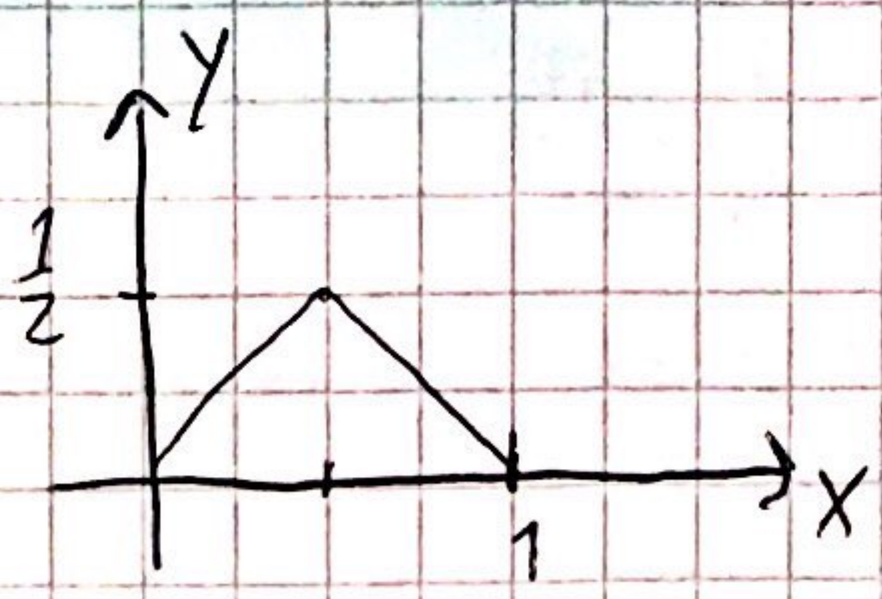
PRACTICO 5

2

a)



$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a)+f(b)] = \frac{1}{2}(0+0) = 0$$

b)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \int_0^{\frac{1}{2}} f(x)dx + \int_{\frac{1}{2}}^1 f(x)dx$$

$$\approx \frac{1}{4}\left[\frac{1}{2}+0\right] + \frac{1}{4}\left[0+\frac{1}{2}\right] = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

c)

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{b+a}{2}\right)+f(b)\right] = \frac{1}{6}\left[0+4\cdot\frac{1}{2}+0\right] = \frac{1}{3}$$

3

a) $h = \frac{b-a}{2}$

$$\int_0^{12} f(x)dx = \frac{h}{3}[2m + 0m + 1m] = \frac{6m}{3}[3m] = 6m^2$$

Luego, $V = 6m^2 \cdot 10m = 60m^3$

b)

$12m \cdot 10m \cdot X = 60m^3$

$X = \frac{60}{120} \frac{m^3}{m^2}$

$\boxed{X = \frac{1}{2} m}$

4 a)

$$\int_{-1}^{1} f(x)dx \approx A_0 f\left(\frac{a}{2}\right) + A_1 f\left(\frac{a+b}{2}\right) + A_2 f(b)$$

$f(x) = 1$ ⟶ grado cero

$$\int_{-1}^{1} f(x)dx = \boxed{2 = A_0 + A_1 + A_2} \quad ①$$

queremos que sea exacto

$f(x) = x$ ⟶ grado 1

$$\int_{-1}^{1} f(x)dx = 0 = A_0 \cdot \frac{-1}{4} + 0 + A_2 \cdot 1 \implies \boxed{\frac{A_0 + A_2}{2} = 0} \quad ②$$

$f(x) = x^2$

$$\int_{-1}^{1} f(x)dx = \frac{2}{3} = A_0 \cdot \frac{1}{4} + 0 A_1 + A_2 \cdot \frac{1}{4} \implies \boxed{\frac{A_0 + A_2}{4} = \frac{2}{3}} \quad ③$$

Para que sea exacta en grado 0,1,2 se tiene que

Cumplir 1, 2, 3

$$\begin{cases} A_0 + A_1 + A_2 = 2 \\ -\frac{A_0}{2} + \frac{A_2}{2} = 0 \implies A_0 = A_2 \\ \frac{A_0 + A_2}{4} = \frac{2}{3} \end{cases}$$

$A_1 + 2A_0 = 2 \implies \underbrace{A_2 = \frac{2 - A_1}{2}}_{\alpha}$

$\frac{2A_2}{4} = \frac{2}{3} \implies \boxed{A_2 = \frac{4}{3}} \implies \boxed{A_0 = \frac{4}{3}}$

