PRACTICO 3

1

a) $X_0 = 2$, $X_1 = 2{,}5$, $X_2 = 4$

Newton (con dif divididas)

$$\begin{array}{c|ccc} X_0 & f[X_0] & f[X_0,X_1] & f[X_0,X_1,X_2] \\ X_1 & f[X_1] & f[X_1,X_2] & \\ X_2 & f[X_2] & & \end{array}$$

$\to$ esta fila son los c_0, c_1, c_2

$f[X_0] = \frac{1}{2}$

$f[X_1] = \frac{2}{5}$

$f[X_2] = \frac{1}{4}$

$$f[X_0,X_1] = \frac{f[X_1]-f[X_0]}{X_1-X_0} = \frac{\frac{2}{5}-\frac{1}{2}}{\frac{5}{2}-2} = \frac{\frac{4-5}{10}}{\frac{5-4}{2}} = -\frac{\frac{1}{10}}{\frac{1}{2}} = -\frac{1}{5}$$

$$f[X_1,X_2] = \frac{f[X_2]-f[X_1]}{X_2-X_1} = \frac{\frac{1}{4}-\frac{2}{5}}{4-\frac{5}{2}} = \frac{\frac{5-8}{20}}{\frac{8-5}{2}} = -\frac{3}{20} : \frac{3}{2} = -\frac{1}{10}$$

$$f[X_0,X_1,X_2] = \frac{f[X_1,X_2]-f[X_0,X_1]}{X_2-X_0} = \frac{-\frac{1}{10}+\frac{1}{5}}{4-2} = \frac{\frac{2}{10}-\frac{1}{10}}{2} = \frac{1}{20}$$

Entonces:

$$P_2(x) = \sum_{i=0}^{2} f[x_0, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

$$= c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$$

$$= \frac{1}{2} + \left(-\frac{1}{5}\right)(x-2) + \frac{1}{20}(x-2)\left(x-\frac{5}{2}\right)$$

Lagrange

$$f(x_0) = \frac{1}{2}$$

$$f(x_1) = \frac{2}{5}$$

$$f(x_2) = \frac{1}{4}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \left(\frac{x-x_j}{x_1-x_j}\right) = \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right) = \left(\frac{x-2}{\frac{5}{2}-2}\right)\left(\frac{x-4}{\frac{5}{2}-4}\right)$$

$$= \frac{8(x-2)(x-4)}{15}$$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \left(\frac{x-x_j}{x_0-x_j}\right) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right) = \left(\frac{x-\frac{5}{2}}{2-\frac{5}{2}}\right)\left(\frac{x-4}{2-4}\right)$$

$$= \left(x-\frac{5}{2}\right)(x-4)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \left(\frac{x-x_j}{x_2-x_j}\right) = \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right) = \left(\frac{x-2}{4-2}\right)\left(\frac{x-\frac{5}{2}}{4-\frac{5}{2}}\right)$$

$$= (x-2)\left(x-\frac{5}{2}\right)$$

Entonces

$$\overline{P}_2(X) = \frac{1}{2}\left(X-\frac{5}{2}\right)(X-4) + \frac{8}{15}(X-2)(X-4) + \frac{1}{12}(X-2)\left(X-\frac{5}{2}\right)$$

i) haciendo las multiplicaciones y sumas de los paréntesis queda que $P_2(X) = \overline{P}_2(X)$. $P, \overline{P}$ son de grado 2.

ii)

$$P_2(3) = \frac{-8(1)(1)}{15} + \frac{\frac{1}{2}(-1)}{2} + \frac{(1)\left(\frac{1}{2}\right)}{12} = \frac{8}{15} - \frac{1}{4} + \frac{1}{24} = \frac{64}{120} - \frac{30}{120} + \frac{5}{120}$$

$$= \frac{39}{120} = \frac{13}{40}$$

$$\overline{P_2(3)} = \frac{1}{2} + \left(-\frac{1}{5}\right) + \frac{1}{20} \cdot \frac{1}{2} = \frac{1}{2} - \frac{1}{5} + \frac{1}{40} = \frac{20-8+1}{40} = \frac{13}{40}$$

2

Sea g una función que interpola a f en $\underbrace{x_0, \dots, x_n}_{n+1 \text{ puntos}}$

Sea $h_n = f_n - g_n$. Entonces h_n es de grado $\leq n$. Además $h_n(X_i) = 0$ pues $f_n(X_i) = g_n(X_i)$. Por lo tanto hay $n+1$ puntos en los que $h_n(X) = 0$, osea $n+1$ raíces. Por teorema fundamental del álgebra, $h_n(X) = 0$, por lo tanto $\boxed{f_n = g_n}$.

