SINGULAR VALUE DECOMPOSITION

Singular Value Decomposition (SVD) is a method for decomposing a matrix into a product of three matrices.

WHY DECOMPOSITION?

Decomposing matrices:

- Can simplify and speed up operations with them
- Can assist in dimensionality reduction (PCA)

COMPARING DECOMPOSITION METHODS

Decomposition Method
Eigendecomposition
LU
Non-negative matrix factorization
SVD

COMPARING DECOMPOSITION METHODS

Decomposition Method	Works with rectangular matrices?	Works with non-diagonalizable matrices?	Works with matrices with negative elements?
Eigendecomposition			
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Non-negative matrix factorization	✅	✅	❌
SVD			

COMPARING DECOMPOSITION METHODS

Decomposition Method	Works with rectangular matrices?	Works with non-diagonalizable matrices?	Works with matrices with negative elements?
Eigendecomposition	❌	❌	✅
LU	❌	✅	✅
Non-negative matrix factorization	✅	✅	❌
SVD	✅	✅	✅

Is SVD perfect?

Is SVD perfect?

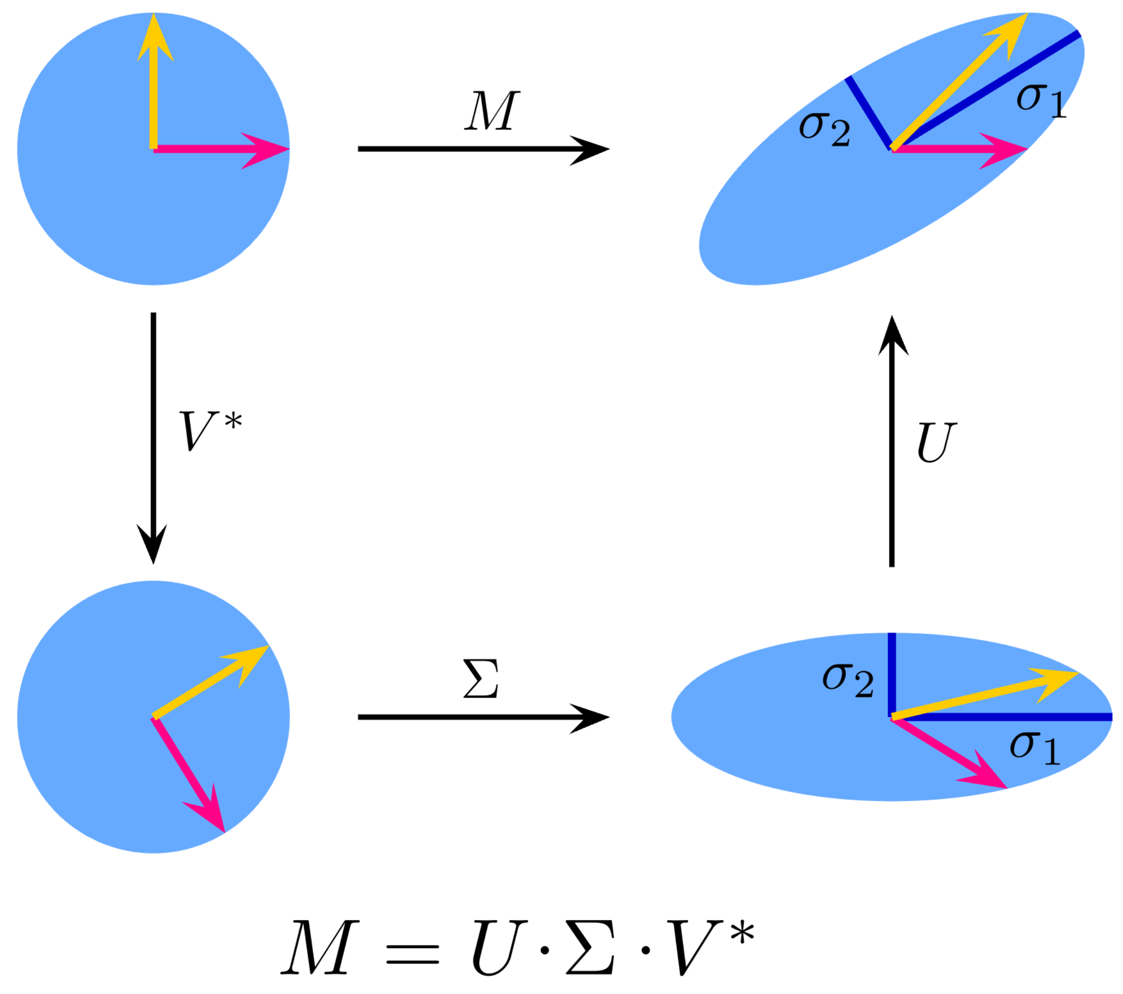
No.

