I.5. **1F** ③ $y = x^{\frac{1}{n}} \Leftrightarrow y^n = x$

Differentiating both sides wrt $x$, $ny^{n-1}\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{n}y^{1-n} = \underline{\frac{1}{n}x^{\frac{1}{n}-1}}$ //

⑤ $\sin x + \sin y = \frac{1}{2}$.

Differentiating both sides wrt $x$, $\cos x + \cos y \frac{dy}{dx} = 0$

Horizontal tangent lines imply $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \underline{\frac{\pi}{2}} + k\pi,\ k \in \mathbb{Z}$.

When k is even, $\sin x = 1 \Rightarrow \sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6} + 2n\pi$ or $\frac{7\pi}{6}\pi + 2n\pi,\ n \in \mathbb{Z}$

When k is odd, $\sin x = -1 \Rightarrow \sin y = \frac{3}{2} \Rightarrow y \notin \mathbb{R}$ [$\because \sin y \in [-1,1]\ \forall y \in \mathbb{R}$].

Thus the points are $\begin{cases} (\frac{\pi}{2} + 2k\pi,\ -\frac{\pi}{6} + 2n\pi), & k, n \in \mathbb{Z} \\ (\underline{\frac{\pi}{2} + 2k\pi,\ \frac{7\pi}{6}} + 2n\pi), & k, n \in \mathbb{Z}. \end{cases}$ //

(8c) Given $c^2 = a^2 + b^2 - 2ab\cos\theta$, differentiating both sides by b, and treating $c, \theta$ as constants,

$0 = 2a\frac{da}{db} + 2b - 2\cos\theta\left(b\frac{da}{db} + a\right)$

$2a\cos\theta - 2b = \frac{da}{db}(2a - 2b\cos\theta)$

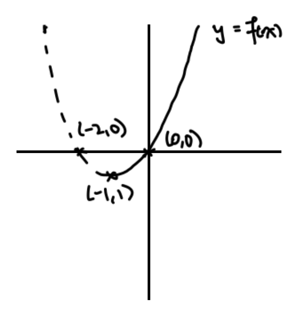
$\Rightarrow \frac{da}{db} = \frac{2a\cos\theta - 2b}{2a - 2b\cos\theta}$ //

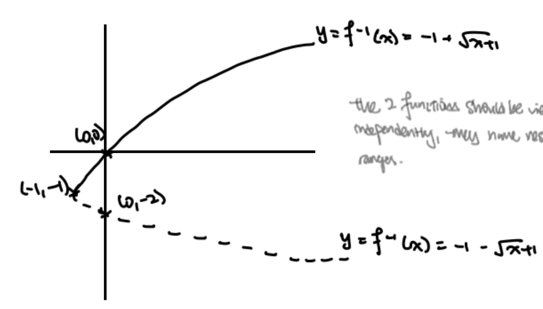
**1A** (5b) Given $y = x^2 + 2x$, let $y = f(x)$.

$y + 1 = x^2 + 2x + 1 = (x+1)^2$

$x + 1 = \pm\sqrt{y+1} \Rightarrow x = -1 \pm \sqrt{y+1}$

Interchange betw. $x$ & $y$, $f^{-1}(x) = -1 \pm \sqrt{x+1}$



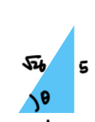




the 2 functions should be viewed independently, they have restricted ranges.

**5A** (1a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

(1b) $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

(1c)




$\sin\theta = \frac{5}{\sqrt{26}}$, $\cos\theta = \frac{1}{\sqrt{26}}$,

$\cot\theta = \frac{1}{5}$, $\mathrm{cosec}\,\theta = \frac{\sqrt{26}}{5}$,

$\sec\theta = \sqrt{26}$.

(3f) $y = \sin^{-1}\left(\frac{a}{x}\right) \Leftrightarrow \sin y = \frac{a}{x}$.