3 Notar que $g(x) = f(x)$ $\forall x \in \{x_0, \dots, x_{n-1}\}$ y $h(x) = f(x)$ $\forall x \in \{x_1, \dots, x_n\}$

$X = X_0$:

$$P(X_0) = g(X_0) + \frac{X_0 - X_0}{X_n - X_0} \cdot (g(X_0) - h(X_0)) = f(X_0) + 0 \cdot (g(X_0) - h(X_0))$$

$= f(x_0) \implies p(x)$ interpola a f en x_0

$\underline{x_i \in \{x_1, \cdots, x_{n-1}\}}$

$$p(x_i) = g(x_i) + \frac{x_0 - x_i}{x_n - x_0}\left(\overset{f(x_i)}{\overset{\shortparallel}{g(x_i)}} - \overset{f(x_i)}{\overset{\shortparallel}{h(x_i)}}\right)$$

$$p(x_i) = f(x_i) + \frac{x_0 - x_i}{x_n - x_0} \cdot (0)$$

$p(x_i) = f(x_i) \implies p(x)$ interpola a $f(x)$ $\forall x \in \{x_1, \cdots, x_{n-1}\}$

$\underline{x = x_n}$

$$p(x_n) = g(x_n) + \frac{x_0 - x_n}{x_n - x_0}\left(g(x_n) - \overset{f(x_n)}{\overset{\shortparallel}{h(x_n)}}\right)$$

$$p(x_n) = g(x_n) - \frac{x_n - x_0}{x_n - x_0}\left(g(x_n) - f(x_n)\right)$$

$$p(x_n) = g(x_n) - g(x_n) + f(x_n) = f(x_n)$$

$\implies p(x)$ interpola a $f(x)$ en x_n

$\therefore p(x)$ interpola a $f(x)$ $\forall x \in \{x_0, \cdots, x_n\}$

4

a) Sea $p(x) = \sum_{k=0}^{n} L_k(x)$, es decir de grado $\leq n$. Sea $h(x) = p(x) - 1$.

Entonces $h(x)$ es de grado $\leq n$.

Recordando que $L_i(x_j) \begin{cases} 1 & si\ i=j \\ 0 & si\ i \neq j \end{cases}$, tenemos:

$$h(x_0) = \sum_{k=0}^{n} L_k(x_0) - 1 = \underbrace{L_0(x_0)}_{=1} + \underbrace{\cdots}_{=0} + \underbrace{L_n(x_0)}_{=0} - 1 = 1 - 1 = 0$$

$\vdots$

$$h(x_n) = \sum_{k=0}^{n} L_k(x_n) - 1 = \underbrace{L_0(x_n)}_{=0} + \underbrace{\cdots}_{=0} + \underbrace{L_n(x_n)}_{=1} - 1 = 1 - 1 = 0$$

Entonces $h(x_k) = 0 \ \forall k = 0, \cdots, n$, entonces tiene $n+1$ raices, pero es de grado $\leq n$, por teorema fundamental del algebra tenemos que $h(x) = 0$. Entonces $p(x) = 1$.

b)

Sea $p(x) = \sum_{k=0}^{n} x_k \cdot L_k(x)$, es decir que es de grado $\leq n$. Sea $h(x) = p(x) - x$, tenemos que $h(x)$ es de grado $\leq n$. Notemos:

$$h(x_0) = \sum_{k=0}^{n} x_k L_k(x_0) - x_0 = x_0 . L_0(x_0) + \cdots + x_k L_k(x_0) - x_0 = x_0 - x_0 = 0$$

$\vdots$

$$h(x_k) = \sum_{k=0}^{n} x_k \cdot L_k(x_k) - x_n = x_0 L_0(x_k) + \cdots + x_k L_k(x_k) - x_k = x_k - x_k = 0$$

Entonces $h(x_k)=0$ $\forall k=0,\dots,n$. Entonces $h(x)$ tiene $n+1$ raíces, pero es de grado $\leq n$. Por teorema fundamental del álgebra, $h(x)=0$, entonces $p(x)=x$

c)

Sea $p(x)=\sum_{k=0}^{n} x_k^m L_k(x)$, es de grado $\leq n$. Sea $h(x)=p(x)-x^m$, osea $h(x)$ es de grado $\leq n$. Notemos que:

$$h(x_0)=x_0^m L_0(x_0)+\cdots+x_k^m L_k(x_0)-x_0^m=x_0^m-x_0^m=0$$

$$\vdots$$

$$h(x_k)=x_0^m L_0(x_k)+\cdots+x_k^m L_k(x_k)-x_k^m=x_k^m-x_k^m=0$$

Entonces $h(x_k)=0$ $\forall k=0,1,\dots,n$. Entonces $h(x)$ tiene $n+1$ raíces, pero es de grado $\leq n$. Por lo tanto por teorema fundamental del álgebra $h(x)=0$, entonces $P(x)=x^m$

