Stability & Equilibrium

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1 Warm-Up Question

1. What are the two requirements for equilibrium?

2. How can an equilibrium collapse?

3. Are all equilibria the same? If not, what is the difference?
(Hint: think about a pen standing on a desk and one lying down)

4. How can you tell whether an equilibrium is stable or not?
(Hint: what happens if you disturb the equilibrium?)

2 Energy

2.1 First Derivative: Equilibrium

In the previous section, we mention the relation between equilibrium and force[1] the net force equal to 0 suggests equilibrium. If there is no force acting, there will be no acceleration, which means no change in velocity[2].

$$\frac{dv}{dx} = 0$$

where x is a parameter.
Kinetic energy can be written as

$$T = \frac{1}{2}mv^2$$

Since

$$\frac{dT}{dx} = mv\frac{dv}{dx}$$

The change in kinetic energy is 0

$$\frac{dT}{dx} = 0$$

Assume ideal states where mechanical energy[3] is conserved; T denotes kinetic energy; V denotes potential energy:

$$T + V = C$$

[1] We don't concern torque at this stage as in most cases objects are modelled as a point mass.
[2] Disambiguity: dv/dx means the change in velocity but is not equivalent to acceleration.
[3] Mechanical energy is the sum of kinetic energy and potential energy

where C is an arbitary constant subject to a real case.

Then, we differentiate both sides with respect to x.

$$\frac{dT}{dx} + \frac{dV}{dx} = 0$$

Since

$$\frac{dT}{dx} = 0$$

Therefore

$$\frac{dT}{dx} = \frac{dV}{dx} = 0$$

The result should be intuitive: no force, no change in energy form.

2.2 Second Derivative: Stability

We have discussed about stability in those warm-up questions; an example of a pen in different states is given. Apparently, the pen in boths states undergoes no change in energy: they just stay over there. However, our common sense tells us that the standing pen has the potential to fall down and becomes a lying pen. So, is it possible to show the stability of a system by using mathematical equations?

Let's start from examining the energy change during the process of a pen falling down and rising. In the first process, the pen gradually loses its gravitational potential energy which is converted into kinetic energy. In the second case, the energy change is the opposite; nevertheless, gravitational potential energy will be converted back to kinetic energy if the external force is removed and the equilibrium (standing upright) has been reached yet. According these two processes, we can tell that potential energy tends to decrease. Therefore, when potential energy has a minimum value[4], the equilibrium is regarded as stable.

Given $V(x)$, if we seek for a minimum value of V, then

$$\frac{dV}{dx} = 0$$

and

$$\frac{d^2V}{dx^2} > 0$$

To memorise this, think about the graph of a cubic function. In fact, the graph is

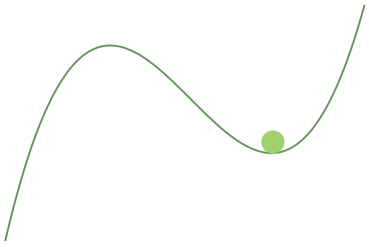


Figure 1: Cubic Graph

another example of stable equilibrium. Imagine the cubic graph is a smooth track and a sphere is placed at the minimum. If the sphere is displaced a bit, it will roll down to the minimum point[5].

[4] Minimum value is not equivalent to zero!
[5] Apparently, this is not true as the track is smooth.

3 Questions

3.1 Q1

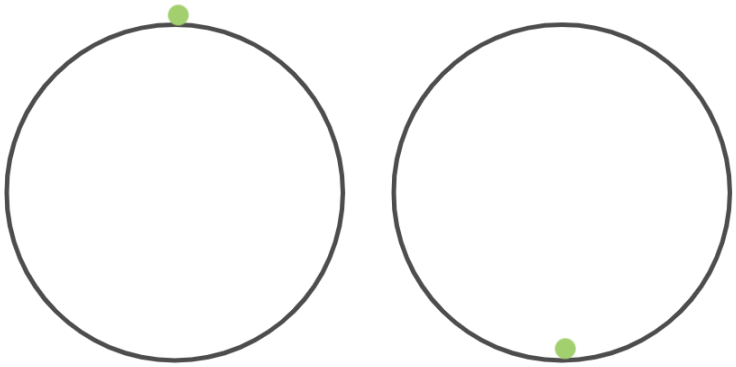
By Considering energy, show that both a standing pen and a lying-down pen are in equilibrium. Then suggest the nature of equilibria. Justify your answers.

3.2 Q2

By considering the energy, show that when there is no load on spring the spring will have its natrual length.

3.3 Q3

A particle P rests at the lowest point inside a hollow sphere centre O and radius r. By considering the energy, show that P is in stable equilibrium. Then, P is lifted to the highest point above the sphere. Show that now P is in unstable equilibrium.



3.4 Q4

By considering the torque on a book, show that it is possible to for the book to tilt at one point. Then state the nature of this equilibrium and jusiify your answer.

3.5 Q5

By considering energy, show that an empty water bottle is less stable than a bottle with certain amount of water.