Differentiating both sides by $x$, $\frac{dy}{dx}\cos y = -\frac{a}{x^2}$

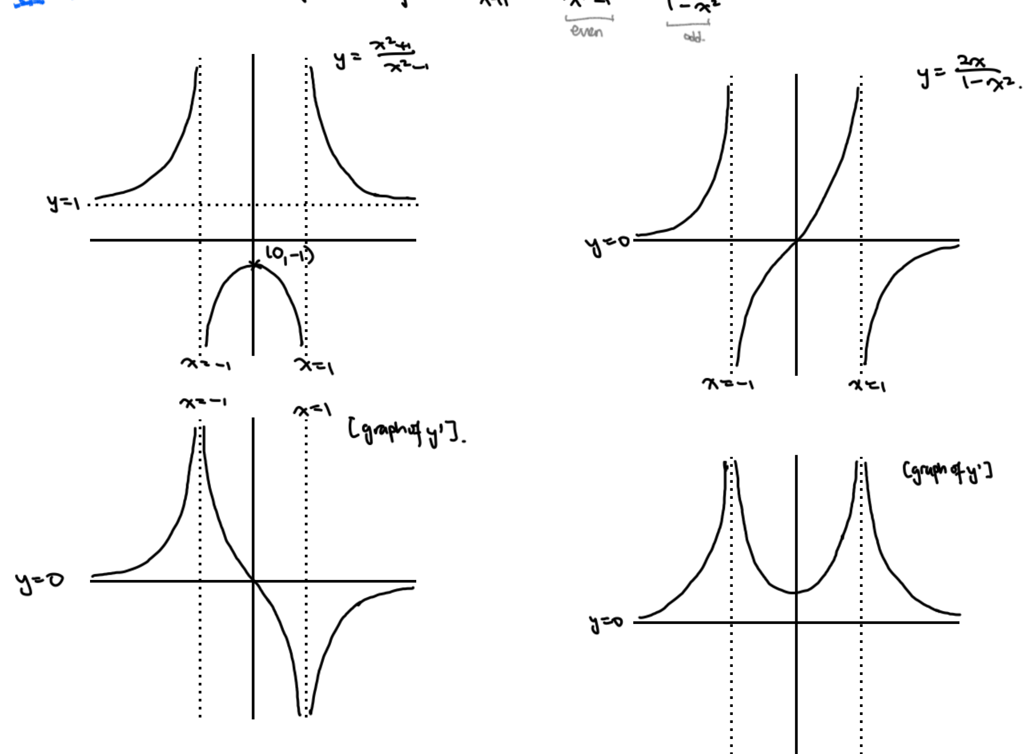
$\therefore \frac{dy}{dx} = -\frac{a}{x^2\cos y} = -\frac{a}{x^2\sqrt{x^2 - a^2}}$ //

(3h) $y = \sin^{-1}\sqrt{1-x} \Leftrightarrow \sin y = \sqrt{1-x}$

Diff. both sides wrt $x$, $\frac{dy}{dx}\cos y = -\frac{1}{2\sqrt{1-x}}$.

$\therefore \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x}} \cdot \sec y = -\frac{1}{2\sqrt{x(1-x)}}$ //

II. 1. In Problem 1, Part II of PS1. we found $\frac{x-1}{x+1} = \underbrace{\frac{x^2+1}{x^2-1}}_{\text{even}} + \underbrace{\frac{2x}{1-x^2}}_{\text{odd}}$




II. 2. (a) $\frac{d}{dx}\tan^3(x^4) = 3\tan^2(x^4) \cdot \sec^2(x^4) \cdot 4x^3$

$= 12x^3\tan^2(x^4)\sec^2(x^4)$. //

(b) $\frac{d}{dy}(\sin^2 y\cos^2 y) = (2\sin y\cos y)\cos^2 y + (-2\sin y\cos y)\sin^2 y$

$= 2\sin y\cos y(\cos^2 y - \sin^2 y)$

$= \sin 2y\cos 2y$.