Por α:

$$\frac{4}{3} = \frac{2 - A_1}{2} \implies \boxed{A_1 = -\frac{2}{3}}$$

b) Por a) se cumple para $n = 0, 1, 2$

$\underbrace{f(x) = 3}$

$$\int_{-1}^{1} x^3 \, dx = 0$$

Con regla: $= \frac{4}{3} \cdot \left(-\frac{1}{8}\right) + \left(-\frac{2}{3}\right) \cdot 0 + \frac{4}{3} \cdot \frac{1}{8} = 0$ ✓

$\underbrace{f(x) = x^4}$

$$\int_{-1}^{1} x^4 \, dx = \frac{2}{5}$$

Con regla: $= \frac{4}{3} \cdot \frac{1}{16} + 0 + \frac{4}{3} \cdot \frac{1}{16} = \frac{1}{6} \neq \frac{2}{5}$

$\therefore$ Precisa hasta $n = 3$

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Trapecio

$$|E| = \left|-\frac{b-a}{12}\right| \cdot |h^2| |f''(\mu)|$$

para algún $\mu \in (a,b)$, $h = \frac{b-a}{n}$

Acotemos $|f''|$:

$f(x) = e^{-x^2}$

$f'(x) = (-2x)\, e^{-x^2}$

$f''(x) = (4x^2 - 2)\, e^{-x^2}$

$f'''(x) = (-8x^3 + 12x)\, e^{-x^2}$

$f''''(x) = (16x^4 - 48x^2 + 12)\, e^{-x^2}$

Necesitamos encontrar su máx y mín en $[0,1]$

$$f'''(x) = 0 = \underbrace{e^{-x^2}}_{\neq 0}(-8x^3 + 12x) \Rightarrow -8x^3 + 12x = 0$$

$$\Rightarrow x(-8x^2 + 12) = 0$$

$$\Rightarrow x_1 = 0 \lor x_2 = \sqrt{\tfrac{3}{2}}$$

Veamos que $f''''(x_{1,2}) \neq 0$ para que no sea un punto de inflexión.

$f''''(0) = 12$

$$f''''\left(\sqrt{\tfrac{3}{2}}\right) = \left(16\left(\tfrac{3}{2}\right)^2 - 48 \cdot \tfrac{3}{2} + 12\right) \cdot e^{-\frac{3}{2}} = \left(\frac{144 - 288 + 48}{4}\right) e^{-\frac{3}{2}} \neq 0$$

Ambos son extremos, ahora veamos cual tiene imagen mayor.

$$|f''(0)| = |-2,1| = 2$$

$$|f''(\sqrt{\tfrac{3}{2}})| = |4 \cdot e^{-\frac{3}{2}}| = 0,89$$

$$\therefore f''(\mu) \leq 2 \quad \forall \mu \in [0,1]$$

Entonces:

$$|E| \leq \left| \frac{b-a}{12} \cdot \frac{(b-a)^2}{n^2} \cdot 2 \right| = \left| \frac{(b-a)^3}{6n^2} \right| = \frac{1}{6n^2} \leq \frac{1}{2}$$

$$\frac{1}{2} 10^{-6} \geq \frac{1}{6n^2}$$

$$n^2 \geq \frac{2}{6} \cdot 10^{6}$$

$$n \geq \sqrt{\frac{10^6}{3}} \simeq 578,03$$

$$\boxed{n = 579}$$

Simpson

$$|E| = \left| \frac{(b-a)}{180} \cdot h^4 f^{(4)}(\mu) \right| = \left| \frac{(b-a)^5}{180 n^5} \cdot f^{(4)}(\mu) \right|$$

De la misma manera que en el anterior, se puede demostrar que $|f^{(4)}(x)| \leq 2 \ \forall x \in [0,1]$

Luego:

$$|E| \leq \left|\frac{(b-a)^5}{180 n^5} \cdot 2\right| = \frac{1}{90 n^5} \leq \frac{1}{2} 10^{-6}$$

$$\Rightarrow \quad n^5 \geq \frac{2}{9} \cdot 10^6$$

$$n \geq \sqrt[5]{\frac{2}{9} \cdot 10^6} \approx 7{,}4$$

$$\boxed{n = 8}$$

6

$$V(t) = \frac{dx(t)}{dt} \Longrightarrow X = \int V(t)\, dt$$

Siendo $h = \frac{b-a}{n}$:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]$$

