Exercise Solutions for
Foundations of Algebraic Geometry
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Part I

Preliminaries

Chapter 1

Some category theory

1.2.A. (a) If the object is $*$, then $\mathrm{Mor}(*,*)$ under the operation of composition has identity and associativity by the definition of a category, and inverses by the definition of a groupoid, so forms a group.

Conversely, given any group G we can define a category with a single object $*$, and $\mathrm{Mor}(*,*) = G$, with composition given by the group operation, which is a category by identity and associativity, and a groupoid since inverses exist.

(b) Take a category with two objects A, B, and no morphisms other than id_A and id_B. The identity morphisms are isomorphisms, so this is a groupoid, but it has more than one object.

1.2.B. Composition on $\mathrm{Mor}(A, A)$ has identity and associativity by the category axioms, and restricting to the invertible morphisms gives a subgroup of this monoid. For **Set** this is the symmetric group $\mathrm{Aut}(X) = \mathrm{Sym}(X)$ of bijections on X, for $\mathbf{Vec}_k$ it's the general linear group $\mathrm{Aut}(V) = \mathrm{GL}(V)$.

If $f : A \to B$ is an isomorphism, we can define a group homomorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$ by $\psi^f = f\psi f^{-1}$, which has inverse given by the same construction with f^{-1}, and hence is an isomorphism $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$. In fact we see that Aut defines a functor from any groupoid to **Grp** with this action on morphisms.

1.2.C. We have linear maps $V \to V^{\vee\vee}$ given by $v \mapsto E_v$, where $E_v(f) = f(v)$. These form a natural transformation (note that V is the identity functor evaluated at V), since if $T : V \to W$ is linear then

$$T^{\vee\vee}(E_v) = T^\vee \circ E_v = E_{T(v)}.$$

Moreover in the finite-dimensional case they are isomorphisms (from the dual basis construction), so this gives a natural isomorphism.

1.2.D. If we take bases B_V for every object V in $\mathbf{f.d.Vec}_k$, then we get a functor $\mathbf{f.d.Vec}_k \to \mathscr{V}$ by the basis-induced isomorphisms $V \cong k^{|B_V|}$. If we take B_{k^n} to be the standard basis, then this even gives a left-inverse to the inclusion functor. Now the isomorphisms $V \cong k^{|B_V|}$ induced by the bases B_V give a natural isomorphism from the identity to the composite functor $\mathbf{f.d.Vec}_k \to \mathbf{f.d.Vec}_k$ (which maps V to $k^{|B_V|}$), essentially by the definition of the functor $\mathbf{f.d.Vec}_k \to \mathscr{V}$; it sends a linear map $T : V \to W$ to the unique linear map $k^{|B_V|} \to k^{|B_W|}$ such that the following square commutes:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d] & W \arrow[d] \\
k^{|B_V|} \arrow[r] & k^{|B_W|}
\end{tikzcd}$$

1.3.A. If A, B are initial objects, then since there is always at least one map from A (resp. B) to another object, we have some $f : A \to B$ (resp. $g : B \to A$). Then since maps out of A (resp. B) to a given target are unique, we have $g \circ f = \mathrm{id}_A$ (resp. $f \circ g = \mathrm{id}_B$). Hence $A \cong B$, and in fact the isomorphism is unique since f and g are the only maps between A and B.

Since final objects are initial objects in the opposite category, and isomorphisms in $\mathscr{C}$ correspond one-to-one with isomorphisms in $\mathscr{C}^{\text{opp}}$, final objects are also uniquely isomorphic.

1.3.B. In **Set** the initial object is $0 = \emptyset$ and the final object is $1 = \{\emptyset\}$. The same is true for **Top**, where the only topologies on these sets make them discrete and indiscrete. In **CRing** the initial object is $\mathbb{Z}$ and the final object is the zero ring. In the poset category $\wp(X)$ (or of a topology) the initial object is $\emptyset$ and the final object is X.

1.3.C. Since $a/1 = 0$ iff $sa = 0$ for some $s \in S$, we have that $\ker(A \to S^{-1}A)$ is non-zero iff S contains a zero-divisor.

1.3.D. Suppose $f : A \to B$ is an A-algebra with $f(S) \subseteq U(B)$. Any A-algebra map $\psi : S^{-1}A \to B$ must have $\psi(a/1) = f(a)$ by definition, and then

$$f(s)\psi(a/s) = \psi(s)\psi(a/s) = \psi(a/1) = f(a),$$

so $\psi(a/s) = f(s)^{-1}f(a)$ since $f(s) \in U(B)$. It remains to show that $\psi(a/s) = f(s)^{-1}f(a)$ is well-defined and a homomorphism. But if $a/s = b/t$, then we have some $u \in S$ with $u(ta - sb) = 0$, so

$$f(u)\big(f(t)f(a) - f(s)f(b)\big) = 0 \implies f(s)^{-1}f(a) = f(t)^{-1}f(b).$$

The rules for arithmetic on fractions make ψ a homomorphism:

$$\begin{aligned}
\psi((a/s) + (b/t)) &= \big(f(s)f(t)\big)^{-1}\big(f(t)f(a) + f(s)f(b)\big) \\
&= f(s)^{-1}f(a) + f(t)^{-1}f(b) = \psi(a/s) + \psi(b/t), \\
\psi((a/s) \times (b/t)) &= \big(f(s)f(t)\big)^{-1}\big(f(a)f(b)\big) \\
&= f(s)^{-1}f(a) \times f(t)^{-1}f(b) \\
&= \psi(a/s) \times \psi(b/t).
\end{aligned}$$

This is clearly equivalent to the condition that every map $f : A \to B$ with $f(A) \subseteq U(B)$ factors uniquely through $A \to S^{-1}A$, since such a factoring is exactly an A-algebra map $S^{-1}A \to B$. Since the composition $A \to S^{-1}A \to B$ for any ring map $S^{-1}A \to B$ must send S into $U(B)$ as units map to units, this shows that any ring map out of $S^{-1}A$ is an instance of the above scenario, so that ring maps out of $S^{-1}A$ correspond precisely to ring maps out of A that send elements of S to units. Finally, $S^{-1}A$-modules are ring maps $S^{-1}A \to \operatorname{End}(M)$ for abelian groups M, which hence correspond to ring maps $A \to \operatorname{End}(M)$ where elements of S map to units. This shows that $S^{-1}A$-modules correspond to A-modules where every element of S acts as an isomorphism.

1.3.E. Define fractions $m/s \in S^{-1}M$ for $m \in M$, $s \in S$ by the equivalence relation $m/s = n/t$ iff there is some $u \in S$ with $u(tm - sn) = 0$. Then define addition by $(m/s) + (n/t) = (tm + sn)/(st)$, and scalar multiplication by $(r/s)(m/t) = (rm)/(st)$. These are well-defined, and give an $S^{-1}A$-module structure; the proofs for well-definedness, commutativity and associativity of addition are identical as for $S^{-1}A$, and scalar multiplication is:

- Well-defined, since if $r'/s' = r/s$ and $m'/t' = m/t$, we have say $u(sr' - s'r) = v(tm' - t'm) = 0$, and then

$$\begin{aligned}
uv(s't'rm - str'm') &= uv(s't'rm - s'trm' + s'trm' - str'm') \\
&= 0 + 0.
\end{aligned}$$

- Associative, since

$$\begin{aligned}
(r_1/s_1) \cdot (r_2m)/(s_2t) &= (r_1r_2m)/(s_1s_2t) \\
&= (r_1r_2)/(s_1s_2) \cdot (m/t).
\end{aligned}$$

- Distributive, since

$$\begin{aligned}(r/s)\cdot((m_1/t_1)+(m_2/t_2)) &= (r/s)\cdot(t_2m_1+t_1m_2)/(t_1t_2)\\ &= (rt_2m_1+rt_1m_2)/(st_1t_2)\\ &= (rst_2m_1+rst_1m_2)/(s^2t_1t_2)\\ &= (r/s)(m_1/t_1)+(r/s)(m_2/t_2).\end{aligned}$$

We can then define $\phi : M \to S^{-1}M$ by $m \mapsto m/1$, which is A-linear since the induced A-module structure on $S^{-1}A$ is given by $A \to S^{-1}A$, meaning $a(m/1) = (a/1)(m/1) = (am)/1$. If $f : M \to N$ is A-linear and S acts on N by isomorphisms (i.e. N is an $S^{-1}A$-module), then a factoring $\bar{f} : S^{-1}M \to N$ through ϕ is uniquely determined: we must have $\bar{f}(m/1) = f(m)$, and

$$\begin{aligned} s\bar{f}(m/s) = \bar{f}(sm/s) = \bar{f}(m/1) = f(m)\\ \implies \bar{f}(m/s) = s^{-1}f(m),\end{aligned}$$

where $s^{-1}(\cdot)$ denotes the inverse of multiplication by s on N. Defining $\bar{f}$ in this way gives a well-defined map, since if $u \in S$ satisfies $u(s'm - sm') = 0$ we get

$$u(s'f(m) - sf(m')) = 0 \implies s^{-1}f(m) = (s')^{-1}f(m'),$$

and it is A-linear since s^{-1} is A-linear. In fact $\bar{f}$ is $S^{-1}A$-linear, being an A-linear map of A-modules that are actually $S^{-1}A$-modules using the characterization from 1.3.D.

1.3.F. (a) Taking products of the natural maps, we have a map $f : M_1 \times \cdots \times M_n \to S^{-1}M_1 \times \cdots S^{-1}M_n$. Then since S acts as isomorphisms on the target (a product of isomorphisms is an isomorphism), this extends to a map $\bar{f} : S^{-1}(M_1 \times \cdots \times M_n) \to S^{-1}M_1 \times \cdots \times S^{-1}M_n$. Then

$$\begin{aligned}\ker(f) &= \{m \in M_1 \times \cdots \times M_n : s_im_i = 0 \text{ for some } s_i \in S \text{ for each } i\}\\ &= \{m \in M_1 \times \cdots \times M_n : sm = 0 \text{ for some } s \in S\}\end{aligned}$$

by taking $s = s_1 \cdots s_n$, and this is exactly the kernel of the map $M_1 \times \cdots \times M_n \to S^{-1}(M_1 \times \cdots M_n)$ so $\bar{f}$ is injective. Surjectivity follows since we have

$$(m_1/s_1, \ldots, m_n/s_n) = \big((s_2 \cdots s_n)m_1, \ldots, (s_1 \cdots s_{n-1})m_n\big)/(s_1 \cdots s_n).$$

Note that both halves of the proof rely on taking a product of almost all s_i's, and hence do not generalize to infinite products.

(b) By the same argument as in (a), the natural map $f : \bigoplus_i M_i \to \bigoplus_i S^{-1}M_i$ induces a map

$$\bar{f} : S^{-1}\bigoplus_i M_i \to \bigoplus_i S^{-1}M_i.$$

Conversely, the inclusions $M_j \to \bigoplus_i M_i$ give maps $S^{-1}M_j \to S^{-1}\bigoplus_i M_i$ by composing with $\bigoplus_i M_i \to S^{-1}\bigoplus_i M_i$ and noting that S then acts by isomorphisms on the target. Hence taking their sum gives a natural map

$$g : \bigoplus_i S^{-1}M_i \to S^{-1}\bigoplus_i M_i.$$

Define the following inclusions and localization maps:

$$\begin{aligned} \theta_j &: M_j \to \bigoplus_i M_i & \lambda_j &: M_j \to S^{-1}M_j\\ \psi_j &: S^{-1}M_j \to \bigoplus_i S^{-1}M_i & \lambda &: \bigoplus_i M_i \to S^{-1}\bigoplus_i M_i.\end{aligned}$$

The definitions give $\bar{f} \circ \lambda \circ \theta_j = \psi_j \circ \lambda_j$ and $g \circ \psi_j \circ \lambda_j = \lambda \circ \theta_j$, so

$$g \circ \bar{f} \circ \lambda \circ \theta_j = \lambda \circ \theta_j \qquad \text{and} \qquad \bar{f} \circ g \circ \psi_j \circ \lambda_j = \psi_j \circ \lambda_j,$$

meaning $g \circ \bar{f} = \mathrm{id}$, $\bar{f} \circ g = \mathrm{id}$ by the universal property of the direct sum. Hence we have an isomorphism.

(c) Taking $S = \mathbb{Z} \setminus 0$ and $M_i = \mathbb{Z}$, we have $S^{-1}\mathbb{Z}^{\mathbb{N}} \to \mathbb{Q}^{\mathbb{N}}$ given by $(a_i)_{i=1}^{\infty}/n \mapsto (a_i/n)_{i=1}^{\infty}$. Since n is a common denominator for the image sequence, but $(1/i)_{i=1}^{\infty} \in \mathbb{Q}^{\mathbb{N}}$ has no common denominator, we see that the map is not surjective.

1.3.G. More generally, $\mathbb{Z}/(n) \otimes \mathbb{Z}/(m) \cong \mathbb{Z}/(d)$ where $(n, m) = (d)$:

Each elementary tensor $a \otimes b = ab(1 \otimes 1)$ for $a, b \in \mathbb{Z}$, so $1 \otimes 1$ generates the tensor product. It has order dividing n (and similarly m) since $n(1 \otimes 1) = n \otimes 1 = 0 \otimes 1 = 0$. Hence it has order dividing d, so we can define a surjection $\mathbb{Z}/(d) \to \mathbb{Z}/(n) \otimes \mathbb{Z}/(m)$ by $1 \mapsto 1 \otimes 1$.

Conversely, since d divides n and m we have homomorphisms $\mathbb{Z}/(n) \to \mathbb{Z}/(d)$ and $\mathbb{Z}/(m) \to \mathbb{Z}/(d)$. Taking their tensor product gives a linear map $\mathbb{Z}/(n) \otimes \mathbb{Z}/(m) \to \mathbb{Z}/(d) \otimes \mathbb{Z}/(d)$, and we can compose this with the multiplication map $\mathbb{Z}/(d) \otimes \mathbb{Z}/(d) \to \mathbb{Z}/(d)$ (which exists since multiplication is $\mathbb{Z}$-bilinear) to get a map $\mathbb{Z}/(n) \otimes \mathbb{Z}/(m) \to \mathbb{Z}/(d)$ which sends $1 \otimes 1 \mapsto 1$.

Since these maps interchange the generators 1 and $1 \otimes 1$ of the respective $\mathbb{Z}$-modules, they form an isomorphism.

1.3.H. Say we have an exact sequence $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$. Taking products with the identity map, we get

$$M' \times N \xrightarrow{f \times \mathrm{id}} M \times N \xrightarrow{g \times \mathrm{id}} M'' \times N,$$

and taking the free modules generated by these as sets, we get

$$F(M' \times N) \xrightarrow{F(f \times \mathrm{id})} F(M \times N) \xrightarrow{F(g \times \mathrm{id})} F(M'' \times N).$$

Since f and id are linear, the map $F(f \times \mathrm{id})$ sends bilinearity relations to bilinearity relations, and hence descends to a map on the tensor products. The same applies to $F(g \times \mathrm{id})$, so we get induced maps as follows:

$$\begin{array}{ccccc} F(M' \times N) & \xrightarrow{F(f \times \mathrm{id})} & F(M \times N) & \xrightarrow{F(g \times \mathrm{id})} & F(M'' \times N) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q''} \\ M' \otimes N & \xrightarrow{f \otimes \mathrm{id}} & M \otimes N & \xrightarrow{g \otimes \mathrm{id}} & M'' \otimes N. \end{array}$$

Since g is surjective, $g \times \mathrm{id}$ is surjective and hence $F(g \times \mathrm{id})$ hits all the generators of $F(M'' \times N)$. Therefore $F(g \times \mathrm{id})$ and the induced map $g \otimes \mathrm{id}$ are both surjective. The horizontal composites are $0 \times \mathrm{id}$ and $0 \otimes \mathrm{id} = 0$, so $\mathrm{im}(f \otimes \mathrm{id}) \subseteq \ker(g \otimes \mathrm{id})$. It remains only to prove the reverse inclusion.

Define $\psi : F(M'' \times N) \to \mathsf{coker}(f \otimes \mathrm{id})$ by $\psi(g(m) \times n) = m \otimes n + \mathrm{im}(f \otimes \mathrm{id})$. This is well-defined, since g is surjective and if $g(m_1) = g(m_2)$ we have $m_1 - m_2 = f(m)$ for some $m \in M'$, so

$$m_1 \otimes n - m_2 \otimes n = (m_1 - m_2) \otimes n = (f \otimes \mathrm{id})(m \otimes n) \in \mathrm{im}(f \otimes \mathrm{id}).$$

Now ψ induces a map $\bar{\psi} : M'' \otimes N \to \mathsf{coker}(f \otimes \mathrm{id})$, since $\psi(\ker q'') = 0$:

$$\begin{aligned} \psi((g(m_1) + g(m_2)) \times n) &= (m_1 + m_2) \otimes n + \mathrm{im}(f \otimes \mathrm{id}) \\ &= m_1 \otimes n + m_2 \otimes n + \mathrm{im}(f \otimes \mathrm{id}) \\ &= \psi(g(m_1) \times n) + \psi(g(m_2) \times n), \end{aligned}$$

$$\begin{aligned} \psi(g(m) \times (n_1 + n_2)) &= g(m) \otimes (n_1 + n_2) + \mathrm{im}(f \otimes \mathrm{id}) \\ &= g(m) \otimes n_1 + g(m) \otimes n_2 + \mathrm{im}(f \otimes \mathrm{id}) \\ &= \psi(g(m) \times n_1) + \psi(g(m) \otimes n_2), \end{aligned}$$

$$\begin{aligned}
\psi(a(g(m)\times n)) &= a\psi(g(m)\times n) \\
&= a(m\otimes n)+\mathrm{im}(f\otimes \mathrm{id}) \\
&= (am)\otimes n+\mathrm{im}(f\otimes \mathrm{id}) &&= m\otimes (an)+\mathrm{im}(f\otimes \mathrm{id}) \\
&= \psi(g(am)\times n) &&= \psi(g(m)\times an) \\
&= \psi(ag(m)\times n) &&= \psi(g(m)\times an).
\end{aligned}$$

Moreover, we have $\bar{\psi}\circ(g\otimes \mathrm{id}) = \pi_f$ where $\pi_f : M\otimes N \to \mathrm{coker}(f\otimes \mathrm{id})$ is the quotient map, essentially by the definition of ψ. Hence $\ker(g\otimes \mathrm{id}) \subseteq \ker \pi_f = \mathrm{im}(f\otimes \mathrm{id})$.

