Wilberforce Pendulum

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1 Introduction

A Wilberforce pendulum was invented by a British physicist Lionel Wilberforce at the end of the 19th century, which consists of a mass suspended by a long helical spring that can rotate about its vertical axis. Once released, it has two motions: bobbing up and down in the vertical direction and rotating back and forth about the vertical axis. With proper adjustments, purely bobbing and purely rotation can happen alternately. This intriguing motion will be investigated in this essay, with experiments carried out and results compared with the prediction given by the mathematical model.

2 Qualitative Explanation

When loaded, the spring will have longitudinal extension. However, the spring itself will have little extension due to its high Young's Modulus. Therefore, the spring's radius will decrease as shown in the diagram above which is resulted from a tangential force[1]. According to Newton's Third Law, there will be a force acting on the pendulum at the bottom, which will generate a torque.

3 Mathematical Model

3.1 Euler-Lagrange Equations

In this section, we ignore the damping effect caused by the air drag and assume a linear coupling relation[REB91]. Therefore, the Lagrangian of a Wilberforce Pendulum system can be written as[2]:

$$\mathcal{L} \equiv \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\delta\theta^2 - \frac{1}{2}\epsilon z\theta$$

This leads to two equations of motion with respect to z and θ, where z is defined as the displacement from the equilibrium point and θ as the angular displacement from the equilibrium point.

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\theta = 0$$

[1] The force is the tension on the spring, which must be tangent to the spring at any point.
[2] Dots above letters denote the derivative with respect to time. The number of dots shows the order of derivative.

$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

Taking the second derivative of the first equation gives:

$$\ddot{\theta} = -\frac{2}{\epsilon}(m\frac{d^4 z}{dt^4} + k\frac{d^2 z}{dt^2})$$

Rearranging the first equation gives:

$$\theta = -\frac{2}{\epsilon}(m\ddot{z} + kz)$$

The second equation can be written as:

$$2mI\frac{d^4 z}{dt^4} + 2(Ik + m\delta)\ddot{z} + (2\delta k - \frac{\epsilon^2}{2})z = 0$$

Define $\omega_z^2 = k/m$ and $\omega_\theta^2 = \delta/I$, and change the equation into[3]:

$$\frac{d^4 z}{dt^4} + (\omega_z^2 + \omega_\theta^2)\frac{d^2 z}{dt^2} + (\omega_z^2\omega_\theta^2 - \frac{\epsilon^2}{4mI})z = 0$$

The equation of θ is in the same form:

$$\frac{d^4 \theta}{dt^4} + (\omega_z^2 + \omega_\theta^2)\frac{d^2 \theta}{dt^2} + (\omega_z^2\omega_\theta^2 - \frac{\epsilon^2}{4mI})\theta = 0$$

3.2 Solving the Equations

Since the equations have been simplified to ordinary differential equations, the solutions are in the form of:

$$\theta(t) = \theta_0 e^{i\omega t}$$
$$z(t) = z_0 e^{i\omega t}$$

Substituting them back into the previous equations generates:

$$\omega^4 - (\omega_z^2 + \omega_\theta^2)\omega^2 + (\omega_z^2\omega_\theta^2 - \frac{\epsilon^2}{4mI}) = 0$$

The quadratic formula gives:

$$w^2 = \frac{\omega_z^2 + \omega_\theta^2 \pm \sqrt{(\omega_z^2 - \omega_\theta^2)^2 + \frac{\epsilon^2}{mI}}}{2}$$

The equations of motion can be written as[4]

$$\theta(t) = A_1 sin(\omega_1 t) + B_1 cos(\omega_1 t) + C_1 sin(\omega_2 t) + D_1 cos(\omega_2 t)$$

$$z(t) = A_2 sin(\omega_1 t) + B_2 cos(\omega_1 t) + C_2 sin(\omega_2 t) + D_2 cos(\omega_2 t)$$

Then, the frequency of beats is[Zha23]

$$f = f_1 - f_2$$

where $f_1 = \omega_1/2\pi$ and $f_2 = \omega_2/2\pi$.

4 Measurements

4.1 Mass of the pendulum

The mass of pendulum is given by the reading on the electronic balance.

$$mass = 0.2695 \pm 0.0001kg$$

[3] ω_z is the natural frequency of a mass-spring system, ω_θ the one of rotational frequency

[4] Though there are four solutions ω_1, ω_2, ω_3, and ω_4, there are only two frequencies. Negative roots can be dealt with trigonometry identities: $sinA = -sin(-A)$ and $cosA = cos(-A)$.