Luego

$$X(t) = \int_0^{24} V(t)\,dt \approx \frac{24}{2.2}\left[38 + 2[41+46+48] + 45\right]$$

$$= 2118 \text{ mts.}$$

⑦

Sea $w(x)=1$ una función de peso en $[-1,1]$, sea $\{1, x, x^2-\frac{1}{3}, x^3-\frac{3}{5}x\}=\{\phi_0,\phi_1,\phi_2,\phi_3\}$ los polinomios de Legendre. Tomando $q=\phi_3(x)$, es de grado $3=(2+1)$, es decir $n=2$, es ortogonal a todo polinomio * de grado $\leq n=2$ con respecto a w. Entonces siendo x_0, x_1, x_2 las 3 raíces de q, entonces la fórmula

* de la base

$$\int_{-1}^{1} f(x)w(x) \simeq \sum_{i=0}^{n} a_i f(x_i) \qquad \text{con } i=0,1,2$$

con $a_i = \int_{-1}^{1} w(x) \cdot \prod_{\substack{i=0 \\ j\neq}}^{n} \left(\frac{x-x_j}{x_i-x_j}\right) dx$, será exacta para todo polinomio de grado menor a $2n+1=5$.

Entonces:

$$A_0 = \int_{-1}^{1} \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2}\, dx = \frac{1}{(x_0-x_1)(x_0-x_2)} \int_{-1}^{1} (x-x_1)(x-x_2)\, dx$$

$$A_1 = \int_{-1}^{1} \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2}\, dx = \frac{1}{(x_1-x_0)(x_1-x_2)} \int_{-1}^{1} (x-x_0)(x-x_2)\, dx$$

$$A_2 = \int_{-1}^{1} \frac{x-x_1}{x_2-x_1} \cdot \frac{x-x_0}{x_2-x_0}\, dx = \frac{1}{(x_2-x_1)(x_2-x_0)} \int_{-1}^{1} (x-x_1)(x-x_0)\, dx$$

Sabemos que x_0, x_1, x_2 son las raíces de q, luego:

$$q(x) = x^3 - \frac{3}{5}x = 0 \implies x(x^2 - \frac{3}{5}) = 0 \implies x_0 = -\sqrt{\frac{3}{5}},\ x_1 = 0,\ x_2 = \sqrt{\frac{3}{5}}$$

Finalmente:

$$A_0 = \frac{1}{\left(-\sqrt{\frac{3}{5}}\right)\left(-2\sqrt{\frac{3}{5}}\right)} \int_{-1}^{1} \left(x^2 + \sqrt{\tfrac{3}{5}}\,x - 0 + 0\right) dx$$

$$= \frac{1}{\frac{6}{5}} \cdot \int_{-1}^{1} \left(x^2 + \sqrt{\tfrac{3}{5}}\,x\right) dx = \frac{5}{6}\left(\frac{x^3}{3} + \sqrt{\tfrac{3}{5}}\frac{x^2}{2} \Big|_{-1}^{1} \right)$$

$$= \frac{5}{6}\left(\left(\frac{1}{3} + \frac{1}{2}\sqrt{\tfrac{3}{5}}\right) - \left(-\frac{1}{3} + \frac{1}{2}\sqrt{\tfrac{3}{5}}\right)\right) = \frac{5}{6}\left(\frac{2}{3}\right) = \frac{5}{9}$$

$$A_1 = \frac{1}{\sqrt{\frac{3}{5}}\left(-\sqrt{\frac{3}{5}}\right)} \int_{-1}^{1} \left(x^2 - \sqrt{\tfrac{3}{5}}\,x + \sqrt{\tfrac{3}{5}}\,x - \frac{3}{5}\right) dx = -\frac{5}{3}\int_{-1}^{1} \left(x^2 - \frac{3}{5}\right) dx$$

$$= -\frac{5}{3}\left(\frac{x^3}{3} - \frac{3}{5}x \Big|_{-1}^{1}\right) = -\frac{5}{3}\left[\left(\frac{1}{3} - \frac{3}{5}\right) - \left(-\frac{1}{3} + \frac{3}{5}\right)\right] = -\frac{5}{3}\left(\frac{2}{3} - \frac{6}{5}\right)$$