6 Notemos que $f\in C^{n+1}[0,5]$, también tenemos un polinomio p de grado $\leq n$ que interpola a f en $n+1$ puntos distintos en $[0,5]$. Por lo tanto, por teorema, para cada $x\in[0,5]$ existe un $\xi=\xi_x\in(0,5)$ tal que:

$$f(x)-p(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

Entonces:

$$\left.\begin{array}{l} f(x) = 2^x \\ f'(x) = \ln(2) \cdot 2^x \\ f''(x) = \ln(2)^2 \cdot 2^x \\ f'''(x) = \ln(2)^3 \cdot 2^x \end{array}\right\} f^{n+1}(x) = \ln(2)^{n+1} \cdot 2^x$$

$$|f^{n+1}(x)| < |1^n \cdot 2^x| = 2^x \text{, dado que } x \in [0,5] \quad |f^{n+1}(x)| \leq 2^5 = 32$$

para todo $x \in [0,5]$, en particular $\forall x \in (0,5)$: $|f^{n+1}(x)| \leq 2^5 = 32$

Por otro lado, dado que $x, x_i \in [0,5]$, $|x - x_i| \leq 5$. Por lo tanto:

$$\prod_{i=0}^{n} |(x - x_i)| \leq 5^n$$

Entonces:

$$|P_n(x) - f(x)| = \frac{|f^{n+1}(\xi)|}{(n+1)!} \cdot \left|\prod_{i=0}^{n} (x - x_i)\right| < \frac{32 \cdot 5^n}{(n+1)!} < \frac{32 \cdot 5^{n+1}}{(n+1)!}$$

7

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\left.\begin{array}{l} f(x) = \cosh(x) \\ f'(x) = \sinh(x) \\ f''(x) = \cosh(x) \\ f'''(x) = \sinh(x) \end{array}\right\} f^{n}(x) = \begin{cases} \cosh(x) & \text{si } n \text{ es par} \\ \sinh(x) & \text{si } n \text{ es impar} \end{cases}$$

$f^{(23)}(x) = \sinh(x)$

$(\sinh(x))' = \cosh(x) = \frac{e^{x}+e^{-x}}{2} > 0 \Longrightarrow \sinh(x)$ es creciente

entonces $\forall x \in [-1,1]$: $\sinh(x) \leq \sinh(1) \approx 1{,}17$

Por otro lado

$$\underbrace{|x-x_0|}_{<2} \cdots \underbrace{|x-x_{22}|}_{<2} < 2^{23}$$

→ pues es la longitud de $[-1,1]$

Entonces:

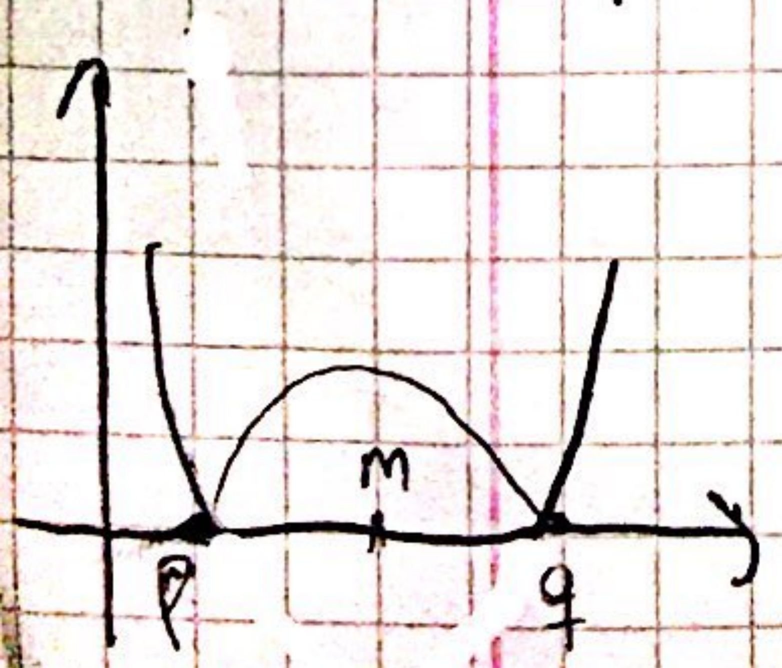
$$|E_{22}(x)| = \frac{|f^{(23)}(\xi)|\,|(x-x_0)\cdots(x-x_{22})|}{23!} \leq \frac{1{,}17\cdot 2^{23}}{23!} \approx 3{,}82\cdot 10^{-16} \leq 5\cdot 10^{-16}$$

