector of the ratios of the risky assets held in the portfolio and constitutes the control process.

On the basis of the DP algorithm (2.82) we have, for $v > 0$,

$$W_N(v) = \max_{\bar{C}_N \le v} u_N(\bar{C}_N) = \log v,$$

and, for $n = N, \dots, 1$,

$$\begin{aligned}
W_{n-1}(v) &= \max_{\bar{\pi}_n, \bar{C}_{n-1}} \left(\log \bar{C}_{n-1} + E \left[W_n \left(G_n \left(v, \mu_n; \bar{\pi}_n, \bar{C}_{n-1} \right) \right) \right] \right) \\
&= \max_{\bar{\pi}_n, \bar{C}_{n-1}} f_{n,v} \left(\bar{\pi}_n, \bar{C}_{n-1} \right),
\end{aligned}$$

where

$$
\begin{aligned}
f_{n,v}(\pi, C) = & \log C \\
& + p_1 W_n\Big((v-C)(1+r) + \pi^1 v(u_1 - 1 - r) + \pi^2 v(u_2 - 1 - r)\Big) \\
& + p_2 W_n\Big((v-C)(1+r) + \pi v\,(m - 1 - r) + \pi^2 v(m_2 - 1 - r)\Big) \\
& + p_3 W_n\Big((v-C)(1+r) + \pi v\,(d - 1 - r) + \pi^2 v(d_2 - 1 - r)\Big).
\end{aligned}
$$

Assuming that $f_{n,v}$ admits a global maximum in $\left(\bar{\pi}_{n,v}, \bar{C}_{n-1,v}\right)$, this determines the optimal strategy[7]:

$$\pi_n^{\max}(v) = \bar{\pi}_{n,v}, \qquad C_{n-1}^{\max}(v) = \bar{C}_{n-1,v}, \qquad v \in \mathbb{R}_+,\ n = 1, \dots, N.$$

Also here we verify case by case a posteriori whether the optimal values found for π and C guarantee the positivity of the portfolio value.

2.4.10 Intermediate consumption in the standard trinomial model: DP method

We use only the DP method to solve the problem of maximization of expected utility from intermediate consumption in the case of a logarithmic utility in a standard trinomial model with N periods (cf. Paragraph 1.4.2), parameters u, m, d and short rate r.

By (2.10) (see also Example 2.35) the dynamics of the portfolio value are given by

$$V_n = \left(V_{n-1} - C_{n-1}\right)(1+r) + \begin{cases} V_{n-1}\pi_n(u - 1 - r) \\ V_{n-1}\pi_n(m - 1 - r) \\ V_{n-1}\pi_n(d - 1 - r), \end{cases}$$

where π denotes the ratio of the risky asset held in the portfolio and C is the consumption process. Analogously to the previous example and contrary to what we had done in the examples with terminal utility, here we do not impose on π and C to guarantee that $V_n > 0$. We shall verify it case by case as in the next Example 2.44.

Following Example 2.35, with the choice of the utility functions

$$u_n(C) = \log C \quad \text{for} \quad n = 0, \dots, N \quad \text{and} \quad u(C) \equiv 0,$$

on the basis of the DP algorithm (2.82) we have, for $v > 0$,

$$W_N(v) = \max_{\bar{C}_N \le v} u_N(\bar{C}_N) = \log v,$$

[7] It holds, furthermore, that $C_N^{\max}(v) = v$.

and, for $n = N, \dots, 1$,

$$W_{n-1}(v) = \max_{\bar{\pi}_n, \bar{C}_{n-1}} \left(\log \bar{C}_{n-1} + E\left[W_n \left(G_n \left(v, \mu_n; \bar{\pi}_n, \bar{C}_{n-1} \right) \right) \right] \right)$$
$$= \max_{\bar{\pi}_n, \bar{C}_{n-1}} f_{n,v}\left(\bar{\pi}_n, \bar{C}_{n-1} \right),$$

where

$$\begin{aligned} f_{n,v}(\pi, C) = \log C &+ p_1 W_n\Big((v-C)(1+r) + \pi v(u-1-r) \Big) \\ &+ p_2 W_n\Big((v-C)(1+r) + \pi v\,(m-1-r) \Big) \\ &+ p_3 W_n\Big((v-C)(1+r) + \pi v\,(d-1-r) \Big). \end{aligned}$$

Assuming that $f_{n,v}$ admits a global maximum in $\left(\bar{\pi}_{n,v}, \bar{C}_{n-1,v}\right)$, this defines the optimal strategy[8]:

$$\pi_n^{\max}(v) = \bar{\pi}_{n,v}, \qquad C_{n-1}^{\max}(v) = \bar{C}_{n-1,v}, \qquad v \in \mathbb{R}_+,\ n = 1, \dots, N.$$

Example 2.44. Consider the following numerical values for the parameters: $N = 2$ and, as in Example 2.41, $r = 0$, $u = 2$, $m = \frac{5}{4}$, $d = \frac{1}{2}$ and $p_1 = p_2 = \frac{1}{3}$. The dynamics of the portfolio value are then

$$V_n = \begin{cases} V_{n-1}(1+\pi_n) - C_{n-1}, \\ V_{n-1}\left(1 + \frac{\pi_n}{4}\right) - C_{n-1}, \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) - C_{n-1}. \end{cases} \tag{2.114}$$

On the basis of the DP algorithm we have

$$\begin{aligned} W_2(v) &= \log v, \\ W_1(v) &= \max_{\bar{\pi}_2, \bar{C}_1} \left(\log \bar{C}_1 + \frac{1}{3}\log\left(v\left(1+\bar{\pi}_2\right) - \bar{C}_1 \right) + \frac{1}{3}\log\left(v\left(1 + \frac{\bar{\pi}_2}{4}\right) - \bar{C}_1 \right) \right. \\ &\qquad \left. + \frac{1}{3}\log\left(v\left(1 - \frac{\bar{\pi}_2}{2}\right) - \bar{C}_1 \right) \right) \\ &= \max_{\bar{\pi}_2, \bar{C}_1} f_{2,v}(\bar{\pi}_2, \bar{C}_1), \end{aligned} \tag{2.115}$$

where

$$\begin{aligned} f_{2,v}(\pi, C) = &\log C + \frac{1}{3}\log\left(v\left(1+\pi\right) - C\right) + \frac{1}{3}\log\left(v\left(1 + \frac{\pi}{4}\right) - C \right) \\ &+ \frac{1}{3}\log\left(v\left(1 - \frac{\pi}{2}\right) - C \right). \end{aligned}$$

[8] It holds, furthermore, that $C_N^{\max}(v) = v$.

To determine the maximum, we compute the critical points by solving the system

$$\begin{cases}\partial_\pi f_{2,v}(\pi,C) = \frac{v}{3}\left(\frac{1}{2C+(\pi-2)v} + \frac{1}{4(v-C)+\pi v} + \frac{1}{v(1+\pi)-C}\right) = 0\\ \partial_C f_{2,v}(\pi,C) = \frac{1}{C} + \frac{2}{6C+3(\pi-2)v} + \frac{1}{3C-3v(1-\pi)} + \frac{4}{12C-3(4+\pi)v} = 0.\end{cases}$$

The solutions of this system are the pairs

$$\left(\frac{1}{2}\left(-1-\sqrt{3}\right), \frac{v}{2}\right) \quad \text{and} \quad \left(\frac{1}{2}\left(-1+\sqrt{3}\right), \frac{v}{2}\right).$$

However, only

$$\left(\pi_{2,v}^{\max}, C_{1,v}^{\max}\right) := \left(\frac{1}{2}\left(-1+\sqrt{3}\right), \frac{v}{2}\right)$$

belongs to the domain of $f_{2,v}$. In this point the function $f_{2,v}$ takes its maximum value

$$f_{2,v}\left(\frac{1}{2}\left(-1+\sqrt{3}\right), \frac{v}{2}\right) = 2\log v + \frac{1}{3}\log\frac{3\sqrt{3}}{256}. \tag{2.116}$$

Notice that $C_{1,v}^{\max} > 0$ and so it belongs to the domain of the function $u(C) = \log C$.

Compute now the optimal expected initial utility: recalling the expression of W_1 in (2.115)-(2.116), we have

$$\begin{aligned} W_0(v) &= \max_{\bar{\pi}_1, \bar{C}_0}\left(\log \bar{C}_0 + E\left[W_1\left(G_1\left(v, \mu_1; \bar{\pi}_1, \bar{C}_0\right)\right)\right]\right) \\ &= \max_{\bar{\pi}_1, \bar{C}_0} f_{1,v}\left(\bar{\pi}_1, \bar{C}_0\right), \end{aligned}$$

where

$$\begin{aligned} f_{1,v}(\pi, C) = \log C &+ \frac{1}{3}\log\frac{3\sqrt{3}}{256} + \frac{2}{3}\log\left(v\left(1+\pi\right) - C\right) \\ &+ \frac{2}{3}\log\left(v\left(1+\frac{\pi}{4}\right) - C\right) + \frac{2}{3}\log\left(v\left(1-\frac{\pi}{2}\right) - C\right). \end{aligned}$$

Also in this case there are two points in which the gradient of the function $f_{1,v}$ vanishes, namely

$$\left(\frac{2}{3}\left(-1-\sqrt{3}\right), \frac{v}{3}\right) \quad \text{and} \quad \left(\frac{2}{3}\left(-1+\sqrt{3}\right), \frac{v}{3}\right).$$

However, only

$$\left(\pi_{1,v}^{\max}, C_{0,v}^{\max}\right) := \left(\frac{2}{3}\left(-1+\sqrt{3}\right), \frac{v}{3}\right)$$

belongs to the domain of $f_{1,v}$ and in this point the function $f_{1,v}$ takes its maximum value

$$f_{1,v}\left(\frac{2}{3}\left(-1+\sqrt{3}\right),\frac{v}{3}\right)=5\log v+\log\frac{3\sqrt{3}}{128}.$$

Here too we have $C_{0,v}^{\max}>0$ which is thus in the domain of the function $u(C)=\log C$.

We finally verify that the value of the optimal strategy is positive no matter what the initial capital $v>0$ is. In fact, using formula (2.114) for the dynamics of the portfolio value, and inserting the optimal values $\left(\pi_{1,v}^{\max},C_{0,v}^{\max}\right)$, in the first period we have

$$V_1^u=\frac{2\sqrt{3}}{3}v,\qquad V_1^m=\frac{3+\sqrt{3}}{6}v,\qquad V_1^d=\left(1-\frac{\sqrt{3}}{3}\right)v.$$

Analogously, inserting the optimal values $\left(\pi_{2,v}^{\max},C_{1,v}^{\max}\right)$ in formula (2.114) and denoting by V_1 the (positive) value in the first period, in the second period we have

$$V_2^u=\frac{\sqrt{3}}{2}V_1,\qquad V_2^m=\frac{3+\sqrt{3}}{8}V_1,\qquad V_2^d=\frac{3-\sqrt{3}}{4}V_1.$$

This shows that the process V is positive. $\square$

2.5 Solved problems

Problem 2.45. In a binomial market model over N periods, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0=v$, for a utility function of the form

$$u(v)=-\frac{1}{v},\qquad v>0.$$

i) Using the martingale method, show that the terminal value of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_N=\frac{v}{\sqrt{\widetilde{L}}E\left[\sqrt{\widetilde{L}}\right]},\tag{2.117}$$

where $\widetilde{L}=B_N^{-1}L$ and $L=\frac{dQ}{dP}$ is the Radon-Nikodym derivative of the martingale measure Q with respect to the physical measure P;

ii) for the case of a single period, i.e. $N=1$, compute the optimal strategy and prove that the optimal ratio (proportion) of wealth to invest in the risky asset is given by

$$\pi_1^{\max}(v)=\frac{1+r}{(u-d)E\left[\sqrt{L}\right]}\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{1-p}{1-q}}\right).\tag{2.118}$$

Solution of Problem 2.45
i) We recall the expression of the Radon-Nikodym derivative (cf. Example 2.19)

$$L = \frac{dQ}{dP} = \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N}, \tag{2.119}$$

where ν_N is the random number of up-movements of the risky asset.

Since $u'(v) = \frac{1}{v^2}$, we have $\mathcal{I}(w) = \frac{1}{\sqrt{w}}$. Then by Theorem 2.18 we have that the optimal terminal value is equal to

$$\bar{V}_N = \frac{1}{\sqrt{\lambda \widetilde{L}}}, \tag{2.120}$$

where λ is determined by the budget equation

$$E^P\left[\mathcal{I}\left(\lambda\widetilde{L}\right)\widetilde{L}\right] = \frac{E^P\left[\sqrt{\widetilde{L}}\right]}{\sqrt{\lambda}} = v, \quad \text{that is} \quad \sqrt{\lambda} = \frac{E^P\left[\sqrt{\widetilde{L}}\right]}{v}.$$

Substituting the expression for λ in (2.120) we get (2.117). We observe that, by (2.119), we have

$$\begin{aligned} E\left[\sqrt{L}\right] &= \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \left(\frac{q}{p}\right)^{\frac{k}{2}} \left(\frac{1-q}{1-p}\right)^{\frac{N-k}{2}} \\ &= \sum_{k=0}^{N} \binom{N}{k} (pq)^{\frac{k}{2}} \left((1-p)(1-q)\right)^{\frac{N-k}{2}}. \end{aligned}$$

ii) In the case of $N = 1$, L takes only the values $\frac{q}{p}$ and $\frac{1-q}{1-p}$, in case of increase and decrease of the underlying respectively. To determine the optimal strategy, we impose the replication condition

$$\alpha_1 S_1 + \beta_1(1+r) = \bar{V}_1$$

which is equivalent to the following system of equations in the unknowns α_1, β_1:

$$\begin{aligned} \alpha_1 u S_0 + \beta_1(1+r) &= \frac{v(1+r)}{E\left[\sqrt{L}\right]}\sqrt{\frac{p}{q}}, \\ \alpha_1 d S_0 + \beta_1(1+r) &= \frac{v(1+r)}{E\left[\sqrt{L}\right]}\sqrt{\frac{1-p}{1-q}}. \end{aligned}$$

We obtain as solutions

$$\bar{\alpha}_1(v) = \frac{v(1+r)}{S_0(u-d)E\left[\sqrt{L}\right]}\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{1-p}{1-q}}\right)$$

from which (2.118) follows. □

Problem 2.46. In a binomial market model with $u = 2$, $d = 1/2$ and $r = 0$, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for a utility function of the form

$$u(v) = 1 - \frac{1}{v}, \qquad v > 0.$$

i) On the basis of the DP algorithm, show by induction that the optimal expected value is of the form

$$W_n(v) = 1 - \frac{M^{N-n}}{v}, \tag{2.121}$$

for a suitable value M dependent on the probability p of an up-movement;

ii) determine the optimal strategy $\pi_n^{\max}$, verifying that it is the same for each n, and determine the values of p such that $\pi_n^{\max} > 0$.

Solution of Problem 2.46

i) The dynamics of the value of a self-financing portfolio is given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = V_{n-1}(1 + \pi_n \mu_n) = \begin{cases} V_{n-1}(1 + \pi_n(u-1)), \\ V_{n-1}(1 + \pi_n(d-1)). \end{cases}$$

Therefore, starting from $V_{n-1} > 0$, the condition $V_n > 0$ is equivalent to

$$\pi_n(u-1) > -1, \qquad \pi_n(d-1) > -1,$$

or, using the fact that $d < 1 < u$,

$$\frac{1}{1-u} < \pi_n < \frac{1}{1-d}.$$

In our case $-1 < \pi_n < 2$.

As suggested, in order to prove (2.121), we proceed by induction on n: for $n = N$ we have

$$W_N(v) = 1 - \frac{1}{v} = 1 - \frac{M^0}{v}.$$

At the generic step $n - 1$, we have

$$W_{n-1}(v) = \max_{\pi_n \in]-1,2[} E\left[W_n(G_n(v, \mu_n; \pi_n))\right] =$$

(by the inductive hypothesis)

$$\begin{aligned} &= \max_{\pi_n \in]-1,2[} \left(p\left(1 - \frac{M^{N-n}}{v(1+\pi_n)}\right) + (1-p)\left(1 - \frac{M^{N-n}}{v(1-\frac{\pi_n}{2})}\right)\right) \\ &= 1 - \frac{M^{N-n}}{v} \min_{\pi_n \in]-1,2[} f(\pi_n), \end{aligned}$$

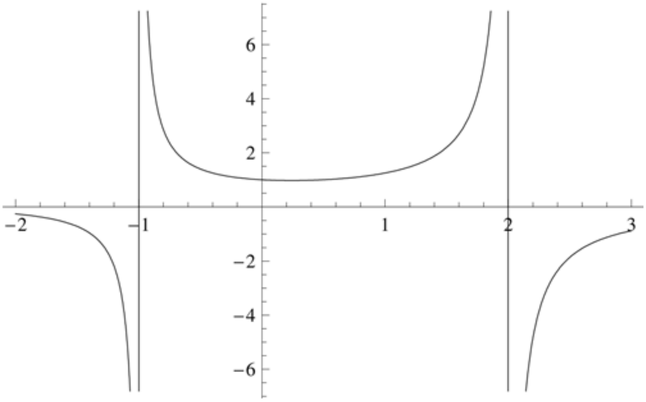

Fig. 2.6. Graph of the function f in (2.122) for $p = \frac{1}{2}$

where

$$f(\pi) = \frac{p}{1+\pi} + \frac{1-p}{1-\frac{\pi}{2}}, \tag{2.122}$$

is the function represented in Figure 2.6, in the case $p = \frac{1}{2}$. We thus obtain (2.121) with

$$M = \min_{\pi \in]-1,2[} f(\pi).$$

ii) Since the optimal strategy $\pi_n^{\max}$ is given by a minimizer of the function f in the interval $]-1,2[$, it is independent of n. To determine it, we compute the derivative of f:

$$f'(\pi) = \frac{2(1-p)}{(-2+\pi)^2} - \frac{p}{(1+\pi)^2},$$

which vanishes at the points

$$\pi^{\pm}(p) := \frac{2 \pm 3\sqrt{2p(1-p)}}{3p-2},$$

under the assumption that $p \neq \frac{2}{3}$. However, since $\pi^+(p)$ does not belong to the interval $]-1,2[$ for $p \in]0,1[$, then the only acceptable solution remains

$$\pi_n^{\max}(p) := \frac{2 - 3\sqrt{2p(1-p)}}{3p-2}.$$

Notice that for $p = \frac{2}{3}$ we have

$$f'(\pi) = \frac{4\pi - 2}{(\pi^2 - \pi - 2)^2}$$

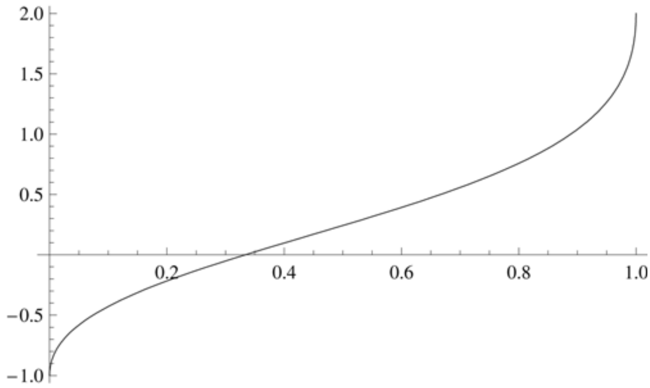

Fig. 2.7. Graph of the function $p \mapsto \pi_n^{\max}(p)$

as the only critical point of f. In this case

$$\frac{1}{2} = \lim_{p \to \frac{2}{3}} \pi^{-}(p) =: \pi_n^{\max}\left(\tfrac{2}{3}\right).$$

The graph of the function $p \mapsto \pi_n^{\max}(p)$, for $p \in]0,1[$, is shown in Figure 2.7. Further, the inequality

$$\pi_n^{\max}(p) = \frac{2 - 3\sqrt{2p(1-p)}}{3p-2} > 0, \qquad p \in]0,1[,$$

is solved by $p \in \left]\frac{1}{3}, 1\right[$. □

Problem 2.47. In a binomial market model over N periods, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the power utility function

$$u(v) = \frac{v^\gamma}{\gamma}, \qquad v > 0,$$

where γ is a given parameter such that $\gamma < 1$ and $\gamma \neq 0$.

i) Assuming that the risk-free rate is zero, $r = 0$, using the DP algorithm show that the optimal expected value is of the form

$$W_n(v) = M^{N-n} \frac{v^\gamma}{\gamma} \tag{2.123}$$

where M is an appropriate constant. Prove also that the optimal proportion of the investment in the risky asset is given by

$$\pi_n^{\max}(v) = \frac{K-1}{u-1+K(1-d)}, \qquad \text{where} \qquad K = \left(\frac{p(u-1)}{(1-p)(1-d)}\right)^{\frac{1}{1-\gamma}}. \tag{2.124}$$

ii) By using the martingale method, show that the terminal value of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_N = vM\widetilde{L}^{\frac{1}{\gamma-1}}, \qquad M = \left(E^P\left[\widetilde{L}^{\frac{\gamma}{\gamma-1}}\right]\right)^{-1}, \tag{2.125}$$

where $\widetilde{L} = B_N^{-1}L$ and $L = \frac{dQ}{dP}$ is the Radon-Nikodym derivative of the martingale measure Q with respect to the physical measure P.

iii) Using again the martingale method and the risk-neutral valuation formula

$$V_n = \frac{1}{(1+r)^{N-n}}E^Q\left[V_N \mid \mathcal{F}_n\right], \tag{2.126}$$

determine the optimal portfolio value and the optimal strategy.

Solution of Problem 2.47

i) Recall from Section 2.4.2 that, when $r = 0$, the dynamics of the value of a self-financing portfolio is given by

$$V_n = V_{n-1}\left(1 + \pi_n\mu_n\right) = \begin{cases} V_{n-1}\left(1 + \pi_n(u-1)\right), \\ V_{n-1}\left(1 + \pi_n(d-1)\right). \end{cases}$$

Assuming that $V_{n-1} > 0$, we have $V_n > 0$ if

$$\pi \in D := \left]\frac{1}{1-u}, \frac{1}{1-d}\right[.$$

Therefore, for a given $v > 0$, the DP algorithm reads as follows:

$$\begin{cases} W_N(v) & = \frac{v^\gamma}{\gamma}, \qquad \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) & = \sup\limits_{\pi_n \in D}\Big(pW_n\left(v\left(1 + \pi_n(u-1)\right)\right) \\ & \quad +(1-p)W_n\left(v\left(1+\pi_n(d-1)\right)\right)\Big). \end{cases}$$

We proceed by induction: the thesis is obviously verified for $n = N$. Moreover, assuming the inductive hypothesis (2.123), by the DP algorithm we have

$$W_{n-1}(v) = M^{N-n}\frac{v^\gamma}{\gamma}\max_\pi f(\pi),$$

where f is the function

$$f(\pi) = p\left(1 + \pi(u-1)\right)^\gamma + (1-p)\left(1 + \pi(d-1)\right)^\gamma. \tag{2.127}$$

The graph of f is displayed in Figure 2.8 for $\gamma = \pm\frac{1}{2}$ and the parameters $u = 2$ and $d = \frac{1}{2}$. Thus the thesis is proved with $M = \max\limits_\pi f(\pi)$.

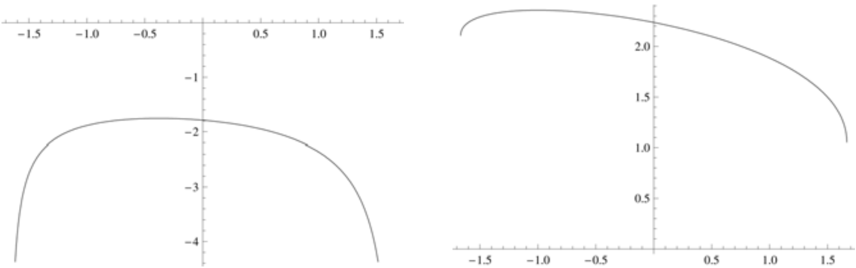

Fig. 2.8. Graph of the function f in (2.127) for $\gamma = \frac{1}{2}$ (on the right) and $\gamma = -\frac{1}{2}$ (on the left)

Finally, to determine the optimal strategy, we seek the maximum point of the function f in (2.127). By imposing

$$f'(\pi) = p\gamma(u-1)\left(1+\pi(u-1)\right)^{\gamma-1} + (1-p)\gamma(d-1)\left(1+\pi(d-1)\right)^{\gamma-1} = 0$$

we get

$$\frac{1+\pi(u-1)}{1+\pi(d-1)} = \left(\frac{p(u-1)}{(1-p)(1-d)}\right)^{\frac{1}{1-\gamma}} =: K,$$

and therefore the only critical point (and thus maximizer) of f is

$$\bar{\pi} = \frac{K-1}{u-1+K(1-d)}$$

which defines the optimal strategy as in (2.124). Let us verify that $\bar{\pi} \in D$: first of all, we have

$$\frac{K-1}{u-1+K(1-d)} < \frac{K}{K(1-d)} = \frac{1}{1-d}.$$

Concerning the other inequality, we note that if $K \geq 1$ then

$$\frac{K-1}{u-1+K(1-d)} \geq 0 > \frac{1}{1-u}.$$

On the other hand, for $K < 1$, we first remark that, since $K > 0$, we have

$$\frac{1}{u-1+K(1-d)} < \frac{1}{u-1+K(1-u)}$$

and therefore

$$\frac{K-1}{u-1+K(1-d)} > \frac{K-1}{u-1+K(1-u)} = \frac{K-1}{(K-1)(1-u)} = \frac{1}{1-u}.$$

ii) Let us recall (cf. (2.37)) the expression of the Radon-Nikodym derivative

$$L = \frac{dQ}{dP} = \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N}$$

where ν_N is the random number of up-movements of the risky asset.

Since $u'(v) = v^{\gamma-1}$, we have $\mathcal{I}(w) = w^{\frac{1}{\gamma-1}}$. Then from Theorem 2.18 we infer that the optimal terminal value is equal to

$$\bar{V}_N = \mathcal{I}\left(\lambda \widetilde{L}\right) = \left(\lambda \widetilde{L}\right)^{\frac{1}{\gamma-1}}, \tag{2.128}$$

where λ is determined by the budget equation

$$v = E^P\left[\left(\lambda \widetilde{L}\right)^{\frac{1}{\gamma-1}} \widetilde{L}\right]$$

equivalent to

$$\lambda^{\frac{1}{\gamma-1}} = \frac{v}{E^P\left[\widetilde{L}^{\frac{\gamma}{\gamma-1}}\right]}.$$

Plugging the above expression into (2.128), we get (2.125). More explicitly we also have

$$\begin{aligned} E\left[\widetilde{L}^{\frac{\gamma}{\gamma-1}}\right] &= \frac{1}{(1+r)^{\frac{N\gamma}{\gamma-1}}} \sum_{k=0}^{N} \binom{N}{k} \left(\frac{q}{p}\right)^{\frac{\gamma k}{\gamma-1}} \left(\frac{1-q}{1-p}\right)^{\frac{\gamma(N-k)}{\gamma-1}} p^k (1-p)^{N-k} \\ &= \frac{1}{(1+r)^{\frac{N\gamma}{\gamma-1}}} \sum_{k=0}^{N} \binom{N}{k} q^{\frac{\gamma k}{\gamma-1}} (1-q)^{\frac{\gamma(N-k)}{\gamma-1}} p^{\frac{k}{\gamma-1}} (1-p)^{\frac{N-k}{\gamma-1}}. \end{aligned} \tag{2.129}$$

iii) We put for brevity $\delta = \frac{1}{\gamma-1}$. We already proved that the optimal terminal value is equal to $\bar{V}_N = vM\widetilde{L}^\delta$ and therefore by (2.126) we have

$$\bar{V}_n = \frac{vM}{(1+r)^{N-n}} E^Q\left[\widetilde{L}^\delta \mid \mathcal{F}_n\right].$$

We recall (cf. Example 2.19) that

$$L_n := E\left[L \mid \mathcal{F}_n\right] = \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n},$$

and notice that, for any $n < N$,

$$\begin{aligned} L &= \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N} \\ &= \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n} \left(\frac{q}{p}\right)^{\nu_N-\nu_n} \left(\frac{1-q}{1-p}\right)^{N-n-(\nu_N-\nu_n)}. \end{aligned}$$

Thus, since ν_n is a $\mathcal{F}_n$-measurable random variable and $\nu_N - \nu_n$ has the same distribution as ν_{N-n} and is independent of $\mathcal{F}_n$ under Q, we have

$$E^Q\left[L^\delta \mid \mathcal{F}_n\right] = L_n^\delta E^Q\left[L_{N-n}^\delta\right]. \tag{2.130}$$

Then we get the following formula

$$\bar{V}_n = \frac{vM_n}{(1+r)^{N(1+\delta)-n}} L_n^\delta, \qquad \text{with} \quad M_n = ME^Q\left[L_{N-n}^\delta\right], \tag{2.131}$$

where M_n has an explicit expression that is easily obtained as in (2.129).

Finally, we determine the optimal strategy by imposing the replication condition

$$\alpha_n S_n + \beta_n B_n = \bar{V}_n$$

equivalent to the system

$$\begin{cases} \alpha_n u S_{n-1} + \beta_n B_n = \frac{vM_n}{(1+r)^{N(1+\delta)-n}} \left(\frac{q}{p}\right)^{(\nu_{n-1}+1)\delta} \left(\frac{1-q}{1-p}\right)^{(n-1-\nu_{n-1})\delta} \\ \alpha_n d S_{n-1} + \beta_n B_n = \frac{vM_n}{(1+r)^{N(1+\delta)-n}} \left(\frac{q}{p}\right)^{(\nu_{n-1})\delta} \left(\frac{1-q}{1-p}\right)^{(n-\nu_{n-1})\delta} \end{cases}$$

from which

$$\begin{aligned} \alpha_n &= \frac{vM_n}{(1+r)^{N(1+\delta)-n}(u-d)S_{n-1}} \cdot \\ &\quad \cdot \left(\left(\frac{q}{p}\right)^\delta - \left(\frac{1-q}{1-p}\right)^\delta\right) \left(\frac{q}{p}\right)^{\nu_{n-1}\delta} \left(\frac{q-1}{p-1}\right)^{(n-1-\nu_{n-1})\delta} \\ &= \frac{vM_n}{(1+r)^{N(1+\delta)-n}(u-d)S_{n-1}} \left(\left(\frac{q}{p}\right)^\delta - \left(\frac{1-q}{1-p}\right)^\delta\right) L_{n-1}^\delta. \end{aligned} \tag{2.132}$$

Using formulas (2.131) and (2.132), we can easily derive the optimal proportion to invest in the risky asset:

$$\pi_n^{\max} = \frac{\alpha_n S_{n-1}}{\bar{V}_{n-1}} = \frac{(1+r)M_n}{(u-d)M_{n-1}} \left(\left(\frac{q}{p}\right)^\delta - \left(\frac{1-q}{1-p}\right)^\delta\right).$$

We conclude by observing that from (2.124) we know that $\pi_n^{\max}$ is independent of the state and the time: this does not seem obvious from the above formula. However, we can directly verify that the ratio $\frac{M_n}{M_{n-1}}$ is independent of n. Indeed, by (2.131) we have

$$\bar{V}_{n-1} = \frac{vM_{n-1}}{(1+r)^{N(1+\delta)-n+1}} L_{n-1}^\delta;$$

on the other hand the risk-neutral valuation formula yields

$$\bar{V}_{n-1} = \frac{1}{1+r} E^Q\left[\bar{V}_n \mid \mathcal{F}_{n-1}\right] =$$

(by (2.131))

$$= \frac{vM_n}{(1+r)^{N(1+\delta)-n+1}} E^Q \left[L_n^\delta \mid \mathcal{F}_{n-1} \right] =$$

(proceeding as in the proof of (2.131))

$$= \frac{vM_n}{(1+r)^{N(1+\delta)-n+1}} L_{n-1}^\delta E^Q \left[L_1^\delta \right].$$

Then, equating the two expressions, we get

$$M_{n-1} = M_n E^Q \left[L_1^\delta \right] = M_n \left(q \left(\frac{q}{p} \right)^\delta - (1-q) \left(\frac{1-q}{1-p} \right)^\delta \right),$$

which proves the thesis. □

Problem 2.48. In a binomial market model over N periods, assume $r = 0$ and consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the exponential utility function

$$u(v) = -e^{-v}, \qquad v \in \mathbb{R}.$$

i) Using the martingale method determine the optimal terminal value of the portfolio;
ii) in the one-period case, i.e. for $N = 1$, compute the optimal strategy and show that the optimal proportion to invest in the risky asset is given by

$$\pi_1^{\max}(v) = \frac{1}{v(u-d)} \log \frac{p(1-q)}{q(1-p)}. \tag{2.133}$$

Solution of Problem 2.48

i) Recall (cf. Example 2.19) that the Radon-Nikodym derivative of the martingale measure Q with respect to the physical measure P is given by

$$L = \frac{dQ}{dP} = \left(\frac{q}{p} \right)^{\nu_N} \left(\frac{1-q}{1-p} \right)^{N-\nu_N}, \tag{2.134}$$

where ν_N is the random variable which denotes the number of up-movements of the risky asset.

Since $u'(v) = e^{-v}$ we have $\mathcal{I}(w) = -\log w$ with $w > 0$. Then by Theorem 2.18 and recalling that $r = 0$, we have that the optimal terminal value is equal to

$$\bar{V}_N = \mathcal{I}(\lambda L) = -\log L - \log \lambda,$$

where λ is determined by the budget equation

$$v = E^Q \left[-\log(\lambda L) \right], \qquad \text{that is} \qquad -\log \lambda = v + E^Q \left[\log L \right].$$

Then we have

$$\bar{V}_N = v - \log L + E^Q [\log L],$$

where

$$E^Q [\log L] = Nq \log\left(\frac{q}{p}\right) + N(1-q) \log\left(\frac{1-q}{1-p}\right).$$

ii) For $N = 1$, L takes only the values $\frac{q}{p}$ and $\frac{1-q}{1-p}$, in case of upward and downward movements of the underlying respectively. Moreover we have

$$\bar{V}_1 = v - \log L + q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p}.$$

To determine the optimal strategy, we impose the replication condition

$$\alpha_1 S_1 + \beta_1 = \bar{V}_1$$

which is equivalent to the following system of equations in the unknowns α_1, β_1:

$$\alpha_1 u S_0 + \beta_1 = v + (1-q)\left(\log \frac{1-q}{1-p} - \log \frac{q}{p}\right),$$
$$\alpha_1 d S_0 + \beta_1 = v + q\left(\log \frac{q}{p} - \log \frac{1-q}{1-p}\right).$$

We get the solution

$$\bar{\alpha}_1(v) = \frac{1}{S_0(u-d)} \log \frac{p(1-q)}{q(1-p)},$$

and, by the definition $\pi_1 = \frac{\alpha_1 S_0}{V_0}$, we have (2.133). □

Problem 2.49. In a binomial market model with $r = 0$, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for a utility function of the form

$$u(v) = -e^{-v}, \qquad v \in \mathbb{R}.$$

i) Using the DP algorithm show by induction that the optimal expected value is of the form

$$W_n(v) = -g(n)e^{-v}, \qquad g(n) = \left(pe^{-c_1} + (1-p)e^{-c_2}\right)^{N-n}, \tag{2.135}$$

where

$$c_1 = (1-q) \log \frac{p(1-q)}{q(1-p)}, \qquad c_2 = q \log \frac{q(1-p)}{p(1-q)},$$

and $q = \frac{1-d}{u-d}$ denotes the probability of an up-movement under the martingale measure. Also show that the optimal strategy is defined by

$$\pi_n^{\max}(v) = \frac{1}{v(u-d)} \log \frac{p(1-q)}{q(1-p)}, \tag{2.136}$$

which, for $n = 1$, coincides with the result obtained by the martingale method in (2.133).

ii) Consider the analogous problem in a standard trinomial model with $m = 1$ and show that

$$W_n(v) = -h(n)e^{-v}, \qquad h(n) = \left(p_1 e^{-(1-\delta)c} + p_2 + p_3 e^{\delta c}\right)^{N-n},$$

where

$$c = \log \frac{p_1(1-\delta)}{p_3 \delta} \quad \text{with} \quad \delta = \frac{1-d}{u-d}.$$

Moreover, prove that the optimal strategy is defined by

$$\pi_n^{\max}(v) = \frac{1}{v(u-d)} \log \frac{p_1(1-\delta)}{p_3\delta}. \tag{2.137}$$

Solution of Problem 2.49

i) We proceed by backward induction. The statement is obviously true for $n = N$. Assuming it true for n, we show it for $n-1$. Recalling that the dynamics of the value of a self-financing portfolio is given by

$$V_n = V_{n-1}\left(1 + \pi_n \mu_n\right) = \begin{cases} V_{n-1}\left(1 + \pi_n(u-1)\right), \\ V_{n-1}\left(1 + \pi_n(d-1)\right), \end{cases}$$

by the DP principle, we have

$$W_{n-1}(v) = -g(n) \min_{\pi_n} E\left[e^{-v(1+\pi_n\mu_n)}\right].$$

We put

$$f_v(\pi) = E\left[e^{-v(1+\pi_n\mu_n)}\right] = pe^{-v(1+\pi(u-1))} + (1-p)e^{-v(1+\pi(d-1))}$$

and set the first derivative of f_v equal to zero to obtain the minimizer:

$$f_v'(\pi) = -pv(u-1)e^{-v(1+\pi(u-1))} - (1-p)v(d-1)e^{-v(1+\pi(d-1))} = 0,$$

namely

$$\frac{p(u-1)}{(1-p)(1-d)} e^{-v\pi(u-d)} = 1$$

from which we have that the only critical point (and thus minimizer) of f_v is

$$\bar{\pi} = \frac{1}{v(u-d)} \log \frac{p(u-1)}{(1-p)(1-d)}.$$

This defines the optimal strategy and hence proves (2.136). Finally, a simple calculation shows that

$$f_v(\bar{\pi}) = pe^{-v-c_1} + (1-p)e^{-v+c_2}$$

from which the result follows.

Notice that for $p = q$, we have $\pi_n^{\max}(v) \equiv 0$ and this fact can be justified as follows: on average, under the martingale measure, the risky asset has the same yield as the bond. Therefore, to minimize the risk, it seems logical that the optimal strategy consists of investing all the wealth in the riskless asset.

ii) Again we proceed by induction and assuming the statement true for n, we prove it for $n-1$: by the DP principle we have

$$\begin{aligned} W_{n-1}(v) &= -h(n) \min_{\pi} E\left[e^{-v(1+\pi\mu_n)}\right] \\ &= -h(n) \min_{\pi} \Big(p_1 e^{-v(1+\pi(u-1))} + p_2 e^{-v(1+\pi(m-1))} \\ &\qquad + p_3 e^{-v(1+\pi(d-1))}\Big). \end{aligned} \tag{2.138}$$

Setting the first derivative equal to zero and recalling the assumption $m = 1$, we get

$$p_1(u-1)e^{-v(1+\pi(u-1))} + p_3(d-1)e^{-v(1+\pi(d-1))} = 0$$

from which

$$\frac{p_1(u-1)}{p_3(1-d)} e^{-v\pi(u-d)} = 1$$

that gives the optimal strategy in (2.137). Finally, using this strategy to calculate the optimal value in (2.138), we get

$$W_{n-1}(v) = -h(n)\left(p_1 e^{-v-(1-\delta)c} + p_2 e^{-v} + p_3 e^{-v+\delta c}\right)$$

and this proves the thesis. □

Problem 2.50. In a binomial market model over N periods, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the "utility function"

$$u(v) = Kv - \frac{v^2}{2}, \qquad v \leq K,$$

where K is a positive constant. Strictly speaking, u is not a utility function according to Definition 2.10: however, it corresponds to the classical mean-variance criterion and still meets the two main conditions **H1** and **H2**.

Recall that the Radon-Nikodym derivative $L = \frac{dQ}{dP}$ of the martingale measure Q with respect to the physical measure P is given by

$$L = \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N}, \tag{2.139}$$

where ν_N is the random number of up-movements of the risky asset up to the terminal period.

In particular, consider the case where

$$S_0 = 1, \quad u = 2, \quad d = \frac{1}{2}, \quad r = 0, \quad p = \frac{2}{5} \text{ and } N = 2.$$

It follows that for the unique martingale measure we have $q = \frac{1}{3}$.

i) Using the martingale method, show that in the case $r = 0$ the terminal value in $N = 2$ of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_2^{\nu_2} = K - \frac{L_2^{\nu_2}}{E^Q\left[L_2^{\nu_2}\right]}(K - v) \le K, \tag{2.140}$$

with

$$E^Q\left[L_2^{\nu_2}\right] = \sum_{k=0}^{2}\binom{2}{k}\left(\frac{q^2}{p}\right)^k\left(\frac{(1-q)^2}{1-p}\right)^{2-k} = \frac{5^2 \cdot 11}{2^2 \cdot 36}. \tag{2.141}$$

ii) Determine the optimal investment strategy (α_n, β_n) in $n = 0$ as well as in $n = 1$ for the two scenarios when the price has risen and when it has fallen respectively (it suffices to determine α_n).

iii) Express the above strategy in terms of the ratio π_n invested in the risky asset.

Solution of Problem 2.50

i) Since $u'(v) = K - v$, we have $\mathcal{I}(w) = K - w$. Then by Theorem 2.18 the optimal terminal value is equal to

$$\bar{V}_2^{\nu_2} = K - \lambda L_2^{\nu_2}, \tag{2.142}$$

where λ is determined by the budget equation

$$E^Q\left[K - \lambda L_2^{\nu_2}\right] = v, \qquad \text{that is} \qquad \lambda = \frac{K - v}{E^Q\left[L_2^{\nu_2}\right]}.$$

Plugging the expression for λ into (2.142), we get (2.140) with $E^Q\left[L_2^{\nu_2}\right]$ following immediately from (2.139).

ii) To determine the optimal investment strategy it is convenient to first determine (see Step P3 in Section 2.2.3) the optimal values $\bar{V}_n$ of a self-financing portfolio and then require that $\alpha_n S_n + \beta_n B_n = \bar{V}_n$. From the martingale property of $\bar{V}_n$ we have

$$\bar{V}_n = \frac{1}{1+r} E^Q\left[\bar{V}_{n+1} \mid \mathcal{F}_n\right].$$

Recalling from (2.139) the expression for $L_2^{\nu_2} = L$, for $N = 2$ we have from (2.140) and (2.141)

$$\bar{V}_2^{\nu_2} = K - (K - v)\,\frac{2^4 \cdot 3^2}{11^2}\left(\frac{3}{4}\right)^{\nu_2}.$$

For $n = 1$ it then follows

$$\begin{aligned}\bar{V}_1^{\nu_1} &= (K - (K - v))\left(\frac{3}{4}\right)^{\nu_1}\left(q\left(\frac{3}{4}\,\frac{2^4 \cdot 3^2}{11^2}\right) + (1-q)\frac{2^4 \cdot 3^2}{11^2}\right)\\ &= (K - (K - v))\left(\frac{3}{4}\right)^{\nu_1}\frac{12}{11}.\end{aligned}$$

Finally, for $n = 0$,

$$\bar{V}_0 = K - (K - v)\left(q\frac{9}{11} + (1-q)\frac{12}{11}\right) = K - (K - v) = v$$

as it should be.

Now we can proceed to determine the optimal investment strategy. For (α_s, β_2) we have in the scenario $S_1^1 = S_0 u$ (to emphasize the scenario we write (α_2^1, β_2^1))

$$\begin{cases} 4\alpha_2^1 + \beta_2^1 = K - (K - v)\frac{3^4}{11^2} \\ \alpha_2^1 + \beta_2^1 = K - (K - v)\frac{2^2 \cdot 3^3}{11^2} \end{cases}$$

so that $\alpha_2^1 = (K - v)\frac{3^2}{11^2}$.

On the other hand, in the scenario $S_1^1 = S_0 d$ (here we write (α_2^0, β_2^0))

$$\begin{cases} 2\alpha_2^0 + \beta_2^0 = K - (K - v)\frac{2^2 \cdot 3^3}{11^2} \\ \frac{1}{4}\alpha_2^0 + \beta_2^0 = K - (K - v)\frac{2^4 \cdot 3^3}{11^2} \end{cases}$$

so that $\alpha_2^0 = (K - v)\frac{2^4 \cdot 3}{11^2}$.

Finally, for (α_1, β_1) we have to require

$$\begin{cases} 2\alpha_1 + \beta_1 = K - (K - v)\frac{9}{11} \\ \frac{1}{2}\alpha_1 + \beta_1 = K - (K - v)\frac{12}{11} \end{cases}$$

so that $\alpha_1 = (K - v)\frac{2}{11}$.

iii) We have

$$\begin{cases} \pi_2^1 = \frac{\alpha_2^1 S_1^1}{\bar{V}_1^1} = \frac{(K-v)\frac{9}{11^2}\cdot 2}{K-(K-v)\frac{9}{11}} = \frac{18(K-v)}{11(2K+9v)} \\ \pi_2^0 = \frac{\alpha_2^0 S_1^0}{\bar{V}_1^0} = \frac{(K-v)\frac{3\cdot 16}{11^2}\cdot\frac{1}{2}}{K-(K-v)\frac{12}{11}} = \frac{24(K-v)}{11(12v-K)} \\ \pi_1 = \frac{\alpha_1 S_0}{\bar{V}_0} = \frac{2}{11}(K-v)\frac{1}{v} = \frac{2}{11}\left(\frac{K}{v} - 1\right). \end{cases}$$

□

Problem 2.51. In a binomial market model with $u = 2$, $d = \frac{1}{2}$, $r = 0$ and $p = \frac{2}{5}$, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the "utility function" (see also Problem 2.50)

$$u(v) = Kv - \frac{v^2}{2}, \qquad v \in [0, K],$$

where K is a positive constant and, by convention, $u(v) = -\infty$ for $v \in \mathbb{R} \setminus [0, K]$.

i) Specify recursive relations of Dynamic Programming to solve the given optimization problem;
ii) determine the optimal strategy in the last period, from $N-1$ to N;
iii) consider next the utility function

$$u_1(v) = Kv - \frac{v^2}{2}, \qquad v \in]-\infty, K],$$

the difference with $u(v)$ being that here we allow also for $v < 0$. On the basis of the DP algorithm, show that the optimal value is of the form

$$W_n(v) = g(n)K^2 + M^{N-n}\left(Kv - \frac{v^2}{2}\right) \tag{2.143}$$

with $M = \frac{54}{55}$ and for a suitable function g. Furthermore prove that the optimal proportion to invest in the risky asset is given by

$$\pi_n^{\max}(v) = \frac{2}{11}\left(\frac{K}{v} - 1\right),$$

where v corresponds to the optimal portfolio value in the period $[t_{n-1}, t_n]$ in the various scenarios, and show that it coincides with the optimal strategy π_n derived in Problem 2.50 iii) by means of the martingale method.

Solution of Problem 2.51

i) Recall that, in the context of the given model, the dynamics of the value of a self-financing portfolio is given by (cf. (2.79))

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = \begin{cases} V_{n-1}\left(1 + \pi_n\right) & \text{if } \mu_n = u - 1, \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) & \text{if } \mu_n = d - 1. \end{cases} \tag{2.144}$$

Now, for $V_{n-1} \in [0, K]$, we have that $V_n \geq 0$ if and only if

$$\begin{cases} 1 + \pi_n \geq 0 \\ 1 - \frac{\pi_n}{2} \geq 0, \end{cases} \qquad \text{that is} \qquad -1 \leq \pi_n \leq 2. \tag{2.145}$$

Furthermore we have that $V_n \leq K$ if and only if[9]

$$\begin{cases} V_{n-1}\left(1+\pi_n\right) \leq K \\ V_{n-1}\left(1-\frac{\pi_n}{2}\right) \leq K, \end{cases} \qquad \text{that is} \qquad -2\left(\frac{K}{V_{n-1}}-1\right) \leq \pi_n \leq \frac{K}{V_{n-1}}-1.$$

Consequently, given $V_{n-1} \in [0, K]$, we have that $V_n \in [0, K]$ if and only if

$$a(V_{n-1}) \leq \pi_n \leq b(V_{n-1})$$

where

$$a(v) = -\min\left\{1, 2\left(\frac{K}{v}-1\right)\right\}, \qquad b(v) = \min\left\{2, \frac{K}{v}-1\right\}. \tag{2.146}$$

Thus the DP algorithm becomes

$$\begin{cases} W_N(v) = u(v), \qquad \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \max\limits_{\pi_n \in [a(v), b(v)]} E\left[W_n\left(G_n(v, \mu_n; \pi_n)\right)\right], \end{cases} \tag{2.147}$$

with a, b as in (2.146).

ii) From (2.147) we get

$$W_{N-1}(v) = \max_{\pi_N \in [a(v), b(v)]} E\left[u\left(G_N(v, \mu_N; \pi_N)\right)\right] = \max_{\pi_N \in [a(v), b(v)]} f_v(\pi_N) \tag{2.148}$$

where

$$f_v(\pi) = \frac{2}{5}\left(K(1+\pi)v - \frac{1}{2}(1+\pi)^2 v^2\right) + \frac{3}{5}\left(K\left(1-\frac{\pi}{2}\right)v - \frac{1}{2}\left(1-\frac{\pi}{2}\right)^2 v^2\right).$$

The graph of f_v is shown in Figure 2.9.

To determine the maximum in (2.148), we calculate the derivative of f_v

$$\begin{aligned} f_v'(\pi) =& \frac{2}{5}\left(Kv - (1+\pi)v^2\right) + \frac{3}{5}\left(-\frac{Kv}{2} - \frac{1}{2}\left(-1+\frac{\pi}{2}\right)v^2\right) \\ =& \frac{v}{20}(2K - (2+11\pi)v) \end{aligned}$$

and observe that the only critical point of f_v is

$$\bar{\pi}_v = \frac{2}{11}\left(\frac{K}{v}-1\right). \tag{2.149}$$

Notice that

$$\lim_{v \to 0^+} \bar{\pi}_v = +\infty$$

[9] If $V_{n-1} = 0$ then $V_n \leq K$ for any $\pi_n \in \mathbb{R}$.

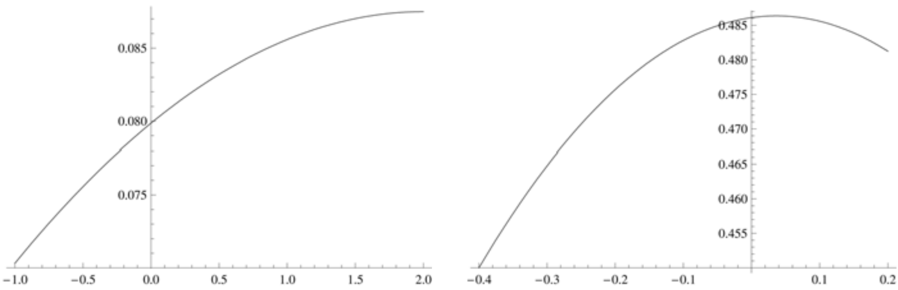

Fig. 2.9. Graph of the function f_v on the interval $[a(v), b(v)]$ with $K = 1$ and $v = \frac{1}{12}$ (on the left) and $v = \frac{10}{12}$ (on the right)

and $\bar{\pi}_v$ belongs to the interval $[a(v), b(v)]$ if and only if $\bar{\pi}_v \leq 2$, i.e. if $v \geq \frac{K}{12}$. Then the maximum point of f_v in the interval $[a(v), b(v)]$ is given by

$$\pi^{\max}(v) = \begin{cases} 2 & \text{if } 0 \leq v \leq \frac{K}{12}, \\ \frac{2}{11}\left(\frac{K}{v} - 1\right) & \text{if } \frac{K}{12} \leq v \leq K, \end{cases}$$

and we have

$$W_{N-1}(v) = \begin{cases} \frac{3v}{5}(2K - 3v) & \text{if } 0 \leq v \leq \frac{K}{12}, \\ \frac{1}{20}\left(K^2 + 18Kv - 9v^2\right) & \text{if } \frac{K}{12} \leq v \leq K. \end{cases}$$

iii) The function $u_1(v) = Kv - \frac{v^2}{2}$ is defined for any $v \in \mathbb{R}$ but it is an increasing function (and verifies the properties of a utility function) only for $v \leq K$. The previously assumed condition $u(v) = -\infty$ for $v < 0$ expresses the fact that the optimal strategy is sought among the strategies that do not expose to loss, i.e. among the strategies such that $V_n \geq 0$ for any n. With the utility function $u_1(v)$ for this point iii) this restriction is removed.

Let $V_{n-1} \leq K$. Recalling formula (2.144) for the dynamics of the value of a self-financing portfolio, we have that $V_n \leq K$ if and only if π_n verifies the following condition:

$$\begin{cases} V_{n-1}(1 + \pi_n) \leq K \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) \leq K, \end{cases} \qquad \text{that is} \qquad -2(K - V_{n-1}) \leq \pi_n V_{n-1} \leq K - V_{n-1}. \tag{2.150}$$

Setting $g(N) = 0$, (2.143) is clearly true for $n = N$. Assuming that (2.143) is true for a generic n, by the DP algorithm we have

$$W_{n-1}(v) = g(n)K^2 + M^{N-n} \max_{-2(K-v) \leq \pi v \leq K-v} f_v(\pi)$$

where

$$f_v(\pi) = \frac{2}{5}\left(K(1+\pi)v - \frac{v^2}{2}(1+\pi)^2\right) + \frac{3}{5}\left(K\left(1 - \frac{\pi}{2}\right)v - \frac{v^2}{2}\left(1 - \frac{\pi}{2}\right)^2\right).$$

To determine the optimal strategy, we set the first derivative of f_v equal to zero:

$$f'_v(\pi) = \frac{2}{5}\left(Kv - v^2(1+\pi)\right) + \frac{3}{5}\left(-\frac{Kv}{2} + \frac{v^2}{2}\left(1 - \frac{\pi}{2}\right)\right) = 0.$$

The only critical point of f_v is

$$\bar{\pi}_v = \frac{2}{11}\left(\frac{K}{v} - 1\right) \tag{2.151}$$

which verifies condition (2.150) and determines the optimal strategy. Thus we have

$$\begin{aligned} W_{n-1}(v) &= g(n)K^2 + M^{N-n} f_v(\bar{\pi}_v) \\ &= g(n)K^2 + M^{N-n}\left(\frac{K^2}{110} + \frac{54}{55}\left(Kv - \frac{v^2}{2}\right)\right), \end{aligned}$$

from which the final result follows.

For completeness, we also give the dynamics of the value of the optimal strategy: setting $\pi_n = \bar{\pi}_{V_{n-1}}$ in (2.144), we have

$$V_n = \begin{cases} \frac{1}{11}\left(9V_{n-1} + 2K\right) & \text{if } \mu_n = u - 1, \\ \frac{1}{11}\left(12V_{n-1} - K\right) & \text{if } \mu_n = d - 1. \end{cases} \tag{2.152}$$

Finally, we show that the strategy in (2.151) coincides with that derived in Problem 2.50 iii) recalling that in each period the value of v has to correspond to the optimal portfolio value in that period in the various scenarios.

Starting from π_1 in the first period, we have $v = V_0$ and so the expressions coincide.

Passing on to $n = 1$, consider first the scenario of an up-movement, for which we use the notation π_2^1. In this case we have from (2.152)

$$V_1^1 = \frac{1}{11}\left(9V_0 + 2K\right)$$

and so, putting $V_0 = v$, from (2.151) we obtain

$$\pi_2^1 = \frac{2}{11}\left(\frac{11K}{9v + 2K} - 1\right) = \frac{18(K - v)}{11(2K + 9v)}$$

which coincides with the corresponding expression in Problem 2.50 iii).

Analogously, in the scenario of a down-movement we have

$$V_1^0 = \frac{1}{11}(12v - K)$$

and so

$$\pi_2^0 = \frac{2}{11}\left(\frac{11K}{12v - K} - 1\right) = \frac{24(K - v)}{11(12v - K)}.$$

□

Problem 2.52. Given is a completed trinomial market model over N periods and with data as in Problem 1.41 and where, letting $p_i := P(h_n = i)$ for $i = 1, 2$, we assume $p_1 = p_2 = p_3 = \frac{1}{3}$. Consider the maximization of expected utility of terminal wealth starting from an initial capital $V_0 = v$ for the utility function

$$u(v) = 1 - \frac{1}{v}, \quad v > 0$$

i) Using the martingale method and putting $N = 1$, show that the terminal value of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_1^{\nu_1^1,\nu_1^2} = \frac{v}{\kappa_0}\left(\sqrt{2}\right)^{\nu_1^1}\left(\frac{2}{3}\right)^{\nu_1^2}$$

where κ_0 is a constant and ν_n^i denotes the random number of events for which, for the driving random factors in the model, we have $h_k = i$ for $k = 1, \ldots, n$ and $i = 1, 2, 3$.

ii) Derive the system of equations to be satisfied by the optimal strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the initial period.