$\frac{d}{dy}(\sin^2 y\cos^2 y) = \frac{d}{dy}(\sin y\cos y)^2$

$= \frac{1}{4}\frac{d}{dy}\sin^2 2y$

$= \frac{1}{4}(2\sin 2y\cos 2y \cdot 2)$

$= \sin 2y\cos 2y$. //

II. 3. (a) Given $y = uv$,

$y' = u'v + uv'$.

$y'' = (u'v)' + (uv')'$

$= u''v + u'v' + u'v' + uv''$

$= u''v + \underline{2u'v'} + uv''$. //

(b) $y''' = (u''v)' + (2u'v')' + (uv'')'$

$= u'''v + u''v' + 2(u''v' + u'v'') + u'v'' + uv'''$

$= u'''v + 3u''v' + 3u'v'' + uv'''$ //

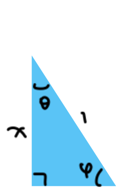
II 4. (a) let $\theta = \cos^{-1}x$, for $0 \le \theta \le \pi$.

$\cos\theta = x$

Diff. both sides wrt $x$, $-\sin\theta \cdot \frac{d\theta}{dx} = 1$

$\therefore \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\cos^2\theta}} = -\frac{1}{\sqrt{1-x^2}}$ //

(b)




let $\theta = \cos^{-1}x$, $\varphi = \sin^{-1}x$.

Then no matter what $x$ is, $\theta + \varphi = \frac{\pi}{2} =$ constant.

$\therefore \frac{d}{dx}\cos^{-1}x + \frac{d}{dx}\sin^{-1}x = \frac{d}{dx}(\cos^{-1}x + \sin^{-1}x)$

$= \frac{d}{dx}(\theta + \varphi)$

$= \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$. //

I.6 **1H** ① (a) When $t = \lambda$, $y_0e^{k\lambda} = \frac{1}{2}y_0$

$\Rightarrow e^{k\lambda} = \frac{1}{2} \Rightarrow \lambda = \underline{\frac{1}{k}\ln\frac{1}{2}}$ //

(b) $y_1 = y_0e^{kt_1}$

Then, when $t = t_1 + \lambda$,

$y = y_0e^{k(t_1+\lambda)}$

$= y_0e^{kt_1}e^{k\lambda} = y_1e^{\ln\frac{1}{2}} = \underline{\frac{1}{2}y_1}$ //

② The formula for pH is $pH = -\log_{10}([H^+])$,

where $[H^+]$ is the concentration of $[H^+]$ ions.

If $[H^+]_{or} = 2[H^+]_{dil}$,

$-\lg([H^+]_{or}) = -\lg(2[H^+]_{dil})$

$= -(\lg([H^+]_{dil}) + \lg 2)$

$\therefore pH_{original} = \underline{pH_{diluted}} - \lg 2$ //

(3a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$ [$\Rightarrow x > 0$].

$\ln[(y+1)(y-1)] = \ln(xe^{2x})$

$y^2 - 1 = xe^{2x}$

$y^2 = xe^{2x} \Rightarrow y = \sqrt{xe^{2x}}$ [$\because y > 1$].

(5b) $y = e^x - e^{-x}$

If we let $u = e^x$, $y = u - \frac{1}{u} \Leftrightarrow u^2 - yu - 1 = 0$

$\Rightarrow u = \frac{y + \sqrt{y^2+4}}{2}$ [rej. -ve branch $\because u = e^x > 0$, and $\sqrt{y^2+4} > \sqrt{y^2} = y$. $\Rightarrow y - \sqrt{y^2+4} < 0$].