1.3.I. Define a category where the objects are pairs (T,t) with T an R-module and $t : M\times N \to T$ a bilinear map, with morphisms $(T,t)\to(T',t')$ the R-linear maps $f : T\to T'$ such that $t' = f\circ t$. This is a category, since $t = \mathrm{id}\circ t$ and if $t'' = g\circ t'$ then $t'' = gf\circ t$.

The given definition is then that the tensor product of M and N is the initial object of this category. Hence uniqueness follows from 1.3.A; initial objects are unique up to unique isomorphism. Here this means that the tensor product (T,t) is unique up to unique isomorphisms $f : T\to T'$ satisfying $t' = f\circ t$.

1.3.J. Suppose $u : M\times N\to T$ is a bilinear map. Let $i : M\times N \to F(M\times N)$ be the set inclusion, and $p : F(M\times N)\to M\otimes N$ the quotient map. Then u extends to a unique linear map $s : F(M\times N)\to T$ satisfying $u = s\circ i$, and since u is bilinear we have $\ker p\subseteq \ker s$, so that s induces a unique linear map $\bar{s} : M\otimes N\to T$ satisfying $s = \bar{s}\circ p$. Putting this together, we have a unique linear map $\bar{s} : M\otimes N \to T$ such that $u = \bar{s}\circ p\circ i$, so $(M\otimes N, p\circ i)$ is a tensor product of M and N ($p\circ i$ is bilinear by construction).

1.3.K. (a) Define scalar multiplication by $b_1\in B$ as a map $B\otimes_A M\to B\otimes_A M$ satisfying $b_1(b_2\otimes m) = (b_1b_2)\otimes m$. Since $(b_2,m)\mapsto (b_1b_2)\otimes m$ is A-bilinear, this gives a well-defined A-linear map by the universal property of the tensor product. Associativity clearly holds for elementary tensors, and hence also in general. Distributivity on one side follows from A-linearity, and on the other side follows from bilinearity of $\otimes$.

This is certainly a functor $\mathbf{Mod}_A\to\mathbf{Mod}_A$, as with any tensor product, by $f\mapsto \mathrm{id}\otimes f$. The objects map to B-modules with the above scalar multiplication, so it suffices to show that $\mathrm{id}\otimes f$ is B-linear with respect to it. But this does hold:

$$\begin{aligned}
(\mathrm{id}\otimes f)\big(b_1(b_2\otimes m)\big) &= (\mathrm{id}\otimes f)\big((b_1b_2)\otimes m\big) \\
&= (b_1b_2)\otimes f(m) \\
&= b_1(b_2\otimes f(m)) \\
&= b_1\big((\mathrm{id}\otimes f)(b_2\otimes m)\big).
\end{aligned}$$

(b) Define a map $B\times C\to \mathrm{End}_A(B\otimes_A C)$ by mapping (b,c) to the composition of the scalar multiplication maps from (a), which commute since they commute on elementary tensors. This map is A-bilinear, since composition on $\mathrm{End}_A(B\otimes_A C)$ is A-bilinear and the scalar multiplication maps are A-linear. Hence we get an A-linear map $\mu : B\otimes_A C\to \mathrm{End}_A(B\otimes_A C)$. This defines a binary operation $x\cdot y = \mu(x)(y)$, which acts on elementary tensors by $(b_1\otimes c_1)\cdot(b_2\otimes c_2) = (b_1b_2)\otimes(c_1c_2)$ by the definition of the scalar multiplications. From this we see that associativity and commutativity hold, since they hold on elementary tensors and the product is A-bilinear. Moreover $1\otimes 1$ is an identity for the operation, so that this gives $B\otimes_A C$ the structure of a ring.

1.3.L. Tensoring with the natural map $M\to S^{-1}M$ gives a map $(S^{-1}A)\otimes_A M\to (S^{-1}A)\otimes_A (S^{-1}M)$. Then since scalar multiplication is bilinear, it gives a map $(S^{-1}A)\otimes_A (S^{-1}M)\to S^{-1}M$. Hence we get a composite map $\mu : (S^{-1}A)\otimes_A M\to S^{-1}M$ which sends $(r/s)\otimes m$ to $(r/s)(m/1) = rm/s$.

Conversely, tensoring with the natural map $A\to S^{-1}A$ gives a map $M\cong A\otimes_A M\to (S^{-1}A)\otimes_A M$, and the target is an $S^{-1}A$-module from 1.3.K, so S acts by isomorphisms on it, and hence there is a unique factored map $\sigma : S^{-1}M\to (S^{-1}A)\otimes_A M$ by the universal property of $S^{-1}M$. This is given by $m/s\mapsto s^{-1}(1\otimes m) = (1/s)\otimes m$, since the composite $M\cong A\otimes_A M\to (S^{-1}A)\otimes_A M$ maps $m\mapsto 1\otimes m$.

Then μ, σ are isomorphisms, since:

$$(\mu \circ \sigma)(m/s) = \mu\big((1/s) \otimes m\big) = m/s,$$
$$(\sigma \circ \mu)\big((r/s) \otimes m\big) = \sigma(rm/s) = (1/s) \otimes rm = (r/s) \otimes m.$$

Hence $(S^{-1}A) \otimes_A M \cong S^{-1}M$ as A-modules. Since both are $S^{-1}A$-modules, the isomorphism is also an isomorphism of $S^{-1}A$-modules; all A-linear maps are $S^{-1}A$-linear.

1.3.M. For each $j \in I$, tensoring the natural map $N_j \to \bigoplus_{i \in I} N_i$ gives a map $M \otimes N_j \to M \otimes \big(\bigoplus_{i \in I} N_i\big)$. Taking the sum of these maps gives

$$\delta : \bigoplus_{i \in I} (M \otimes N_i) \to M \otimes \big(\bigoplus_{i \in I} N_i\big).$$

Conversely, the map $M \times \big(\bigoplus_{i \in I} N_i\big) \to \bigoplus_{i \in I} (M \otimes N_i)$ given by

$$(m, n_{i_1} + \cdots + n_{i_k}) \mapsto (m \otimes n_{i_1}) + \cdots + (m \otimes n_{i_k})$$

is bilinear, and hence gives a map

$$\phi : M \otimes \big(\bigoplus_{i \in I} N_i\big) \to \bigoplus_{i \in I} (M \otimes N_i)\,.$$

We have $\phi \circ \delta = \mathrm{id}$, since for each $j \in J$ it is given on $M \otimes N_j$ by $m \otimes n \mapsto \phi(m \otimes n) = m \otimes n$, and $\delta \circ \phi = \mathrm{id}$, since

$$\begin{aligned} m \otimes (n_{i_1} + \cdots + n_{i_k}) &\mapsto \delta\big((m \otimes n_{i_1}) + \cdots + (m \otimes n_{i_k})\big) \\ &= m \otimes n_{i_1} + \cdots + m \otimes n_{i_k} \\ &= m \otimes (n_{i_1} + \cdots + n_{i_k}). \end{aligned}$$

1.3.N. By definition, the evident maps p_X and p_Y agree after mapping to Z. Suppose W is a set with maps $f : W \to X$, $g : W \to Y$ such that $\alpha \circ f = \beta \circ g$. Define $h : W \to X \times_Z Y$ by $h(w) = (f(w), g(w))$, which is well-defined since $\alpha \circ f = \beta \circ g$. By construction $f = p_X \circ h$ and $g = p_Y \circ h$.

1.3.O. Since maps are determined by their source and target in this category, the only interesting condition is the existence of a map $W \to X \times_Z Y$ whenever maps $W \to X$, $W \to Y$ exist (as well as the existence of the maps p_X, p_Y). This says that $X \times_Z Y \subseteq W$ whenever $X \subseteq W$ and $Y \subseteq W$. Hence $X \times_Z Y$ is minimal with respect to containing X and Y, i.e. $X \times_Z Y = X \cup Y$; the (fibered) product is the union.

1.3.P. Firstly, for any objects X, Y we have unique maps $X \to Z$, $Y \to Z$ since Z is final, and so there is a unique way to consider the fibered product $X \times_Z Y$. Now $X \times Y$ is defined as the final object of the category of objects W with maps $p_X : W \to X$ and $p_Y : W \to Y$, where morphisms are maps $f : W \to W'$ such that $p_X = p'_X \circ f$ and $p_Y = p'_Y \circ f$. The fibered product $X \times_Z Y$ is defined as the final object of the full subcategory where we only consider the objects such that p_X and p_Y agree after mapping to Z. But maps to Z are unique, so in this case the subcategory is the same as the original category, and hence $X \times_Z Y$ is final in the same category as $X \times Y$. Therefore there is a unique isomorphism between $X \times_Z Y$ and $X \times Y$ in said category, i.e. a unique isomorphism that preserves the projections to X and Y.

1.3.Q. Label the diagram:

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d, "b"] & V \arrow[d, "c"] \\ W \arrow[r, "d"] \arrow[d, "e"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

If we have maps $u : A \to V$, $v : A \to Y$ with $fcu = gv$, then since W is a fiber product over Z there is a unique map $\eta : A \to W$ with $cu = d\eta$ and $v = e\eta$. Then since U is a fiber product over X, and $d\eta = cu$, there is a unique map $\zeta : A \to U$ with $\eta = b\zeta$ and $u = a\zeta$. It follows that $v = e\eta = eb\zeta$, so ζ factors u and v. If $\zeta' : A \to U$ also factors u and v, then $b\zeta'$ factors cu and v, so $b\zeta = b\zeta'$ since W is a fiber product over Z. Then ζ and ζ' both factor u and η, so $\zeta = \zeta'$ since U is a fiber product over X. Since we have both existence and uniqueness of ζ, this shows that U is a fiber product over Z.

1.3.R. The projection maps $X_1 \times_Y X_2 \to X_i$ agree on Y, and hence also on Z after composing with $Y \to Z$. By the definition of $X_1 \times_Z X_2$ there is therefore a map $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ factoring said compositions.

1.3.S. Label the following maps: (Note that r is defined so that $p_i \circ r$ are the projections of $X_1 \times_Y X_2$, and then $f = q_i \circ p_i \circ r$. Also g and d are defined by their composites $y_i \circ g = q_i \circ p_i$, $y_i \circ d = \mathrm{id}$.)

$$X_1 \xleftarrow{p_1} X_1 \times_Z X_2 \xrightarrow{p_2} X_2$$

$$X_1 \xrightarrow{q_1} Y \xleftarrow{q_2} X_2$$

$$\begin{tikzcd} Y \times_Z Y \arrow[r, "y_1"] \arrow[d, "y_2"] & Y \arrow[d, "\lambda"] \\ Y \arrow[r, "\lambda"] & Z \end{tikzcd} \qquad \begin{tikzcd} X_1 \times_Y X_2 \arrow[r, "r"] \arrow[d, "f"] & X_1 \times_Z X_2 \arrow[d, "g"] \\ Y \arrow[r, "d"] & Y \times_Z Y \end{tikzcd}$$

The magic square commutes, since

$$(y_i \circ g) \circ r = q_i \circ p_i \circ r = f = \mathrm{id} \circ f = (y_i \circ d) \circ f,$$

and maps to $Y \times_Z Y$ are determined by their compositions with y_1 and y_2. Now if we are given maps $a : W \to Y$, $b : W \to X_1 \times_Z X_2$ such that $d \circ a = g \circ b$, then

$$q_i \circ p_i \circ b = y_i \circ g \circ b = y_i \circ d \circ a = \mathrm{id} \circ a = a.$$

Hence $q_1 \circ (p_1 \circ b) = q_2 \circ (p_2 \circ b)$, so we have a map $c : W \to X_1 \times_Y X_2$ such that $p_i \circ b = p_i \circ r \circ c$. But then $r \circ c = b$, since maps to $X_1 \times_Z X_2$ are uniquely determined by their compositions with p_1 and p_2. Moreover

$$f \circ c = (q_i \circ p_i \circ r) \circ c = q_i \circ p_i \circ b = a,$$

so c factors a and b. Uniqueness holds, since whenever $r \circ c = b$, we have $(p_i \circ r) \circ c = p_i \circ b$, and c is determined by its compositions with the projections $p_i \circ r$. (In fact this shows that r is a monomorphism.)

1.3.T. Suppose X and Y are sets, and let $X \amalg Y$ be the disjoint union $(X \times \{0\}) \cup (Y \times \{1\})$ with inclusions $i_X : X \to X \amalg Y$, $x \mapsto (x, 0)$ and $i_Y : Y \to X \amalg Y$, $y \mapsto (y, 1)$. If we have maps $f : X \to Z$ and $g : Y \to Z$, then there is a unique map $e : X \amalg Y \to Z$ satisfying $e \circ i_X = f$ and $e \circ i_Y = g$, since these force $e(x, 0) = f(x)$ and $e(y, 1) = g(y)$, which covers all of $X \amalg Y$, and shows existence since e is well-defined in this manner.

1.3.U. Tensoring the map $A \to C$ with B gives an A-linear map $B \cong B \otimes_A A \to B \otimes_A C$. This maps $b \mapsto b \otimes 1$ since that's what the isomorphism $B \cong B \otimes_A A$ does. This is in fact a ring homomorphism, since $(b_1 \otimes 1)(b_2 \otimes 1) = (b_1 b_2) \otimes 1$. The given square commutes since $a \otimes 1 = a(1 \otimes 1) = 1 \otimes a$.

Now suppose we have any A-algebra W, and A-algebra homomorphisms $f : B \to W$, $g : C \to W$. Then by A-linearity we get an A-linear map $B \otimes_A C \to W \otimes_A W$ given by $b \otimes c \mapsto f(b) \otimes g(c)$, which we can compose with the A-bilinear multiplication on W (a map $W \otimes_A W \to W$) to get a map $B \otimes_A C \to W$ given by $b \otimes c \mapsto f(b)g(c)$. In fact this is a ring homomorphism, since $f(b_1 b_2)g(c_1 c_2) = f(b_1)g(c_1)f(b_2)g(c_2)$. It factors f and g through the natural maps since $f(1) = g(1) = 1$. Any other such homomorphism must send $b \otimes 1$ to $f(b)$ and $1 \otimes c$ to $g(c)$, so $b \otimes c = (b \otimes 1)(1 \otimes c) \mapsto f(b)g(c)$ and hence we have uniqueness.

1.3.V. If m_1, m_2 are monomorphisms, and $m_1 \circ m_2 \circ f = m_1 \circ m_2 \circ g$, then $m_2 \circ f = m_2 \circ g$ since m_1 is a monomorphism, and $f = g$ since m_2 is a monomorphism.

1.3.W. If π is a monomorphism, then the following is a fiber product:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd} \qquad (*)$$

If maps to X agree after π then they must be the same, and trivially factor through the identity. With respect to this product, the induced map $X \to X$ is just the identity, which is certainly an isomorphism.

Conversely, if $X \times_Y X$ exists, and the induced map $X \to X \times_Y X$ is an isomorphism, then composing with this isomorphism expresses X as a fiber product along π, and the composed projection maps are the identity by definition, so again $(*)$ is a fiber product. Then if maps to X agree after π, they factor through a single map to X as compositions with the identity, and hence are equal.

An equivalent formulation is then that π is a monomorphism iff $(*)$ is a fiber product.

1.3.X. With the notation of 1.3.S, by 1.3.W the magic square reduces to

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r, "r"] \arrow[d] & X_1 \times_Z X_2 \arrow[d, "g"] \\ Y \arrow[r, "\mathrm{id}"] & Y \end{tikzcd}$$

Now g and id agree after composition with id and g, and so induce a map $s : X_1 \times_Z X_2 \to X_1 \times_Y X_2$ with $r \circ s = \mathrm{id}$. Then $r \circ s \circ r = r \circ \mathrm{id}$, so $s \circ r = \mathrm{id}$ since in 1.3.S we saw that r is a monomorphism. Hence r is an isomorphism.

1.3.Y. (a) If g has this property, then $g = i_A(\mathrm{id}_A)$ so g is unique. But for any $u : A \to C$ we have $i_C(\mathrm{id}_A \circ u) = i_A(\mathrm{id}_A) \circ u$, since by assumption the i_C's commute with right-composition, and hence $i_C(u) = i_A(\mathrm{id}_A) \circ u$.