8 a) Sea $x \in [p,q]$

Dado que p, q son sus raíces, por lo tanto tiene su máximo en $x = \frac{p+q}{2}$. Entonces:

$$|(x-p)(x-q)| \leq \left|\left(\frac{p+q}{2}-p\right)\left(\frac{p+q}{2}-q\right)\right| = \left|\left(\frac{q-p}{2}\right)\left(\frac{p-q}{2}\right)\right|$$

$$= \left(\frac{p-q}{2}\right)^2 \leq \left(\frac{b-a}{2}\right)^2 \qquad (*)_1$$



Pues $p, q \in [a,b] \Longrightarrow |p-q| \leq |b-a|$

(*1) Ej 8 a)

$|f(x)| = |(x-p)(x-q)|$

1. Si $x \in [p,q] \Longrightarrow |f(x)| \leq f(m)$ no ya probamos

2. Si $x \in [a,p] \Longrightarrow |f(x)| \leq f(a)$

$f(a) = (a-p)(a-q) = (a-m+h)(a-m-h)$

Además: $a-m = a - \frac{a+b}{2} = \frac{a-b}{2}$

Luego: $f(a) = \left(\frac{a-b}{2}+h\right)\left(\frac{a-b}{2}-h\right) = \frac{(a-b)^2}{4} - h^2 \leq \frac{(b-a)^2}{4}$

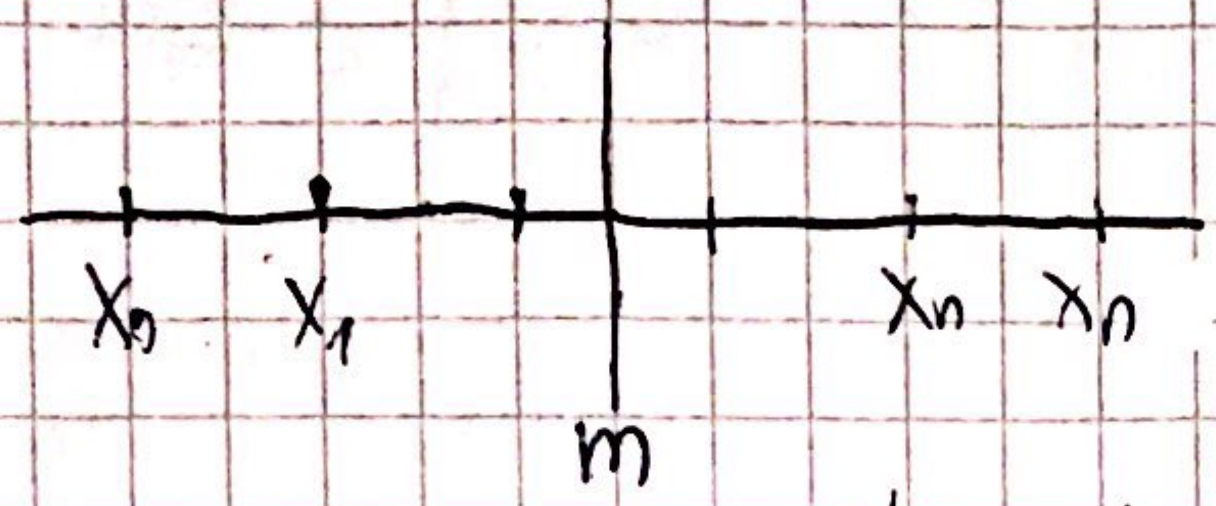
3. Si $x \in [q,b] \Longrightarrow |f(x)| \leq f(b)$

$f(b) = (b-p)(b-q) = (b-m+h)(b-m-h)$

$= b - \frac{a+b}{2} = \frac{b-a}{2}$

Luego $f(b) = \left(\frac{b-a}{2}+h\right)\left(\frac{b-a}{2}-h\right) = \frac{(b-a)^2}{4} - h^2 \leq \frac{(b-a)^2}{4}$

b) <u>n Impar</u> (Tengo n+1 puntos, osea una cantidad par).



↳ punto medio no es un nodo

Entonces, si asociamos de a 2 factores:

$$\underbrace{|(x-x_0)(x-x_n)|}\cdot\underbrace{|(x-x_1)(x-x_{n-1})|}\cdot\underbrace{|(x-x_2)(x-x_{n-2})|}\cdots$$