Solution of Problem 2.52

i) Since $u'(v) = \frac{1}{v^2}$, we have $\mathcal{I}(w) = \frac{1}{\sqrt{w}}$. Since $r = 0$, by Theorem 2.18 it then follows that

$$\bar{V}_1^{\nu_1^1,\nu_1^2} = \frac{1}{\sqrt{\lambda}\sqrt{L_1^{\nu_1^1,\nu_1^2}}}$$

where λ is determined by the budget equation

$$v = E^P\left[\frac{1}{\sqrt{\lambda}}\left(L_1^{\nu_1^1,\nu_1^2}\right)^{-\frac{1}{2}} L_1^{\nu_1^1,\nu_1^2}\right] = \frac{1}{\sqrt{\lambda}}E^P\left[\sqrt{L_1^{\nu_1^1,\nu_1^2}}\right].$$

Notice now that from the fact that

$$L_1^{\nu_1^1,\nu_1^2} = \left(\frac{q_1}{p_1}\right)^{\nu_1^1}\left(\frac{q_2}{p_2}\right)^{\nu_1^2}\left(\frac{q_3}{p_3}\right)^{1-\nu_1^1-\nu_1^2}$$

we obtain

$$\begin{aligned}
E^P\left[\sqrt{L_1^{\nu_1^1,\nu_1^2}}\right] &= \sum_{k_1=0}^{1}\sum_{k_2=0}^{1-k_1}\begin{pmatrix} & 1 & \\ k_1 & k_2 & 1-k_1-k_2\end{pmatrix}\cdot \\
&\quad \cdot\left(\frac{q_1}{p_1}\right)^{\frac{k_1}{2}}\left(\frac{q_2}{p_2}\right)^{\frac{k_2}{2}}\left(\frac{q_3}{p_3}\right)^{\frac{1-k_1-k_2}{2}} p_1^{k_1}p_2^{k_2}p_3^{1-k_1-k_2} \\
&= \sqrt{q_1p_1} + \sqrt{q_2p_2} + \sqrt{q_3p_3} \\
&= \frac{1}{\sqrt{6}} + \frac{1}{3\sqrt{2}} + 1 =: \kappa_0.
\end{aligned} \tag{2.153}$$

The budget equation then yields

$$\frac{1}{\sqrt{\lambda}} = \frac{v}{E^P\left[\sqrt{L_1^{\nu_1^1,\nu_1^2}}\right]} = \frac{v}{\kappa_0}$$

so that

$$\begin{aligned}\bar{V}_1^{\nu_1^1,\nu_1^2} &= \frac{v}{\kappa_0}\left(\sqrt{\frac{p_1}{q_1}}\right)^{\nu_1^1}\left(\sqrt{\frac{p_2}{q_2}}\right)^{\nu_1^2}\left(\sqrt{\frac{p_3}{q_3}}\right)^{1-\nu_1^1-\nu_1^2}\\ &= \frac{v}{\kappa_0}\left(\sqrt{2}\right)^{\nu_1^1}\left(\frac{2}{3}\right)^{\nu_1^2}.\end{aligned}$$

ii) The system to be satisfied by $(\alpha_1^1, \alpha_1^2, \beta_1)$ is

$$\begin{cases}\alpha_1^1 S_0^1 u_1 + \alpha_1^2 S_0^2 u_2 + \beta_1(1+r) = \bar{V}_1^{1,0}\\ \alpha_1^1 S_0^1 m_1 + \alpha_1^2 S_0^2 m_2 + \beta_1(1+r) = \bar{V}_1^{0,1}\\ \alpha_1^1 S_0^1 d_1 + \alpha_1^2 S_0^2 d_2 + \beta_1(1+r) = \bar{V}_1^{0,0}\end{cases}$$

namely

$$\begin{cases}2\alpha_1^1 + \frac{8}{3}\alpha_1^2 + \beta_1 = \frac{v}{\kappa_0}\sqrt{2}\\ \alpha_1^1 + \frac{8}{9}\alpha_1^2 + \beta_1 = \frac{v}{\kappa_0}\sqrt{\frac{2}{3}}\\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \beta_1 = \frac{v}{\kappa_0}\end{cases}$$

with κ_0 as in (2.153). □

Problem 2.53. In a completed trinomial market model over N periods and with data as in Example 2.40, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the power utility function

$$u(v) = \frac{v^\gamma}{\gamma}, \qquad v > 0,$$

where γ is a given parameter such that $\gamma < 1$ and $\gamma \neq 0$.

i) Using the Dynamic Programming algorithm, show that the optimal expected value is of the form

$$W_n(v) = M^{N-n}\frac{v^\gamma}{\gamma}, \tag{2.154}$$

for a suitable constant M. Show also that the optimal proportions to be invested in the two risky assets are given by

$$\bar{\pi}_n^1 = -\frac{2\left(5 + 2^{4+\frac{1}{\gamma-1}} - 7\cdot 3^{\frac{\gamma}{\gamma-1}}\right)}{1 + 2^{\frac{\gamma}{\gamma-1}} + 3^{\frac{\gamma}{\gamma-1}}}, \qquad \bar{\pi}_n^2 = \frac{18\left(1 + 2^{\frac{\gamma}{\gamma-1}}\right)}{1 + 2^{\frac{\gamma}{\gamma-1}} + 3^{\frac{\gamma}{\gamma-1}}} - 9, \tag{2.155}$$

and thus they are independent of the state and the period;

ii) by using the martingale method and recalling the assumption $r = 0$, show that the terminal value of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_N = vML^{\frac{1}{\gamma-1}}, \quad \text{with} \quad M = \left(E^P\left[L^{\frac{\gamma}{\gamma-1}}\right]\right)^{-1}, \tag{2.156}$$

where $L = \frac{dQ}{dP}$ is the Radon-Nikodym derivative of the martingale measure Q with respect to the physical measure P;

iii) using the risk-neutral valuation formula (with $r = 0$)

$$V_n = E^Q\left[V_N \mid \mathcal{F}_n\right], \tag{2.157}$$

determine the value of the optimal portfolio and the optimal strategy.

Solution of Problem 2.53

i) Recall from Example 2.40 that if $r = 0$ the dynamics of the value of a self-financing portfolio is given by

$$V_n = \begin{cases} V_{n-1}\left(1 + \pi_n^1 + \frac{5\pi_n^2}{3}\right) \\ V_{n-1}\left(1 - \frac{\pi_n^2}{9}\right) \\ V_{n-1}\left(1 - \frac{\pi_n^1}{2} - \frac{2\pi_n^2}{3}\right). \end{cases}$$

Again by Example 2.40, assuming $V_{n-1} > 0$, we have that $V_n > 0$ if

$$\pi_n \in D = \left\{(\pi^1, \pi^2) \mid \pi^1 + \frac{5\pi^2}{3} > -1, \ -\pi^1 - \frac{4\pi^2}{3} > -2\right\}.$$

Notice that D is a bounded domain.

Therefore, for any fixed $v > 0$, the DP algorithm becomes

$$\begin{cases} W_N(v) \quad = \frac{v^\gamma}{\gamma}, \qquad \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup\limits_{(\pi_n^1, \pi_n^2) \in D} \frac{1}{3}\Big(W_n\left(v\left(1 + \pi_n^1 + \frac{5}{3}\pi_n^2\right)\right) \\ \qquad +W_n\left(v\left(1 - \frac{1}{9}\pi_n^2\right)\right) + W_n\left(v\left(1 - \frac{1}{2}\pi_n^1 - \frac{2}{3}\pi_n^2\right)\right)\Big). \end{cases}$$

We proceed by induction: the claim is obviously true for $n = N$. Then we assume the inductive hypothesis (2.154): by the DP algorithm we have

$$W_{n-1}(v) = M^{N-n}\frac{v^\gamma}{3\gamma} \max_{(\pi^1, \pi^2) \in D} f(\pi^1, \pi^2),$$

where f is the function

$$f(\pi^1, \pi^2) = \left(1 + \pi_n^1 + \frac{5}{3}\pi_n^2\right)^\gamma + \left(1 - \frac{1}{9}\pi_n^2\right)^\gamma + \left(1 - \frac{1}{2}\pi_n^1 - \frac{2}{3}\pi_n^2\right)^\gamma. \tag{2.158}$$

To determine the maximum of f, we compute the partial derivatives:

$$\begin{aligned}\partial_{\pi^1} f(\pi^1,\pi^2) =& \gamma\left(1+\pi^1+\frac{5}{3}\pi^2\right)^{\gamma-1} - \frac{\gamma}{2}\left(1-\frac{\pi^1}{2}-\frac{2\pi^2}{3}\right)^{\gamma-1},\\ \partial_{\pi^2} f(\pi^1,\pi^2) =& \frac{5\gamma}{3}\left(1+\pi^1+\frac{5}{3}\pi^2\right)^{\gamma-1} - \frac{\gamma}{9}\left(1-\frac{\pi^2}{9}\right)^{\gamma-1}\\ &- \frac{2\gamma}{3}\left(1-\frac{\pi^1}{2}-\frac{2\pi^2}{3}\right)^{\gamma-1}.\end{aligned}$$

Putting the gradient equal to zero we obtain a system of equations equivalent to

$$\begin{cases} 1+\pi^1+\frac{5}{3}\pi^2 = \left(\frac{1}{2}\right)^{\frac{1}{\gamma-1}}\left(1-\frac{\pi^1}{2}-\frac{2\pi^2}{3}\right) \\ 1+\pi^1+\frac{5}{3}\pi^2 = \left(\frac{1}{3}\right)^{\frac{1}{\gamma-1}}\left(1-\frac{\pi^2}{9}\right) \end{cases}$$

which is a linear system of two equations with two unknowns whose solution $(\bar{\pi}^1,\bar{\pi}^2)$ is given by (2.155). Since D is bounded, the critical point $(\bar{\pi}^1,\bar{\pi}^2)$ is also a global maximum for f and we have

$$\begin{aligned}M = \frac{1}{3}\max f =& \frac{1}{3}f(\bar{\pi}^1,\bar{\pi}^2) = \frac{2^\gamma}{3}\left(1+\left(\frac{3}{2}\right)^{\frac{\gamma}{1-\gamma}}+3^{\frac{\gamma}{1-\gamma}}\right)^{-\gamma}\\ &+\frac{1}{3}\left(\frac{2^{4+\frac{1}{\gamma-1}}-52^{\frac{\gamma}{\gamma-1}}}{1+2^{\frac{\gamma}{\gamma-1}}+3^{\frac{\gamma}{\gamma-1}}}\right)^{\gamma}+\frac{6^\gamma}{3}\left(1+2^{\frac{\gamma}{\gamma-1}}+3^{\frac{\gamma}{\gamma-1}}\right)^{-\gamma}.\end{aligned}$$

ii) By Theorem 2.18 and recalling that $r=0$ and hence $B_N=1$, we have (see also (2.128) in Problem 2.47)

$$\bar{V}_N = \mathcal{I}(\lambda L) = (\lambda L)^{\frac{1}{\gamma-1}},$$

where (cf. (2.40)),

$$L(\omega) = \left(\frac{q_1}{p_1}\right)^{\nu_N^1(\omega)}\left(\frac{q_2}{p_2}\right)^{\nu_N^2(\omega)}\left(\frac{q_3}{p_3}\right)^{N-\nu_N^1(\omega)-\nu_N^2(\omega)}$$

and by the budget equation

$$v = E^Q\left[\mathcal{I}(\lambda L)\right] = \lambda^{\frac{1}{\gamma-1}}E^Q\left[L^{\frac{1}{\gamma-1}}\right] = \lambda^{\frac{1}{\gamma-1}}E^P\left[L^{\frac{\gamma}{\gamma-1}}\right].$$

Therefore, setting[10]

$$M := \left(E^P\left[L^{\frac{\gamma}{\gamma-1}}\right]\right)^{-1},$$

[10] M can be computed explicitly as in (2.129).

we get $\lambda = (vM)^{\gamma-1}$ and finally

$$\bar{V}_N = v\, M\, L^{\frac{1}{\gamma-1}}.$$

iii) For brevity, we put $\delta = \frac{1}{\gamma-1}$. Since $\bar{V}_N = vML^{\delta}$, by (2.126) we have

$$\bar{V}_n = vM\, E^Q\left[L^{\delta} \mid \mathcal{F}_n\right].$$

By (2.40) we have

$$\begin{aligned}
L &= \left(\frac{q_1}{p_1}\right)^{\nu_N^1} \left(\frac{q_2}{p_2}\right)^{\nu_N^2} \left(\frac{q_3}{p_3}\right)^{N-\nu_N^1-\nu_N^2} \\
&= \left(\frac{q_1}{p_1}\right)^{\nu_n^1} \left(\frac{q_2}{p_2}\right)^{\nu_n^2} \left(\frac{q_3}{p_3}\right)^{n-\nu_n^1-\nu_n^2} \cdot \\
&\quad \cdot \left(\frac{q_1}{p_1}\right)^{\nu_N^1-\nu_n^1} \left(\frac{q_2}{p_2}\right)^{\nu_N^2-\nu_n^2} \left(\frac{q_3}{p_3}\right)^{N-n-(\nu_N^1-\nu_n^1)-(\nu_N^2-\nu_n^2)}.
\end{aligned}$$

Thus, since ν_n^1, ν_n^2 are $\mathcal{F}_n$-measurable random variables and $\nu_N^i - \nu_n^i$, for $i = 1, 2$, has the same Q-distribution as ν_{N-n}^i regardless to $\mathcal{F}_n$, we have

$$E^Q\left[L^{\delta} \mid \mathcal{F}_n\right] = L_n^{\delta} E^Q\left[L_{N-n}^{\delta}\right]. \tag{2.159}$$

Finally, we get the following formula

$$\bar{V}_n = vM_n L_n^{\delta}, \qquad \text{with} \ \ M_n = ME^Q\left[L_{N-n}^{\delta}\right], \tag{2.160}$$

where M_n has an explicit expression that can be calculated similarly to (2.129).

Concerning the optimal strategy, for $n \le N$ we have the replication condition

$$\alpha_n^1 S_n^1 + \alpha_n^2 S_n^2 + \beta_n B_n = \bar{V}_n,$$

equivalent to the system

$$\begin{aligned}
2\alpha_n^1 S_{n-1}^1 + \frac{8}{3}\alpha_n^2 S_{n-1}^2 + \beta_n = vM_n \left(\frac{q_1}{p_1}\right)^{(\nu_{n-1}^1+1)\delta} \left(\frac{q_2}{p_2}\right)^{\nu_{n-1}^2\delta} \cdot \\
\cdot \left(\frac{q_3}{p_3}\right)^{(n-1-\nu_{n-1}^1-\nu_{n-1}^2)\delta}
\end{aligned}$$

$$\alpha_n^1 S_{n-1}^1 + \frac{8}{9}\alpha_n^2 S_{n-1}^2 + \beta_n = vM_n \left(\frac{q_1}{p_1}\right)^{\nu_{n-1}^1\delta} \cdot \\ \cdot \left(\frac{q_2}{p_2}\right)^{(\nu_{n-1}^2+1)\delta} \left(\frac{q_3}{p_3}\right)^{(n-1-\nu_{n-1}^1-\nu_{n-1}^2)\delta}$$

$$\frac{1}{2}\alpha_n^1 S_{n-1}^1 + \frac{1}{3}\alpha_n^2 S_{n-1}^2 + \beta_n = vM_n \left(\frac{q_1}{p_1}\right)^{\nu_{n-1}^1\delta} \cdot \\ \cdot \left(\frac{q_2}{p_2}\right)^{\nu_{n-1}^2\delta} \left(\frac{q_3}{p_3}\right)^{(n-\nu_{n-1}^1-\nu_{n-1}^2)\delta} . \qquad \square$$

Problem 2.54. Analogously to Problem 2.52 let a completed trinomial market model over N periods be given with data as in Problem 1.41 and where, letting $p_i := P(h_n = i)$ for $i = 1, 2$, we assume $p_1 = p_2 = p_3 = \frac{1}{3}$. Consider the maximization of expected utility of terminal wealth starting from an initial capital $V_0 = v$ for the utility function

$$u(v) = 1 - e^{-v}, \quad v \in \mathbb{R}.$$

i) Using the martingale method and putting $N = 1$, show that the terminal value of the portfolio that achieves the maximum expected utility is given by
$$\bar{V}_1^{\nu_1^1,\nu_1^2} = v + \kappa + \nu_1^1 \log 2 - \nu_1^2 \log \frac{3}{2} \tag{2.161}$$
where κ is a constant and, as before, ν_n^i denotes the random number of events for which, for the driving random factors in the model, we have $h_k = i$ for $k = 1, \ldots, n$ and $i = 1, 2, 3$.

ii) Derive the system of equations to be satisfied by the optimal strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the initial period and verify that the solution is given by
$$\alpha_1^1 = 7\log\frac{2}{3} - \frac{5\log 2}{3}, \quad \alpha_1^2 = \frac{9}{2}\log\frac{3}{2} + \frac{3}{2}\log 2, \quad \beta_1 = v + \kappa + 2\log\frac{3}{2} + \frac{\log 2}{3}.$$
Verify it also via $\alpha_1^1 S_0^1 + \alpha_1^2 S_0^2 + \beta_1 = v$.

Solution of Problem 2.54

i) Being $u'(v) = e^{-v}$, we have $\mathcal{I}(w) = -\log w$. Since $r = 0$, by Theorem 2.18 it then follows that
$$\bar{V}_1^{\nu_1^1,\nu_1^2} = -\log\lambda - \log L_1^{\nu_1^1,\nu_1^2}$$
where λ is determined by the budget equation
$$v = E^Q\left[-\log\lambda - \log L_1^{\nu_1^1,\nu_1^2}\right].$$

Recall now that
$$L_1^{\nu_1^1,\nu_1^2} = \left(\frac{q_1}{p_1}\right)^{\nu_1^1}\left(\frac{q_2}{p_2}\right)^{\nu_1^2}\left(\frac{q_3}{p_3}\right)^{1-\nu_1^1-\nu_1^2}$$

and that $E^Q[\nu_1^1] = q_1$, $E^Q[\nu_1^2] = q_2$. We thus obtain

$$\begin{aligned}
E^Q\left[\log L_1^{\nu_1^1,\nu_1^2}\right] &= E^Q\left[\nu_1^1 \log\left(\frac{q_1}{p_1}\right) + \nu_1^2 \log\left(\frac{q_2}{p_2}\right) + (1-\nu_1^1-\nu_1^2)\log\left(\frac{q_3}{p_3}\right)\right] \\
&= q_1 \log\left(\frac{q_1}{p_1}\right) + q_2 \log\left(\frac{q_2}{p_2}\right) + (1-q_1-q_2)\log\left(\frac{q_3}{p_3}\right) \\
&= -\frac{1}{6}\log 2 + \frac{1}{2}\log 3 - \frac{1}{2}\log 2 \\
&= -\frac{2}{3}\log 2 + \frac{\log 3}{2} =: \kappa. \qquad (2.162)
\end{aligned}$$

The budget equation then yields

$$-\log\lambda = v + \kappa$$

and therefore (2.161) follows.

ii) The system to be satisfied by $(\alpha_1^1, \alpha_1^2, \beta_1)$ is

$$\begin{cases}
\alpha_1^1 S_0^1 u_1 + \alpha_1^2 S_0^2 u_2 + \beta_1(1+r) = \bar{V}_1^{1,0} \\
\alpha_1^1 S_0^1 m_1 + \alpha_1^2 S_0^2 m_2 + \beta_1(1+r) = \bar{V}_1^{0,1} \\
\alpha_1^1 S_0^1 d_1 + \alpha_1^2 S_0^2 d_2 + \beta_1(1+r) = \bar{V}_1^{0,0}
\end{cases}$$

namely

$$\begin{cases}
2\alpha_1^1 + \frac{8}{3}\alpha_1^2 + \beta_1 = v + \kappa + \log 2 \\
\alpha_1^1 + \frac{8}{9}\alpha_1^2 + \beta_1 = v + \kappa + \log\frac{2}{3} \\
\frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \beta_1 = v + \kappa
\end{cases}$$

that, indeed, leads to the given solution. Using the expression of κ in (2.162), we can easily verify that

$$\alpha_1^1 S_0^1 + \alpha_1^2 S_0^2 + \beta_1 = \alpha_1^1 + \alpha_1^2 + \beta_1 = v. \qquad \square$$

Problem 2.55. For a completed trinomial market model over N periods let the data be as in Problem 1.42 and, letting $p_i := P(h_n = i)$ for $i = 1, 2$, assume that $p_1 = p_2 = p_3 = \frac{1}{3}$. Consider the maximization of expected utility of terminal wealth starting from an initial capital $V_0 = v$ for the "utility function" (see Problem 2.50)

$$u(v) = Kv - \frac{v^2}{2}, \qquad v \le K, \qquad (K > 0 \text{ given}).$$

i) Using the martingale method and putting $N = 2$, show that the terminal value of the portfolio that achieves the maximum expected utility is given by

$$\bar{V}_2^{\nu_2^1,\nu_2^2} = K - \frac{L_2^{\nu_1^1,\nu_1^2}}{E^Q\left[L_2^{\nu_1^1,\nu_1^2}\right]}(K - B_2 v)$$

where $B_2 = (1+r)^2 = \frac{9}{4}$, $E^Q\left[L_2^{\nu_1^1,\nu_1^2}\right] = E^Q\left[\left(\frac{3}{2}\right)^{\nu_2^1}\left(\frac{1}{2}\right)^{\nu_2^2}\right]$ and, as before, ν_n^i denotes the random number of events for which, for the driving random factors in the model, we have $h_k = i$ for $k = 1,\dots,n$ and $i = 1,2,3$.

ii) Denoting, analogously to $\bar{V}_2^{\nu_2^1,\nu_2^2}$, by $\bar{V}_1^{\nu_1^1,\nu_1^2}$ the value in $n = 1$ of the optimal self-financing portfolio corresponding to (ν_1^1,ν_1^2), determine the expression of $\bar{V}_1^{\nu_1^1,\nu_1^2}$ in each of the three possible scenarios.

iii) Derive the system of equations to be satisfied by the optimal strategy $(\alpha_1^1,\alpha_1^2,\beta_1)$ for the initial period.

Solution of Problem 2.55

i) Being $u'(v) = K - v$ and thus $\mathcal{I}(w) = K - w$, by Theorem 2.18 we have

$$\bar{V}_2^{\nu_2^1,\nu_2^2} = K - \lambda \widetilde{L}_2^{\nu_2^1,\nu_2^2} \tag{2.163}$$

where λ is determined by the budget equation

$$v = E^P\left[(K - \lambda\widetilde{L}_2)\widetilde{L}_2\right] = E^Q\left[B_2^{-1}K - \lambda B_2^{-2}L_2\right] \tag{2.164}$$

which implies

$$\lambda = \frac{B_2K - B_2^2 v}{E^Q\left[L_2^{\nu_2^1,\nu_2^2}\right]}. \tag{2.165}$$

Plugging this last expression into (2.163), we obtain

$$\bar{V}_2^{\nu_2^1,\nu_2^2} = K - \frac{L_2^{\nu_2^1,\nu_2^2}}{E^Q\left[L_2^{\nu_2^1,\nu_2^2}\right]}(K - B_2 v).$$

For the last statement recall that, for a generic period,

$$L_n^{\nu_n^1,\nu_n^2} = \left(\frac{q_1}{p_1}\right)^{\nu_n^1}\left(\frac{q_2}{p_2}\right)^{\nu_n^2}\left(\frac{q_3}{p_3}\right)^{1-\nu_n^1-\nu_n^2} = \left(\frac{3}{2}\right)^{\nu_n^1}\left(\frac{1}{2}\right)^{\nu_n^2}.$$

ii) From the martingale property of $\bar{V}_n$ we have

$$\bar{V}_n = \frac{1}{1+r}E^Q\left[\bar{V}_{n+1} \mid \mathcal{F}_n\right],$$

in particular,

$$\begin{aligned}
\bar{V}_1^{\nu_1^1,\nu_1^2} &= \frac{2}{3}E^Q\left[K - \frac{L_2^{\nu_2^1,\nu_2^2}}{E^Q\left[L_2^{\nu_2^1,\nu_2^2}\right]}(K - B_2 v) \mid \mathcal{F}_1^{\nu_1^1,\nu_1^2}\right] \\
&= \frac{2}{3}K - \frac{2}{3}\left(K - \frac{9}{4}v\right)\frac{E^Q\left[L_2^{\nu_2^1,\nu_2^2} \mid \mathcal{F}_1^{\nu_1^1,\nu_1^2}\right]}{E^Q\left[L_2^{\nu_2^1,\nu_2^2}\right]}.
\end{aligned}$$

Being $E^Q[L_2] = E^Q\left[E^Q[L_2 \mid \mathcal{F}_1]\right]$, we compute $E^Q[L_2 \mid \mathcal{F}_1]$ in the three scenarios corresponding to $\mathcal{F}_1 = \mathcal{F}_1^{\nu_1^1,\nu_1^2}$:

a) $h_1 = 1$

$$E^Q\left[L_2 \mid \mathcal{F}_1^{1,0}\right] = \left(\frac{3}{2}\right)^2 q_1 + \frac{3}{2}\cdot\frac{1}{2}q_2 + \frac{3}{2}q_3 = \frac{7}{4},$$

b) $h_1 = 2$

$$E^Q\left[L_2 \mid \mathcal{F}_1^{0,1}\right] = \frac{3}{2}\cdot\frac{1}{2}q_1 + q_2\left(\frac{1}{2}\right)^2 + \frac{1}{2}q_3 = \frac{7}{12},$$

c) $h_1 = 3$

$$E^Q\left[L_2 \mid \mathcal{F}_1^{0,0}\right] = \frac{3}{2}q_1 + \frac{1}{2}q_2 + q_3 = \frac{7}{6},$$

which implies

$$E^Q\left[L_2^{\nu_2^1,\nu_2^2}\right] = \frac{7}{4}q_1 + \frac{7}{12}q_2 + \frac{7}{6}q_3 = \frac{49}{36}.$$

Consequently we have

$$\begin{cases} \bar{V}_1^{1,0} = \frac{2}{3}K - \frac{2}{3}\left(K - \frac{9}{4}v\right)\frac{7}{4}\cdot\frac{36}{49} = \frac{2}{3}K - \frac{6}{7}\left(K - \frac{9}{4}v\right) \\ \bar{V}_1^{0,1} = \frac{2}{3}K - \frac{2}{3}\left(K - \frac{9}{4}v\right)\frac{7}{12}\cdot\frac{36}{49} = \frac{2}{3}K - \frac{2}{7}\left(K - \frac{9}{4}v\right) \\ \bar{V}_1^{0,0} = \frac{2}{3}K - \frac{2}{3}\left(K - \frac{9}{4}v\right)\frac{7}{6}\cdot\frac{36}{49} = \frac{2}{3}K - \frac{4}{7}\left(K - \frac{9}{4}v\right). \end{cases}$$

ii) The system to be satisfied by the optimal $(\alpha_1^1, \alpha_1^2, \beta_1)$ is

$$\begin{cases} \alpha_1^1 S_0^1 u_1 + \alpha_1^2 S_0^2 u_2 + \beta_1(1+r) = \bar{V}_1^{1,0} \\ \alpha_1^1 S_0^1 m_1 + \alpha_1^2 S_0^2 m_2 + \beta_1(1+r) = \bar{V}_1^{0,1} \\ \alpha_1^1 S_0^1 d_1 + \alpha_1^2 S_0^2 d_2 + \beta_1(1+r) = \bar{V}_1^{0,0} \end{cases}$$

namely

$$\begin{cases} \frac{7}{3}\alpha_1^1 + \frac{22}{9}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{27}{14}v - \frac{4}{21}K \\ \alpha_1^1 + \alpha_1^2 + \frac{3}{2}\beta_1 = \frac{9}{14}v + \frac{8}{21}K \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{9}{7}v - \frac{2}{21}K. \end{cases}$$

□

Problem 2.56. In a standard trinomial market model over N periods, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for the power utility function

$$u(v) = \frac{v^\gamma}{\gamma}, \qquad v > 0,$$

where γ is a given constant such that $\gamma < 1$ and $\gamma \neq 0$.

i) Assuming that the risk-free rate is zero, $r = 0$, show by the DP algorithm that the optimal expected value is of the form

$$W_n(v) = M^{N-n} \frac{v^\gamma}{\gamma} \tag{2.166}$$

for some constant M. Also prove that, if $m = 1$, then the optimal proportion to be invested in the risky asset is given by

$$\pi_n^{\max}(v) = \frac{K-1}{u-1+K(1-d)}, \quad \text{with} \quad K = \left(\frac{p_1(u-1)}{p_3(1-d)} \right)^{\frac{1}{1-\gamma}}; \tag{2.167}$$

ii) Consider a market model where μ_n is a binomial random variable with the same values for u and d as in i). Show that, assuming

$$p_1 = P(1+\mu_n = u) \quad \text{and} \quad p_3 = 1 - p_1 = P(1+\mu_n = d),$$

the same results hold as in i).

Solution of Problem 2.56

i) Recall from Section 2.4.5 that, in the case of $r = 0$, the dynamics of the value of a self-financing portfolio is given by

$$V_n = \begin{cases} V_{n-1}\left(1+\pi_n(u-1)\right) & \text{if } \mu_n = u-1, \\ V_{n-1}\left(1+\pi_n(m-1)\right) & \text{if } \mu_n = m-1, \\ V_{n-1}\left(1+\pi_n(d-1)\right) & \text{if } \mu_n = d-1. \end{cases}$$

Thus, starting from $V_{n-1} > 0$, we have that $V_n > 0$ if (see (2.88))

$$\pi \in D := \left] \frac{1}{1-u}, \frac{1}{1-d} \right[,$$

and, for a fixed $v > 0$, the DP algorithm becomes

$$\begin{cases} W_N(v) = \frac{v^\gamma}{\gamma}, \qquad \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup\limits_{\pi_n \in D} \Big(p_1 W_n\left(v\left(1+\pi_n(u-1)\right)\right) \\ \qquad +p_2 W_n\left(v\left(1+\pi_n(m-1)\right)\right) + p_3 W_n\left(v\left(1+\pi_n(d-1)\right)\right) \Big). \end{cases}$$

We proceed by induction: the claim is obviously true for $n = N$. Then we assume the inductive hypothesis (2.166): by the DP algorithm we have

$$W_{n-1}(v) = M^{N-n} \frac{v^\gamma}{\gamma} \max_\pi f(\pi), \tag{2.168}$$

where f is the function

$$f(\pi) = p_1\left(1+\pi(u-1)\right)^\gamma + p_2\left(1+\pi(m-1)\right)^\gamma + p_3\left(1+\pi(d-1)\right)^\gamma, \tag{2.169}$$

and this proves the thesis with $M = \max\limits_\pi f(\pi)$.

Finally, to determine the optimal strategy, we seek the maximum point of the function f in (2.169). By imposing

$$f'(\pi) = p_1\gamma(u-1)\left(1+\pi(u-1)\right)^{\gamma-1} + p_2\gamma(m-1)\left(1+\pi(m-1)\right)^{\gamma-1} \\ + p_3\gamma(d-1)\left(1+\pi(d-1)\right)^{\gamma-1} = 0$$

and, recalling that by assumption $m = 1$, we get

$$\frac{1+\pi(u-1)}{1+\pi(d-1)} = \left(\frac{p_1(u-1)}{p_3(1-d)}\right)^{\frac{1}{1-\gamma}} =: K.$$

Hence the only critical point (and maximizer) of f is

$$\bar{\pi} = \frac{K-1}{u-1+K(1-d)}$$

which, as for the logarithmic utility (see Example 2.41) is independent of n and v, and defines the optimal strategy as in (2.167). To verify that $\bar{\pi} \in D$ we can use the same argument as in Problem 2.47.

Plugging the expression of $\bar{\pi}$ into the recursive formula of the Dynamic Programming, we get (see (2.168) and (2.169))

$$W_{n-1}(v) = M^{N-n}\frac{v^\gamma}{\gamma}\cdot \\ \cdot\left[p_1\left(1+\tfrac{(K-1)(u-1)}{u-1+K(1-d)}\right)^\gamma + p_2 + p_3\left(1+\tfrac{(K-1)(d-1)}{u-1+K(1-d)}\right)^\gamma\right]$$

where the last term in brackets represents the value of $M = \max_\pi f(\pi)$.

ii) Just set $p_2 = 0$ in the trinomial model: then everything else remains the same for the binomial model with the results matching those of Problem 2.47-i). □

Problem 2.57. Consider a standard trinomial market model with data $u^1 = \frac{7}{3}$, $m^1 = 1$ and $d^1 = \frac{1}{2}$. Assuming $r = 0$ and that the physical measure is defined by $p_1 = p_2 = p_3 = \frac{1}{3}$, consider the problem of maximizing the expected utility of terminal wealth starting from an initial capital $V_0 = v$, for a "utility function" of the form (see Problem 2.50)

$$u(v) = Kv - \frac{v^2}{2}, \qquad v \le K,$$

where K is a positive constant and, by convention, we put $u(v) = -\infty$ for $v > K$.

i) Specify the recursive relation of Dynamic Programming to solve the given optimization problem and determine the optimal strategy $\pi_N^{\max}(v)$;
ii) in the completed trinomial market model with a second asset such that $u^2 = \frac{22}{9}$, $m^2 = 1$ and $d^2 = \frac{1}{3}$, specify the recursive relation of Dynamic Programming in the one-period case, $N = 1$.

Solution of Problem 2.57

i) Recall that, in the context of the assigned model, the dynamics of the value of a self-financing portfolio is given by (cf. (2.79))

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = V_{n-1}\left(1 + \pi_n \mu_n\right)$$
$$= \begin{cases} V_{n-1}\left(1 + \frac{4}{3}\pi_n\right) & \text{if } \mu_n = u^1 - 1, \\ V_{n-1} & \text{if } \mu_n = m^1 - 1, \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) & \text{if } \mu_n = d^1 - 1. \end{cases} \tag{2.170}$$

Consequently, if $V_{n-1} \le K$, we have that $V_n \le K$ for the values of π_n such that

$$\pi_n \in D(V_{n-1}) := \left\{ \pi \mid -2\left(K - V_{n-1}\right) \le \pi V_{n-1} \le \frac{3}{4}\left(K - V_{n-1}\right) \right\}. \tag{2.171}$$

Then the DP algorithm becomes

$$\begin{cases} W_N(v) = u(v), & \text{and, for } n = N, \dots, 1, \\ W_{n-1}(v) = \sup\limits_{\pi_n \in D(v)} E\left[W_n\left(G_n(v, \mu_n; \pi_n)\right)\right]. \end{cases} \tag{2.172}$$

In particular, recalling that $u(v) = -\infty$ for $v > K$, we have

$$W_{N-1}(v) = \sup_{\pi_N \in D(v)} E\left[u\left(G_N(v, \mu_N; \pi)\right)\right] = \frac{1}{3} \max_{\pi_N \in D(v)} f_v(\pi_N)$$

where

$$f_v(\pi) = Kv\left(1 + \frac{4\pi}{3}\right) - \frac{v^2}{2}\left(1 + \frac{4\pi}{3}\right)^2$$
$$+ Kv - \frac{v^2}{2} + Kv\left(1 - \frac{\pi}{2}\right) - \frac{v^2}{2}\left(1 - \frac{\pi}{2}\right)^2.$$

If $v = 0$ then $f_v \equiv 0$, while for $v > 0$ we have

$$f_v'(\pi) = \frac{4Kv}{3} - \frac{4v^2}{3}\left(1 + \frac{4\pi}{3}\right) - \frac{Kv}{2} + \left(1 - \frac{\pi}{2}\right)\frac{v^2}{2}$$
$$= \frac{1}{36} v(30K - (30 + 73\pi)v),$$

and therefore

$$\bar{\pi}_v = \frac{30(K - v)}{73v}$$

is the only critical point and maximizer of f_v. Notice that $\bar{\pi}_v$ verifies condition (2.171) and thus defines the optimal strategy $\pi_N^{\max}(v)$.

Also notice that

$$\lim_{v \to 0^+} \bar{\pi}_v = +\infty.$$

Inserting the expression of $\pi_N^{\max}(v)$ into (2.170) for $n = N$, we obtain the dynamics of the portfolio corresponding to the optimal strategy in the last period:

$$V_N = \begin{cases} \frac{1}{73}(40K + 33V_{N-1}), \\ V_{N-1}, \\ \frac{1}{73}(-15K + 88V_{N-1}) \end{cases}.$$

ii) For simplicity, we put

$$a^i = u^i - 1, \qquad b^i = m^i - 1, \qquad c^i = d^i - 1, \qquad i = 1, 2.$$

The DP algorithm is analogous to (2.172) where now

$$G_n(v, \mu_n; \pi_n^1, \pi_n^2) = \begin{cases} v\left(1 + \pi_n^1 a^1 + \pi_n^2 a^2\right), \\ v\left(1 + \pi_n^1 b^1 + \pi_n^2 b^2\right), \\ v\left(1 + \pi_n^1 c^1 + \pi_n^2 c^2\right). \end{cases}$$

For $N = 1$ we have

$$W_0(v) = \sup_{(\bar{\pi}^1, \bar{\pi}^2) \in D(v)} E\left[u\left(G_n(v, \mu_1; \bar{\pi}^1, \bar{\pi}^2)\right)\right] = \frac{1}{3} \max_{(\bar{\pi}^1, \bar{\pi}^2) \in D(v)} f_v\left(\bar{\pi}^1, \bar{\pi}^2\right)$$

where $D(v)$ denotes the set of pairs (π^1, π^2) such that $G_n(v, \mu_n; \pi^1, \pi^2) \leq K$ for the different values taken by μ_n and

$$\begin{aligned} f_v\left(\pi^1, \pi^2\right) = {} & Kv\left(1 + \pi^1 a^1 + \pi^2 a^2\right) - \frac{v^2}{2}\left(1 + \pi^1 a^1 + \pi^2 a^2\right)^2 \\ & + Kv\left(1 + \pi^1 b^1 + \pi^2 b^2\right) - \frac{v^2}{2}\left(1 + \pi^1 b^1 + \pi^2 b^2\right)^2 \\ & + Kv\left(1 + \pi^1 c^1 + \pi^2 c^2\right) - \frac{v^2}{2}\left(1 + \pi^1 c^1 + \pi^2 c^2\right)^2. \end{aligned}$$

Setting the gradient of the function f_v equal to zero, we obtain the system of linear equations

$$\begin{cases} Ka^1 - va^1\left(1 + \pi^1 a^1 + \pi^2 a^2\right) \\ +Kb^1 - vb^1\left(1 + \pi^1 b^1 + \pi^2 b^2\right) + Kc^1 - vc^1\left(1 + \pi^1 c^1 + \pi^2 c^2\right) = 0 \\ Ka^2 - va^2\left(1 + \pi^1 a^1 + \pi^2 a^2\right) \\ +Kb^2 - vb^2\left(1 + \pi^1 b^1 + \pi^2 b^2\right) + Kc^2 - vc^2\left(1 + \pi^1 c^1 + \pi^2 c^2\right) = 0 \end{cases}$$

with solution

$$\bar{\pi}_v^1 = \frac{38(K - v)}{3v}, \qquad \bar{\pi}_v^2 = \frac{11(v - K)}{v}.$$

A direct calculation shows that

$$G_n\left(v, \mu_n; \bar{\pi}_n^1, \bar{\pi}_n^2\right) = \begin{cases} K & \text{if } \mu_n^1 = u^1 - 1,\ \mu_n^2 = u^2 - 1, \\ v & \text{if } \mu_n^1 = m^1 - 1,\ \mu_n^2 = m^2 - 1, \\ K & \text{if } \mu_n^1 = d^1 - 1,\ \mu_n^2 = d^2 - 1 \end{cases}$$

and therefore $\left(\bar{\pi}_v^1, \bar{\pi}_v^2\right) \in D(v)$. Then, since the graph of the function

$$\left(\pi^1, \pi^2\right) \mapsto f_v\left(\pi^1, \pi^2\right)$$

is a paraboloid facing downward, we can conclude that $\left(\bar{\pi}_n^1, \bar{\pi}_n^2\right)$ is a global maximum point. □

Remark 2.58. *Assume that in Problem 2.57 the risk-free rate is not zero but $r = \frac{1}{2}$. Then the dynamics of a self-financing portfolio is given by*

$$\begin{aligned} V_n = G_n(V_{n-1}, \mu_n; \pi_n) &= V_{n-1}\left(1 + r + \pi_n\left(\mu_n - r\right)\right) \\ &= \begin{cases} V_{n-1}\left(\frac{3}{2} + \frac{5}{6}\pi_n\right) & \text{if } \mu_n = u^1 - 1, \\ V_{n-1}\left(\frac{3}{2} - \frac{\pi_n}{2}\right) & \text{if } \mu_n = m^1 - 1, \\ V_{n-1}\left(\frac{3}{2} - \pi_n\right) & \text{if } \mu_n = d^1 - 1. \end{cases} \end{aligned} \tag{2.173}$$

Hence, if $V_{n-1} \leq \frac{2}{3}K$ then $V_n \leq K$ for the values of π_n such that

$$-\left(K - \frac{3}{2}V_{n-1}\right) \leq \pi_n V_{n-1} \leq \frac{6}{5}\left(K - \frac{3}{2}V_{n-1}\right).$$

However, if $V_{n-1} > \frac{2}{3}K$ then V_n takes values greater than K for any π_n; in particular, assuming that $u = -\infty$ in $]K, +\infty[$, then the maximization problem starting from $v > \frac{2}{3}K$ has no solution. □

Problem 2.59. Given is a binomial market model over $N = 2$ periods with the following data

$$S_0 = 1, \quad u = 2, \quad d = \frac{1}{2}, \quad r = 0, \quad p = \frac{1}{2},$$

so that the unique martingale measure corresponds to $q = \frac{1}{3}$. Consider the problem of maximizing the expected utility from intermediate consumption starting from an initial capital $V_0 = v$ and for a utility function of the form

$$u_n(C) \equiv u(C) = 1 - \frac{1}{C}, \qquad C > 0,\ n \geq 0.$$

i) Using the martingale method, show that the optimal consumption in $n = 0, 1, 2$ is given by

$$\bar{C}_0 = \frac{v}{\kappa}, \qquad \bar{C}_1^{\nu_1} = \frac{v}{\kappa}\frac{\sqrt{3}}{2}(\sqrt{2})^{\nu_1}, \qquad \bar{C}_2^{\nu_2} = \frac{v}{\kappa}\frac{3}{4}(\sqrt{2})^{\nu_2}. \tag{2.174}$$

where κ is a positive constant and ν_n denotes the random number of up-movements of the asset price up to and including the n-th period (with obvious meaning, we use here the notations $\bar{C}_n^{\nu_n}$ as well as $\bar{V}_n^{\nu_n}$ and $\mathcal{F}_n^{\nu_n}$);
ii) for $n = 1$ and in the scenario $\nu_1 = 0$ determine the value $\bar{V}_1^0$ of the self-financing portfolio corresponding to the optimal consumption process;

iii) determine the optimal investment strategy (α_2, β_2) in $n = 1$ in the scenario $\nu_1 = 0$ and verify its correctness with respect to the optimal values $\bar{V}_1^0$ and $\bar{C}_1^0$ obtained in points i) and ii).

Solution of Problem 2.59

i) Since $u'(v) = \frac{1}{v^2}$, we have $\mathcal{I}(w) = \frac{1}{\sqrt{w}}$. Being $r = 0$, from Theorem 2.24 we then have

$$\bar{C}_n = \mathcal{I}(\lambda L_n) = \lambda^{-1} L_n^{-\frac{1}{2}}$$

where λ is determined by the budget equation

$$v = E^P\left[\sum_{n=0}^{2} L_n \mathcal{I}(\lambda L_n)\right] = \frac{1}{\sqrt{\lambda}}\left(1 + \sum_{n=1}^{2} E^P\left[\sqrt{L_n}\right]\right).$$

Recall now that

$$L_n^{\nu_n} = \left(\frac{q}{p}\right)^{\nu_n}\left(\frac{1-q}{1-p}\right)^{n-\nu_n}$$

which, for the given data, becomes

$$L_n^{\nu_n} = \left(\frac{4}{3}\right)^n 2^{-\nu_n}.$$

The budget equation then yields

$$v = \frac{1}{\sqrt{\lambda}}\left(1 + E^P\left[\frac{2}{\sqrt{3}} 2^{-\frac{\nu_1}{2}} + \frac{4}{3} 2^{-\frac{\nu_2}{2}}\right]\right) = \frac{\kappa}{\sqrt{\lambda}}$$

where

$$\kappa = 1 + \frac{1}{2}\left(\frac{2}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\right) + \frac{1}{3}\left(\frac{3}{2} + \sqrt{2}\right).$$

Then we have $\lambda = \frac{\kappa^2}{v^2}$ from which (2.174) follows.

ii) From the replicating condition $\bar{V}_2 = \bar{C}_2$, on the basis of equation (2.61) we obtain

$$\bar{V}_1^0 = E^Q\left[\bar{V}_2 \mid \mathcal{F}_1^0\right] + \bar{C}_1^0 = \frac{3v}{4\kappa}\left(\frac{2}{3} + \frac{1}{3}\sqrt{2}\right) + \frac{\sqrt{3}}{2\kappa} v.$$

iii) The system that has to be satisfied by the optimal (α_2, β_2) is

$$\begin{cases} \alpha_2 S_1^0 u + \beta_2 = \bar{V}_2^1 = \bar{C}_2^1 \\ \alpha_2 S_1^0 d + \beta_2 = \bar{V}_2^0 = \bar{C}_2^0 \end{cases}$$

namely

$$\begin{cases} \alpha_2 + \beta_2 = \frac{3v}{4\kappa}\sqrt{2} \\ \frac{1}{4}\alpha_2 + \beta_2 = \frac{3v}{4\kappa} \end{cases}$$

from which $\alpha_2 = \frac{v}{\kappa}\left(\sqrt{2} - 1\right)$ and $\beta_2 = \frac{v}{\kappa}\left(1 - \frac{\sqrt{2}}{4}\right)$.

The correctness of the result can be checked by verifying whether (see the definition of the self-financing condition (2.1) in Definition 2.1)

$$\alpha_n S_{n-1} + \beta_n B_{n-1} = \bar{V}_{n-1} - \bar{C}_{n-1}.$$

In the specific case this becomes, for $n = 2$,

$$\frac{1}{2}\alpha_2 + \beta_2 = \bar{V}_1^0 - \bar{C}_1^0 = \frac{v}{4\kappa}\left(2 + \sqrt{2}\right)$$

and can be seen to be satisfied by the values for α_2 and β_2 obtained above.

□

Problem 2.60. Given is a binomial market model over $N = 2$ periods with the same data as in the previous problem, namely

$$S_0 = 1, \quad u = 2, \quad d = \frac{1}{2}, \quad r = 0, \quad p = \frac{1}{2}$$

so that the unique martingale measure corresponds to $q = \frac{1}{3}$. Consider the problem of maximizing the expected utility from intermediate consumption starting from an initial capital $V_0 = v$ and for a utility function of the form

$$u_n(C) \equiv u(C) = 2\sqrt{C}, \qquad C > 0,\ n \geq 0.$$

i) Using the martingale method show that the optimal consumption in $n = 0, 1, 2$ is given by

$$\bar{C}_0 = \frac{64}{217}v, \quad \bar{C}_1^{\nu_1} = 4^{\nu_1}\frac{36}{217}v, \quad \bar{C}_2^{\nu_2} = 4^{\nu_2}\frac{81}{868}v,$$

where ν_n denotes the random number of up-movements of the asset price up to and including the n-th period;

ii) show that the value in $n = 1$ of the self-financing portfolio corresponding to the optimal consumption process is given by

$$\bar{V}_1^{\nu_1} = \begin{cases} \frac{306}{217}v & \text{if } \nu_1 = 1, \\ \frac{153}{434}v & \text{if } \nu_1 = 0; \end{cases}$$

iii) determine the optimal investment strategy (α_2, β_2) for the second period in the scenario $\nu_1 = 1$ and verify its correctness with respect to the optimal values $\bar{V}_1^1$ and $\bar{C}_1^1$ obtained in points i) and ii).

Solution of Problem 2.60

i) Since $u'(v) = v^{-\frac{1}{2}}$, we have $\mathcal{I}(w) = \frac{1}{w^2}$. Being $r = 0$, from Theorem 2.24 we then have

$$\bar{C}_n = \mathcal{I}(\lambda L_n) = \lambda^{-2} L_n^{-2}$$

where λ is determined by the budget equation

$$v = E^P\left[\sum_{n=0}^{2} L_n \mathcal{I}(\lambda L_n)\right] = \frac{1}{\lambda^2}\left(1 + \sum_{n=1}^{2} E^P\left[L_n{}^{-1}\right]\right).$$

Recall now that

$$L_n^{\nu_n} = \left(\frac{q}{p}\right)^{\nu_n}\left(\frac{1-q}{1-p}\right)^{n-\nu_n}$$

which, for the given data, becomes

$$L_n^{\nu_n} = \left(\frac{4}{3}\right)^n 2^{-\nu_n}.$$

The budget equation then yields

$$\begin{aligned}
v &= \frac{1}{\lambda^2}\left(1 + E^P\left[\frac{3}{4}2^{\nu_1} + (\frac{3}{4})^2 2^{\nu_2}\right]\right) \\
&= \frac{1}{\lambda^2}\left(1 + \left(\frac{3}{8} + \frac{3}{4}\right) + \left(\frac{9}{64} + \frac{9}{16} + \frac{9}{16}\right)\right) \\
&= \frac{1}{\lambda^2}\frac{217}{64}
\end{aligned}$$

from which $\frac{1}{\lambda^2} = \frac{64}{217}v$ and therefore

$$\begin{aligned}
\bar{C}_0 &= \frac{64}{217}v, \\
\bar{C}_1^{\nu_1} &= \frac{64}{217}v\frac{9}{16}4^{\nu_1} = 4^{\nu_1}\frac{36}{217}v, \\
\bar{C}_2^{\nu_2} &= \frac{8}{53}v\left(\frac{9}{16}\right)^2 4^{\nu_2} = 4^{\nu_2}\frac{81}{868}v.
\end{aligned}$$

ii) From the replicating condition $\bar{V}_2 = \bar{C}_2$, on the basis of equation (2.61) we obtain

$$\begin{aligned}
\bar{V}_1^{\nu_1} &= E^Q\left[\bar{V}_2 \mid \mathcal{F}_1^{\nu_1}\right] + \bar{C}_1^{\nu_1} \\
&= v\frac{81}{868}\left(\frac{1}{3}4^{\nu_1+1} + \frac{2}{3}4^{\nu_1}\right) + \frac{36}{217}4^{\nu_1}v = 4^{\nu_1}\frac{153}{434}v.
\end{aligned}$$

One may check the correctness of this result on the basis of the relation (see again (2.61))

$$v = \bar{V}_0 = E^Q\left[\bar{V}_1\right] + \bar{C}_0.$$

Computing the right hand side we obtain

$$\left(\frac{1}{3}\cdot\frac{306}{217} + \frac{2}{3}\cdot\frac{153}{434} + \frac{64}{217}\right)v$$

which, in fact, coincides with v.

iii) In the scenario $\nu_1 = 1$ the price of the underlying is $S_1 = 2$ and the strategy has to be chosen so as to replicate $\bar{V}_2^{\nu_2} = \bar{C}_2^{\nu_2}$ in $n = 2$. Being $r = 0$ and therefore $B_n \equiv B_0 = 1$, we have

$$\begin{cases} 4\alpha_2 + \beta_2 = \frac{81}{868}4^2 v = \frac{324}{217}v \\ \alpha_2 + \beta_2 = \frac{81}{868}4v = \frac{81}{217}v \end{cases}$$

with solution

$$\alpha_2 = \frac{81}{217}v, \qquad \beta_2 = 0.$$

The correctness of the result can again be checked on the basis of the relation

$$\alpha_n S_{n-1} + \beta_n B_{n-1} = \bar{V}_{n-1} - \bar{C}_{n-1}.$$

In the specific case, for $n = 2$, we now have

$$\begin{aligned} \alpha_2 S_1 + \beta_2 B_1 &= \frac{162}{217}v \\ \bar{V}_1 - \bar{C}_1 &= \frac{306}{217}v - \frac{144}{217}v = \frac{162}{217}v, \end{aligned}$$

thus confirming the correctness of the result. □

Problem 2.61. In a binomial market model over N periods, consider the problem of maximizing the expected utility from intermediate consumption starting from an initial capital $V_0 = v$, for a utility function of the form

$$u_n(C) = \sqrt{C}, \qquad C \geq 0, \; n \leq N.$$

Assuming the following data $u = 2$, $d = \frac{1}{2}$, $r = 0$ and $p = \frac{1}{2}$, use the DP algorithm to prove that the optimal expected utility is of the form

$$W_n(v) = k_n \sqrt{v}, \tag{2.175}$$

for some positive constants k_n. Determine also the optimal strategy of investment and consumption.

Solution of Problem 2.61

The dynamics of the value of a self-financing strategy with consumption is given by

$$\begin{aligned} V_n = G_n(V_{n-1}, \mu_n; \pi_n, C_{n-1}) &= (V_{n-1} - C_{n-1})(1 + r) + V_{n-1}\pi_n(\mu_n - r) \\ &= \begin{cases} V_{n-1}(1 + \pi_n) - C_{n-1} \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) - C_{n-1}, \end{cases} \end{aligned}$$

where π denotes the proportion of the risky asset in the portfolio and C is the consumption process.

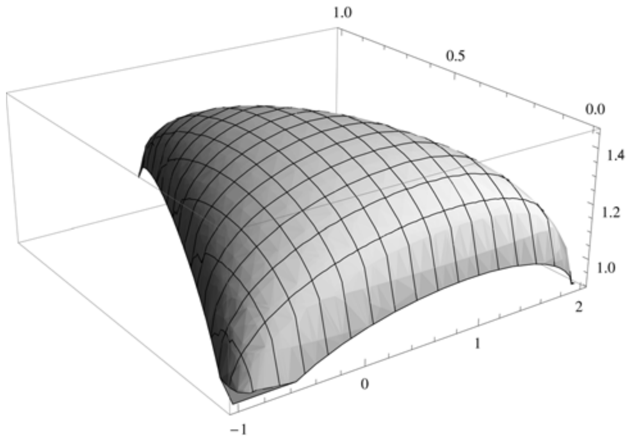

Fig. 2.10. Graph of the function $f_{n,v}$ in (2.177) for $v = 1$

We prove the thesis by induction: for $n = N$, it is obviously true with $k_N = 1$. Assuming (2.175), by the DP algorithm (cf. (2.82)) we have

$$\begin{aligned} W_{n-1}(v) &= \max_{\pi_n, C_{n-1}} \left(\sqrt{C_{n-1}} + k_n E\left[G_n(v, \mu_n; \pi_n, C_{n-1})^{\frac{1}{2}} \right] \right) \\ &= \max_{\pi, C} f_{n,v}(\pi, C) \end{aligned} \tag{2.176}$$

where the function

$$f_{n,v}(\pi, C) = \sqrt{C} + \frac{k_n}{2}\sqrt{-C + v + \pi v} + \frac{k_n}{2}\sqrt{-C + v - \frac{\pi v}{2}} \tag{2.177}$$

is defined on the set D of the pairs (π, C) with $C \geq 0$ and $G_n(v, \mu_n; \pi, C) \geq 0$. The graph of the function $f_{n,v}$, with $v = 1$, is shown in Figure 2.10.

We set the gradient of $f_{n,v}$ equal to zero to determine the critical points:

$$\begin{aligned} \partial_\pi f_{n,v}(\pi, C) &= -\frac{k_n v}{8\sqrt{-C + v - \frac{\pi v}{2}}} + \frac{k_n v}{4\sqrt{-C + v + \pi v}} = 0 \\ \partial_C f_{n,v}(\pi, C) &= \frac{1}{2\sqrt{C}} - \frac{k_n}{4\sqrt{-C + v - \frac{\pi v}{2}}} - \frac{k_n}{4\sqrt{-C + v + \pi v}} = 0. \end{aligned}$$

This system has a unique solution that is the maximum point of $f_{n,v}$ and defines the optimal strategy:

$$\pi_n^{\max}(v) = \frac{9k_n^2}{8 + 9k_n^2}, \qquad C_{n-1}^{\max}(v) = \frac{8v}{8 + 9k_n^2}.$$

Moreover, by (2.176) we have

$$\begin{aligned}
W_{n-1}(v) &= f_{n,v}(\pi_n^{\max}(v), C_{n-1}^{\max}(v)) \\
&= 2\sqrt{2}\sqrt{\frac{v}{8+9k_n^2}} + \frac{k_n}{2}\sqrt{v - \frac{8v}{8+9k_n^2} - \frac{9k_n^2 v}{2\,(8+9k_n^2)}} \\
&\quad + \frac{k_n}{2}\sqrt{v - \frac{8v}{8+9k_n^2} + \frac{9k_n^2 v}{8+9k_n^2}} \\
&= \frac{\sqrt{8+9k_n^2}}{2\sqrt{2}}\sqrt{v},
\end{aligned}$$

and this proves the thesis with the sequence k_n defined recursively by $k_N = 1$ and

$$k_{n-1} = \frac{\sqrt{8+9k_n^2}}{2\sqrt{2}}.$$

Notice that the maximization has been performed on the domain of the function $f_{n,v}$ and in particular $(\pi_n^{\max}, C_{n-1}^{\max}) \in D$. □

Problem 2.62. In a binomial market model over N periods, consider the problem of maximizing the expected utility from intermediate consumption for the power utility function

$$u_n(C) = \frac{C^\gamma}{\gamma}, \qquad C > 0,\ n \le N,$$

where γ is a fixed parameter such that $\gamma < 1$ and $\gamma \neq 0$.

i) Using the martingale method, show that the optimal consumption process is given by

$$\bar{C}_n = Mv\widetilde{L}_n^{\frac{1}{\gamma-1}}, \tag{2.178}$$

where M is a positive constant (to be determined) and $\widetilde{L}_n = B_n^{-1}L_n$ with

$$L_n = E^P\left[\frac{dQ}{dP} \mid \mathcal{F}_n\right],$$

that is (L_n) is the Radon-Nikodym derivative process of the martingale measure Q with respect to the physical measure P;

ii) using the risk-neutral valuation formula

$$V_{n-1} = \frac{1}{1+r}E^Q\left[V_n \mid \mathcal{F}_{n-1}\right] + C_{n-1}, \qquad n = 1, \dots, N, \tag{2.179}$$

prove by induction that the value of the optimal portfolio is of the form

$$\bar{V}_n = M_n\bar{C}_n \tag{2.180}$$

where $M_0, \dots, M_N$ are positive constants to be determined;

iii) determine the process $\bar{\alpha}$ of the optimal strategy.