(b) The map $i_A(\mathrm{id}_A)$ has inverse given by $i_A^{-1}(\mathrm{id}_{A'})$:

$$i_A^{-1}(\mathrm{id}_{A'}) \circ i_A(\mathrm{id}_A) = i_A^{-1}\big(\mathrm{id}_{A'} \circ i_A(\mathrm{id}_A)\big) = i_A^{-1}\big(i_A(\mathrm{id}_A)\big) = \mathrm{id}_A,$$
$$i_A(\mathrm{id}_A) \circ i_A^{-1}(\mathrm{id}_{A'}) = i_A\big(\mathrm{id}_A \circ i_A^{-1}(\mathrm{id}_{A'})\big) = i_A\big(i_A^{-1}(\mathrm{id}_{A'})\big) = \mathrm{id}_{A'}.$$

1.3.Z. (a) From 1.3.Y with arrows reversed, associating a natural transformation $\eta : h^A \to h^B$ with the morphism $\eta(\mathrm{id}_A) : B \to A$ gives a bijection.

(b) Natural transformations $h_A \to h_B$ are in bijection with morphisms $A \to B$, as proved in 1.3.Y.

(c) If $\eta : h^A \to F$, consider $\eta(\mathrm{id}_A) \in F(A)$. For any $f : A \to B$, we have

$$\eta(f) = \eta\big(h^A(f)(\mathrm{id}_A)\big) = F(f)\big(\eta(\mathrm{id}_A)\big)$$

by naturality. Hence η is completely determined by $\eta(\mathrm{id}_A)$. Moreover given any $x \in F(A)$, we can define a natural transformation $\eta_x : h^A \to F$ by $\eta_x(f) = F(f)(x)$, and so this sets up a bijection between natural transformations $h^A \to F$ and elements of $F(A)$.

1.4.A. Given a diagram $F : \mathscr{J} \to \mathscr{C}$, the object $F(e)$ together with the maps $F(\lambda)$ for $\lambda : i \to e$ in $\mathscr{J}$ gives a limit for the diagram, since any collection of commuting maps from the diagram includes a map from $F(e)$ which factors all the other maps since e is initial.

1.4.B. Suppose we have maps $f_i : W \to A_i$ for $i \in \mathscr{I}$ which commute with each $F(m)$ for $m \in \mathrm{Mor}(\mathscr{I})$. Then the map $w \mapsto (f_i(w))_{i \in \mathscr{I}}$ into the given set is well-defined, since $F(m)(f_i(w)) = f_j(w)$ by assumption. Moreover this clearly gives each f_i after composition with the projection onto A_i, and hence the universal property of $\varprojlim_{\mathscr{I}} A_i$ is satisfied.

1.4.C. (a) If we consider $\mathbb{N} \setminus 0$ as a poset with the relation of divisibility, then it indexes a diagram of abelian groups $\frac{1}{n}\mathbb{Z}$ with the inclusion maps $\frac{1}{n}\mathbb{Z} \to \frac{1}{m}\mathbb{Z}$ for $n \mid m$. Then $\mathbb{Q}$ is the colimit of this diagram, since anything with maps from $\frac{1}{n}\mathbb{Z}$ for each n, that commute with these inclusions, factors uniquely through $\mathbb{Q}$; if maps $f_n : \frac{1}{n}\mathbb{Z} \to G$ have this property, and $\frac{a}{n} = \frac{b}{m}$, then

$$f_n\left(\tfrac{a}{n}\right) = f_{nm}\left(\tfrac{am}{nm}\right) = f_{nm}\left(\tfrac{nb}{nm}\right) = f_m\left(\tfrac{b}{m}\right),$$

so we can define the image of $\frac{a}{n}$ in G without conflicts.

(b) A collection of subsets $U_i \subseteq X$ form a subcategory of the poset category $\wp(X)$, and subcategories are diagrams indexed by themselves. The union $\bigcup_i U_i$ is the colimit of this diagram, since $\bigcup_i U_i$ contains each U_i, and is contained in anything containing each U_i.

1.4.D. The relation $\sim$ is reflexive due to the identity maps in $\mathscr{I}$, and clearly symmetric. For transitivity, suppose a_i and a_j have equal images under $f_n : A_i \to A_n$ and $g_n : A_j \to A_n$, while a_j and a_k have equal images under $g_m : A_j \to A_m$ and $h_m : A_k \to A_m$. Since $\mathscr{I}$ is filtered, there are maps $\alpha : A_n \to A_l$ and $\beta : A_m \to A_l$ in the diagram by (i), and we may assume that $\alpha \circ g_n = \beta \circ g_m$ by (ii). Then a_i and a_k have equal images under $\alpha \circ f_n$ and $\beta \circ g_m$.

Hence $\sim$ is an equivalence relation, so the quotient set is well-defined. If there are maps $f_i : A_i \to W$ that commute with everything in the diagram, then the map $f = \amalg_{i\in\mathscr{I}} f_i$; $(a_i, i) \mapsto f_i(a_i)$ descends to the quotient, since if a_i and a_j have equal images under $f : A_i \to A_k$ and $g : A_j \to A_k$ then

$$f_i(a_i) = f_k\big(f(a_i)\big) = f_k\big(g(a_j)\big) = f_j(a_j).$$

Any factorization of the f_i's through the quotient must take these values, and hence the universal property is satisfied.

1.4.E. Addition is independent of the choice of u, v and l, since if we also have arrows $u' : i \to l'$ and $v' : j \to l'$, then since $\mathscr{I}$ is filtered there are arrows $w : l \to k$ and $w' : l' \to k$ with $w \circ u = w' \circ u'$ and $w \circ v = w' \circ v'$, so

$$\begin{aligned} F(w)\big(F(u)(m_i) + F(v)(m_j)\big) &= F(wu)(m_i) + F(wv)(m_j) \\ &= F(w'u')(m_i) + F(w'v')(m_j) \\ &= F(w')\big(F(u')(m_i) + F(v')(m_j)\big). \end{aligned}$$

Moreover it is independent of the choice of m_i and m_j, since if $m_{i'}$ and $m_{j'}$ are equivalent representatives then since $\mathscr{I}$ is filtered there are maps $u : i \to k$, $v : j \to k$, $u' : i' \to k$ and $v' : j' \to k$ with $F(u)(m_i) = F(u')(m_{i'})$ and $F(v)(m_j) = F(v')(m_{j'})$, so

$$F(u)(m_i) + F(v)(m_j) = F(u')(m_{i'}) + F(v')(m_{j'}).$$

Hence addition is well-defined, and clearly inherits commutativity and associativity. If we define 0 as the image of any $0 \in M_j$, then $(m_i, i) \sim 0$ iff the described condition holds, i.e. iff m_i is sent to 0 by some map in the diagram (in particular the choice of j is irrelevant). This gives an identity for addition since $F(u)(m_i) + 0 \sim m_i$, and we have existence of inverses by taking the inverse of a representative.

Scalar multiplication on $\amalg_{i\in\mathscr{I}} A_i$ descends to the quotient, since the maps are A-linear; if m_i and m_j have equal images under f and g then $f(am_i) = af(m_i) = ag(m_j) = g(am_j)$. Associativity and distributivity are then inherited from the original modules. Hence we do have an A-module structure on the given set, which by construction makes the inclusions A-linear.

Now any module with A-linear maps $f_i : A_i \to M$ that commute with the diagram gets a unique map of sets f from the above module factoring them, and this map of sets is in fact A-linear since its composition with each inclusion is A-linear:

$$f(am_i) = f_i(am_i) = af_i(m_i) = af(m_i),$$

and

$$\begin{aligned} f\big(F(u)(m_i)+F(v)(m_j)\big) &= f_k\big(F(u)(m_i)+F(v)(m_j)\big) \\ &= f_k\big(F(u)(m_i)\big)+f_k\big(F(v)(m_j)\big) \\ &= f_i(m_i)+f_j(m_j) \\ &= f(m_i)+f(m_j). \end{aligned}$$

Hence this is a colimit in $\mathbf{Mod}_A$.

1.4.F. We can define a category with objects $s \in S$ and a single morphism $s \to t$ iff $s \mid t$. Then the inclusions $\frac{1}{s}A \to \frac{1}{t}A$ give a diagram in $\mathbf{Mod}_A$ indexed by this category. These inclusions all commute with the inclusions $\frac{1}{s}A \to S^{-1}A$, since everything is just given by the identity map on $\mathrm{Frac}(A)$. Now if an A-module M has maps $f_s : \frac{1}{s}A \to M$ commuting with this diagram, then since $\bigcup_{s\in S}\frac{1}{s}A = S^{-1}A$ a factoring of these maps through some $f : S^{-1}A \to M$ is unique, and we have existence since if $a/s = b/t$ then

$$\begin{aligned} f_s(a/s) &= f_{st}(at/st) \\ &= f_{st}(sb/st) \\ &= f_t(b/t) \end{aligned}$$

so we can define $f(a/s) = f_s(a/s)$. This gives an A-linear map, as we can compute $(a/s)+(b/t)$ in $\frac{1}{st}A$. Hence $S^{-1}A$ is the colimit of this diagram in $\mathbf{Mod}_A$.

Now assume A is no longer a domain. In $\mathbf{Alg}_A$, morphisms $S^{-1}A \to B$ are unique, and exist iff the image of S in B consists of units. Therefore if we have morphisms $A_s \to B$ for each $s \in S$, then each s maps to a unit in B, and hence we have a unique morphism $S^{-1}A \to B$. It follows that $S^{-1}A$ is the colimit of the diagram whose objects are the A_s for $s \in S$ and whose morphisms are all the morphisms between them.

1.4.G. Let N be the submodule generated by the given relations, and let $L = \bigoplus_{i\in\mathscr{J}} M_i/N$. The natural maps $M_i \to L$ commute with the $F(n)$ by the construction of N. If we have maps $f_i : M_i \to M$, then a factoring through $\bigoplus_{i\in\mathscr{J}} M_i$ exists and is unique by the universal property of the direct sum. If the f_i also commute with the $F(n)$, then this map descends to the quotient L again uniquely. Hence L satisfies the universal property of the colimit.

1.5.A. The commuting square for τ to be a natural transformation in the second argument is:

$$\begin{tikzcd} \mathrm{Mor}_{\mathscr{B}}(F(A),B) \arrow[r, "g_*"] \arrow[d, "\tau_{AB}"] & \mathrm{Mor}_{\mathscr{B}}(F(A),B') \arrow[d, "\tau_{AB'}"] \\ \mathrm{Mor}_{\mathscr{A}}(A,G(B)) \arrow[r, "Gg_*"] & \mathrm{Mor}_{\mathscr{A}}(A,G(B')) \end{tikzcd}$$

1.5.B. Let $\eta_A = \tau_{A\,F(A)}\big(\mathrm{id}_{F(A)}\big)$, and $\epsilon_B = \tau^{-1}_{G(B)\,B}\big(\mathrm{id}_{G(B)}\big)$. If $g : F(A) \to B$ and $f : A \to G(B)$, then

$$\tau_{AB}(g) = (\tau_{AB}\circ g_*)\big(\mathrm{id}_{F(A)}\big) = \big(Gg_*\circ\tau_{A\,F(A)}\big)\big(\mathrm{id}_{F(A)}\big) = Gg\circ\eta_A$$

and

$$\tau_{AB}(\epsilon_B\circ Ff) = (\tau_{AB}\circ Ff^*)(\epsilon_B) = \big(f^*\circ\tau_{G(B)\,B}\big)(\epsilon_B) = f.$$

1.5.C. An element of $\mathrm{Hom}(M\otimes N,P)$ corresponds to a bilinear map $f : M\times N \to P$ by the universal property of $M\otimes N$. Linearity in the second argument means that $n \mapsto f(m,n)$ is A-linear for each $m \in M$, and linearity in the first argument means $m \mapsto (n \mapsto f(m,n))$ is a linear map $M \to \mathrm{Hom}(N,P)$. This gives an association $\mathrm{Hom}(M\otimes N,P) \to \mathrm{Hom}(M,\mathrm{Hom}(N,P))$. Conversely, if $\Phi \in \mathrm{Hom}(M,\mathrm{Hom}(N,P))$, then $(m,n) \mapsto \Phi(m)(n)$ is linear in the second argument by the definition of $\mathrm{Hom}(N,P)$ and linear in the first argument since Φ is linear. Hence it gives a map $M\otimes N \to P$. This gives an inverse to the previous association, setting up the bijection.

1.5.D. To show naturality, suppose $f : M \to M'$ and $g : P \to P'$. Then $(f \otimes N)^*$ acts on bilinear maps $M' \times N \to P$ to give a bilinear map $M \times N \to P$ by applying f to the first argument. This corresponds to applying f^* to the associated map $M \to \operatorname{Hom}(N, P)$, so the square for f commutes. Correspondingly, $\operatorname{Hom}(g, N)_*$ acts on maps $M \to \operatorname{Hom}(N, P)$ by composition to give maps $M \to \operatorname{Hom}(N, P')$, which gives the same result as composing g with the associated map $M \otimes N \to P$ to get a map $M \otimes N \to P'$ before unwrapping the bilinear mapping $M \to \operatorname{Hom}(N, P')$. Hence the square for g commutes also.

1.5.E. Let $f : N \to N \otimes_B A$ map n to $n \otimes 1$. Then $f^* : \operatorname{Hom}_A(N \otimes_B A, M) \to \operatorname{Hom}_B(N, M_B)$ since f is B-linear and A-linear maps to M give B-linear maps to M_B. Conversely, let $g : M_B \otimes_B A \to M$ be given by $m \otimes a \mapsto am$. Then for $\phi \in \operatorname{Hom}_B(N, M_B)$, we have an A-linear map $g \circ (\phi \otimes_B \operatorname{id}_A) : N \otimes_B A \to M$. Call these associations F_{NM} and G_{NM}.

- If $\psi \in \operatorname{Hom}_A(N \otimes_B A, M)$, then

$$(G_{NM} \circ F_{NM})(\psi)(n \otimes a) = g\big(\psi(f(n)) \otimes a\big) = a\psi(n \otimes 1) = \psi(n \otimes a),$$

  so $G_{NM} \circ F_{NM} = \operatorname{id}$.
- If $\phi \in \operatorname{Hom}_B(N, M_B)$, then

$$(F_{NM} \circ G_{NM})(\phi)(n) = g\big((\phi \otimes_B \operatorname{id}_A)(f(n))\big) = g(\phi(n) \otimes 1) = \phi(n),$$

  so $F_{NM} \circ G_{NM} = \operatorname{id}$.
- If $\alpha : M \to M'$ is an A-linear map, applying F_{NM} and composing with $\alpha_B : M_B \to M'_B$ is the same as composing with α and then applying $F_{NM'}$, since both are given by composition with $f : N \to N \otimes_B A$ and α (as a function of sets $\alpha_B = \alpha$).
- If $\beta : N' \to N$ is a B-linear map, applying F_{NM} and then composing with β is the same as composing with $\beta \otimes_B A$ and then applying $F_{N'M}$, since both are given by composition with $n' \mapsto \beta(n') \otimes 1$.

1.5.F. If it's already a group then the target of the identity map is a group, and we always have unique factorization through identity maps.

1.5.G. Define $H(S) = (S \times S)/\sim$ with $(a, b) \sim (c, d)$ iff $a + d = c + b$, and denote the equivalence class of (a, b) by $[a, b]$. The abelian semigroup structure on $S \times S$ descends to this quotient, since

$$\begin{aligned}(a, b) \sim (a', b') &\implies a + b' = a' + b \\ &\implies a + c + b' + d = a' + c + b + d \\ &\implies (a + c, b + d) \sim (a' + c, b' + d).\end{aligned}$$

Since S is non-empty, we have some $a \in S$, and then $[a, a]$ is an identity since $(a + b, a + c) \sim (b, c)$. Moreover inverses exist, since $[a, b] + [b, a] = [a + b, a + b]$ which was just seen to be an identity element. Hence $H(S)$ is an abelian group. We define a map $i_S : S \to H(S)$ by $a \mapsto [a + a, a]$. This is a map of semigroups, since $S \to S;\ a \mapsto a + a$ is. Note that $(a, b) + (b + b, b) \sim (a + a, a)$ so $[a, b] = i_S(a) - i_S(b)$.

Now we show that $H(S)$ with i_S has the universal property of a groupification. Suppose $f : S \to G$ is a semigroup map with G an abelian group. Define $g : H(S) \to G$ by $[a, b] \mapsto f(a) - f(b)$, which is well-defined since

$$\begin{aligned}(a, b) \sim (a', b') &\implies f(a) + f(b') = f(a') + f(b) \\ &\implies f(a) - f(b) = f(a') - f(b').\end{aligned}$$

This gives a map of (semi)groups, since $f(a + c) - f(b + d) = f(a) - f(b) + f(c) - f(d)$. Moreover it factors f through the i_S, since $g(a + a, a) = f(a) + f(a) - f(a) = f(a)$. Uniqueness holds, since any map $g : H(S) \to G$ of (semi)groups factoring f must have

$$g([a, b]) = g(i_S(a) - i_S(b)) = f(a) - f(b).$$

Note that H is a functor, since a map $S \to S'$ composes with $i_{S'}$ to give a map $S \to H(S')$, which then has a unique factoring $H(S) \to H(S')$ since $H(S')$ is a group. From the above construction, we see that $H(\phi)(a,b) = (\phi(a), \phi(b))$, i.e. $H(\phi)$ is the map induced by $\phi \times \phi$.