$$= -\frac{5}{3}\left(\frac{10-18}{15}\right) = \frac{40}{45} = \frac{8}{9}$$

$$A_2 = \frac{1}{\sqrt{\frac{3}{5}} \cdot 2\sqrt{\frac{3}{5}}} \int_{-1}^{1} \left(x^2 + \sqrt{\tfrac{3}{5}}\,x\right) dx = \frac{5}{6}\left(\frac{x^3}{3} + \sqrt{\tfrac{3}{5}}\frac{x^2}{2}\Big|_{-1}^{1}\right)$$

$$= \frac{5}{6}\left[\left(\frac{1}{3} + \sqrt{\tfrac{3}{5}} \cdot \frac{1}{2}\right) - \left(-\frac{1}{3} + \sqrt{\tfrac{3}{5}} \cdot \frac{1}{2}\right)\right] = \frac{5}{6} \cdot \frac{2}{3} = \frac{5}{9}$$

Luego:

$$\int_{-1}^{1} \cos(x) = 1,68294\ldots \quad \sim \text{o exacto}$$

Con la regla:

$$\int_{-1}^{1} \cos(x)\,dx \approx \frac{5}{9}\cos\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9}\cos(0) + \frac{5}{9}\cos\left(\sqrt{\tfrac{3}{5}}\right) = 1,6830\ldots$$

8

Del práctico 11: $\{\phi_0, \phi_1, \phi_2\} = \{1, x, x^2 - \frac{3}{5}\}$, es decir $n=1$, por lo tanto la regla será exacta para polinomios de grado $\leq 2.1+1 = 3$. Sabemos que x_0, x_1 son las raíces de ϕ_2:

$$x^2 - \frac{3}{5} = 0 \implies x_0 = -\sqrt{\frac{3}{5}}, \; x_1 = \sqrt{\frac{3}{5}}$$

Luego:

$$A_0 = \int_{-1}^{1} x^2 \frac{x - x_1}{x_0 - x_1} dx = \int_{-1}^{1} \frac{x^2(x - \sqrt{\frac{3}{5}})dx}{-2\sqrt{\frac{3}{5}}} = -\frac{1}{2}\sqrt{\frac{5}{3}} \int_{-1}^{1} (x^3 - \sqrt{\tfrac{3}{5}}x^2) dx$$

$$= -\frac{1}{2}\sqrt{\frac{5}{3}} \left(\frac{x^4}{4} - \sqrt{\frac{3}{5}} \frac{x^3}{3} \Big|_{-1}^{1} \right) = -\frac{1}{2}\sqrt{\frac{5}{3}} \left[\left(\frac{1}{4} - \frac{1}{3}\sqrt{\frac{3}{5}} \right) - \left(\frac{1}{4} + \frac{1}{3}\sqrt{\frac{3}{5}} \right) \right]$$

$$= -\frac{1}{2}\sqrt{\frac{5}{3}} \left(-\frac{2}{3}\sqrt{\frac{3}{5}} \right) = \frac{1}{3}$$

$$A_1 = \int_{-1}^{1} x^2 \left(\frac{x - x_0}{x_1 - x_0} \right) dx = \int_{-1}^{1} x^2 \frac{x + \sqrt{\frac{3}{5}}}{2\sqrt{\frac{3}{5}}} dx = \frac{1}{2}\sqrt{\frac{5}{3}} \int_{-1}^{1} (x^3 + \sqrt{\tfrac{3}{5}}x^2) dx$$

$$= \frac{1}{2}\sqrt{\frac{5}{3}} \left(\frac{x^4}{4} + \sqrt{\frac{3}{5}} \frac{x^3}{3} \Big|_{-1}^{1} \right) = \frac{1}{2}\sqrt{\frac{5}{3}} \left[\left(\frac{1}{4} + \frac{1}{3}\sqrt{\frac{3}{5}} \right) - \left(\frac{1}{4} - \frac{1}{3}\sqrt{\frac{3}{5}} \right) \right]$$

$$= \frac{1}{2}\sqrt{\frac{5}{3}} \left(\frac{2}{3}\sqrt{\frac{3}{5}} \right) = \frac{1}{3}$$

Entonces

$$\int_{-1}^{1} f(x) x^2 dx \approx \frac{1}{3} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{1}{3} f\left(\sqrt{\frac{3}{5}}\right)$$ será exacta para polinomios de grado ≤ 3