$\therefore e^x = \frac{y + \sqrt{y^2+4}}{2}$

$\Rightarrow x = \ln\left(\frac{y + \sqrt{y^2+4}}{2}\right)$ //

**1I** ① (c) $\frac{d}{dx}e^{-x^2} = (e^{-x^2})(-2x) = \underline{-2xe^{-x^2}}$ //

(d) $\frac{d}{dx}(x\ln x - x) = \ln x + \frac{x}{x} - 1 = \underline{\ln x}$ //

(e) $\frac{d}{dx}\ln(x^2) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$ //

(f) $\frac{d}{dx}[\ln x]^2 = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$ //

(m) $\frac{d}{dx}\left[\frac{1-e^x}{1+e^x}\right] = \frac{-e^x(1+e^x) - e^x(1-e^x)}{(1+e^x)^2}$

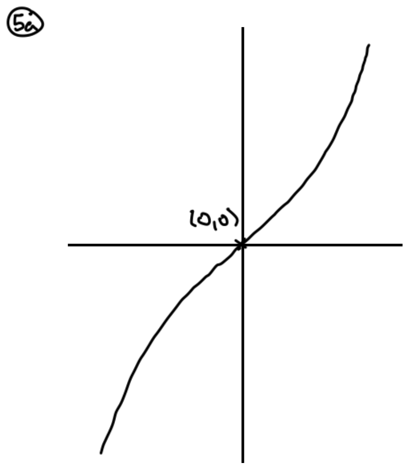
$= -\frac{2e^x}{(1+e^x)^2}$ //

(4a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^3 = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^3$

$= e^3$ //

I. 7

**5A** (5a)




$y = \sinh x$

$= \frac{1}{2}(e^x - e^{-x})$

$y' = \frac{1}{2}(e^x + e^{-x})$

$y' = 0 \Leftrightarrow x \notin \mathbb{R}$

$y'' = \frac{1}{2}(e^x - e^{-x})$

$y'' = 0 \Leftrightarrow x = 0 \Rightarrow$ pt. of inflection.

$\begin{cases} y'' > 0 \Rightarrow x > 0, \\ y'' < 0 \Rightarrow x < 0 \end{cases}$

Then $\sinh x = -\sinh(-x)$, so odd function, symmetric abt. origin.

as $x \to \infty$, $y \to \frac{1}{2}(\infty - 0) = \infty$.

as $x \to -\infty$, $y \to \frac{1}{2}(0 - \infty) = -\infty$.

(5b) let $y = \sinh^{-1}x \Leftrightarrow x = \sinh y$

$= \frac{1}{2}(e^y - e^{-y})$

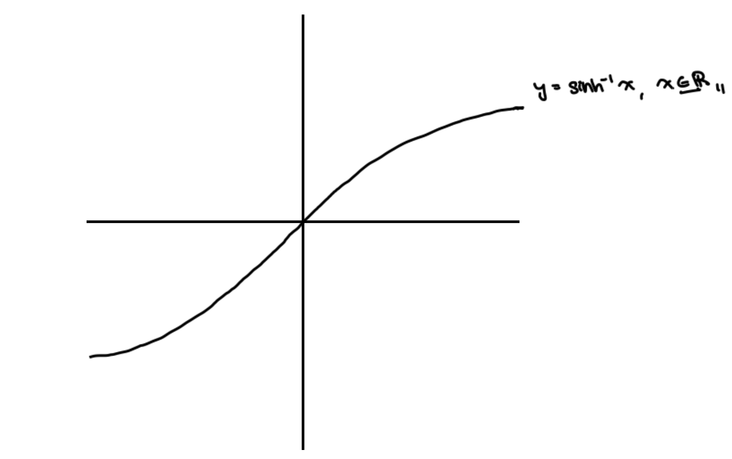
letting $u = e^y$, $2x = u - \frac{1}{u} \Leftrightarrow u^2 - 2xu - 1 = 0$

$\Rightarrow u = e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$= x \pm \sqrt{x^2+1}$

$\therefore y = \ln(x + \sqrt{x^2+1})$ [take +ve branch, as $\ln k \in \mathbb{R} \Rightarrow k > 0$]