Solution of Problem 2.62

i) We recall the expression in (2.38) of the process L_n:

$$L_n = \left(\frac{q}{p}\right)^{\nu_n} \left(\frac{1-q}{1-p}\right)^{n-\nu_n},$$

where ν_n is the random number of up-movements of the risky asset in the first n periods. Moreover we have $u'(v) = v^{\gamma-1}$ and then $\mathcal{I}(w) = w^{\frac{1}{\gamma-1}}$.

By Theorem 2.24 we have that the optimal consumption is equal to

$$\bar{C}_n = \left(\lambda \widetilde{L}_n\right)^{\frac{1}{\gamma-1}}, \tag{2.181}$$

where λ is determined by the budget equation

$$v = E^P\left[\sum_{n=0}^{N} \widetilde{L}_n \mathcal{I}\left(\lambda \widetilde{L}_n\right)\right] = \lambda^{\frac{1}{\gamma-1}} \sum_{n=0}^{N} E^P\left[\widetilde{L}_n^{\frac{\gamma}{\gamma-1}}\right].$$

Thus we have

$$\lambda^{\frac{1}{\gamma-1}} = Mv$$

where M is the constant defined by

$$M^{-1} = \sum_{n=0}^{N} E^P\left[\widetilde{L}_n^{\frac{\gamma}{\gamma-1}}\right].$$

Inserting the expression of λ into (2.181), we get (2.178):

$$\bar{C}_n = Mv\widetilde{L}_n^{\frac{1}{\gamma-1}}.$$

Notice that

$$\begin{aligned} M^{-1} &= \sum_{n=0}^{N} \frac{1}{(1+r)^{\frac{n\gamma}{\gamma-1}}} E\left[L_n^{\frac{\gamma}{\gamma-1}}\right] \\ &= \sum_{n=0}^{N} \frac{1}{(1+r)^{\frac{n\gamma}{\gamma-1}}} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{q}{p}\right)^{k\frac{\gamma}{\gamma-1}} \left(\frac{1-q}{1-p}\right)^{(n-k)\frac{\gamma}{\gamma-1}} p^k (1-p)^{n-k}. \end{aligned}$$

For instance, for $N = 2$ we have

$$\begin{aligned} M^{-1} = 1 &+ \frac{1}{(1+r)^{\frac{\gamma}{\gamma-1}}} \left(p\left(\frac{q}{p}\right)^{\frac{\gamma}{\gamma-1}} + (1-p)\left(\frac{1-q}{1-p}\right)^{\frac{\gamma}{\gamma-1}} \right) \\ &+ \frac{1}{(1+r)^{\frac{2\gamma}{\gamma-1}}} \left(p^2\left(\frac{q}{p}\right)^{\frac{2\gamma}{\gamma-1}} + 2p(1-p)\left(\frac{q(1-q)}{p(1-p)}\right)^{\frac{\gamma}{\gamma-1}} \right. \\ &\left. + (1-p)^2\left(\frac{1-q}{1-p}\right)^{\frac{2\gamma}{\gamma-1}} \right). \end{aligned}$$

ii) We proceed by backward induction: since $\bar{V}_N = \bar{C}_N$, then (2.180) is true for $n = N$ with $M_N = 1$. Let us now assume that

$$\bar{V}_n = M_n \bar{C}_n$$

for a suitable constant M_n: then by the risk-neutral valuation formula we have

$$\bar{V}_{n-1} = \frac{M_n}{1+r} E^Q \left[\bar{C}_n \mid \mathcal{F}_{n-1} \right] + \bar{C}_{n-1}. \tag{2.182}$$

Now, from (2.178) it follows that

$$\begin{aligned}
E^Q \left[\bar{C}_n \mid \mathcal{F}_{n-1} \right] &= M v E^Q \left[\widetilde{L}_n^{\frac{1}{\gamma-1}} \mid \mathcal{F}_{n-1} \right] \\
&= \frac{Mv}{(1+r)^{\frac{n}{\gamma-1}}} \left(q \left(\frac{q}{p} \right)^{\frac{\nu_{n-1}+1}{\gamma-1}} \left(\frac{1-q}{1-p} \right)^{\frac{n-1-\nu_{n-1}}{\gamma-1}} \right. \\
&\qquad \left. + (1-q) \left(\frac{q}{p} \right)^{\frac{\nu_{n-1}}{\gamma-1}} \left(\frac{1-q}{1-p} \right)^{\frac{n-\nu_{n-1}}{\gamma-1}} \right) \\
&= \frac{\bar{C}_{n-1}}{(1+r)^{\frac{1}{\gamma-1}}} \left(q \left(\frac{q}{p} \right)^{\frac{1}{\gamma-1}} + (1-q) \left(\frac{1-q}{1-p} \right)^{\frac{1}{\gamma-1}} \right) = \bar{C}_{n-1} E^Q \left[\widetilde{L}_1^{\frac{1}{\gamma-1}} \right].
\end{aligned}$$

Plugging the last expression into (2.182), we get

$$\bar{V}_{n-1} = \left(1 + \frac{M_n}{1+r} E^Q \left[\widetilde{L}_1^{\frac{1}{\gamma-1}} \right] \right) \bar{C}_{n-1}$$

which proves the claim with the constants M_n defined recursively by

$$M_N = 1, \qquad M_{n-1} = \left(1 + \frac{M_n}{1+r} E^Q \left[\widetilde{L}_1^{\frac{1}{\gamma-1}} \right] \right),$$

or, more explicitly,

$$M_{N-n} = \sum_{k=0}^{n} \frac{1}{(1+r)^k} \left(E^Q \left[\widetilde{L}_1^{\frac{1}{\gamma-1}} \right] \right)^k.$$

iii) To determine the optimal strategy for the n-th period, we impose the replication condition

$$\alpha_n S_n + \beta_n B_n = \bar{V}_n,$$

and use the expression of $\bar{V}_n$ calculated in the previous step, i.e.

$$\bar{V}_n = M M_n v \widetilde{L}_n^{\frac{1}{\gamma-1}}.$$

Then we have

$$\begin{cases}
\alpha_n u S_{n-1} + \beta_n B_n = \frac{M M_n v}{(1+r)^{\frac{n}{\gamma-1}}} \left(\frac{q}{p} \right)^{\frac{\nu_{n-1}+1}{\gamma-1}} \left(\frac{1-q}{1-p} \right)^{\frac{n-1-\nu_{n-1}}{\gamma-1}} \\
\alpha_n d S_{n-1} + \beta_n B_n = \frac{M M_n v}{(1+r)^{\frac{n}{\gamma-1}}} \left(\frac{q}{p} \right)^{\frac{\nu_{n-1}}{\gamma-1}} \left(\frac{1-q}{1-p} \right)^{\frac{n-\nu_{n-1}}{\gamma-1}}
\end{cases}$$

from which

$$\alpha_n = \frac{MM_n v}{(u-d)S_{n-1}(1+r)^{\frac{n}{\gamma-1}}} \cdot \\ \cdot \left(\left(\frac{q}{p}\right)^{\frac{1}{\gamma-1}} - \left(\frac{1-q}{1-p}\right)^{\frac{1}{\gamma-1}} \right) \left(\frac{q}{p}\right)^{\frac{\nu_{n-1}}{\gamma-1}} \left(\frac{q-1}{p-1}\right)^{\frac{n-1-\nu_{n-1}}{\gamma-1}} \\ = \frac{MM_n v}{(u-d)S_{n-1}(1+r)^{\frac{n}{\gamma-1}}} \left(\left(\frac{q}{p}\right)^{\frac{1}{\gamma-1}} - \left(\frac{1-q}{1-p}\right)^{\frac{1}{\gamma-1}} \right) L_{n-1}^{\frac{1}{\gamma-1}}. \qquad \square$$

Problem 2.63. In a binomial market model over N periods, consider the problem of maximizing the expected utility from intermediate consumption for a utility function of the form

$$u_n(C) = -e^{-C}, \qquad C \in \mathbb{R},\ n \leq N.$$

Putting $u = 2$, $d = \frac{1}{2}$, $r = 0$ and $p = \frac{1}{2}$, use the Dynamic Programming algorithm to prove that the optimal expected utility is of the form

$$W_n(v) = -h_n e^{-k_n v}, \tag{2.183}$$

for suitable positive constants h_n, k_n. Determine also the optimal strategy of investment and consumption.

Solution of Problem 2.63

As in Problem 2.61, the dynamics of the value of a self-financing strategy with consumption is given by

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n, C_{n-1}) = (V_{n-1} - C_{n-1})(1+r) + V_{n-1}\pi_n(\mu_n - r) \\ = \begin{cases} V_{n-1}(1+\pi_n) - C_{n-1} \\ V_{n-1}\left(1 - \frac{\pi_n}{2}\right) - C_{n-1}, \end{cases}$$

where π denotes the proportion of the risky asset in the portfolio and C is the consumption process.

Proceed by induction: the claim is obviously true for $n = N$ with $h_N = k_N = 1$. Assuming (2.183), by the DP algorithm we have

$$W_{n-1}(v) = \max_{\pi_n, C_{n-1}} \left(-e^{-C_{n-1}} + h_n E\left[e^{-k_n G_n(v,\mu_n;\pi_n,C_{n-1})} \right] \right) \\ = \max_{\pi, C} f_{n,v}(\pi, C) \tag{2.184}$$

where the function

$$f_{n,v}(\pi, C) = -e^{-C} - \frac{h_n}{2}\left(e^{-k_n(-C+v+\pi v)} + e^{-k_n\left(-C+v-\frac{\pi v}{2}\right)} \right), \tag{2.185}$$

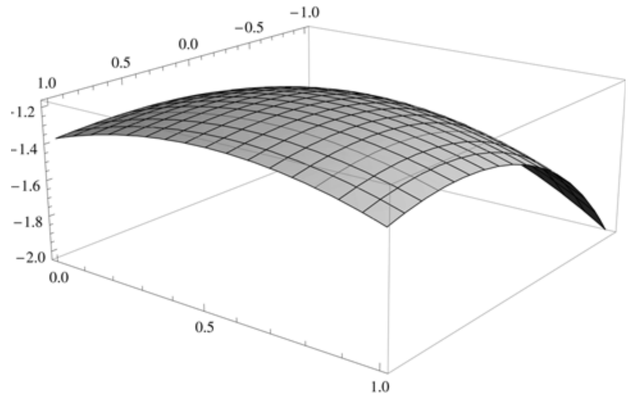

Fig. 2.11. Graph of the function $f_{n,v}$ in (2.185) with $v = k_n = h_n = 1$

is defined for any $C \geq 0$ and $\pi \in \mathbb{R}$. The graph of the function $f_{n,v}$ is shown in Figure 2.11.

We set the gradient of $f_{n,v}$ equal to zero to determine the critical points: we obtain the system

$$\partial_\pi f_{n,v}(\pi, C) = -\frac{k_n h_n v}{4} e^{-k_n\left(-C+v-\frac{\pi v}{2}\right)} + \frac{k_n h_n v}{2} e^{-k_n(-C+v+\pi v)} = 0$$
$$\partial_C f_{n,v}(\pi, C) = e^{-C} - \frac{k_n h_n}{2} e^{-k_n\left(-C+v-\frac{\pi v}{2}\right)} - \frac{k_n h_n}{2} e^{-k_n(-C+v+\pi v)} = 0,$$

which is equivalent to

$$e^{\frac{3k_n \pi v}{2}} = 2, \qquad k_n h_n e^{(1+k_n)C - k_n(1+\pi)v} \left(1 + e^{\frac{3k_n \pi v}{2}}\right) = 2.$$

This system has a unique solution which is the maximizer of $f_{n,v}$ and defines the optimal strategy

$$\pi_n^{\max}(v) = \frac{2\log 2}{3k_n v}, \qquad C_{n-1}^{\max}(v) = \frac{k_n v + \log\left(\frac{2}{3}\frac{2^{2/3}}{k_n h_n}\right)}{1 + k_n}. \tag{2.186}$$

Furthermore, from (2.184) we get

$$W_{n-1}(v) = f_{n,v}(\pi_n^{\max}(v), C_{n-1}^{\max}(v)) = -h_{n-1} e^{-k_{n-1} v},$$

where the sequences (h_n) and (k_n) are recursively defined by $h_N = k_N = 1$ and

$$h_{n-1} = \frac{1 + k_n}{k_n} \left(2^{-\frac{5}{3}} 3 k_n h_n\right)^{\frac{1}{1+k_n}}, \qquad k_{n-1} = \frac{k_n}{1 + k_n}. \tag{2.187}$$

Since the exponential utility function is defined over $\mathbb{R}$, we must verify that the process $C^{\max}$ in (2.186) is a consumption process and, in particular, it is such that

$$C_n^{\max} \geq 0, \qquad 0 \leq n \leq N. \tag{2.188}$$

It is easily proved that

$$0 < K_{n-1} \leq k_n \leq 1,$$

and consequently, by the definition (2.186), for (2.188) to hold, it is sufficient to show that

$$y_n := k_n h_n \leq \frac{2}{3} 2^{\frac{2}{3}}, \qquad 0 \leq n \leq N.$$

On the other hand, by definition (2.187), we have

$$y_{n-1} = \left(2^{-\frac{5}{3}} 3 y_n\right)^{\frac{1}{1+k_n}} \leq \sqrt{2^{-\frac{5}{3}} 3 y_n} < \sqrt{y_n},$$

so that $y_n \leq 1 < \frac{2}{3} 2^{\frac{2}{3}}$. □

Problem 2.64. Consider a completed trinomial market model over N periods, with

$$u_1 = 2, \quad m_1 = 1, \quad d_1 = \frac{1}{2}, \quad u_2 = \frac{8}{3}, \quad m_2 = \frac{8}{9}, \quad d_2 = \frac{1}{3},$$
$$S_0^1 = S_0^2 = 1, \quad r = 0 \quad \text{and} \quad p_1 = p_2 = p_3 = \frac{1}{3}.$$

Use the martingale method to solve the problem of maximizing the expected utility from intermediate consumption starting from an initial capital $V_0 = v$, for a utility function of the form

$$u_n(C) = 2\sqrt{C}, \qquad C > 0, \; n \leq N.$$

i) Show that the unique equivalent martingale measure Q is defined by

$$q_1 = \frac{1}{6}, \quad q_2 = \frac{1}{2}, \quad q_3 = \frac{1}{3},$$

and that the Radon-Nikodym derivative of the martingale measure with respect to the physical measure is given by

$$L_n = \left(\frac{1}{2}\right)^{\nu_n^1} \left(\frac{3}{2}\right)^{\nu_n^1 2},$$

where ν_n^1 and ν_n^2 denote the random number of movements occurring in the first n periods and corresponding to u and m respectively;

ii) in the case of $N = 1$, prove that

$$\bar{C}_0 = \frac{9}{20} v, \qquad \bar{C}_1 = \begin{cases} \frac{9}{5} v & \text{if } \nu_1^1 = 1, \\ \frac{1}{5} v & \text{if } \nu_1^2 = 1, \\ \frac{9}{20} v & \text{if } \nu_1^1 = \nu_1^2 = 0. \end{cases}$$

How can one check directly the correctness of these numerical results?

iii) Knowing that $V_1 = \bar{C}_1$, write the system of equations satisfied by the hedging strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$ in the first period. How can one check the correctness of the solution?

Solution of Problem 2.64

i) The values of q_i are given by formula (1.46). Concerning L_n, we have the general expression (see (2.41))

$$L_n = \left(\frac{q_1}{p_1}\right)^{\nu_n^1} \left(\frac{q_2}{p_2}\right)^{\nu_n^2} \left(\frac{q_3}{p_3}\right)^{n-\nu_n^1-\nu_n^2}.$$

In our case, since $q_3 = p_3$, we easily get the result.

ii) By Theorem 2.24 the optimal consumption is equal to

$$\bar{C}_n = \mathcal{I}_n(\lambda L_n), \quad \text{with} \quad \mathcal{I}_n = (u_n')^{-1},$$

and the budget equation is

$$E^P\left[\sum_{n=0}^{N} L_n \mathcal{I}_n(\lambda L_n)\right] = v.$$

In our case $\mathcal{I}_n(y) = \frac{1}{y^2}$ so that

$$\bar{C}_n = \lambda^{-2} 4^{\nu_n^1} \left(\frac{4}{9}\right)^{\nu_n^2}$$

and the budget equation becomes

$$v = \lambda^{-2} E^P\left[1 + 2^{\nu_1^1}\left(\frac{2}{3}\right)^{\nu_1^2}\right] = \lambda^{-2}\left(1 + \frac{2}{3} + \frac{2}{9} + \frac{1}{3}\right) = \frac{20}{9\lambda^2}.$$

Hence we have $\lambda^{-2} = \frac{9}{20} v$ and

$$\bar{C}_n = \frac{9}{20} 4^{\nu_n^1} \left(\frac{4}{9}\right)^{\nu_n^2} v.$$

In particular, for $n = 0$ and $n = 1$, the result follows.

A verification can be obtained on the basis of the following relation, which is a consequence of the self-financing condition (2.2), equation $V_1 = \bar{C}_1$ and the assumption $r = 0$:

$$E^Q\left[\bar{C}_1\right] = v - \bar{C}_0.$$

The left hand side of the previous equation is equal to

$$v\left(\frac{1}{6} \cdot \frac{9}{5} + \frac{1}{2} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{9}{20}\right) = \frac{11}{20} v$$

and this value coincides with that of the right side, that is

$$v - \bar{C}_0 = v\left(1 - \frac{9}{20}\right) = \frac{11}{20}v.$$

iii) The system of equations, satisfied by the hedging strategy $(\alpha_1^1, \alpha_1^2, \beta_1)$, is

$$\begin{cases} 2\alpha_1^1 + \frac{8}{3}\alpha_1^2 + \beta_1 = \frac{9}{5}v \\ \alpha_1^1 + \frac{8}{9}\alpha_1^2 + \beta_1 = \frac{1}{5}v \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \beta_1 = \frac{9}{20}v. \end{cases}$$

To verify the correctness of the numerical solution we note that, with the data from this problem, we have

$$v = V_0 = \alpha_1^1 + \alpha_1^2 + \beta_1.$$

On the other hand, as already seen in ii), we necessarily have

$$v - \bar{C}_0 = E^Q\left[\bar{C}_1\right].$$

□

3

American options

The American options generalize the European options in the sense that they can be exercised at any moment prior to maturity. They are part of the more general category of American-type derivatives that we shall define in Subsection 3.1 as a sequence $X = (X_n)$ of random variables that are adapted to a given filtration $(\mathcal{F}_n)$, typically generated by the prices of the underlyings. The value of X_n is the premium/payoff paid to the holder of the derivative if he/she exercises the option at time t_n.

In addition to the typical problems for European derivatives, for the American derivatives we have also the problem of determining the optimal exercise strategy. The latter is an optimization problem, and for this reason we have made the chapter on American options to follow that on portfolio optimization. We shall therefore deal with the following problems:

i) determining the *arbitrage price* of an American option/derivative;
ii) determining an *optimal exercise strategy*;
iii) determining a *hedging strategy* for the American derivative.

The basic tool for the solution of these problems is given by the so-called *Snell envelope* of a given sequence of random variables, in our case $X = (X_n)$, which is defined below by (3.9) for discounted values and by (3.19) for undiscounted ones. We shall see that the Snell envelope of the American derivative $X = (X_n)$ allows one to obtain, in addition to the arbitrage price of the option, also two optimal exercise strategies, the so-called minimal and maximal ones. Notice also that the form of (3.9), and of (3.19) respectively, recalls the Dynamic Programming algorithm.

With respect to the case of European derivatives, the pricing of American derivatives is made difficult by the fact that, in general, the process $\widetilde{X} = (\widetilde{X}_n)$ of discounted payoff values is not a martingale and it is thus not possible to determine a replicating strategy (α, β), namely such that $V_n^{(\alpha,\beta)} = X_n$ a.s., which on the contrary was one of the fundamentals of arbitrage free pricing in the European case. While in the majority of the textbooks (see e.g. [16] and

Pascucci A., Runggaldier W.J.: Financial Mathematics. Theory and Problems for Multi-period Models. DOI 10.1007/978-88-470-2538-7_3, © Springer-Verlag Italia 2012

[20]) the arbitrage price of an American derivative is defined inductively by means of the Snell envelope of $\widetilde{X} = (\widetilde{X}_n)$, here in the Sections 3.1.1 and 3.1.2 we follow [17] and define the price of an American derivative as the unique value that does not generate arbitrage possibilities; we then show that it coincides with the price defined on the basis of the Snell envelope.

While for pricing and hedging it is natural to assume that the market is complete, as we shall do also in this chapter, market completeness is irrelevant when determining an optimal exercise strategy. As we shall see, this fact follows directly from the algorithm based on the Snell envelope for which the existence of a martingale measure suffices. On the other hand, in the problem section we shall consider only complete markets.

The question arises naturally on whether there exists a relationship between American and European options and we shall deal with this issue towards the end of the theoretical part, namely in Section 3.2.

In the section on solved problems we shall deal with the three problems of pricing, hedging and optimal exercising for various types of American options in complete market models, namely the binomial and the completed trinomial models. To allow for an easier transition from the theoretical part to that of the problems as such, we have included Section 3.3.1 of "Preliminaries".

3.1 American derivatives and early exercise strategies

Consider a market model (B, S) of the type introduced in Chapter 1 and suppose that there is absence of arbitrage opportunities or, equivalently, there exists at least one martingale measure Q.

An American derivative is characterized by the possibility of an early exercise at a time instant t_n, $0 \leq n \leq N$ during the lifetime of the contract. To describe an American derivative it is therefore necessary to specify the premium (the payoff) that has to be paid to the holder in case he/she decides to exercise the option at time t_n with $n \leq N$. As for the European derivatives, also the American ones are relative to one or more underlyings, which typically are risky assets and of which the price is generically denoted by S. For example, in the case of an American Call with underlying S and strike K, the payoff at time t_n is given by $X_n = (S_n - K)^+$. Even if in this book we consider $\mathcal{F}_n = \mathcal{F}_n^\mu = \mathcal{F}_n^S$ for each n, to emphasize that a given derivative depends on the underlying S, by analogy to Definition 1.17 of a European derivative, in the following definition of American derivative we denote the filtration explicitly by $\left(\mathcal{F}_n^S\right)$.

Definition 3.1. *An American derivative is a non-negative stochastic process $X = (X_n)$ adapted to the filtration $(\mathcal{F}_n^S)$.*

Remark 3.2. *Recall that we had assumed an underlying probability space in which Ω has a finite number of elements. Consequently, all expected values of X that we shall introduce in the sequel exist and are finite.* □

Since the choice of the time instant at which to exercise an American option has to depend only on the currently available information, which typically is given by the observations of the underlying asset's prices, the following definition of *exercise strategy* appears to be natural.

Definition 3.3. *An exercise strategy (or time) is a stopping time, namely a random variable*

$$\nu : \ \Omega \longrightarrow \{0, 1, \dots, N\},$$

such that

$$\{\nu = n\} \in \mathcal{F}_n, \qquad n = 0, \dots, N. \tag{3.1}$$

We denote by $\mathcal{T}_0$ the family of all exercise strategies.

Intuitively, for each trajectory $\omega \in \Omega$ of the underlying market, the number $\nu(\omega)$ represents the time instant in which one decides to exercise the American derivative. Condition (3.1) expresses the fact that the decision to exercise at the n-th time instant depends only on $\mathcal{F}_n$, namely on the information that is available at time t_n.

Definition 3.4. *Given an American derivative X and an exercise strategy $\nu \in \mathcal{T}_0$, the random variable X_ν defined by*

$$(X_\nu)(\omega) = X_{\nu(\omega)}(\omega), \qquad \omega \in \Omega,$$

is called the payoff of X relative to the strategy ν. Given a martingale measure Q, we say that an exercise strategy ν_0 is optimal for X in Q if

$$E^Q\left[\widetilde{X}_{\nu_0}\right] = \max_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right]. \tag{3.2}$$

One may consider the random variable $\widetilde{X}_\nu$ as discounted payoff of an *European option*. The expectation $E^Q\left[\widetilde{X}_\nu\right]$ then represents the risk neutral price of the option (in the martingale measure Q), for the choice of ν as early exercise.

3.1.1 Arbitrage pricing

In a market that is free of arbitrage and complete, the price of a European derivative with payoff X_N is by definition equal to the value of a replicating strategy: in particular, *the discounted arbitrage price is a Q-martingale.* In pricing an American derivative we have to take into account the fact that, in general, it is not possible to determine a replicating strategy, namely a strategy $(\alpha, \beta) \in \mathcal{A}$ such that $V_n^{(\alpha,\beta)} = X_n$, a.s. for all n. The reason is simple: while $\widetilde{V}^{(\alpha,\beta)}$ is a Q-martingale, $\widetilde{X}$ is in general only an adapted process.

On the other hand, on the basis of arbitrage considerations it is possible to determine an upper and a lower bound for the price of X. To fix ideas, denote by H_0 a possible initial price of X and, by analogy to the development in Section 1.3.2, define the family of super-replicating strategies for X:

$$\mathcal{A}_X^+ = \{(\alpha, \beta) \in \mathcal{A} \mid V_n^{(\alpha,\beta)} \geq X_n,\ n = 0, \dots, N\}.$$

To avoid introducing arbitrage opportunities, the price H_0 has to be less than or equal to the initial value $V_0^{(\alpha,\beta)}$ of any $(\alpha, \beta) \in \mathcal{A}_X^+$, namely

$$H_0 \leq \inf_{(\alpha,\beta)\in\mathcal{A}_X^+} V_0^{(\alpha,\beta)}.$$

Analogously, define

$$\mathcal{A}_X^- = \{(\alpha, \beta) \in \mathcal{A} \mid \text{ there exists } \nu \in \mathcal{T}_0 \text{ s.t. } X_\nu \geq V_\nu^{(\alpha,\beta)}\}.$$

Intuitively, an element (α, β) of $\mathcal{A}_X^-$ represents an investment strategy on which to take a short position in order to receive funds with which to buy the American option and to exploit the fact that there exists an exercise strategy ν which guarantees a payoff X_ν greater than or equal to $V_\nu^{(\alpha,\beta)}$: such a payoff is therefore sufficient to close the short position on (α, β). To avoid creating an arbitrage opportunity, the initial price H_0 of X has necessarily to be greater or equal to $V_0^{(\alpha,\beta)}$ for any $(\alpha, \beta) \in \mathcal{A}_X^-$ and so we have

$$\sup_{(\alpha,\beta)\in\mathcal{A}_X^-} V_0^{(\alpha,\beta)} \leq H_0.$$

Summing up, we have determined an interval for the possible arbitrage prices of X. We show now that the risk neutral value of the payoff relative to an optimal exercise strategy belongs to this interval.

Proposition 3.5. *In a market free of arbitrage, for each martingale measure Q we have*

$$\sup_{(\alpha,\beta)\in\mathcal{A}_X^-} V_0^{(\alpha,\beta)} \leq \max_{\nu\in\mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right] \leq \inf_{(\alpha,\beta)\in\mathcal{A}_X^+} V_0^{(\alpha,\beta)}. \tag{3.3}$$

Proof. For each $(\alpha, \beta) \in \mathcal{A}_X^-$ there exists $\nu_0 \in \mathcal{T}_0$ such that $V_{\nu_0}^{(\alpha,\beta)} \leq X_{\nu_0}$. Furthermore, $\widetilde{V}^{(\alpha,\beta)}$ is a Q-martingale and therefore by the Optional Sampling Theorem[1] we have

$$V_0^{(\alpha,\beta)} = \widetilde{V}_0^{(\alpha,\beta)} = E^Q\left[\widetilde{V}_{\nu_0}^{(\alpha,\beta)}\right] \leq E^Q\left[\widetilde{X}_{\nu_0}\right] \leq \sup_{\nu\in\mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right],$$

from which, by the arbitrariness of $(\alpha, \beta) \in \mathcal{A}_X^-$ we obtain the first inequality in (3.3).

[1]The Optional Sampling Theorem states that if M is a martingale (a sub-martingale) and ν is a bounded stopping time then $E[M_\nu] = M_0$ ($E[M_\nu] \geq M_0$ respectively). For the proof see for example Theorem A.129 in [17].

On the other hand, if $(\alpha, \beta) \in \mathcal{A}_X^+$ then, again by the Optional Sampling Theorem, for each $\nu \in \mathcal{T}_0$ we have

$$V_0^{(\alpha,\beta)} = E^Q \left[\widetilde{V}_\nu^{(\alpha,\beta)} \right] \geq E^Q \left[\widetilde{X}_\nu \right],$$

from which we obtain the second inequality in (3.3), given the arbitrariness of $(\alpha, \beta) \in \mathcal{A}_X^+$ and $\nu \in \mathcal{T}_0$. □

3.1.2 Arbitrage price in a complete market

By analogy to what happens for European-type derivatives, in order to be able to give a unique definition of arbitrage price for an American option, assume that the market is complete, namely that there exists a unique equivalent martingale measure. To introduce now the definition of arbitrage price of an American derivative we need the following preliminary result.

Theorem 3.6 (Doob's decomposition theorem). *Every adapted process Y can be decomposed in a unique way into the sum*

$$Y = M + A \tag{3.4}$$

where M is a martingale such that $M_0 = Y_0$ and A is a predictable process with $A_0 = 0$. Furthermore, Y is a super-martingale if and only if A is decreasing.

Proof. Define recursively the processes M and A by putting

$$\begin{cases} M_0 = Y_0, \\ M_n = M_{n-1} + Y_n - E\left[Y_n \mid \mathcal{F}_{n-1}\right], & n \geq 1, \end{cases} \tag{3.5}$$

and

$$\begin{cases} A_0 = 0, \\ A_n = A_{n-1} - \left(Y_{n-1} - E\left[Y_n \mid \mathcal{F}_{n-1}\right]\right), & n \geq 1. \end{cases} \tag{3.6}$$

More explicitly, we have

$$M_n = Y_n + \sum_{k=0}^{n-1} \left(Y_k - E\left[Y_{k+1} \mid \mathcal{F}_k\right]\right), \tag{3.7}$$

and

$$A_n = -\sum_{k=0}^{n-1} \left(Y_k - E\left[Y_{k+1} \mid \mathcal{F}_k\right]\right). \tag{3.8}$$

It is then easy to verify that M is a martingale, A is predictable and (3.4) holds. One can show the statement also by induction and in this case it suffices to use formulae (3.5) and (3.6).

Concerning the uniqueness of the decomposition, if (3.4) holds then we have also

$$Y_n - Y_{n-1} = M_n - M_{n-1} + A_n - A_{n-1},$$

and considering conditional expectations (under the assumption that M is a martingale and A is predictable), we have

$$E\left[Y_n \mid \mathcal{F}_{n-1}\right] - Y_{n-1} = A_n - A_{n-1},$$

from which it follows that A has necessarily to be defined by formula (3.6). Finally, from (3.4) and from the previous relation, we may conclude that M is uniquely defined by the recursive relation (3.5). □

To obtain the arbitrage price as well as the hedging strategy in a complete market, and to determine an optimal exercise strategy in an arbitrage free market, a fundamental tool is given by the so-called *Snell envelope* of the process X that defines a given American derivative.

In general, having fixed a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_n)$, we give the following:

Definition 3.7. *Given an adapted process X, we call Snell envelope of X the smallest super-martingale that dominates X.*

We have now the following:

Lemma 3.8. *Given an American derivative X and denoting by $\widetilde{X}$ its discounted value, the process $\widetilde{H}$ defined recursively by*

$$\widetilde{H}_n = \begin{cases} \widetilde{X}_N, & n = N, \\ \max\left\{\widetilde{X}_n, E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right]\right\}, & n = 0, \dots, N-1, \end{cases} \tag{3.9}$$

is the smallest super-martingale that dominates $\widetilde{X}$ and is thus the Snell envelope of $\widetilde{X}$.

Proof. Evidently $\widetilde{H}$ is an adapted and non-negative process. Furthermore, for each n we have

$$\widetilde{H}_n \geq E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right], \tag{3.10}$$

namely $\widetilde{H}$ is a *Q-super-martingale.* Furthermore $\widetilde{H}$ *is the smallest super-martingale that dominates $\widetilde{X}$:* in fact, if Y is a Q-super-martingale such that $Y_n \geq \widetilde{X}_n$, then first of all we have

$$\widetilde{H}_N = \widetilde{X}_N \leq Y_N.$$

The statement then follows by induction: assuming that $\widetilde{H}_n \leq Y_n$, we obtain

$$\begin{aligned} \widetilde{H}_{n-1} &= \max\left\{\widetilde{X}_{n-1}, E^Q\left[\widetilde{H}_n \mid \mathcal{F}_{n-1}\right]\right\} \\ &\leq \max\left\{\widetilde{X}_{n-1}, E^Q\left[Y_n \mid \mathcal{F}_{n-1}\right]\right\} \\ &\leq \max\left\{\widetilde{X}_{n-1}, Y_{n-1}\right\} = Y_{n-1}. \end{aligned}$$

□

Under the assumption that the market is arbitrage free and complete, the following result is preliminary to the definition of arbitrage price of the American option X.

Theorem 3.9. *Assume that there exists a unique martingale measure Q. Then there exists a strategy $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$ and so we have:*

i) $V_n^{(\alpha,\beta)} \geq X_n, \ n = 0, \dots, N$;
ii) there exists $\nu_0 \in \mathcal{T}_0$ *such that* $X_{\nu_0} = V_{\nu_0}^{(\alpha,\beta)}$.

Furthermore

$$E^Q\left[\widetilde{X}_{\nu_0}\right] = V_0^{(\alpha,\beta)} = \max_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_{\nu}\right], \tag{3.11}$$

and this value defines the initial arbitrage price of X.

Remark 3.10. *On the basis of what had been remarked in Section 3.1.1, the initial price of X, as defined in* (3.11), *is the unique value to be assigned to X in order to avoid introducing arbitrage possibilities. As in the European case, this price corresponds to the initial value of a hedging strategy for X. Furthermore, it is also equal to the expected value under Q of the payoff that is obtained by an optimal early exercise strategy.* □

Remark 3.11. *The existence of an optimal exercise strategy ν_0 is obtained here only for complete markets. In Section 3.1.3 we shall show the existence of an optimal exercise strategy under the only assumption of absence of arbitrage opportunities.* □

Proof. The proof is constructive and consists of two main steps, namely:

1) construct the Snell envelope of the process $\widetilde{X}$;
2) use Doob's decomposition theorem to separate out the martingale part of the process $\widetilde{H}$ for the purpose of determining the strategy $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$.

We then conclude the proof of the theorem by showing that $\widetilde{H}_0 = \widetilde{V}_0^{(\alpha,\beta)} = V_0^{(\alpha,\beta)}$ and that (3.11) holds.

Step 1. Introduce the process H by putting $H_n = B_n \widetilde{H}_n$ where $\widetilde{H}$ is defined recursively in (3.9). We show (cf. (3.14)) that H defines the arbitrage price of the American derivative X. This definition has a clear intuitive meaning: at maturity, the option X has in fact the value $H_N = X_N$ and, at time t_{N-1} the value

- X_{N-1} in case it is exercised;
- $\frac{1}{1+r} E^Q\left[H_N \mid \mathcal{F}_{N-1}\right]$ equal to the price of a European Call option with payoff H_N and maturity N, in case it is not exercised.

It appears then reasonable to define

$$H_{N-1} = \max\left\{X_{N-1}, \frac{1}{1+r} E^Q\left[H_N \mid \mathcal{F}_{N-1}\right]\right\}, \tag{3.12}$$

and, repeating this argument backwards, we obtain for $\widetilde{H}_n = B_n^{-1} H_n$ the equation (3.9).

The fact that $\widetilde{H}$ is a Q-super-martingale means that $\widetilde{H}$ "decreases on average" and, intuitively, this corresponds to the fact that, as time goes by, the advantage of an early exercise possibility diminishes.

Step 2. We prove here that there exists $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$. Since $\widetilde{H}$ is a Q-super-martingale, by Doob's decomposition theorem we have

$$\widetilde{H} = M + A$$

where M is a Q-martingale such that $M_0 = \widetilde{H}_0$ and A is a decreasing predictable process with zero initial value.

Since by assumption the market is complete, there exists a strategy $(\alpha, \beta) \in \mathcal{A}$ which replicates the European derivative with payoff M_N, in the sense that $\widetilde{V}_N(\alpha, \beta) = M_N$. Furthermore, since M and $\widetilde{V} := \widetilde{V}^{(\alpha,\beta)}$ are Q-martingales with the same final value, they are equal:

$$\widetilde{V}_n = E^Q\left[\widetilde{V}_N \mid \mathcal{F}_n\right] = E^Q\left[M_N \mid \mathcal{F}_n\right] = M_n. \tag{3.13}$$

Therefore $(\alpha, \beta) \in \mathcal{A}_X^+$ since $A_n \leq 0$. Furthermore, being $A_0 = 0$, we have

$$V_0 = M_0 = H_0. \tag{3.14}$$

To verify that $(\alpha, \beta) \in \mathcal{A}_X^-$, put

$$\nu_0(\omega) = \min\{n \mid \widetilde{H}_n(\omega) = \widetilde{X}_n(\omega)\}, \qquad \omega \in \Omega. \tag{3.15}$$

Since

$$\{\nu_0 = n\} = \{\widetilde{H}_0 > \widetilde{X}_0\} \cap \cdots \cap \{\widetilde{H}_{n-1} > \widetilde{X}_{n-1}\} \cap \{\widetilde{H}_n = \widetilde{X}_n\} \in \mathcal{F}_n$$

for all n, ν_0 is an exercise strategy. Furthermore, ν_0 is the first time instant at which $\widetilde{X}_n \geq E^Q\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right]$ and so it represents intuitively the first moment when it is advantageous to exercise the option.

On the basis of Doob's decomposition theorem, for $n = 1, \dots, N$ we have (see in particular (3.7))

$$M_n = \widetilde{H}_n + \sum_{k=0}^{n-1} \left(\widetilde{H}_k - E^Q\left[\widetilde{H}_{k+1} \mid \mathcal{F}_k\right]\right),$$

and consequently

$$M_{\nu_0} = \widetilde{H}_{\nu_0} \tag{3.16}$$

since

$$\widetilde{H}_k = E^Q\left[\widetilde{H}_{k+1} \mid \mathcal{F}_k\right] \quad \text{on} \quad \{k < \nu_0\}.$$

Then, by (3.13), we have (recall that we had put $\widetilde{V} = \widetilde{V}^{(\alpha,\beta)}$)

$$\widetilde{V}_{\nu_0} = M_{\nu_0} =$$

(by (3.16))

$$= \widetilde{H}_{\nu_0} =$$

(by the definition of ν_0)

$$= \widetilde{X}_{\nu_0}, \tag{3.17}$$

and this proves that $(\alpha, \beta) \in \mathcal{A}_X^-$.

Conclusion. We finally verify that ν_0 is an optimal exercise strategy. Since $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$, by (3.3) of Proposition 3.5 we obtain

$$V_0 = V_0^{(\alpha,\beta)} = \max_{\nu \in \mathcal{T}_0} E^Q \left[\widetilde{X}_\nu \right].$$

On the other hand, by (3.17) and by the Optional Sampling Theorem, we have

$$V_0 = E^Q \left[\widetilde{X}_{\nu_0} \right]$$

and this concludes the proof. □

Remark 3.12. *Theorem 3.9 is important also from an applied point of view since it leads to an algorithm to compute:*

i) *the arbitrage price of X. In fact, by* (3.14) *the initial price of X is equal to H_0 and it can be computed by the iterative formula* (3.9) *(or also by* (3.19) *below). Notice that this formula is a particular case of the Dynamic Programming algorithm;*

ii) *an optimal exercise strategy, given by ν_0 in* (3.15)*, for which we have*

$$E^Q \left[\widetilde{X}_{\nu_0} \right] = \max_{\nu \in \mathcal{T}_0} E^Q \left[\widetilde{X}_\nu \right].$$

To this effect see also Section 3.1.3 where, without requiring market completeness, we show that there may be more optimal exercise strategies, of which ν_0 is the one that exercises prior to all the others;

iii) *a hedging strategy for X given by $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$ and such that $V_n^{(\alpha,\beta)} \geq X_n$ for every n: by* (3.14) *the initial cost of this strategy is H_0 and it is equal to the arbitrage price of the derivative according to the definition in* (3.11)*. Recall that (α, β) is defined as a replicating strategy for the European payoff M_N. The problem of determining (α, β) will be studied more comprehensively in Section 3.1.4.* □

Remark 3.13. *Having fixed $n \leq N$, put*

$$\mathcal{T}_n = \{\nu \in \mathcal{T}_0 \mid \nu \geq n\}.$$

Intuitively one may think of $\mathcal{T}_n$ *as a family of exercise strategies for an American option bought at time* t_n. *An exercise strategy* $\nu_n \in \mathcal{T}_n$ *is called optimal for* X *in* Q *if*

$$E^Q\left[\widetilde{X}_{\nu_n} \mid \mathcal{F}_n\right] = \max_{\nu\in\mathcal{T}_n} E^Q\left[\widetilde{X}_\nu \mid \mathcal{F}_n\right].$$

If $\widetilde{H}$ *is the process in* (3.9), *put*

$$\nu_n(\omega) = \min\{k \geq n \mid \widetilde{H}_k(\omega) = \widetilde{X}_k(\omega)\}, \qquad \omega \in \Omega.$$

Theorem 3.9 can be extended and we can show that ν_n *is the first optimal exercise time following* n. *More precisely we have*

$$\widetilde{H}_n = E^Q\left[\widetilde{X}_{\nu_n} \mid \mathcal{F}_n\right] = \max_{\nu\in\mathcal{T}_n} E^Q\left[\widetilde{X}_\nu \mid \mathcal{F}_n\right]. \tag{3.18}$$

We point out that for the undiscounted values $H_n = B_n\widetilde{H}_n$ *we have (see also (3.12))*

$$H_n = \begin{cases} X_N, & n = N, \\ \max\left\{X_n, \frac{1}{1+r}E^Q\left[H_{n+1} \mid \mathcal{F}_n\right]\right\}, & n = 0,\dots,N-1. \end{cases} \tag{3.19}$$

□

On the basis of Theorem 3.9 and of Remark 3.13 it is natural to give the following:

Definition 3.14. *The process* H *in* (3.19) *is called arbitrage price of the American derivative* X.

Remark 3.15. *From (3.18) it follows that the price* H_n *is* $\mathcal{F}_n$*−measurable. Recall that we had considered* $(\mathcal{F}_n) = \left(\mathcal{F}_n^S\right)$ *namely the filtration generated by the prices* S *of the assets in the market. If* S *is a Markov process and* $X_n = f_n(S_n)$ *for some deterministic function* f_n, *then, again by (3.18), it follows that*

$$H_n = H_n(S_n) \tag{3.20}$$

namely the price at time n *is a function of the prices of the underlyings at the same time instant.* □

Remark 3.16. *For the American Put in the binomial model there exists a so-called critical price. More precisely we have the following (for more details see Section 2.5.6 in [17]): let* $H_n(x)$ *be the (undiscounted) price of an American Put at time* n *when the price of the underlying is* $S_n = x$. *If the parameter* d *in the binomial model is less than 1, then* $x \mapsto H_n(x)$ *is continuous, convex and decreasing and there exists* $x_n^* \in (0,K)$ *such that*

$$\begin{cases} H_n(x) = (K-x)^+ & \text{if } x \in [0, x_n^*] \\ H_n(x) > (K-x)^+ & \text{if } x \in]x_n^*, Kd^{-(N-n)}[\\ H_n(x) = 0 & \text{if } x \geq Kd^{-(N-n)}. \end{cases} \tag{3.21}$$

□

We conclude this section by stating[2] a result that will be used in the sequel. Given a process $H = (H_n)$ and a stopping time ν, denote by $H^\nu = (H_n^\nu)$ the process defined by

$$H_n^\nu(\omega) = H_{n\wedge\nu(\omega)}(\omega), \qquad \omega \in \Omega,$$

namely the process H stopped at time ν.

Lemma 3.17. *If H is adapted then also H^ν is an adapted process. If H is a martingale (resp. super-, sub-martingale) then also H^ν is a martingale (resp. super-, sub-martingale).*

3.1.3 Optimal exercise strategies

The optimal exercise strategy for an American derivative X is in general not unique. The purpose of this section is to characterize the optimal exercise strategies and to determine the minimal and maximal ones among them.

While for the univocal definition of arbitrage price we have assumed that the market is complete, to determine the optimal exercise strategy we shall only assume that the market is free of arbitrage. Let us then fix a martingale measure Q and consider the Snell envelope $\widetilde{H}$ of $\widetilde{X}$ *relative to* Q (see (3.9)).

On the basis of Definition 3.4, an exercise strategy $\bar{\nu} \in \mathcal{T}_0$ is optimal for X in Q if

$$E^Q\left[\widetilde{X}_{\bar{\nu}}\right] = \max_{\nu\in\mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right].$$

Lemma 3.18. *For any $\nu \in \mathcal{T}_0$ we have*

$$E^Q\left[\widetilde{X}_\nu\right] \le H_0. \tag{3.22}$$

Furthermore $\nu \in \mathcal{T}_0$ is optimal for X in Q if and only if

$$E^Q\left[\widetilde{X}_\nu\right] = H_0. \tag{3.23}$$

Proof. We have

$$E^Q\left[\widetilde{X}_\nu\right] \overset{(1)}{\le} E^Q\left[\widetilde{H}_\nu\right] = E^Q\left[\widetilde{H}_N^\nu\right] \overset{(2)}{\le} H_0 \tag{3.24}$$

where inequality (1) is a consequence of the fact that $X_\nu \le H_\nu$ and (2) is a consequence of the fact that $\widetilde{H}$ (and thus by Lemma 3.17 also $\widetilde{H}^\nu$) is a Q-super-martingale, on the basis of the definition of Snell envelope.

On the basis of (3.22), it becomes clear that (3.23) is a sufficient condition for the optimality of ν. To prove that (3.23) is also necessary, it suffices to verify that there exists at least one strategy for which this equality holds:

[2]For the proof see, for instance, Lemma A.125 in [17].

two such strategies will be constructed explicitly in Proposition 3.20 below. The reader may verify that the proof of that proposition is independent of the result that we are about to prove so that there is no risk of using a cyclic argument. Notice also that, under the assumption of market completeness, in Theorem 3.9 we had defined an exercise strategy for which (3.23) holds. □

Corollary 3.19. *If $\nu \in \mathcal{T}_0$ is such that*

i) $\widetilde{X}_\nu = \widetilde{H}_\nu$*;*
ii) $\widetilde{H}^\nu$ *is a Q-martingale,*

then ν is an optimal exercise strategy for X in Q.

Proof. Conditions $i)$ and $ii)$ guarantee that equality holds in (1) and (2) of formula (3.24) respectively. It follows that $E^Q\left[\widetilde{X}_\nu\right] = H_0$ and thus, by Lemma 3.18, ν is optimal for X in Q. □

For convenience we introduce the process E defined by

$$E_n = \frac{1}{1+r} E^Q \left[H_{n+1} \mid \mathcal{F}_n\right], \qquad n \le N-1. \tag{3.25}$$

Putting by convention $E_N = -1$, we have (see (3.19)) that

$$H_n = \max\{X_n, E_n\}, \qquad n \le N,$$

and, furthermore, the sets $\{n \mid X_n \ge E_n\}$ and $\{n \mid X_n > E_n\}$ are nonempty since $X_N \ge 0$ by assumption. Consequently, the following definitions of exercise strategies are well-posed:

$$\nu_{\min} = \min\{n \mid X_n \ge E_n\}, \tag{3.26}$$
$$\nu_{\max} = \min\{n \mid X_n > E_n\}. \tag{3.27}$$

Proposition 3.20. *The exercise strategies $\nu_{\min}$ and $\nu_{\max}$ are optimal for X in Q.*

Proof. We prove the optimality of $\nu_{\min}$ and $\nu_{\max}$ by verifying the conditions $i)$ and $ii)$ of Corollary 3.19. From the definition (3.26)-(3.27) it follows that

$$H_{\nu_{\min}} = \max\{X_{\nu_{\min}}, E_{\nu_{\min}}\} = X_{\nu_{\min}},$$
$$H_{\nu_{\max}} = \max\{X_{\nu_{\max}}, E_{\nu_{\max}}\} = X_{\nu_{\max}},$$

which proves $i)$. Recall then that by Doob's decomposition theorem

$$\widetilde{H}_n = M_n + A_n, \qquad n \le N,$$

where M is a Q-martingale such that $M_0 = H_0$ and A is a decreasing predictable process with $A_0 = 0$. Specifically we have (see (3.8))

$$A_n = -\sum_{k=0}^{n-1} \left(\widetilde{H}_k - \widetilde{E}_k\right), \qquad n \le N.$$

Form definition (3.26)-(3.27) we have

$$H_n = E_n \quad \text{for} \quad n \le \nu_{\max} - 1,$$

so that

$$A_n = 0 \quad \text{for} \quad n \le \nu_{\max}, \tag{3.28}$$

and

$$A_n < 0 \quad \text{for} \quad n \ge \nu_{\max} + 1. \tag{3.29}$$

It follows then that

$$\widetilde{H}_n = M_n \quad \text{for} \quad n \le \nu_{\max}, \tag{3.30}$$

and therefore, being clearly $\nu_{\min} \le \nu_{\max}$, we have

$$\widetilde{H}^{\nu_{\min}} = M^{\nu_{\min}}, \qquad \widetilde{H}^{\nu_{\max}} = M^{\nu_{\max}}.$$

Consequently, by Lemma 3.17, we have that the processes $\widetilde{H}^{\nu_{\min}}, \widetilde{H}^{\nu_{\max}}$ are Q-martingales: this proves *ii*) of Corollary 3.19 and concludes the proof. □

Notice that the previous proof extends partially Theorem 3.9 where we had proved the optimality of ν_0 in (3.15) under the assumption of market completeness. Notice finally that $\nu_{\min}$ and $\nu_{\max}$ are the *first* and *last* optimal exercise strategy for X in Q respectively.

Proposition 3.21. *If $\nu \in \mathcal{T}_0$ is optimal for X in Q then*

$$\nu_{\min} \le \nu \le \nu_{\max} \qquad P\text{-}a.s.$$

Proof. Suppose that

$$P(\nu < \nu_{\min}) > 0. \tag{3.31}$$

To prove that ν cannot be optimal it suffices to show that in this case (1) in (3.24) is a strict inequality. Now, being P and Q equivalent, from (3.31) follows

$$Q\left(\widetilde{X}_\nu < \widetilde{H}_\nu\right) > 0,$$

and therefore, since $\widetilde{X}_\nu \le \widetilde{H}_\nu$, we have

$$E^Q\left[\widetilde{X}_\nu\right] < E^Q\left[\widetilde{H}_\nu\right].$$

On the other hand suppose that

$$P(\nu > \nu_{\max}) > 0. \tag{3.32}$$

To prove that ν cannot be optimal it suffices to show that (2) in (3.24) is a strict inequality. Now, being P, Q equivalent and the process A decreasing and non-positive, from (3.29) it follows that

$$E^Q[A_\nu] < 0.$$

Consequently we have

$$E^Q\left[\widetilde{H}_\nu\right] = E^Q[M_\nu] + E^Q[A_\nu] < M_0 = H_0. \qquad \square$$

Remark 3.22. *The previous results extend easily to exercise strategies that are posterior to a given n. In particular (cf. Lemma 3.18) for all $\nu \in \mathcal{T}_n$ we have*

$$E^Q\left[\widetilde{X}_\nu \mid \mathcal{F}_n\right] \le H_n,$$

and $\nu \in \mathcal{T}_n$ is optimal for X in Q if and only if $E^Q\left[\widetilde{X}_\nu \mid \mathcal{F}_n\right] = H_n$. Furthermore (cf. Corollary 3.19) if $\nu \in \mathcal{T}_n$ is such that

i) $\widetilde{X}_\nu = \widetilde{H}_\nu$;
ii) $\widetilde{H}^\nu$ *is a Q-martingale,*

then ν is an optimal exercise strategy for X in Q. Finally (cf. Propositions 3.20 and 3.21) the exercise strategies defined by

$$\nu_{n,\min} = \min\{k \ge n \mid X_k \ge E_k\},$$
$$\nu_{n,\max} = \min\{k \ge n \mid X_k > E_k\},$$

are the first and last optimal exercise strategies respectively that are posterior to time n. $\square$

3.1.4 Hedging strategies

Consider an American derivative X in a complete market in which Q denotes the martingale measure. From a theoretical point of view the hedging problem for the derivative X is solved by Theorem 3.9 (see also Remark 3.12): in the proof of that theorem a sub- and super-replicating strategy (α, β) (namely a strategy $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$) is defined in terms of a replicating strategy for the European derivative M_N. More precisely, if M denotes the martingale process in Doob's decomposition of $\widetilde{H}$, which is the Snell envelope of $\widetilde{X}$, by the completeness of the market there exists a strategy $(\alpha, \beta) \in \mathcal{A}$ such that

$$\widetilde{V}_N^{(\alpha,\beta)} = M_N$$

and it can be shown that $(\alpha, \beta) \in \mathcal{A}_X^+ \cap \mathcal{A}_X^-$. Recall that, once we have determined $\widetilde{H}$ as in (3.9), on the basis of (3.5) the process M can be computed by the (forwards) recursion

$$M_0 = H_0, \qquad M_{n+1} = M_n + \widetilde{H}_{n+1} - E\left[\widetilde{H}_{n+1} \mid \mathcal{F}_n\right], \tag{3.33}$$

and, consequently, the hedging strategy can be constructed by proceeding as in the European case. Notice however that from (3.33), in particular from the conditioning with respect to $\mathcal{F}_n$, it follows that the payoff M_N depends on the individual trajectories of the underlying S also in the case in which X does not. As a consequence (we shall verify it directly in Problem 3.28), this method to compute the hedging strategy turns out to be extremely burdensome, in particular when the number of time periods is large.

On the other hand the process M depends on the trajectory of the underlying only because it has to keep track of the possible early exercises, but at the moment when the derivative is being exercised there is no more need to hedge it: this elementary remark suggests how the hedging problem can be considerably simplified. We recall in fact (cf. (3.30)) that

$$\widetilde{H}_n = M_n \quad \text{for} \quad n \le \nu_{\max}.$$

In particular, prior to $\nu_{\max}$ *the hedging strategy can be computed by using directly the process* $\widetilde{H}$ *instead of* M*:* the advantage is that, if X is Markovian[3] then also $\widetilde{H}$ is. For example, consider the case of a binomial model and denote by

$$S_{n,k} = u^k d^{n-k} S_0, \qquad n = 0, \dots, N, \quad k = 0, \dots, n, \tag{3.34}$$

the value of the underlying, identified by the Markovian coordinates n (time) and k (number of up-movements). Analogously, denote by $H_{n,k}$ the value of H in the node of coordinates (n, k) on the binomial tree. The hedging strategy for the n-th period with $n \le \nu_{\max}$ is then simply given by

$$\alpha_{n,k} = \frac{H_{n,k+1} - H_{n,k}}{(u-d) S_{n-1,k}}, \qquad k = 0, \dots, n-1, \tag{3.35}$$

exactly as in the European case[4].