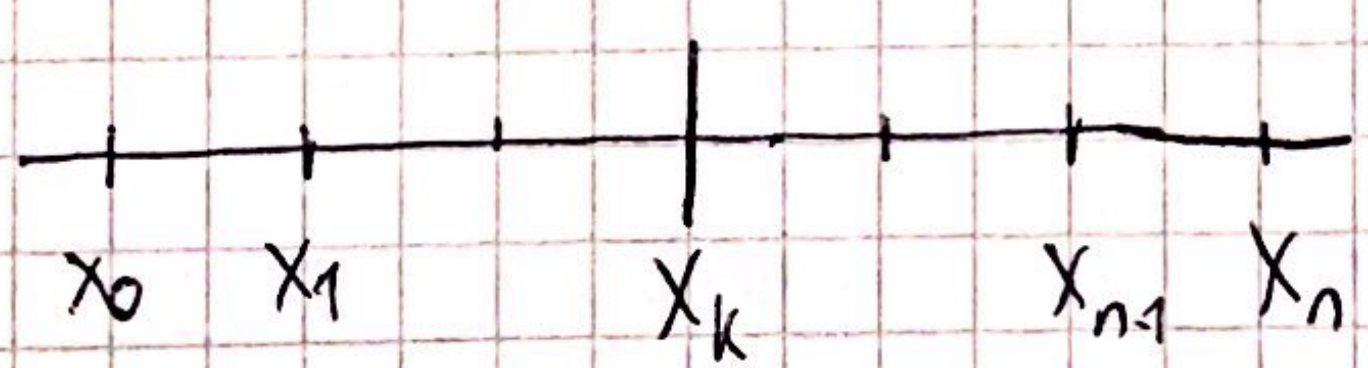
Son cuadráticas con raíces en x_j, x_{n-j} para $j=0,\cdots,\frac{n-1}{2}$

Entonces cada factor es $\leq\left(\frac{b-a}{2}\right)^2$ por ej anterior.

Y tenemos $\frac{n+1}{2}$ pares de factores.

$$\leq\left(\frac{b-a}{2}\right)^2\cdots\left(\frac{b-a}{2}\right)^2=\left(\left(\frac{b-a}{2}\right)^2\right)^{\frac{n+1}{2}}=\frac{(b-a)^{n+1}}{2^{n+1}}$$

<u>n Par</u>



↳ el punto medio es un nodo.

Entonces asociando de a 2 factores excepto el punto medio:

$$|(x-x_0)(x-x_n)|\cdot|(x-x_1)(x-x_{n-1})|\cdots|x-x_k|$$

Cada doble factor son cuadráticas con raíces en x_j, x_{n-j}

para $j=0,\dots,\frac{n}{2}$. Por lo anterior dichos factores son $\leq\left(\frac{b-a}{2}\right)^2$

Es de notar que estos son $\frac{n}{2}$ factores.

Por otro lado, dado que $x\in[a,b]$, $|x-x_k|\leq\frac{b-a}{2}$ $\forall x$.

Entonces:

$$\leq\left(\frac{b-a}{2}\right)^2\cdots\left(\frac{b-a}{2}\right)^2\cdot\frac{(b-a)}{2}=\left(\left(\frac{b-a}{2}\right)^2\right)^{\frac{n}{2}}\cdot\frac{b-a}{2}=\frac{(b-a)^{n+1}}{2^{n+1}}$$

9

a) $f(x) = \cos(\pi x)$

$$x_0 = -1$$
$$x_1 = 0$$
$$x_2 = 1$$

x_0	$f(x_0)$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_2]$
x_1	$f(x_1)$	$f[x_1,x_2]$	$f[x_1,x_2,x_2]$	
x_2	$f(x_2)$	$f'(x_2)$		
x_2	$f(x_2)$			

$$f(x_0) = -1$$
$$f(x_1) = 1$$
$$f(x_2) = -1$$

$$f'(x_2) = -\pi . \sin(\pi) = 0$$

$$f[x_1,x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{-1-1}{1-0} = -2$$

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{1+1}{1} = 2$$

$$f[x_1,x_2,x_2] = \frac{f[x_2,x_2]-f[x_1,x_2]}{x_2-x_1} = \frac{0+2}{1} = 2$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{-2-2}{2} = -2$$

$$f[x_0,x_1,x_2,x_2] = \frac{f[x_1,x_2,x_2]-f[x_0,x_1,x_2]}{x_2-x_0} = \frac{2+2}{2} = 2$$

$\Rightarrow c_0 = -1,\ c_1 = 2,\ c_2 = -2,\ c_3 = 2$

$P(X) = -1 + 2(X+1) - 2(X+1)(X) + 2(X+1)(X)(X-1)$

b)

