Central Forces

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1 Introduction

A central force is by definition a force that points radially and whose magnitude depends only on the distance from the source (that is, not on the angle around the source). Equivalently, we may say that a central force is one whose potential depends only on the distance from the source [Mor07].

People other than physicists or mathematicians may not hear about the central force before; nonetheless, it is worth investigating. First, central forces play a significant role in some well known phenomena. For example, circular motions, planetary motions and some simple harmonic motions. Second, it turns out not be a difficult topic and will only be complicated when solving equations.

2 Crucial Theorems

2.1 Conservation of Angular Momentum

For a point mass, *angular momentum* $\mathbf{L}$ is defined by

$$\mathbf{L} = \mathbf{p} \times \mathbf{r}$$

Theorem 1 *If a particle is only subject to a central force, then its angular momentum will be conserved.*

Proof. To prove that the angular momentum is conserved, we can prove that the change of angular momentum is zero. That is

$$\frac{d\mathbf{L}}{dt} = 0$$

Applying the definition of *angular momentum* here, we will have

$$\frac{d\mathbf{L}}{dt} \equiv \frac{d}{dt}(\mathbf{p} \times \mathbf{r})$$

Using product rule to expand the right hand side gives

$$\frac{d\mathbf{L}}{dt} \equiv \frac{d\mathbf{p}}{dt}\mathbf{r} + \frac{d\mathbf{r}}{dt}\mathbf{p}$$

$$\frac{d\mathbf{L}}{dt} \equiv \mathbf{F}\mathbf{r} + \mathbf{v}(m\mathbf{v})$$

Since the cross product of two parallel vectors is always 0

$$\frac{d\mathbf{L}}{dt} = 0$$

□

2.2 Orbits in a Plane

Kepler's first law: each planet's orbit about the Sun is an ellipse. The Sun's center is always located at one focus of the orbital ellipse. Let's consider one thing, whether gravity, the force allowing planets to orbit around the sun, is a kind of central forces or not.

$$\mathbf{F} = -G\frac{Mm}{r^2}\vec{r_1}$$

According to the formula above, the gravitation between two given objects only depends on the distance. Hence, gravitation is a kind of central force.

It is noticable that all planets in the solar system have their own elliptical orbits. This implies they all move in a plane. So, is this just a coincidence? The answer is Yes. At a given time, the velocity, displacement and acceleration can all lie in the same plane. This holds because the displacement and the acceleration are always collinear and two lines define a plane. Owing to symmetry, if the particle can slip out of the plane to one of its sides, there will be no reason that the particle will not appear on the other side. Therefore, the particle should always remain on P.

Theorem 2 *If a particle is subject to a central force only, then its motion takes place in a plane.*

Proof. At a given instant t_0, consider the plane P containing the position vector $\mathbf{r}_0$ (with the source of the potential taken to be the origin) and the velocity vector $\mathbf{v}_0$. We claim that $\mathbf{r}$ lies in P at all times. This is true because P is defined as the plane orthogonal to the vector $\mathbf{n}_0 \equiv \mathbf{r}_0 \times \mathbf{v}_0$. But in the proof of Theorem 1, we showed that the vector $\mathbf{r}\times\mathbf{v}\equiv(\mathbf{r}\times\mathbf{v})\frac{}{m}$ does not change with time. Therefore, $\mathbf{r}\times\mathbf{v}=\mathbf{n}_0$ for all t. Since $\mathbf{r}$ is certainly orthogonal to $\mathbf{r} \times\mathbf{v}$, we see that $\mathbf{r}$ is orthogonal to $\mathbf{n}_0$ for all t. Hence, $\mathbf{r}$ must always lie in P [Mor07]. □

3 Gravitational Force

3.1 The Equation of Motion

The Lagrangian is defined as:

$$\mathcal{L} \equiv T - V$$

where T is the kinetic energy and V is the potential energy. If a particle, with mass m, is only subject to a central force, we can write its potential energy as V(r). Let r and θ be the polar coordinates in the plane of motion (we can do this because the motion takes place in a plane), the Lagrangian can be written as:

$$\mathcal{L} = \frac{m}{2}(\dot{\mathbf{r}}^2 + \mathbf{r}^2\dot{\theta}^2) - V(r)$$