At time $\nu_{\max}$ there is no need to compute the hedging strategy $\alpha_{\nu_{\max}}$ (for the $(\nu_{\max}+1)$-st period) because $\nu_{\max}$ is the last instant at which it is possible to exercise optimally the American derivative. If the holder of the option exercises erroneously at a time after $\nu_{\max}$, then the seller could make a sure profit without risk (namely an arbitrage): in fact, since the value of the previously constructed hedging strategy is equal to $M_{\nu_{\max}}$, for the seller it suffices to use in the $(\nu_{\max}+1)$-st period the strategy

$$\alpha_{\nu_{\max}+1,k} = \frac{M_{\nu_{\max}+1,k+1} - M_{\nu_{\max}+1,k}}{(u-d) S_{\nu_{\max},k}},$$

to obtain in the following period the amount

$$M_{\nu_{\max}+1} > \widetilde{H}_{\nu_{\max}+1} \ge \widetilde{X}_{\nu_{\max}+1},$$

[3] In the sense that X_n depends on the underlying through the values of a Markov process which (see Remark 3.15) may be the value S_n of the underlying itself at time n if S is Markovian.

[4] Notice that, although in (3.35) $\alpha_{n,k}$ is determined on the basis of the undiscounted process H_n, the corresponding strategy nevertheless replicates the discounted process $\widetilde{H}_n$, as can be seen from (1.41) in Chapter 1.

which is strictly greater than the payoff. We refer to Problem 3.29 for an actual example of construction of an arbitrage strategy in case the holder of the option decides to exercise non-optimally.

3.2 American and European options

In a market that is free of arbitrage and complete, denote by H the process described by the arbitrage price of an American derivative X. Furthermore, denote by H^E the price process of the European derivative with payoff X_N. Recall (see (3.18)) that we have

$$\widetilde{H}_n = \max_{\nu \in \mathcal{T}_n} E^Q \left[\widetilde{X}_\nu \mid \mathcal{F}_n \right], \qquad \widetilde{H}_n^E = E^Q \left[\widetilde{X}_N \mid \mathcal{F}_n \right], \qquad n = 0, \dots, N,$$

where, as usual, Q denotes the martingale measure; furthermore $\widetilde{H}$ is the smallest Q-super-martingale greater than or equal to $\widetilde{X}$.

In this section we study the relationship between the prices H and H^E. In particular we present two ways to show that, in the absence of dividends and assuming $r \geq 0$, an American Call option has the same value as the corresponding European one and therefore an optimal exercise strategy is given by $\nu = \nu_{\max} = N$.

Proposition 3.23. *We have*

$$H_n \geq H_n^E, \qquad n = 0, \dots, N. \tag{3.36}$$

Furthermore, if $H_n^E \geq X_n$ for all n, then $H = H^E$ and $\nu = N$ is an optimal exercise strategy.

Proof. The first statement is a consequence of the super-martingale property of $\widetilde{H}$ since

$$\widetilde{H}_n \geq E^Q \left[\widetilde{H}_N \mid \mathcal{F}_n \right] = E^Q \left[\widetilde{X}_N \mid \mathcal{F}_n \right] = \widetilde{H}_n^E.$$

This is intuitively clear since an American option, compared with the corresponding European one, allows the holder to have more freedom in choosing the exercise time and so it has more value.

For what concerns the second statement, the assumption $H_n^E \geq X_n$ for all n implies that $\widetilde{H}^E$ is a martingale (and thus also a super-martingale) that dominates $\widetilde{X}$: since on the other hand (see Lemma 3.8) $\widetilde{H}$ is the smallest super-martingale that dominates $\widetilde{X}$, we have the equality $\widetilde{H} = \widetilde{H}^E$ from which also $H = H^E$ follows. □

Corollary 3.24. *In the case when $X_n = (S_n - K)^+$ and $r \geq 0$, we have $H_n^E \geq (S_n - K)^+$ and so $H = H^E$: in other words, an American Call has the same value as the corresponding European Call.*

Proof. Putting $B_n = (1+r)^n$, first of all we have

$$\widetilde{H}_n^E = \frac{1}{B_N} E^Q \left[(S_N - K)^+ \mid \mathcal{F}_n \right] \geq \frac{1}{B_N} E^Q \left[S_N - K \mid \mathcal{F}_n \right] = \widetilde{S}_n - \frac{K}{B_N}.$$

Consequently, since $r \geq 0$, we have

$$H_n^E \geq S_n - K \frac{B_n}{B_N} \geq S_n - K,$$

and since $H_n^E \geq 0$, we also have

$$H_n^E \geq (S_n - K)^+ . \qquad \square$$

We give now the second sufficient condition for which an American option has the same value as the corresponding European one.

Proposition 3.25. *If $\widetilde{X}$ is a Q-sub-martingale, namely we have*

$$\widetilde{X}_n \leq E^Q \left[\widetilde{X}_{n+1} \mid \mathcal{F}_n \right], \qquad n = 0, \dots, N-1,$$

then $H = H^E$.

Proof. By the assumption of sub-martingality and by the Optional Sampling Theorem we have

$$E^Q \left[\widetilde{X}_\nu \mid \mathcal{F}_n \right] \leq E^Q \left[\widetilde{X}_N \mid \mathcal{F}_n \right] \tag{3.37}$$

for any n and $\nu \in \mathcal{T}_n$. From (3.37) it follows that N is an optimal exercise time and also that

$$\widetilde{H}_n = \max_{\nu \in \mathcal{T}_n} E^Q \left[\widetilde{X}_\nu \mid \mathcal{F}_n \right] \leq E^Q \left[\widetilde{X}_N \mid \mathcal{F}_n \right] = \widetilde{H}_n^E,$$

and so, by (3.36), we can conclude that $H = H^E$.

We may also give the following direct proof of (3.37): for each $\nu \in \mathcal{T}_n$ we have

$$\begin{aligned}
E^Q \left[\widetilde{X}_\nu \mid \mathcal{F}_n \right] &= \sum_{k=n}^{N} E^Q \left[\mathbb{1}_{\{\nu = k\}} \widetilde{X}_k \mid \mathcal{F}_n \right] \\
&\leq \sum_{k=n}^{N} E^Q \left[\mathbb{1}_{\{\nu = k\}} E^Q \left[\widetilde{X}_N \mid \mathcal{F}_k \right] \mid \mathcal{F}_n \right] \\
&= \sum_{k=n}^{N} E^Q \left[E^Q \left[\mathbb{1}_{\{\nu = k\}} \widetilde{X}_N \mid \mathcal{F}_k \right] \mid \mathcal{F}_n \right] \\
&= \sum_{k=n}^{N} E^Q \left[\mathbb{1}_{\{\nu = k\}} \widetilde{X}_N \mid \mathcal{F}_n \right] = E^Q \left[\widetilde{X}_N \mid \mathcal{F}_n \right].
\end{aligned} \qquad \square$$

We present now the following alternative proof of Corollary 3.24. On the basis Proposition 3.25, it suffices to show that, if $X_n = (S_n - K)^+$ and $r \geq 0$, then $\widetilde{X}$ is a Q-sub-martingale: we have in fact

$$
\begin{aligned}
E^Q\left[\widetilde{X}_{n+1} \mid \mathcal{F}_n\right] &= \frac{1}{B_{n+1}} E^Q\left[(S_{n+1} - K)^+ \mid \mathcal{F}_n\right] \\
&\geq \frac{1}{B_{n+1}} E^Q\left[S_{n+1} - K \mid \mathcal{F}_n\right] = \widetilde{S}_n - \frac{K}{B_{n+1}} \geq
\end{aligned}
$$

(being $r \geq 0$)

$$\geq \widetilde{S}_n - \frac{K}{B_n},$$

from which the thesis follows since $E^Q\left[\widetilde{X}_{n+1} \mid \mathcal{F}_n\right] \geq 0$.

We give now a more general criterion for the validity of the equality $H = H^E$, which is based on the convexity property of the payoff function.

Corollary 3.26. *If $\widetilde{X}_n = g(\widetilde{S}_n)$ with g a convex function, then $\widetilde{X}$ is a Q-sub-martingale. In particular, in the case when $r = 0$, not only the Call but also the American Put has the same value as the corresponding European Put.*

Proof. The statement is a consequence of Jensen's inequality: we have in fact

$$E^Q\left[\widetilde{X}_{n+1} \mid \mathcal{F}_n\right] = E^Q\left[g(\widetilde{S}_{n+1}) \mid \mathcal{F}_n\right] \geq g\left(E^Q\left[\widetilde{S}_{n+1} \mid \mathcal{F}_n\right]\right) = g(\widetilde{S}_n) = \widetilde{X}_n.$$

Finally, if $r = 0$, it is clear that the payoff of the American Put is of the form $X_n = \widetilde{X}_n = g(\widetilde{S}_n) = g(S_n)$ where $g(x) = (K - x)^+$ is a convex function and so the statement follows from Proposition 3.25. □

At this point, given that in the "undiscounted" version both the Call and the Put are convex functions of the underlying, one may ask oneself what distinguishes the Put from the Call to make the American version of the Put not equivalent to the European one (at least in the case when $r \neq 0$). Under the assumption that $X_n = \varphi(S_n)$ for all n and that $r \geq 0$, the following result gives a condition on φ, in addition to convexity, which guarantees that $H = H^E$.

Corollary 3.27. *Let $r \geq 0$ and suppose that $X_n = \varphi(S_n)$ for all n. If φ is a convex function such that $\varphi(0) = 0$, then $\widetilde{X}$ is a Q-sub-martingale and consequently $H = H^E$.*

Proof. Note preliminarily that, by the convexity of φ, we have

$$\varphi(\alpha x + (1-\alpha)y) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y), \qquad x, y \in \mathbb{R}^d,\ \alpha \in [0,1],$$

and, in particular, for $y = 0$, we have

$$\varphi(\alpha x) \leq \alpha\varphi(x), \qquad x \in \mathbb{R}^d. \tag{3.38}$$

Thus we obtain

$$X_n = \varphi(S_n) = \varphi\left(\frac{1}{1+r}E^Q\left[S_{n+1} \mid \mathcal{F}_n\right]\right) \leq$$

(by (3.38) with $\alpha = \frac{1}{1+r} \in]0,1]$, being $r \geq 0$)

$$\leq \frac{1}{1+r}\varphi\left(E^Q\left[S_{n+1} \mid \mathcal{F}_n\right]\right) \leq$$

(by Jensen's inequality)

$$\leq \frac{1}{1+r}E^Q\left[\varphi(S_{n+1}) \mid \mathcal{F}_n\right] = \frac{1}{1+r}E^Q\left[X_{n+1} \mid \mathcal{F}_n\right],$$

from which the statement follows. $\square$

To conclude notice that the Call and the Put are also characterized by the fact that in the Call the φ is monotonically increasing and in the Put decreasing. Notice however that the only properties of φ being convex and monotonically increasing are in general not sufficient to guarantee that, having put $X_n = \varphi(S_n)$, the process $\widetilde{X}$ is a Q-sub-martingale. For example, the function $\varphi(x) = 1 + x$, with $x \in \mathbb{R}$, is convex, monotonically increasing and such that, in the case of $r > 0$, the process $\widetilde{X}$ is a *Q-super-martingale*: we have in fact

$$\begin{aligned} X_n &= 1 + S_n = 1 + \frac{1}{1+r}E^Q\left[S_{n+1} \mid \mathcal{F}_n\right] \\ &> \frac{1}{1+r}E^Q\left[1 + S_{n+1} \mid \mathcal{F}_n\right] = \frac{1}{1+r}E^Q\left[X_{n+1} \mid \mathcal{F}_n\right]. \end{aligned}$$

In this context see also Problem 3.30.

3.3 Solved problems

3.3.1 Preliminaries

We introduce the notations that we shall systematically use in the solution of the problems. Basically, the two models under consideration are the binomial and trinomial models: in the binomial case, as in Section 3.1.4, we denote by $Y_{n,k}$ the price of a security Y identified on the binomial tree by the coordinates n (time) and k (number of up-movements).

More generally, when it is necessary to identify a complete trajectory of the asset process for the binomial model, we adopt a notation of the form

$$Y_n^{ud\dots uu} \tag{3.39}$$

where, as usual, the subscript n indicates the time, while the superscript denotes the sequence of upward and downward movements (denoted by u and d respectively) of the trajectory, ordered from the initial time until time n.

In the case of the trinomial model we use a notation similar to (3.39) where now the superscript contains not only the letters u and d but also m which corresponds to the "intermediate" movement.

In the theoretical part we have seen that the price H of an American option X is defined by the recursive formula

$$H_n = \begin{cases} X_N, & n = N, \\ \max\{X_n, E_n\}, & n = 0, \dots, N-1, \end{cases} \tag{3.40}$$

where

$$E_n = \begin{cases} -1, & n = N, \\ \frac{1}{1+r} E^Q\left[H_{n+1} \mid \mathcal{F}_n\right], & n = 0, \dots, N-1. \end{cases} \tag{3.41}$$

By the Markov property of the binomial and trinomial price processes, if the payoff is of the form $X_n = X_n(S_n)$, we have (cf. Remark 3.15) that, at any time n, also H_n can be expressed as a function of S_n.

Once we have determined the process E in (3.41), the first and last optimal exercise strategies are defined respectively by

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} \tag{3.42}$$

$$\nu_{\max} = \min\{n \mid X_n > E_n\}. \tag{3.43}$$

Finally, concerning the issue of hedging, we have observed in Section 3.1.4, that before time $\nu_{\max}$ the hedging strategy can be directly calculated using the process H instead of M, the martingale part of the Doob decomposition of $\widetilde{H}$. Then, for any $n \leq \nu_{\max}$, the hedging strategy (α_n, β_n) to be built at time $n-1$ for the n-th period, is determined by the replication condition

$$\alpha_n S_n + \beta_n B_n = H_n. \tag{3.44}$$

On the other hand, at time $\nu_{\max}$ there is no need to compute the hedging strategy because $\nu_{\max}$ is the last time of optimal exercise of the American derivative.

3.3.2 Solved problems

Problem 3.28. In a binomial market model over three periods (i.e. $N = 3$), consider an American Put option with payoff

$$X_n = (K - S_n)^+.$$

Assume the numerical data $u = 2, d = \frac{1}{2}, S_0 = 1, K = \frac{1}{2}$ and take the risk-free rate r as arbitrary. Determine:

i) the process H of the option price and the minimal and maximal optimal exercise strategies;
ii) the martingale M of the Doob decomposition of $\widetilde{H}$ and, for $r > 0$, the hedging strategy of the American derivative. *(This point of the problem is intended to show the burden of calculating the hedging strategy on the basis of the replication of the martingale process M calculated as in (3.33) and the advantage of the alternative procedure described in Section 3.1.4.)*

Solution of Problem 3.28

i) The binomial tree of the asset prices is shown in Figure 3.1. The model is arbitrage-free under the condition $d < 1 + r < u$ which in our case becomes

$$-\frac{1}{2} < r < 1. \tag{3.45}$$

Taking (3.45) for granted, the martingale measure is defined by

$$q = \frac{1 + r - d}{u - d} = \frac{1}{3}(1 + 2r), \qquad 1 - q = \frac{2}{3}(1 - r).$$

Recall that the arbitrage price process of the American derivative is defined recursively by

$$H_n = \begin{cases} \left(\frac{1}{2} - S_N\right)^+, & n = N, \\ \max\left\{\left(\frac{1}{2} - S_n\right)^+, E_n\right\}, & n = 0, \dots, N-1, \end{cases} \tag{3.46}$$

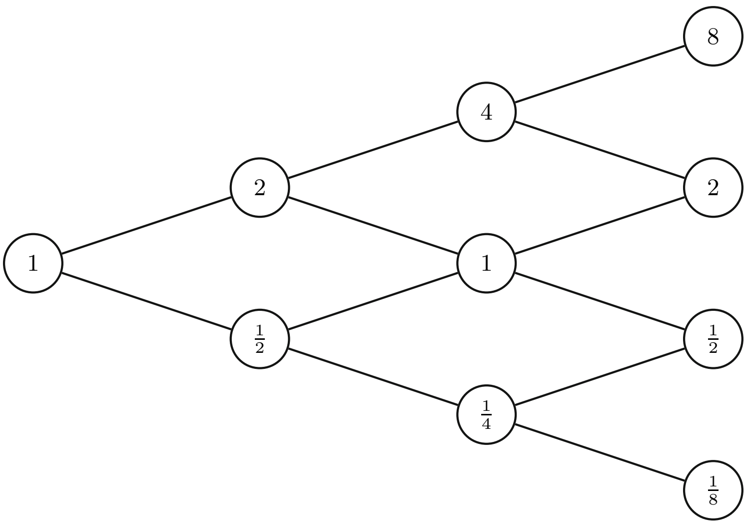

Fig. 3.1. Binomial tree of the underlying asset prices

where E is the process in (3.41), that is, for $n < N$,

$$E_n = \frac{1}{1+r} E^Q \left[H_{n+1} \mid \mathcal{F}_n \right].$$

Adopting the notations of Section 3.3.1, at the last time we have

$$\begin{cases} H_{3,3} = X_{3,3} = \left(\frac{1}{2} - 8\right)^+ = 0, \\ H_{3,2} = X_{3,2} = \left(\frac{1}{2} - 2\right)^+ = 0, \\ H_{3,1} = X_{3,1} = \left(\frac{1}{2} - \frac{1}{2}\right)^+ = 0, \\ H_{3,0} = X_{3,0} = \left(\frac{1}{2} - \frac{1}{8}\right)^+ = \frac{3}{8}. \end{cases}$$

Consequently, by (3.46) we have

$$X_{2,2} = X_{2,1} = E_{2,2} = E_{2,1} = 0,$$

from which $H_{2,2} = H_{2,1} = 0$. Furthermore

$$X_{2,0} = \left(\frac{1}{2} - \frac{1}{4}\right)^+ = \frac{1}{4},$$

and

$$E_{2,0} = \frac{1}{1+r} \left(qH_{3,1} + (1-q)H_{3,0}\right) = \frac{3}{8}\left(\frac{1-q}{1+r}\right) = \frac{1}{4}\left(\frac{1-r}{1+r}\right).$$

We thus obtain

$$\begin{aligned} H_{2,0} &= \max\left\{X_{2,0}, E_{2,0}\right\} \\ &= \max\left\{\frac{1}{4}, \frac{1}{4}\left(\frac{1-r}{1+r}\right)\right\} = \begin{cases} \frac{1}{4} & \text{if } \ 0 < r < 1, \\ \frac{1}{4}\left(\frac{1-r}{1+r}\right) & \text{if } \ -\frac{1}{2} < r \le 0. \end{cases} \end{aligned}$$

At time $n = 1$ we have $X_{1,1} = E_{1,1} = H_{1,1} = 0$ and

$$\begin{aligned} E_{1,0} &= \frac{1}{1+r} \left(qH_{2,1} + (1-q)H_{2,0}\right) \\ &= \begin{cases} \frac{1}{4}\left(\frac{1-q}{1+r}\right) = \frac{1}{6}\left(\frac{1-r}{1+r}\right) & \text{if } \ 0 < r < 1, \\ \frac{1}{4}\frac{(1-q)(1-r)}{(1+r)^2} = \frac{1}{6}\left(\frac{1-r}{1+r}\right)^2 & \text{if } \ -\frac{1}{2} < r \le 0. \end{cases} \end{aligned}$$

In any case $X_{1,0} = 0$ and therefore $H_{1,0} = E_{1,0}$. Finally, at the initial time we have $X_{0,0} = 0$ and therefore

$$\begin{aligned} H_{0,0} = E_{0,0} &= \frac{1}{1+r} \left(qH_{1,1} + (1-q)H_{1,0}\right) \\ &= \begin{cases} \frac{1}{6}\frac{(1-q)(1-r)}{(1+r)^2} = \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 & \text{if } \ 0 < r < 1, \\ \frac{1}{6}\frac{(1-q)(1-r)^2}{(1+r)^3} = \frac{1}{9}\left(\frac{1-r}{1+r}\right)^3 & \text{if } \ -\frac{1}{2} < r \le 0. \end{cases} \end{aligned}$$

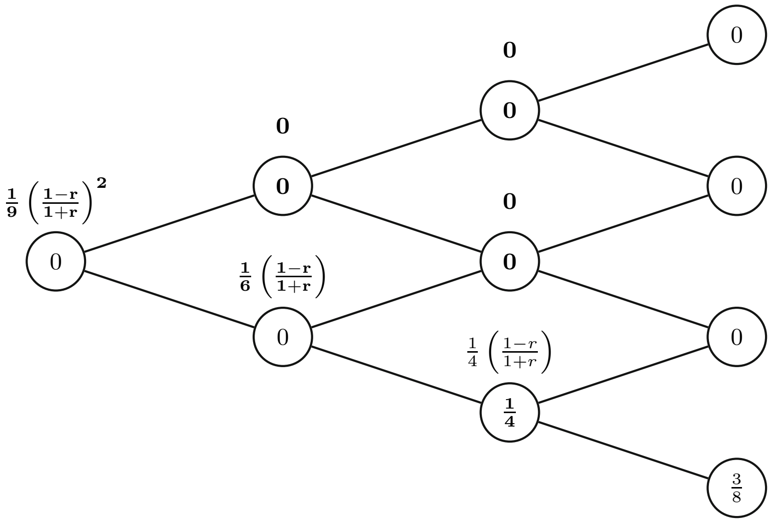

Fig. 3.2. Values of processes X (inside the circles) and E (above the circles) in the case of $0 < r < 1$

To facilitate the calculations that follow, in Figure 3.2 we depict the values of the processes X (inside the circles) and E (above the circles) in the case of $0 < r < 1$: the maximum of the two, which is equal to the price H of the American option, is marked in bold.

Looking at Figure 3.2, we can easily determine the minimal and the maximal optimal exercise strategies for $0 < r < 1$. Indeed, by definition we have

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_{1,1}\} \\ 2 & \text{on } \{S_1 = S_{1,0}\}. \end{cases}$$

Similarly, we have

$$\nu_{\max} = \min\{n \mid X_n > E_n\} = \begin{cases} 2 & \text{on } \{S_2 = S_{2,0}\} \\ 3 & \text{otherwise.} \end{cases}$$

The minimal and the maximal optimal exercise strategies are shown in Figure 3.3 in the case of $0 < r < 1$.

Next we consider the case of $-\frac{1}{2} < r \leq 0$: Figure 3.4 displays the values of the processes X (inside the circles) and E (above the circles) and the maximum of the two, which is equal to the price H of the American option, is marked in bold.

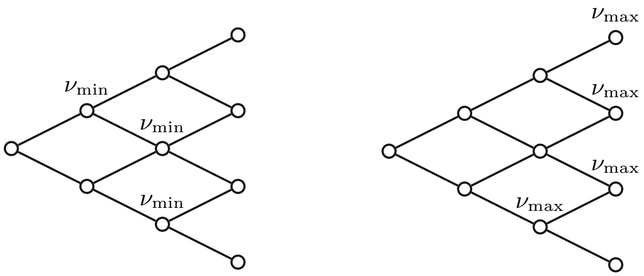

Fig. 3.3. Minimal (left) and maximal (right) optimal exercise strategies for $0 < r < 1$

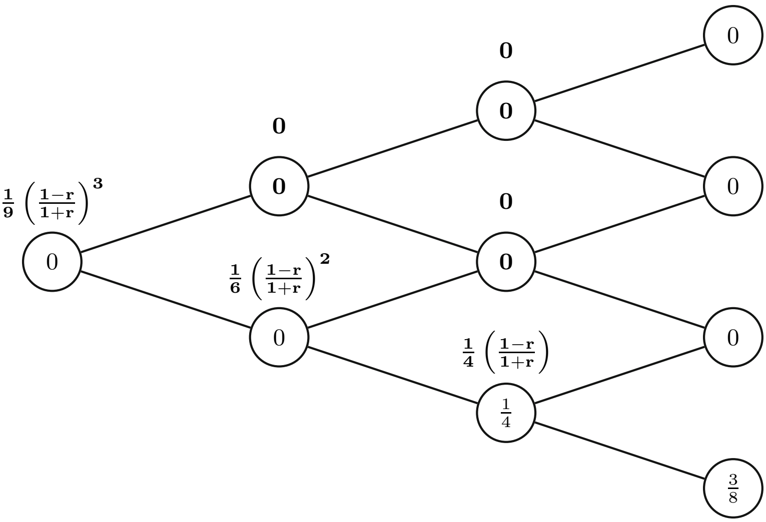

Fig. 3.4. Values of processes X (inside the circles) and E (above the circles) in the case of $-\frac{1}{2} < r < 0$

With regard to the optimal exercise strategies, we first consider the case of $r < 0$: by definition we have

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_{1,1}\}, \\ 2 & \text{on } \{S_1 = S_{1,0}\} \cap \{S_2 = S_{2,1}\}, \\ 3 & \text{on } \{S_2 = S_{2,0}\}, \end{cases}$$

and

$$\nu_{\max} = \min\{n \mid X_n > E_n\} = 3.$$

The minimal and maximal optimal exercise strategies are shown in Figure 3.5.

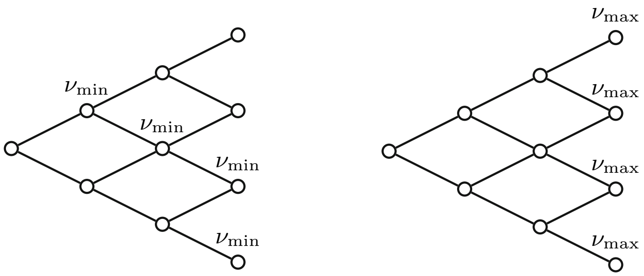

Fig. 3.5. Minimal (left) and maximal (right) optimal exercise strategies in the case of $-\frac{1}{2} < r < 0$

Lastly, when $r = 0$, we have $\nu_{\max} = 3$ and

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_{1,1}\} \\ 2 & \text{on } \{S_1 = S_{1,0}\}. \end{cases}$$

Notice that in this case (as for $r < 0$) $N = \nu_{\max}$ is an optimal exercise time and therefore the American derivative is worth as much as the corresponding European derivative, in accordance with Corollary 3.26.

ii) We determine the process M using the forward recursive formula (3.33) that in this context becomes

$$M_0 = H_0, \qquad M_{n+1} = M_n + \widetilde{H}_{n+1} - \widetilde{E}_n.$$

Since the process M depends on the trajectory of the underlying, the notation $M_{n,k}$ is no longer appropriate because the values of M at time n do not depend only on the number of up-movements of the underlying, but also on the order in which they occurred. Therefore below we adopt the notation (3.39).

We first consider the case of $0 \leq r < 1$. Looking at Figure 3.2, we can easily determine the process M:

$$\begin{aligned}
M_0 &= H_0 = \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2, \\
M_1^u &= \widetilde{H}_{1,1} + M_0 - E_0 = 0 + \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 - \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 = 0, \\
M_1^d &= \widetilde{H}_{1,0} + M_0 - E_0 = \frac{1-r}{6(1+r)^2} + \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 - \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 \\
&= \frac{1-r}{6(1+r)^2}.
\end{aligned}$$

At time t_2 (i.e. $n = 2$) we have

$$\begin{aligned}
M_2^{uu} &= \widetilde{H}_{2,2} + M_1^u - \widetilde{E}_{1,1} = 0,\\
M_2^{ud} &= \widetilde{H}_{2,1} + M_1^u - \widetilde{E}_{1,1} = 0,\\
M_2^{du} &= \widetilde{H}_{2,1} + M_1^d - \widetilde{E}_{1,0} = 0 + \frac{1-r}{6(1+r)^2} - \frac{1-r}{6(1+r)^2} = 0,\\
M_2^{dd} &= \widetilde{H}_{2,0} + M_1^d - \widetilde{E}_{1,0} = \frac{1}{4(1+r)^2} + \frac{1-r}{6(1+r)^2} - \frac{1-r}{6(1+r)^2}\\
&= \frac{1}{4(1+r)^2},
\end{aligned}$$

and at time t_3 (i.e. $n = 3$) we have

$$\begin{aligned}
M_3^{uuu} &= \widetilde{H}_{3,3} + M_2^{uu} - \widetilde{E}_{2,2} = 0,\\
M_3^{uud} &= \widetilde{H}_{3,2} + M_2^{uu} - \widetilde{E}_{2,2} = 0,\\
M_3^{udu} &= \widetilde{H}_{3,2} + M_2^{ud} - \widetilde{E}_{2,1} = 0,\\
M_3^{duu} &= \widetilde{H}_{3,2} + M_2^{du} - \widetilde{E}_{2,1} = 0,\\
M_3^{udd} &= \widetilde{H}_{3,1} + M_2^{ud} - \widetilde{E}_{2,1} = 0,\\
M_3^{dud} &= \widetilde{H}_{3,1} + M_2^{du} - \widetilde{E}_{2,1} = 0,\\
M_3^{ddu} &= \widetilde{H}_{3,1} + M_2^{dd} - \widetilde{E}_{2,0} = 0 + \frac{1}{4(1+r)^2} - \frac{1-r}{4(1+r)^3} = \frac{r}{2(1+r)^3},\\
M_3^{ddd} &= \widetilde{H}_{3,0} + M_2^{dd} - \widetilde{E}_{2,0} = \frac{3}{8(1+r)^3} + \frac{1}{4(1+r)^2} - \frac{1-r}{4(1+r)^3}\\
&= \frac{1}{2(1+r)^3}\left(\frac{3}{4} + r\right).
\end{aligned}$$

We can directly verify that, as we observed in Section 3.1.4, we simply have

$$M_n = \widetilde{H}_n \quad \text{for } n \le \nu_{\max}. \tag{3.47}$$

In particular, in the case of $r < 0$, since $\nu_{\max} = N$, we have that the process M coincides with $\widetilde{H}$.

Finally we determine the hedging strategy in the case of $r > 0$. In retrospect, as already mentioned in Section 3.1.4, *to compute the hedging strategy is not necessary to determine the process* M: in fact, we recall that it suffices to calculate the strategy for $n \le \nu_{\max}$ and therefore, by (3.47), we can directly use the standard replication formula (3.35) which is given here for convenience

$$\alpha_{n,k} = \frac{H_{n,k+1} - H_{n,k}}{(u-d)S_{n-1,k}}, \qquad k = 0, \dots, n-1.$$

Here $\alpha_{n,k}$ denotes the hedging strategy for the n-th period, built at time $n-1$ when $S_{n-1} = S_{n-1,k}$. The strategy for the first period is equal to

$$\alpha_1 = \frac{H_{1,1} - H_{1,0}}{(u-d)S_0} = \frac{0 - \frac{1}{6}\left(\frac{1-r}{1+r}\right)}{\frac{2}{3}} = -\frac{1}{4}\left(\frac{1-r}{1+r}\right).$$

From the self-financing condition $H_0 = \alpha_1 S_0 + \beta_1 B_0$, we can easily obtain also the amount of non-risky assets, which is equal to

$$\beta_1 = H_0 - \alpha_1 = \frac{1}{9}\left(\frac{1-r}{1+r}\right)^2 + \frac{1}{4}\left(\frac{1-r}{1+r}\right) = \frac{4r^2 - 17r + 13}{36(1+r)^2}.$$

Thus the hedging strategy requires taking a short position in the risky asset for an amount equal to $\frac{1}{4}\left(\frac{1-r}{1+r}\right)$ units of S and to invest it, together with the amount received from the sale of the derivative, in the bond.

In the second period, the strategy is given by

$$\alpha_{2,1} = \frac{H_{2,2} - H_{2,1}}{(u-d)S_{1,1}} = 0,$$
$$\alpha_{2,0} = \frac{H_{2,1} - H_{2,0}}{(u-d)S_{1,0}} = \frac{0 - \frac{1}{4}}{\frac{2}{3} \cdot \frac{1}{2}} = -\frac{1}{3}.$$

Hence, in case of an up-movement, the entire wealth is invested in the bond; otherwise, another $\frac{1}{3}$ units of risky asset are sold. Finally, in the last period we have to compute the strategy only in case of $S_2 = S_{2,2}$ and $S_2 = S_{2,1}$ because when $S_2 = S_{2,0}$, an early exercise occurs (see the tree on the right in Figure 3.3). We have

$$\alpha_{3,2} = \frac{H_{3,3} - H_{3,2}}{(u-d)S_{2,2}} = 0,$$
$$\alpha_{3,1} = \frac{H_{3,2} - H_{3,1}}{(u-d)S_{2,1}} = 0.$$

The hedging strategy α is as shown in Figure 3.6. □

Problem 3.29. In a binomial market model consider an American option with payoff

$$X_n = \min\{\max\{S_n, K_1\}, K_2\},$$

which is usually called a "collar"-type option. The numerical data are as follows

$$u = 2, \quad d = \frac{1}{2}, \quad S_0 = 1, \quad K_1 = 1, \quad K_2 = 2, \quad r = \frac{1}{2},$$

and the time horizon is two periods, i.e. $N = 2$.

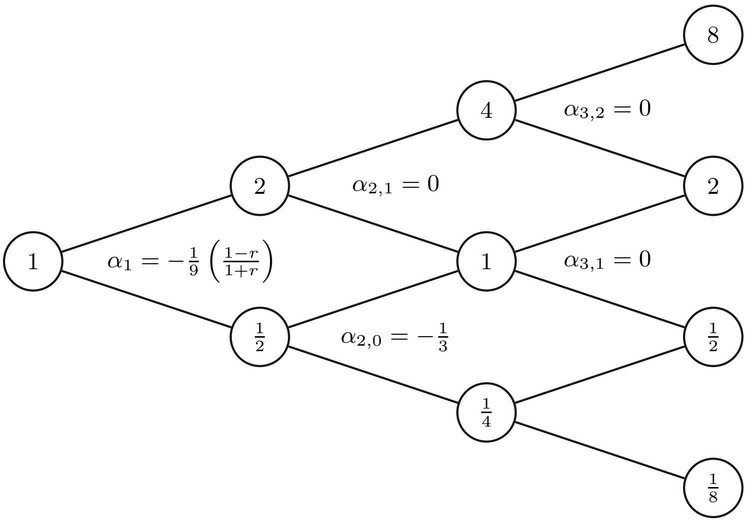

Fig. 3.6. Hedging strategy in the case of $0 < r < 1$

i) Determine the process of the American option price and the minimal and maximal optimal exercise strategies;
ii) determine the hedging strategy;
iii) show what would happen if the customer exercised the option incorrectly at time $n = 2$.

Solution of Problem 3.29

i) We first note that, on the basis of the numerical data, the martingale measure is given by

$$q = \frac{1 + r - d}{u - d} = \frac{2}{3}.$$

Figure 3.7 shows the binomial tree of the underlying prices (inside the circles) and of the American option payoff (above the circles).

At the last time, the arbitrage price H of the derivative is equal to

$$\begin{cases} H_2^{uu} = X_2^{uu} = \min\{\max\{4, 1\}, 2\} = 2, \\ H_2^{ud} = X_2^{ud} = \min\{\max\{1, 1\}, 2\} = 1, \\ H_2^{dd} = X_2^{dd} = \min\left\{\max\left\{\frac{1}{4}, 1\right\}, 2\right\} = 1. \end{cases}$$

We now calculate the arbitrage price at time $n = 1$: by definition we have

$$\begin{aligned} H_1^u &= \max\{X_1^u, E_1^u\} = \max\left\{2, \frac{1}{1+r}\left(qH_2^{uu} + (1-q)H_2^{ud}\right)\right\} \\ &= \max\left\{2, \frac{2}{3}\left(\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 1\right)\right\} = \max\left\{2, \frac{10}{9}\right\} = 2, \end{aligned}$$

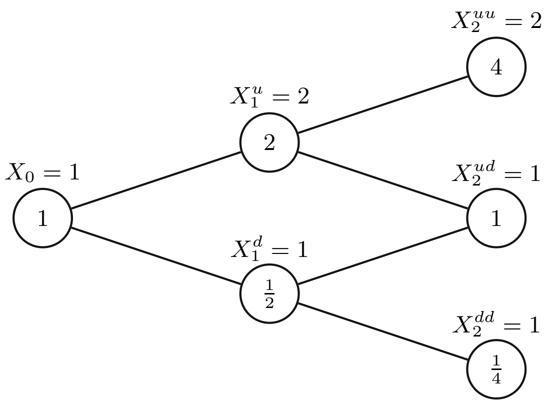

Fig. 3.7. Binomial tree of the prices of S (inside the circles) and of the values of the payoff X (above the circles)

$$\begin{aligned} H_1^d &= \max\left\{X_1^d, E_1^d\right\} = \max\left\{1, \frac{1}{1+r}\left(qH_2^{du} + (1-q)H_2^{dd}\right)\right\} \\ &= \max\left\{1, \frac{2}{3}\left(\frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 1\right)\right\} = \max\left\{1, \frac{2}{3}\right\} = 1. \end{aligned}$$

Moreover, at the initial time we have

$$\begin{aligned} H_0 &= \max\left\{X_0, E_0\right\} = \max\left\{1, \frac{1}{1+r}\left(qH_1^{u} + (1-q)H_1^{d}\right)\right\} \\ &= \max\left\{1, \frac{2}{3}\left(\frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 1\right)\right\} = \max\left\{1, \frac{10}{9}\right\} = \frac{10}{9}. \end{aligned}$$

Comparing the values of X and E just calculated, we get

$$\bar{\nu} := \nu_{\min} = \min\{n \mid X_n \geq E_n\} = \nu_{\max} = \min\{n \mid X_n > E_n\} = 1,$$

and therefore in this case the American option is not equivalent to a European one. This is also in agreement with Proposition 3.25, since here $\widetilde{X}$ is not a Q-sub-martingale. Indeed, for instance we have

$$\widetilde{X}_1^u = \frac{4}{3} > E^Q\left[\widetilde{X}_2 \mid \mathcal{F}_1^u\right] = \frac{2}{3}\cdot\frac{8}{9} + \frac{1}{3}\cdot\frac{4}{9} = \frac{20}{27}.$$

The optimal exercise strategy is shown in Figure 3.8.

ii) As discussed in Section 3.1.4, since $\nu_{\max} = 1$, it is sufficient to calculate the hedging strategy only for the first period. In addition, this strategy coincides with the hedging strategy of H_1 and therefore we obtain it by imposing the replication condition

$$\alpha_1 S_1 + \beta_1 B_1 = H_1,$$

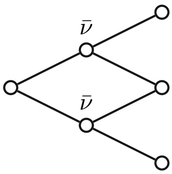

Fig. 3.8. Optimal exercise strategy

which provides the system of linear equations

$$\begin{cases} \alpha_1 u S_0 + \beta_1(1+r) = H_1^u, \\ \alpha_1 d S_0 + \beta_1(1+r) = H_1^d, \end{cases}$$

equivalent to

$$\begin{cases} 2\alpha_1 + \frac{3}{2}\beta_1 = 2, \\ \frac{1}{2}\alpha_1 + \frac{3}{2}\beta_1 = 1. \end{cases}$$

The solution of this system is given by

$$\alpha_1 = \frac{2}{3}, \qquad \beta_1 = \frac{4}{9}. \tag{3.48}$$

Let us now verify that the initial cost of this strategy is equal to the initial price of the American option: indeed, recalling that $S_0 = B_0 = 1$, we have

$$\frac{2}{3} S_0 + \frac{4}{9} B_0 = \frac{10}{9} = H_0.$$

Alternatively, we could also calculate the strategy using formula (3.35) which here becomes

$$\alpha_1 = \frac{H_1^u - H_1^d}{S_0(u-d)}, \qquad \beta_1 = H_0 - \alpha_1 S_0.$$

iii) Assume now that the holder does not exercise the option in a rational way at time $n = 1$. For example, suppose that for $S_1 = S_1^u = 2$ the holder does not exercise the option. In this case it is necessary to hedge the payoff X_2 and to this end we construct the hedging strategy (α_2, β_2) such that

$$\alpha_2 S_2 + \beta_2 B_2 = X_2,$$

or, more explicitly,

$$\begin{cases} \alpha_2 u S_1^u + \beta_2(1+r)^2 = X_2^{uu}, \\ \alpha_2 d S_1^u + \beta_2(1+r)^2 = X_2^{ud}. \end{cases}$$

Then we get

$$\begin{cases} 4\alpha_2 + \frac{9}{4}\beta_2 = 2, \\ \alpha_2 + \frac{9}{4}\beta_2 = 1, \end{cases}$$

with solution

$$\alpha_2 = \frac{1}{3}, \qquad \beta_2 = \frac{8}{27}.$$

Notice however that the cost for the construction of such a strategy when $S_1 = S_1^u = 2$ is equal to

$$V_1^u := \alpha_2 S_1^u + \beta_2 B_1 = \frac{1}{3} \cdot 2 + \frac{8}{27} \cdot \frac{3}{2} = \frac{10}{9}$$

that is strictly less than $H_1^u = 2$, which is value of the hedging strategy constructed at the beginning. In short, the seller of the option has to use only $\frac{10}{9}$ to hedge the option, compared with a capital equal to 2 (resulting from the initial sale of the option and the investment in the hedging strategy (3.48) in the first period). The result is a sure profit without risk for the seller, equal to

$$H_1^u - V_1^u = \frac{8}{9}.$$

Similarly, if the holder does not exercise when $S_1 = S_1^d = \frac{1}{2}$, we construct the hedging strategy such that

$$\begin{cases} \alpha_2 u S_1^d + \beta_2 (1+r)^2 = X_2^{ud}, \\ \alpha_2 d S_1^d + \beta_2 (1+r)^2 = X_2^{dd}. \end{cases}$$

This system is equivalent to

$$\begin{cases} \alpha_2 + \frac{9}{4}\beta_2 = 1, \\ \frac{1}{4}\alpha_2 + \frac{9}{4}\beta_2 = 1, \end{cases}$$

with solution

$$\alpha_2 = 0, \qquad \beta_2 = \frac{4}{9}.$$

In this case the cost for the construction of the strategy at $S_1 = S_1^d = \frac{1}{2}$ is equal to

$$V_1^d = \alpha_2 S_1^d + \beta_2 B_1 = \frac{4}{9} \cdot \frac{3}{2} = \frac{2}{3}$$

which is strictly less than $H_1^d = 1$, equal to the value of the hedging strategy constructed at the initial time. The result is again a sure profit without risk for the seller, equal to

$$H_1^d - V_1^d = \frac{1}{3}. \qquad \square$$

Problem 3.30. For a binomial market model with time horizon $N = 2$ and $r \geq 0$, consider the following two American options with payoffs

$$X_n^k := (S_n - 1)^{2k}, \quad k \in \mathbb{N},$$

and

$$Y_n^k := \max\left\{1, (S_n - 1)^{2k+1}\right\}, \quad k \in \mathbb{N} \cup \{0\},$$

respectively.

i) In what situations do the two American options coincide with the corresponding European ones?
ii) Considering the specific case with the following data

$$S_0 = 1, \quad u = 2, \quad d = \frac{1}{2}, \quad r = 0 \quad \text{so that} \quad q = \frac{1}{3},$$

determine the minimal and maximal optimal exercise strategies for each of the two options and put the result in relation with that of point i).

Solution of Problem 3.30

i) In Section 3.2 we have seen various sufficient conditions that we may verify in order to check whether or not the American options coincide with the corresponding European ones.

a) The first condition is $H_n^E \geq X_n$ in Proposition 3.23. For the option X_n^k we have the following: using the inequality (3.38) for $\alpha = \frac{1}{B_N}$, $x = S_N - 1$ and $g(x) = x^{2k}$ as well as Jensen's inequality, we obtain

$$\begin{aligned} \widetilde{H}_n^E &= E^Q\left[\frac{(S_N-1)^{2k}}{B_N} \mid \mathcal{F}_n\right] \geq E^Q\left[\left(\frac{S_N}{B_N} - \frac{1}{B_N}\right)^{2k} \mid \mathcal{F}_n\right] \\ &\geq \left(E^Q\left[\frac{S_N}{B_N} - \frac{1}{B_N} \mid \mathcal{F}_n\right]\right)^{2k} = \left(\frac{S_n}{B_n} - \frac{1}{B_N}\right)^{2k}. \end{aligned} \tag{3.49}$$

From this, using once more the inequality (3.38) and the fact that $B_n \leq B_N$ because $r \geq 0$, we get

$$H_n^E \geq B_n\left(\frac{S_n}{B_n} - \frac{1}{B_n}\right)^{2k} \geq (S_n - 1)^{2k} = X_n^k \tag{3.50}$$

and so the American option X_n^k reduces to the corresponding European one.

Coming next to Y_n^k, notice first that the payoff can be rewritten as

$$Y_n^k = \max\left\{1, \left((S_n-1)\mathbb{1}_{\{S_n-1\geq 0\}}\right)^{2k+1}\right\}.$$

Assume this time that $r = 0$ and thus $B_n \equiv 1$. Using Jensen's inequality we then have

$$\begin{aligned} H_n^E &= E^Q\left[\max\left\{1, \left((S_N-1)\mathbb{1}_{\{S_N-1\geq 0\}}\right)^{2k+1}\right\} \mid \mathcal{F}_n\right] \\ &\geq \max\left\{1, \left(E^Q\left[(S_N-1)\mathbb{1}_{\{S_N-1\geq 0\}} \mid \mathcal{F}_n\right]\right)^{2k+1}\right\} \\ &\geq \max\left\{1, \left(E^Q\left[(S_N-1) \mid \mathcal{F}_n\right]\right)^{2k+1}\right\} \\ &= \max\left\{1, (S_n-1)^{2k+1}\right\} = Y_n^k \end{aligned} \tag{3.51}$$

and so the result is true also for the option Y^k in any complete (binomial) market model provided we assume $r = 0$.

b) The next sufficient condition is the Q-submartingale property of $\widetilde{X}$ given in Proposition 3.25. Starting again from the option X_n^k, we may follow the same considerations as in (3.49), but in so doing we face the problem of having to apply twice the inequality (3.38), whereby the second time it leads to the inverse inequality. If however we assume $r = 0$ (i.e. $B_n \equiv 1$), then

$$\begin{aligned} E^Q\left[X_{n+1} \mid \mathcal{F}_n\right] &= E^Q\left[(S_{n+1}-1)^{2k} \mid \mathcal{F}_n\right] \\ &\geq \left(E^Q\left[S_{n+1} \mid \mathcal{F}_n\right]-1\right)^{2k} = (S_n - 1)^{2k} = X_n^k. \end{aligned}$$

Assuming $r = 0$ also for the payoff Y^k we have, analogously to (3.51)

$$\begin{aligned} E^Q\left[Y_{n+1}^k \mid \mathcal{F}_n\right] &= E^Q\left[\max\left\{1, (S_{n+1}-1)^{2k+1}\right\} \mid \mathcal{F}_n\right] \\ &= E^Q\left[\max\left\{1, \left((S_{n+1}-1)\mathbb{1}_{\{S_{n+1}-1\geq 0\}}\right)^{2k+1}\right\} \mid \mathcal{F}_n\right] \\ &\geq \max\left\{1, \left(E^Q\left[(S_{n+1}-1)\mathbb{1}_{\{S_{n+1}-1\geq 0\}} \mid \mathcal{F}_n\right]\right)^{2k+1}\right\} \\ &\geq \max\left\{1, \left(E^Q\left[(S_{n+1}-1) \mid \mathcal{F}_n\right]\right)^{2k+1}\right\} \\ &= \max\left\{1, (S_n - 1)^{2k+1}\right\} = Y_n^k. \end{aligned}$$

c) The third sufficient condition is the one expressed in Corollary 3.27. Since this is first of all a sufficient condition for the Q-submartingality of $\widetilde{X}$ that we have already shown in point b) to hold for $r = 0$, we may check whether the present one is an easier condition to apply in the case of $r > 0$. Notice now that, analogously to point a), the inequality (3.38) holds more generally for a function $\varphi(x)$ with $x = \psi(S)$ provided $\varphi(\psi(S)) = 0$ whenever $\psi(S) = 0$. Notice furthermore that for the first type of options we may write

$$X_n^k = \varphi(\psi(S))$$

with $\psi(S) = (S-1)$ and $\varphi(x) = x^{2k}$ whereby φ is convex and $\varphi(0) = 0$ so that the assumptions of the corollary hold in the above generalized sense implying that $\widetilde{X}$ is a Q-submartingale and so we again have as in point a) that the American option X_n^k reduces to the corresponding European one independently on whether $r = 0$ or not. More explicitly, we have that for the above mentioned generalization of the inequality (3.38), the proof of Corollary 3.27 becomes in the case of the options X_n^k

$$\begin{aligned} X_n^k = (S_n - 1)^{2k} &= \left(\frac{1}{1+r}E^Q\left[S_{n+1} \mid \mathcal{F}_n\right] - 1\right)^{2k} \\ &\leq \frac{1}{1+r}\left(E^Q\left[(S_{n+1}-1) \mid \mathcal{F}_n\right]\right)^{2k} \leq \frac{1}{1+r}E^Q\left[(S_{n+1}-1)^{2k} \mid \mathcal{F}_n\right]. \end{aligned} \tag{3.52}$$

Coming to the options of the type Y_n^k, we find that it becomes considerably more difficult to check whether the conditions of Corollary 3.27 are satisfied unless $r = 0$.

Concluding point c) it appears that, with respect to Proposition 3.25, the conditions of Corollary 3.27 may be applicable to a greater variety of situations; still, also these conditions are not always easy to verify.

ii) Start from X_n^k. At the last time we have

$$\begin{aligned} H_2^{uu} &= X_2^{uu} = 3^{2k} \\ H_2^{ud} &= X_2^{ud} = 0 \\ H_2^{dd} &= X_2^{dd} = \left(\frac{3}{4}\right)^{2k}. \end{aligned}$$

For $n = 1$ we then have

$$\begin{aligned} H_1^u &= \max\{X_1^u, E_1^u\} = \max\left\{1, \frac{1}{3}\, 3^{2k} + \frac{2}{3} \cdot 0\right\} = 3^{2k-1} \\ H_1^d &= \max\{X_1^d, E_1^d\} = \max\left\{\left(\frac{1}{2}\right)^{2k}, \frac{1}{3} \cdot 0 + \left(\frac{2}{3}\right)^{2k}\right\} = \frac{2}{3}\left(\frac{3}{4}\right)^{2k}. \end{aligned}$$

Finally, for $n = 0$,

$$H_0 = \max\{X_0, E_0\} = \max\left\{1, \frac{1}{3} 3^{2k-1} + \left(\frac{2}{3}\right)^2 \left(\frac{3}{4}\right)^{2k}\right\} = 3^{2k-2} + \frac{4}{9}\left(\frac{3}{4}\right)^{2k}.$$

We can now compute

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = 2, \qquad \nu_{\max} = \min\{n \mid X_n > E_n\} = 2,$$

and so in this case the American option coincides with the corresponding European one, as already established in point i). Furthermore, $\nu_{\min} = \nu_{\max} = 2$ a.s.

Next consider Y^k. At the last time we have

$$\begin{aligned} H_2^{uu} &= Y_2^{uu} = 3^{2k+1} \\ H_2^{ud} &= Y_2^{ud} = 1 \\ H_2^{dd} &= Y_2^{dd} = 1. \end{aligned}$$

At time $n = 1$ we then have

$$\begin{aligned} H_1^u &= \max\{Y_1^u, E_1^u\} = \max\left\{1, \frac{1}{3} 3^{2k+1} + \frac{2}{3}\right\} = \frac{3^{2k+1} + 2}{3} \geq \frac{5}{3} \\ H_1^d &= \max\{Y_1^d, E_1^d\} = \max\left\{1, \frac{1}{3} + \frac{2}{3}\right\} = 1, \end{aligned}$$

and at the initial time,

$$H_0 = \max\{Y_0, E_0\} = \max\left\{1, \frac{3^{2k+1}+2}{3^2} + \frac{2}{3}\right\} = \frac{8+3^{2k+1}}{9} \geq \frac{11}{9}.$$

We can then compute

$$\nu_{\min} = \min\{n \mid Y_n \geq E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_1^d\}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\nu_{\max} = \min\{n \mid Y_n > E_n\} = 2,$$

so that also in this case the American option coincides with the corresponding European one, again as already established in point i) since here we have assumed $r = 0$. However, contrary to what happened for X^k, here we do not have the full equality $\nu_{\min} = \nu_{\max}$ a.s. □

Problem 3.31. In a binomial market model consider an American "up-and-out" Call option with payoff

$$X_n = \begin{cases} (S_n - K)^+ & \text{if } S_k \leq 3 \ \text{ for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Assuming the following numerical data

$$u = 2, \quad d = \frac{1}{2}, \quad r = 0, \quad S_0 = 1, \quad K = \frac{1}{3},$$

and the time horizon of three periods, i.e. $N = 3$, determine:

i) the process H of the option price;
ii) the minimal and maximal optimal exercise strategies;
iii) the hedging strategy of the option for the first period.

Solution of Problem 3.31

i) We have a barrier option whose payoff is path-dependent, that is X_n depends on the trajectory of the underlying and not only on S_n. In particular, for each n it is necessary to consider all the different trajectories, for which at a time prior or equal to n, the price of the underlying asset is greater than the barrier 3: in that case the payoff is zero. Figure 3.9 shows the binomial tree of the asset price (inside the circles) and of the payoff of the American option (above the circles).

At the last time, the arbitrage price H of the derivative is equal to

$$\begin{cases} H_3^{uuu} = X_3^{uuu} = 0, \\ H_3^{uud} = X_3^{uud} = 0, \\ H_3^{udu} = X_3^{udu} = \left(2 - \frac{1}{3}\right)^+ = \frac{5}{3}, \\ H_3^{udd} = X_3^{udd} = \left(\frac{1}{2} - \frac{1}{3}\right)^+ = \frac{1}{6}, \\ H_3^{ddd} = X_3^{ddd} = \left(\frac{1}{8} - \frac{1}{3}\right)^+ = 0. \end{cases}$$

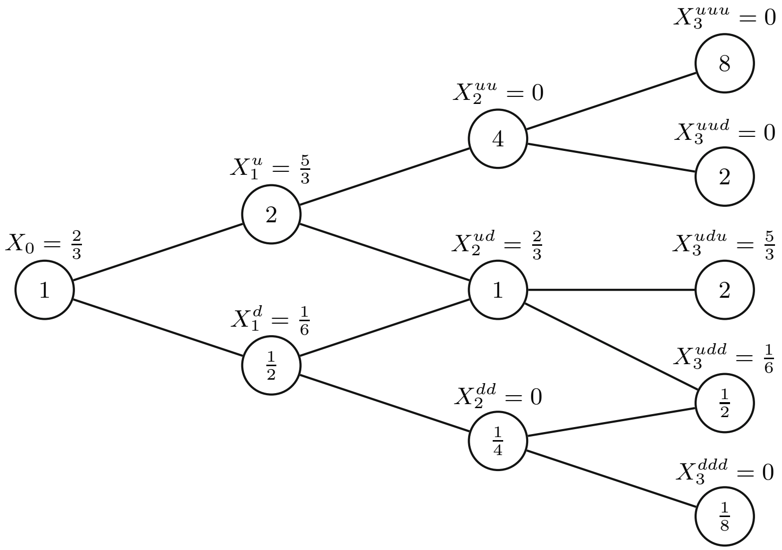

Fig. 3.9. Binomial tree of the prices of the underlying S (inside the circles) and of the values of the payoff X (above the circles)

Next we compute the arbitrage price at time $n = 2$: by definition we have

$$\begin{aligned}
H_2^{uu} &= \max\left\{X_2^{uu}, E_2^{uu}\right\} = \max\left\{0, qH_3^{uuu} + (1-q)H_3^{uud}\right\} = 0,\\
H_2^{ud} &= \max\left\{X_2^{ud}, E_2^{ud}\right\} = \max\left\{\frac{2}{3}, qH_3^{udu} + (1-q)H_3^{udd}\right\}\\
&= \max\left\{\frac{2}{3}, \frac{1}{3}\cdot\frac{5}{3} + \frac{2}{3}\cdot\frac{1}{6}\right\} = \max\left\{\frac{2}{3}, \frac{2}{3}\right\} = \frac{2}{3},\\
H_2^{dd} &= \max\left\{X_2^{dd}, E_2^{dd}\right\} = \max\left\{0, qH_3^{ddu} + (1-q)H_3^{ddd}\right\}\\
&= \frac{1}{3}\cdot\frac{1}{6} + \frac{2}{3}\cdot 0 = \frac{1}{18}.
\end{aligned}$$

At time $n = 1$, we have

$$\begin{aligned}
H_1^{u} &= \max\left\{X_1^{u}, E_1^{u}\right\} = \max\left\{\frac{5}{3}, qH_2^{uu} + (1-q)H_2^{ud}\right\}\\
&= \max\left\{\frac{5}{3}, \frac{1}{3}\cdot 0 + \frac{2}{3}\cdot\frac{2}{3}\right\} = \max\left\{\frac{5}{3}, \frac{4}{9}\right\} = \frac{5}{3},\\
H_1^{d} &= \max\left\{X_1^{d}, E_1^{d}\right\} = \max\left\{\frac{1}{6}, qH_2^{du} + (1-q)H_2^{dd}\right\}\\
&= \max\left\{\frac{1}{6}, \frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{18}\right\} = \max\left\{\frac{1}{6}, \frac{7}{27}\right\} = \frac{7}{27}.
\end{aligned}$$

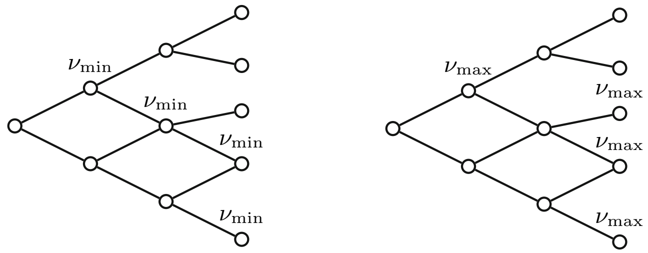

Fig. 3.10. Minimal (left) and maximal (right) optimal exercise strategies

Finally, at the initial time we have

$$\begin{aligned} H_0 &= \max\{X_0, E_0\} = \max\left\{\frac{2}{3}, qH_1^u + (1-q)H_1^d\right\} \\ &= \max\left\{\frac{2}{3}, \frac{1}{3}\cdot\frac{5}{3} + \frac{2}{3}\cdot\frac{7}{27}\right\} = \max\left\{\frac{2}{3}, \frac{59}{81}\right\} = \frac{59}{81}. \end{aligned}$$

ii) Comparing the values of X and E computed in the previous step, we can easily determine the minimal and maximal optimal exercise strategies. In fact, by definition we have

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_1^u\}, \\ 2 & \text{on } \{S_2 = S_2^{du}\}, \\ 3 & \text{otherwise,} \end{cases}$$

and

$$\nu_{\max} = \min\{n \mid X_n > E_n\} = \begin{cases} 1 & \text{on } \{S_1 = S_1^u\}, \\ 3 & \text{otherwise.} \end{cases}$$

The minimal and maximal optimal exercise strategies are shown in Figure 3.10. Notice that the fact that it is optimal to exercise early if $S_1 = uS_0$, is due to the presence of the barrier. For values of S "far" from the barrier, early exercise is not optimal just as in the standard case of an American Call with no barrier that is equivalent to the European Call.

iii) We calculate the hedging strategy for the first period: since $\nu_{\max} \geq 1$, as discussed in Section 3.1.4, this strategy coincides with the hedging strategy of H and therefore we obtain it by imposing the replication condition

$$\alpha_1 S_1 + \beta_1 B_1 = H_1,$$

which provides the system of linear equations

$$\begin{cases} \alpha_1 uS_0 + \beta_1(1+r) = H_1^u, \\ \alpha_1 dS_0 + \beta_1(1+r) = H_1^d, \end{cases}$$

equivalent to (recall that $r=0$)

$$\begin{cases} 2\alpha_1+\beta_1=\frac{5}{3},\\ \frac{1}{2}\alpha_1+\beta_1=\frac{7}{27}. \end{cases}$$

The solution of this system is

$$\alpha_1=\frac{76}{81}, \qquad \beta_1=-\frac{17}{81}.$$

Finally, we check that the initial cost of the strategy is equal to the initial price of the American option: recalling that $S_0=B_0=1$, we have

$$\frac{76}{81}S_0-\frac{17}{81}B_0=\frac{59}{81}=H_0.$$

Alternatively, we could also calculate the strategy using formula (3.35) which here becomes

$$\alpha_1=\frac{H_1^u-H_1^d}{S_0(u-d)}, \qquad \beta_1=H_0-\alpha_1 S_0. \qquad \square$$

Problem 3.32. In a completed trinomial market model (cf. Section 1.4.2), consider an American Put option on a basket of two assets, with payoff

$$X_n=\left(2-\frac{S_n^1+S_n^2}{2}\right)^+.$$

Assume that the parameters of the price process of the risky assets $S=(S^1,S^2)$ are given by

$$u_1=\frac{11}{6},\ u_2=\frac{5}{6},\ m_1=m_2=1,\ d_1=\frac{1}{2},\ d_2=2,\ S_0^1=S_0^2=1,\ r=\frac{1}{4}.$$

Consider the time horizon of two periods, i.e. $N=2$, and notice that, according to the numerical data, the unique equivalent martingale measure Q is defined by

$$Q(h=1)=q_1=\frac{1}{2}, \quad Q(h=2)=q_2=\frac{1}{6}, \quad Q(h=3)=q_3=\frac{1}{3},$$

with h as in Section 1.4.2. Determine:

i) the price process of the American option;
ii) the minimal and maximal optimal exercise strategies;
iii) the hedging strategy.