4.2 Stiffness of the spring

The stiffness was calculated by using the gradient of mass-extension graph times the gravitational acceleration.

$$k = 0.705 \times 9.81 = 6.92 N/m$$

4.3 Moment of inertia

Moment of inertia cannot be measured directly; it needs to be calculated. A light gate was used here with a rotatble base connected to a torsional spring at the bottom so that the time taken for each oscillation can be measured. A cylinder was placed on the base with the axis it rotates about aligning with the rotational axis of the base, and the average period of oscillations was measured for reference[5]. Then, the same process was conducted with the pendulum replacing the cylinder. Afterwards, the masses of the cylinder was measured.

If the oscillations can be described as simple (damped) harmonic motion[6], then the period of oscillations is proportional the square root of moment of inertia.

$$T \propto \sqrt{I}$$

$$I_{cylinder} = \frac{1}{2}MR^2 = \frac{1}{2} \times 0.4396 \times 0.05^2 = 5.495 \times 10^{-4} \pm 2.037\% \; kg \cdot m^2$$

The average period of oscillations of the base without loads is $0.641s$, and that of the base plus the cylinder $0.932s$.

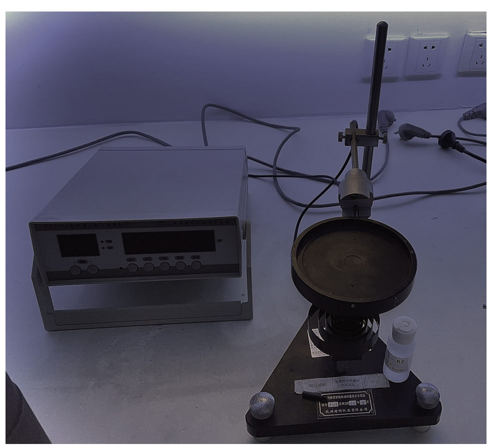
$$\frac{\sqrt{I_0}}{0.641} = \frac{\sqrt{I_0 + 5.495 \times 10^{-4}}}{0.932}$$

$$I_0 = 4.931 \times 10^{-4} \pm 2.564\% \; kg \cdot m^2$$

The average period of oscillations of the base plus the pendulum is $0.668s$.

$$\frac{\sqrt{I_0}}{0.641} = \frac{\sqrt{I_0 + I_{pendulum}}}{0.668}$$

$$I_{pendulum} = 4.242 \times 10^{-5} \pm 3.175\% \; kg \cdot m^2$$



4.4 Displacement

The displacement was tracked by an ultrasonic distance sensor connected to a laptop and placed beneath the pendulum. Then, the laptop used Pasco to plot a graph of position against time.

[5] A cylinder is chosen for reference as it is in a regular geometric shape so that the moment of inertia can be easily calculated simply by using the dimensions.

[6] Here, an assumption is made that the restoring force is proportional to angular displacement at small angular displacements.

4.5 Rotation and its Period

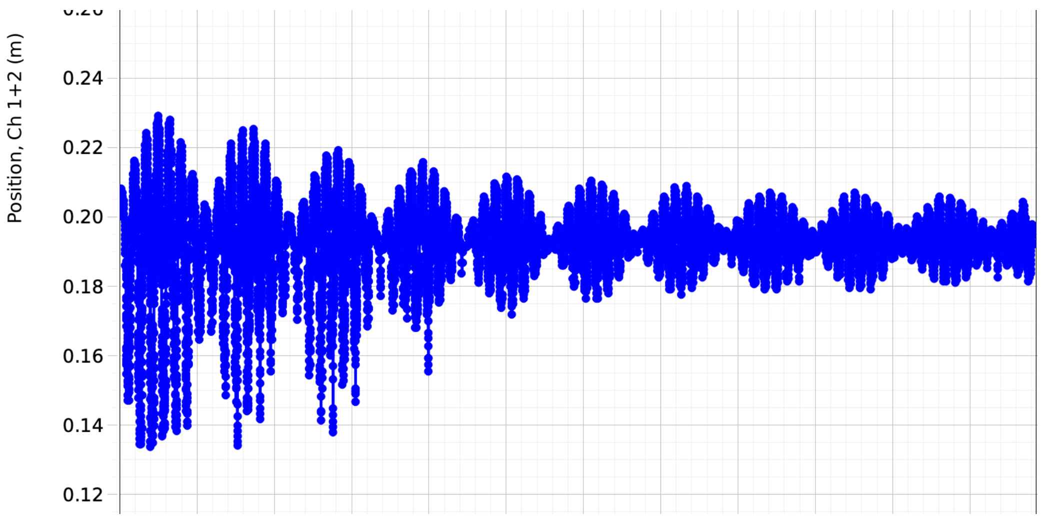
The period of rotation was measured by timing 20 rotations and dividing the total time by 20. This process was repeated for another five times to reduce random error.