The equation of motion can be obtained by:

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}) = \frac{\partial\mathcal{L}}{\partial\mathbf{r}}$$

The left hand side should be:

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\mathbf{r}}) = \frac{d(m\dot{\mathbf{r}})}{dt}$$

$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\mathbf{r}}) = m\ddot{\mathbf{r}}$$

The right hand side should be:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{r}} = m\mathbf{r}\dot{\theta}^2 - V'(r)$$

Hence,

$$m\ddot{\mathbf{r}} = m\mathbf{r}\dot{\theta}^2 - V'(r)$$

Since the angular momentum is conserved, we can use $\mathbf{L}$ to indicate that:

$$\mathbf{L} \equiv m\mathbf{r}^2\dot{\theta}$$

Eliminate $\dot{\theta}$ from the equation:

$$m\ddot{\mathbf{r}} = \frac{\mathbf{L}^2}{m\mathbf{r}^3} - V'(r)$$

Multiply by dr and integrate both sides:

$$\frac{m}{2}\dot{\mathbf{r}}^2 + \frac{L^2}{2mr^2} + V(r) = E$$

where E is merely a integration constant. In order to have a function $\mathbf{r}$ in terms of θ, we need to remove dt from the equation above. We can attain this by using $\mathbf{L}$ again:

$$(\frac{d\mathbf{r}}{d\theta})^2 = \frac{2m\mathbf{r}^4}{\mathbf{L}^2}(E - V(r)) - \mathbf{r}^2$$

V(r), in this case, is just the gravitational potential energy:

$$V(r) = -G\frac{Mm}{r}$$

Put that in the equation:

$$(\frac{d\mathbf{r}}{d\theta})^2 = \frac{2m\mathbf{r}^4}{\mathbf{L}^2}(E + G\frac{Mm}{r}) - \mathbf{r}^2$$

In theory, we can just take a square root, separate variables, integrate to find θ(r) and then invert to find $\mathbf{r}(\theta)$. This method, although straight forward, is rather messy. So let's solve for $\mathbf{r}(\theta)$ in a slick way [Mor07]. We can substitute $\mathbf{r}$ by letting $y \equiv 1/\mathbf{r}$. And since the gravitational potential is only dependent on $\mathbf{r}$, we can let $A \equiv GMm$. Therefore, with the chain rule:

$$(\frac{dy}{d\theta})^2 = \frac{2m}{\mathbf{L}^2}(E + Ay) - y^2$$

Complete the square:

$$(\frac{dy}{d\theta})^2 = -(y - \frac{mA}{L^2})^2 + \frac{2mE}{L^2} + (\frac{mA}{L^2})^2$$

Let $z \equiv y - mA/L^2$:

$$\frac{dz}{dy} = 1$$

$$(\frac{dz}{d\theta})^2 = -z^2 + (\frac{mA}{L^2})^2(1 + \frac{2EL^2}{mA^2})$$

Define $B \equiv \frac{mA}{L^2}\sqrt{1 + \frac{2EL^2}{mA^2}}$

$$(\frac{dz}{d\theta})^2 = -z^2 + B^2$$

Rearrange the equation and integrate both sides:

$$\int d\theta = \int \frac{dz}{\sqrt{-z^2 + B^2}}$$

Use trigonometry substitution:

$$\theta = arccos(\frac{z}{B}) + \theta_0$$

where θ_0 is a constant. If a proper axis of the polar coordinates is chosen, θ_0 will be 0. Now by retracing the definition of z and y, we can link z and r: $z \equiv 1/r - mA/L^2$. Substitute the relation back into the equation above, and rearrange the new equation:

$$\mathbf{r} = \frac{L^2}{mA(1 + ecos\theta)}$$

where eccentricity $e \equiv \sqrt{1 + \frac{2EL^2}{mA^2}}$.