Solution of Problem 3.32

i) We use the notations (3.39), (3.41) and show in Figure 3.11 the trinomial tree (of the prices of the underlying assets) that in this case is not "recombining" because $u_i d_i \neq m_i^2$. Notice that the up-movement of one of the two assets corresponds to the down-movement of the other one.

At the last time we have

$$\begin{cases} H_2^{uu} = X_2^{uu} = \left(2 - \frac{\frac{121}{36} + \frac{25}{36}}{2}\right)^+ = 0, \\ H_2^{um} = X_2^{um} = \left(2 - \frac{\frac{11}{6} + \frac{5}{6}}{2}\right)^+ = \frac{2}{3}, \\ H_2^{ud} = X_2^{ud} = \left(2 - \frac{\frac{11}{12} + \frac{5}{3}}{2}\right)^+ = \frac{17}{24}, \\ H_2^{mm} = X_2^{mm} = 0, \\ H_2^{md} = X_2^{md} = \left(2 - \frac{\frac{1}{2}+2}{2}\right)^+ = \frac{3}{4}, \\ H_2^{dd} = X_2^{dd} = \left(2 - \frac{\frac{1}{4}+4}{2}\right)^+ = 0. \end{cases}$$

Then, by definition of arbitrage price, we have

$$H_1 = \max\{X_1, E_1\},$$

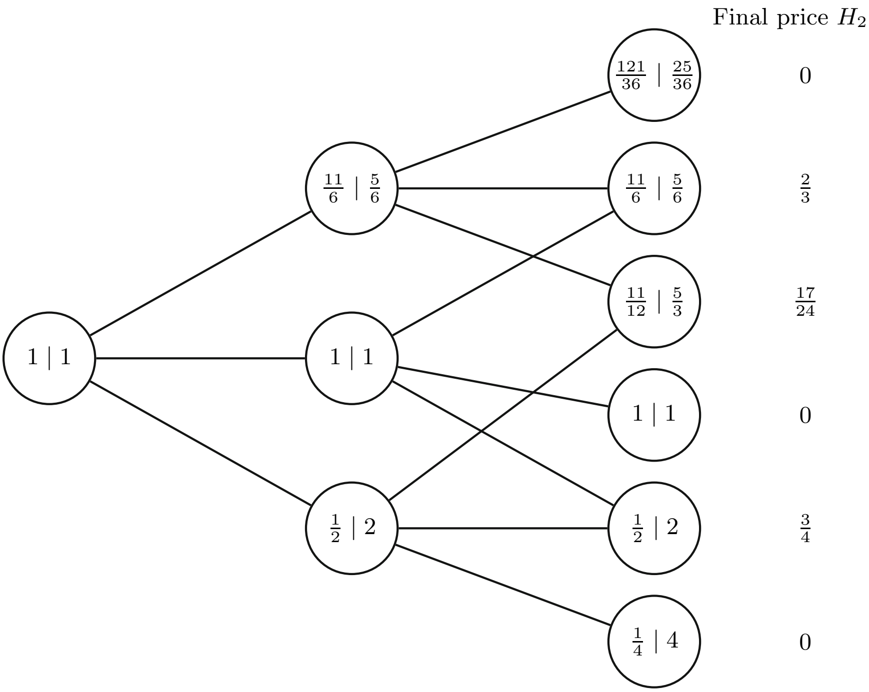

Fig. 3.11. Two-period trinomial tree: prices of the assets S^1, S^2 and final price $H_2 = X_2$ of the Put option

where

$$\begin{cases} X_1^u = \left(2 - \frac{\frac{11}{6}+\frac{5}{6}}{2}\right)^+ = \frac{2}{3}, \\ X_1^m = 0, \\ X_1^d = \left(2 - \frac{\frac{1}{2}+2}{2}\right)^+ = \frac{3}{4}. \end{cases}$$

Moreover

$$E_1 = \frac{1}{1+r} E^Q \left[H_2 \mid \mathcal{F}_1 \right]$$

and in particular

$$E_1^u = \frac{1}{1+r}\left(\frac{1}{2}H_2^{uu} + \frac{1}{6}H_2^{um} + \frac{1}{3}H_2^{ud}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot 0 + \frac{1}{6}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{17}{24}\right) = \frac{5}{18},$$
$$E_1^m = \frac{1}{1+r}\left(\frac{1}{2}H_2^{mu} + \frac{1}{6}H_2^{mm} + \frac{1}{3}H_2^{md}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot\frac{3}{4}\right) = \frac{7}{15},$$
$$E_1^d = \frac{1}{1+r}\left(\frac{1}{2}H_2^{du} + \frac{1}{6}H_2^{dm} + \frac{1}{3}H_2^{dd}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{17}{24} + \frac{1}{6}\cdot\frac{3}{4} + \frac{1}{3}\cdot 0\right) = \frac{23}{60}.$$

It follows that

$$H_1^u = X_1^u = \frac{2}{3} > E_1^u, \qquad H_1^m = E_1^m = \frac{7}{15}, \qquad H_1^d = X_1^d = \frac{3}{4} > E_1^d.$$

Finally, we have $X_0 = 0$ and

$$E_0 = \frac{1}{1+r}\left(\frac{1}{2}H_1^u + \frac{1}{6}H_1^m + \frac{1}{3}H_1^d\right) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{2}{3} + \frac{1}{6}\cdot\frac{7}{15} + \frac{1}{3}\cdot\frac{3}{4}\right) = \frac{119}{225},$$

and thus the initial price of the American option is equal to

$$H_0 = \max\{X_0, E_0\} = \frac{119}{225}.$$

ii) Concerning the optimal exercise strategies, by definition we have

$$\nu_{\max} = \min\{n \mid X_n > E_n\} = \begin{cases} 1 & \text{on } \{h_1 = 1\} \cup \{h_1 = 3\}, \\ 2 & \text{otherwise,} \end{cases} \tag{3.53}$$

and we easily verify that

$$\nu_{\min} = \nu_{\max} =: \bar{\nu}.$$

The optimal strategy $\bar{\nu}$ is displayed in Figure 3.12.

iii) We recall that, as discussed in Section 3.1.4, it is sufficient to determine the hedging strategy (α_n, β_n) of the American option only for $n \le \nu_{\max}$ and in this case the strategy coincides with the hedging strategy of H.

Given the expression of $\nu_{\max}$ in (3.53), it is therefore sufficient to calculate the initial strategy (α_1, β_1) for the first period and then the strategy (α_2, β_2)

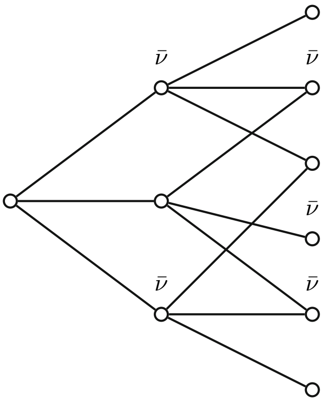

Fig. 3.12. Optimal exercise strategy

for the second period only in the case of $h_1 = 2$, that is for $S_1^1 = S_1^2 = 1$. In fact $n = 1$ is the last time of optimal exercise when $h_1 = 1$ or $h_1 = 3$: in other words, if the assets rise or decrease in the first period then the option is exercised and therefore we do not need to determine the hedging strategy for the second period.

We determine the hedging strategy for the first period by imposing the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1.$$

We get

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases}$$

which provides the system

$$\begin{cases} \frac{11}{6}\alpha_1^1 + \frac{5}{6}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{2}{3}, \\ \alpha_1^1 + \alpha_1^2 + \frac{5}{4}\beta_1 = \frac{7}{15}, \\ \frac{1}{2}\alpha_1^1 + 2\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{3}{4}, \end{cases}$$

with solution

$$\alpha_1^1 = \frac{89}{270}, \qquad \alpha_1^2 = \frac{121}{270}, \qquad \beta_1 = -\frac{56}{225}.$$

We verify that the initial cost of this strategy is equal to the initial price of the American option: recalling that $S_0^1 = S_0^2 = B_0 = 1$, we have

$$\frac{89}{270} S_0^1 + \frac{121}{270} S_0^2 - \frac{56}{225} B_0 = \frac{119}{225} = H_0.$$

Next we determine the hedging strategy for the second period when $S_1^1 = S_1^2 = 1$, by imposing the replication condition

$$\alpha_2^1 S_2^1 + \alpha_2^2 S_2^2 + \beta_2 B_2 = H_2.$$

We get

$$\begin{cases} \alpha_2^1 u_1 + \alpha_2^2 u_2 + \beta_2 (1+r)^2 = H_2^{mu}, \\ \alpha_2^1 m_1 + \alpha_2^2 m_2 + \beta_2 (1+r)^2 = H_2^{mm}, \\ \alpha_2^1 d_1 + \alpha_2^2 d_2 + \beta_2 (1+r)^2 = H_2^{md}, \end{cases}$$

which provides the system

$$\begin{cases} \frac{11}{6}\alpha_2^1 + \frac{5}{6}\alpha_2^2 + \frac{25}{16}\beta_2 = \frac{2}{3}, \\ \alpha_2^1 + \alpha_2^2 + \frac{25}{16}\beta_2 = 0, \\ \frac{1}{2}\alpha_2^1 + 2\alpha_2^2 + \frac{25}{16}\beta_2 = \frac{3}{4}, \end{cases}$$

with solution

$$\alpha_2^1 = \frac{19}{18}, \qquad \alpha_2^2 = \frac{23}{18}, \qquad \beta_2 = -\frac{112}{75}.$$

Again we see that the value of this strategy is equal to the price of the American option: indeed, since $S_1^1 = S_1^2 = 1$ and $B_1 = \frac{5}{4}$, we have

$$\frac{19}{18} S_1^1 + \frac{23}{18} S_1^2 - \frac{112}{75} B_1 = \frac{7}{15} = H_1^m.$$

Note that, although it is a Put option, the hedging strategy consists of taking a long position on the risky assets: this is due to the contravariant dynamics of the risky assets, that is the fact that an up-movement of one of two assets corresponds to the decrease of the other asset. □

Problem 3.33. In a completed trinomial market model, consider an American Call option on a basket of two assets, with payoff

$$X_n = S_n^2 - m_n \quad \text{where} \quad m_n := \min\left\{S_n^1, S_n^2\right\}.$$

Assume that the parameters for the price process of the risky assets $S = \left(S^1, S^2\right)$ are again given by

$$u_1 = \frac{11}{6},\ u_2 = \frac{5}{6},\ m_1 = m_2 = 1,\ d_1 = \frac{1}{2},\ d_2 = 2,\ S_0^1 = S_0^2 = 1,\ r = \frac{1}{4}.$$

Consider a time horizon of two periods, i.e. $N = 2$, and notice that, according to the numerical data, the unique equivalent martingale measure Q is defined by

$$Q(h=1) = q_1 = \frac{1}{2}, \quad Q(h=2) = q_2 = \frac{1}{6}, \quad Q(h=3) = q_3 = \frac{1}{3},$$

with h as in Section 1.4.2. Determine:

i) the price process of the American option;
ii) the minimal and maximal optimal exercise strategies;
iii) the hedging strategy for the second period in the scenarios where $\nu_{\max} > 1$.

Solution of Problem 3.33

i) We use the notations (3.39), (3.41). Figure 3.13 shows the trinomial tree (of the prices of the underlying assets), that in this case is not "recombining" because $u_i d_i \neq m_i^2$.

At the last time we have

$$\begin{cases} H_2^{uu} = X_2^{uu} = \frac{25}{36} - \frac{25}{36} = 0, \\ H_2^{um} = X_2^{um} = \frac{5}{6} - \frac{5}{6} = 0, \\ H_2^{ud} = X_2^{ud} = \frac{5}{3} - \frac{11}{12} = \frac{3}{4}, \\ H_2^{mm} = X_2^{mm} = 0, \\ H_2^{md} = X_2^{md} = 2 - \frac{1}{2} = \frac{3}{2}, \\ H_2^{dd} = X_2^{dd} = 4 - \frac{1}{4} = \frac{15}{4}. \end{cases}$$

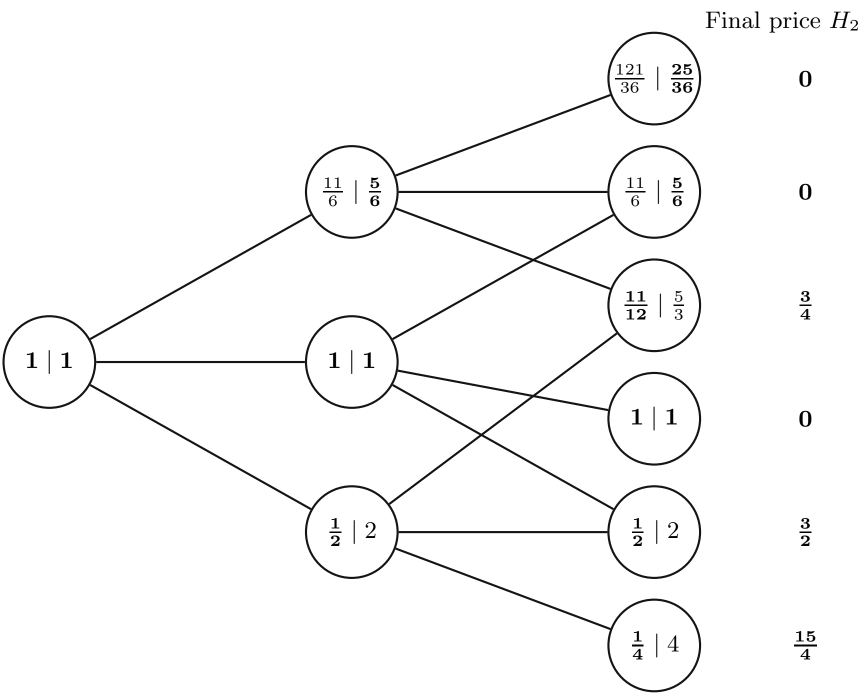

Fig. 3.13. Two-period trinomial tree: prices of the assets S^1, S^2 (we mark in bold the minimum of the two, equal to the value of m_n) and the final price $H_2 = X_2$ of the Put option

Then, by definition of arbitrage price, we obtain

$$H_1 = \max\{X_1, E_1\},$$

where

$$\begin{cases} X_1^u = \frac{5}{6} - \frac{5}{6} = 0, \\ X_1^m = 1 - 1 = 0, \\ X_1^d = 2 - \frac{1}{2} = \frac{3}{2}. \end{cases}$$

Moreover, we have

$$E_1 = \frac{1}{1+r} E^Q \left[H_2 \mid \mathcal{F}_1 \right]$$

and, specifically,

$$E_1^u = \frac{1}{1+r}\left(\frac{1}{2}H_2^{uu} + \frac{1}{6}H_2^{um} + \frac{1}{3}H_2^{ud}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot 0 + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot\frac{3}{4}\right) = \frac{1}{5},$$

$$E_1^m = \frac{1}{1+r}\left(\frac{1}{2}H_2^{mu} + \frac{1}{6}H_2^{mm} + \frac{1}{3}H_2^{md}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot 0 + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot\frac{3}{2}\right) = \frac{2}{5},$$

$$E_1^d = \frac{1}{1+r}\left(\frac{1}{2}H_2^{du} + \frac{1}{6}H_2^{dm} + \frac{1}{3}H_2^{dd}\right) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{3}{4} + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot\frac{15}{4}\right) = \frac{13}{10}.$$

Thus

$$H_1^u = E_1^u = \frac{1}{5} > X_1^u, \qquad H_1^m = E_1^m = \frac{2}{5} > X_1^m, \qquad H_1^d = X_1^d = \frac{3}{2} > E_1^d.$$

Finally, we have $X_0 = 0$ and

$$E_0 = \frac{1}{1+r}\left(\frac{1}{2}H_1^u + \frac{1}{6}H_1^m + \frac{1}{3}H_1^d\right) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{1}{5} + \frac{1}{6}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{3}{2}\right) = \frac{8}{15},$$

from which we deduce that the initial price of the American option is equal to

$$H_0 = \max\{X_0, E_0\} = E_0 = \frac{8}{15}.$$

ii) With regard to the optimal exercise strategies, we have

$$\nu_{\min} = \nu_{\max} = \begin{cases} 1 & \text{on } \{h_1 = 3\}, \\ 2 & \text{otherwise.} \end{cases} \tag{3.54}$$

iii) Given the expression of $\nu_{\max}$ in (3.54), it is sufficient to calculate the strategy (α_2, β_2) for the second period, when $h_1 = 1$ or $h_1 = 2$.

We begin with the case of $S_1^1 = S_1^2 = 1$ (i.e. $h_1 = 2$): by imposing the replication condition

$$\alpha_2^1 S_2^1 + \alpha_2^2 S_2^2 + \beta_2 B_2 = H_2,$$

we get

$$\begin{cases} \alpha_2^1 u_1 + \alpha_2^2 u_2 + \beta_2(1+r)^2 = H_2^{mu}, \\ \alpha_2^1 m_1 + \alpha_2^2 m_2 + \beta_2(1+r)^2 = H_2^{mm}, \\ \alpha_2^1 d_1 + \alpha_2^2 d_2 + \beta_2(1+r)^2 = H_2^{md}, \end{cases}$$

which provides the system

$$\begin{cases} \frac{11}{6}\alpha_2^1 + \frac{5}{6}\alpha_2^2 + \frac{25}{16}\beta_2 = 0, \\ \alpha_2^1 + \alpha_2^2 + \frac{25}{16}\beta_2 = 0, \\ \frac{1}{2}\alpha_2^1 + 2\alpha_2^2 + \frac{25}{16}\beta_2 = \frac{3}{2}, \end{cases}$$

with solution

$$\alpha_2^1 = \frac{1}{3}, \qquad \alpha_2^2 = \frac{5}{3}, \qquad \beta_2 = -\frac{32}{25}.$$

We check that the value of this strategy is equal to the price of the American option: indeed, since $S_1^1 = S_1^2 = 1$ and $B_1 = \frac{5}{4}$, we have

$$\frac{1}{3}S_1^1 + \frac{5}{3}S_1^2 - \frac{32}{25}B_1 = \frac{2}{5} = H_1^m.$$

Let us now consider the case of $S_1^1 = \frac{11}{6}$, $S_1^2 = \frac{5}{6}$ (i.e. $h_1 = 1$): by imposing the replication condition

$$\alpha_2^1 S_2^1 + \alpha_2^2 S_2^2 + \beta_2 B_2 = H_2,$$

we get

$$\begin{cases} \frac{11}{6}\alpha_2^1 u_1 + \frac{5}{6}\alpha_2^2 u_2 + \beta_2(1+r)^2 = H_2^{uu}, \\ \frac{11}{6}\alpha_2^1 m_1 + \frac{5}{6}\alpha_2^2 m_2 + \beta_2(1+r)^2 = H_2^{um}, \\ \frac{11}{6}\alpha_2^1 d_1 + \frac{5}{6}\alpha_2^2 d_2 + \beta_2(1+r)^2 = H_2^{ud}, \end{cases}$$

which provides the system

$$\begin{cases} (\frac{11}{6})^2\alpha_2^1 + (\frac{5}{6})^2\alpha_2^2 + \frac{25}{16}\beta_2 = 0, \\ \frac{11}{6}\alpha_2^1 + \frac{5}{6}\alpha_2^2 + \frac{25}{16}\beta_2 = 0, \\ \frac{11}{6} \cdot \frac{1}{2}\alpha_2^1 + 2 \cdot \frac{5}{6}\alpha_2^2 + \frac{25}{16}\beta_2 = \frac{3}{4}, \end{cases}$$

with solution

$$\alpha_2^1 = \frac{1}{11}, \qquad \alpha_2^2 = 1, \qquad \beta_2 = -\frac{16}{25}.$$

Again we see that the value of this strategy is equal to the price of the American option: indeed, since $S_1^1 = \frac{11}{6}, S_1^2 = \frac{5}{6}$ and $B_1 = \frac{5}{4}$, we have

$$\frac{1}{11}S_1^1 + S_1^2 - \frac{16}{25}B_1 = \frac{1}{5} = H_1^u. \qquad \square$$

Problem 3.34. In a completed trinomial market model (cf. Section 1.4.2) with a bond and two risky assets S_n^1, S_n^2, consider an exchange option of American style with payoff process

$$X_n = (K + S_n^2 - S_n^1)^+.$$

The numerical data are

$$u_1 = \frac{7}{3},\ u_2 = \frac{22}{9},\ m_1 = m_2 = 1,\ d_1 = \frac{1}{2},\ d_2 = \frac{1}{3},\ S_0^1 = S_0^2 = 1,\ r = \frac{1}{2},$$

and the time horizon is two periods, i.e. $N = 2$. Note that, according to the numerical data, the unique equivalent martingale measure Q is given by

$$Q(h = 1) = q_1 = \frac{1}{2}, \quad Q(h = 2) = q_2 = \frac{1}{6}, \quad Q(h = 3) = q_3 = \frac{1}{3},$$

with h as in Section 1.4.2.

i) In the case of $K = 0$, compute the price of the American option. Determine the minimal and maximal optimal exercise strategies and verify that this American option is equivalent to the corresponding European one;
ii) in the case of $K \leq 0$, using the property of convexity of the payoff function, check that this American option reduces to a European one;
iii) for $K = \frac{1}{10}$, compute the price of the American option and verify that at the initial time it is greater than the price of the corresponding European one. Also determine the minimal and maximal optimal exercise strategies. Finally, specify the system of equations that has to be satisfied by the hedging strategy in the first period.

Solution of Problem 3.34

i) We use the notations (3.39), (3.41) and show in Figure 3.14 the trinomial tree of the prices of the underlyings. At the last time we have

$$\begin{cases} H_2^{uu} = X_2^{uu} = \left(\frac{484}{81} - \frac{49}{9}\right)^+ = \frac{43}{81}, \\ H_2^{um} = X_2^{um} = \left(\frac{22}{9} - \frac{7}{3}\right)^+ = \frac{1}{9}, \\ H_2^{ud} = X_2^{ud} = \left(\frac{22}{27} - \frac{7}{6}\right)^+ = 0, \\ H_2^{mm} = X_2^{mm} = (1 - 1)^+ = 0, \\ H_2^{md} = X_2^{md} = \left(\frac{1}{3} - \frac{1}{2}\right)^+ = 0, \\ H_2^{dd} = X_2^{dd} = \left(\frac{1}{9} - \frac{1}{4}\right)^+ = 0. \end{cases}$$

Then, by definition of arbitrage price, we have

$$H_1 = \max\{X_1, E_1\},$$

where $X_1 = \left(S_1^2 - S_1^1\right)^+$ and in particular

$$X_1^u = \left(\frac{22}{9} - \frac{7}{3}\right)^+ = \frac{1}{9}, \quad X_1^m = (1 - 1)^+ = 0, \quad X_1^d = \left(\frac{1}{3} - \frac{1}{2}\right)^+ = 0.$$

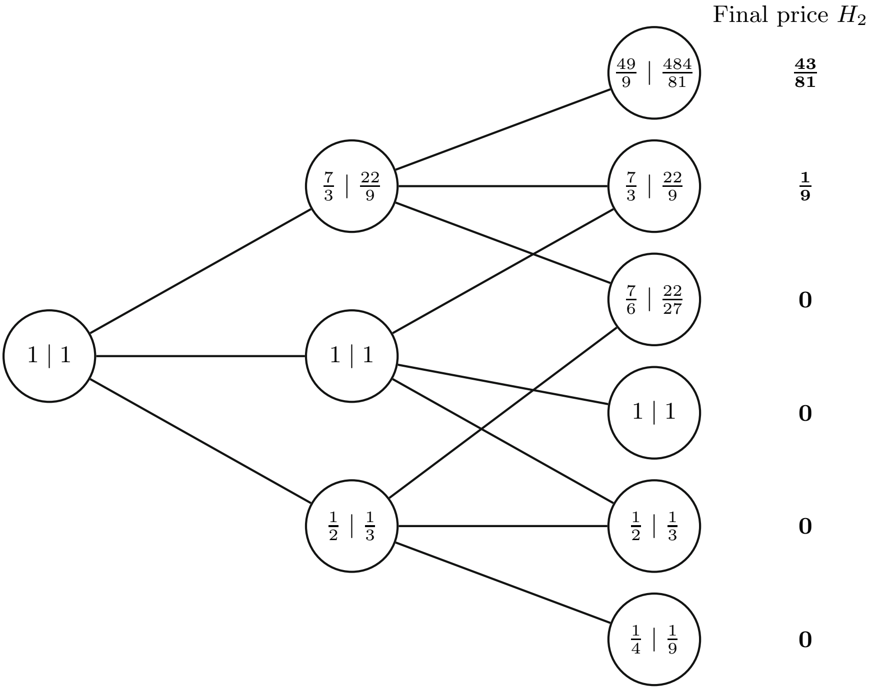

Fig. 3.14. Two-period trinomial tree: prices of the assets S^1, S^2 and final price H_2 of the exchange option in the case of $K = 0$

Moreover we have $E_1 = \frac{1}{1+r} E^Q [H_2 \mid \mathcal{F}_1]$ and in particular

$$\begin{aligned}
E_1^u &= \frac{1}{1+r}\left(\frac{1}{2}H_2^{uu} + \frac{1}{6}H_2^{um} + \frac{1}{3}H_2^{ud}\right) \\
&= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{43}{81} + \frac{1}{6}\cdot\frac{1}{9} + \frac{1}{3}\cdot 0\right) = \frac{46}{243}, \\
E_1^m &= \frac{1}{1+r}\left(\frac{1}{2}H_2^{mu} + \frac{1}{6}H_2^{mm} + \frac{1}{3}H_2^{md}\right) \\
&= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{1}{9} + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 0\right) = \frac{1}{27}, \\
E_1^d &= \frac{1}{1+r}\left(\frac{1}{2}H_2^{du} + \frac{1}{6}H_2^{dm} + \frac{1}{3}H_2^{dd}\right) = \frac{2}{3}\left(\frac{1}{2}\cdot 0 + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 0\right) = 0.
\end{aligned}$$

Therefore

$$H_1^u = E_1^u = \frac{46}{243}, \qquad H_1^m = E_1^m = \frac{1}{27}, \qquad H_1^d = X_1^d = 0.$$

Finally, $X_0 = (1-1)^+ = 0$ and

$$\begin{aligned} E_0 &= \frac{1}{1+r}\left(\frac{1}{2}H_1^u + \frac{1}{6}H_1^m + \frac{1}{3}H_1^d\right) \\ &= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{46}{243} + \frac{1}{6}\cdot\frac{1}{27} + \frac{1}{3}\cdot 0\right) = \frac{49}{729}, \end{aligned}$$

from which we deduce that the initial price of the American option is equal to

$$H_0 = \max\{X_0, E_0\} = \frac{49}{729}.$$

With regard to the optimal exercise strategies, by definition we have

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\} = \begin{cases} 1 & \text{on } \{h_1 = h_1^d\}, \\ 2 & \text{otherwise,} \end{cases}$$

and

$$\nu_{\max} = \min\{n \mid X_n > E_n\} = 2.$$

Since $N = \nu_{\max} = 2$ is an optimal exercise strategy, according to the definition of arbitrage price, we have

$$H_0 = \max_{\nu \in \mathcal{T}_0} E^Q\left[\widetilde{X}_\nu\right] = E^Q\left[\widetilde{X}_{\nu_{\max}}\right] = E^Q\left[\widetilde{X}_2\right]$$

and thus the American option is equivalent to the European one with payoff X_2. The minimal and maximal optimal exercise strategies are displayed in Figure 3.15.

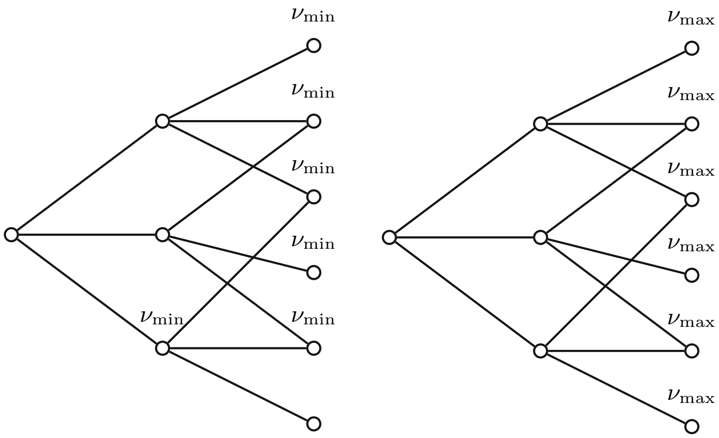

Fig. 3.15. Minimal (left) and maximal (right) optimal exercise strategies in the case of $K = 0$

ii) The payoff is equal to

$$X_n = g\left(S_n^1, S_n^2\right) = \left(K + S_n^1 - S_n^2\right)^+$$

where g is a convex function and $g(0) = 0$ because $K \le 0$. From Corollary 3.27 it follows that $\widetilde{X}_n$ is a Q-sub-martingale and therefore a possible optimal exercise consists of continuing up to maturity. We can also check directly that $\widetilde{X}_n$ is a Q-sub-martingale: indeed, we have

$$\begin{aligned}
E^Q\left[\widetilde{X}_{n+1} \mid \mathcal{F}_n\right] &= E^Q\left[\left(\frac{K}{B_{n+1}} + \frac{S_{n+1}^2}{B_{n+1}} - \frac{S_{n+1}^1}{B_{n+1}}\right)^+ \mid \mathcal{F}_n\right] \\
&\ge \left(E^Q\left[\frac{K}{B_{n+1}} + \frac{S_{n+1}^2}{B_{n+1}} - \frac{S_{n+1}^1}{B_{n+1}} \mid \mathcal{F}_n\right]\right)^+ \\
&= \left(\frac{K}{B_{n+1}} + \frac{S_n^2}{B_n} - \frac{S_n^1}{B_n}\right)^+ \ge \left(\frac{K}{B_n} + \frac{S_n^2}{B_n} - \frac{S_n^1}{B_n}\right)^+ = \widetilde{X}_n
\end{aligned}$$

where, in the second step, we used Jensen's inequality and in the fourth step, the fact that $K \le 0$ and $r \ge 0$. Thus in this case the conclusion follows from Proposition 3.25.

iii) Assume $K = \frac{1}{10}$: at the last time we have

$$\begin{cases}
H_2^{uu} = X_2^{uu} = \left(\frac{1}{10} + \frac{484}{81} - \frac{49}{9}\right)^+ = \frac{511}{810}, \\
H_2^{um} = X_2^{um} = \left(\frac{1}{10} + \frac{22}{9} - \frac{7}{3}\right)^+ = \frac{19}{90}, \\
H_2^{ud} = X_2^{ud} = \left(\frac{1}{10} + \frac{22}{27} - \frac{7}{6}\right)^+ = 0, \\
H_2^{mm} = X_2^{ud} = \left(\frac{1}{10} + 1 - 1\right)^+ = \frac{1}{10}, \\
H_2^{md} = X_2^{md} = \left(\frac{1}{10} + \frac{1}{3} - \frac{1}{2}\right)^+ = 0, \\
H_2^{dd} = X_2^{dd} = \left(\frac{1}{10} + \frac{1}{9} - \frac{1}{4}\right)^+ = 0.
\end{cases} \tag{3.55}$$

Then, by definition of arbitrage price, we have

$$H_1 = \max\left\{X_1, E_1\right\}, \tag{3.56}$$

where $X_1 = \left(\frac{1}{10} + S_1^2 - S_1^1\right)^+$: in particular

$$\begin{cases}
X_1^u = \left(\frac{1}{10} + \frac{22}{9} - \frac{7}{3}\right)^+ = \frac{19}{90}, \\
X_1^m = \left(\frac{1}{10} + 1 - 1\right)^+ = \frac{1}{10}, \\
X_1^d = \left(\frac{1}{10} + \frac{1}{3} - \frac{1}{2}\right)^+ = 0,
\end{cases} \tag{3.57}$$

and

$$\begin{cases}
E_1^u = \frac{1}{1+r}\left(\frac{1}{2}H_2^{uu} + \frac{1}{6}H_2^{um} + \frac{1}{3}H_2^{ud}\right) \\
\qquad = \frac{2}{3}\left(\frac{1}{2}\cdot\frac{511}{810} + \frac{1}{6}\cdot\frac{19}{90} + \frac{1}{3}\cdot 0\right) = \frac{284}{1215}, \\
E_1^m = \frac{1}{1+r}\left(\frac{1}{2}H_2^{mu} + \frac{1}{6}H_2^{mm} + \frac{1}{3}H_2^{md}\right) \\
\qquad = \frac{2}{3}\left(\frac{1}{2}\cdot\frac{19}{90} + \frac{1}{6}\cdot\frac{1}{10} + \frac{1}{3}\cdot 0\right) = \frac{11}{135}, \\
E_1^d = \frac{1}{1+r}\left(\frac{1}{2}H_2^{du} + \frac{1}{6}H_2^{dm} + \frac{1}{3}H_2^{dd}\right) \\
\qquad = \frac{2}{3}\left(\frac{1}{2}\cdot 0 + \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 0\right) = 0.
\end{cases} \tag{3.58}$$

Thus we have

$$\begin{aligned} H_1^u &= \max\{X_1^u, E_1^u\} = E_1^u = \frac{284}{1215}, \\ H_1^m &= \max\{X_1^m, E_1^m\} = X_1^m = \frac{1}{10}, \\ H_1^d &= \max\left\{X_1^d, E_1^d\right\} = E_1^d = X_1^d = 0. \end{aligned}$$

Finally, $X_0 = (1-1)^+ = 0$ and

$$\begin{aligned} E_0 &= \frac{1}{1+r}\left(\frac{1}{2}H_1^u + \frac{1}{6}H_1^m + \frac{1}{3}H_1^d\right) \\ &= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{284}{1215} + \frac{1}{6}\cdot\frac{1}{10} + \frac{1}{3}\cdot 0\right) = \frac{649}{7290}, \end{aligned} \tag{3.59}$$

and therefore the initial price of the American option is equal to

$$H_0 = \max\{X_0, E_0\} = \frac{649}{7290}.$$

We now verify that this value is greater than the price H_0^E of the corresponding European option: indeed, by (3.55), we have

$$\begin{aligned} H_0^E &= \frac{1}{(1+r)^2}E^Q\left[\widetilde{X}_2\right] \\ &= \frac{4}{9}\left(q_1^2\frac{511}{810} + 2q_1q_2\frac{19}{20} + q_2^2\frac{1}{10}\right) = \frac{317}{3645} < H_0. \end{aligned}$$

Next we determine the minimal and maximal optimal exercise strategies: by definition (3.42)-(3.43) we simply have to compare the values of X and E previously computed in (3.56), (3.57), (3.58) and (3.59), to get

$$\nu_{\min} = \begin{cases} 1 & \text{on } \{h_1 = h_1^d, h_1^m\}, \\ 2 & \text{otherwise}, \end{cases}, \qquad \nu_{\max} = \begin{cases} 1 & \text{on } \{h_1 = h_1^m\}, \\ 2 & \text{otherwise}, \end{cases},$$

as shown in Figure 3.16.

Finally we determine the initial hedging strategy: since $\nu_{\max} \geq 1$, the hedging strategy of the American option in the first period coincides with the hedging strategy of H_1, in agreement with what we observed in Section 3.1.4. Hence we impose the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1$$

which is equivalent to

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases}$$

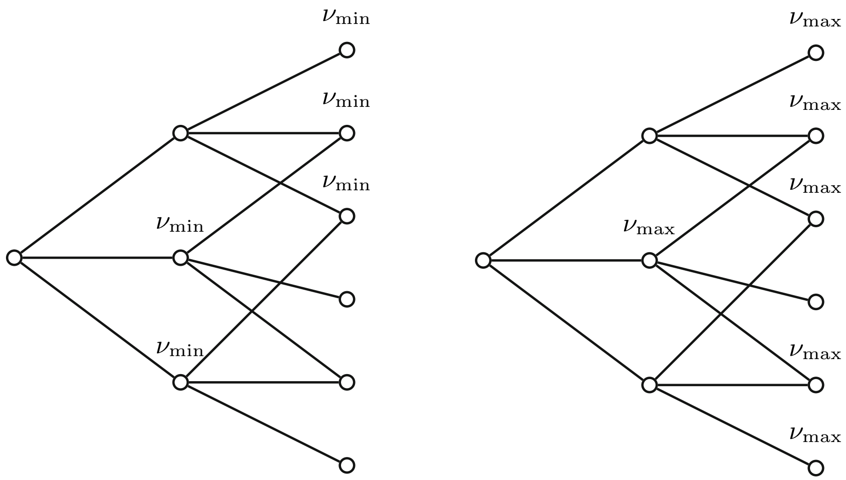

Fig. 3.16. Optimal exercise strategies for the American exchange option with payoff $X_n = \left(\frac{1}{10} + S_n^2 - S_n^1\right)^+$

and provides the system

$$\begin{cases} \frac{7}{3}\alpha_1^1 + \frac{22}{9}\alpha_1^2 + \frac{3}{2}\beta_1 = \frac{284}{1215}, \\ \alpha_1^1 + \alpha_1^2 + \frac{3}{2}\beta_1 = \frac{1}{10}, \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{3}{2}\beta_1 = 0. \end{cases}$$

□

Problem 3.35. Consider a completed trinomial market model with two risky assets S^1 and S^2 (in addition to a riskless one). The numerical data are

$$u_1 = 2,\ m_1 = 1,\ d_1 = \frac{1}{2},\ u_2 = \frac{7}{3},\ m_2 = \frac{7}{9},\ d_2 = \frac{1}{3},\ S_0^1 = S_0^2 = 1,\ r = \frac{1}{4}.$$

It turns out that the unique equivalent martingale measure Q is defined by

$$Q(h=1) = q_1 = \frac{3}{8}, \quad Q(h=2) = q_2 = \frac{3}{8}, \quad Q(h=3) = q_3 = \frac{1}{4},$$

with h as in Section 1.4.2. For a time horizon of two periods, i.e. $N = 2$, consider an American-Asian Put option with floating strike, whose payoff is given by

$$X_n = \left(A_n - S_n^1\right)^+, \qquad A_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k^1.$$

Determine:

i) the price process of the option;
ii) the minimal and maximal optimal exercise strategies and the hedging strategy for the first period.

Solution of Problem 3.35

i) The option depends only on the first asset and the payoff is path-dependent, that is X_n depends on the trajectory of the underlying up to time n and not only on the price S_n^1. The trinomial tree is represented in Figure 3.17: we distinguish the individual trajectories and put the price S^1 and the corresponding average A inside the circles; the values of the payoff X are outside the circles.

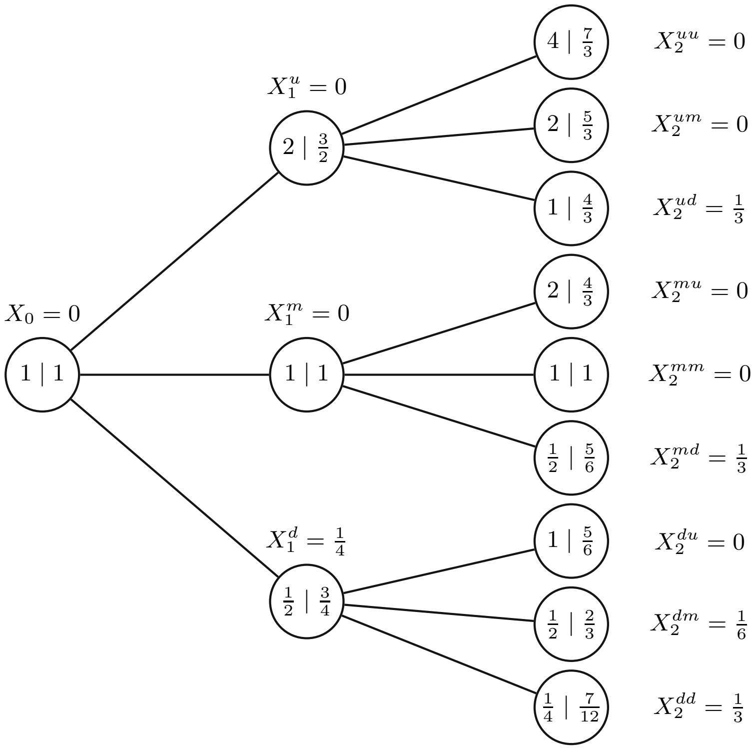

Fig. 3.17. Two-period trinomial tree: the values of S^1 (left) and of the average A (right) are shown inside the circles. The values of payoff X are put outside the circles

At the last time, the arbitrage price H of the derivative is equal to

$$\begin{cases} H_2^{uu} = X_2^{uu} = \left(\frac{7}{3} - 4\right)^+ = 0, \\ H_2^{um} = X_2^{um} = \left(\frac{5}{3} - 2\right)^+ = 0, \\ H_2^{ud} = X_2^{ud} = \left(\frac{4}{3} - 1\right)^+ = \frac{1}{3}, \\ H_2^{mu} = X_2^{mu} = \left(\frac{4}{3} - 2\right)^+ = 0, \\ H_2^{mm} = X_2^{mm} = 0, \\ H_2^{md} = X_2^{md} = \left(\frac{5}{6} - \frac{1}{2}\right)^+ = \frac{1}{3}, \\ H_2^{du} = X_2^{du} = \left(\frac{5}{6} - 1\right)^+ = 0, \\ H_2^{dm} = X_2^{dm} = \left(\frac{2}{3} - \frac{1}{2}\right)^+ = \frac{1}{6}, \\ H_2^{dd} = X_2^{dd} = \left(\frac{7}{12} - \frac{1}{4}\right)^+ = \frac{1}{3}. \end{cases}$$

We now calculate the arbitrage price at time $n = 1$: by definition we have

$$\begin{aligned} H_1^u &= \max\left\{X_1^u, E_1^u\right\} = \max\left\{0, \frac{1}{1+r}\left(q_1 X_2^{uu} + q_2 X_2^{um} + q_3 X_2^{ud}\right)\right\} \\ &= \frac{4}{5}\left(\frac{3}{8} \cdot 0 + \frac{3}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{3}\right) = \frac{1}{15}, \\ H_1^m &= \max\left\{X_1^m, E_1^m\right\} = \max\left\{0, \frac{1}{1+r}\left(q_1 X_2^{mu} + q_2 X_2^{mm} + q_3 X_2^{md}\right)\right\} \\ &= \frac{4}{5}\left(\frac{3}{8} \cdot 0 + \frac{3}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{3}\right) = \frac{1}{15}, \\ H_1^d &= \max\left\{X_1^d, E_1^d\right\} = \max\left\{\frac{1}{4}, \frac{1}{1+r}\left(q_1 X_2^{du} + q_2 X_2^{dm} + q_3 X_2^{dd}\right)\right\} \\ &= \max\left\{\frac{1}{4}, \frac{4}{5}\left(\frac{3}{8} \cdot 0 + \frac{3}{8} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{3}\right)\right\} = \max\left\{\frac{1}{4}, \frac{7}{60}\right\} = \frac{1}{4}. \end{aligned}$$

Finally, at the initial time we have

$$\begin{aligned} H_0 &= \max\left\{X_0, E_0\right\} = \max\left\{0, \frac{1}{1+r}\left(q_1 H_1^u + q_2 H_1^m + q_3 H_1^d\right)\right\} \\ &= \frac{4}{5}\left(\frac{3}{8} \cdot \frac{1}{15} + \frac{3}{8} \cdot \frac{1}{15} + \frac{1}{4} \cdot \frac{1}{4}\right) = \frac{9}{100}. \end{aligned}$$

ii) With regard to the optimal exercise strategies, recalling the definition

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\}, \qquad \nu_{\max} = \min\{n \mid X_n > E_n\},$$

we get

$$\nu_{\max} = \begin{cases} 1 & \text{on } \{h_1 = 3\}, \\ 2 & \text{otherwise}, \end{cases} \tag{3.60}$$

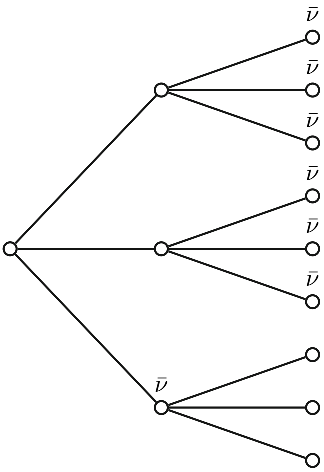

Fig. 3.18. Optimal exercise strategy

and it is easily verified that $\nu_{\min} = \nu_{\max} =: \bar{\nu}$. Figure 3.18 shows the optimal exercise strategy $\bar{\nu}$.

Finally, we determine the hedging strategy for the first period: noting that $\nu_{\max} \geq 1$ by (3.60), this strategy coincides with the hedging strategy of H (cf. Section 3.1.4) and therefore it can be determined by imposing the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1. \tag{3.61}$$

We observe that, in the completed trinomial model, even if the option depends only on the asset S^1, the hedging strategy requires nevertheless to invest in both risky assets. As a matter of fact, with the possibility of investing only in S^1, the market would be incomplete.

Equation (3.61) yields the system of linear equations

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases}$$

equivalent to

$$\begin{cases} 2\alpha_1^1 + \frac{7}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{15}, \\ \alpha_1^1 + \frac{7}{9}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{15}, \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{4}. \end{cases}$$

The solution of this system is

$$\alpha_1^1 = -\frac{77}{90}, \qquad \alpha_1^2 = \frac{11}{20}, \qquad \beta_1 = \frac{89}{225}.$$

Thus the hedging strategy of the Asian Put, requires to take a short position on the underlying and at the same time to buy a number of units of the second risky asset S^2 and of the bond B equal to $\frac{11}{20}$ and $\frac{89}{225}$ respectively. We verify that the initial cost of this strategy is equal to the initial price of the American option: indeed, recalling that $S_0^1 = S_0^2 = B_0 = 1$, we have

$$-\frac{77}{90}S_0^1 + \frac{11}{20}S_0^2 + \frac{89}{225}B_0 = \frac{9}{100} = H_0. \qquad \square$$

Problem 3.36. Consider a completed trinomial market model with two risky assets S^1 and S^2 (in addition to a riskless one). With the numerical data of the previous problem, namely

$$u_1 = 2,\ m_1 = 1,\ d_1 = \frac{1}{2},\ u_2 = \frac{7}{3},\ m_2 = \frac{7}{9},\ d_2 = \frac{1}{3},\ S_0^1 = S_0^2 = 1,\ r = \frac{1}{4},$$

the unique equivalent martingale measure Q is defined by

$$Q(h=1) = q_1 = \frac{3}{8}, \quad Q(h=2) = q_2 = \frac{3}{8}, \quad Q(h=3) = q_3 = \frac{1}{4},$$

with h as in Section 1.4.2. For a time horizon of two periods, i.e. $N = 2$, consider a Backward American Put option with payoff process

$$X_n = M_n - S_n^1, \qquad M_n = \max_{k \le n} S_k^1.$$

Determine:

i) the price process of the option;
ii) the minimal and maximal optimal exercise strategies and the hedging strategy for the first period.

Solution of Problem 3.36

i) As in Problem 3.35, the option depends only on the first asset and the payoff is path-dependent, that is X_n depends on the trajectory of the underlying up to time n and not only on the price S_n^1. The trinomial tree is represented in Figure 3.19: we distinguish the individual trajectories and put the price S^1 and the corresponding maximum M inside the circles; the values of the payoff X are outside the circles.

At the last time, the arbitrage price H of the derivative is equal to

$$\begin{cases} H_2^{uu} = X_2^{uu} = 4 - 4 = 0, \\ H_2^{um} = X_2^{um} = 2 - 2 = 0, \\ H_2^{ud} = X_2^{ud} = 2 - 1 = 1, \\ H_2^{mu} = X_2^{mu} = 2 - 2 = 0, \\ H_2^{mm} = X_2^{mm} = 1 - 1 = 0, \\ H_2^{md} = X_2^{md} = 1 - \frac{1}{2} = \frac{1}{2}, \\ H_2^{du} = X_2^{du} = 1 - 1 = 0, \\ H_2^{dm} = X_2^{dm} = 1 - \frac{1}{2} = \frac{1}{2}, \\ H_2^{dd} = X_2^{dd} = 1 - \frac{1}{4} = \frac{3}{4}. \end{cases}$$

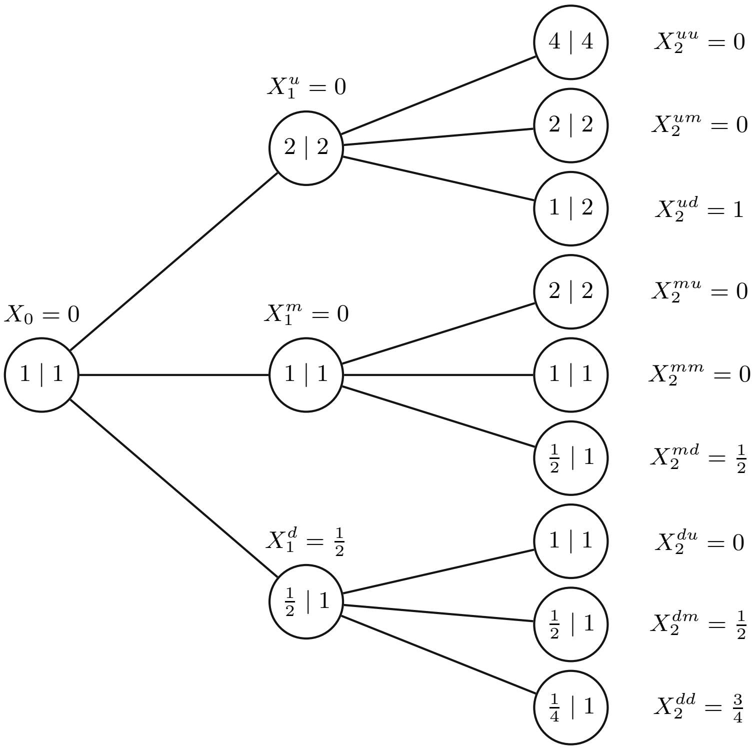

Fig. 3.19. Two-period trinomial tree: the values of S^1 (left) and of the maximum M (right) are shown inside the circles. The values of payoff X are put outside the circles

We determine the arbitrage price at time $n = 1$: by definition we have

$$
\begin{aligned}
H_1^u &= \max\{X_1^u, E_1^u\} = \max\left\{0, \frac{1}{1+r}\left(q_1 X_2^{uu} + q_2 X_2^{um} + q_3 X_2^{ud}\right)\right\} \\
&= \frac{4}{5}\left(\frac{3}{8}\cdot 0 + \frac{3}{8}\cdot 0 + \frac{1}{4}\cdot 1\right) = \frac{1}{5}, \\
H_1^m &= \max\{X_1^m, E_1^m\} = \max\left\{0, \frac{1}{1+r}\left(q_1 X_2^{mu} + q_2 X_2^{mm} + q_3 X_2^{md}\right)\right\} \\
&= \frac{4}{5}\left(\frac{3}{8}\cdot 0 + \frac{3}{8}\cdot 0 + \frac{1}{4}\cdot \frac{1}{2}\right) = \frac{1}{10}, \\
H_1^d &= \max\left\{X_1^d, E_1^d\right\} = \max\left\{\frac{1}{2}, \frac{1}{1+r}\left(q_1 X_2^{du} + q_2 X_2^{dm} + q_3 X_2^{dd}\right)\right\} \\
&= \max\left\{\frac{1}{2}, \frac{4}{5}\left(\frac{3}{8}\cdot 0 + \frac{3}{8}\cdot \frac{1}{2} + \frac{1}{4}\cdot \frac{3}{4}\right)\right\} = \max\left\{\frac{1}{2}, \frac{3}{10}\right\} = \frac{1}{2}.
\end{aligned}
$$

Finally, at the initial time we have

$$H_0 = \max\{X_0, E_0\} = \max\left\{0, \frac{q_1 H_1^u + q_2 H_1^m + q_3 H_1^d}{1+r}\right\}$$
$$= \frac{4}{5}\left(\frac{3}{8}\cdot\frac{1}{5} + \frac{3}{8}\cdot\frac{1}{10} + \frac{1}{4}\cdot\frac{1}{2}\right) = \frac{19}{100}.$$

ii) With regard to the optimal exercise strategies, recalling the definition

$$\nu_{\min} = \min\{n \mid X_n \geq E_n\}, \qquad \nu_{\max} = \min\{n \mid X_n > E_n\},$$

we get

$$\nu_{\max} = \begin{cases} 1 & \text{on } \{h_1 = 3\}, \\ 2 & \text{otherwise}, \end{cases} \tag{3.62}$$

and therefore we have $\nu_{\min} = \nu_{\max}$ denoting the common value by $\bar{\nu}$. The representation of the optimal strategy $\bar{\nu}$ is the same as in Figure 3.18 of Problem 3.35.

Finally we compute the hedging strategy for the first period: noting that $\nu_{\max} \geq 1$ by (3.62), this strategy coincides with the hedging strategy of H (cf. Section 3.1.4) and therefore it can be determined by imposing the replication condition

$$\alpha_1^1 S_1^1 + \alpha_1^2 S_1^2 + \beta_1 B_1 = H_1. \tag{3.63}$$

We observe again that in the trinomial model, even if the option depends only on the asset S^1, the hedging strategy requires nevertheless to invest in both risky assets.