The universal property gives a bijection $\mathrm{Mor}_{\mathbf{Semigrp}}(S, F(G)) \to \mathrm{Mor}_{\mathbf{Grp}}(H(S), G)$ by composition with i_S. For naturality, it suffices to show that this commutes with (semi)group maps $\phi : S \to S'$, since it clearly commutes with group maps $G \to G'$ (composition on the left doesn't affect composition on the right). This holds, since

$$\begin{aligned}(i_{S'} \circ \phi)(s) &= [\phi(s) + \phi(s), \phi(s)] \\ &= [\phi(s+s), \phi(s)] \\ &= H(\phi)(s+s, s) \\ &= \big(H(\phi) \circ i_S\big)(s).\end{aligned}$$

1.5.H. The claim is that the localization functor $\mathbf{Mod}_A \to \mathbf{Mod}_{S^{-1}A}$ is a left-adjoint to the forgetful functor $\mathbf{Mod}_{S^{-1}A} \to \mathbf{Mod}_A$. In other words, that we have a natural bijection

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}N, M) \to \mathrm{Hom}_A(N, M)$$

for A-modules N and $S^{-1}A$-modules M. But the universal property of localization already stipulates that A-linear maps $S^{-1}N \to M$ correspond bijectively to A-linear maps $N \to M$ whenever M is an $S^{-1}A$-module, and of course all A-linear maps in this situation are actually $S^{-1}A$-linear. Naturality is the only thing that needs checking. It is clear for M, since composition on the left is unaffected by localizing the domain. For N, we have naturality since if $\phi : N' \to N$ and $f : S^{-1}N \to M$, then $f\big(\phi(n')/1\big) = f\big((S^{-1}\phi)(n'/1)\big)$.

1.6.A. It is clear that the image $\operatorname{im} f^i$ includes into A^{i+1}, and the quotient $A^{i+1}/\operatorname{im} f^i$ is $\operatorname{coker} f^i$. Now the inclusion $\ker f^i \to A^i$ descends to an inclusion $H^i(A^\bullet) = \ker f^i/\operatorname{im} f^{i-1} \to A^i/\operatorname{im} f^{i-1} = \operatorname{coker} f^{i-1}$, with quotient $A/\ker f^i \cong \operatorname{im} f^i$, giving the second sequence.

1.6.B. By 1.6.5.3 we have

$$\begin{aligned} h^i(A^\bullet) &= \dim\ker d^i - \dim\operatorname{im} d^{i-1} \\ \dim\operatorname{im} d^i &= \dim A^i - \dim\ker d^i, \end{aligned}$$

so

$$\begin{aligned}\sum_{i=1}^n (-1)^i h^i(A^\bullet) &= \sum_{i=1}^n (-1)^i \left[\dim\ker d^i - \dim\operatorname{im} d^{i-1}\right] \\ &= -\dim\operatorname{im} d^0 - \dim\operatorname{im} d^n + \sum_{i=1}^n (-1)^i \left[\dim\ker d^i + \dim\operatorname{im} d^i\right] \\ &= 0 + \sum_{i=1}^n (-1)^i \dim A^i.\end{aligned}$$

1.6.C. The morphisms $\mathrm{Hom}(A^\bullet, B^\bullet)$ are a subgroup of $\prod_i \mathrm{Hom}(A^i, B^i)$, and hence inherit the abelian group structure with distributivity over composition. The chain $\cdots \to 0 \to 0 \to 0 \to \cdots$ gives a zero object, since uniqueness of maps in and out of it is inherited component-wise, and the relevant diagrams commute because every composed morphism is zero. Given complexes $(A^\bullet, f^\bullet)$ and $(B^\bullet, g^\bullet)$, the complex $(A^\bullet \times B^\bullet, f^\bullet \times g^\bullet)$ is a product, since existence and uniqueness of maps holds for each term, and the diagrams commute in both variables. Hence $\mathbf{Com}_{\mathscr{C}}$ is an additive category.

Given a morphism $\phi^\bullet : A^\bullet \to B^\bullet$, let $k^i : Z^i \to A^i$ be a kernel for ϕ^i. Since $\phi^\bullet$ is a morphism, $\phi^i \circ f^{i-1} \circ k^{i-1} = g^i \circ \phi^{i-1} \circ k^{i-1} = 0$, so there is a unique factoring map $d^{i-1} : Z^{i-1} \to Z^i$ such that $f^{i-1} \circ k^{i-1} = k^i \circ d^{i-1}$. This gives a complex $(Z^\bullet, d^\bullet)$ such that $k^\bullet$ is a morphism. Now suppose we have another complex $(W^\bullet, e^\bullet)$, and a morphism $l^\bullet : W^\bullet \to A^\bullet$ such that $\phi^\bullet \circ l^\bullet = 0$. For each i, there is a unique factoring of l^i through k^i, since k^i is a kernel of ϕ^i. Say $l^i = k^i \circ m^i$. Then

$$\begin{aligned} l^i \circ e^{i-1} &= d^{i-1} \circ l^{i-1} \\ \implies k^i \circ m^i \circ e^{i-1} &= d^{i-1} \circ k^{i-1} \circ m^{i-1} \\ &= k^i \circ d^{i-1} \circ m^{i-1}, \end{aligned}$$

so $m^i \circ e^{i-1} = d^{i-1} \circ m^{i-1}$ since kernels are monomorphisms (factorizations through a kernel are unique by definition). Hence $m^\bullet$ is a morphism, and we see that $k^\bullet$ is a kernel of $\phi^\bullet$. The same argument with arrows reversed shows that cokernels exist, so $\mathbf{Com}_{\mathscr{C}}$ satisfies axiom (1).

Suppose now that $\phi^\bullet$ is a monomorphism. Then each ϕ^i is a monomorphism; if $a : X \to A^i$ we can construct the following chain map $F^\bullet(a)$:

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f^i \circ a} & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & A^{i-2} & \xrightarrow{f^{i-2}} & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & A^{i+2} & \xrightarrow{f^{i+2}} & A^{i+3} & \longrightarrow & \cdots \end{array}$$

and if $\phi^i \circ a = \phi^i \circ b$ then $\phi^\bullet \circ F^\bullet(a) = \phi^\bullet \circ F^\bullet(b)$, so since $\phi^\bullet$ is a monomorphism we get $F^\bullet(a) = F^\bullet(b)$, and hence $a = b$. Now let $k^\bullet : B^\bullet \to C^\bullet$ be the cokernel for $\phi^\bullet$, and suppose $\psi^\bullet : Y^\bullet \to B^\bullet$ is such that $k^\bullet \circ \psi^\bullet = 0$. Since ϕ^i is a monomorphism, and $\mathscr{C}$ an abelian category, we have unique maps $l^i : Y^i \to A^i$ with $\psi^i = \phi^i \circ l^i$. It remains to show that $l^\bullet$ is a morphism. Say $y^\bullet$ is the coboundary operator of $Y^\bullet$. Then

$$\begin{aligned} \phi^i \circ l^i \circ y^{i-1} &= \psi^i \circ y^{i-1} \\ &= g^{i-1} \circ \psi^{i-1} \\ &= g^{i-1} \circ \phi^{i-1} \circ l^{i-1} \\ &= \phi^i \circ f^{i-1} \circ l^{i-1}, \end{aligned}$$

so $l^i \circ y^{i-1} = f^{i-1} \circ l^{i-1}$ since ϕ^i is a monomorphism. This shows axiom (2).

When $\phi^\bullet$ is an epimorphism, reversing arrows gives a proof of axiom (3). For example, given an individual map $a : A^i \to X$, the constructed chain map is as follows:

$$\begin{array}{ccccccccccccccc} \cdots & \longrightarrow & A^{i-3} & \xrightarrow{f^{i-3}} & A^{i-2} & \xrightarrow{f^{i-2}} & A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & A^{i+2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle a \circ f^{i-1}} & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Hence $\mathbf{Com}_{\mathscr{C}}$ is an abelian category.

1.6.D. If $\phi^\bullet : A^\bullet \to B^\bullet$ is a morphism, then ϕ^i maps $\ker f^i$ into $\ker g^i$ since $g^i \circ \phi^i = \phi^{i+1} \circ f^i$. Moreover ϕ^i maps $\operatorname{im} f^{i-1}$ into $\operatorname{im} g^{i-1}$ since $\phi^i \circ f^{i-1} = g^{i-1} \circ \phi^{i-1}$. Hence ϕ^i descends to a map $H^i(A^\bullet) \to H^i(B^\bullet)$. The induction of maps in this way clearly respects composition and identities, so this gives a functor.

1.6.E. Left-exactness means that $F(\ker g)$ is a kernel of $F(g)$, and right-exactness means that $F(\operatorname{coker} f)$ is a cokernel of $F(f)$. Hence if a functor is exact, it commutes with kernels and cokernels. Then such a functor sends exact sequences to exact sequences, since the definition of an exact sequence is in terms of kernels and cokernels.

1.6.F. (a) Since localization is tensoring with $S^{-1}A$, by 1.3.H it is right-exact (note that exactness is unaffected by changing which scalar multiplication is considered). Now if $f : M \to N$ is a map of A-modules, then

$$\begin{aligned} \ker(S^{-1}f) &= \{m/s : f(m)/s = 0 \text{ in } S^{-1}N\} \\ &= \{m/s : uf(m) = 0 \text{ for some } u \in S\} \\ &= \{m/s : um \in \ker f \text{ for some } u \in S\} \\ &= \{m/s : m \in \ker f\} = S^{-1} \ker f \end{aligned}$$

since $m/s = (um)/(us)$. Therefore localization preserves kernels, and is left-exact.

(b) We want to show that $(\cdot) \otimes_A M$ sends cokernels to cokernels. Suppose $f : N' \to N$ has a cokernel $g : N \to N''$. Then if $t : N \otimes_A M \to K$ is such that $t \circ (f \otimes \mathrm{id}) = 0$, the map $t_m : n \mapsto t(n \otimes m)$ satisfies $t_m \circ f = 0$ for all $m \in M$. Hence there are unique factor maps $l_m : N'' \to K$ such that $t_m = l_m \circ g$. Now since $m \mapsto t_m$ is linear, $m \mapsto l_m$ is also linear by the uniqueness of l_m and linearity of $l_m \mapsto l_m \circ g$. Hence we get a map $l : N'' \otimes_A M \to K$ such that $l(n \otimes m) = l_m(n)$, and so $t = l \circ (g \otimes \mathrm{id})$. Since l_m is uniquely determined for all $m \in M$, this map l is uniquely determined. Therefore $g \otimes \mathrm{id}$ is a cokernel for $f \otimes \mathrm{id}$ as desired.

(c) That $\mathrm{Hom}(C, \cdot)$ is a functor $\mathscr{C} \to \mathbf{Ab}$ follows from distributivity of composition over addition of morphisms. Then if $f : A \to B$ is a morphism in $\mathscr{C}$, we have $g \in \ker\big(\mathrm{Hom}(C, f)\big)$ iff $f \circ g = 0$, iff g factors through $\ker f$, i.e. g is in the image of $\mathrm{Hom}(C, \ker f)$. Moreover $\mathrm{Hom}(C, \ker f)$ is a monomorphism, since $\ker f$ is, so $\mathrm{Hom}(C, \ker f)$ is a kernel of $\mathrm{Hom}(C, f)$. Hence $\mathrm{Hom}(C, \cdot)$ is left-exact.

The special case for modules is a consequence, since sequences in $\mathbf{Mod}_A$ are exact iff their images in $\mathbf{Ab}$ are exact. The only difference is that $\mathrm{Hom}(M, N)$ is an A-module for A-modules M, N, rather than just an abelian group, which is clear and doesn't affect the argument.

(d) For the same reason as above it suffices to consider the general case. If $f : A \to B$ is a morphism in $\mathscr{C}$, then $g \in \ker\big(\mathrm{Hom}(f, C)\big)$ iff $g \circ f = 0$, iff g factors through the cokernel of f, i.e. g is in the image of $\mathrm{Hom}(\mathrm{coker}\, f, C)$. Moreover $\mathrm{Hom}(\mathrm{coker}\, f, C)$ is a monomorphism, since $\mathrm{coker}\, f$ is an epimorphism, so $\mathrm{Hom}(\mathrm{coker}\, f, C)$ is a kernel of $\mathrm{Hom}(f, C)$. Hence $\mathrm{Hom}(\cdot, C)$ is left-exact.

1.6.G. Since $\mathrm{Hom}_A(\cdot, N)$ is left-exact we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^{\oplus p}, N) \to \mathrm{Hom}_A(A^{\oplus q}, N),$$

and since localization is exact we get an exact sequence

$$0 \to S^{-1}\,\mathrm{Hom}_A(M, N) \to S^{-1}\,\mathrm{Hom}_A(A^{\oplus p}, N) \to S^{-1}\,\mathrm{Hom}_A(A^{\oplus q}, N). \tag{A}$$

Conversely, applying $S^{-1}(\cdot)$ followed by $\mathrm{Hom}_{S^{-1}A}(\cdot, S^{-1}N)$ gives an exact sequence

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to \mathrm{Hom}_{S^{-1}A}\big((S^{-1}A)^{\oplus p}, S^{-1}N\big) \to \mathrm{Hom}_{S^{-1}A}\big((S^{-1}A)^{\oplus q}, S^{-1}N\big). \tag{B}$$

Now localization commutes with direct sums, and $\mathrm{Hom}(A, N) \cong N$ by evaluation at 1, so using the universal property of the direct sum (all the products are direct sums since q is finite) gives

$$\begin{aligned} S^{-1}\,\mathrm{Hom}_A(A^{\oplus q}, N) &\cong S^{-1}\big(\mathrm{Hom}_A(A, N)^{\oplus q}\big) \\ &\cong S^{-1}(N^{\oplus q}) \\ &\cong (S^{-1}N)^{\oplus q} \\ &\cong \mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N)^{\oplus q} \\ &\cong \mathrm{Hom}_{S^{-1}A}\big((S^{-1}A)^{\oplus q}, S^{-1}N\big). \end{aligned}$$

Hence the exact sequences (A) and (B) can be written as

$$0 \to S^{-1}\,\mathrm{Hom}_A(M, N) \to (S^{-1}N)^{\oplus p} \to (S^{-1}N)^{\oplus q}$$

and

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to (S^{-1}N)^{\oplus p} \to (S^{-1}N)^{\oplus q}.$$

Now the terminal maps of both sequences are given as multiplication by a single $p \times q$ matrix with entries in A; the transpose of that for the map $A^{\oplus q} \to A^{\oplus p}$. (Localizing the map doesn't change that it is given by this matrix, and applying Hom only transposes it. The same applies in the other order.) Hence our two modules are isomorphic, since they are both kernels of the same map.

1.6.H. $\star$ skipped.

1.6.I. Write $k_i : \ker h_i \to A_i$. Then if $\lambda : i \to j$, we have

$$k_j \circ a(\lambda) \circ k_i = b(\lambda) \circ h_i \circ k_i = 0,$$

so there is a unique map $z(\lambda) : \ker h_i \to \ker h_j$ such that $a(\lambda) \circ k_i = k_j \circ z(\lambda)$. This makes the kernels into a diagram, with the k_i's giving a natural transformation from it to a.

Now suppose we have limits Z, A, B for these diagrams, with maps $\gamma_i : Z \to \ker h_i$, $\alpha_i : A \to A_i$, $\beta_i : B \to B_i$. The natural transformations induce unique maps $k : Z \to A$, $h : A \to B$ given by $\alpha_i \circ k = k_i \circ \gamma_i$ and $\beta_i \circ h = h_i \circ \alpha_i$. The claim is then that k is a kernel for h.

By definition, a map $\phi : X \to A$ is equivalent to having maps $\phi_i : X \to A_i$ that commute with the diagram, and the composite $h \circ \phi$ is zero iff the individual composites $h_i \circ \phi_i$ are all zero. In this case each ϕ_i factors uniquely through a map $l_i : X \to \ker h_i$, and these maps commute with the diagram; if $\lambda : i \to j$ we have

$$\begin{aligned} k_j \circ z(\lambda) \circ l_i &= a(\lambda) \circ k_i \circ l_i \\ &= a(\lambda) \circ \phi_i \\ &= \phi_j, \end{aligned}$$

so $z(\lambda) \circ l_i$ satisfies the same property that uniquely defined l_j. Hence there is a unique factorization $l : X \to Z$ such that $\gamma_i \circ l = l_i$. Any other map $r : X \to Z$ with $\phi = k \circ r$ must satisfy $\phi_i = k_i \circ \gamma_i \circ r$, so since the k_i's are monomorphisms we have $\gamma_i \circ r = \gamma_i \circ l$. But then $r = l$ by the universal property of Z. Therefore k is a kernel of h as required.