$$T_{pendulum} = 1.535 \pm 0.02s$$

[7]

5 Data Analysis

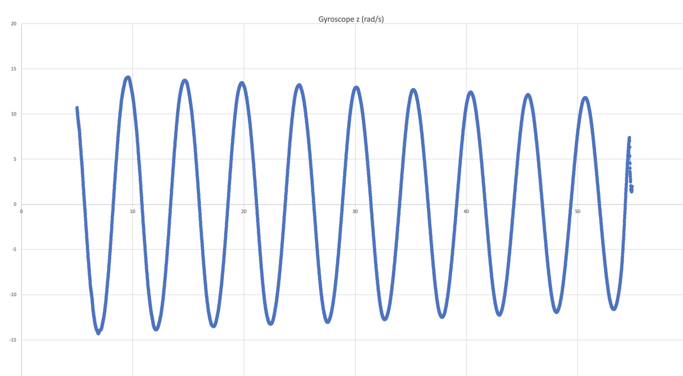
5.1 Displacement against Time



$$T = 10.9 \pm 0.3s$$

5.2 Predicted Period of Rotation

Unfortunately, a gyroscope cannot be attached to the pendulum during the motion as ithe pendulum will swing in all directions, which stops the ultrasonic distance sensor from tracking the position. As such, the gyroscope was only used to measure the rotation of the pendulum at the same level.



The average period of oscillations is $T_G = 5.14s$. The same method in the $Section 4.3$ was then used to determine the moment of inertia of the pendulum and the gyroscope as an entity.

$$\frac{\sqrt{I_0}}{0.641} = \frac{\sqrt{I_0 + I_G}}{0.888}$$

[7] $\pm 0.02s$ is due to human reaction.

$$I_G = 4.534 \times 10^{-4} \pm 3.101\%\ kg \cdot m^2$$

At small angular displacements, the rotation can be approximated by harmonic motion. Therefore, there is a mathematical relation:

$$T \propto \sqrt{I}$$

$$\frac{T_G}{\sqrt{I_G}} = \frac{T_{pendulum}}{\sqrt{I_{pendulum}}}$$

$$T_{pendulum} = 1.575 \pm 3.333\%\ s$$

Compared with the measured period, the percentage uncertainty is:

$$\frac{1.575 - 1.535}{1.535} \times 100\% = 2.606\%$$

which is within the uncertainty.

5.3 Coupling Effect

If there is no coupling relation between the bobbing and the rotation, the period of beat will be the inverted number of the difference between two frequencies. The equation in $Section 3.1$ can be reduced to:

$$w^2 = \frac{\omega_z^2 + \omega_\theta^2 \pm \sqrt{(\omega_z^2 - \omega_\theta^2)^2}}{2}$$

which gives two solutions of ω:

$$\omega_1 = \omega_z$$

$$\omega_2 = \omega_\theta$$

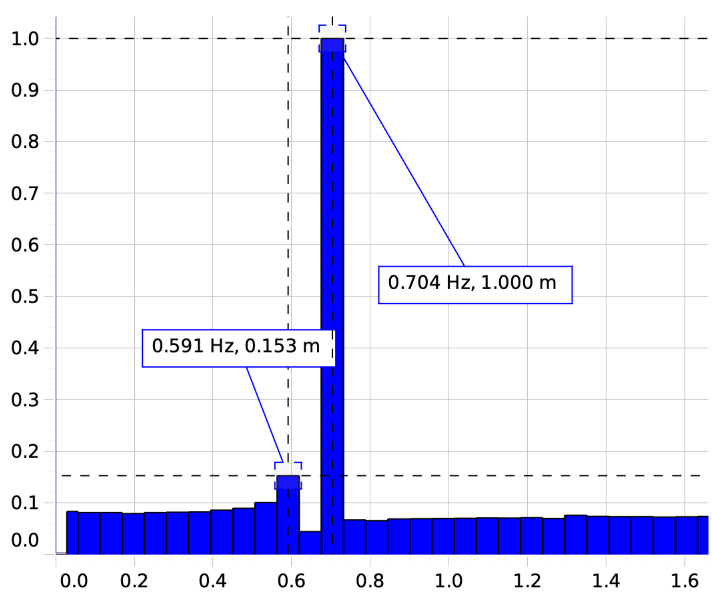
Hence, the period of beat can be calculated:

$$f_z = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 0.945Hz$$

$$f_\theta = \frac{1}{T_{pendulum}} = 0.651Hz$$

$$T = \frac{1}{f_z - f_\theta} = 3.401s$$

The Fast Fourier Transformation tool in Pasco gives the frequencies of two normal modes of the



motion. The beat frequency calculated shouild be:

$$T = \frac{1}{f_1 - f_2} = 8.850s$$

which is closer to the measured value and proves the coupling effect between two motions.

6 Conclusion

In the end, even though the prediction of the mathematical model has differences greater than the uncertainties of measurements, it stills indicates the dependence between two motions. In fact, only the bobbing and rotation were examined in the experiment. The swings of the pendulum in different directions were neglected; the air drag not taken into account. Further investigations can be carried out to address these problems.

7 Further Investigation

7.1 Air Drag

Dynamic equations of the system can be written as:

$$m\ddot{z} + C\dot{z} + kz + \frac{1}{2}\epsilon\theta = 0$$

$$I\ddot{\theta} + \mathcal{C}\dot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

which can be simplified to fourth order linear differential equation:

$$A\frac{d^4\theta}{dt^4} + B\frac{d^3\theta}{dt^3} + D\frac{d^2\theta}{dt^2} + E\frac{d\theta}{dt} + F\theta = 0$$

$$A\frac{d^4 z}{dt^4} + B\frac{d^3 z}{dt^3} + D\frac{d^2 z}{dt^2} + E\frac{dz}{dt} + Fz = 0$$

where $A = mI$, $B = m\mathcal{C} + CI$, $D = m\delta + Ik + C\mathcal{C}$, $E = \mathcal{C}k + C\delta$, $F = k\delta - \frac{\epsilon^2}{4}$. A general solution to ordinary differential equation can be written in the form of:

$$\theta(t) = \theta_0 e^{i\omega t}$$

$$z(t) = z_0 e^{i\omega t}$$

Putting this back into the equation gives:

$$A\omega^4 - Bi\omega^3 - D\omega^2 + Ei\omega + F = 0$$

7.2 More than Bobbing and Rotating

In reality, the pendulum's motion is more complicated as it will also swing. Here, four variables are needed to describe the motion including the swings, so the lagrangian of the system should be rewritten as:

$$\mathcal{L}(z,\theta,\phi,\gamma) \equiv \frac{1}{2}m[\dot{z}^2 + (l_0+z)^2\dot{\phi}^2 + (l_0+z)^2\dot{\gamma}^2] + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\delta\theta^2 - \frac{1}{2}\epsilon z\theta - V(z,\phi,\gamma)$$

where z is the extension of the spring in its longitudinal direction; θ is the angular displacement of the pendulum itself; ϕ is the angular deflection of the pendulum from the vertical in one plane; γ is the angular deflection of the pendulum from the vertical in the plane perpendicular to the plane of ϕ[8]; V is the gravitational potential energy.

[8] Two planes intersect at the line where the pendulum rests.

This will give four Euler-Lagrangian equations for the motion:

$$m(l_0 + z)(\dot{\phi}^2 + \dot{\gamma}^2) - kx - \frac{1}{2}\epsilon\theta + \frac{1}{sec^2\gamma + sec^2\theta - 1} = m\ddot{z}$$

$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

$$(l_0 + z)\ddot{\gamma} = \frac{2gsec^2\gamma tan\gamma}{(sec^2\gamma + sec^2\phi - 1)^2}$$

$$(l_0 + z)\ddot{\phi} = \frac{2gsec^2\phi tan\phi}{(sec^2\gamma + sec^2\phi - 1)^2}$$

References

[REB91] Todd S. marshall Richard E. Berg. Wilberforce pendulum oscillations and normal modes. *American Journal of Physics*, 59, Jan 1991.

[Zha23] Dina Zhabinskaya. 8.6: Beats. *Physics LibreTexts*, Jan 2023.