Formula (3.63) yields the following system of linear equations

$$\begin{cases} \alpha_1^1 u_1 S_0^1 + \alpha_1^2 u_2 S_0^2 + \beta_1(1+r) = H_1^u, \\ \alpha_1^1 m_1 S_0^1 + \alpha_1^2 m_2 S_0^2 + \beta_1(1+r) = H_1^m, \\ \alpha_1^1 d_1 S_0^1 + \alpha_1^2 d_2 S_0^2 + \beta_1(1+r) = H_1^d, \end{cases}$$

equivalent to

$$\begin{cases} 2\alpha_1^1 + \frac{7}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{5}, \\ \alpha_1^1 + \frac{7}{9}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{10}, \\ \frac{1}{2}\alpha_1^1 + \frac{1}{3}\alpha_1^2 + \frac{5}{4}\beta_1 = \frac{1}{2}. \end{cases}$$

The solution of this system is

$$\alpha_1^1 = -2, \qquad \alpha_1^2 = \frac{27}{20}, \qquad \beta_1 = \frac{21}{25}.$$

Finally, we check that the initial cost of this strategy is equal to the initial price of the American option: indeed, since $S_0^1 = S_0^2 = B_0 = 1$, we have

$$-2S_0^1 + \frac{27}{20}S_0^2 + \frac{21}{25}B_0 = \frac{19}{100} = H_0.$$

□

4

Interest rates

In this chapter we consider the term structure of interest rates and the interest rate derivatives. The interest rates are closely related to the bond market; we therefore introduce the interest rates in connection with the simplest assets on the bond market, namely the so-called T-bonds which are contracts that guarantee a unitary amount at a given maturity T and their prices express the expectations of the market on the future value of money.

To determine various quantities related to the interest rates we need stochastic evolution models for the rates themselves. By analogy to standard modeling in continuous time, also in discrete time we consider two classes of models: "short" and "forward"-type of models.

Among the short-type models a particular role is played by the so-called "affine" models. For a general treatment of such models see e.g. [6], [8], [9]. Our treatment for the specific case of discrete time is inspired by [10]. For the forward models we base ourselves on the original paper [13] (see also [7]).

The stochasticity in the evolution of the rates constitutes a risk factor for future payments of interest, whether paid or received. To confine this kind of risk, by analogy to the derivatives in the stock market (where the underlyings are risky assets), interest rate derivatives have been introduced and they are the object of study in the last part of this chapter. For the solution of the pricing problem for the interest rate derivatives we have tried to reduce all the calculations to expressions that involve only T-bonds together with their recursive relations. Even if recursive calculations may be demanding in terms of the amount of computations, nevertheless they constitute a unifying approach to all interest rate derivatives differently from what happens in continuous time, where the so-called "market models" have been introduced and these models may vary with the type of derivative that is being considered (for example the LIBOR and the Swap Market Models). For an introductory treatment of interest rate derivatives we refer to Chapter 25 in [3]. A more exhaustive treatment can be found e.g. in [4].

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Before actually solving the problems in the problem Section 4.6, in Section 4.6.1 we synthesize some specific properties of our two models and this particularly in view of their applications in the solution of the proposed problems. The problems themselves are grouped according to the topics treated in the theoretical part, beginning from bond options and then treating successively Caps and Floors, Swap Rates and Forward Swaps, and finally Swaptions.

4.1 Bonds and interest rates

In this section we introduce and define the economic quantities that are relevant for the description of the interest rate markets[1].

We start by illustrating the principal characteristics, on the basis of which one usually classifies the rates. In the following, $t < T < S$ denote three time instants: an interest rate r, relative to the time interval $[T, S]$, may be of one of the following types:

- **Simple or compounded.** r is a simple or compounded rate if it is defined on the basis of the simple or compounded capitalization rule respectively;
- **Annualized or on the basis of the interval $[T, S]$.** r is an annualized rate if it is evaluated on an annual basis. More precisely, r is an annualized compounded rate if the following capitalization formula holds

  $$C_S = C_T e^{(S-T)r},$$

  where C_t denotes the value of the capital at time t; analogously, r is a simple annualized rate if instead we have as capitalization formula

  $$C_S = C_T \left(1 + (S-T)r\right).$$

  On the other hand, the compounded capitalization formula

  $$C_S = C_T e^{r}$$

  defines the compounded rate r on the basis of the interval $[T, S]$ and the formula

  $$C_S = C_T \left(1 + r\right)$$

  defines the simple rate r on the basis of the interval $[T, S]$;
- **Spot o forward.** r is a spot rate if it is evaluated at T, namely at the beginning of the reference interval; r is instead a forward rate if it is evaluated at a time instant $t < T$ prior to the reference interval.

[1] We shall deal with the modeling of interest rate markets starting from the next Section 4.2, where we shall analyze various approaches to assign a stochastic dynamics to interest rates and to corresponding assets.

Since the interest rates are not directly traded on the financial markets, it is appropriate to first analyze the simplest related assets called T**-bonds**. A T-bond is a contract that guarantees to the holder the payment of a unit monetary amount (for example 1 Euro) at time T, called maturity. Essentially, the prices of T-bonds express the expectations of the market on the future value of money.

In the following we consider a time interval $[0, \bar{T}]$ divided into a number $\bar{N}$ of sub-intervals of equal length $\Delta = \frac{\bar{T}}{\bar{N}}$: in our discrete time models we assume that the transactions occur only at the dates $t_n = n\Delta$ with $0 \leq n \leq \bar{N}$. For $0 \leq n \leq N \leq \bar{N}$ we denote by $p(n, N)$ the price at time $t_n = n\Delta$ of the T-bond with maturity $T = N\Delta$. A T-bond is sometimes also called **zero coupon bond** with maturity T, because it is a contract that does not imply the payment of coupons prior to maturity.

Notation 4.1. *For convenience, and if it does not lead to ambiguity, in the following we shall refer to the n-th time instant indifferently by t_n or by n. Analogously, the n-th period will be denoted by $[t_{n-1}, t_n]$ or more often simply by $[n-1, n]$.*

Notice that by definition we have $p(N, N) = 1$. Furthermore, at time $n = 0$ (today), the family of prices $p(0, N)$ varying with $N = 1, 2, \dots, \bar{N}$ is known (observable on the market) and constitutes the term structure of the zero coupon bonds, in other words, the family of initial data for the price processes $p(\cdot, N)$ according to the maturity N. In what follows we denote by $p^*(0, N)$, $N \leq \bar{N}$, the term structure that is observable on the market in order to distinguish it from the theoretical prices $p(0, N)$, $N \leq \bar{N}$, generated by a given mathematical model.

Starting from the prices of the T-bonds it is possible to define different types of interest rates. Suppose $n < N < M$ and consider an investment at time n which consists in selling one unit of $p(n, N)$ in order to buy $\frac{p(n,N)}{p(n,M)}$ units of $p(n, M)$, according to the following scheme:

n	N	M
sell a T-bond with $T = N\Delta$ collect $p(n, N)$		
buy T-bonds with $T = M\Delta$ in no. of $\frac{p(n,N)}{p(n,M)}$	pay 1	collect $\frac{p(n,N)}{p(n,M)}$

The investment has zero cost at time n and consists in paying 1 at time N in order to receive $\frac{p(n,N)}{p(n,M)}$ at time M. Define then the following rates coherently with the prices of the zero coupon bonds:

- $L(n; N, M)$ is the **simple annualized forward rate**, for the period $[t_N, t_M]$, evaluated in n, and it is defined by the capitalization formula
$$\frac{p(n,N)}{p(n,M)} = 1 + L(n;N,M)(M-N)\Delta,$$
or equivalently by
$$L(n;N,M) = \frac{1}{(M-N)\Delta}\left(\frac{p(n,N)}{p(n,M)} - 1\right). \tag{4.1}$$
This definition is based on the fact that the investment of 1 Euro in N at the simple rate $L(n;N,M)$ has to yield the same result as the above described investment that involves only T-bonds. We also denote by
$$L(n,N) := L(n;N,N+1) = \frac{1}{\Delta}\left(\frac{p(n,N)}{p(n,N+1)} - 1\right), \tag{4.2}$$
the simple annualized forward rate evaluated in n for the period $[t_N, t_{N+1}]$.

- $R(n; N, M)$ is the **compounded forward rate** for the period $[t_N, t_M]$, evaluated in n, and it is defined by the capitalization formula
$$\frac{p(n,N)}{p(n,M)} = e^{R(n;N,M)}. \tag{4.3}$$
Also here the definition is based on the fact that the investment of 1 Euro in N at the compounded rate $R(n;N,M)$ has to yield the same result as the above described investment that involves only T-bonds. We also denote by
$$R(n,N) = R(n;N,N+1) = \log\frac{p(n,N)}{p(n,N+1)} \tag{4.4}$$
the compounded forward rate evaluated in n for the period $[t_N, t_{N+1}]$.

- $r_n := R(n,n)$ is the **compounded spot rate** (on the basis Δ) relative to the period $[t_n, t_{n+1}]$. We shall simply call r the **short rate**. Note that by definition we have
$$p(n,n+1) = e^{-r_n}. \tag{4.5}$$

Having introduced the short rate, we denote as usual by B the value of the **money market account**, namely of an investment that consists in revaluing in each individual period the initial capital at the short rate: more precisely, the dynamics of B is given by the recursive formula
$$B_{n+1} = B_n e^{r_n},$$
or more generally by
$$B_N = B_n \exp\left(\sum_{k=n}^{N-1} r_k\right), \qquad 0 \le n < N, \tag{4.6}$$
where by convention we suppose that $B_0 = 1$.

We stress the fact that r_n denotes a *random* value that becomes known at time n: in particular, differently from the previous chapters where the interest rate was supposed deterministic if not constant, starting from the next paragraph r (and consequently B) will be described by a stochastic process. It is also said that B is a "locally riskless" asset since an investment in the money market account at time n leads to a return that is certain and without risk in the immediately following period $[t_n, t_{n+1}]$, given that r_n is known at time t_n.

Remark 4.2. *Notice the difference between the quantities $p(n,N)$ and*

$$D(n,N) := \exp\left(-\sum_{k=n}^{N-1} r_k\right), \qquad 0 \le n < N \le \bar{N}, \tag{4.7}$$

usually called **discount factor** *over the period $[t_n, t_N]$: both represent the value at time n of a monetary unit delivered at time N. However $p(n,N)$ is an observable value at time n because it represents the price of a contract that is traded on the market at time n; on the contrary $D(n,N)$ is not known at time n, because it is a random value that depends on the evolution of the rates up to maturity. This observation will be made rigorous in mathematical terms in the next section (in particular see formula* (4.10)*).* □

4.2 Market models for interest rates

There exist various approaches to the stochastic modeling of the interest rates in discrete time. Here we examine two among the principal classes of models, called of the *short* and *forward* type respectively. We start by presenting in general terms some basic ideas.

- **Short models.** In a short model the dynamics of the short rate r is given by means of a suitable stochastic process. The T-bonds $p(\cdot, N)$ are considered *derivatives of the underlying r with maturity N and payoff equal to* 1, on the basis of the condition $p(N,N) = 1$. The idea is to use the classical theory of arbitrage pricing to obtain the prices of the T-bonds from the dynamics of r. Some aspects of this approach are as follows:
  i) the short rate is not an asset that is traded on the market: therefore a model of the short type is generally, also in the most simple cases, an *incomplete* market model, namely the martingale measure is not unique;
  ii) since the initial term structure $p^*(0,N)$, $N = 1, \dots, \bar{N}$, consists of market data that have to be reproduced by the model, it is necessary to determine explicitly conditions on the process r so that the theoretical prices of the T-bonds $p(\cdot, N)$ satisfy the condition

  $$p(0,N) = p^*(0,N), \qquad N \le \bar{N}. \tag{4.8}$$

  Usually one assigns the dynamics of r directly *under a martingale measure* (cf. Definition 4.3) by means of a stochastic process that depends on some parameters. Imposing (4.8), one then tries to obtain the value of these parameters (this procedure is called *model calibration*) and in this way one solves indirectly also the problem of the choice of the martingale measure. An example of calibration of a short rate model is given in Section 4.3.2;

iii) it is generally considered to be non realistic that the dynamics of the price processes $p(\cdot, N)$ for all maturities are "driven" by the only stochastic process given by the short rate.

We describe next the forward-type models, for which some of the above-mentioned problems do not arise.

- **Forward models.** In a forward model one assigns directly the dynamics of the price processes $p(\cdot, N)$ for each $N = 1, \dots, \bar{N}$. In this way the T-bonds are considered as primary assets of a discrete time market model of the type studied in Chapter 1. In this case the initial term structure $p^*(0, N)$, $N = 1, \dots, \bar{N}$, is automatically assumed as initial data of the price processes. On the other hand notice that, assigning the dynamics of all the prices of the T-bonds, it is necessary to verify whether the model is free of arbitrage: in other words, the problem arises to establish conditions that ensure the existence (and possibly uniqueness) of the martingale measure.

Suppose now to have a discrete time market model (of the short or forward type), given on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_n))$, where we assume that r and $p(\cdot, N)$, $N = 1, \dots, \bar{N}$, are *adapted* processes and the price processes $p(\cdot, N)$ are *positive*. Recalling the definition (4.6) of money market account denote by

$$\widetilde{p}(n, N) = \frac{p(n, N)}{B_n}, \qquad 0 \le n \le N,$$

the discounted price processes of the T-bonds.

Definition 4.3. *A martingale measure with numeraire B is a probability measure Q equivalent to P, with respect to which the discounted price processes of the T-bonds are martingales, namely it holds that*

$$\widetilde{p}(n, N) = E^Q \left[\widetilde{p}(n + 1, N) \mid \mathcal{F}_n \right], \qquad 0 \le n < N \le \bar{N}. \tag{4.9}$$

The following lemma translates into mathematical terms the contents of Remark 4.2.

Lemma 4.4. *The martingale condition* (4.9) *is equivalent to*[2]

$$p(n, N) = E^Q \left[D(n, N) \mid \mathcal{F}_n \right], \qquad 0 \le n \le N \le \bar{N}, \tag{4.10}$$

where $D(n, N)$ is the discount factor defined in (4.7).

[2]By convention we put $D(n, n) = 1$.

Proof. From the martingale property (4.9) it follows that

$$\widetilde{p}(n,N) = E^Q \left[\widetilde{p}(N,N) \mid \mathcal{F}_n\right], \qquad n \le N,$$

and, being $p(N,N) = 1$, also that

$$\frac{p(n,N)}{B_n} = E^Q \left[B_N^{-1} \mid \mathcal{F}_n\right], \qquad n \le N.$$

Then, since r is an adapted process, we have

$$p(n,N) = E^Q \left[B_n B_N^{-1} \mid \mathcal{F}_n\right], \qquad n \le N,$$

from which (4.10) follows.

Conversely, from (4.10) and being r adapted, we have

$$\begin{aligned} p(n,N) &= E^Q \left[e^{-r_n} D(n+1,N) \mid \mathcal{F}_n\right] \\ &= E^Q \left[e^{-r_n} E^Q \left[D(n+1,N) \mid \mathcal{F}_{n+1}\right] \mid \mathcal{F}_n\right] \\ &= E^Q \left[e^{-r_n} p(n+1,N) \mid \mathcal{F}_n\right], \end{aligned}$$

and the thesis follows by dividing the terms by B_n. □

The following result contains a characterization of the martingale measure expressed in terms of an important relation among the prices of the T-bonds.

Proposition 4.5. *Let Q be a measure equivalent to P. Then Q is a martingale measure if and only if*

$$\frac{p(n,N)}{p(n,n+1)} = E^Q \left[p(n+1,N) \mid \mathcal{F}_n\right], \qquad 0 \le n < N \le \bar{N}. \tag{4.11}$$

Proof. We use Lemma 4.4 and show that (4.11) is equivalent to (4.10): in fact, since r_n is $\mathcal{F}_n$-measurable, (4.10) is equivalent to

$$p(n,N) = e^{-r_n} E^Q \left[D(n+1,N) \mid \mathcal{F}_n\right] =$$

(by (4.5))

$$\begin{aligned} &= p(n,n+1) E^Q \left[E^Q \left[D(n+1,N) \mid \mathcal{F}_{n+1}\right] \mid \mathcal{F}_n\right] \\ &= p(n,n+1) E^Q \left[p(n+1,N) \mid \mathcal{F}_n\right], \end{aligned}$$

which in turn is equivalent to (4.11). □

In view of the applications to the study of the interest rate derivatives (cf. Section 4.5) we prove also the following:

Corollary 4.6. *Under a martingale measure Q, if a process $X = (X_n)_{n \le N}$ verifies*

$$X_n = E^Q \left[D(n,N) X_N \mid \mathcal{F}_n \right], \qquad n \le N,$$

then $\widetilde{X} = \left(\frac{X_n}{B_n} \right)$ is a Q-martingale. In particular it holds that

$$X_k = E^Q \left[D(k,n) X_n \mid \mathcal{F}_k \right], \qquad k \le n \le N, \tag{4.12}$$

and, in the case $n = k+1$,

$$X_k = p(k,k+1) E^Q \left[X_{k+1} \mid \mathcal{F}_k \right], \qquad k < N. \tag{4.13}$$

Proof. Since the interest rate process r is adapted, for each $k \le n$ one has

$$\begin{aligned} E^Q \left[\widetilde{X}_n \mid \mathcal{F}_k \right] &= \frac{1}{B_k} E^Q \left[E^Q \left[\frac{D(n,N) B_k}{B_n} X_N \mid \mathcal{F}_n \right] \mid \mathcal{F}_k \right] \\ &= \frac{1}{B_k} E^Q \left[D(k,N) X_N \mid \mathcal{F}_k \right] = \widetilde{X}_k. \end{aligned}$$

Equation (4.12) then follows immediately from the martingale property and from the fact that r is adapted; finally, (4.13) follows by combining (4.12) with (4.5). □

4.3 Short models

As mentioned in Section 4.2, in a short model one assigns the dynamics of the interest rate r by means of a suitable discrete time stochastic process. Furthermore, one usually assumes that the dynamics of r is directly given under a martingale measure Q. As mentioned in the introduction to the chapter, in this section we shall base ourselves mainly on [10].

In the following we shall consider the case in which r is a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, Q, (\mathcal{F}_n))$. We denote by Q_n the transition kernel of r_n, seen as a Markov chain under the measure Q: more precisely we put

$$(Q_n \varphi)(r_n) := E^Q \left[\varphi(r_{n+1}) \mid \mathcal{F}_n \right] = \int_{\mathbb{R}} \varphi(\varrho) Q_n (r_n, d\varrho), \tag{4.14}$$

for each integrable real valued function φ on $\mathbb{R}$. Notice that we allow the space state to be $\mathbb{R}$ and thus not necessarily discrete as in the previous sections.

Example 4.7 (Hull-White model in discrete time). Suppose given the following recursive dynamics for the process r:

$$r_{n+1} = r_n + (\Phi_n - a_n r_n) \Delta + \sigma_n \sqrt{\Delta} W_n, \qquad n = 0, \dots, \bar{N} - 1 \tag{4.15}$$

where a_n, σ_n and Φ_n are non-negative parameters, $\Delta = \frac{\bar{T}}{\bar{N}}$ is the length of each sub-interval of $[0, \bar{T}]$ and (W_n) is a sequence of independent random variables with standard normal distribution, $W_n \sim \mathcal{N}_{0,1}$ with respect to a measure Q. Then we have

$$r_{n+1} \sim \mathcal{N}_{r_n + (\Phi_n - a_n r_n)\Delta, \sigma_n^2 \Delta} \tag{4.16}$$

namely

$$\begin{aligned}(Q_n \varphi)(r_n) &= \int_{\mathbb{R}} \varphi(\varrho) \mathcal{N}_{r_n + (\Phi_n - a_n r_n)\Delta, \sigma_n^2 \Delta}(d\varrho) \\ &= \frac{1}{\sigma\sqrt{2\pi\Delta}} \int_{\mathbb{R}} \varphi(\varrho) \exp\left(-\frac{(\varrho - r_n - (\Phi_n - a_n r_n)\Delta)^2}{2\sigma_n^2 \Delta}\right) d\varrho.\end{aligned}$$

Notice that the recursive formula (4.15) corresponds to an Euler-Maruyama discretization of the stochastic differential equation proposed by Hull-White in [14]:

$$dr_t = (\Phi(t) - a(t) r_t)\, dt + \sigma(t) dW_t, \tag{4.17}$$

where $\sigma = \sigma(t)$ is the (deterministic) volatility function, $a = a(t)$ is the speed (or rate) of *mean reversion*, Φ is a function that governs the long rate mean and W is a real Brownian motion.

We stress the fact that the process r in (4.15) may take arbitrarily large negative values. □

From the dynamics (4.14), we see now how to obtain the prices of the T-bonds. To this effect we use Lemma 4.4 which gives the expression for the prices $p(n, N)$ in terms of a conditional expectation under Q of the short rates r_k with $n \le k < N$. Before stating the next result, we introduce some notations that we shall use systematically in the sequel. Put

$$\varphi_0(r) := e^{-r}$$

and introduce the family of functions $\left(\varphi_n^N\right)$, with $0 \le n < N \le \bar{N}$, by means of the recursive definition:

$$\begin{cases} \varphi_{N-1}^N(r) = 1, \\ \varphi_{n-1}^N(r) = Q_{n-1}\left(\varphi_0 \varphi_n^N\right)(r), \qquad 1 \le n \le N-1, \end{cases} \tag{4.18}$$

for each $N \le \bar{N}$ and $r \in \mathbb{R}$. One then has the following result.

Proposition 4.8. *Suppose* (4.14) *holds under the martingale measure* Q*, then*

$$p(n, N) = e^{-r_n} \varphi_n^N(r_n), \qquad 0 \le n < N \le \bar{N}, \tag{4.19}$$

with φ_n^N *defined in* (4.18).

Proof. Given N, we prove the statement by backwards induction over n. In the case $n = N-1$ one has $p(N-1, N) = e^{-r_{N-1}}$ and so the statement follows from the fact that, by definition, $\varphi^N_{N-1} \equiv 1$.

Assume now that, by the induction hypothesis, (4.19) holds: since Q is supposed to be a martingale measure, by Proposition 4.5 we have

$$p(n-1, N) = p(n-1, n)E^Q\left[p(n, N) \mid \mathcal{F}_{n-1}\right] =$$

(by the induction hypothesis)

$$\begin{aligned} &= e^{-r_{n-1}} E^Q\left[e^{-r_n}\varphi^N_n(r_n) \mid \mathcal{F}_{n-1}\right] \\ &= e^{-r_{n-1}} Q_{n-1}\left(\varphi_0 \varphi^N_n\right)(r_{n-1}) = e^{-r_{n-1}}\varphi^N_{n-1}(r_{n-1}). \end{aligned}$$ □

4.3.1 Affine models

In particular cases (among them, as we shall see shortly, also that of Example 4.7) it is possible to make the expression for the prices of the T-bonds more explicit. For this purpose we introduce the following *moment generating function*:

$$\widetilde{Q}_n(r, \lambda) := \int_{\mathbb{R}} e^{-\lambda\varrho} Q_n(r, d\varrho), \qquad r, \lambda \in \mathbb{R}. \tag{4.20}$$

Since the exponential function is positive, the integral in (4.20) is well defined (possibly equal to $+\infty$) for each $r, \lambda \in \mathbb{R}$.

Proposition 4.9. *If there exist functions f_n, g_n such that*

$$\widetilde{Q}_n(r, \lambda) = \exp\left(-f_n(\lambda) - g_n(\lambda)r\right), \qquad r, \lambda \in \mathbb{R}, \tag{4.21}$$

for each n, $0 \le n < \bar{N}$, then the functions φ^N_n in (4.18) have the following expression

$$\varphi^N_n(r) = \exp\left(-A^N_n - B^N_n r\right), \qquad r \in \mathbb{R}, \tag{4.22}$$

with the constants A^N_n, B^N_n defined by the recursions

$$\begin{cases} A^N_{N-1} = B^N_{N-1} = 0, \\ A^N_{n-1} = A^N_n + f_{n-1}\left(1 + B^N_n\right), \\ B^N_{n-1} = g_{n-1}\left(1 + B^N_n\right). \end{cases} \tag{4.23}$$

In particular the following formula holds for the prices of the T-bonds:

$$p(n, N) = \exp\left(-A^N_n - \left(1 + B^N_n\right) r_n\right), \qquad 0 \le n < N \le \bar{N}. \tag{4.24}$$

Definition 4.10. *We say that a short model, for which (4.24) holds, is an affine term structure model*[3]*.*

[3] In general, in an affine model the price of the T-bonds is the exponential of a linear (affine) function of the rate r.

Proof (of Proposition 4.9). Given N, suppose that (4.21) holds; we prove the statement by backwards induction over n. In the case $n = N-1$ one has $\varphi_{N-1}^N(r) = 1$ which is coherent with (4.22)-(4.23) since by definition $A_{N-1}^N = B_{N-1}^N = 0$.

Supposing now that the statement holds for n, we prove it for $n-1$:

$$\varphi_{n-1}^N(r) = Q_{n-1}\left(\varphi_0\varphi_n^N\right)(r) = \int_{\mathbb{R}} e^{-\varrho}\varphi_n^N(\varrho)Q_{n-1}(r, d\varrho) =$$

(by the induction hypothesis)

$$\begin{aligned} &= \int_{\mathbb{R}} \exp\left(-A_n^N - \varrho\left(1+B_n^N\right)\right) Q_{n-1}(r, d\varrho) \\ &= e^{-A_n^N}\widetilde{Q}_{n-1}\left(r, 1+B_n^N\right) = \end{aligned}$$

(by the assumption (4.21))

$$= \exp\Big(- A_n^N - f_{n-1}\left(1+B_n^N\right) - g_{n-1}\left(1+B_n^N\right) r\Big).$$

This proves the statement as well as the recursive formulae (4.23). Finally, formula (4.24) follows directly from Proposition 4.8. □

Corollary 4.11. *Under the assumptions of Proposition 4.9 one also has*

$$\begin{aligned} L(n,N) &= \frac{1}{\Delta}\Big(\exp\left(\left(A_n^{N+1} - A_n^N\right) + \left(B_n^{N+1} - B_n^N\right) r_n\right) - 1\Big), \\ R(n,N) &= A_n^{N+1} - A_n^N + \left(B_n^{N+1} - B_n^N\right) r_n. \end{aligned}$$

Proof. It suffices to combine the expression (4.24) for $p(n,N)$ with the formulae (4.2) and (4.4) for the forward rates. □

4.3.2 Discrete time Hull-White model

In the section we consider again the model of Example 4.7 and show that it is an affine model. For simplicity we consider only the case where volatility and mean reversion speed are constant, namely $a_n \equiv a$ and $\sigma_n \equiv \sigma$. We study furthermore the problem of calibrating the model to the initial term structure.

First we compute the moment generating function of the normal distribution $\mathcal{N}_{\mu,\sigma^2}$.

Lemma 4.12. *We have*

$$\int_{\mathbb{R}} e^{-\lambda\varrho}\mathcal{N}_{\mu,\sigma^2}(d\varrho) = e^{-\lambda\mu+\frac{\lambda^2\sigma^2}{2}}. \tag{4.25}$$

Proof. A simple calculation shows that

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-\lambda\varrho}\mathcal{N}_{\mu,\sigma^2}(d\varrho)\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\lambda\varrho}\exp\left(-\frac{(\varrho-\mu)^2}{2\sigma^2}\right)d\varrho\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{2\sigma^2\lambda\varrho+\varrho^2-2\varrho\mu+\mu^2}{2\sigma^2}\right)d\varrho\\
&= \exp\left(\frac{-\mu^2+\left(\lambda\sigma^2-\mu\right)^2}{2\sigma^2}\right)\int_{\mathbb{R}}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\left(\varrho+\lambda\sigma^2-\mu\right)^2}{2\sigma^2}\right)d\varrho\\
&= \exp\left(-\lambda\mu+\frac{\lambda^2\sigma^2}{2}\right).
\end{aligned}$$

□

Returning to the discrete time Hull-White model assume that the short rate has the following recursive dynamics:

$$r_{n+1} = r_n + (\Phi_n - a r_n)\,\Delta + \sigma\sqrt{\Delta}W_n, \qquad n = 0,\dots,\bar{N}-1. \tag{4.26}$$

Then

$$\widetilde{Q}_n\left(r_n,\lambda\right) = \int_{\mathbb{R}} e^{-\lambda\varrho}\mathcal{N}_{r_n+(\Phi_n-ar_n)\Delta,\sigma^2\Delta}(d\varrho) =$$

(by Lemma 4.12)

$$\begin{aligned}
&= \exp\left(-\lambda\left(r_n+(\Phi_n-ar_n)\,\Delta\right)+\frac{\lambda^2\sigma^2\Delta}{2}\right)\\
&= \exp\left(-f_n(\lambda)-g_n(\lambda)r_n\right)
\end{aligned}$$

with

$$f_n(\lambda) = \Phi_n\Delta\lambda - \frac{\sigma^2\Delta}{2}\lambda^2 \quad\text{and}\quad g_n(\lambda) = \lambda\left(1-a\Delta\right).$$

Therefore, by Proposition 4.9, the discrete time Hull-White model is a model possessing an affine term structure and so we have the following expression for the prices of the T-bonds:

$$p(n,N) = e^{-A_n^N-\left(1+B_n^N\right)r_n}, \qquad 0\le n<N\le\bar{N}, \tag{4.27}$$

where

$$\begin{cases}
A_{N-1}^N = B_{N-1}^N = 0,\\
A_{n-1}^N = A_n^N + \Phi_{n-1}\Delta\left(1+B_n^N\right) - \frac{\sigma^2\Delta}{2}\left(1+B_n^N\right)^2,\\
B_{n-1}^N = \left(1+B_n^N\right)\left(1-a\Delta\right),
\end{cases} \tag{4.28}$$

or, more explicitly, for $n \leq N-2$,

$$\begin{cases} A_n^N = \sum_{k=n}^{N-2} \left(\Phi_k \Delta \left(1 + B_{k+1}^N\right) - \frac{\sigma^2 \Delta}{2} \left(1 + B_{k+1}^N\right)^2 \right), \\ B_n^N = (1 - a\Delta)^{N-n-2} \left(N - n - 1 - a\Delta\right). \end{cases} \tag{4.29}$$

Corollary 4.11 gives also the expressions for the forward rates $L(n, N)$ and $R(n, N)$ as functions of the constants A_n^N and B_n^N.

Remark 4.13. *In the case $a = 0$, combining the previous formulae with*

$$\begin{aligned} A_n^N - A_{n+1}^N &= (N - n - 1)\Phi_n \Delta - \frac{(N-n-1)^2}{2} \sigma^2 \Delta, \\ (1 + B_n^N) r_n - (1 + B_{n+1}^N) r_{n+1} &= (N - n) r_n \\ &\quad - (N - n - 1) \left(r_n + \Phi_n \Delta + \sigma \Delta W_n\right), \\ r_n &= -\frac{\log p(n, N) + A_n^N}{N - n}, \end{aligned}$$

we obtain the following recursive relation for the prices of the T-bonds:

$$\begin{aligned} p(n+1, N) = p(n, N) \exp \Bigg(&- \frac{\log p(n, N) + A_n^N}{N - n} \\ &- \frac{\sigma^2 \Delta}{2} (N - n - 1)^2 - (N - n - 1) \sigma \Delta W_n \Bigg). \end{aligned} \tag{4.30}$$

□

We show now that it is possible to calibrate the discrete time Hull-White model to the initial term structure $p^*(0, N)$, $N \leq \bar{N}$ that is observed on the market. For simplicity we consider only the case $a = 0$.

In this model, for each choice of the volatility parameter σ, we are able to reproduce the initial term structure by choosing appropriately the parameters Φ_n. In fact, the following proposition holds:

Proposition 4.14. *In the discrete time Hull-White model with dynamics*

$$r_{n+1} = r_n + \Phi_n \Delta + \sigma \sqrt{\Delta} W_n,$$

one has

$$\Phi_n = \frac{R(0, n+1) - R(0, n)}{\Delta} + \sigma^2 \left(n + \frac{1}{2}\right). \tag{4.31}$$

Remark 4.15. *On the basis of Proposition 4.14, for each choice of σ, the calibration of the model consists simply in putting*

$$\Phi_n = \frac{R^*(0, n+1) - R^*(0, n)}{\Delta} + \sigma^2 \left(n + \frac{1}{2}\right),$$

where R^* is the market forward rate, commonly defined as

$$R^*(0,n) = \log \frac{p^*(0,n)}{p^*(0,n+1)}.$$

Notice the analogy of the calibration formula (4.31) *with the one of the corresponding continuous time model (see e.g. Section 22.4.2 in [3] which concerns the calibration of the Ho-Lee model which is obtained from that of Hull-White by putting $a = 0$).* □

Proof (of Proposition 4.14). Notice above all that by (4.29) with $a = 0$ one has $B_n^N = N - n - 1$ and

$$A_0^n = \sum_{k=0}^{n-2} \left(\Phi_k \Delta (N-k-1) - \frac{\sigma^2 \Delta}{2} (N-k-1)^2 \right),$$

from which

$$A_0^{n+1} - A_0^n = \Delta \sum_{k=0}^{n-1} \Phi_k - \frac{\sigma^2 \Delta}{2} n^2 \quad \text{and} \quad B_0^{n+1} - B_0^n = 1. \tag{4.32}$$

Then by the forward rate formula of Corollary 4.11 and by (4.32), one has

$$R(0,n+1) = \Delta \sum_{k=0}^{n} \Phi_k - \frac{\sigma^2 \Delta}{2} (n+1)^2 + r_0,$$

$$R(0,n) = \Delta \sum_{k=0}^{n-1} \Phi_k - \frac{\sigma^2 \Delta}{2} n^2 + r_0.$$

Subtracting the second equation from the first one leads to

$$\Phi_n \Delta = R(0,n+1) - R(0,n) + \sigma^2 \Delta n + \frac{\sigma^2 \Delta}{2},$$

from which the statement follows. □

Remark 4.16. *There exists a further type of distribution which, like the normal distribution in Lemma 4.12, has a moment generating function that allows to obtain an affine model: it is the non central chi-square distribution which is the distribution of the square of one or more normal random variables. The interest in this type of distribution originates from the need to have an affine model in which the rate r is positive: recall in fact that in the Hull-White model r may take arbitrarily large negative values.*

A famous example of a continuous time affine model in which $r_t \geq 0$, is the one proposed by Cox, Ingersoll and Ross [5] in which the dynamics of the short rate is described by the following stochastic differential equation

$$dr_t = (\Phi - a r_t) dt + \sigma \sqrt{r_t} dW_t, \tag{4.33}$$

where W is a Brownian motion. In (4.33) *a drift term is present with mean reversion as in the dynamics* (4.17) *of the Hull-White model but, contrary to the latter one, the solution r_t of* (4.33) *is non-negative due to the square root in the diffusion term.*

It is well known (see for example [16], Section 6.2.2) that the moment generating function of r_t in (4.33) coincides with that of a suitable non central chi-square distribution. Notice however that it is not possible to obtain a discrete time affine model with dynamics equivalent to (4.33) *simply by discretizing the equation with an Euler scheme as in the Hull-White model: for further details concerning the existence of discrete time CIR-type models we refer to [12].*

We recall that the non central chi-square distribution (with one degree of freedom) is the distribution of X^2 where X is a random variable with normal distribution, $X \sim \mathcal{N}_{\mu,\sigma^2}$. The corresponding moment generating function has the following expression (see, for example, [1]):

$$E\left[e^{-\lambda X^2}\right] = \frac{e^{-\frac{\mu^2\lambda}{1+2\sigma^2\lambda}}}{1+2\sigma^2\lambda}, \qquad \lambda > -\frac{1}{2\sigma^2}. \tag{4.34}$$

Consequently, the simplest way to modify (4.15) *to obtain a non-negative process r that preserves the affine structure, appears to be that of considering the dynamics*

$$r_{n+1} = X_{n+1}^2 \quad \textit{where} \quad X_{n+1} \sim \mathcal{N}_{\sqrt{r_n+(\Phi_n - a_n r_n)\Delta},\sigma_n\sqrt{\Delta}}$$

by analogy to (4.16).

In this case, by (4.34) *one has*

$$\widetilde{Q}_n\left(r_n,\lambda\right) = \frac{e^{-\frac{\lambda(r_n+(\Phi_n - a r_n)\Delta)}{1+2\sigma_n\sqrt{\Delta}\lambda}}}{1+2\sigma_n\sqrt{\Delta}\lambda}, \qquad \lambda > -\frac{1}{2\sigma_n\sqrt{\Delta}},$$

which shows that the model is affine. □

4.4 Forward models

In this section we illustrate the "forward" approach which, as already mentioned, consists in assigning directly the price processes $p(\cdot, N)$, $N = 1, \dots, \bar{N}$.

Examining formula (4.11) it seems natural to introduce a recursive dynamics of the following type

$$p(n,N) = \frac{p(n-1,N)}{p(n-1,n)}\,\mu_{n,N}, \qquad 1 \le n \le N \le \bar{N}, \tag{4.35}$$

where $\mu_{n,N}$ are random variables defined on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_n))$ and we suppose that, for each $N \le \bar{N}$, the processes $n \mapsto \mu_{n,N}$

are adapted. Notice that for $n = N$, from (4.35) follows that necessarily $\mu_{n,n} = 1$ for each n.

Equation (4.35) provides a natural and convenient relation in the sense that it allows to simplify the martingale condition (4.11) which becomes (4.36). Since this latter formula implies that the discounted prices of the T-bonds are martingales in the probability measure Q, it is the basis for the study of the existence and uniqueness of the martingale measure (or of the properties of absence of arbitrage opportunity and completeness of the model).

Lemma 4.17. *Let Q be a measure equivalent to P. Then Q is a martingale measure if and only if*

$$E^Q\left[\mu_{n,N} \mid \mathcal{F}_{n-1}\right] = 1, \qquad 1 \le n \le N \le \bar{N}. \tag{4.36}$$

Proof. Equation (4.36) can be obtained by simply combining (4.35) with (4.11). □

To illustrate how Lemma 4.17 can be used, in Sections 4.4.1 and 4.4.2 we analyze the problem of the absence of arbitrage opportunities (or of the existence of a martingale measure) in the two significant cases of the forward binomial and multinomial models.

Equation (4.35) defines the processes of the T-bonds in a recursive way. The following result gives the explicit expression for the price processes in terms of the initial data.

Proposition 4.18. *The following formula for the price processes of the T-bonds holds*

$$p(n, N) = \frac{p(0, N)}{p(0, n)} \prod_{k=1}^{n} \frac{\mu_{k,N}}{\mu_{k,n}}, \tag{4.37}$$

where $1 \le n \le N \le \bar{N}$.

Notice that (4.37) expresses the price of a T-bond in terms of the initial prices $p(0, \cdot)$ (observable on the market) and of the stochastic factors $\mu_{n,N}$ of the considered model. Therefore, as already pointed out, a convenient characteristic of forward models is the fact that the initial term structure $p^*(0, N)$, $N = 1, \dots, \bar{N}$, is automatically assumed as initial data of the price process.

Proof. Equation (4.37) can be proved by either applying repeatedly (4.35) or by induction: in fact

$$p(n, N) = p(n-1, N) \frac{1}{p(n-1, n)} \mu_{n,N} =$$

(substituting the expressions of $p(n-1,N)$ and $p(n-1,n)$ obtained from (4.35))

$$= \left(\frac{p(n-2,N)}{p(n-2,n-1)}\mu_{n-1,N}\right)\frac{p(n-2,n-1)}{p(n-2,n)\mu_{n-1,n}}\,\mu_{n,N}$$

$$= p(n-2,N)\frac{1}{p(n-2,n)}\frac{\mu_{n-1,N}\mu_{n,N}}{\mu_{n-1,n}\mu_{n,n}} = \cdots = \frac{p(0,N)}{p(0,n)}\prod_{k=1}^{n}\frac{\mu_{k,N}}{\mu_{k,n}}. \qquad \square$$

Combining the recursive formula (4.35) for the prices of the T-bonds with the definitions (4.2), (4.4) and (4.5) of the rates, we obtain directly the following recursive formulae for the processes of the rates.

Corollary 4.19. *One has*

$$L(n,N) = \left(L(n-1,N) + \frac{1}{\Delta}\right)\frac{\mu_{n,N}}{\mu_{n,N+1}} - \frac{1}{\Delta}, \tag{4.38}$$

$$R(n,N) = R(n-1,N) + \log\frac{\mu_{n,N}}{\mu_{n,N+1}}, \tag{4.39}$$

$$r_n = -r_{n-1} - \log p(n-1,n+1) - \log\mu_{n,n+1}. \tag{4.40}$$

Analogously, by combining the expression (4.37) for the prices of the T-bonds with the definitions (4.2), (4.4) and (4.5) of the rates, one obtains directly the explicit formulae in the next:

Corollary 4.20. *The following holds*

$$L(n,N) = \frac{1}{\Delta}\left(\frac{p(0,N)}{p(0,N+1)}\prod_{k=1}^{n}\frac{\mu_{k,N}}{\mu_{k,N+1}} - 1\right), \tag{4.41}$$

$$R(n,N) = \log\frac{p(0,N)}{p(0,N+1)} + \sum_{k=1}^{n}\log\frac{\mu_{k,N}}{\mu_{k,N+1}}, \tag{4.42}$$

$$r_n = \log\frac{p(0,n)}{p(0,n+1)} + \sum_{k=1}^{n}\log\frac{\mu_{k,n}}{\mu_{k,n+1}}. \tag{4.43}$$

4.4.1 Binomial forward model

Consider a forward model of the type (4.35) in which $\mu_{n,N}$ are random variables that may take two values

$$\mu_{n,N} = \begin{cases} u_{n,N} \\ d_{n,N} \end{cases} \tag{4.44}$$

where $0 < d_{n,N} < u_{n,N}$, with $1 \leq n < N \leq \bar{N}$, are suitably chosen parameters. Proceeding as in the proof of Theorem 1.26 (relative to the standard binomial model) one can prove that, if the following condition holds

$$d_{n,N} < 1 < u_{n,N}, \qquad 1 \leq n < N \leq \bar{N}, \tag{4.45}$$

then there exists a martingale measure Q. In fact Q is a martingale measure if and only if (4.36) holds, where this latter condition becomes, under the assumption (4.44),

$$\begin{aligned} 1 &= E^Q \left[\mu_{n,N} \mid \mathcal{F}_{n-1}\right] \\ &= u_{n,N} Q\left(\mu_{n,N} = u_{n,N} \mid \mathcal{F}_{n-1}\right) + d_{n,N}\left(1 - Q\left(\mu_{n,N} = u_{n,N} \mid \mathcal{F}_{n-1}\right)\right), \end{aligned}$$

from which

$$q_{n,N} := Q(\mu_{n,N} = u_{n,N}) = 1 - Q(\mu_{n,N} = d_{n,N}) = \frac{1 - d_{n,N}}{u_{n,N} - d_{n,N}}. \tag{4.46}$$

It follows in particular that, since the conditional probability $q_{n,N}$ is constant, $\mu_{n,N}$ and $\mathcal{F}_{n-1}$ are independent under Q: notice however that, contrary to the binomial case of Section 1.4.1, *this fact is not sufficient to uniquely identify*[4] Q since, given n, we cannot prove the independence under Q of the random variables $\mu_{n,N}$ for a changing maturity N. Thus, under the condition (4.45) *the binomial forward model is free of arbitrage, but incomplete.*

As an example, consider the binomial forward model with $\bar{N} = 3$ and assume the dynamics (4.35) with $u_{n,N} \equiv u = 2$ and $d_{n,N} \equiv d = \frac{1}{2}$ for $1 \leq n < N \leq 3$. In Figure 4.1 we represent the prices of the T-bonds.

In this case the elementary events are represented by triples of the form

$$\omega = (\mu_{1,2}, \mu_{1,3}, \mu_{2,3}) \quad \text{with} \quad \mu_{1,2}, \mu_{1,3}, \mu_{2,3} \in \{u, d\},$$

and so the probability space contains 8 elements:

$$\begin{array}{llll} \omega_1 = (u,u,u), & \omega_2 = (u,u,d), & \omega_3 = (u,d,u), & \omega_4 = (u,d,d), \\ \omega_5 = (d,u,u), & \omega_6 = (d,u,d), & \omega_7 = (d,d,u), & \omega_8 = (d,d,d). \end{array}$$

	$p(0,N)$	$p(1,N)$	$p(2,N)$	$p(3,N)$
$N=1$	$p(0,1)$	1		
$N=2$	$p(0,2)$	$p(1,2) = \frac{p(0,2)}{p(0,1)}\mu_{1,2}$	1	
$N=3$	$p(0,3)$	$p(1,3) = \frac{p(0,3)}{p(0,1)}\mu_{1,3}$	$p(2,3) = \frac{p(0,3)}{p(0,2)}\frac{\mu_{1,3}\mu_{2,3}}{\mu_{1,2}}$	1

Fig. 4.1. Prices of the T-bonds in the binomial forward model with $\bar{N} = 3$

[4]In the binomial model of Section 1.4.1 the martingale measure Q is uniquely identified as a product measure thanks to the independence property under Q of the sequence of random variables μ_n.

We use the notation $q_k = Q(\{\omega_k\})$ for $k = 1, \dots, 8$. Imposing the martingale condition (4.36) and taking into account (4.46), we obtain

$$Q(\{\mu_{1,2} = u\}) = Q(\{\mu_{1,3} = u\}) = Q(\{\mu_{2,3} = u\}) = \frac{1}{3}. \tag{4.47}$$

If we had the independence of the random variables $\mu_{1,2}, \mu_{1,3}, \mu_{2,3}$, the preceding conditions (4.47) would allow to uniquely identify q_k, $k = 1, \dots, 8$. In our case we can however only state that the conditions (4.47) are equivalent to the system

$$\begin{cases} q_1 + q_2 + q_3 + q_4 = \frac{1}{3}, \\ q_1 + q_2 + q_5 + q_6 = \frac{1}{3}, \\ q_1 + q_3 + q_5 + q_7 = \frac{1}{3}. \end{cases}$$

From the condition of the independence of $\mu_{2,3}$ from $\mu_{1,2}$ and $\mu_{1,3}$ (which follows from the fact that $q_{n,N}$ in (4.46) is constant with respect to $\mathcal{F}_{n-1}$ and in the present case equal to $\frac{1}{3}$) we furthermore have that

$$q_1 = Q(\{\mu_{1,2} = \mu_{1,3} = u\})Q(\{\mu_{2,3} = u\}) = \frac{1}{3}(q_1 + q_2)$$

from which $q_2 = 2q_1$. Analogously one also finds that

$$q_4 = 2q_3, \qquad q_6 = 2q_5, \qquad q_8 = 2q_7.$$

In summary we obtain a system of 7 linear equations in the unknowns q_k which has infinitely many solutions of the form

$$\begin{aligned} &q_1 = q - \frac{1}{9}, \qquad q_3 = \frac{2}{9} - q = q_5, \qquad q_7 = q, \\ &q_2 = 2q_1, \qquad q_4 = 2q_3, \qquad q_6 = 2q_5, \qquad q_8 = 2q_7, \end{aligned}$$

with $q \in]0, 1[$ and it leads to a generic martingale measure, parametrized by q. It is easy to verify that, corresponding to the value $q = \frac{4}{27}$, we obtain the unique martingale measure with respect to which $\mu_{1,2}, \mu_{1,3}, \mu_{2,3}$ are independent random variables.

4.4.2 Multinomial forward model

Consider a forward model of the type (4.35) in which $\mu_{n,N}$ are of the form

$$\mu_{n,N} = \varphi_{n,N}(\xi_n), \qquad 1 \le n < N \le \bar{N}, \tag{4.48}$$

where ξ_n are random variables that take the values $1, 2, \dots, H_n$ with $H_n \in \mathbb{N}$ given and $\varphi_{n,N}$ are suitably chosen functions.

We want to point out explicitly that in this model there exists *a unique process* (ξ_n) that drives the price movements of the T-bonds *for all maturities* $n < N \le \bar{N}$: from this point of view the model appears to possess the same flexibility as the short models. Notice also that the binomial model of Section 4.4.1 is not necessarily of the form (4.48); thus (4.48) does not lead to the most general expression of a multinomial model and so one could call it more appropriately a multinomial "single factor" forward model.

To study the existence of a martingale measure we use the relation (4.36): given n, we obtain

$$1 = E^Q \left[\mu_{n,N} \mid \mathcal{F}_{n-1} \right] = \sum_{h=1}^{H_n} \varphi_{n,N}(h) Q \left(\xi_n = h \mid \mathcal{F}_{n-1} \right) \tag{4.49}$$

which has to hold for each N such that $n < N \leq \bar{N}$. Putting for simplicity $q_h^{(n)} = Q\left(\xi_n = h \mid \mathcal{F}_{n-1}\right)$, equation (4.49) together with the condition

$$\sum_{h=1}^{H_n} q_h^{(n)} = 1, \tag{4.50}$$

leads to a linear system[5] of $\bar{N} - n + 1$ equations in the unknowns $q_1^{(n)}, \dots, q_{H_n}^{(n)}$. In the case when the equations in (4.49)-(4.50) are linearly independent, in order that the system has a solution it is necessary that

$$H_n \geq \bar{N} - n + 1.$$

If there exists a solution $q_1^{(n)}, \dots, q_{H_n}^{(n)}$ such that $q_h^{(n)} > 0$ for each h, then the market is *free of arbitrage but could be incomplete*.

In the particular case in which[6]

$$H_n = \bar{N} - n + 1,$$

one has that, under suitable assumptions (analogous to (4.45) of the binomial case), the linear system (4.49)-(4.50) admits as unique solution a vector $\left(q_1^{(n)}, \dots, q_{H_n}^{(n)}\right)$ of real *positive* numbers that define, as n varies, the unique martingale measure Q with respect to which the random variables $\xi_1, \dots, \xi_{N-1}$ are *independent*. In this case *the model is free of arbitrage and complete.*

The preceding treatment is interesting from the theoretical point of view, but in practice it is not easy to determine explicit conditions on the functions $\varphi_{n,N}$ that assure the solvability of the linear systems (4.49)-(4.50) and at the same time the existence of a martingale measure. For this reason, from an operative point of view it is more convenient to start from assigning a

[5]The coefficients of the system depend on the choice of the functions $\varphi_{n,N}$, namely on the parameters of the model.

[6]The choice of $H_n = \bar{N} - n + 1$ appears as a natural one since at time n one has $\bar{N} - n + 1$ assets that are exactly the $\bar{N} - n$ zero coupon bonds relative to the maturities $N = n + 1, n + 2, \dots, \bar{N}$ and the money market account B.

martingale measure Q and then obtain a posteriori the values of the functions $\varphi_{n,N}$ that assure the validity of the martingale conditions (4.49).

For example, to fix the ideas, we make the particular choice

$$H_n = H \quad \text{and} \quad q_h^{(n)} = q_h, \quad \text{for each } n, \tag{4.51}$$

with $H \in \mathbb{N}$ and $q_h \in]0,1[$, such that $q_1 + \cdots + q_H = 1$, are given parameters. Notice that the choice of $q_h^{(n)} = q_h$ is equivalent to assuming that ξ_n are mutually independent. Notice also that the choice (4.51) is not particularly restrictive and, in any case, the arguments that follow can easily be adapted to the case when H and q_h depend on n. Therefore (4.49) reduces to

$$\sum_{h=1}^{H} q_h \varphi_{n,N}(h) = 1, \tag{4.52}$$

and so, given some *arbitrary* positive constants $c_{n,N}^{(h)}$, with $h = 1, \dots, H$, by putting

$$\varphi_{n,N}(h) = \frac{c_{n,N}^{(h)}}{\sum\limits_{k=1}^{H} q_k c_{n,N}^{(k)}}$$

the martingale condition (4.52) is clearly satisfied.

The practical aspects of this model will be recalled in Section 4.6.1. For this purpose, to conclude the present section, we furthermore point out that the following choice of the constants $c_{n,N}^{(h)}$ is particularly significant because it allows one to obtain a "recombining" tree:

$$c_{n,N}^{(h)} = \delta_h^{N-n} \quad \text{from which} \quad \varphi_{n,N}(h) = \frac{\delta_h^{N-n}}{\sum\limits_{k=1}^{H} q_k \delta_k^{N-n}}, \tag{4.53}$$

where δ_h, $h = 1, \dots, H$, are positive real numbers. For convenience in what follows, given $k \in \mathbb{N}$, we use the notation

$$q \cdot \delta^k = \sum_{h=1}^{H} q_h \delta_h^k. \tag{4.54}$$

We show in just a particular case that the tree is recombining: we prove that the pairs $(\xi_1 = 1, \xi_2 = 2)$ and $(\xi_1 = 2, \xi_2 = 1)$ lead to the same price for the T-bonds. In fact, a simple calculation shows that in both cases, based on (4.37), one has

$$p(2,N) = \frac{p(0,N)}{p(0,2)} \frac{\mu_{1,N}\mu_{2,N}}{\mu_{1,2}\mu_{2,2}} = \frac{p(0,N)}{p(0,2)} \frac{\delta_1^{N-2}\delta_2^{N-2}}{(q \cdot \delta^{N-1})\,(q \cdot \delta^{N-2})}\,(q \cdot \delta)\,.$$

Combining the expression (4.53) with the formulae of Corollary 4.19, we obtain:

Corollary 4.21. *The following recursive formulae hold*[7]

$$L(n,N) = \left(L(n-1,N) + \frac{1}{\Delta} \right) \frac{q \cdot \delta^{N+1-n}}{q \cdot \delta^{N-n}} \sum_{h=1}^{H} \delta_h^{-1} \mathbb{1}_{\{\xi_n = h\}} - \frac{1}{\Delta}, \qquad (4.55)$$

$$R(n,N) = R(n-1,N) + \log \left(\frac{q \cdot \delta^{N+1-n}}{q \cdot \delta^{N-n}} \right) - \sum_{h=1}^{H} (\log \delta_h) \mathbb{1}_{\{\xi_n = h\}}, \qquad (4.56)$$

$$r_n = -r_{n-1} - \log p(n-1, n+1) + \log q \cdot \delta - \sum_{h=1}^{H} (\log \delta_h) \mathbb{1}_{\{\xi_n = h\}}. \qquad (4.57)$$

Furthermore, the following explicit formulae hold

$$L(n,N) = \frac{1}{\Delta} \left(\frac{p(0,N)}{p(0,N+1)} \frac{q \cdot \delta^N}{q \cdot \delta^{N-n}} \prod_{k=1}^{n} \sum_{h=1}^{H} \delta_h \mathbb{1}_{\{\xi_k = h\}} - 1 \right), \qquad (4.58)$$

$$R(n,N) = \log \frac{p(0,N)}{p(0,N+1)} + \log \frac{q \cdot \delta^N}{q \cdot \delta^{N-n}} + \sum_{k=1}^{n} \sum_{h=1}^{H} \mathbb{1}_{\{\xi_k = h\}} \log \delta_h, \qquad (4.59)$$

$$r_n = \log \frac{p(0,n)}{p(0,n+1)} + \log (q \cdot \delta^n) + \sum_{k=1}^{n} \sum_{h=1}^{H} \mathbb{1}_{\{\xi_k = h\}} \log \delta_h. \qquad (4.60)$$

4.5 Interest rate derivatives

4.5.1 Caps and Floors

Consider a sequence of maturities[8] $(N_j)_{j=1,\dots,J}$ with $N_j \in \mathbb{N}$ and

$$1 \le N_0 < N_1 < \cdots < N_J \le \bar{N}.$$

Suppose that an individual or an institution that we shall call "agent" has to pay interest on a given capital (that we shall assume unitary) in each of the intervals $[N_{j-1}, N_j]$. More precisely suppose that, for each interval, payments are made at the end while the interest rate is established at the beginning and, relative to the j-th interval, it corresponds to the rate

$$L_j := L(N_{j-1}; N_{j-1}, N_j)$$

[7] We denote by $\mathbb{1}_A$ the indicator function of the set A.
[8] Recall Notation 4.1.

where $L(n; N, M)$ is the annualized simple rate defined in (4.1). We recall that the rate L_j is supposed to be known at time N_{j-1} and it holds that

$$L_j = \frac{1}{\alpha_j}\left(\frac{1}{p\left(N_{j-1}, N_j\right)} - 1\right) \tag{4.61}$$

where, for convenience, we denote by

$$\alpha_j = \left(N_j - N_{j-1}\right)\Delta, \qquad j = 1, \dots, J,$$

the length of the j-th interval. In summary, the agent has to pay at the maturity N_j, namely at the end of the j-th interval, the amount $\alpha_j L_j$ equal to the interest on the unitary capital for the period $[N_{j-1}, N_j]$.