1.6.J. Suppose $\mathscr{I}$, $\mathscr{J}$ are small categories, and we have a diagram $\pi : \mathscr{I} \times \mathscr{J} \to \mathscr{C}$ in $\mathscr{C}$, where $\mathscr{I} \times \mathscr{J}$ has objects (resp. morphisms) given by ordered pairs of those for $\mathscr{I}$ and $\mathscr{J}$. Suppose that for each $i \in \mathscr{I}$, and each $j \in \mathscr{J}$, the diagrams $\pi(\cdot, j)$ and $\pi(i, \cdot)$ have limits. (Note that these are diagrams indexed by $\mathscr{I}$ and $\mathscr{J}$; a morphism λ is mapped to $\pi(\mathrm{id}_i, \lambda)$ or $\pi(\lambda, \mathrm{id}_j)$.) Write $L_j = \varprojlim \pi(\cdot, j)$ with $l_{ji} : L_j \to \pi(i, j)$, and $R_i = \varprojlim \pi(i, \cdot)$ with $r_{ij} : R_i \to \pi(i, j)$. If $\lambda : j_1 \to j_2$, then the morphisms $\pi(\cdot, \lambda)$ give a natural transformation $\pi(\cdot, j_1) \to \pi(\cdot, j_2)$. Similarly if $\mu : i_1 \to i_2$, then the morphisms $\pi(\mu, \cdot)$ give a natural transformation $\pi(i_1, \cdot) \to \pi(i_2, \cdot)$. As previously, these natural transformations give morphisms $l(\lambda) : L_{j_1} \to L_{j_2}$ and $r(\mu) : R_{i_1} \to R_{i_2}$ by $l_{j_2 i} \circ l(\lambda) = \pi(\mathrm{id}_i, \lambda) \circ l_{j_1 i}$ and $r_{i_2 j} \circ r(\mu) = \pi(\mu, \mathrm{id}_j) \circ r_{i_1 j}$. In this way the L_j's and R_i's form diagrams indexed by $\mathscr{J}$ and $\mathscr{I}$ in $\mathscr{C}$. Suppose we have a limit $A = \varprojlim L_j$ with $a_j : A \to L_j$. The claim is that A gives a limit $\varprojlim_{\mathscr{I} \times \mathscr{J}} \pi(i, j)$ of the whole product-indexed diagram. By symmetry, A is then also a limit of the R_i's, so the limits commute.

Suppose we have maps $f_{ij} : X \to \pi(i, j)$ that commute with the diagram, so $\pi(\lambda, \mu) \circ f_{i_1 j_1} = f_{i_2 j_2}$. Since $\pi(\lambda, \mathrm{id}_j) \circ f_{i_1 j} = f_{i_2 j}$, we get maps $g_j : X \to L_j$ with $f_{ij} = l_{ji} \circ g_j$. Then

$$\begin{aligned} l_{j_2 i} \circ l(\lambda) \circ g_{j_1} &= \pi(\mathrm{id}_i, \lambda) \circ l_{j_1 i} \circ g_{j_1} \\ &= \pi(\mathrm{id}_i, \lambda) \circ f_{i j_1} \\ &= f_{i j_2} \\ &= l_{j_2 i} \circ g_{j_2}, \end{aligned}$$

so $l(\lambda) \circ g_{j_1} = g_{j_2}$ and hence we have a map $h : X \to A$ with $g_j = a_j \circ h$, so $f_{ij} = (l_{ji} \circ a_j) \circ h$. Any other factoring $f_{ij} = (l_{ji} \circ a_j) \circ h'$ must satisfy $a_j \circ h' = a_j \circ h$ by the universal property of L_j, and then $h' = h$ by the universal property of A, so we see that A is a limit as required.

1.6.K. Suppose $\mathscr{I}$ is a small filtered category. We want to show that if $a : \mathscr{I} \to \mathbf{Mod}_A$, $b : \mathscr{I} \to \mathbf{Mod}_A$ are diagrams, and $h_i : a(i) \to b(i)$ are maps that commute with the diagrams (i.e. a natural transformation), then $\varinjlim \ker h_i \cong \ker(\varinjlim A_i \to \varinjlim B_i)$. In other words, kernels commute with colimits.

Let $A = \varinjlim a(i)$, $\alpha_i : a(i) \to A$ and $B = \varinjlim b(i)$, $\beta_i : b(i) \to B$. Then let $h : A \to B$ be the map defined by $h \circ \alpha_i = \beta_i \circ h_i$. Note that if $\lambda : i \to j$, then $a(\lambda)$ maps $\ker h_i$ to $\ker h_j$, so restriction makes the kernels $\ker h_i$ into a diagram.

Suppose we have maps $f_i : \ker h_i \to X$ which commute with the diagram. Define $f : \ker h \to X$ by $f(\alpha_i(x)) = f_i(x)$, which gives a well-defined function of sets since:

- The colimit A is spanned by the images of the α_i's (see 1.4.C/D), $\ker h \subseteq A$, and $\alpha_i(\ker h_i) \subseteq \ker h$.
- If $\alpha_i(x) = \alpha_j(y)$, then we have maps $\lambda : i \to k$, $\mu : j \to k$ with $a(\lambda)(x) = a(\mu)(y)$ (see 1.4.C/D). Hence

$$\begin{aligned} f_i(x) &= f_k\big(a(\lambda)(x)\big) \\ &= f_k\big(a(\mu)(y)\big) \\ &= f_j(y) \end{aligned}$$

  since the f_i's commute with the diagram.

Moreover f is A-linear, since

$$f(a \cdot \alpha_i(x)) = f(\alpha_i(ax)) = f_i(ax) = af_i(x) = af(\alpha_i(x)),$$

and

$$\begin{aligned} f(\alpha_i(x) + \alpha_j(y)) &= f\big(\alpha_k(a(\lambda)(x)) + \alpha_k(a(\mu)(y))\big) \\ &= (f \circ \alpha_k)\big(a(\lambda)(x) + a(\mu)(y)\big) \\ &= (f \circ \alpha_k)\big(a(\lambda)(x)\big) + (f \circ \alpha_k)\big(a(\mu)(y)\big) \\ &= f(\alpha_i(x)) + f(\alpha_j(y)). \end{aligned}$$

Clearly f is the unique map factoring the f_i's through the α_i's, since this factoring was its exact definition, and hence $\ker h$ is a colimit of the $\ker h_i$'s.

1.6.L. By 1.6.K, filtered colimits are exact, and by the FHHF theorem exact functors commute with homology.

1.6.M. As in 1.6.I and other exercises, the maps between limits are the unique factorizations of the given maps of terms; we have composed maps $\varprojlim A_n \to A_n \to B_n$ which commute to give a map $\varprojlim A_n \to \varprojlim B_n$, and similarly for $\varprojlim B_n \to \varprojlim C_n$. Recalling that elements of the limit $\varprojlim A_n$ are sequences $(a_n)_{n\geq 0}$ with $a_n \in A_n$ (see 1.4.A), these maps are simply given by applying the horizontal maps term-wise.

By 1.6.I the sequence (1.6.13.1) is exact on the left, so it suffices only to show that $\varprojlim B_n \to \varprojlim C_n$ is surjective. Take an element $(c_n)_{n\geq 0} \in \varprojlim C_n$. For each $n \geq 0$, since $B_n \to C_n$ is surjective we have some β_n mapping to c_n. Now define an element $(b_n)_{n\geq 0}$ of $\varprojlim B_n$ by induction:

Take $b_0 = \beta_0$. Now suppose we have $b_0, \ldots, b_{n-1}$ that are pre-images of $c_0, \ldots, c_{n-1}$ with b_k mapping to b_{k-1}. The difference between b_{n-1} and the image of β_n is given by some element of A_{n-1}, since this difference maps to $c_{n-1} - c_{n-1} = 0$ in C_{n-1}. Then by surjectivity of the transition maps, we can find a pre-image $a_n \in A_n$ of this element. Define $b_n = \beta_n - a_n$, so that by construction b_n maps to b_{n-1}, and also to $c_n - 0 = c_n$ in C_n.

Chapter 2

Sheaves

2.1.A. If $f \notin \mathfrak{m}_p$, then $f(p) \neq 0$. By continuity, $f(x) \neq 0$ on some neighbourhood of p, so that $1/f$ is a continuous function (and gives a multiplicative inverse for f) on this neighbourhood. If f is differentiable, then $1/f$ is differentiable where it is defined, so we get a genuine inverse for f in the ring of germs $\mathscr{O}_p$.

Since units are contained in no maximal ideal, this shows that no maximal ideal meets $\mathscr{O}_p \setminus \mathfrak{m}_p$, i.e. they can only be contained in (and hence equal to) $\mathfrak{m}_p$.

2.1.B. Let $x_i \in \mathscr{O}_p$ be the ith coordinate function, and $p_i = x_i(p)$. If $f \in \mathfrak{m}_p$, then $f(x) = Df_p(x-p)+o(x-p)$. In the case where $Df_p = 0$, we have $f(x) = o(x-p)$, so $f(x)/|x-p| \to 0$ as $x \to p$. It follows that $f(x)/(x_i - p_i) \to 0$ as $x \to p$, so that they give elements $g_i \in \mathfrak{m}_p$ with $f = (x_i - p_i)g_i$, and therefore $f \in \mathfrak{m}_p^2$. Conversely if $f \in \mathfrak{m}_p^2$, then $Df_p = 0$ since a product of two linear functions vanishes to quadratic order (if L is linear then $L(x-p)/|x-p|$ is bounded).

Hence the elements of $\mathfrak{m}_p/\mathfrak{m}_p^2$ are determined precisely by their derivatives at p, which are nothing more than the linear functionals on the tangent space at p. So $\mathfrak{m}_p/\mathfrak{m}_p^2$ is canonically isomorphic to the cotangent space.

2.2.A. The data $\mathscr{F}(U)$ is the functor on objects, the data $\mathrm{res}_{U,V}$ is the functor on morphisms, and the remaining two conditions are the two properties a functor is required to satisfy; respecting identities and composition of morphisms. The functor is contravariant since by convention we had morphisms $U \hookrightarrow V$ when $U \subseteq V$, whereas $\mathrm{res}_{U,V} : \mathscr{F}(V) \to \mathscr{F}(U)$.

2.2.B.
(a) Restriction of functions clearly makes this a presheaf, but taking the functions $f_n(z) = n$ on the open sets $\{z \in \mathbb{C} : n < |z| < n+1\}$ gives a collection of functions with no (bounded) gluing despite agreeing on intersections.

(b) Again, restriction of functions clearly makes this a presheaf (of sets). However, the squaring map $z \mapsto z^2$ has a holomorphic square root on the two open sets $\mathbb{C} \setminus (-\infty, 0]$ and $\mathbb{C} \setminus [0, \infty)$, despite having no holomorphic square root on their union $\mathbb{C} \setminus 0$.

2.2.C. It is the limit of the diagram of $\mathscr{F}(U_i)$'s and $\mathscr{F}(U_i \cap U_j)$'s with the restriction maps $\mathscr{F}(U_i), \mathscr{F}(U_j) \to \mathscr{F}(U_i \cap U_j)$; the axioms say that elements of $\mathscr{F}(\cup_{i \in I} U_i)$ exist and are uniquely determined by elements of the $\mathscr{F}(U_i)$ (and $\mathscr{F}(U_i \cap U_j)$) that "commute" (as maps from a singleton set) with these restriction maps, and this generalizes to all morphisms in **Set**. Since the forgetful functor commutes with limits, this gives an accurate description in any category of sets with additional structure.

2.2.D.
(a) The identity axiom clearly holds, since equality of functions is a local property; they are equal iff they are equal at all points. Gluing is clear for functions, and then also for continuous or differentiable functions since continuity and differentiability are local properties; if a function is continuous (resp. differentiabile) in an open neighbourhood of every point then it is continuous (resp. differentiable) itself.

(b) The argument that real-valued continuous functions on a manifold or $\mathbb{R}^n$ form a sheaf applies unchanged to a general topological space.

2.2.E. The continuous maps $U \to S$ for any topological space S form a sheaf, as the argument in 2.2.D didn't rely on the target being $\mathbb{R}$; it is still the case that a map $U \to S$ is continuous iff it is continuous on a neighbourhood of each point. The constant sheaf is the special case for S a discrete space.

2.2.F. See above. Let's work it out in full. It forms a presheaf:

- If $V \subseteq U$, restriction of functions $f \mapsto f|_V$ gives a map $\mathscr{F}(U) \to \mathscr{F}(V)$, since the restriction of a continuous function is continuous: $f|_V^{-1}(O) = V \cap f^{-1}(O)$ is open for open sets O since f is continuous and intersections of open sets are open.
- Restriction $\mathscr{F}(U) \to \mathscr{F}(U)$ is the identity by definition, and if $W \subseteq V \subseteq U$ then the composite $\mathscr{F}(U) \to \mathscr{F}(V) \to \mathscr{F}(W)$ is $(f|_V)|_W = f|_W$; both given by f on points.

It forms a sheaf:

- If $U = \cup_{i \in I} U_i$, $f, g \in \mathscr{F}(U)$, and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f(x) = g(x)$ for any point $x \in U$ since $x \in U_i$ for some $i \in I$. Hence $f = g$.
- If $U = \cup_{i \in I} U_i$, $f_i \in \mathscr{F}(U_i)$ for each $i \in I$, and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then one can define $f : U \to Y$ by
$$f(x) = f_i(x) \quad \text{if } x \in U_i,$$
which is well-defined since if $x \in U_i$ and $x \in U_j$ then $f_i(x) = f_j(x)$ by assumption. Then f is continuous; if $V \subseteq Y$ is open then
$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i = \bigcup_{i \in I} f_i^{-1}(V)$$
is open since each f_i is continuous.

2.2.G. (a) Restriction descends to sections, since if $V \subseteq U$ we have $\mu \circ s|_V = (\mu \circ s)|_V = (\mathrm{id}|_U)|_V = \mathrm{id}_V$. Hence we get a presheaf. The identity axiom holds as for any other presheaf of functions with restriction, and gluing holds since s is a section iff it is a section at each point, so the function obtained by gluing section maps remains a section.

(b) From 2.2.F they form a sheaf of sets, and each $\mathscr{F}(U)$ has the structure of a group by $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$. Then the restriction maps are group homomorphisms, since $(f_1 \cdot f_2)|_V = f_1|_V \cdot f_2|_V$ (both are given by $x \mapsto f_1(x) f_2(x)$ for $x \in V$). This doesn't even require continuity of the group structure.

2.2.H. The map $V \mapsto \pi^{-1}(V)$ gives a map from the open sets in Y to the open sets in X since π is continuous, and this map is monotone; if $U \subseteq V$ then $\pi^{-1}(U) \subseteq \pi^{-1}(V)$. Hence it gives a covariant functor of the poset categories of open sets in Y and X. Since $\mathscr{F}$ is a contravariant functor from the category of open sets in X, its composite with this π^{-1} functor is a contravariant functor from the category of open sets in Y. Hence $\pi_* \mathscr{F}$ is a presheaf on Y.

If $\mathscr{F}$ is a sheaf, then $\pi_* \mathscr{F}$ is also a sheaf since
$$U = \bigcup_{i \in I} U_i \implies \pi^{-1}(U) = \bigcup_{i \in I} \pi^{-1}(U_i)$$
so identity and gluing are inherited directly.

2.2.I. If an element of $(\pi_* \mathscr{F})_q$ is represented by (f, U), where U is an open set containing q and $f \in \mathscr{F}(\pi^{-1}(U))$, then $(f, \pi^{-1}(U))$ represents an element of $\mathscr{F}_p$, since $p \in \pi^{-1}(U)$. This gives a well-defined map $(\pi_* \mathscr{F})_q \to \mathscr{F}_p$: if (g, V) represents the same element as (f, U), we have some open set $q \in W \subseteq U \cap V$ with $\mathrm{res}_{U,W}(f) = \mathrm{res}_{V,W}(g)$, so $\mathrm{res}_{\pi^{-1}(U), \pi^{-1}(W)}(f) = \mathrm{res}_{\pi^{-1}(V), \pi^{-1}(W)}(g)$, so that $(f, \pi^{-1}(U))$ and $(g, \pi^{-1}(V))$ represent the same element of $\mathscr{F}_p$. This will generally be a morphism, since it acts trivially on representatives.

Using the universal property, a morphism $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$ is the same as a collection of morphisms $\pi_*\mathscr{F}(U) \to \mathscr{F}_p$ for all open sets $U \ni q$ that commute with restriction. We have the canonical isomorphism $\pi_*\mathscr{F}(U) \cong \mathscr{F}(\pi^{-1}(U))$, and this can be composed with the map $\mathscr{F}(\pi^{-1}(U)) \to \mathscr{F}_p$ that exists since $p \in \pi^{-1}(U)$. These morphisms commute with restriction by the definition of $\mathscr{F}_p$, so this gives the desired morphism $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$.

2.2.J. Suppose we have elements of $\mathscr{O}_{X,p}$ and $\mathscr{F}_p$ represented by (r, U), (m, V) respectively. Then

$$(\mathrm{res}_{U,U\cap V}(r) \cdot \mathrm{res}_{V,U\cap V}(m), U \cap V)$$

gives an element of $\mathscr{F}_p$. This is a well-defined operation: if (r', U') and (m', V') are alternative representatives, we can get by intersection an open set $W \subseteq U \cap U' \cap V \cap V'$ with $\mathrm{res}_{U,W}(r) = \mathrm{res}_{U',W}(r')$ and $\mathrm{res}_{V,W}(m) = \mathrm{res}_{V',W}(m')$, so

$$\begin{aligned}
\mathrm{res}_{U\cap V,W}\big(\mathrm{res}_{U,U\cap V}(r) \cdot \mathrm{res}_{V,U\cap V}(m)\big) &= \mathrm{res}_{U,W}(r) \cdot \mathrm{res}_{V,W}(m) \\
&= \mathrm{res}_{U,W}(r') \cdot \mathrm{res}_{V,W}(m') \\
&= \mathrm{res}_{U'\cap V',W}\big(\mathrm{res}_{U',U'\cap V'}(r') \cdot \mathrm{res}_{V',U'\cap V'}(m')\big).
\end{aligned}$$

Since the restriction maps are morphisms, this operation gives a $\mathscr{O}_{X,p}$-module structure on $\mathscr{F}_p$.