X_0	$f(X_0)$	$f[X_0,X_1]$	$f[X_0,X_1,X_2]$	$f[X_0,X_1,X_2,X_2]$	$f[X_0,X_1,X_2,X_2,X_2]$
X_1	$f(X_1)$	$f[X_1,X_2]$	$f[X_1,X_2,X_2]$	$f[X_1,X_2,X_2,X_2]$	
X_2	$f(X_2)$	$f'(X_2)$	$\frac{f''(X_2)}{2!}$		
X_2	$f(X_2)$	$f'(X_2)$			
X_2	$f(X_2)$				

$$\frac{f''(X_2)}{2!} = \frac{-\pi \cdot \cos(\pi)}{2} = \frac{\pi^2}{2}$$

$$f[X_1,X_2,X_2,X_2] = \frac{f[X_2,X_2,X_2] - f[X_1,X_2,X_2]}{X_2 - X_1} = \frac{\frac{f''(X_2)}{2} - 2}{1} = \frac{\pi^2 - 4}{2}$$

$$f[X_0,X_1,X_2,X_2] = \frac{f[X_1,X_2,X_2] - f[X_0,X_1,X_2]}{X_2 - X_0} = \frac{2+2}{2} = 2$$

$$f[X_0,X_1,X_2,X_2,X_2] = \frac{f[X_1,X_2,X_2,X_2] - f[X_0,X_1,X_2,X_2]}{X_2 - X_0} = \frac{\pi^2 - 8}{4}$$

$$P(X) = -1 + 2(X+1) - 2(X+1)X + 2(X+1)X(X-1) + \frac{\pi^2-8}{4}(X+1)X(X-1)^2$$

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10

a)

el error del spline lineal está dado por:

$$|e(x)| = \frac{|f''(x)|}{2!} \cdot \frac{(x_{i+1}-x_i)^2}{4}$$

$I=[x_0,x_n]$

$$\left.\begin{array}{l} x_0 = 1+\frac{0}{n} = 1 \\ x_n = 1+\frac{n}{n} = 2 \end{array}\right\} I=[1,2]$$

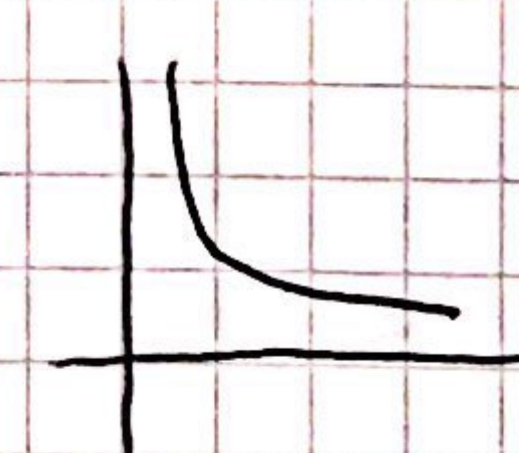
Por otro lado:

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4\sqrt{x^3}}$

$|f''(x)|$ tiene la pinta:



es decir es decreciente

Entonces $|f''(x)| \le |f''(1)| \quad \forall x \in I$

$|f''(1)| = \frac{1}{4}$

Además:

$$(X_{i+1}-X_i)^2=\left(\left(1+\frac{i+1}{n}\right)-\left(1+\frac{i}{n}\right)\right)^2=\left(\frac{1}{n}\right)^2=\frac{1}{n^2}$$

Queremos que $|e(x)| \leq 5 \cdot 10^{-8}$, entonces:

$$\underbrace{\frac{|f''(x)|}{2!}\cdot\frac{\left(\frac{1}{n}\right)^2}{4}}_{\leq \frac{1}{4}\cdot\frac{1}{2!}\cdot\frac{1}{4n^2}} \leq 5 \cdot 10^{-8}$$

Entonces:

$$\frac{1}{32}\cdot\frac{1}{n^2} \leq 5 \cdot 10^{-8}$$

$$\frac{1}{32.5}\cdot 10^8 \leq n^2$$

$$\frac{1}{160}\cdot 10^8 \leq n^2$$

$$790{,}56 \simeq \frac{1}{\sqrt{160}}\cdot 10^4 \leq n$$

$\circledast\ h=\frac{b-a}{n}$

$\therefore$ El número necesario de nodos es 791

$$h=\frac{1}{791}$$

b) ~~Todo el análisis es análogo a lo anterior, porque básicamente es un spline lineal, lo único que cambia es el término cuadrático del error.~~

b) $i = 0, 2, 4, \dots, n$

$X_0 = 1$

$X_n = 2$

$$|e_x| \le \frac{|f'''(x)|}{8} \cdot |X_{i+2} - X_i|^2$$

$$X_{i+2} - X_i = 1 + \frac{i+2}{n} - \left(1 + \frac{i}{n}\right) = \frac{2}{n}$$

$$\Longrightarrow \quad |e_x| \le \frac{1}{4} \cdot \frac{1}{8} \cdot \left(\frac{2}{n}\right)^2 = \frac{1}{8n^2}$$