Let us now fix j. Since at the time points prior to N_{j-1} the rate L_j is unknown (random), to be protected against an increase in this rate the agent signs a contract that locks the payment at a given maximum “cap” rate K (called *Cap*). In different terms, the value at time N_j of such a contract (called *Caplet*) is equal to

$$\alpha_j \left(L_j - K\right)^+.$$

In fact, in the case in which $L_j \geq K$, this value is equal to the difference between the interest that the agent has to pay at the rate L_j and the interest paid at the maximal rate K; if however $L_j < K$, then the Caplet has value zero. In summary, with a Cap the agent is assured to have to pay interest at a rate at most equal to K.

Consider now a market model of interest rates free of arbitrage and under a given martingale measure Q. Coherently with the arbitrage pricing formulas of Chapter 1 we give the following:

Definition 4.22. *Denote by*

$$Caplet_j(n) = E^Q\left[D(n, N_j)\alpha_j \left(L_j - K\right)^+ \mid \mathcal{F}_n\right] \tag{4.62}$$

the price at time $n \leq N_{j-1}$ of the Caplet relative to the j-th interval $[N_{j-1}, N_j]$ and to the measure Q.

We are now going to see how it is possible to express the price of a Caplet in terms of the prices of the T-bonds.

Proposition 4.23. *The price of the j-th Caplet is given by*

$$Caplet_j(N_{j-1}) = (1 + K\alpha_j)\left(\frac{1}{1 + K\alpha_j} - p(N_{j-1}, N_j)\right)^+ \tag{4.63}$$

and, recursively, by

$$Caplet_j(n) = p(n, n+1)E^Q\left[Caplet_j(n+1) \mid \mathcal{F}_n\right] \tag{4.64}$$

for $n < N_{j-1}$.

Proof. We prove equation (4.63): since L_j is $\mathcal{F}_{N_{j-1}}$-measurable, one has

$$\mathit{Caplet}_j(N_{j-1}) = \alpha_j \left(L_j - K\right)^+ p(N_{j-1}, N_j) =$$

(by (4.61))

$$= \left(1 - p(N_{j-1}, N_j)\left(1 + \alpha_j K\right)\right)^+$$

from which the statement follows.

Finally, (4.64) follows directly from the definition (4.62) and from formula (4.13) of Corollary 4.6 applied to $X_N = \alpha_j \left(L_j - K\right)^+$ and with $N = N_j$. □

Remark 4.24. *Using* (4.12) *from Corollary 4.6 we obtain also the more general expression*

$$\mathit{Caplet}_j(n) = \left(1 + K\alpha_j\right) E^Q \left[D\left(n, N_{j-1}\right) \left(\frac{1}{1 + K\alpha_j} - p(N_{j-1}, N_j) \right)^+ \mid \mathcal{F}_n \right]$$

and we observe in particular that the j-th Caplet corresponds to a Put option with underlying a bond and with maturity N_{j-1}. □

Since a Cap corresponds to a sequence of Caplets, one for each maturity N_j, $j = 1, \cdots, J$, the price of a Cap is defined as the sum of the prices of the corresponding Caplets.

Definition 4.25. *Denote by*

$$\begin{aligned} \mathit{Cap}(n) &= \sum_{j=1}^{J} E^Q \left[D(n, N_j)\alpha_j \left(L_j - K\right)^+ \mid \mathcal{F}_n \right] \\ &= \sum_{j=1}^{J} \mathit{Caplet}_j(n), \end{aligned} \tag{4.65}$$

the price of the Cap at time $n \le N_0$, under the measure Q.

Clearly the computation of the price of a Cap can be reduced to the computation of the prices of the individual Caplets that make up the Cap and is usually performed by using the recursive formulae of Proposition 4.23.

On the other hand, the price of a *Floor* contract is defined by

$$\mathit{Floor}(n) = \sum_{j=1}^{J} E^Q \left[D(n, N_j)\alpha_j \left(K - L_j\right)^+ \mid \mathcal{F}_n \right], \qquad n \le N_0. \tag{4.66}$$

On the basis of such a contract the agent (who in this case receives from the counterparty the interests at rate L) receives at each maturity N_j also an amount equal to

$$\alpha_j \left(K - L_j\right)^+.$$

In the case when $L_j < K$, this latter amount is equal to the difference of the interest that the agent receives at rate L_j and the interest at rate K; if however $L_j \geq K$ then this latter amount is zero. In conclusion, with a Floor the agent receives the interests at a rate that is guaranteed of being at least equal to K.

The individual terms in the sum in (4.66) are called *Floorlets* and so the price of a Floor is simply the sum of the prices of the corresponding Floorlets. One can compute the price of a Floorlet by means of a procedure that is analogous to the one for a Caplet. One may however also use the "Floor-Cap Parity" relation, analogous to the "Put-Call Parity", which we shall describe at the end of the next section.

4.5.2 Interest Rate Swaps

Caps and Floors concern a series of exchanges of future payments which take place only if these exchanges turn out to be convenient to the holder of such a contract. One may however consider also exchanges that take place in any case between two series of interest payments, one at a fixed rate and the other one at a floating rate (the value then results from the same expression as for Caps or Floors but without the positive part). Such exchanges are called *Interest Rate Swaps (IRS)* and they may be of two types

- **Payer Forward Swap (*PFS*).** It is a contract that obliges the holder to pay at each maturity N_j, $j = 1, \dots, J$, the interests on a unitary capital at a fixed rate K and in exchange he receives the interests at the floating rate L_j. In conclusion, at each maturity N_j the holder receives (or pays in case the amount is negative)
$$\alpha_j(L_j - K).$$
- **Receiver Forward Swap (*RFS*).** It is a contract analogous to *PFS* in which the holder receives at the fixed rate K and pays at the variable rate L. Thus at each maturity N_j the holder receives (or pays in case the amount is negative)
$$\alpha_j(K - L_j).$$

Consider now a market model of interest rates free of arbitrage and under a given martingale measure Q. According to the arbitrage pricing formulae of Chapter 1, we define the prices *PFS* and *RFS* in the following way.

Definition 4.26. *The price of a Payer Forward Swap at time $n \leq N_0$ and under the measure Q is*

$$PFS(n) = \sum_{j=1}^{J} E^Q \left[D(n, N_j)\alpha_j \left(L_j - K \right) \mid \mathcal{F}_n \right]. \tag{4.67}$$

The price of the corresponding Receiver Forward Swap is

$$RFS(n) = -PFS(n).$$

The pricing of $PFS(n)$ can be reduced to the computation of the individual terms in the sum in (4.67). We thus use the notation

$$PFS_j(n) := E^Q\left[D(n, N_j)\alpha_j\left(L_j - K\right) \mid \mathcal{F}_n\right] \tag{4.68}$$

for each $j = 1, \dots, J$.

The difficulty in the computation of $PFS_j(n)$ consists in the fact that the discount factor $D(n, N_j)$ is stochastic: thus, contrary to the stock market case treated in Chapter 1 and in which we had assumed a *deterministic* short rate, here the factor $D(n, N_j)$ cannot be taken out from the conditional expectation.

This difficulty can be overcome with a technique based on the change of numeraire. More precisely, given j, we shall see shortly that for the computation of $PFS_j(n)$ it is convenient to use $p(\cdot, N_j)$ as numeraire instead of B. We are thus going to perform a change of martingale measure in formula (4.68), from the measure Q relative to the numeraire B to the measure Q^j relative to the numeraire $p(\cdot, N_j)$.

To this end it appears to be useful to recall preliminarily the results on the change of numeraire shown in Section 1.6.2 (in particular we recall the formulae (1.55) and (1.56)), that we state here in the form that is appropriate to the present context.

Lemma 4.27 (Change of numeraire). *In an arbitrage free market model of interest rates, let Q be a martingale measure with numeraire B and $(Y_n)_{n \le N}$ a positive process such that $\widetilde{Y} = \left(\frac{Y_n}{B_n}\right)_{n \le N}$ is a Q-martingale (Y represents the price of a traded asset to be taken as the new numeraire). Then the martingale measure Q^Y with numeraire Y is defined by*

$$E^{Q^Y}\left[X \mid \mathcal{F}_n\right] = E^Q\left[X \frac{Y_N}{Y_n}\left(\frac{B_N}{B_n}\right)^{-1} \mid \mathcal{F}_n\right], \qquad n \le N, \tag{4.69}$$

for each integrable random variable X and we have

$$E^Q\left[D(n, N)X \mid \mathcal{F}_n\right] = Y_n E^{Q^Y}\left[\frac{X}{Y_N} \mid \mathcal{F}_n\right]. \tag{4.70}$$

The advantage of using the measure Q^j (relative to the numeraire $p(\cdot, N_j)$) consists in the fact that, by definition, the prices of the T-bonds, normalized with respect to $p(\cdot, N_j)$, are Q^j-martingales: more precisely, for each $N \le N_j$, the processes

$$n \longmapsto \frac{p(n, N)}{p(n, N_j)}, \qquad n \le N,$$

are Q^j-martingales. Furthermore, since the forward rate is defined as a linear function of $\frac{p(n, N_{j-1})}{p(n, N_j)}$ and more precisely by

$$L(n; N_{j-1}, N_j) = \frac{1}{\alpha_j}\left(\frac{p(n, N_{j-1})}{p(n, N_j)} - 1\right),$$

we have the following result.

Lemma 4.28. *The process $n \mapsto L(n; N_{j-1}, N_j)$ is a Q^j-martingale. Consequently, the following formula holds*

$$\begin{aligned} E^{Q^j}\left[L_j \mid \mathcal{F}_n\right] &= E^{Q^j}\left[L(N_{j-1}; N_{j-1}, N_j) \mid \mathcal{F}_n\right] \\ &= L(n; N_{j-1}, N_j) = \frac{1}{\alpha_j}\left(\frac{p(n, N_{j-1})}{p(n, N_j)} - 1\right), \end{aligned} \tag{4.71}$$

and it expresses the conditional expectation of L_j at time n, in terms of the prices of the T-bonds that are observable at time n.

In the following result we express the price of a *PFS* in terms of prices of T-bonds.

Proposition 4.29. *The price of a Payer Forward Swap at time $n \le N_0$ with payment dates $N_1, \dots, N_J$ and fixed rate K, is equal to*

$$PFS(n) = p(n, N_0) - p(n, N_J) - K \sum_{j=1}^{J} \alpha_j p(n, N_j). \tag{4.72}$$

Proof. For each fixed j, we compute $PFS_j(n)$ in (4.68). Denote by Q^j the martingale measure relative to the numeraire $Y = p(\cdot, N_j)$ defined as in (4.69) of Lemma 4.27. It holds that

$$PFS_j(n) = E^{Q}\left[D(n, N_j)\alpha_j \left(L_j - K\right) \mid \mathcal{F}_n\right] =$$

(by (4.70) and recalling that $p(N_j, N_j) = 1$)

$$= \alpha_j p(n, N_j) E^{Q^j}\left[L_j - K \mid \mathcal{F}_n\right] =$$

(by (4.71))

$$\begin{aligned} &= \alpha_j p(n, N_j)\left(\frac{1}{\alpha_j}\left(\frac{p(n, N_{j-1})}{p(n, N_j)} - 1\right) - K\right) \\ &= p(n, N_{j-1}) - p(n, N_j)\left(1 + \alpha_j K\right). \end{aligned} \tag{4.73}$$

Summing over j from 1 to J the previous expression, we obtain finally (4.72). □

As a simple consequence of the relation

$$L - K = (L - K)^+ - (K - L)^+$$

and of the definitions (4.65) of Cap, (4.66) of Floor and (4.67) of Payer Forward Swap we have:

Proposition 4.30. *The following relation holds between the prices at time $n \le N_0$ of Caps, Floors and PFS with payment dates $N_1, \dots, N_J$ and fixed rate K:*

$$Cap(n) - Floor(n) = PFS(n). \tag{4.74}$$

4.5.3 Swaptions and Swap Rate

A *Swaption* (or Swap option) is a contract that gives the holder the right, but not the obligation, to enter an Interest Rate Swap at a future date N_0. For example, if we consider a Payer Forward Swap with maturities N_j, $j = 1, \dots, J$ and fixed rate K, then a Swaption is an option with maturity N_0 and payoff $(PFS(N_0))^+$. In fact, $PFS(N_0)$ is the price/value of the given Payer Forward Swap at time N_0. If this value is positive, the Swap is favorable for the holder, if negative, it is favorable for the counterparty. The holder will therefore exercise the option only if the value of the Payer Forward Swap in N_0 is positive. Putting for simplicity

$$C(n) = \sum_{j=1}^{J} \alpha_j p(n, N_j), \qquad n \le N_0, \tag{4.75}$$

we recall that by Proposition 4.29 one has

$$PFS(n) = p(n, N_0) - p(n, N_J) - KC(n). \tag{4.76}$$

In an arbitrage free market model of interest rates in which a martingale measure Q is given, we then define the price of a Swaption in the following way.

Definition 4.31. *The price of a Swaption at time $n \le N_0$, relative to the measure Q, is*

$$\begin{aligned} Swaption(n) &= E^Q\left[D(n, N_0)\,(PFS(N_0))^+ \mid \mathcal{F}_n\right] \\ &= E^Q\left[D(n, N_0)\,(1 - p(N_0, N_J) - KC(N_0))^+ \mid \mathcal{F}_n\right]. \end{aligned} \tag{4.77}$$

For practical purposes the price of a Swaption may be computed by means of the following iterative algorithm.

Proposition 4.32. *We have*

$$Swaption(N_0) = (1 - p(N_0, N_J) - KC(N_0))^+ \tag{4.78}$$

and for each $n < N_0$

$$Swaption(n) = p(n, n+1) E^Q\left[Swaption(n+1) \mid \mathcal{F}_n\right]. \tag{4.79}$$

Proof. Equation (4.78) follows from the definition (4.77) of the price for $n = N_0$. Equation (4.79) on the other hand follows from formula (4.13) of Corollary 4.6 applied to $X_N = (1 - p(N_0, N_J) - KC(N_0))^+$ and for $N = N_0$. $\square$

Remark 4.33. *Over a single period (case $J = 1$) the Swaption reduces to a Cap. In fact, by* (4.78) *one has*

$$\begin{aligned} Swaption(N_0) &= (1 - p(N_0, N_1) - K\alpha_1 p(N_0, N_1))^+ \\ &= (1 + K\alpha_1)\left(\frac{1}{1 + K\alpha_1} - p(N_0, N_1)\right)^+ \end{aligned}$$

according to formula (4.63) *for the payoff of a Caplet. Therefore, the iterative formulae* (4.64) *for a Cap and* (4.79) *for a Swaption, lead to the same price.* □

Definition 4.34. *The Swap Rate $swr(n)$ at time $n \le N_0$ and relative to the payment dates $N_0, \dots, N_J$ is the value of the fixed rate K which makes the corresponding $PFS(n)$ a fair contract, namely such that*

$$PFS(n) = 0.$$

Notice that for such a value we also have that $RFS(n) = 0$ so that the definition turns out to be independent of whether one considers a Payer or Receiver Forward Swap.

On the basis of Proposition 4.29 (see also formula (4.76)), one has

$$swr(n) = \frac{p(n, N_0) - p(n, N_J)}{C(n)}, \qquad n \le N_0, \tag{4.80}$$

which gives the value of the Swap Rate as a function of the prices of the T-bonds. Notice furthermore that for the Payer Forward Swap with fixed rate K one has

$$\begin{aligned} PFS(n) &= p(n, N_0) - p(n, N_J) - KC(n) \\ &= \underbrace{p(n, N_0) - p(n, N_J) - swr(n)C(n)}_{=0} + (swr(n) - K)\, C(n) \\ &= (swr(n) - K) \sum_{j=1}^{J} \alpha_j p(n, N_j). \end{aligned}$$

Since $C(n) \ge 0$, we may rewrite the payoff of a Swaption in the following way

$$Swaption(N_0) = C(N_0)\, (swr(N_0) - K)^+ ,$$

emphasizing the fact that the Swaption is being exercised only if $swr(N_0) > K$.

Combining Definition 4.31 with the preceding formula, we obtain the following expression for the price of the Swaption:

$$Swaption(n) = E^Q \left[D(n, N_0) C(N_0)\, (swr(N_0) - K)^+ \mid \mathcal{F}_n \right], \qquad n \le N_0. \tag{4.81}$$

This pricing formula requires the computation of the conditional expected value of the product of three terms that in general are correlated. Consequently, also in this case it turns out to be convenient to adopt the technique of change of numeraire: more precisely, we apply Lemma 4.27 choosing as numeraire $Y_n = C(n)$ in (4.75), to obtain from (4.81) the following expression

$$Swaption(n) = C(n) E^{0,J} \left[(swr(N_0) - K)^+ \mid \mathcal{F}_n \right], \tag{4.82}$$

where $E^{0,J}$ denotes the expectation under the martingale measure $Q^{0,J}$ relative to the numeraire C. This formula allows one to compute *Swaption*(n) as the price of a Call option on the Swap Rate, provided one possesses the dynamics of *swr* in $Q^{0,J}$.

We point out that the discounted price processes with respect to the numeraire C

$$n \longmapsto \frac{p(n, N_j)}{C(n)}, \qquad j = 1, \dots, J$$

are $Q^{0,J}$-martingales: consequently, from formula (4.80) it then follows that also *the process of the Swap Rate is a $Q^{0,J}$-martingale.*

4.6 Solved problems

4.6.1 Recalling the basic models

In view of their practical use in the solution of the problems, we recall here some aspects that we have developed in the theoretical part. We shall do it for each of the two basic models that we have introduced, namely the discrete time Hull-White model of Section 4.3.2 (we shall call it *Model 1*) and for the multinomial forward model of Section 4.4.2 (that we shall refer to as *Model 2*).

Recalling Model 1 (Section 4.3.2)

Consider the discrete time Hull-White model for the short rate namely

$$r_{n+1} = r_n + (\varPhi_n - a r_n)\,\varDelta + \sigma\sqrt{\varDelta} W_n. \tag{4.83}$$

For simplicity in the problems we shall take $\varPhi_n \equiv 1, a = 0, \sigma = 1$ and $\varDelta = 1$. Then, putting $\xi_{n+1} = W_n \sim \mathcal{N}_{0,1}$, from formula (4.30) one has

$$\begin{aligned} p(n+1, N) = p(n, N)^{\frac{N-n-1}{N-n}} \exp\left(-\frac{A_n^N}{N-n} - \frac{\sigma^2}{2}(N-n-1)^2 \right) \cdot \\ \cdot \exp\left(-(N-n-1)\sigma\xi_{n+1}\right). \end{aligned}$$

In particular one has

$$\begin{cases} p(1,2) = p(0,2)^{\frac{1}{2}} \exp\left(-\frac{A_0^2}{2} - \frac{\sigma^2}{2}\right) e^{-\sigma\xi_1} \\ p(1,3) = p(0,3)^{\frac{2}{3}} \exp\left(-\frac{A_0^3}{3} - 2\sigma^2\right) e^{-2\sigma\xi_1} \\ p(2,3) = p(1,3)^{\frac{1}{2}} \exp\left(-\frac{A_1^3}{2} - \frac{\sigma^2}{2}\right) e^{\sigma\xi_2} \\ \qquad\quad = p(0,3)^{\frac{1}{3}} \exp\left(-\frac{A_0^3}{6} - \frac{A_1^3}{2} - \frac{3}{2}\sigma^2\right) e^{-2(\sigma\xi_1+\xi_2)}, \end{cases} \tag{4.84}$$

where, by the formulae (4.29), it holds that

$$\begin{cases} A_0^1 = B_0^1 = 0, & \\ A_0^2 = 1 - \frac{1}{2} = \frac{1}{2}, & B_0^2 = 1, \\ A_0^3 = 2 + 1 - \frac{1}{2}(4+1) = \frac{1}{2}, & B_0^3 = 2, \\ A_1^3 = 1 - \frac{1}{2} = \frac{1}{2}, & \end{cases} \tag{4.85}$$

so that (4.84) becomes

$$\begin{cases} p(1,2) = p(0,2)\frac{1}{2}e^{-\frac{3}{4}}e^{-\xi_1} \\ p(1,3) = p(0,3)\frac{2}{3}e^{-\frac{13}{6}}e^{-2\xi_1} \\ p(2,3) = p(0,3)\frac{1}{3}e^{-\frac{11}{6}}e^{-(\xi_1+\xi_2)}. \end{cases} \tag{4.86}$$

Remark 4.35. *To be coherent with the model, in particular with the chosen values of the model parameters in (4.83) (see the line following (4.83)), the initial term structure $p(0,N)$ cannot be chosen arbitrarily. By* (4.27) *the following relation has in fact to hold*

$$p(0,N) = \exp\left(-A_0^N - (1+B_0^N)r_0\right), \qquad N \le \bar{N},$$

which, on the basis of (4.85), yields

$$p(0,1) = e^{-r_0}, \qquad p(0,2) = e^{-\frac{1}{2}(1+4r_0)}, \qquad p(0,3) = e^{-\frac{1}{2}(1+6r_0)}. \tag{4.87}$$

Choosing then for instance $p(0,1) = \frac{1}{2}e^{\frac{1}{4}}$ this implies $r_0 = \log 2 - \frac{1}{4}$ and consequently we obtain

$$p(0,1) = \frac{1}{2}e^{\frac{1}{4}}, \qquad p(0,2) = \frac{1}{4}, \qquad p(0,3) = \frac{1}{8}e^{\frac{1}{4}}, \tag{4.88}$$

which are the values that we shall use in the problems.

In real applications one has to proceed in the inverse order: starting from the initial term structure as given by the market, one has first to calibrate the model, namely one has to determine the model parameters Φ_n, a, σ so that the theoretical prices, computed on the basis of the model, possibly match the market prices. □

Remark 4.36. *For the problems for Model 1 we shall use the following integration results where Φ denotes the cumulative distribution function of the standard Normal $\mathcal{N}_{0,1}$:*

$$\int_{\{\xi>\gamma\}} e^{-\xi}\mathcal{N}_{0,1}(d\xi) = \frac{1}{\sqrt{2\pi}}\int_{\{\xi>\gamma\}} e^{-\xi}e^{-\frac{\xi^2}{2}}d\xi = \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\{\xi>\gamma\}} e^{-\frac{1}{2}(\xi+1)^2}d\xi \tag{4.89}$$

$$= \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{\{y>\gamma+1\}} e^{-\frac{1}{2}y^2}dy = e^{\frac{1}{2}}\int_{\{y>\gamma+1\}}\mathcal{N}_{0,1}(dy) = e^{\frac{1}{2}}\Phi(-\gamma-1),$$

$$\int_{\{\xi>\gamma\}} e^{\xi} \mathcal{N}_{0,1}(d\xi) = \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{\{\xi>\gamma\}} e^{-\frac{1}{2}(\xi-1)^2} d\xi = e^{\frac{1}{2}} \Phi(-\gamma+1), \tag{4.90}$$

$$\int_{\{\xi>\gamma\}} e^{-2\xi} \mathcal{N}_{0,1}(d\xi) = \frac{e^{2}}{\sqrt{2\pi}} \int_{\{\xi>\gamma\}} e^{-\frac{1}{2}(\xi+2)^2} d\xi = e^{2} \Phi(-\gamma-2). \tag{4.91}$$

$\square$

Recalling Model 2 (Section 4.4.2)

The evolution of the prices of the bonds is given by (see (4.35) and (4.48))

$$p(n,N) = \frac{p(n-1,N)}{p(n-1,n)} \varphi_{n,N}(\xi_n) \tag{4.92}$$

where ξ_n are i.i.d. with values in $\{1,\dots,H\}$ and where, putting $q_h = Q(\xi_n = h)$ and recalling the notation (4.54), one assumes that (see (4.53))

$$\varphi_{n,N}(h) = \frac{\delta_h^{N-n}}{q \cdot \delta^{N-n}}. \tag{4.93}$$

In the problems we shall choose $\delta_h = h$ for $h = 1,\dots,H$, so that

$$\varphi_{n,N}(\xi_n) = \frac{\xi_n^{N-n}}{\sum\limits_{h=1}^{H} q_h h^{N-n}}. \tag{4.94}$$

By formula (4.37) with $\mu_{n,N} = \varphi_{n,N}(\xi_n)$, one then has

$$\begin{aligned} p(n,N) &= \frac{p(0,N)}{p(0,n)} \prod_{k=1}^{n} \frac{\varphi_{k,N}(\xi_k)}{\varphi_{k,n}(\xi_k)} \\ &= \frac{p(0,N)}{p(0,n)} \prod_{k=1}^{n} \xi_k^{N-n} \left(\sum_{h=1}^{H} q_h h^{n-k}\right) \left(\sum_{h=1}^{H} q_h h^{N-k}\right)^{-1}. \end{aligned} \tag{4.95}$$

In most of the problems we put $q_h = \frac{1}{H}$ for $h = 1,\dots,H$ and we shall take $N \leq 3$. Using then the relations

$$\sum_{h=1}^{H} h = \frac{H(H+1)}{2} \qquad \text{and} \qquad \sum_{h=1}^{H} h^2 = \frac{H(H+1)(2H+1)}{6},$$

we obtain in particular

$$
\begin{aligned}
p(1,2) &= \frac{p(0,2)}{p(0,1)}\xi_1 \frac{\sum_{h=1}^{H} q_h}{\sum_{h=1}^{H} q_h h} = \frac{p(0,2)}{p(0,1)} \frac{2\xi_1}{H+1}, \\
p(1,3) &= \frac{p(0,3)}{p(0,1)}\xi_1^2 \frac{\sum_{h=1}^{H} q_h}{\sum_{h=1}^{H} q_h h^2} = \frac{p(0,3)}{p(0,1)} \frac{6\xi_1^2}{(H+1)(2H+1)}, \\
p(2,3) &= \frac{p(0,3)}{p(0,2)}\xi_1\xi_2 \frac{\sum_{h=1}^{H} q_h h}{\sum_{h=1}^{H} q_h h^2} \frac{\sum_{h=1}^{H} q_h}{\sum_{h=1}^{H} q_h h} = \frac{p(0,3)}{p(0,2)} \frac{6\xi_1\xi_2}{(H+1)(2H+1)}.
\end{aligned}
\tag{4.96}
$$

If not specified otherwise, in the sequel we shall assume

$$
p(0,1) = \frac{4}{5}, \qquad p(0,2) = \frac{3}{5}, \qquad p(0,3) = \frac{2}{5}. \tag{4.97}
$$

Remark 4.37. *An alternative version of Model 2 is obtained by choosing*

$$
\varphi_{n,N}(k) = \frac{k^{N-n}}{q_k \sum_{h=1}^{H} h^{N-n}}. \tag{4.98}
$$

Also in this case Q is an equivalent martingale measure since one has (see (4.11) in Proposition 4.5)

$$
\begin{aligned}
p(n,n+1)E^Q[p(n+1,N) \mid \mathcal{F}_n] &= p(n,N)E^Q[\varphi_{n+1,N}(\xi_{n+1})] \\
&= p(n,N)\sum_{k=1}^{H} q_k \frac{k^{N-n}}{q_k \sum_{h=1}^{H} h^{N-n}} = p(n,N).
\end{aligned}
$$

□

4.6.2 Options on T-bonds

Put on T-bonds

In general, a Put with maturity N_0 on a bond with maturity N_1 is priced at time $n = 0$ by means of

$$
P_0 = E^Q\left[\frac{1}{B_{N_0}}(K - p(N_0,N_1))^+\right] = E^Q\left[e^{-\sum_{n=0}^{N_0-1} r_n}(K - p(N_0,N_1))^+\right].
$$

Recalling that (cf. (4.5))

$$r_n = -\log p(n, n+1),$$

one obtains the formula

$$P_0 = E^Q \left[\prod_{n=0}^{N_0-1} p(n, n+1)(K - p(N_0, N_1))^+ \right].$$

Problem 4.38 (Model 1). Determine P_0 in the case of $N_0 = 1, N_1 = 2$ and $K = 1$.

Solution of Problem 4.38
On the basis of (4.86) we have

$$\begin{aligned} P_0 &= E^Q \left[p(0,1)(1 - p(1,2))^+ \right] \\ &= p(0,1) E^Q \left[\left(1 - p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} e^{-\xi_1} \right)^+ \right] \\ &= p(0,1) \int_D (1 - p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} e^{-\xi_1}) \mathcal{N}_{0,1}(d\xi_1), \end{aligned}$$

where

$$D = \{\xi_1 \mid p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} e^{-\xi_1} < 1\} = \{\xi_1 \mid \xi_1 > \gamma\}$$

with $\gamma = \frac{1}{2} \log p(0,2) - \frac{3}{4}$. Therefore, using (4.89), we obtain

$$\begin{aligned} P_0 &= p(0,1) \left(\int_{\{\xi > \gamma\}} \mathcal{N}_{0,1}(d\xi) - p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} e^{\frac{1}{2}} \int_{\{y > \gamma + 1\}} \mathcal{N}_{0,1}(d\xi) \right) \\ &= p(0,1) \left(\Phi(-\gamma) - p(0,2)^{\frac{1}{2}} e^{-\frac{1}{4}} \Phi(-\gamma - 1) \right) \\ &= p(0,1) \left(\Phi\left(-\frac{1}{2} \log p(0,2) + \frac{3}{4} \right) - p(0,2)^{\frac{1}{2}} e^{-\frac{1}{4}} \Phi\left(-\frac{1}{2} \log p(0,2) - \frac{1}{4} \right) \right). \end{aligned}$$

Assuming (cf. (4.88))

$$p(0,1) = \frac{1}{2} e^{\frac{1}{4}} \qquad \text{and} \qquad p(0,2) = \frac{1}{4}$$

so that

$$p(0,2)^{\frac{1}{2}} = \frac{1}{2}, \qquad \log p(0,2) = -2 \log 2,$$

it results that

$$P_0 = \frac{1}{2} e^{\frac{1}{4}} \left(\Phi\left(\log 2 + \frac{3}{4} \right) - \frac{1}{2} e^{-\frac{1}{4}} \Phi\left(\log 2 - \frac{1}{4} \right) \right).$$

□

Problem 4.39 (Model 1). Determine P_0 in the context of Problem 4.38, but with $N_0 = 2, N_1 = 3$.

Solution of Problem 4.39
We have

$$
\begin{aligned}
P_0 &= E^Q\left[p(0,1)p(1,2)(1-p(2,3))^+\right] \\
&= p(0,1)p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}E^Q\left[e^{-\xi_1}\left(1-p(0,3)^{\frac{1}{3}}e^{-\frac{1}{16}}e^{-(\xi_1+\xi_2)}\right)^+\right] \\
&= p(0,1)p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}\frac{1}{2\pi}\iint e^{-x_1}\left(1-p(0,3)^{\frac{1}{3}}e^{-\frac{11}{16}}e^{-(x_1+x_2)}\right)^+\cdot \\
&\qquad\qquad \cdot e^{-\frac{1}{2}(x_1^2+x_2^2)}dx_1dx_2.
\end{aligned}
$$

Even if at this point one could use the computations in (4.89)-(4.91), here the integral can more conveniently be computed via an approximation by means of a discretization of the Normal random variable. □

Problem 4.40 (Model 2). Determine P_0 in the case of $N_0 = 2$, $N_1 = 3$, $K = 1$ and putting $q_h = \frac{1}{H}$ for $H = 2$.

Solution of Problem 4.40
i) *Direct method (using (4.96) and putting then $H = 2$)*

$$
\begin{aligned}
P_0 &= E^Q\left[p(0,1)p(1,2)(1-p(2,3))^+\right] \\
&= E^Q\left[\frac{2}{H+1}p(0,2)\xi_1\left(1-\frac{p(0,3)}{p(0,2)}\frac{6\xi_1\xi_2}{(H+1)(2H+1)}\right)^+\right] \\
&= E^Q\left[\frac{2}{H+1}\xi_1\left(p(0,2)-\frac{6\xi_1\xi_2}{(H+1)(2H+1)}p(0,3)\right)^+\right] \\
&= E^Q\left[\frac{2\xi_1}{15}\left(3-\frac{4}{5}\xi_1\xi_2\right)^+\right] \\
&= \frac{1}{4}\left(\frac{2}{15}\left(3-\frac{4}{5}\right)^+ + \frac{2}{15}\left(3-\frac{8}{5}\right)^+\right. \\
&\qquad \left. + \frac{4}{15}\left(3-\frac{8}{5}\right)^+ + \frac{4}{15}\left(3-\frac{16}{5}\right)^+\right) \\
&= \frac{1}{4}\left(\frac{2}{15}\cdot\frac{11}{5}+\frac{2}{15}\cdot\frac{7}{5}+\frac{4}{15}\cdot\frac{7}{5}\right) = \frac{16}{75}.
\end{aligned}
$$

ii) *Recursive method*
By Corollary 4.6, the following recursive formula holds

$$
\begin{aligned}
P_0 &= p(0,1)E^Q[P_1], \\
(P_1)_{|\xi_1} &= p(1,2)_{|\xi_1}E^Q\left[(1-p(2,3))^+ \mid \xi_1\right],
\end{aligned}
$$

where $(\cdot)_{|\xi_1}$ denotes the conditioning on a given value of ξ_1. We then have

$$
\begin{aligned}
(P_1)_{|\xi_1} &= \frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H+1}E^Q\left[\left(1-\frac{p(0,3)}{p(0,2)}\frac{6\xi_1\xi_2}{(H+1)(2H+1)}\right)^+ \mid \xi_1\right]\\
&= \frac{1}{p(0,1)}\frac{2\xi_1}{H+1}E^Q\left[\left(p(0,2)-\frac{6\xi_1\xi_2}{(H+1)(2H+1)}p(0,3)\right)^+ \mid \xi_1\right]
\end{aligned}
$$

which, for $H=2$, yields

$$
\begin{aligned}
(P_1)_{|\xi_1=1} &= \frac{1}{p(0,1)}\frac{2}{3}\cdot\frac{1}{2}\left(\left(\frac{3}{5}-\frac{4}{25}\right)^+ + \left(\frac{3}{5}-\frac{8}{25}\right)^+\right)\\
&= \frac{1}{p(0,1)}\frac{1}{3}\cdot\frac{18}{25},\\
(P_1)_{|\xi_1=2} &= \frac{1}{p(0,1)}\frac{4}{3}\cdot\frac{1}{2}\left(\left(\frac{3}{5}-\frac{8}{25}\right)^+ + \left(\frac{3}{5}-\frac{16}{25}\right)^+\right)\\
&= \frac{1}{p(0,1)}\frac{2}{3}\cdot\frac{7}{25},\\
P_0 &= \frac{p(0,1)}{2}\left((P_1)_{|\xi_1=1}+(P_1)_{|\xi_1=2}\right) = \frac{1}{6}\cdot\frac{18}{25}+\frac{2}{6}\cdot\frac{7}{25} = \frac{16}{75}.
\end{aligned}
$$

□

Call on T-bonds

Problem 4.41 (Model 1). In Model 1 with initial term structure $p(0,1)=\frac{1}{2}e^{\frac{1}{4}}, p(0,2)=\frac{1}{4}$ as in (4.88), consider the Call option with maturity at time $N=1$ and payoff equal to

$$
C_1 = \left(p(1,2)-\frac{3}{4}\right)^+.
$$

i) Determine the initial price C_0;
ii) being the market incomplete, for the hedging use the criterion of quadratic risk, namely minimize with respect to α and β[9] the following quantity

$$
E^P\left[\left(\left(p(1,2)-\frac{3}{4}\right)^+ - V_1\right)^2\right],
$$

where P is the "physical" measure under which we assume that $\xi_1 \sim \mathcal{N}(0,1)$ and V_1 is the value in $n=1$ of a self financing portfolio with given initial value V_0.

[9] α represents the number of units invested in the underlying bond that has maturity in $\bar{N}=2$ and β is the amount invested in the non-risky asset.

Solution of Problem 4.41

i) We have

$$
\begin{aligned}
C_0 &= p(0,1)E^Q\left[\left(p(1,2)-\frac{3}{4}\right)^+\right]\\
&= p(0,1)\int_D\left(p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi}-\frac{3}{4}\right)\mathcal{N}_{0,1}(d\xi),
\end{aligned}
$$

where

$$
D := \left\{\xi \mid p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi}>\frac{3}{4}\right\} = \{\xi \mid \xi<\delta\},
$$

with

$$
\delta = \frac{1}{2}\log p(0,2)-\frac{3}{4}-\log\frac{3}{4} = \log\frac{2}{3}-\frac{3}{4}.
$$

Using the technique that has led to (4.89)-(4.91) we then have

$$
\begin{aligned}
C_0 =& p(0,1)\left(-\frac{3}{4}\Phi(\delta)+p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}\int_{\{\xi_1<\delta\}}e^{-\xi_1}\mathcal{N}_{0,1}(d\xi_1)\right)\\
=& p(0,1)\left(-\frac{3}{4}\Phi(\delta)+p(0,2)^{\frac{1}{2}}e^{-\frac{1}{4}}\int_{\{y<\delta+1\}}\mathcal{N}_{0,1}(dy)\right)\\
=& p(0,1)\left(-\frac{3}{4}\Phi(\delta)+p(0,2)^{\frac{1}{2}}e^{-\frac{1}{4}}\Phi(\delta+1)\right)\\
=& -\frac{3}{8}e^{\frac{1}{4}}\Phi\left(\log\frac{2}{3}-\frac{3}{4}\right)+\frac{1}{4}\Phi\left(\log\frac{2}{3}+\frac{1}{4}\right)
\end{aligned}
$$

where Φ the cumulative distribution function of the standard Normal.

ii) We have

$$
p(1,2) = p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi_1} =: Ae^{-\xi_1},
$$

and so

$$
V_1 = V_0+\alpha p(1,2)+\frac{\beta}{p(0,1)} = V_0+\alpha Ae^{-\xi_1}+2\beta.
$$

For the same δ as in point a) we then have

$$
\begin{aligned}
&E^P\left[\left(\left(p(1,2)-\frac{3}{4}\right)^+-V_1\right)^2\right]\\
&= E^P\left[\left(\left(Ae^{-\xi_1}-\frac{3}{4}\right)^+-V_0-\alpha Ae^{-\xi_1}-2\beta\right)^2\right]\\
&= \int_{\{\xi_1<\delta\}}\left((1-\alpha)Ae^{-\xi_1}-\left(2\beta+V_0+\frac{3}{4}\right)\right)^2\mathcal{N}_{0,1}(d\xi_1).
\end{aligned}
$$

Putting for brevity $B(\beta, V_0) := 2\beta + V_0 + \frac{3}{4}$, and using (4.89)-(4.91), we have to minimize with respect to α and β, for V_0 given, the quantity (in the first step we replace A and $B(\beta, V_0)$ with their explicit expressions):

$$
\begin{aligned}
&\int_{\{\xi_1<\delta\}} (1-\alpha)^2 A^2 e^{-2\xi_1} \mathcal{N}_{0,1}(d\xi_1) + \int_{\{\xi_1<\delta\}} B^2(\beta, V_0) \mathcal{N}_{0,1}(d\xi_1) \\
&- \int_{\{\xi_1<\delta\}} 2(1-\alpha) A B(\beta, V_0) e^{-\xi_1} \mathcal{N}_{0,1}(d\xi_1) \\
&= (1-\alpha)^2 p(0,2) e^{\frac{1}{2}} \int_{\{z<\delta+2\}} \mathcal{N}_{0,1}(dz) \\
&+ \left(2\beta + V_0 + \frac{3}{4}\right)^2 \int_{\{\xi<\delta\}} \mathcal{N}_{0,1}(d\xi) \\
&- 2\left(2\beta + V_0 + \frac{3}{4}\right)(1-\alpha) p(0,2)^{\frac{1}{2}} e^{-\frac{1}{4}} \int_{\{y<\delta+1\}} \mathcal{N}_{0,1}(dy) \\
&= \frac{(1-\alpha)^2}{4} e^{\frac{1}{2}} \Phi\left(\log\frac{2}{3} + \frac{5}{4}\right) + \left(2\beta + V_0 + \frac{3}{4}\right)^2 \Phi\left(\log\frac{2}{3} - \frac{3}{4}\right) \\
&- \left(2\beta + V_0 + \frac{3}{4}\right)(1-\alpha) e^{-\frac{1}{4}} \Phi\left(\log\frac{2}{3} + \frac{1}{4}\right),
\end{aligned}
$$

which turns out to be a quadratic function in α and β that can be minimized by searching for the critical points. $\square$

Problem 4.42 (Model 2). Consider the Call option of Problem 4.41 for Model 2 with $q_h = \frac{1}{H}$, $H = 2$ and with initial term structure as in (4.97), namely $p(0,1) = \frac{4}{5}$ and $p(0,2) = \frac{3}{5}$:

i) determine the initial price C_0;
ii) on the basis of the Put-Call Parity determine the initial price P_0 of the corresponding Put option;
iii) determine the hedging strategy for the Call option at time $n = 0$.

Solution of Problem 4.42

i)

$$
\begin{aligned}
C_0 =& E^Q\left[e^{-r_0} C_1\right] = p(0,1) E^Q[C_1] \\
=& p(0,1) E^Q\left[\left(\frac{p(0,2)}{p(0,1)} \frac{2\xi_1}{H+1} - \frac{3}{4}\right)^+\right] \\
=& \frac{4}{5} \cdot \frac{1}{2}\left(\left(\frac{3}{4} \cdot \frac{2}{3} - \frac{3}{4}\right)^+ + \left(\frac{3}{4} \cdot \frac{4}{3} - \frac{4}{3}\right)^+\right) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}.
\end{aligned}
$$

ii) The Put-Call Parity is based on

$$(K-S)^+ = (S-K)^+ + K - S$$

from which, being here $S = p(1,2)$, by means of the same formula that has led to the Call we have

$$\begin{aligned} P_0 &= p(0,1)E^Q\left[\left(\frac{3}{4} - p(1,2)\right)^+\right] \\ &= p(0,1)E^Q\left[\left(p(1,2) - \frac{3}{4}\right)^+\right] + \frac{3}{4}p(0,1) - p(0,1)E^Q[p(1,2)] \\ &= C_0 + \frac{3}{5} - p(0,1)E^Q[p(1,2)] = \end{aligned}$$

(by (4.11) from Proposition 4.5)

$$= C_0 + \frac{3}{5} - p(0,2).$$

iii) In this case the market model is complete, as one may also verify directly. Equating then the portfolio value at time $n = 1$ with the claim, it turns out that the following has to hold

$$\alpha \frac{p(0,2)}{p(0,1)} \frac{2}{3}\xi_1 + \beta e^{r_0} = \left(\frac{p(0,2)}{p(0,1)} \frac{2}{3}\xi_1 - \frac{3}{4}\right)^+$$

whatever the value (among the two possible ones) of ξ_1 may be. Taking furthermore into account that $e^{r_0} = p(0,1)^{-1}$, one finds

$$\begin{cases} \frac{1}{2}\alpha + \frac{5}{4}\beta = 0, \\ \alpha + \frac{5}{4}\beta = \frac{1}{4}, \end{cases}$$

from which $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{5}$. The result may be verified by

$$C_0 = \alpha p(0,2) + \beta = \frac{1}{2} \cdot \frac{3}{5} - \frac{1}{5} = \frac{1}{10}.$$

□

Problem 4.43 (Model 2). The context is the same as that of the previous Problem 4.42, except that now we put $H = 3$ so that the market becomes incomplete.

i) assuming that $q_h = \frac{1}{H} = \frac{1}{3}$, determine the initial price C_0;
ii) for the hedging use the quadratic risk criterion according to which one has to minimize with respect to α and β the expression

$$E^P\left[\left(\left(p(1,2) - \frac{3}{4}\right)^+ - V_1\right)^2\right]$$

where P is the "physical" measure for which we assume $P(\xi_1 = h) = \frac{1}{3}$ for $h = 1, 2, 3$ and V_1 is the value in $n = 1$ of a self financing portfolio having a given initial value V_0.

Solution of Problem 4.43
i) We have

$$\begin{aligned} C_0 &= p(0,1)E^Q\left[\left(\frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H+1} - \frac{3}{4}\right)^+\right] \\ &= \frac{4}{5}\cdot\frac{1}{3}\left(\left(\frac{3}{4}\cdot\frac{1}{2} - \frac{3}{4}\right)^+ + \left(\frac{3}{4} - \frac{3}{4}\right)^+\left(\frac{3}{4}\cdot\frac{3}{2} - \frac{3}{4}\right)^+\right) \\ &= \frac{4}{15}\cdot\frac{3}{8} = \frac{1}{10}. \end{aligned}$$

ii) Since

$$p(1,2) = \frac{p(0,2)}{p(0,1)}\frac{1}{2}\xi_1 = \frac{3}{8}\xi_1,$$

one finds

$$V_1 = V_0 + \alpha p(1,2) + \frac{\beta}{p(0,1)} = V_0 + \frac{3}{8}\alpha\xi_1 + \frac{5}{4}\beta.$$

Then

$$\begin{aligned} &E^P\left[\left(\left(p(1,2) - \frac{3}{4}\right)^+ - V_1\right)^2\right] = \frac{1}{3}\left(\left(\frac{3}{8} - \frac{3}{4}\right)^+ - V_0 - \frac{3}{8}\alpha - \frac{5}{4}\beta\right)^2 \\ &\quad + \frac{1}{3}\left(\left(\frac{3}{4} - \frac{3}{4}\right)^+ - V_0 - \frac{3}{4}\alpha - \frac{5}{4}\beta\right)^2 \\ &\quad + \frac{1}{3}\left(\left(\frac{9}{8} - \frac{3}{4}\right)^+ - V_0 - \frac{9}{8}\alpha - \frac{5}{4}\beta\right)^2 \\ &= \frac{1}{3}\left(V_0 + \frac{3}{8}\alpha + \frac{5}{4}\beta\right)^2 + \frac{1}{3}\left(V_0 + \frac{3}{4}\alpha + \frac{5}{4}\beta\right)^2 \\ &\quad + \frac{1}{3}\left((V_0 - \frac{3}{8}) + \frac{9}{8}\alpha + \frac{5}{4}\beta\right)^2, \end{aligned}$$

which is a quadratic function in α and β and can be minimized by means of the usual tools from differential calculus for more than one real variable. □

4.6.3 Caps and Floors

Since Caps and Floors consist of Caplets and Floorlets, the problems will concern only the latter. To simplify the expressions, assume a unitary reference capital and also a unitary time step, namely $\varDelta = 1$. In the problems of this section we shall always consider at most three periods ($n = 0, 1, 2, 3$) and the sequence of maturities

$$N_0 = 1, \;\; N_1 = 2 \;\; \text{and } N_2 = 3. \tag{4.99}$$

As follows from Proposition 4.23, the computation of a Caplet reduces essentially to the computation of a Put option on a T-bond (analogously, the computation of a Floorlet reduces to that of a Call option) and thus the procedure will be analogous to that used for the Put options.

Problem 4.44 (Model 1). Recall that in Model 1 the prices of the T-bonds are given by the formulae in (4.86) where ξ_1, ξ_2 are i.i.d. standard normal random variables under martingale measure Q. Determine the initial prices $Caplet_1(0)$ and $Caplet_2(0)$ of the Caplets relative to the intervals $[1,2]$ and $[2,3]$ respectively, for a rate $K = 1$.

Solution of Problem 4.44

For what concerns $Caplet_1(0)$, using (4.63)-(4.64) one obtains

$$\begin{aligned} Caplet_1(0) &= p(0,1)E^Q\left[(1-2p(1,2))^+\right] \\ &= \frac{1}{2}e^{\frac{1}{4}}\int_D \left(1 - e^{-\frac{3}{4}}e^{-\xi_1}\right)\mathcal{N}_{0,1}(d\xi_1), \end{aligned}$$

with

$$D := \left\{\xi \mid 1 - e^{-\frac{3}{4}}e^{-\xi} > 0\right\} = \left\{\xi \mid \xi > -\frac{3}{4}\right\}.$$

Thus

$$\begin{aligned} Caplet_1(0) &= \frac{1}{2}e^{\frac{1}{4}}\left(\int_{\{\xi > -\frac{3}{4}\}}\mathcal{N}_{0,1}(d\xi) - e^{-\frac{3}{4}}e^{\frac{1}{2}}\int_{\{y > \frac{1}{4}\}}\mathcal{N}_{0,1}(dy)\right) \\ &= \frac{1}{2}e^{\frac{1}{4}}\left(\varPhi\left(\frac{3}{4}\right) - e^{-\frac{1}{4}}\varPhi\left(-\frac{1}{4}\right)\right). \end{aligned}$$

For what concerns $Caplet_2(0)$, we need the following values, for which we consider the initial term structure in (4.88):

$$\begin{aligned} p(1,2) &= p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi_1} = \frac{1}{2}e^{-\frac{3}{4}}e^{-\xi_1}, \\ p(2,3) &= p(0,3)^{\frac{1}{3}}e^{-\frac{11}{6}}e^{-(\xi_1+\xi_2)} = \frac{1}{2}e^{-\frac{19}{12}}e^{-(\xi_1+\xi_2)}. \end{aligned}$$

We then have

$$Caplet_2^{\xi_1}(1) = \frac{1}{2}e^{-\frac{3}{4}}e^{-\xi_1}E^Q\left[\left(1 - \frac{1}{2}e^{-\frac{19}{12}}e^{-(\xi_1+\xi_2)}\right)^+ \mid \xi_1\right]$$

and thus

$$\begin{aligned} Caplet_2(0) &= \frac{1}{2}e^{-\frac{3}{4}}E^Q\left[e^{-\xi_1}\left(1-\frac{1}{2}e^{-\frac{19}{12}}e^{-(\xi_1+\xi_2)}\right)^+\right] \\ &= \frac{1}{4\pi}e^{-\frac{3}{4}}\iint e^{-x_1}\left(1-\frac{1}{2}e^{-\frac{19}{12}}e^{-(x_1+x_2)}\right)^+ e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}dx_1dx_2, \end{aligned}$$

which, as for the options on T-bonds, may be computed approximately by means of a discretization of the normal random variable. □

Problem 4.45 (Model 2). Consider Model 2 with $p(0,1)=\frac{4}{5}$, $p(0,2)=\frac{3}{5}$, $p(0,3)=\frac{2}{5}$. Determine the initial price $Caplet_2(0)$ of the Caplet with $K=1$, relative to the interval $[2,3]$ in the following cases:

i) $H=2$ and $q_h=Q(\xi_n=h)=\frac{1}{2}$;
ii) $H=2$ and $q=Q(\xi_n=1)\in\,]0,1[$;
iii) $H=3$ and $q_h=Q(\xi_n=h)=\frac{1}{3}$.

Solution of Problem 4.45

i) Following Proposition 4.23, we have

$$Caplet_2(2)=(1-2p(2,3))^+ \tag{4.100}$$

and, for $n<2$,

$$Caplet_2(n)=p(n,n+1)E^Q[Caplet_2(n+1)\mid\mathcal{F}_n].$$

Using the symbol $Caplet_2^h(n)$ to denote the price of the Caplet in n if $\xi_n=h$, we have

$$\begin{aligned} Caplet_2^1(1) &= p(1,2)\frac{1}{2}\left(Caplet_2^1(2)+Caplet_2^2(2)\right) \\ &= \frac{1}{4}\left(\left(1-\frac{8}{15}\right)^+ + \left(1-\frac{16}{15}\right)^+\right) = \frac{7}{4\cdot 15}, \\ Caplet_2^2(1) &= \frac{1}{2}\left(\left(1-\frac{16}{15}\right)^+ + \left(1-\frac{32}{15}\right)^+\right) = 0. \end{aligned}$$

Thus

$$Caplet_2(0)=p(0,1)\frac{1}{2}\cdot\frac{7}{4\cdot 15}=\frac{2}{5}\cdot\frac{7}{4\cdot 15}=\frac{7}{150}.$$

ii) We first need the values of $p(1,2)$ and $p(2,3)$ which we can obtain from the formulae (4.96): in particular we have

$$\begin{aligned} p(1,2) &= \frac{p(0,2)}{p(0,1)}\frac{1}{q+2(1-q)}\xi_1=\frac{3}{4}\frac{\xi_1}{2-q}, \\ p(2,3) &= \frac{p(0,3)}{p(0,2)}\frac{1}{q+4(1-q)}\xi_1\xi_2=\frac{2}{3}\frac{\xi_1\xi_2}{4-3q}. \end{aligned}$$

It then follows that

$$Caplet_2^1(1) = \frac{3}{4}\frac{1}{2-q}\left(q\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ + (1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+\right)$$
$$Caplet_2^2(1) = \frac{3}{2}\frac{1}{2-q}\left(q\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+ + (1-q)\left(1-\frac{16}{3}\frac{1}{4-3q}\right)^+\right),$$

and thus

$$\begin{aligned}Caplet_2(0) = &\frac{4}{5}\cdot\frac{3}{2}\frac{1}{2-q}\left(\frac{q^2}{2}\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ + \frac{(1-q)q}{2}\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+\right.\\ &\left. + q(1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+ + (1-q)^2\left(1-\frac{16}{3}\frac{1}{4-3q}\right)^+\right)\\ = &\frac{6}{5}\frac{1}{2-q}\left(\frac{q^2}{2}\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ + \frac{3}{2}q(1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+\right),\end{aligned}$$

being $\left(1-\frac{16}{3}\frac{1}{4-3q}\right)^+ = 0$ for any $q \in [0,1]$.

We verify this result by putting $q = \frac{1}{2}$: one has

$$Caplet_2(0) = \frac{4}{5}\cdot\frac{1}{8}\left(1-\frac{8}{15}\right)^+ = \frac{1}{10}\cdot\frac{7}{15} = \frac{7}{150}$$

which coincides with what was obtained in point a).

iii) One has

$$\begin{aligned}Caplet_2^1(1) &= p(1,2)\frac{1}{3}\left(Caplet_2^1(2) + Caplet_2^2(2) + Caplet_2^3(2)\right)\\ &= \frac{1}{8}\left(\left(1-\frac{2}{7}\right)^+ + \left(1-\frac{4}{7}\right)^+ + \left(1-\frac{6}{7}\right)^+\right) = \frac{1}{8}\cdot\frac{9}{7} = \frac{9}{56}\\ Caplet_2^2(1) &= \frac{1}{4}\left(\left(1-\frac{4}{7}\right)^+ + \left(1-\frac{8}{7}\right)^+ + \left(1-\frac{12}{7}\right)^+\right) = \frac{1}{4}\cdot\frac{3}{7} = \frac{3}{28}\\ Caplet_2^3(1) &= \frac{3}{8}\left(\left(1-\frac{6}{7}\right)^+ + \left(1-\frac{12}{7}\right)^+ + \left(1-\frac{18}{7}\right)^+\right) = \frac{3}{8}\cdot\frac{1}{7} = \frac{3}{56}.\end{aligned}$$

Thus

$$Caplet_2(0) = \frac{4}{5}\cdot\frac{1}{3}\cdot\frac{9+6+3}{56} = \frac{3}{35}. \qquad \square$$

Problem 4.46 (Model 2). Consider Model 2 with $H = 2$, $p(0,1) = \frac{4}{5}$, $p(0,2) = \frac{3}{5}$, rate $K = 1$, and let the maturities be given as in (4.99):

i) determine the initial price $Caplet_1^{(q)}(0)$ of the Caplet relative to the interval $[1,2]$, for all possible values of $q = Q(\xi_1 = 1) \in]0,1[$, thereby verifying that, for $q \geq \frac{1}{2}$, one has $Caplet_1^{(q)}(0) = 0$ and that, for $q < \frac{1}{2}$, it follows that
$$Caplet_1^{(q)}(0) = \frac{4}{5}q\left(1 - \frac{3}{2}\frac{1}{2-q}\right);$$

ii) assuming $q < \frac{1}{2}$, does there exist a value of q for which $Caplet_1^{(q)}(0) = \frac{1}{35}$? Is this value of $Caplet_1^{(q)}(0)$ sufficient to fully calibrate the model to the market?

Solution of Problem 4.46

i) By Proposition 4.23 with $K = 1$ and $\alpha_j = 1$, one has

$$Caplet_1^{\xi_1}(1) = 2\left(\frac{1}{2} - p(1,2)|_{\xi_1}\right)^+ = (1 - 2p(1,2)|_{\xi_1})^+$$
$$Caplet_1^{(q)}(0) = p(0,1)\left(q\,Caplet_1^1(1) + (1-q)\,Caplet_1^2(1)\right).$$

Then, by the expression of $p(1,2)$, one has (see also Problem 4.45-b)

$$Caplet_1^1(1) = \left(1 - \frac{3}{2}\frac{1}{2-q}\right)^+, \qquad Caplet_1^2(1) = \left(1 - \frac{3}{2-q}\right)^+ = 0,$$

since for all values of $q \in]0,1[$ one has $\frac{3}{2-q} > 1$. Consequently

$$Caplet_1^{(q)}(0) = \frac{4}{5}q\left(1 - \frac{3}{2}\frac{1}{2-q}\right)^+.$$

Notice that $1 - \frac{3}{2}\frac{1}{2-q}$ is a decreasing function of q which vanishes in $q = \frac{1}{2}$. It follows that, for $q \geq \frac{1}{2}$, one has $Caplet_1^{(q)}(0) = 0$ while, for $q < \frac{1}{2}$, it follows that $Caplet_1^{(q)}(0) = \frac{4}{5}q\left(1 - \frac{3}{2}\frac{1}{2-q}\right)$.

ii) Assuming $q < \frac{1}{2}$, the condition on q becomes $\frac{4}{5}q\left(1 - \frac{3}{2}\frac{1}{2-q}\right) = \frac{1}{35}$ which leads to the following second order equation

$$28q^2 - 15q + 2 = 0,$$

with solutions

$$q_{1,2} = \frac{15 \pm \sqrt{225 - 224}}{56},$$

namely $q_1 = \frac{1}{4}$ and $q_2 = \frac{2}{7}$.