2.3.A. This essentially follows as in chapter 1 from the view that limits are functors. The maps $\mathscr{F}(U) \to \mathscr{G}(U)$ given by ϕ commute with restriction by the definition of a morphism of presheaves, and composing them with the limit maps $\mathscr{G}(U) \to \mathscr{G}_p$ when $p \in U$ results in maps $\mathscr{F}(U) \to \mathscr{G}_p$ commuting with restriction (since restriction on $\mathscr{G}$ is trivialized in the stalk $\mathscr{G}_p$). Hence we get a map $\mathscr{F}_p \to \mathscr{G}_p$ by the universal property of $\mathscr{F}_p$.

2.3.B. Given a morphism $\phi : \mathscr{F} \to \mathscr{G}$ of sheaves on X, we get a morphism $\pi_*\phi : \pi_*\mathscr{F} \to \pi_*\mathscr{G}$ given by the maps $\mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U))$ given by ϕ on $\pi^{-1}(U)$. This inherits naturality from ϕ to give a morphism of sheaves, and we have both $\pi_*(\mathrm{id}) = \mathrm{id}$ and $\pi_*(\phi \circ \psi) = \pi_*\phi \circ \pi_*\psi$, so that π_* becomes a functor in this way.

2.3.C. It is a presheaf: if $U \subseteq V$ and $\phi : \mathscr{F}|_V \to \mathscr{G}|_V$, then restricting ϕ to just the open subsets of U gives a morphism $\mathrm{res}_{V,U}\phi : \mathscr{F}|_U \to \mathscr{G}|_U$. Restrictions in this way satisfy the presheaf axioms since in all cases the result is just defined by ϕ on the relevant open subsets.

It is a sheaf:

- If $U = \cup_{i\in I} U_i$, and $\mathrm{res}_{U,U_i}\phi = \mathrm{res}_{U,U_i}\psi$ for all $i \in I$, then for any $V \subseteq U$ and $x \in \mathscr{F}(V)$, we have

$$\begin{aligned}
\mathrm{res}_{V,V\cap U_i}\big(\phi(V)(x)\big) &= \phi(V \cap U_i)(\mathrm{res}_{V,V\cap U_i}(x)) \\
&= \psi(V \cap U_i)(\mathrm{res}_{V,V\cap U_i}(x)) \\
&= \mathrm{res}_{V,V\cap U_i}\big(\psi(V)(x)\big),
\end{aligned}$$

  so $\phi(V)(x) = \psi(V)(x)$ by identity on $V = \cup_{i\in I}(V \cap U_i)$. Hence $\phi(V) = \psi(V)$ for arbitrary $V \subseteq U$, so $\phi = \psi$.
- If $U = \cup_{i\in I} U_i$, and we have $\phi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ such that $\mathrm{res}_{U_i,U_i\cap U_j}\phi_i = \mathrm{res}_{U_i,U_i\cap U_j}\phi_j$, then for any $V \subseteq U$ define $\phi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ as follows: given $x \in \mathscr{F}(V)$, the values

$$y_i = \phi_i(V \cap U_i)(\mathrm{res}_{V,V\cap U_i}(x))$$

  satisfy

$$\begin{aligned}
\mathrm{res}_{V\cap U_i,V\cap U_i\cap U_j}(y_i) &= \phi_i(V \cap U_i \cap U_j)(\mathrm{res}_{V,V\cap U_i\cap U_j}(x)) \\
&= \phi_j(V \cap U_i \cap U_j)(\mathrm{res}_{V,V\cap U_i\cap U_j}(x)) \\
&= \mathrm{res}_{V\cap U_j,V\cap U_i\cap U_j}(y_j),
\end{aligned}$$

  so by gluing there is a (unique) $y \in \mathscr{G}(V)$ with $\mathrm{res}_{V,V\cap U_i}(y) = y_i$. We let $\phi(V)(x) = y$. By construction $\mathrm{res}_{U,U_i}\phi = \phi_i$.

2.3.D. (a) For each $U \subseteq X$, a morphism $\phi : \underline{\{p\}}|_U \to \mathscr{F}|_U$ specifies for each $V \subseteq U$ a morphism $\phi(V) : \{p\} \to \mathscr{F}(V)$. This is the same as giving an element $\phi(V)(p) \in \mathscr{F}(V)$, and naturality of ϕ implies that these elements are just given by $\operatorname{res}_{U,V}\big(\phi(U)(p)\big)$. Hence $\operatorname{Mor}(\underline{\{p\}}|_U, \mathscr{F}|_U) \cong \mathscr{F}(U)$, so that $\mathscr{H}\!om(\underline{\{p\}}, \mathscr{F}) \cong \mathscr{F}$.

(b) For each $U \subseteq X$, a morphism $\phi : \underline{\mathbb{Z}}|_U \to \mathscr{F}|_U$ specifies for each $\emptyset \neq V \subseteq U$ a morphism $\phi(V) : \mathbb{Z} \to \mathscr{F}(V)$. This is the same as giving an element $\phi(V)(1) \in \mathscr{F}(V)$, and naturality of ϕ implies that these elements are just given by $\operatorname{res}_{U,V}\big(\phi(U)(1)\big)$. Hence $\operatorname{Mor}(\underline{\mathbb{Z}}|_U, \mathscr{F}|_U) \cong \mathscr{F}(U)$, so that $\mathscr{H}\!om_{\mathbf{Ab}_X}(\underline{\mathbb{Z}}, \mathscr{F}) \cong \mathscr{F}$. The key point here is that $\mathbb{Z}$ is the free abelian group on one element.

(c) For each $U \subseteq X$, a morphism $\phi : \mathscr{O}_X|_U \to \mathscr{F}|_U$ specifies for each $V \subseteq U$ a morphism (of $\mathscr{O}_X(V)$-modules) $\phi(V) : \mathscr{O}_X(V) \to \mathscr{F}(V)$. This is the same as giving an element $\phi(V)(1) \in \mathscr{F}(V)$, and naturality of ϕ implies that these elements are just given by $\operatorname{res}_{U,V}\big(\phi(U)(1)\big)$. Hence $\operatorname{Mor}(\mathscr{O}_X|_U, \mathscr{F}|_U) \cong \mathscr{F}(U)$, so that $\mathscr{H}\!om_{\mathbf{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$. The key point here is that $\mathscr{O}_X(V)$ is the free $\mathscr{O}_X(V)$-module on one element.

2.3.E. The restriction maps essentially come from the view of kernels as limits, and limits as functors. For $V \subseteq U$, we have

$$\begin{tikzcd} 0 \arrow[r] & \ker\phi(U) \arrow[r] \arrow[d, dashed] & \mathscr{F}(U) \arrow[r] \arrow[d, "\operatorname{res}_{U,V}"] & \mathscr{G}(U) \arrow[d, "\operatorname{res}_{U,V}"] \\ 0 \arrow[r] & \ker\phi(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V) \end{tikzcd}$$

where the dashed arrow is uniquely induced from the universal property of the kernel, since the square on the right commutes by the definition of a morphism of presheaves. Really $\ker\phi(U)$ is just a subobject of $\mathscr{F}(U)$ for each U, and the restriction maps $\operatorname{res}_{U,V}$ of $\mathscr{F}$ map $\ker\phi(U)$ into $\ker\phi(V)$ because of this commuting square, so $\ker_{\text{pre}}\phi$ becomes a sub-presheaf of $\mathscr{F}$; the composition of restriction maps behaves correctly as inherited from $\mathscr{F}$.

2.3.F. Let's prove a general result, tying this in with 1.6.C. If $\mathscr{I}$ is a small category, and $\mathscr{C}$ is an abelian category, then the functors $\mathscr{I} \to \mathscr{C}$ together with natural transformations between them form a functor category $\mathscr{C}^{\mathscr{I}}$. If O is a set of objects in $\mathscr{I}$, we can consider the full subcategory of functors that send every object in O to 0 in $\mathscr{C}$. The claim is that this forms an abelian category, where kernels and cokernels can be computed term-wise. Viewing presheaves as functors, this shows what we want for $\ker_{\text{pre}}$ and $\operatorname{coker}_{\text{pre}}$, and we can get the result of 1.6.C by using a non-empty O to characterize chain complexes.

It is an additive category:

- For functors F and G in the category, we have a subgroup relation

$$\operatorname{Hom}(F,G) \leq \prod_{i \in \mathscr{I}} \operatorname{Hom}_{\mathscr{C}}(F(i), G(i)),$$

  since the zero morphism is natural (the relevant composite morphisms are all zero, so everything commutes), and linearity of composition in $\mathscr{C}$ means naturality is preserved by addition. Hence $\operatorname{Hom}(F,G)$ has an abelian group structure, which distributes over composition since composition applies term-wise.

- The constant zero functor is an object of the category, and gives a zero object since $\prod \operatorname{Hom}_{\mathscr{C}}(0,\cdot)$ and $\prod \operatorname{Hom}_{\mathscr{C}}(\cdot,0)$ are trivial.

- We have a product $F \times G$ given by $(F \times G)(i) = F(i) \times G(i)$ and $(F \times G)(\lambda) = F(\lambda) \times G(\lambda)$: the universal property holds term-wise, the transformations $F \times G \to F$ and $F \times G \to G$ are natural, and a transformation $\phi : H \to F \times G$ is natural iff the projections $H \to F$ and $H \to G$ are. Since $0 \times 0 = 0$ in $\mathscr{C}$ this still respects O.

It is an abelian category, with kernels and cokernels computed term-wise:

- If $\phi : F \to G$ is a natural transformation in the category, then for $i \in \mathscr{I}$ we have kernels in $\mathscr{C}$ say $\ker \phi(i) : Z(i) \to F(i)$. Then Z actually gives a functor (axiom of choice!): if $\lambda : i \to j$, then we have

$$\begin{tikzcd} 0 \arrow[r] & Z(i) \arrow[r, "\ker \phi(i)"] \arrow[d, dashed] & F(i) \arrow[r, "\phi(i)"] \arrow[d, "F(\lambda)"] & G(i) \arrow[d, "G(\lambda)"] \\ 0 \arrow[r] & Z(j) \arrow[r, "\ker \phi(j)"] & F(j) \arrow[r, "\phi(j)"] & G(j) \end{tikzcd}$$

and a unique map $Z(\lambda) : Z(i) \to Z(j)$ exists by the universal property of $\ker \phi(j)$, since the square commutes. Note that for $i \in O$, we have $F(i) = 0$ so that $Z(i) = 0$ as required. By the construction of Z, the maps $\ker \phi$ are a natural transformation $Z \to F$. We now prove that this gives a kernel for ϕ. Suppose $\psi : H \to F$ is a natural transformation with $\phi \circ \psi = 0$. Then for each $i \in \mathscr{I}$, there is a unique map $f(i) : H(i) \to Z(i)$ such that $\psi(i) = \ker \phi(i) \circ f(i)$. If $\lambda : i \to j$, we have

$$\begin{aligned} \ker \phi(j) \circ f(j) \circ H(\lambda) &= \psi(j) \circ H(\lambda) \\ &= F(\lambda) \circ \psi(i) \\ &= F(\lambda) \circ \ker \phi(i) \circ f(i) \\ &= \ker \phi(j) \circ Z(\lambda) \circ f(i), \end{aligned}$$

so $f(j) \circ H(\lambda) = Z(\lambda) \circ f(i)$ since $\ker \phi(j)$ is a monomorphism (being a kernel). Hence f is natural as required. Reversing arrows, we see that cokernels also exist, and are given term-wise in the same manner.

- Now suppose $\phi : F \to G$ is a monomorphism in our functor category. Since the category is additive, we have $\ker \phi = 0$. From above, this means that $\ker \phi(i) = 0$ for all $i \in \mathscr{I}$. Hence each $\phi(i)$ is a monomorphism. From above, we have a cokernel $\operatorname{coker} \phi : G \to H$, and $\operatorname{coker} \phi(i)$ is a cokernel for $\phi(i)$. Suppose $\psi : K \to G$ is such that $\operatorname{coker} \phi \circ \psi = 0$. Then $\psi(i) \circ \operatorname{coker} \phi(i) = 0$, so since $\phi(i)$ is a kernel for $\operatorname{coker} \phi(i)$ there is a unique factoring $f(i) : K(i) \to F(i)$ such that $\psi(i) = \phi(i) \circ f(i)$. Now if $\lambda : i \to j$, we have

$$\begin{aligned} \phi(j) \circ f(j) \circ K(\lambda) &= \psi(j) \circ K(\lambda) \\ &= G(\lambda) \circ \psi(i) \\ &= G(\lambda) \circ \phi(i) \circ f(i) \\ &= \phi(j) \circ F(\lambda) \circ f(i), \end{aligned}$$

so $f(j) \circ K(\lambda) = F(\lambda) \circ f(i)$ since $\phi(j)$ is a monomorphism. Hence f is a natural transformation, and we see that ϕ is a kernel for its cokernel. Reversing arrows, we see that epimorphisms are cokernels for their kernels, and hence the axioms for an abelian category are satisfied.

2.3.G. Suppose $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ is an exact sequence in $\mathbf{Ab}_X^{\text{pre}}$, so $\ker g = \operatorname{im} f$. Evaluation at U gives $\mathscr{F}(U) \xrightarrow{f(U)} \mathscr{G}(U) \xrightarrow{g(U)} \mathscr{H}(U)$ in $\mathbf{Ab}$. From the previous result (viewing presheaves as functors, and presheaf morphisms as natural transformations) we have $(\ker g)(U) = \ker(g(U))$, and

$$\begin{aligned} (\operatorname{im} f)(U) &= \big(\ker(\operatorname{coker} f)\big)(U) \\ &= \ker\big((\operatorname{coker} f)(U)\big) \\ &= \ker\big(\operatorname{coker}(f(U))\big) \\ &= \operatorname{im}(f(U)), \end{aligned}$$

so $\ker(g(U)) = \operatorname{im}(f(U))$, and hence $\mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is exact.

2.3.H. From 2.3.G we have the forward implication. It suffices to consider a sequence $\mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$; suppose $\mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ is exact for every U. Then $(\operatorname{im} f)(U) = \operatorname{im}(f(U))$ is a kernel for $g(U)$ for every U, and from the result in 2.3.F we see that $\operatorname{im} f$ is a kernel for g. Hence $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact.

2.3.I. Since $\mathbf{Ab}_X$ is a full subcategory of the abelian category $\mathbf{Ab}_X^{\text{pre}}$, the kernel $\ker_{\text{pre}}\phi$ is a kernel in the additive category $\mathbf{Ab}_X$ iff its domain is in fact a sheaf. ($\mathbf{Ab}_X$ is additive since $\underline{0}$ is a sheaf, and products of sheaves are sheaves.)

Now $\ker_{\text{pre}}\phi(U)$ is a subgroup of $\mathscr{F}(U)$, and the restriction morphisms on $\ker_{\text{pre}}\phi$ are the restrictions of those from $\mathscr{F}$, so identity is inherited directly from $\mathscr{F}$. To show gluing, it suffices to show that a section $f \in \mathscr{F}(U)$ is in $\ker_{\text{pre}}\phi(U)$ if its restriction to U_i is in $\ker_{\text{pre}}\phi(U_i)$ for each i, where $U = \cup_{i\in I}U_i$. But by identity on $\mathscr{G}$, the image $\phi(f)$ is determined by each $\text{res}_{U,U_i}(\phi(f)) = \phi(\text{res}_{U,U_i}f)$, and hence $\phi(f) = 0$ iff $\phi(\text{res}_{U,U_i}f) = 0$ for each i.

2.3.J. The given morphisms are well-defined, since locally constant maps are certainly holomorphic, and $2\pi i f$ is a holomorphic logarithm for $e^{2\pi i f}$. Now for any open $U \subseteq \mathbb{C}$, the map $\underline{\mathbb{Z}}(U) \to \mathscr{O}_X(U)$ is certainly injective, and for $f \in \mathscr{O}_X(U)$ we have $e^{2\pi i f} = 1$ (note that $\mathscr{F}(U)$ is a multiplicative abelian group) iff $f(z) \in \mathbb{Z}$ for all $z \in U$, i.e. iff f is a continuous function to the discrete space $\mathbb{Z}$. This shows exactness at $\underline{\mathbb{Z}}$ and O_X. Finally, if $g \in \mathscr{F}(U)$ then we have some $f \in \mathscr{O}_X(U)$ with $g = e^f$. Hence $g = e^{2\pi i(f/2\pi i)}$ is in the image of $\mathscr{O}_X(U) \to \mathscr{F}(U)$, showing exactness at $\mathscr{F}$.

Now the identity map $z \mapsto z$ is present in $\in \mathscr{F}(\mathbb{C} \setminus [0,\infty))$ and $\mathscr{F}(\mathbb{C} \setminus (-\infty, 0])$, since the natural logarithm has holomorphic branches on these sets. However the unique gluing on their union $\mathbb{C} \setminus 0$ has no holomorphic logarithm, so $\mathscr{F}$ does not satisfy gluing and is not a sheaf.

2.4.A. If sections $f, g \in \mathscr{F}(U)$ have the same germs, then for every $p \in U$ there is some open neighbourhood $p \in W_p \subseteq U$ with $\text{res}_{U,W_p}(f) = \text{res}_{U,W_p}(g)$, by the concrete definition of $\mathscr{F}_p$. Then since $U = \cup_{p\in U}W_p$, we have $f = g$ by identity.