- SVD is computationally expensive.
- However, SVD is **widely applicable** and **powerful**.

WHAT IS SVD?

Breaks a linear transformation (matrix) into three* steps:

- V^* - Rotate
  - * = Transposed
  - Generalizes to higher dimensional space
- Σ
  - Expand/reduce dimensionality
  - Scale each axis
- U - Rotate



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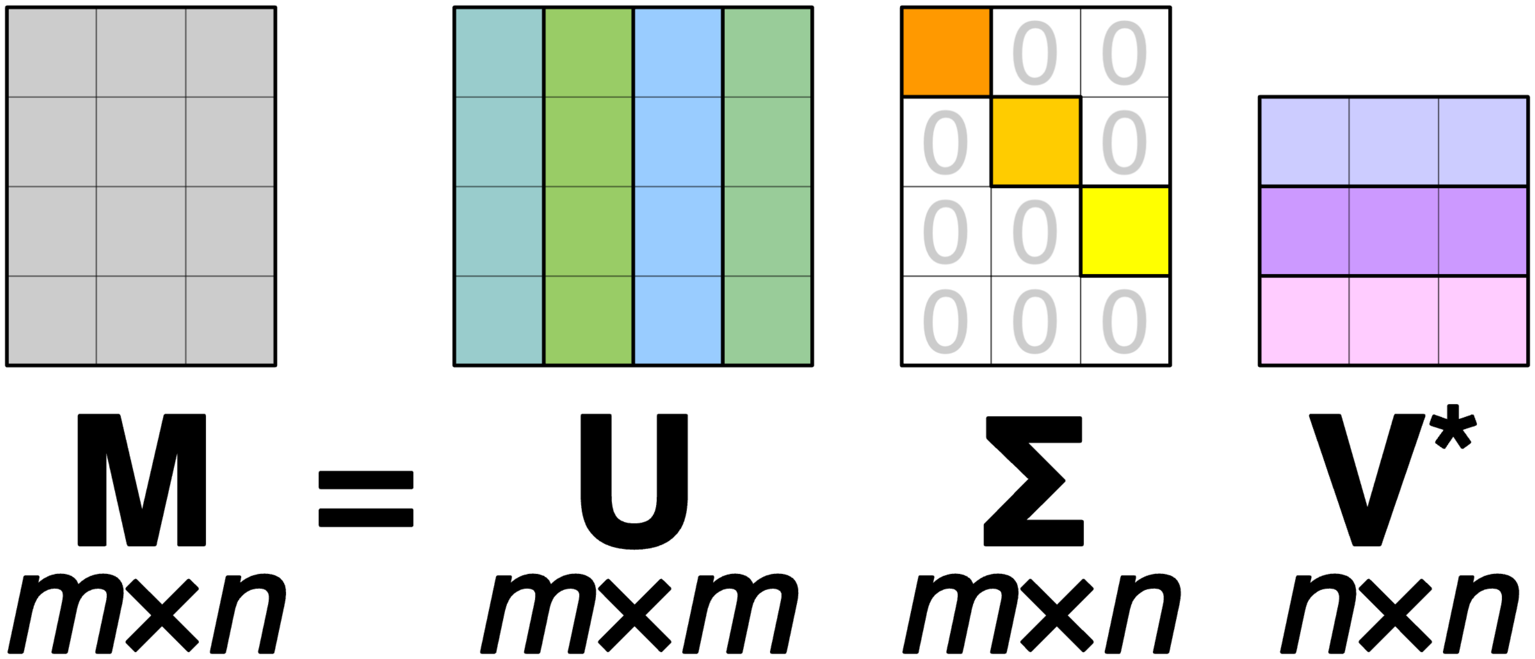
SVD VISUALIZED