Both values are compatible with $q < \frac{1}{2}$ and so the given market data are not sufficient to fully calibrate the model. □

Remark 4.47. *Like the formula* (4.74) *for the Floor-Cap Parity, one can also prove the following Floorlet-Caplet Parity according to which*

$$Floorlet_j = Caplet_j - PFS_j$$

where $Floorlet_j$ *denotes the price of the Floorlet relative to the period* $[N_{j-1}, N_j]$, $Caplet_j$ *that of the corresponding Caplet and* PFS_j *that of the Payer Forward Swap defined in* (4.68). □

Problem 4.48 (Model 1). Consider Model 1 with the values of the initial term structure given in (4.88). Determine the initial price $Floorlet_1(0)$ of the Floorlet relative to the interval $[1, 2]$, for a fixed rate $K = 1$.

Solution of Problem 4.48

We use the Floorlet-Caplet Parity. Recalling the choice (4.99) of maturities, we thus determine first the value of the Caplet by means of (4.63)-(4.64):

$$Caplet_1(0) = p(0, 1)E^Q\left[(1 - 2p(1, 2))^+\right].$$

The corresponding calculations were already made in Problem 4.44, according to which

$$Caplet_1(0) = \frac{1}{2}e^{\frac{1}{4}}\left(\Phi\left(\frac{3}{4}\right) - e^{-\frac{1}{4}}\Phi\left(-\frac{1}{4}\right)\right).$$

Notice, furthermore, that in this context $PFS_j(0)$ is given by (see (4.73) in the proof of Proposition 4.29):

$$PFS_j(0) = p(0, N_{j-1}).$$

Concluding we have

$$\begin{aligned} Floorlet_1(0) &= Caplet_1(0) - PFS_1(0) = Caplet_1(0) - p(0, 1) \\ &= \frac{1}{2}e^{\frac{1}{4}}\left(\Phi\left(\frac{3}{4}\right) - 1 - e^{-\frac{1}{4}}\Phi\left(-\frac{1}{4}\right)\right). \end{aligned}$$ □

Problem 4.49 (Model 2). In the context of Problem 4.45-a), consider Model 2 with $q_h = \frac{1}{H}$ and $H = 2$. Determine the price $Floor(0)$ in $n = 0$ of the Floor relative to the periods $[1, 2]$ and $[2, 3]$, for the fixed rate $K = 1$.

Solution of Problem 4.49

Recall the choice (4.99) of the maturities and, as previously, denote by $Caplet_j$ the price of the Caplet relative to the interval $[N_{j-1}, N_j] = [j, j + 1]$, $j = 1, 2$. In order to use the Floor-Cap Parity we first need the price of the Cap over the periods $[1, 2]$ and $[2, 3]$, which is given by

$$Cap(0) = Caplet_1(0) + Caplet_2(0)$$

where (see Problem 4.45-a)) $\mathit{Caplet}_2(0) = \frac{7}{150}$. On the other hand, by (4.64) one has

$$\mathit{Caplet}_1(0) = p(0,1)E^Q\left[\mathit{Caplet}_1(1)\right] = \frac{p(0,1)}{2}\left(\mathit{Caplet}_1^1(1) + \mathit{Caplet}_1^2(1)\right),$$

where, we recall, $\mathit{Caplet}_1^h(n)$ denotes the price of the Caplet in n if $\xi_n = h$. Therefore, using Proposition 4.23, as well as the formulae in (4.96), we obtain

$$\mathit{Caplet}_1^1(1) = 2\left(\frac{1}{2} - p(1,2)|_{\xi_1=1}\right)^+ = 0,$$
$$\mathit{Caplet}_1^2(1) = 2\left(\frac{1}{2} - p(1,2)|_{\xi_1=2}\right)^+ = 0,$$

by which $\mathit{Caplet}_1(0) = 0$. Finally, the PFS with payment dates $N_1 = 2$ and $N_2 = 3$ is, by Proposition 4.29,

$$\mathit{PFS}(0) = p(0,1) - p(0,3) - (p(0,2) + p(0,3)) = -\frac{3}{5}.$$

In conclusion

$$\mathit{Floor}(0) = \frac{7}{150} + \frac{3}{5} = \frac{97}{150}.$$

□

Problem 4.50 (Model 2). Consider the alternative Model 2 of Remark 4.37 with the stochastic factors $\varphi_{n,N}$ as in (4.98) for $H = 2$ and put $q = Q(\xi_n = 1)$ (consequently $1 - q = Q(\xi_n = 2)$). Furthermore, let $p(0,1) = \frac{4}{5}, p(0,2) = \frac{3}{5}$ and $p(0,3) = \frac{2}{5}$:

i) consider a Caplet for the period $[1,2]$ with $K = 1$. Show that, if the observed value is $\mathit{Caplet}_1(0) = \frac{1}{5}$, then necessarily $q = \frac{3}{4}$;
ii) with the so calibrated value of q determine the price $\mathit{Caplet}_1(0)$ in the case of $K = \frac{1}{2}$;
iii) what is the value of q if at time $n = 1$ one observes for a T-bond with maturity 2 a price of $p(1,2) = \frac{3}{4}$? Having observed such a value, are we able to establish whether it was $\xi_1 = 1$ or $\xi_1 = 2$? What can be said if, always in $n = 1$, we observe also $p(1,3) = \frac{3}{5}$?

Solution of Problem 4.50

i) Recalling the choice (4.99) of maturities, by Proposition 4.23 one has

$$\begin{aligned}
&\mathit{Caplet}_1(0) = p(0,1)E^Q\left[(1 - 2p(1,2))^+\right] \\
&= p(0,1)\left(q\left(1 - 2\frac{p(0,2)}{p(0,1)}\varphi_{1,2}(1)\right)^+ + (1-q)\left(1 - 2\frac{p(0,2)}{p(0,1)}\varphi_{1,2}(2)\right)^+\right) \\
&= q\left(p(0,1) - \frac{2p(0,2)}{q(1+2)}\right)^+ + (1-q)\left(p(0,1) - \frac{4p(0,2)}{(1-q)(1+2)}\right)^+ \\
&= \left(\frac{4}{5}q - \frac{2}{5}\right)^+ + \left(\frac{4}{5}(1-q) - \frac{4}{5}\right)^+.
\end{aligned}$$

Given that for no value of $q \in]0,1[$ one has $\frac{4}{5}(1-q) - \frac{4}{5} > 0$, it follows that $Caplet_1(0) = \left(\frac{4}{5}q - \frac{2}{5}\right)^+$ which, equated to $\frac{1}{5}$, yields $(4q-2)^+ = 1$. This is equivalent to the system

$$\begin{cases} 4q - 2 > 0, \\ 4q - 2 = 1, \end{cases}$$

which is satisfied by the only value $q = \frac{3}{4}$.

ii) With $K = \frac{1}{2}$ and $q = \frac{3}{4}$ one has, using the calculations from point a),

$$\begin{aligned} Caplet_1(0) &= p(0,1)E^Q\left[\left(1 - \frac{3}{2}p(1,2)\right)^+\right] \\ &= \left(\frac{4}{5}q - \frac{3}{10}\right)^+ + \left(\frac{4}{5}(1-q) - \frac{3}{5}\right)^+ = \end{aligned}$$

(for $q = \frac{3}{4}$)

$$= \left(\frac{3}{5} - \frac{3}{10}\right)^+ + \left(\frac{1}{5} - \frac{3}{5}\right)^+ = \frac{3}{10}.$$

iii) The relation

$$p(1,2) = \frac{p(0,2)}{p(0,1)}\varphi_{1,2}(\xi),$$

implies, for $p(1,2) = \frac{3}{4}$, that $\varphi_{1,2}(\xi_1) = 1$. The possible values of $\varphi_{1,2}(\xi_1)$ are $\frac{1}{3q}$ for $\xi_1 = 1$ and $\frac{2}{3(1-q)}$ for $\xi_1 = 2$. One can see that $q = \frac{1}{3}$ satisfies both requirements, but it is not possible to establish whether it was $\xi_1 = 1$ or $\xi_1 = 2$. On the other hand, the relation

$$p(1,3) = \frac{p(0,3)}{p(0,1)}\varphi_{1,3}(\xi_1)$$

yields, for $p(1,3) = \frac{3}{5}$, the condition $\frac{3}{5} = \frac{1}{2}\varphi_{1,3}(\xi_1)$. Being, for $q = \frac{1}{3}$,

$$\varphi_{1,3}(1) = \frac{1}{5q} = \frac{3}{5}, \qquad \varphi_{1,3}(2) = \frac{4}{5(1-q)} = \frac{6}{5},$$

the above condition is satisfied only for $\xi_1 = 2$. □

Problem 4.51 (Model 2). Consider the same context as in Problem 4.50, but with $H = 3$ and put $q_h = Q(\xi_n = h)$ for $h = 1,2,3$ for each n:

i) assume that for the Caplet over the period $[1,2]$ and with $K = 3$ one observes a value of

$$Caplet_1(0) = \frac{1}{20}. \tag{4.101}$$

Show that then it has to be $q_1 = \frac{9}{16}$;

ii) assume one observes a further Caplet with $K = \frac{1}{2}$ and price

$$Caplet_1(0) = \frac{13}{40}.$$

Show that this determines completely the values of q_1, q_2 and q_3 and thus the measure Q.

Solution of Problem 4.51

i) Under the given conditions, on the basis of Proposition 4.23 one has

$$\begin{aligned}
Caplet_1(0) &= p(0,1)E^Q\left[(1-4p(1,2))^+\right] \\
&= p(0,1)\sum_{i=1}^{3} q_i\left(1-4\frac{p(0,2)}{p(0,1)}\varphi_{1,2}(i)\right)^+ \\
&= \sum_{i=1}^{3} q_i\left(p(0,1)-4p(0,2)\frac{i}{6q_i}\right)^+ = \sum_{i=1}^{3}\left(\frac{4}{5}q_i-\frac{2}{5}i\right)^+.
\end{aligned}$$

Noticing that the terms for $i = 2, 3$ are equal to zero for any value of $q_1, q_2 \in]0,1[$, imposing the condition (4.101) one has $\left(\frac{4}{5}q_1-\frac{2}{5}\right)^+ = \frac{1}{20}$ which gives $q_1 = \frac{9}{16}$.

ii) For $K = \frac{1}{2}$ one has on the other hand

$$Caplet_1(0) = \sum_{i=1}^{3} q_i\left(p(0,1)-\frac{3}{2}p(0,2)\frac{i}{6q_i}\right)^+ = \sum_{i=1}^{3}\left(\frac{4}{5}q_i-\frac{3i}{20}\right)^+.$$

Since we already know that $q_1 = \frac{9}{16}$, the condition becomes

$$\frac{6}{20}+\left(\frac{4}{5}q_2-\frac{6}{20}\right)^+ + \left(\frac{4}{5}q_3-\frac{9}{20}\right)^+ = \frac{13}{40}$$

namely

$$\left(\frac{4}{5}q_2-\frac{6}{20}\right)^+ + \left(\frac{4}{5}q_3-\frac{9}{20}\right)^+ = \frac{1}{40}.$$

Since $q_2 + q_3 = 1 - \frac{9}{16} = \frac{7}{16}$, the value of each of the probabilities q_2, q_3 cannot exceed $\frac{7}{16}$, but even for such a maximal value for q_3 the second term in the condition above is zero and so the condition becomes

$$\left(\frac{4}{5}q_2-\frac{6}{20}\right)^+ = \frac{1}{40}$$

which yields $q_2 = \frac{13}{32}$. Concluding, we have

$$q_1 = \frac{9}{16}, \qquad q_2 = \frac{13}{32}, \qquad q_3 = \frac{1}{32}.$$

□

Remark 4.52. *If, in the previous problem for $K = \frac{1}{2}$ (point b)) one would have observed $Caplet_1(0) = \frac{7}{20}$, one would have obtained*

$$q_1 = \frac{9}{16}, \qquad q_2 = \frac{7}{16}, \qquad q_3 = 0.$$

The fact that $q_3 = 0$ not only does not lead to a measure Q equivalent to P (consider that $p_3 \neq 0$), but not even to a martingale measure since then one would not have $E^Q[\varphi_{1,2}(\xi_1)] = 1$. Observing $Caplet_1(0) = \frac{7}{20}$ for $K = \frac{1}{2}$ would thus not be coherent with the model. □

4.6.4 Swap Rates and Payer Forward Swaps

For simplicity we shall consider only three periods ($n = 0, 1, 2, 3$) and two types of Swap Rates:

i) the Swap Rate relative to a single period that may be $[1, 2]$ or $[2, 3]$: relative to the period $[1, 2]$, we put $N_0 = 1$ and $N_1 = 2$; relative to the period $[2, 3]$, we put $N_0 = 2$ and $N_1 = 3$. In any case we shall denote the price of the Swap Rate by $swr_{0,1}(n)$. Recall that by formula (4.80) one has

$$swr_{0,1}(n) = \frac{p(n, N_0) - p(n, N_1)}{p(n, N_1)}, \qquad n \leq N_0; \tag{4.102}$$

ii) the Swap Rate relative to the periods $[1, 2]$ and $[2, 3]$ for which we put $N_0 = 1$, $N_1 = 2$ and $N_2 = 3$. We denote this Swap Rate by $swr_{0,2}(n)$ and by formula (4.80) one has

$$swr_{0,2}(n) = \frac{p(n, 1) - p(n, 3)}{p(n, 2) + p(n, 3)}, \qquad n = 0, 1. \tag{4.103}$$

Problem 4.53 (Model 1). Consider Model 1 with the values of the initial term structure given in (4.88). Determine the Swap Rate $swr_{0,1}(n)$, $n = 0, 1$, relative to the single period $[N_0, N_1] = [1, 2]$ and the corresponding martingale measure $Q^{0,1}$ that turns $swr_{0,1}(\cdot)$ into a martingale.

Solution of Problem 4.53

Recall that Model 1 results from a discretization of the continuous time Hull-White model. Recall also that, in continuous time, a measure change implies a translation of the Wiener processes and it is thus reasonable to assume that, while under Q one has $\xi_1 \sim \mathcal{N}(0, 1)$, under $Q^{0,1}$ the random variable ξ_1 has distribution $\mathcal{N}(m, 1)$: determining $Q^{0,1}$ becomes then equivalent to determining the constant m. Considering the maturities $N_0 = 1$ and $N_1 = 2$, by formula (4.80) we have

$$\begin{aligned} swr_{0,1}(0) &= \frac{p(0,1) - p(0,2)}{p(0,2)} = \frac{p(0,1)}{p(0,2)} - 1, \\ swr_{0,1}(1) &= \frac{p(1,1) - p(1,2)}{p(1,2)} = \frac{1}{p(1,2)} - 1 = p(0,2)^{-\frac{1}{2}} e^{\frac{3}{4}} e^{\xi} - 1, \end{aligned}$$

and so, using the moment generating function of a normal random variable, the martingale condition becomes

$$\frac{p(0,1)}{p(0,2)} = p(0,2)^{-\frac{1}{2}} e^{\frac{3}{4}} E^{Q^{0,1}}[e^{\xi}] = p(0,2)^{-\frac{1}{2}} e^{\frac{3}{4}} e^{m+\frac{1}{2}},$$

from which

$$m = \log p(0,1) - \frac{1}{2} \log p(0,2) - \frac{5}{4}.$$

Finally, with the choice of the initial term structure as in (4.88)) one obtains

$$m = \frac{1}{4} - \frac{5}{4} = -1. \qquad \square$$

Problem 4.54 (Model 2). Consider Model 2 with $H = 2$, $p(0,1) = \frac{4}{5}, p(0,2) = \frac{3}{5}$ and $p(0,3) = \frac{2}{5}$. Determine the Swap Rate $swr_{0,1}$ relative to the period $[2,3]$ and the measure that turns $swr_{0,1}$ into a martingale.

Solution of Problem 4.54
By (4.102) and (4.96) we have

$$\begin{aligned}
swr_{0,1}(0) &= \frac{p(0,2)}{p(0,3)} - 1 = \frac{1}{2}, \\
swr_{0,1}(1) &= \frac{p(1,2)}{p(1,3)} - 1 = \frac{p(0,2)}{p(0,3)} \frac{2H+1}{3\xi_1} - 1 = (swr_{0,1}(0)+1)\frac{5}{3\xi_1} - 1, \\
swr_{0,1}(2) &= \frac{1}{p(2,3)} - 1 = \frac{p(0,2)}{p(0,3)} \frac{(H+1)(2H+1)}{6\xi_1\xi_2} - 1 \\
&= (swr_{0,1}(1)+1)\frac{H+1}{2\xi_2} - 1 = (swr_{0,1}(0)+1)\frac{5}{2\xi_1\xi_2} - 1.
\end{aligned}$$

Denote by $Q^{0,1}$ the measure that martingalizes $swr_{0,1}$ and by $E^{0,1}$ the corresponding expectation. Since $H = 2$ and $\xi_n \in \{1,2\}$, put

$$q_1 = Q^{0,1}(\xi_1 = 1), \qquad q_2 = Q^{0,1}(\xi_1 = 2 \mid \mathcal{F}_1).$$

The martingale condition implies for q_1 the following relation

$$\begin{aligned}
swr_{0,1}(0) &= E^{0,1}[swr_{0,1}(1)] = E^{0,1}\left[(swr_{0,1}(0)+1)\frac{5}{3\xi_1} - 1\right] \\
&= q_1\left(swr_{0,1}(0)\frac{5}{3} + \frac{5}{3} - 1\right) + (1-q_1)\left(swr_{0,1}(0)\frac{5}{6} + \frac{5}{6} - 1\right) \\
&= swr_{0,1}(0)\left(\frac{5}{6} + \frac{5}{6}q_1\right) + \left(\frac{5}{6} + \frac{5}{6}q_1\right) - 1,
\end{aligned}$$

which becomes $\frac{5}{6} + \frac{5}{6}q_1 = 1$ and thus

$$q_1 = \frac{1}{5}.$$

For q_2 the condition becomes instead

$$\begin{aligned}
swr_{0,1}(1) &= E^{0,1}\left[swr_{0,1}(2) \mid \mathcal{F}_1\right] = E^{0,1}\left[(swr_{0,1}(1)+1)\frac{3}{2\xi_2} - 1 \mid \mathcal{F}_1\right] \\
&= q_2\left(swr_{0,1}(1)\frac{3}{2} + \frac{3}{2} - 1\right) + (1-q_2)\left(swr_{0,1}(1)\frac{3}{4} + \frac{3}{4} - 1\right) \\
&= swr_{0,1}(1)\left(\frac{3}{4} + \frac{3}{4}q_2\right) + \left(\frac{3}{4} + \frac{3}{4}q_2\right) - 1
\end{aligned}$$

namely $\frac{3}{4} + \frac{3}{4}q_2 = 1$ and thus $q_2 = \frac{1}{3}$ independently of $\mathcal{F}_1$. It follows that ξ_1 and ξ_2 are independent also under $Q^{0,1}$. □

Problem 4.55 (Model 2). In the context of Problem 4.54, determine the Swap Rate $swr_{0,2}$ over the two periods $[1,2]$ and $[2,3]$ as well as the corresponding martingale measure $Q^{0,2}$.

Solution of Problem 4.55

We need to determine $swr_{0,2}(n)$ for $n = 0, 1$: by (4.103) we have

$$\begin{aligned}
swr_{0,2}(0) &= \frac{p(0,1) - p(0,3)}{p(0,2) + p(0,3)} = \frac{\frac{4}{5} - \frac{2}{5}}{\frac{3}{5} - \frac{2}{5}} = \frac{2}{5}, \\
swr_{0,2}(1) &= \frac{p(1,1) - p(1,3)}{p(1,2) + p(1,3)} = \frac{p(0,1) - p(0,3)\frac{2}{5}\xi_1^2}{p(0,2)\frac{2}{3}\xi_1 + p(0,3)\frac{2}{5}\xi_1^2}.
\end{aligned}$$

For what concerns the martingale measure, put $q = Q^{0,2}(\xi_1 = 1)$. The martingale condition becomes

$$swr_{0,2}(0) = E^{0,2}\left[swr_{0,2}(1)\right],$$

which implies

$$\begin{aligned}
\frac{2}{5} = E^{0,2}\left[swr_{0,2}(1)\right] &= q\frac{p(0,1) - \frac{2}{5}p(0,3)}{\frac{2}{3}p(0,2) + \frac{2}{5}p(0,3)} + (1-q)\frac{p(0,1) - \frac{8}{5}p(0,3)}{\frac{4}{3}p(0,2) + \frac{8}{5}p(0,3)} \\
&= q\frac{12 - \frac{12}{5}}{6 + \frac{12}{5}} + (1-q)\frac{12 - \frac{48}{5}}{12 + \frac{48}{5}},
\end{aligned}$$

from which we obtain $q = \frac{91}{325}$. □

(Payer) Forward Swaps

As in the case of the Swap Rates, for simplicity we consider only three periods ($n = 0, 1, 2, 3$), $K = \alpha_j = 1$ and two types of Forward Swaps:

i) the Forward Swap relative to the period $[2,3]$ for which we put $N_0 = 2$ and $N_1 = 3$. Denote this Forward Swap by $PFS_{0,1}$. Recall that by formula (4.72), one has

$$PFS_{0,1}(n) = p(n,2) - 2p(n,3), \qquad n = 0, 1, 2; \tag{4.104}$$

ii) the Forward Swap relative to the periods $[1,2]$ and $[2,3]$ for which we put $N_0 = 1$, $N_1 = 2$ and $N_2 = 3$. Denote this Forward Swap by $PFS_{0,2}(n)$ and by formula (4.72) we have

$$PFS_{0,2}(n) = p(n,1) - p(n,2) - 2p(n,3), \qquad n = 0,1. \tag{4.105}$$

Problem 4.56 (Model 1). Consider Model 1 with the initial values of the term structure as in (4.88). Determine the initial price of the Forward Swap $PFS_{0,1}(0)$ over a single period.

Solution of Problem 4.56

i) *(Direct method)* By (4.104) we have

$$PFS_{0,1}(0) = p(0,2) - 2p(0,3) = \frac{1}{4}\left(1 - e^{\frac{1}{4}}\right).$$

ii) *(Recursive method)* Recall that ξ_1, ξ_2 are i.i.d. standard normals , $\xi_1, \xi_2 \sim \mathcal{N}(0,1)$. Then we have, analogously to what was made for the Caplets in Proposition 4.23,

$$\begin{aligned}
PFS_{0,1}(2) &= 1 - 2p(2,3),\\
PFS_{0,1}(1) &= p(1,2)E^Q[PFS_{0,1}(2) \mid \xi_1],\\
PFS_{0,1}(0) &= p(0,1)E^Q[p(1,2)PFS_{0,1}(2)]\\
&= p(0,1)E^Q\left[p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi_1}\left(1 - 2p(0,3)^{\frac{1}{3}}e^{-\frac{11}{6}}e^{-(\xi_1+\xi_2)}\right)\right]\\
&= p(0,1)p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}\Big(E^Q\left[e^{-\xi_1}\right]\\
&\quad - 2p(0,3)^{\frac{1}{3}}e^{-\frac{11}{6}}E^Q\left[e^{-2\xi_1}\right]E^Q\left[e^{-\xi_2}\right]\Big)\\
&= \frac{1}{2}e^{\frac{1}{4}}\frac{1}{2}e^{-\frac{3}{4}}\left(e^{\frac{1}{2}} - e^{\frac{1}{12}}e^{-\frac{11}{6}}e^{2}e^{\frac{1}{2}}\right) = \frac{1}{4}\left(1 - e^{\frac{1}{4}}\right).
\end{aligned}$$

□

Problem 4.57 (Model 1). In the context of Problem 4.56, determine the initial price of the Forward Swap $PFS_{0,2}(0)$ over two periods.

Solution of Problem 4.57

i) *(Direct method)* By (4.105) one has

$$\begin{aligned}
PFS_{0,2}(0) =& p(0,1) - p(0,2) - 2p(0,3)\\
=& \frac{1}{2}e^{\frac{1}{4}} - \frac{1}{4} + \frac{1}{4}e^{\frac{1}{4}} = \frac{1}{4}\left(e^{\frac{1}{4}} - 1\right).
\end{aligned}$$

ii) *(Recursive method)* Recalling that $\xi_1 \sim \mathcal{N}(0,1)$ and using again (4.105) one has

$$\begin{aligned}
PFS_{0,2}(1) &= 1 - p(1,2) - 2p(1,3), \\
PFS_{0,2}(0) &= p(0,1)E^Q\left[PFS_{0,2}(1)\right] = p(0,1)E^Q\left[1 - p(1,2) - 2p(1,3)\right] \\
&= p(0,1)\left(1 - p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}E^Q\left[e^{-\xi_1}\right] - 2p(0,3)^{\frac{2}{3}}e^{-\frac{13}{6}}E^Q\left[e^{-2\xi_1}\right]\right) \\
&= p(0,1)\left(1 - p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{\frac{1}{2}} - 2p(0,3)^{\frac{2}{3}}e^{-\frac{13}{6}}e^{2}\right) \\
&= \frac{1}{2}e^{\frac{1}{4}}\left(1 - \frac{1}{2}e^{-\frac{1}{4}} - \frac{1}{2}\right) = \frac{1}{4}\left(e^{\frac{1}{4}} - 1\right).
\end{aligned}$$

□

Problem 4.58 (Model 2). Consider Model 2 with $H = 2$ and $p(0,1) = \frac{4}{5}, p(0,2) = \frac{3}{5}$ and $p(0,3) = \frac{2}{5}$. Determine the initial price of the Forward Swap $PFS_{0,1}(0)$ over a single period.

Solution of Problem 4.58
i) *(Direct method)* By (4.104) one has

$$PFS_{0,1}(0) = p(0,2) - 2p(0,3) = -\frac{1}{5}.$$

ii) *(Recursive method)* Assuming $q_h = \frac{1}{H}$, one has

$$\begin{aligned}
PFS_{0,1}(2) &= 1 - 2p(2,3), \\
PFS_{0,1}(1) &= p(1,2)E^Q\left[PFS_{0,1}(2) \mid \xi_1\right], \\
PFS_{0,1}(0) &= p(0,1)E^Q\left[p(1,2)PFS_{0,1}(2)\right], \\
&= p(0,1)E^Q\left[\frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H+1}\left(1 - 2\frac{p(0,3)}{p(0,2)}\frac{6\xi_1\xi_2}{(H+1)(2H+1)}\right)\right] \\
&= E^Q\left[\frac{2p(0,2)}{H+1}\xi_1 - p(0,3)\frac{24\xi_1^2\xi_2}{(H+1)^2(2H+1)}\right] \\
&= \frac{1}{H^2(H+1)}\sum_{h_1,h_2=1}^{H} h_1\left(2Hp(0,2) - p(0,3)\frac{24h_1^2h_2}{(H+1)(2H+1)}\right) \\
&= \frac{1}{H^2(H+1)}\sum_{h_1=1}^{H}\left(2h_1Hp(0,2) - p(0,3)\frac{12h_1^2H}{2H+1}\right) \\
&= \frac{1}{H^2(H+1)}\left(p(0,2)H^2(H+1) - 2p(0,3)H^2(H+1)\right) \\
&= p(0,2) - 2p(0,3) = -\frac{1}{5}.
\end{aligned}$$

□

Problem 4.59 (Model 2). In the context of Problem 4.58, determine the initial price of the Forward Swap $PFS_{0,2}(0)$ over two periods.

Solution of Problem 4.59
i) *(Direct method)* By (4.105) one has

$$PFS_{0,2}(0) = p(0,1) - p(0,2) - 2p(0,3) = \frac{4}{5} - \frac{3}{5} - \frac{4}{5} = -\frac{3}{5}.$$

ii) *(Recursive method)* Assuming $q_h = \frac{1}{H}$, we have

$$\begin{aligned}
PFS_{0,2}(1) &= 1 - p(1,2) - 2p(1,3), \\
PFS_{0,2}(0) &= p(0,1)E^Q\left[PFS_{0,2}(1)\right] \\
&= E^Q\left[p(0,1) - \frac{2p(0,2)}{H+1}\xi_1 - 12p(0,3)\frac{\xi_1^2}{(H+1)(2H+1)}\right] \\
&= p(0,1) - \frac{2p(0,2)}{H+1}\frac{1}{H}\sum_{h=1}^{H} h \\
&\quad - 12p(0,3)\frac{1}{H(H+1)(2H+1)}\sum_{h=1}^{H} h^2 \\
&= p(0,1) - p(0,2) - 2p(0,3) = -\frac{3}{5}.
\end{aligned}$$

□

4.6.5 Swaptions

As was done previously, consider also here just three periods ($n = 0,1,2,3$), $K = \alpha_j = 1$ and two types of Swaptions:

i) the Swaption relative to a single period that may be $[1,2]$ or $[2,3]$. Coherently with the notations in the previous problems, in the case of a Swaption over the period $[1,2]$, put $N_0 = 1$, $N_1 = 2$; in the case of the Swaption over the period $[2,3]$, put $N_0 = 2$, $N_1 = 3$. In any case denote the price of the Swaption by $Swaption_{0,1}(n)$ for $n \le N_0$.
For example, in the case of the period $[2,3]$, we recall that by Proposition 4.32 one has

$$Swaption_{0,1}(2) = (1 - 2p(2,3))^+, \tag{4.106}$$

and, for $n = 0,1$,

$$Swaption_{0,1}(n) = p(n,n+1)E^Q\left[Swaption_{0,1}(n+1) \mid \mathcal{F}_n\right]. \tag{4.107}$$

Recall also Remark 4.33 according to which, in the case of a single period, the Swaption reduces to a Cap which, in turn, reduces to a Put with underlying the T-bond; this may be seen directly also from the expression of the payoff (4.106);

ii) the Swaption relative to the periods $[1,2]$ and $[2,3]$ for which we put $N_0 = 1$, $N_1 = 2$ and $N_2 = 3$. Denote the price of this Swaption by $Swaption_{0,2}(n)$ for $n = 0,1$, and by Proposition 4.32, one has

$$Swaption_{0,2}(1) = (1 - p(1,2) - 2p(1,3))^+, \tag{4.108}$$

$$Swaption_{0,2}(0) = p(0,1)E^Q\left[Swaption_{0,2}(1)\right]. \tag{4.109}$$

The price of a Swaption may be computed as expected value under the martingale measure Q, but also as expected value under the martingale measure $Q^{0,J}$ with numeraire

$$C_{0,J}(n) := \sum_{j=1}^{J} p(n, N_j), \qquad J = 1,2.$$

In fact, based on formula (4.82), one has for a generic K the following: in the case of a single period $[N_0, N_1] = [2,3]$,

$$Swaption_{0,1}(0) = p(0,3)E^{0,1}\left[(swr_{0,1}(2) - K)^+\right], \tag{4.110}$$

and in the case of two periods,

$$Swaption_{0,2}(0) = (p(0,2) + p(0,3))\, E^{0,2}\left[(swr_{0,2}(1) - K)^+\right], \tag{4.111}$$

where $swr_{0,J}(\cdot)$, for $J = 1,2$, is the Swap Rate defined in (4.102) and (4.103) for $N_0 = 1$, $N_2 = 3$.

Problem 4.60 (Model 1). Consider Model 1 with the initial values of the term structure as given in (4.88). Determine the initial price $Swaption_{0,1}(0)$ of the Swaption over the single period $[N_0, N_1] = [1,2]$, by computing it as expected value under the measure Q.

Solution of Problem 4.60

Since the Swaption reduces to a Put with underlying a T-bond, the calculations are analogous to those of Problem 4.38 (the only difference is the factor $2 = K + 1$ which multiplies $p(1,2)$). One has

$$\begin{aligned} Swaption_{0,1}(1) &= (1 - 2p(1,2))^+, \\ Swaption_{0,1}(0) &= p(0,1)E^Q\left[\left(1 - 2p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi_1}\right)^+\right] \\ &= p(0,1)\int_D \left(1 - 2p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-x}\right)\mathcal{N}_{0,1}(dx), \end{aligned}$$

where

$$D = \left\{x \mid 1 - 2p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-x} > 0\right\} = \{x \mid x > \gamma\},$$

with $\gamma = \log 2 + \frac{1}{2}\log p(0,2) - \frac{3}{4}$. Thus, using (4.89), one obtains

$$\begin{aligned}
Swaption_{0,1}(0) &= p(0,1)\Bigg(\int_{\{x>\gamma\}} \mathcal{N}_{0,1}(dx) \\
&\qquad - 2p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} e^{\frac{1}{2}} \int_{\{y>\gamma+1\}} \mathcal{N}_{0,1}(dy) \Bigg) \\
&= p(0,1)\left(\Phi(-\gamma) - 2p(0,2)^{\frac{1}{2}} e^{-\frac{1}{4}} \Phi(-\gamma-1) \right) \\
&= p(0,1)\Bigg(\Phi\left(-\log 2 - \frac{1}{2}\log p(0,2) + \frac{3}{4} \right) \\
&\qquad - 2p(0,2)^{\frac{1}{2}} e^{-\frac{1}{4}} \Phi\left(-\log 2 - \frac{1}{2}\log p(0,2) - \frac{1}{4} \right) \Bigg).
\end{aligned}$$

With the values (cf. (4.88)) $p(0,1) = \frac{1}{2}e^{\frac{1}{4}}$ and $p(0,2) = \frac{1}{4}$ one then obtains

$$\begin{aligned}
Swaption_{0,1}(0) &= \frac{1}{2} e^{\frac{1}{4}} \left(\Phi\left(\frac{3}{4}\right) - e^{-\frac{1}{4}} \Phi\left(-\frac{1}{4}\right) \right) \\
&= \frac{1}{2}\left(e^{\frac{1}{4}} \Phi\left(\frac{3}{4}\right) - \Phi\left(-\frac{1}{4}\right) \right).
\end{aligned}$$

□

Problem 4.61 (Model 1). In the same context of Problem 4.60, determine the price $Swaption_{0,1}(0)$ but with the calculations made under the measure $Q^{0,1}$ which martingalizes the corresponding Swap Rate $swr_{0,1}(\cdot)$.

Solution of Problem 4.61

From Problem 4.53 we have that, under the measure $Q^{0,1}$, the random variable ξ_1 has distribution $\mathcal{N}_{m,1}$ with $m = -1$. Being in this case $C(0) = p(0,2)$, we obtain from (4.82) (recall that we consider here $K = 1$)

$$\begin{aligned}
Swaption_{0,1}(0) &= C(0) E^{Q^{0,1}}\left[(swr_{0,1}(1) - K)^+ \right] \\
&= p(0,2) E^{Q^{0,1}}\left[\left(\frac{1}{p(1,2)} - 2 \right)^+ \right] \\
&= p(0,2) \int_D \left(p(0,2)^{-\frac{1}{2}} e^{\frac{3}{4}} e^{\xi} - 2 \right) \mathcal{N}_{m,1}(d\xi),
\end{aligned}$$

where

$$D = \left\{ \xi \mid e^{\xi} > 2p(0,2)^{\frac{1}{2}} e^{-\frac{3}{4}} \right\} = \{ \xi \mid \xi > \gamma \}$$

with $\gamma = \log 2 + \frac{1}{2}\log p(0,2) - \frac{3}{4}$. It then follows (see also (4.90))

$$
\begin{aligned}
&Swaption_{0,1}(0) = \\
&= p(0,2)\left(p(0,2)^{-\frac{1}{2}}e^{\frac{3}{4}}\int_{\{\xi>\gamma\}} e^{\xi}\mathcal{N}_{m,1}(d\xi) - 2\int_{\{\xi>\gamma\}}\mathcal{N}_{m,1}(d\xi)\right) \\
&= p(0,2)\left(p(0,2)^{-\frac{1}{2}}e^{\frac{3}{4}}e^{m+\frac{1}{2}}\int_{\{\xi>\gamma\}}\mathcal{N}_{m+1,1}(d\xi) - 2\int_{\{\xi>\gamma\}}\mathcal{N}_{m,1}(d\xi)\right) \\
&= p(0,2)\left(p(0,2)^{-\frac{1}{2}}e^{m+\frac{5}{4}}\int_{\{y>\gamma-m-1\}}\mathcal{N}_{0,1}(dy) - 2\int_{\{z>\gamma-m\}}\mathcal{N}_{0,1}(dz)\right).
\end{aligned}
$$

With the value of $m = -1$ and $p(0,1) = \frac{1}{2}e^{\frac{1}{4}}, p(0,2) = \frac{1}{4}$, we obtain finally

$$
Swaption_{0,1}(0) = \frac{1}{2}e^{\frac{1}{4}}\Phi(-\gamma) - \frac{1}{2}\Phi(-\gamma-1) = \frac{1}{2}\left(e^{\frac{1}{4}}\Phi\left(\frac{3}{4}\right) - \Phi\left(-\frac{1}{4}\right)\right),
$$

as in Problem 4.60. □

Problem 4.62 (Model 1). Determine the initial price $Swaption_{0,2}(0)$ of the Swaption over the two periods $[1,2]$ and $[2,3]$, computing it by means of formulae (4.108)-(4.109) as expected value under the measure Q.

Solution of Problem 4.62
Using also (4.89) and (4.91) we have

$$
\begin{aligned}
&Swaption_{0,2}(1) = (1 - p(1,2) - 2p(1,3))^+, \\
&Swaption_{0,2}(0) = p(0,1)E^Q\left[(1 - p(1,2) - 2p(1,3))^+\right] \\
&\quad = p(0,1)E^Q\left[\left(1 - 2p(0,2)^{\frac{1}{2}}e^{-\frac{3}{4}}e^{-\xi_1} - 2p(0,3)^{\frac{2}{3}}e^{-\frac{13}{6}}e^{-2\xi_1}\right)^+\right] \\
&\quad := p(0,1)E^Q\left[\left(1 - Ae^{-\xi_1} - Be^{-2\xi_1}\right)^+\right] \\
&\quad = p(0,1)\left(\int_D \mathcal{N}_{0,1}(d\xi) - Ae^{\frac{1}{2}}\int_D \mathcal{N}_{-1,1}(d\xi) - Be^2\int_D \mathcal{N}_{-2,1}(d\xi)\right),
\end{aligned}
$$

where A and B are positive constants and, putting $y = e^{-\xi}$,

$$
\begin{aligned}
D &= \left\{\xi \mid 1 - Ae^{-\xi} - Be^{-2\xi} > 0\right] \\
&= \{y \mid 1 - Ay - By^2 > 0\} \\
&= \{y \mid y_1 \le y \le y_2\},
\end{aligned}
$$

with y_1, y_2 the two solutions of $1 - Ay - By^2 = 0$ of which $y_1 < 0$ and $y_2 > 0$.
Since y has to be positive, one finds

$$
D = \{\xi \mid \xi > -\log y_2\} = \{\xi \mid \xi > \gamma\}
$$

with $\gamma = -\log y_2$. It then follows that

$$\begin{aligned}
Swaption_{0,2}(0) &= p(0,1)\Bigg(\int_{\{\xi>\gamma\}} \mathcal{N}_{0,1}(d\xi) \\
&\quad - Ae^{\frac{1}{2}}\int_{\{y>\gamma+1\}} \mathcal{N}_{0,1}(dy) - Be^2 \int_{\{z>\gamma+2\}} \mathcal{N}_{0,1}(dz)\Bigg) \\
&= p(0,1)\left(\Phi(-\gamma) - Ae^{\frac{1}{2}}\Phi(-\gamma-1) - Be^2\Phi(-\gamma-2)\right).
\end{aligned}$$

Assuming the initial term structure as in in (4.88), we finally obtain

$$\begin{aligned}
&Swaption_{0,2}(0) = \\
&= \frac{1}{2}e^{\frac{1}{4}}\left(\Phi(\log y_2) - \frac{1}{2}e^{-\frac{3}{4}}e^{\frac{1}{2}}\Phi(\log y_2 - 1) - \frac{1}{2}e^{\frac{1}{6}}e^{-\frac{13}{6}}e^2\Phi(\log y_2 - 2)\right) \\
&= \frac{1}{2}e^{\frac{1}{4}}\Phi(\log y_2) - \frac{1}{4}\Phi(\log y_2 - 1) - \frac{1}{4}e^{\frac{1}{4}}\Phi(\log y_2 - 2).
\end{aligned}$$

Notice that, alternatively, the computation may also be performed via an approximation by means of a discretization of the Normal random variable.□

Problem 4.63 (Model 2). In Model 2 with $q_h = \frac{1}{H}$ and $H = 2$, determine the initial price $Swaption_{0,1}(0)$ of the Swaption over the period $[2,3]$ computing it as expected value under the measure Q according to formulae (4.106)-(4.107).

Solution of Problem 4.63
One has

$$\begin{aligned}
Swaption_{0,1}(2) &= \left(1 - 2\frac{p(0,3)}{p(0,2)}\frac{6\xi_1\xi_2}{(H+1)(2H+1)}\right)^+, \\
Swaption_{0,1}(1) &= p(1,2)E^Q\left[Swaption_{0,1}(2) \mid \xi_1\right] \\
&= p(1,2)\frac{1}{H}\sum_{h_2=1}^{H}\left(1 - 2\frac{p(0,3)}{p(0,2)}\frac{6\xi_1 h_2}{(H+1)(2H+1)}\right)^+ \\
&= \frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H(H+1)}\sum_{h_2=1}^{H}\left(1 - 2\frac{p(0,3)}{p(0,2)}\frac{6\xi_1 h_2}{(H+1)(2H+1)}\right)^+, \\
Swaption_{0,1}(0) &= p(0,1)E^Q\left[Swaption_{0,1}(1)\right] \\
&= p(0,1)\frac{2}{H^2(H+1)}\frac{p(0,2)}{p(0,1)}\sum_{h_1,h_2=1}^{H} h_1\left(1 - 2\frac{p(0,3)}{p(0,2)}\frac{6h_1h_2}{(H+1)(2H+1)}\right)^+ \\
&= \frac{1}{H^2(H+1)}\sum_{h_1,h_2=1}^{H} h_1\left(2p(0,2) - p(0,3)\frac{24h_1h_2}{(H+1)(2H+1)}\right)^+,
\end{aligned}$$

which, in the case of $H = 2$, becomes

$$\begin{aligned} Swaption(0) &= \frac{1}{12}\left(\left(\frac{6}{5} - \frac{2}{5}\cdot\frac{24}{15}\right)^+ + \left(\frac{6}{5} - \frac{2}{5}\cdot\frac{48}{15}\right)^+\right. \\ &\quad \left. + 2\left(\frac{6}{5} - \frac{2}{5}\cdot\frac{48}{15}\right)^+ + 2\left(\frac{6}{5} - \frac{2}{5}\cdot\frac{96}{15}\right)^+\right) \\ &= \frac{1}{12}\left(\frac{90-48}{15\cdot 5}\right)^+ = \frac{1}{12}\cdot\frac{42}{15\cdot 5} = \frac{7}{150}. \end{aligned}$$

□

Problem 4.64 (Model 2). Consider Model 2 with $p(0,1) = \frac{4}{5}, p(0,2) = \frac{3}{5}, p(0,3) = \frac{2}{5}$ and a Swaption over the period $[2,3]$:

i) determine the initial price $Swaption_{0,1}(0)$ as expected value under the measure Q in the case of $H = 2$ and $q = Q(\xi_n = 1) \in]0,1[$. Verify the correctness of the result by comparing it with that of Problem 4.63 for $q = \frac{1}{2}$;
ii) verify that, for $q \in \left]\frac{4}{9}, \frac{8}{9}\right[$, one has

$$Swaption_{0,1}(0) = \frac{3}{5}\frac{q^2}{2-q}\left(1 - \frac{4}{3}\frac{1}{4-3q}\right);$$

iii) assuming $q \in \left]\frac{4}{9}, \frac{8}{9}\right[$, does there exist a value of q, for which $Swaption_{0,1}(0) = \frac{7}{150}$? Does this value $Swaption_{0,1}(0)$ suffice to fully calibrate the model to the market?

Solution of Problem 4.64

i) Based on (4.106), (4.107) and the expressions for $p(1,2)$ and $p(2,3)$ as computed in Problem 4.45-b), we have

$$Swaption_{0,1}(2) = (1 - 2p(2,3))^+ = \left(1 - \frac{4}{3}\frac{\xi_1\xi_2}{4-3q}\right)^+$$

$$\begin{aligned} Swaption_{0,1}(1) &= p(1,2)E^Q\left[Swaption_{0,1}(2) \mid \xi_1\right] \\ &= \frac{3}{4}\frac{\xi_1}{2-q}\left(q\left(1 - \frac{4}{3}\frac{\xi_1}{4-3q}\right)^+ + (1-q)\left(1 - \frac{8}{3}\frac{\xi_1}{4-3q}\right)^+\right) \end{aligned}$$

$$\begin{aligned} Swaption_{0,1}(0) &= p(0,1)E^Q\left[Swaption_{0,1}(1)\right] \\ &= \frac{4}{5}\left(\frac{3}{4}\frac{q}{2-q}\left(q\left(1 - \frac{4}{3}\frac{1}{4-3q}\right)^+ + (1-q)\left(1 - \frac{8}{3}\frac{1}{4-3q}\right)^+\right)\right. \\ &\quad \left. + \frac{3}{4}\frac{2(1-q)}{2-q}\left(q\left(1 - \frac{4}{3}\frac{2}{4-3q}\right)^+ + (1-q)\left(1 - \frac{8}{3}\frac{2}{4-3q}\right)^+\right)\right) \end{aligned}$$

$$= \frac{3}{5}\frac{1}{2-q}\left(q^2\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ + q(1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+\right.$$
$$\left. + 2q(1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+ + 2(1-q)^2\left(1-\frac{16}{3}\frac{1}{4-3q}\right)^+\right)$$
$$= \frac{3}{5}\frac{1}{2-q}\left(q^2\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ + 3q(1-q)\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+\right)$$

being $\left(1-\frac{16}{3}\frac{1}{4-3q}\right)^+ = 0$ for each $q \in [0,1]$ (see also Problem 4.45-b)).

To verify the result we compute $Swaption_{0,1}(0)$ for $q = \frac{1}{2}$. One has

$$Swaption_{0,1}(0) = \frac{2}{5}\left(\frac{1}{4}\left(1-\frac{8}{15}\right)^+ + \frac{3}{4}\left(1-\frac{16}{15}\right)^+\right) = \frac{1}{10}\cdot\frac{7}{15} = \frac{7}{150}$$

which coincides with the result obtained in Problem 4.63.

ii) Based on the result in a) it suffices to verify that, for $q \in \left]\frac{4}{9}, \frac{8}{9}\right[$ on one hand one has $\left(1-\frac{4}{3}\frac{1}{4-3q}\right)^+ = \left(1-\frac{4}{3}\frac{1}{4-3q}\right)$ and, on the other hand, that $\left(1-\frac{8}{3}\frac{1}{4-3q}\right)^+ = 0$. Given $q \in]0,1[$, the first statement is true if $\frac{4}{3}\frac{1}{4-3q} < 1$ and this inequality leads to $0 < q < \frac{8}{9}$. The second statement is true if $\frac{8}{3}\frac{1}{4-3q} > 1$ and this inequality leads in fact to $\frac{4}{9} < q < 1$.

iii) On the basis of the verification in point a), we know that $q = \frac{1}{2} \in \left]\frac{4}{9}, \frac{8}{9}\right[$ is a possible value. For what has been seen in point b), given that we consider $q \in \left]\frac{4}{9}, \frac{8}{9}\right[$, the possible values of q have to satisfy the equation

$$\frac{3}{5}\frac{q^2}{2-q}\left(1-\frac{4}{3}\frac{1}{4-3q}\right) = \frac{7}{150},$$

which is equivalent to the third order equation

$$270q^3 - 219q^2 - 70q + 56 = 0.$$

Since we already know that $q = \frac{1}{2}$ is a solution, dividing the previous equation by $q - \frac{1}{2}$ one finds that the two possible remaining values of q have to satisfy the equation

$$135q^2 - 42q - 56 = 0.$$

The latter has two solutions

$$q^{(1)} = \frac{7-\sqrt{889}}{45}, \qquad q^{(2)} = \frac{7+\sqrt{889}}{45},$$

of which however only $q^{(2)}$ belongs to the interval $\left]\frac{4}{9}, \frac{8}{9}\right[$, being $q^{(1)} < 0$. It follows that, under the assumption $\left]\frac{4}{9}, \frac{8}{9}\right[$, the market value $Swaption_{0,1}(0) = \frac{7}{150}$ is not sufficient to uniquely calibrate the model. □

Problem 4.65 (Model 2). In the same context as Problem 4.63, determine the initial price $Swaption_{0,1}(0)$ of the Swaption over the period $[2,3]$ with the calculations made under the martingale measure $Q^{0,1}$ which martingalizes $swr_{0,1}$.

Solution of Problem 4.65
Putting

$$q^1 := Q^{0,1}(\xi_1 = 1), \qquad q_2 := Q^{0,1}(\xi_2 = 1 \mid \mathcal{F}_1),$$

we have seen in Problem 4.54 that $q_1 = \frac{1}{5}$, $q_2 = \frac{1}{3}$ and that ξ_1 and ξ_2 are independent under $Q^{0,1}$.

Using formula (4.110) compute first

$$\begin{aligned}(swr_{0,1}(2) - 1)^+ &= \left(\frac{1 - p(2,3)}{p(2,3)} - 1\right)^+ = \left(\frac{1}{p(2,3)} - 2\right)^+ \\ &= \left(\frac{p(0,2)}{p(0,3)} \frac{(H+1)(2H+1)}{6\xi_1\xi_2} - 2\right)^+,\end{aligned}$$

which, for $H = 2$ and the given initial term structure in (4.97), becomes

$$(swr_{0,1}(2) - 1)^+ = \left(\frac{15}{4\xi_1\xi_2} - 2\right)^+.$$

The price of the Swaption is then

$$\begin{aligned}Swaption_{0,1}(0) = p(0,3)\Bigg(&q_1 q_2 \left(\frac{15}{4} - 2\right)^+ + q_1(1-q_2)\left(\frac{15}{8} - 2\right)^+ \\ &+ (1-q_1)q_2\left(\frac{15}{8} - 2\right)^+ + (1-q_1)(1-q_2)\left(\frac{15}{16} - 2\right)^+\Bigg) \\ = &\frac{2}{5} q_1 q_2 \frac{15-8}{4} = \frac{7}{150}\end{aligned}$$

coherently with the result of Problem 4.63. □

Problem 4.66 (Model 2). The Swaption in the previous Problems 4.63 and 4.65 is being exercised if $swr_{0,1}(2) \geq 1$. Assuming that $q_h = \frac{1}{H}$ with $H = 5$ and that the initial term structure is the one assigned in (4.97), determine for which values of ξ_1, ξ_2 one has $swr_{0,1}(2) \geq 1$.

Solution of Problem 4.66
On the basis of Problem 4.54 we have

$$\begin{aligned}swr_{0,1}(2) &= (swr_{0,1}(0) + 1)\frac{(H+1)(2H+1)}{6\xi_1\xi_2} - 1 \\ &= \left(\frac{p(0,2)}{p(0,3)} - 1 + 1\right)\frac{(H+1)(2H+1)}{6\xi_1\xi_2} - 1 = \frac{33}{2\xi_1\xi_2} - 1.\end{aligned}$$

The condition is thus satisfied if

$$\frac{33}{2\xi_1\xi_2} \geq 2 \quad \text{namely} \quad \xi_1\xi_2 \leq \frac{33}{4}.$$

The following table for the different values of ξ_1 and $\xi_2 = j$ then results:

$$\begin{array}{lll} \text{for} \quad \xi_1 = 1 & \longrightarrow & \text{all} \;\; j \in \{1, \cdots, 5\}, \\ \text{for} \quad \xi_1 = 2 & \longrightarrow & j \in \{1, \cdots, 4\}, \\ \text{for} \quad \xi_1 \in \{3, 4\} & \longrightarrow & j \in \{1, 2\}, \\ \text{for} \quad \xi_1 = 4 & \longrightarrow & j = 1. \end{array}$$

□

Problem 4.67 (Model 2). Determine the initial price $Swaption_{0,2}(0)$ of the Swaption over the two periods $[1, 2]$ and $[2, 3]$, by computing it as expected value under the measure Q (see (4.108)-(4.109)) and putting $q_h = \frac{1}{H}$ for $H = 2$ and $H = 3$. The initial term structure is supposed to be the one in (4.97).

Solution of Problem 4.67
By (4.108) one has

$$\begin{aligned} &Swaption_{0,2}(1) = (1 - p(1, 2) - 2p(1, 3))^+ \\ &= \frac{1}{p(0, 1)} \left(p(0, 1) - p(0, 2) \frac{2\xi_1}{H + 1} - p(0, 3) \frac{12\xi_1^2}{(H + 1)(2H + 1)} \right)^+, \end{aligned}$$

and by (4.109)

$$\begin{aligned} &Swaption_{0,2}(0) = p(0, 1)E^Q \left[Swaption_{0,2}(1) \right] \\ &= \frac{1}{H} \sum_{h=1}^{H} \left(p(0, 1) - p(0, 2) \frac{2h}{H + 1} - p(0, 3) \frac{12h^2}{(H + 1)(2H + 1)} \right)^+ \end{aligned}$$

which, for $H = 2$, becomes

$$\begin{aligned} Swaption_{0,2}(0) = &\frac{1}{2} \left(\frac{4}{5} - \frac{3}{5} \cdot \frac{2}{3} - \frac{2}{5} \cdot \frac{4}{5} \right)^+ \\ &+ \frac{1}{2} \left(\frac{4}{5} - \frac{3}{5} \cdot \frac{4}{3} - \frac{2}{5} \cdot \frac{16}{5} \right)^+ = \frac{1}{25}, \end{aligned}$$

while, for $H = 3$, it becomes

$$\begin{aligned} Swaption_{0,2}(0) = &\frac{1}{3} \Bigg(\left(\frac{4}{5} - \frac{3}{5} \cdot \frac{1}{2} - \frac{2}{5} \cdot \frac{3}{7} \right)^+ + \left(\frac{4}{5} - \frac{3}{5} \cdot 1 - \frac{2}{5} \cdot \frac{12}{7} \right)^+ \\ &+ \left(\frac{4}{5} - \frac{3}{5} \cdot \frac{6}{4} - \frac{2}{5} \cdot \frac{3 \cdot 9}{7} \right)^+ \Bigg) = \frac{23}{210}. \end{aligned}$$

□

Problem 4.68 (Model 2). In the same context as Problem 4.67, determine the initial price $Swaption_{0,2}(0)$ but with the calculations performed under the measure $Q^{0,2}$ which martingalizes $swr_{0,2}$ and considering only $H = 2$.

Solution of Problem 4.68
Putting

$$q = Q^{0,2}\left(\xi_1 = 1\right), \qquad 1 - q = Q^{0,2}\left(\xi_1 = 2\right),$$

we have seen in Problem 4.55 that $q = \frac{91}{325}$.

Using (4.103), we have first of all

$$\begin{aligned}
&(swr_{0,2}(1) - 1)^+ = \left(\frac{1 - p(1,3)}{p(1,2) + p(1,3)} - 1\right)^+ \\
&= \frac{1}{p(1,2) + p(1,3)}(1 - p(1,2) - 2p(1,3))^+ \\
&= \frac{1}{\frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H+1} + \frac{p(0,3)}{p(0,1)}\frac{6\xi_1^2}{(H+1)(2H+1)}} \cdot \\
&\quad \cdot \left(1 - \frac{p(0,2)}{p(0,1)}\frac{2\xi_1}{H+1} - 2\frac{p(0,3)}{p(0,1)}\frac{6\xi_1^2}{(H+1)(2H+1)}\right)^+ \\
&= \frac{(H+1)(2H+1)}{p(0,2)(2H+1)2\xi_1 + p(0,3)6\xi_1^2} \cdot \\
&\quad \cdot \frac{\left((H+1)(2H+1)p(0,1) - p(0,2)(2H+1)2\xi_1 - p(0,3)12\xi_1^2\right)^+}{(H+1)(2H+1)} \\
&= \frac{(60 - 30\xi_1 - 24\xi_1^2)^+}{30\xi_1 + 12\xi_1^2},
\end{aligned}$$

and thus, on the basis of (4.111),

$$\begin{aligned}
Swaption_{0,2}(0) = E^{0,2}\left[\frac{(60 - 30\xi_1 - 24\xi_1^2)^+}{30\xi_1 + 12\xi_1^2}\right] &= \frac{91}{325}\frac{(60 - 54)^+}{42} \\
+ \left(1 - \frac{91}{325}\right)\frac{(60 - 156)^+}{108} &= \frac{1}{25},
\end{aligned}$$

as in Problem 4.67. □

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