2.4.B. If $s_p = 0$, then there is some open neighbourhood $p \in U$ with $\text{res}_{X,U}(s) = 0$ by the concrete definition of $\mathscr{F}_p$. Then $s_q = 0$ for all $q \in U$, so we see that the complement of $\operatorname{Supp} s$ is open.

2.4.C. For every $p \in U$, we have some open set $U_p \subseteq U$ and a section $t_p \in \mathscr{F}(U_p)$ such that for all $q \in U_p$, the germ $(t_p)_q = s_q$. Then $\text{res}_{U_p,U_p\cap U_q}(t_p)$ and $\text{res}_{U_q,U_p\cap U_q}(t_q)$ have the same germs, and hence are equal by the previous exercise. By gluing, we get a section $s \in \mathscr{F}(U)$ such that $\text{res}_{U,U_p}(s) = t_p$ for all $p \in U$. Then if $p \in U$, the germ of s at p equals the germ of t_p at p, which is s_p.

2.4.D. Given $f \in \mathscr{F}(U)$, the germs of $\phi_i(f)$ are determined by taking the germs of f and applying the maps at the level of stalks, because of the commuting square:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi_i} & \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \prod_{p\in U}\mathscr{F}_p & \longrightarrow & \prod_{p\in U}\mathscr{G}_p \end{array}$$

Since ϕ_1 and ϕ_2 induce the same maps of stalks, this shows that $\phi_1(f)$ and $\phi_2(f)$ have the same germs. By 2.4.A we get $\phi_1(f) = \phi_2(f)$, so that $\phi_1 = \phi_2$.

2.4.E. Clearly isomorphisms of sheaves induce isomorphisms on stalks, since the induction of maps on stalks is functorial. Conversely, suppose $\phi : \mathscr{F} \to \mathscr{G}$ is a map of sheaves, and $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for all $p \in X$. Observing the diagram 2.4.4.1, where now both vertical arrows are injective, the fact that the bottom map is an isomorphism immediately implies that the top map is injective. Now suppose we have a section $s \in \mathscr{G}(U)$. For $p \in U$, the germ $\phi_p^{-1}(s_p)$ has a representative $t^p \in \mathscr{F}(U^p)$ say. Then

$$\phi(U^p)(t^p)_p = \phi_p\big((t^p)_p\big) = \phi_p\big(\phi_p^{-1}(s_p)\big) = s_p,$$

so we can replace U^p by a smaller open set so that the restricted t^p satisfies $\phi(U^p)(t^p) = \text{res}_{U,U^p}(s)$. Then for $q \in U^p$ we have

$$\phi_q\big((t^p)_q\big) = \phi(U^p)(t^p)_q = s_q,$$

and hence $(t^p)_q = \phi_q^{-1}(s_q)$. This shows that the germs $\phi_p^{-1}(s_p)$ form a compatible collection of germs, so that we have some section $f \in \mathscr{F}(U)$ with $f_p = \phi_p^{-1}(s_p)$. Then $\phi(U)(f) = s$ since they have the same germs.

Hence for each U we have that $\phi(U)$ is injective, surjective, and therefore an isomorphism.

2.4.F. (a) Take the discrete space $X = \{0, 1\}$, and the presheaf of sets $\mathscr{F}(U) = U$ (for U non-empty), with the only possible restriction maps. The two global sections have the same germs, since only one germ is possible at each point.

(b) In the same scenario as (a), there is a non-trivial morphism $\mathscr{F} \to \mathscr{F}$ given by sending both global sections to 0. Since the stalks are final objects in **Set**, this induces the same maps on stalks as the identity morphism.

(c) See (b), where the morphism is not an isomorphism on global sections, but is an isomorphism on both stalks.

2.4.G. A sheafification is by definition an initial object in the category of presheaf morphisms from $\mathscr{F}$ to sheaves, and hence is unique up to unique isomorphism in this category. If $\mathscr{F}$ is a sheaf then $\mathrm{id}_{\mathscr{F}}$ is actually in said category, and factorization through the identity is always unique so that it is initial.

2.4.H. The composite $\mathscr{F} \to \mathscr{G} \to \mathscr{G}^{\mathrm{sh}}$ has a unique factorization $\mathscr{F}^{\mathrm{sh}} \to \mathscr{G}^{\mathrm{sh}}$. It is clear that this preserves identities, since factorization through the identity is trivial. If $\phi : \mathscr{F} \to \mathscr{G}$ and $\psi : \mathscr{G} \to \mathscr{H}$, then we can combine the commuting squares:

$$\begin{tikzcd}
\mathscr{F}^{\mathrm{sh}} \arrow[r, "\phi^{\mathrm{sh}}"] & \mathscr{G}^{\mathrm{sh}} \arrow[r, "\psi^{\mathrm{sh}}"] & \mathscr{H}^{\mathrm{sh}} \\
\mathscr{F} \arrow[u] \arrow[r, "\phi"] & \mathscr{G} \arrow[u] \arrow[r, "\psi"] & \mathscr{H} \arrow[u]
\end{tikzcd}$$

so $(\psi \circ \phi)^{\mathrm{sh}} = \psi^{\mathrm{sh}} \circ \phi^{\mathrm{sh}}$ because it composes correctly.

2.4.I. Since the sections are set theoretic functions (with some poorly restrictive codomain) with restriction maps given by function restriction, it certainly forms a separated presheaf. Gluing holds, since unions of compatible collections of germs remain compatible; the condition for a germ at $x \in U$ to be locally given by a section is unaffected by making U larger, and the compatibility needs only be checked at each point, where each point is by assumption contained in some set from the union.

2.4.J. For $f \in \mathscr{F}(U)$, take the element $(f_x)_{x \in U} \in \mathscr{F}^{\mathrm{sh}}(U)$. This is well-defined since the germs f_x are compatible (f is a section), and gives a morphism of sheaves since germs are unaffected by restriction.

2.4.K. Suppose $\phi : \mathscr{F} \to \mathscr{G}$ is a map of presheaves, with $\mathscr{G}$ a sheaf. Define $\psi : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ as follows: given a section $(f_p)_{p \in U}$ of $\mathscr{F}^{\mathrm{sh}}$, take its image under ψ to be the gluing of the compatible germs $\big(\phi_p(f_p)\big)_{p \in U}$. This is well-defined: the germs f_p are compatible (locally given by sections), so taking the image of these sections under ϕ shows that the germs $\phi_p(f_p)$ are also compatible. They then have a (unique) gluing since $\mathscr{G}$ is a sheaf. It gives a morphism, since restriction preserves germs, and ϕ commutes with restriction. Moreover this morphism is unique by 2.4.M and 2.4.D.

2.4.L. The statement of adjointness is that maps of presheaves $\mathscr{F} \to \mathscr{G}$ for $\mathscr{G}$ a sheaf correspond naturally to maps of sheaves $\mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$. The universal property gives an association from maps $\mathscr{F} \to \mathscr{G}$ to maps $\mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$, and the inverse is given by composition with $\mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$. Naturality in $\mathscr{G}$ is clear, since composition on the left commutes with composition on the right. Naturality in $\mathscr{F}$ follows from the definition of how sh acts on morphisms (the individual commuting squares in 2.4.H).

2.4.M. For $p \in X$, the maps $\mathscr{F}^{\mathrm{sh}}(U) \to \mathscr{F}_p$ given by $(f_x)_{x \in U} \mapsto f_p$ descend to a map of stalks $\mathscr{F}^{\mathrm{sh}}_p \to \mathscr{F}_p$, since restriction to an open set containing p leaves f_p unchanged. This gives an inverse to $\mathrm{sh}_p : \mathscr{F}_p \to \mathscr{F}^{\mathrm{sh}}_p$, since:

- If $f \in \mathscr{F}(U)$ is a representative for f_p, then $\mathrm{sh}(U)(f) = (f_x)_{x \in U}$ is a representative for $\mathrm{sh}_p(f_p)$, so $\mathrm{sh}_p(f_p)$ is mapped to f_p.
- Since $(f_x)_{x \in U}$ consists of compatible germs, we may restrict U so that it is given by a section $f \in \mathscr{F}(U)$. Then $\mathrm{sh}_p(f_p)$ has $\mathrm{sh}(U)(f) = (f_x)_{x \in U}$ as a representative, so $\mathrm{sh}_p(f_p)$ is the germ of $(f_x)_{x \in U}$ at p.

2.4.N.

(c) $\implies$ (b): If we have germs at p represented by $f, g \in \mathscr{F}(U)$, and $\phi_p(f_p) = \phi_p(g_p)$, then $\phi(U)(f)$ and $\phi(U)(g)$ have the same germ at p, so we can restrict U so that $\phi(U)(f) = \phi(U)(g)$. Hence $f = g$ on this restricted U, so $f_p = g_p$.

(b) $\implies$ (a): If $\phi \circ \psi_1 = \phi \circ \psi_2$, where $\psi_1, \psi_2 : \mathscr{H} \to \mathscr{F}$ are maps of sheaves, then for $p \in X$ we have $\phi_p \circ (\psi_1)_p = \phi_p \circ (\psi_2)_p$, and so $(\psi_1)_p = (\psi_2)_p$. Hence $\psi_1 = \psi_2$ by 2.4.D.

(a) $\implies$ (c): If we have sections $f_1, f_2 \in \mathscr{F}(U)$ with $\phi(U)(f_1) = \phi(U)(f_2)$, then let $\mathscr{H}$ be the sheaf

$$\mathscr{H}(V) = \begin{cases} \{*\} & \text{if } V \subseteq U \\ \emptyset & \text{otherwise} \end{cases}$$

and consider the maps $\psi_1, \psi_2 : \mathscr{H} \to \mathscr{F}$ given by $* \mapsto f_1, f_2$ respectively. Since $\phi \circ \psi_1 = \phi \circ \psi_2$ we get $\psi_1 = \psi_2$, and hence $f_1 = f_2$.

2.4.O. Reversing the arrows from 2.4.N gives a proof that (b) implies (a). For the reverse implication, suppose $f_1, f_2 : \mathscr{G}_p \to X$ are functions of sets, with $f_1 \circ \phi_p = f_2 \circ \phi_p$. Then if $\mathscr{H}$ is the skyscraper sheaf with stalk X at p, we can define morphisms $\psi_1, \psi_2 : \mathscr{G} \to \mathscr{H}$ by

$$\psi_i(U)(g) = \begin{cases} f_i(g_p) & \text{if } p \in U \\ * & \text{otherwise} \end{cases}$$

Then $\psi_1 \circ \phi = \psi_2 \circ \phi$, so $\psi_1 = \psi_2$ and hence $f_1 = f_2$.

2.4.P. Suppose a germ in $(O_X^*)_p$ is represented by $f \in \mathscr{O}_X^*(U)$. Since f has no zeros, we can define a holomorphic logarithm $\log f(z)$ on an open ball containing p, by integrating $f'(z)/f(z)$. Hence the restriction of f to this ball is in the image of the exponential map, so its germ at p is in the image of $\exp_p$. This shows surjectivity on stalks, so exp is an epimorphism by 2.4.O.

2.5.A. If $x \in X$, suppose we have maps $f_i : \mathscr{F}(B_i) \to X$ for $x \in B_i$ that commute with the restriction maps. For any open set U containing x, since U is a union of B_i's there is some $B_i \subseteq U$ containing x. Then $f_i \circ \mathrm{res}_{U,B_i} : \mathscr{F}(U) \to X$, and if $B_j \subseteq U$ also contains x we have some $B_k \subseteq B_i \cap B_j$ since $B_i \cap B_j$ is open and non-empty, so

$$\begin{aligned} f_i \circ \mathrm{res}_{U,B_i} &= f_k \circ \mathrm{res}_{B_i,B_k} \circ \mathrm{res}_{U,B_i} \\ &= f_k \circ \mathrm{res}_{U,B_k} \\ &= f_k \circ \mathrm{res}_{B_j,B_k} \circ \mathrm{res}_{U,B_j} = f_j \circ \mathrm{res}_{U,B_j}. \end{aligned}$$

Hence these give a well-defined collection of maps $\mathscr{F}(U) \to X$, and they commute with restrictions $\mathscr{F}(U) \to \mathscr{F}(V)$ since if $B_i \subseteq V$ then $B_i \subseteq U$. Because of this, the universal property of $\varinjlim_{x \in U} \mathscr{F}(U)$ is equivalent to the universal property of $\varinjlim_{x \in B_i} \mathscr{F}(B_i)$. Hence we can recover stalks (up to unique isomorphism) from the given data.

Now suppose we have a collection of germs $(f_x)_{x \in U}$, where $f_x \in \mathscr{F}_x$ and U is open. Since any open neighbourhood of x contains a basic open neighbourhood of x, these germs are compatible iff for each x there is a basic open neighbourhood $x \in B_i$ such that the germs $(f_x)_{x \in B_i}$ are given by a section $f \in \mathscr{F}(B_i)$.

This shows that both stalks and compatability of germs can be recovered from the given data. Since $\mathscr{F} \cong \mathscr{F}^{\mathrm{sh}}$, we are then able to recover the sheaf $\mathscr{F}$ completely; simply take the sections on U to be collections $(f_x)_{x \in U}$, where the $f_x \in \varinjlim_{x \in B_i} \mathscr{F}(B_i)$ are compatible in the sense that each x has some $x \in B_i$ with $(f_x)_{x \in B_i}$ given by a section $f \in \mathscr{F}(B_i)$. The restriction maps are trivial in this representation.

2.5.B. The base gluability axiom guarantees surjectivity, and injectivity holds since if $f_p = g_p$ for all p, then by the concrete description of the colimit we have $\mathrm{res}_{B,B_i}(f) = \mathrm{res}_{B,B_i}(g)$ for some $p \in B_i$. So $f = g$ on B by the base identity axiom applied to these B_i's.

2.5.C. (a) Since stalks are determined by the base (see 2.5.A), and morphisms are determined by stalks, morphisms are determined by the base (see 2.4.D).

(b) If $\phi : F \to G$ is a morphism of (pre-)sheaves on the base, we get morphisms $\phi_x : F_x \to G_x$ on stalks, from the view of (co-)limits as functors (since presheaves on the base are contravariant functors from the category of basic open sets). Concretely, one can simply check that mapping an element of F_x represented by $f \in F(B)$ to the image of $\phi(B)(f)$ in G_x is well-defined.

Then if $\mathscr{F}$ and $\mathscr{G}$ are the induced sheaves, we can define $\bar{\phi} : \mathscr{F} \to \mathscr{G}$ by

$$\bar{\phi}(U)\big((f_x)_{x\in U}\big) = (\phi_x(f_x))_{x\in U},$$

which is well-defined since if $(f_x)_{x\in B}$ is given by $f \in F(B)$ then $(\phi_x(f_x))_{x\in B}$ is given by $\phi(B)(f)$, so compatability is preserved.

2.5.D. We have a base given by the open sets B with $B \subseteq U_i$ for some i. For such B, choose a witnessing value i_B (so $B \subseteq U_{i_B}$). Define a sheaf on the base by $F(B) = \mathscr{F}_{i_B}(B)$, with restriction maps $\phi_{i_B i_{B'}}(B') \circ \operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B'}$ for $B' \subseteq B$.

This gives a presheaf on the base:

- Restriction $F(B) \to F(B)$ is given by $\phi_{i_B i_B}(B) \circ \operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B} = \mathrm{id} \circ \mathrm{id}$.
- If $B_3 \subseteq B_2 \subseteq B_1$, then the composite $F(B_1) \to F(B_2) \to F(B_3)$ is given by

$$\begin{aligned}\left(\phi_{i_{B_2} i_{B_3}}(B_3) \circ \operatorname{res}^{(\mathscr{F}_{i_{B_2}})}_{B_2,B_3}\right) &\circ \left(\phi_{i_{B_1} i_{B_2}}(B_2) \circ \operatorname{res}^{(\mathscr{F}_{i_{B_1}})}_{B_1,B_2}\right) \\ &= \phi_{i_{B_2} i_{B_3}}(B_3) \circ \phi_{i_{B_1} i_{B_2}}(B_3) \circ \operatorname{res}^{(\mathscr{F}_{B_1})}_{B_2,B_3} \circ \operatorname{res}^{(\mathscr{F}_{B_1})}_{B_1,B_2} \\ &= \phi_{i_{B_1} i_{B_3}}(B_3) \circ \operatorname{res}^{(\mathscr{F}_{B_1})}_{B_1,B_3}\end{aligned}$$

  using the fact that $\phi_{i_{B_1} i_{B_2}}$ is a sheaf morphism, and the cocycle condition.