3.2 Kepler' Laws: the triumph of ellipses

Johannes Kepler: The ways by which men arrive at knowledge of the celestial things are hardly less wonderful than the nature of these things themselves.

3.2.1 Kepler's First Law

Kepler's first law states that each planet about the sun is an ellipse and the sun lies at one focus of the orbit.

For convenience only, define $C \equiv \frac{L^2}{mA}$. So:

$$\mathbf{r} = \frac{C}{1 + ecos\theta}$$

Circle ($e = 0$)
Obviously, when $e = 0$, $\mathbf{r} = C$, where C is just constant here.

Ellipse ($0 < e < 1$)

3.2.2 Kepler's Second Law

Kepler's second law states that a line joining any planet to the Sun sweeps out equal areas in equal time intervals. Though it sounds fascinating, it is nothing more than the conservation of angular momentum.

Proof. The area swept in dt can be calculated as:

$$dA = \frac{r^2 d\theta}{2}$$

Introduce dt to the equation:

$$\frac{dA}{dt} = \frac{r^2 \dot{\theta}}{2}$$

By the definition of angular momentum:

$$\frac{dA}{dt} = \frac{L}{2m}$$

□

3.2.3 Kepler's Third Law

Kepler's third law states that the square of the period of an orbit, T , is proportional to the cube of the semi-major-axis length, a.

Proof. Start from Kepler's second law:

$$\frac{dA}{dt} = \frac{L}{2m}$$

Integrate both sides with respect to t:

$$A = \frac{Lt}{2m}$$

Here, A denotes the area of the ellipse. The eccentricity of an ellipse will be given by $b^2 = a^2(1 - e^2)$ [ABB+18], where a is the semi-major axis and b is the semi-minor axis. The Area of an ellipse:

$$A = \pi ab$$

$$A = \pi a^2 \sqrt{1 - e^2}$$

Substitute the area into the equation above:

$$\pi a^2 \sqrt{1 - e^2} = \frac{Lt}{2m}$$

Rearrange the equation to put A and t on the same side and square both sides:

$$\frac{a^4}{t^2} = \frac{L^2}{4m^2\pi^2(1-e^2)}$$

Recall $C = \frac{L^2 r}{mGMm}$:

$$\frac{a^4}{t^2} = \frac{GM}{4\pi^2(1-e^2)}C$$

Recall $a = \frac{C}{1-e^2}$:

$$\frac{a^3}{t^2} = \frac{GM}{4\pi^2}$$

□

4 Simple Harmonic Motion: Mass-Spring System

Any motion that repeats at regular intervals is called periodic motion or harmonic motion. Motions that can be expressed by a sine or a cosine of time are called simple harmonic motion [HRW20].

Before moving on to the mass-spring system, I need to define central force again but from a physicist's perspective. The definition from a physicist's point of view shows the dependence solely on the distance from the source [Mor07]. Obvious, the force on a spring, which obeys Hooke's Law $F = -k\delta x$, is a kind of central force.

4.1 The Equation of Motion

The equation can be obtained by solving the second order differential equation. According to Newton's second law $F = ma$:

$$-kx = m\ddot{x}$$

where x denotes the displacement from the equilibrium point. Solving this gives:

$$x = Acos(\omega t)$$

where $\omega = \sqrt{\frac{m}{k}}$ and A is just a constant which depends on the initial condition and denotes the amplitude.

4.2 Frequency and Period

ω in the equation above shows the frequency, and the period T can be written as $\frac{2\pi}{\omega}$. According to the expression of ω, the frequency only depends on the spring constant and the mass. Now I want to demonstrate this from the perspective of energy.