$$\frac{1}{8n^2} \le 5 \cdot 10^{-8}$$

$$\frac{1}{40} \cdot 10^8 \le n^2$$

$$n \ge \sqrt{\frac{10^8}{40}} \approx 1581,\dots$$

$$\therefore n = 1582, \quad h = \frac{1}{1582}$$

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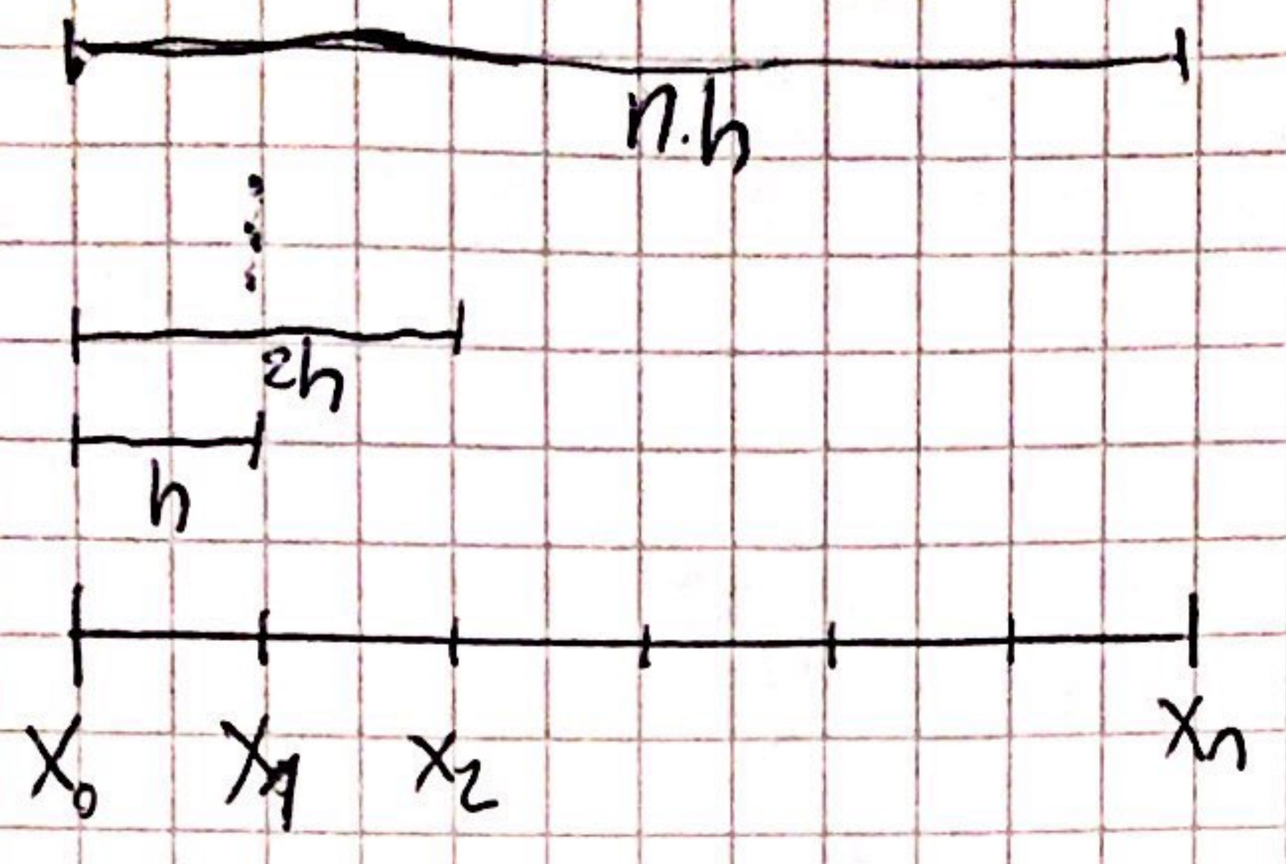
11



$x_0 = a$

$x_1 = a+h$

$x_2 = a+2h$

$\vdots$

$\boxed{x_n = a + i \cdot h}$

Dado que $[a,b]=[0,2\pi] \Longrightarrow \boxed{h = \frac{2\pi}{n}}$

Por lo tanto $x_i = a + i \cdot \frac{2\pi}{n}$

El error del spline lineal: $e(x) = \frac{|f''(x)|}{2!} \frac{(x_{i+1}-x_i)^2}{4}$

Por un lado:

$$(x_{i+1}-x_i)^2 = \left[\left(a + i\frac{2\pi}{n} + \frac{2\pi}{n}\right) - \left(a + i \cdot \frac{2\pi}{n}\right)\right]^2 = \frac{4\pi^2}{n^2}$$

Por otro lado

$$\begin{cases} f(x) = \cos(x) \\ f'(x) = -\sin(x) \\ f''(x) = -\cos(x) \end{cases} \rightsquigarrow |f''(x)| \leq 1 \ \forall x \Longrightarrow \forall x \in [0,2\pi]$$