Animation Attribution: By Lucas Vieira - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=11416486

ANATOMY OF SVD



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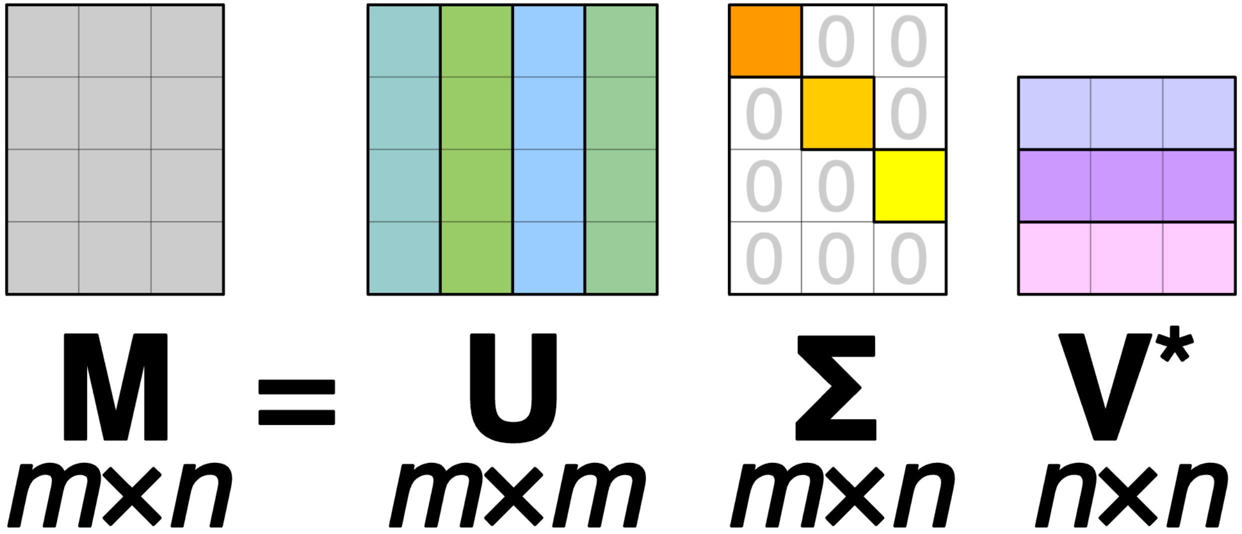
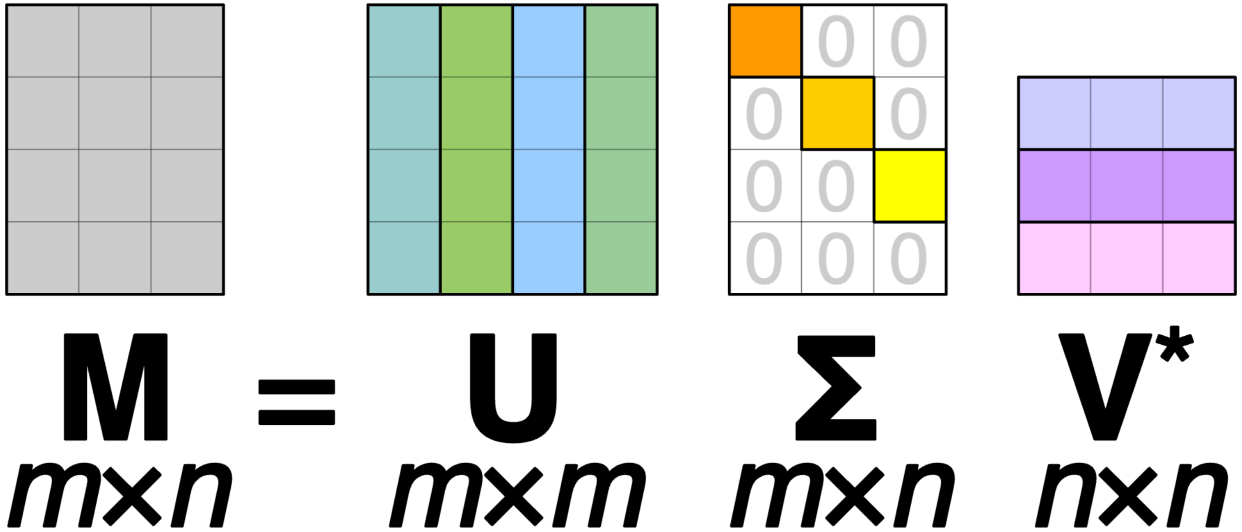