Thus, $\underline{\sinh^{-1}x} = \ln(x + \sqrt{x^2+1})$ //




(5c) $y = \sinh^{-1}x = \ln(x + \sqrt{x^2+1})$

$\Rightarrow y' = \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(1 + \frac{1}{2\sqrt{x^2+1}}(2x)\right)$

$= \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$

$= \frac{x + \sqrt{x^2+1}}{(x + \sqrt{x^2+1})(\sqrt{x^2+1})}$

$= \frac{1}{\sqrt{x^2+1}}$ //

or

$y = \sinh^{-1}x \Leftrightarrow x = \sinh y$

$\frac{d}{dx}(x = \sinh y)$

$\Rightarrow 1 = (\cosh y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

Using $\cosh^2 y - \sinh^2 y = 1$, where $\cosh y \ge 1$,

$\frac{dy}{dx} = \frac{1}{1 + \sinh^2 y} = \frac{1}{1 + x^2}$ //

II 5.

**8.2** (8a) let $M_1$ and $M_2$ be the magnitudes of the 2 earthquakes,

where $M_1 = M_2 + 1$

since $M = \frac{2}{3}\lg\frac{E}{E_0} \Leftrightarrow E = E_0 \cdot 10^{3/2 \cdot M}$,

$\frac{E_1}{E_2} = \frac{10^{\frac{3}{2}M_1}}{10^{\frac{3}{2}M_2}} = 10^{(3/2)(M_1 - M_2)} = \underline{10^{3/2}}$ //

(8c) For an earthquake of magnitude 6, $E = E_0 \cdot 10^9 = 7 \cdot 10^6$ kWh

$\therefore$ no. of days supplied $= \frac{7 \cdot 10^6}{3 \cdot 10^5} = \frac{70}{3}$ days //

(10) We assume that $\log_3 2$ is rational, ie. $\log_3 2 = \frac{p}{q}$, for $p, q \in \mathbb{Z}$.

$\Rightarrow q\log_3 2 = p$

$2^q = 3^p$ ($\Rightarrow\Leftarrow$)

As 2 and 3 are both prime, there are no integers p and q that will satisfy the equation above.

Thus, $\log_3 2$ must be irrational.

(11) $1 < 2$ is true, but since $\underline{\log\left(\frac{1}{2}\right)} < 0$,

this must mean $\log\left(\frac{1}{2}\right) > 2\log\left(\frac{1}{2}\right)$ //

**8.4** (18) Given $y = \sqrt[3]{(x+1)(x-2)(2x+7)}$, and $\ln y = \frac{1}{3}[\ln(x+1) + \ln(x-2) + \ln(2x+7)]$.

$\frac{d}{dx}\left(\ln y = \frac{1}{3}[\ln(x+1) + \ln(x-2) + \ln(2x+7)]\right)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right]$

$\therefore \frac{dy}{dx} = \frac{\sqrt[3]{(x+1)(x-2)(2x+7)}}{3}\left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{2x+7}\right)$ //

(19a) Given $y = \frac{e^x(x^2-1)}{\sqrt{6x-2}}$,

$\ln y = x + \ln(x^2-1) - \frac{1}{2}\ln(6x-2)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 + \frac{2x}{x^2-1} - \frac{3}{6x-2}$

$\therefore \frac{dy}{dx} = \frac{e^x(x^2-1)}{\sqrt{6x-2}} + \frac{2xe^x}{\sqrt{6x-2}} - \frac{3e^x(x^2-1)}{\sqrt{(6x-2)^3}}$ //

II 6. let $y = u_1u_2\cdots u_n$, $\ln y = \ln(u_1u_2\cdots u_n)$

$= \ln u_1 + \ln u_2 + \cdots + \ln u_n$

diff both sides wrt $x$,

$\frac{y'}{y} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}$

$\therefore y' = (u_1u_2\cdots u_n)\left(\frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}\right)$

$= u_1'u_2\cdots u_n + u_1u_2'\cdots u_n + u_1u_2\cdots u_n'$ //