Define $V(x)$ as the potential energy and expand V(x) in a Taylor series around the equilibrium point:

$$V(x) \approx V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2!}V''(X_0)(x-x_0)^2 + \frac{1}{3!}V'''(x_0)(x-x_0)^3 + \cdots$$

$V(x_0)$ is merely a constant and can be neglected. According to Hooke's Law, at the equilibrium point, the force on the spring should be 0, which means $V'(x_0) = 0$. Higher-order terms can be passed over because of the increasing power of $(x - x_0)$. Hence, what we have in the end is:

$$V(x) \approx \frac{1}{2!}V''(X_0)(x-x_0)^2$$

This is exactly the same as Hooke's-Law potential. The second derivative of $V(x)$ is just the spring constant. Hence, the frequency can be written as:

$$\omega = \sqrt{\frac{V''(x_0)}{m}}$$

4.3 Velocity

The expression of velocity can be gained by applying the conservation of mechanical energy [Bre15].

$$E = E_k + V$$

where E denotes the total mechanical energy; E_k denotes the kinetic energy of the mass attached to the spring; V denotes the elastic potential energy of the spring. The total mechanical energy equals to the elastic energy when $x = A$.

$$E = \frac{1}{2}kA^2$$

The general expressions of the kinetic energy and the elastic potential energy are:

$$E_k = \frac{1}{2}mv^2$$

$$V = \frac{1}{2}kx^2$$

Apply the definition of ω here and rearrange the equation:

$$v^2 = \omega^2(A^2 - x^2)$$

$$v = \pm\omega\sqrt{A^2 - x^2}$$

The advantage of the equation above is that we do not have to know the formula of motion. However, the equation does not tell us the direction of the motion. Therefore, instead of using time and period to riddle out the status of motion, we can simply take the first derivative of the equation of motion supposed that we have already known it.

$$x = Acos(\omega t)$$

$$\dot{x} = -A\omega sin(\omega t)$$

5 Bertrand's Theorem

The only central forces that result in closed orbits for all bound particles are the inverse-square law and Hooke's law [GOL11].
The first one is inverse-square central force:

$$F(r) = -\frac{k}{r^2}$$

The second one is force proportional to distance:

$$F(r) = -kr$$

Proof. The equation of motion for the radius r of a particle of mass m moving in a central potential V(r) is given by motion equations:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{dV}{dr}$$

The first term on the left should be 0 given a circular orbit. Recall $L \equiv mr^2\dot{\theta}$ and express the second term in L:

$$m\ddot{r} - \frac{L^2}{mr^3} = -\frac{dV}{dr}$$

Multiply both sides by $\frac{mr^2}{L^2}$:

$$\frac{d^2r}{d\theta^2}\frac{1}{r^2} - \frac{1}{r} = -\frac{mr^2}{L^2}\frac{dV}{dr}$$

Define $u \equiv \frac{1}{r}$:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{dV}{du}$$

Define $J(u) \equiv -\frac{m}{L^2}\frac{dV}{du}$

$$J(u) \equiv -\frac{m}{L^2}\frac{dV(\frac{1}{u})}{du}$$

$$J(u) \equiv -\frac{m}{L^2u^2}f(\frac{1}{u})$$

where f indicates the force. As mentioned above, if the first term in the equation becomes 0, the orbit will become circular, in which case:

$$J(u_0) = u_0$$

With tiny perturbations $u = u_0 + \eta$, the orbit will not be perfectly circular.

$$\frac{d^2\eta}{d\theta^2} + u_0 + \eta = J(u)$$

Expand J in a Taylor series:

$$J(u) \approx J(u_0) + \eta J'(u_0) + \frac{1}{2}\eta^2 J''(u_0) + \frac{1}{6}\eta^3 J'''(u_0) + \cdots$$

Subtracting the constants from both sides and rearranging the equation give:

$$\frac{d^2\eta}{d\theta^2} + \beta^2\eta = \frac{1}{2}\eta^2 J''(u_0) + \frac{1}{6}\eta^3 J'''(u_0) + \cdots \tag{1}$$

where $\beta^2 \equiv 1 - J'(u_0)$. Due to small perturbations, we can simply pass over all terms on the right. Solve the second order differential equation:

$$\eta = h_1 cos(\beta\theta)$$

where h_1 is just an integral constant. Notice that β^2 needs to be positive and definite so that η can describe a bounded stable oscillation.