A real or complex $m \times n$ matrix.

ANATOMY OF SVD



An orthogonal $m \times m$ matrix composed of the left **singular vectors** of M.

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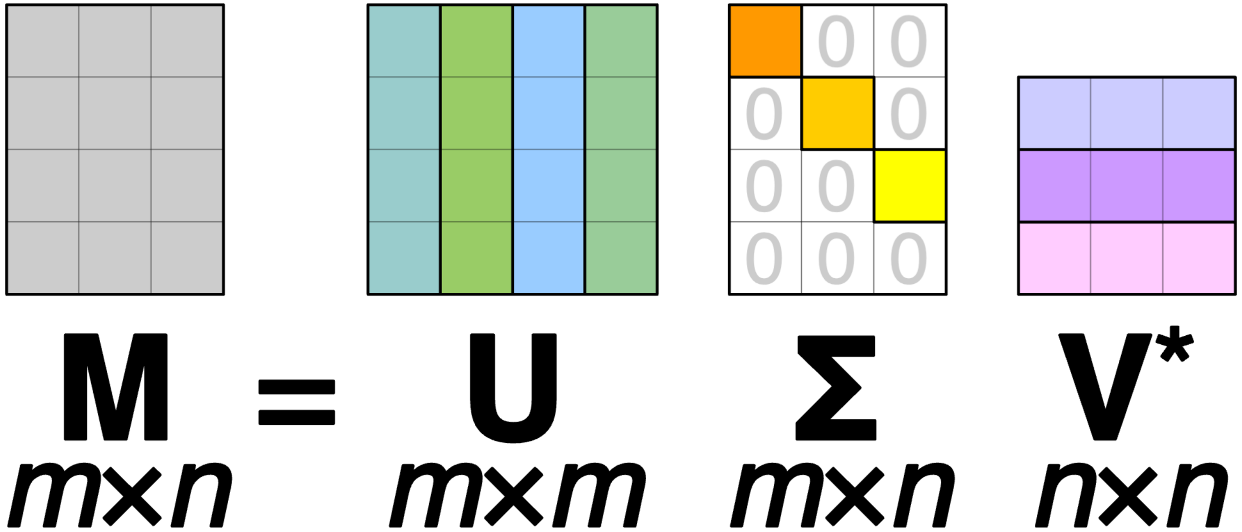
ANATOMY OF SVD



A rectangular diagonal $m \times n$ matrix composed of the **singular values** of M.