Por lo tanto:

$$e(x) \leq \frac{1}{2} \cdot \frac{\frac{4\pi^2}{n^2}}{4} = \frac{\pi^2}{2n^2} \leq 5 \cdot 10^{-7}$$

$$\frac{\pi^2 \cdot 10^7}{10} \leq n^2$$

$$n \geq \pi\sqrt{10^6} = \pi 10^3 = 3141{,}59$$

Scanned with CamScanner

$\therefore$ Son necesarios 3142 nodos

$$h = \frac{2\pi}{3142}$$

12 a)

i $S_0(1) = S_1(1)$

$$\alpha + \gamma = -\alpha + \beta - 5\alpha + 1$$

$$\boxed{7\alpha + \gamma = \beta} \quad (1)$$

ii $S_0'(1) = S_1'(1)$

$$\begin{cases} S_0'(x) = 3\alpha x^2 + \gamma \\ S_1'(x) = -3\alpha x^2 + 2\beta x - 5\alpha \end{cases}$$

$$3\alpha + \gamma = -3\alpha + 2\beta - 5\alpha$$

$$\boxed{11\alpha + \gamma = 2\beta} \quad (2)$$

iii $S_0''(1) = S_1''(1)$

$$\begin{cases} S_0''(x) = 6\alpha x \\ S_1''(x) = -6\alpha x + 2\beta \end{cases}$$

$$6\alpha = -6\alpha + 2\beta$$

$$\boxed{12\alpha = 2\beta} \quad (3)$$

Por (3):

$$\boxed{6\alpha = \beta}$$

Reemplazo en (1)

$$7\alpha + \gamma = 6\alpha + 1$$

$$\boxed{\alpha = -\gamma + 1}$$

$$\gamma = \frac{1}{2}$$

Reemplazo en (2)

$$11\alpha + 1 - \alpha = 12\alpha$$

$$\alpha = \frac{1}{2}$$

$$\beta = 3$$

b)

$S(0) = S_0(0) = 0$

$S(1) = S_0(1) = S_1(1) = \frac{1}{2} + \frac{1}{2} = 1$

$S(2) = S_1(2) = -\frac{1}{2} \cdot 8 + 3 \cdot 4 - 5 \cdot \frac{1}{2} \cdot 2 + 1 = -4 + 12 - 5 + 1 = 4$

$f(0) = 1 + 0 + 0 + 1 = 2$

$f(1) = 2 + \frac{1}{2} - \frac{1}{2} - 1 = 1$

$f(2) = 4 + \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot 2 - 1 = 4 + 2 - 1 - 1 = 4$

$\therefore$ Si interpola

13

$$\begin{cases} S_0'(x) = 3d_1x^2 + 2c_1x + d_1 \\ S_0''(x) = 6d_1x + 2c_1 \\ S_1'(x) = b_2 + 2c_2(x-1) + d_2(3x^2 - 6x + 3) \\ S_1''(x) = 2c_2 + d_2(6x - 6) \end{cases}$$

$x_0 = 0$
$x_1 = 1$
$x_2 = 2$

Condiciones de interpolación

$S_0(x_0) = f(x_0)$
$1 = 1$

$S_0(x_1) = f(x_1)$
① $\boxed{1 + b_1 + c_1 + d_1 = 0}$

$S_1(x_2) = f(x_2)$
② $\boxed{b_2 + c_2 + d_2 = 3}$

Condiciones de continuidad

$$S_1'(X_1) = S_2'(X_1)$$

$$\boxed{3d_1 + 2c_1 + b_1 = b_2} \quad ③$$

$$S_1''(X_1) = S_2''(X_1)$$

$$\boxed{6d_1 + 2c_1 = 2c_2} \quad ④$$

Condiciones Naturales

$$S_0''(X_0) = 0 \quad \wedge \quad S_1''(X_2) = 0$$

$$2c_1 = 0 \Rightarrow \boxed{c_1 = 0} \quad ⑥$$

$$\boxed{2c_2 + 6d_2 = 0} \quad ⑤$$

Por ①②③④⑤⑥ tenemos:

$$\begin{cases} b_1 + d_1 = -1 \\ b_2 + c_2 + d_2 = 3 \\ b_1 + 3d_1 - b_2 = 0 \\ 6d_1 - 2c_2 = 0 \\ 2c_2 + 6d_2 = 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccccc} b_1 & d_1 & b_2 & c_2 & d_2 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 6 & 0 \end{array}\right) \end{array} \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Entonces:

$$S(x) = \begin{cases} 1 - 2x + x^3 & [0,1] \\ (x-1) + 3(x-1)^2 - (x-1)^3 & [1,2] \end{cases}$$