$$1 - \beta^2 \equiv \frac{d}{du_0}(-\frac{mf}{L^2u^2})$$

Applying product rule here:

$$J'(u) = \frac{2mf}{L^2u^3} - \frac{m}{L^2u^2}\frac{df}{du}$$

Applying the definition of J here again:

$$J'(u) = -\frac{2J(u)}{u} + \frac{J(u)}{f}\frac{df}{du}$$

$$1 - \beta^2 = -\frac{2J(u_0)}{u_0} + \frac{u_0}{f_0}\frac{df_0}{du_0}$$

u_0, indeed, can be any value. Therefore,

$$1 - \beta^2 = -2 + \frac{u}{f}\frac{df}{du}$$

Applying chain rule here to substitute u by r:

$$1 - \beta^2 = -2 - \frac{r}{f}\frac{df}{dr}$$

Rearranging the equation and integrating both sides:

$$\int (3 - \beta^2)\frac{dr}{r} = \int -\frac{df}{f}$$

$$(3 - \beta^2)ln(r) = -ln(f) + c$$

$$f = -kr^{\beta^2-3} \tag{2}$$

If the orbit deviates more from the circular orbit (the higher terms cannot be simply omitted), θ may be expanded in Fourier series:

$$\eta \approx h_0 + h_1 cos\beta\theta + h_2 cos2\beta\theta + h_3 cos\beta\theta \tag{3}$$

When approaching the circular orbit, η will tend to $h_1 cos\beta\theta$. This suggests that h_0, h_2 and other coefficients before cosine should vanish faster than h_1. Consider the right hand side of equation (1), since h_3 is small, the factor $cos3\beta\theta$ in x^2 can be dropped. For the same reason, only $cos\beta\theta$ will remain in x^3. By applying the following two identities [GLG]:

$$cos\beta\theta cos2\beta\theta = \frac{1}{2}(cos\beta\theta + cos3\beta\theta)$$

$$cos^3\beta\theta = \frac{1}{4}(3cos\beta\theta + cos3\beta\theta)$$

Substitute equation(3) into equation(1) and rearrange the new equation:

$$\beta^2 a_0 - 3\beta^2 a_2 cos2\beta\theta - 8\beta^3 a_3 cos3\beta\theta$$

$$= \frac{a_1^2}{4}J'' + (\frac{2a_1a_0 + a_1a_2}{2}J'' + \frac{J'''a_1^3}{8})cos\beta\theta + \frac{a_1^2}{4}J''cos2\beta\theta + (\frac{a_1a_2}{2}J'' + \frac{J'''a_1^3}{24})cos3\beta\theta$$

The coefficients between each cosine should be the same:

$$a_0 = \frac{a_1^2 J''}{4\beta^2}$$

$$a_2 = -\frac{a_1^2 J''}{12\beta^2}$$

$$0 = \frac{2a_1a_0 + a_1a_2J''}{2} + \frac{J'''a_1^3}{8} \tag{4}$$

$$a_3 = -\frac{1}{8\beta^3}(\frac{a_1a_2J''}{2} + \frac{J'''a_1^3}{24})$$

Recall the definition of J, with equation (2), J can be expressed as:

$$J(u) = \frac{mk}{L^2}u^{1-\beta^2}$$

Recall the definition of a circular orbit:

$$J(u) = u$$

Therefore, the derivatives of $J(u)$ can be written as:

$$J''(u) = -\frac{\beta^2(1-\beta^2)}{u}$$

$$J'''(u) = \frac{\beta^2(1-\beta^2)(1+\beta^2)}{u^2}$$

Substitute these derivatives into equation (4) and simplify the new equation:

$$0 = \beta^2(1-\beta^2)(4-\beta^2)$$

Notice that β cannot be 0. Hence, the only solutions are

$$\beta = 1$$

$$\beta = 2$$

□

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