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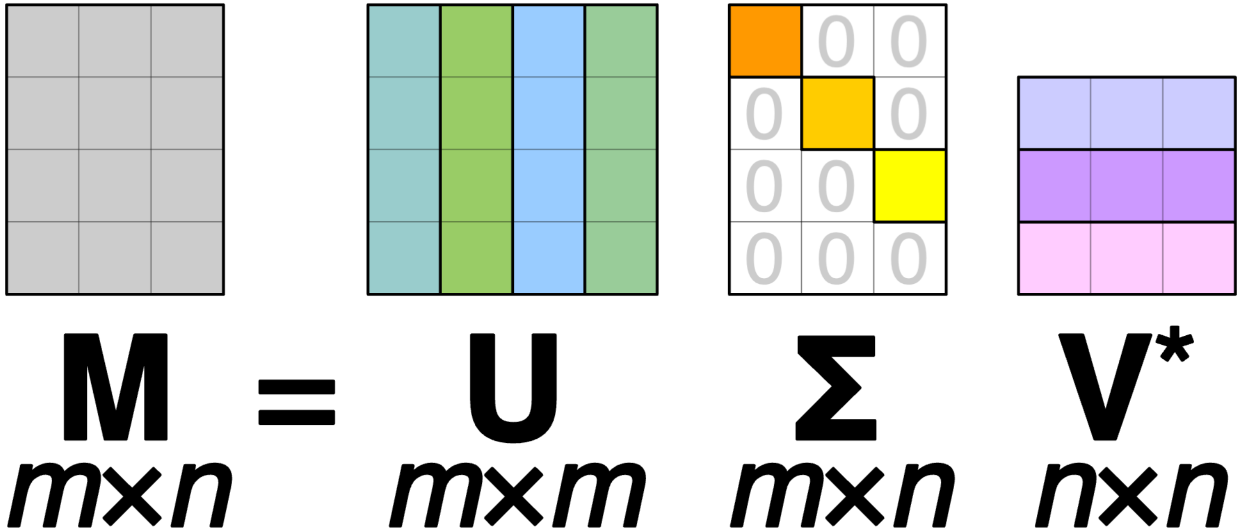
ANATOMY OF SVD



An orthogonal n×n matrix composed of the right singular vectors of M.

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SINGULAR VALUES AND VECTORS

- What are **singular values** and **singular vectors**?
- Think **eigenvalues** and **eigenvectors** for rectangular matrices
  - Rectangular matrix - for $m \times n$ matrix, $m \neq n$
- **Formally**, for a matrix M, the eigenvalues and eigenvectors of MM^* and M^*M.

How do we get the singular values and singular vectors of a matrix?

COMPUTING SYMMETRIC MATRICES

$$M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$(m \times n, m = 2, n = 3)$$

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COMPUTING SYMMETRIC MATRICES

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$$MM^* \text{ or } S_L = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$
$$(m \times m,\ m = 2)$$

$$M^*M \text{ or } S_R = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$
$$(n \times n,\ n = 3)$$

Now, compute the eigenvalues and eigenvectors of both.

SINGULAR VALUES

Left Symmetric Matrix

1. Calculate $S_L - \lambda I$:
$$\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}17-\lambda & 8\\ 8 & 17-\lambda\end{bmatrix}$$
2. Get the determinant of $S_L - \lambda I$:
$$\text{determinant}\left(\begin{bmatrix}17-\lambda & 8\\ 8 & 17-\lambda\end{bmatrix}\right)$$
$$= (17-\lambda)(17-\lambda) - (8)(8)$$
$$= \lambda^2 - 34x + 225$$
3. Solve the characteristic equation:
$$\lambda^2 - 34x + 225 = 0$$
$$(\lambda - 9)(\lambda - 25) = 0$$
$$\lambda_1 = 9, \lambda_2 = 25$$
4. Get singular values from eigenvalues:
$$s_1 = \sqrt{25} = 5$$
$$s_2 = \sqrt{9} = 3$$

Right Symmetric Matrix

1. Calculate $S_R - \lambda I$:
$$\begin{bmatrix}13 & 12 & 2\\ 12 & 13 & -2\\ 2 & -2 & 8\end{bmatrix} - \begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} = \begin{bmatrix}13-\lambda & 12 & 2\\ 12 & 13-\lambda & -2\\ 2 & -2 & 8-\lambda\end{bmatrix}$$
2. Get the determinant of $S_R - \lambda I$:
$$\text{determinant}(S_R - \lambda I)$$
$$= \cdots$$
$$= -\lambda^3 + 34\lambda^2 - 225\lambda$$
3. Solve the characteristic equation:
$$-\lambda^3 + 34\lambda^2 - 225\lambda = 0$$
$$-\lambda(\lambda - 9)(\lambda - 25) = 0$$
$$\lambda_1 = 0, \lambda_2 = 9, \lambda_3 = 25$$
4. Get singular values from eigenvalues:
$$s_1 = \sqrt{25} = 5$$
$$s_2 = \sqrt{9} = 3$$
Third eigenvalue dropped

NOTES ABOUT SINGULAR VALUES

- Singular values are sorted in **descending order**.
- Singular values must be the **same** between S_L and S_R.
  - If an eigenvalue is not an eigenvalue of both matrices, it is **dropped**.

COMPUTING SIGMA

- Now that we have our singular values, we can compute Σ.
- We know Σ is 2x3.
- $s_1 = 5, s_2 = 3$
- $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

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- Note:

$$\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ reduces from 3 dimensions into 2 dimensions.
- $\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ scales x-axis by 5 and y-axis by 3.

COMPUTING U

- U is composed of the **normalized eigenvectors** of S_L
- First eigenvector $(\lambda_1 = 9)$: $(S_L - 9I)\begin{bmatrix} x \\ y \end{bmatrix} = 0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- Second eigenvector $(\lambda_2 = 25)$: $(S_L - 25I)\begin{bmatrix} x \\ y \end{bmatrix} = 0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Normalize eigenvectors:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \div \sqrt{(-1)^2 + (1)^2} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \;\&\; \begin{bmatrix} 1 \\ 1 \end{bmatrix} \div \sqrt{(1)^2 + (1)^2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Arrange into U **based on the order of corresponding singular values:**

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

COMPUTING V*

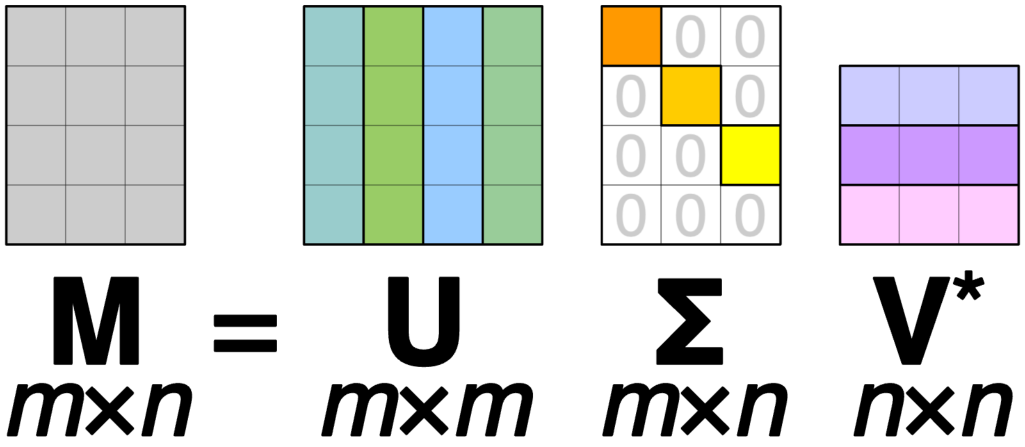
- V^* is composed of the **normalized eigenvectors** of S_R, transposed.
- First eigenvector ($\lambda_1 = 0$): $(S_R)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$
- Second eigenvector ($\lambda_2 = 9$): $(S_R - 9I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$
- Third eigenvector ($\lambda_3 = 25$): $(S_R - 25I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- Normalize eigenvectors:

$$\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \to \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \to \begin{bmatrix} -\frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \\ -\frac{\sqrt{8}}{3} \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

- Arrange into V and **transpose:**

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{bmatrix}, V^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & -\frac{\sqrt{8}}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

SVD



$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} & -\frac{\sqrt{8}}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

THANK YOU