Moreover it gives a sheaf on the base:

- If $B = \cup_\alpha B_\alpha$, and we have $f, g \in F(B)$ agreeing on each B_α, then

$$\begin{aligned}\operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B_\alpha}(f) &= \phi_{i_B i_{B_\alpha}}(B_\alpha)^{-1}\big(\operatorname{res}^{(F)}_{B,B_\alpha}(f)\big) \\ &= \phi_{i_B i_{B_\alpha}}(B_\alpha)^{-1}\big(\operatorname{res}^{(F)}_{B,B_\alpha}(g)\big) = \operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B_\alpha}(g),\end{aligned}$$

  so $f = g$ by identity on $\mathscr{F}_{i_B}$.
- If we have $f_\alpha \in F(B_\alpha)$, where $B = \cup_\alpha B_\alpha$, such that f_α and f_β agree on $B_\alpha \cap B_\beta$ (which is basic), then

$$\begin{aligned}\operatorname{res}^{(\mathscr{F}_{i_B})}_{B_\alpha,B_\alpha\cap B_\beta}\big(\phi_{i_{B_\alpha} i_B}(B_\alpha)(f_\alpha)\big) &= \phi_{i_{B_\alpha} i_B}(B_\alpha \cap B_\beta)\big(\operatorname{res}^{(\mathscr{F}_{i_{B_\alpha}})}_{B_\alpha,B_\alpha\cap B_\beta}(f_\alpha)\big) \\ &= \phi_{i_{B_\alpha\cap B_\beta} i_B}(B_\alpha \cap B_\beta)\big(\operatorname{res}^{(F)}_{B_\alpha,B_\alpha\cap B_\beta}(f_\alpha)\big) \\ &= \phi_{i_{B_\alpha\cap B_\beta} i_B}(B_\alpha \cap B_\beta)\big(\operatorname{res}^{(F)}_{B_\beta,B_\alpha\cap B_\beta}(f_\beta)\big) \\ &= \phi_{i_{B_\beta} i_B}(B_\alpha \cap B_\beta)\big(\operatorname{res}^{(\mathscr{F}_{i_{B_\beta}})}_{B_\beta,B_\alpha\cap B_\beta}(f_\beta)\big) \\ &= \operatorname{res}^{(\mathscr{F}_{i_B})}_{B_\beta,B_\alpha\cap B_\beta}\big(\phi_{i_{B_\beta} i_B}(B_\beta)(f_\beta)\big).\end{aligned}$$

  Hence by gluing on $\mathscr{F}_{i_B}$, we get an $f \in \mathscr{F}_{i_B}(B)$ with $\operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B_\alpha}(f) = \phi_{i_{B_\alpha} i_B}(B_\alpha)(f_\alpha)$, so

$$\begin{aligned}\operatorname{res}^{(F)}_{B,B_\alpha}(f) &= \phi_{i_B i_{B_\alpha}}(B_\alpha)\big(\operatorname{res}^{(\mathscr{F}_{i_B})}_{B,B_\alpha}(f)\big) \\ &= \big(\phi_{i_B i_{B_\alpha}}(B_\alpha) \circ \phi_{i_{B_\alpha} i_B}(B_\alpha)\big)(f_\alpha) = f_\alpha,\end{aligned}$$

  since the cocycle condition guarantees $\phi_{ij} = \phi_{ji}^{-1}$.

Let $\mathscr{F}$ be the induced sheaf. Then for $B \subseteq U_i$, which is automatically basic, we have the natural isomorphism $\mathscr{F}(B) \cong F(B)$. Hence $\mathscr{F}|_{U_i}$ is given by the data of $F|_{U_i}$, and is isomorphic to $\mathscr{F}_i$ by the maps $\phi_{i_B i}(B)$ for $B \subseteq U_i$; they give a morphism of sheaves by the cocycle condition and the definition of restriction on F. The composite isomorphism

$$\mathscr{F}_i|_{U_i \cap U_j} \cong \mathscr{F}|_{U_i \cap U_j} \cong \mathscr{F}_j|_{U_i \cap U_j}$$

is then given by $\phi_{j i_B}(B) \circ \phi_{i_B i}(B)$ for $B \subseteq U_i \cap U_j$, which is just ϕ_{ij} by the cocycle condition.

Now suppose $\mathscr{G}$ is a sheaf on X, with isomorphisms $\mathscr{G}|_{U_i} \cong \mathscr{F}_i$ such that the induced isomorphisms

$$\mathscr{F}_i|_{U_i \cap U_j} \cong \mathscr{G}|_{U_i \cap U_j} \cong \mathscr{F}_j|_{U_i \cap U_j}$$

are given by ϕ_{ij}. There is a unique isomorphism $\mathscr{F} \cong \mathscr{G}$ that restricts to the composite $\mathscr{F}|_{U_i} \cong \mathscr{F}_i \cong \mathscr{G}|_{U_i}$ on each U_i, since these isomorphisms agree on overlaps (being given by ϕ_{ij}); this follows from the fact that $\mathscr{H}\!om(\mathscr{F}, \mathscr{G})$ is a sheaf. Hence $\mathscr{F}$ is unique up to unique isomorphism in this sense.

2.5.E. Given surjectivity on the base, we get surjectivity on the stalks (as calculated from the data on the base); any germ is represented by some element of $G(B_i)$, which is the image of some element of $F(B_i)$, and hence the former's germ is the image of the latter's germ. Then the whole map is an epimorphism by 2.4.O.

2.6.A. Since $\ker(\mathscr{F} \to \mathscr{G})$ is a subsheaf of $\mathscr{F}$, it gives an injection $(\ker(\mathscr{F} \to \mathscr{G}))_x \hookrightarrow \mathscr{F}_x$. The image is contained in $\ker(\mathscr{F}_x \to \mathscr{G}_x)$, since if a section maps to zero then so do all of its germs. If an element of $\ker(\mathscr{F}_x \to \mathscr{G}_x)$ is represented by the section $f \in \mathscr{F}(U)$, then since the germ of f maps to zero we can restrict to a smaller U and assume $f \in \ker(\mathscr{F}(U) \to \mathscr{G}(U)) \cong \ker(\mathscr{F} \to \mathscr{G})(U)$. Then f_x is in the image of the above map, so we have surjectivity also.

2.6.B. The stalk functor $\mathscr{F} \mapsto \mathscr{F}_x$ on *presheaves* is right-exact, because cokernels commute with colimits by 1.6.I (the stalk is a colimit and morphisms of sheaves give natural transformations) and cokernels of presheaves are computed open set by open set (2.3.F). Then the stalk functor on *sheaves* is right-exact, because sheafification preserves stalks (2.4.M).

2.6.C. Since sheafification is a left-adjoint (2.4.L) it commutes with limits, and in particular kernels. Hence

$$(\mathrm{im}_{\mathrm{pre}}\, \phi)^{\mathrm{sh}} = (\ker_{\mathrm{pre}}(\mathrm{coker}_{\mathrm{pre}}\, \phi))^{\mathrm{sh}} = \ker((\mathrm{coker}_{\mathrm{pre}}\, \phi)^{\mathrm{sh}}) = \ker(\mathrm{coker}\, \phi) = \mathrm{im}\, \phi.$$

Since taking stalks commutes with both kernels and cokernels, it commutes with images.

2.6.D. This is the content of 2.6.A and 2.6.B; exact functors are precisely those functors which preserve kernels and cokernels.

2.6.E. Firstly, let's prove the implied claim that exactness can be checked at the level of stalks. Suppose $\mathscr{F} \xrightarrow{\phi} \mathscr{G} \xrightarrow{\psi} \mathscr{H}$ is such that $\mathscr{F}_p \to \mathscr{G}_p \to \mathscr{H}_p$ is exact for each $p \in X$. Then $(\psi \circ \phi)_p = \phi_p \circ \psi_p = 0$ for each p, and hence $\psi \circ \phi = 0$ by 2.4.D. This gives a natural map $\mathrm{im}\, \phi \to \ker \psi$, and at the level of stalks it gives the isomorphism $(\mathrm{im}\, \phi)_x \cong \mathrm{im}(\phi_x) \cong \ker(\psi_x) \cong (\ker \psi)_x$. Hence the map actually gives an isomorphism $\mathrm{im}\, \phi \cong \ker \psi$ by 2.4.E. Therefore $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact.

Now for the actual question. At a point $z \in \mathbb{C}$, taking stalks gives

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} (\mathscr{O}_X)_z \xrightarrow{\exp} (\mathscr{O}_X^*)_z \to 1.$$

The map $\mathbb{Z} \to (\mathscr{O}_X)_z$ is certainly injective, since constant germs are determined by their values at the point. Moreover the map $(\mathscr{O}_X)_z \to (\mathscr{O}_X^*)_z$ is surjective, since non-zero holomorphic maps locally have a logarithm (so some representative of their germ at z has a logarithm). Finally, a holomorphic map composed with exp is locally constant (with value 1) iff it is locally constant (with value in $2\pi i\mathbb{Z}$), since exp is locally invertible. Hence the sequence is exact.

2.6.F. By 2.4.N, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a monomorphism since $\mathscr{F} \to \mathscr{G}$ is. Now $\mathscr{F} \to \mathscr{G} = \ker(\mathscr{G} \to \mathscr{H})$, so since kernels are computed open set by open set we have $\mathscr{F}(U) \to \mathscr{G}(U) = \ker(\mathscr{G}(U) \to \mathscr{H}(U))$. This shows left-exactness.

For a counter-example to right-exactness, take U to be an open annulus in 2.4.10.1. We get

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathscr{O}_X(U) \xrightarrow{\exp} \mathscr{O}_X^*(U) \to 1,$$

and $\mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$ is not surjective, since the section given by $z \mapsto z$ in $\mathscr{O}_X^*(U)$ has no holomorphic logarithm.

2.6.G. Since $\ker = \ker_{\text{pre}}$, a sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact in $\mathbf{Ab}_X$ iff it is exact in $\mathbf{Ab}_X^{\text{pre}}$. Now if $V \subseteq Y$ is open, then the sequence

$$0 \to \pi_*\mathscr{F}(V) \to \pi_*\mathscr{G}(V) \to \pi_*\mathscr{H}(V)$$

is nothing more than

$$0 \to \mathscr{F}(\pi^{-1}(V)) \to \mathscr{G}(\pi^{-1}(V)) \to \mathscr{H}(\pi^{-1}(V)),$$

which is exact since $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact by 2.3.H. Hence $0 \to \pi_*\mathscr{F} \to \pi_*\mathscr{G} \to \pi_*\mathscr{H}$ is exact by 2.3.H again.

2.6.H. As above, left-exactness can be checked open set by open set. In particular, if $0 \to \mathscr{G}' \to \mathscr{G} \to \mathscr{G}''$ is exact, then $0 \to \mathscr{G}'|_U \to \mathscr{G}|_U \to \mathscr{G}''|_U$ is exact for each U (since this simply restricts the open sets that need checking). Hence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}'|_U) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}''|_U) \arrow[d, equal] \\
& \mathscr{H}om(\mathscr{F}, \mathscr{G}')(U) & \mathscr{H}om(\mathscr{F}, \mathscr{G})(U) & \mathscr{H}om(\mathscr{F}, \mathscr{G}'')(U)
\end{tikzcd}$$

is exact by 1.6.F (working in the abelian category $\mathbf{Ab}_U$). Therefore

$$0 \to \mathscr{H}om(\mathscr{F}, \mathscr{G}') \to \mathscr{H}om(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om(\mathscr{F}, \mathscr{G}'')$$

is exact, again since left-exactness can be checked open set by open set.

On the other hand, suppose $\mathscr{G}' \to \mathscr{G} \to \mathscr{G}'' \to 0$ is exact. Since coker is the sheafification of $\operatorname{coker}_{\text{pre}}$, we get

$$\begin{tikzcd}
& & \operatorname{coker}_{\text{pre}} \phi \arrow[r] \arrow[d, "\text{sh}"] & 0 \\
\mathscr{G}' \arrow[r, "\phi"] & \mathscr{G} \arrow[ur, "\bar{\psi}"] \arrow[r, "\psi"] & \mathscr{G}'' \arrow[r] & 0
\end{tikzcd}$$

where the top sequence is exact in $\mathbf{Ab}_X^{\text{pre}}$. Since $\operatorname{coker}_{\text{pre}}$ is computed open set by open set, and sheafification commutes with restriction (this is clear from its construction), for each U we get

$$\begin{tikzcd}
& & (\operatorname{coker}_{\text{pre}} \phi)|_U \arrow[r] \arrow[d, "\text{sh}"] & 0 \\
\mathscr{G}'|_U \arrow[r] & \mathscr{G}|_U \arrow[ur] \arrow[r] & \mathscr{G}''|_U \arrow[r] & 0
\end{tikzcd}$$

with the top sequence remaining exact. But this exactness means that $\mathscr{G}|_U \to \mathscr{G}''|_U$ is given by

$$(\operatorname{coker}_{\text{pre}}(\mathscr{G}'|_U \to \mathscr{G}|_U))^{\text{sh}} = \operatorname{coker}(\mathscr{G}'|_U \to \mathscr{G}|_U),$$

so the bottom sequence is also exact. Hence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(\mathscr{G}'|_U, \mathscr{F}|_U) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{G}|_U, \mathscr{F}|_U) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(\mathscr{G}''|_U, \mathscr{F}|_U) \arrow[d, equal] \\
& \mathscr{H}om(\mathscr{G}', \mathscr{F})(U) & \mathscr{H}om(\mathscr{G}, \mathscr{F})(U) & \mathscr{H}om(\mathscr{G}'', \mathscr{F})(U)
\end{tikzcd}$$

is exact by 1.6.F (again working in $\mathbf{Ab}_U$), and so

$$0 \to \mathscr{H}om(\mathscr{G}', \mathscr{F}) \to \mathscr{H}om(\mathscr{G}, \mathscr{F}) \to \mathscr{H}om(\mathscr{G}'', \mathscr{F})$$

is exact open set by open set.

2.6.I. They form an additive category:

- For $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, the morphisms $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ are a subset of the group of underlying $\mathbf{Ab}_X$ homomorphisms. Moreover they form a subgroup, since the zero morphism is $\mathscr{O}_X$-linear and sums of $\mathscr{O}_X$-linear maps are $\mathscr{O}_X$-linear by distributivity of scalar multiplication. Distributivity of composition is inherited.
- The zero object $\underline{0}$ in $\mathbf{Ab}_X$ has a unique $\mathscr{O}_X$-module structure, and since the morphisms of $\mathscr{O}_X$-modules are a subgroup of the morphisms of underlying groups it remains a zero object.
- If $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-modules, then their product $\mathscr{F} \times \mathscr{G}$ in $\mathbf{Ab}_X$ has an $\mathscr{O}_X$-module structure given by $r(f, g) = (rf, rg)$ for $r \in \mathscr{O}_X(U)$ and $(f, g) \in (\mathscr{F} \times \mathscr{G})(U)$. The projection maps $\mathscr{F} \times \mathscr{G} \to \mathscr{F}$ and $\mathscr{F} \times \mathscr{G} \to \mathscr{G}$ in $\mathbf{Ab}_X$ become $\mathscr{O}_X$-linear with this structure, and if $\phi : \mathscr{H} \to \mathscr{F}$, $\psi : \mathscr{H} \to \mathscr{G}$ are $\mathscr{O}_X$-linear then $(\phi, \psi) : \mathscr{H} \to \mathscr{F} \times \mathscr{G}$ is also $\mathscr{O}_X$-linear. It follows that this gives a product in $\mathbf{Mod}_{\mathscr{O}_X}$, since the morphisms of $\mathscr{O}_X$-modules are a subgroup of the morphisms of underlying groups.

They form an abelian category:

- If $\phi : \mathscr{F} \to \mathscr{G}$ is $\mathscr{O}_X$-linear, then the subgroup $\ker \phi(U)$ of $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-submodule. Moreover the restriction morphisms on $\ker \phi$ are $\mathscr{O}_X$-linear, since they are just given by those of $\mathscr{F}$. Hence the kernel $\ker \phi$ in $\mathbf{Ab}_X$ gets an $\mathscr{O}_X$-module structure, and since $\ker \phi \to \mathscr{F}$ is just given by inclusion maps, it retains the universal property in $\mathbf{Mod}_{\mathscr{O}_X}$.
- If $\phi : \mathscr{F} \to \mathscr{G}$ is $\mathscr{O}_X$-linear, then the quotient group $\mathrm{coker}_{\mathrm{pre}}\, \phi(U)$ of $\mathscr{G}(U)$ is a quotient by an $\mathscr{O}_X(U)$-submodule, and hence gets an $\mathscr{O}_X(U)$-module structure. Moreover the restriction morphisms are induced from those of $\mathscr{G}$, and so are $\mathscr{O}_X$-linear. Hence we get an $\mathscr{O}_X$-module structure on $\mathrm{coker}_{\mathrm{pre}}\, \phi$, and in particular on all of its stalks. Then on its sheafification $\mathrm{coker}\, \phi$, we can define an $\mathscr{O}_X$-module structure by $r(f_x)_{x \in U} = (rf_x)_{x \in U}$. This respects compatability since the restriction morphisms are $\mathscr{O}_X$-linear; if $(f_x)_{x \in U}$ is given by a section f, then $(rf_x)_{x \in U}$ is given by rf. With this structure the map $\mathscr{G} \to \mathrm{coker}\, \phi$ is $\mathscr{O}_X$-linear, and it retains the universal property in $\mathbf{Mod}_{\mathscr{O}_X}$ since when starting with $\mathscr{O}_X$-linear maps, the induced morphisms from $\mathrm{coker}_{\mathrm{pre}}$ and sheafification remain $\mathscr{O}_X$-linear.
- Since the category is additive, an $\mathscr{O}_X$-linear map is a monomorphism (resp. epimorphism) iff its kernel (resp. cokernel) is zero, and hence iff it is a monomorphism (resp. epimorphism) in $\mathbf{Ab}_X$. Then because kernels and cokernels in $\mathbf{Mod}_{\mathscr{O}_X}$ have underlying maps given by the kernels and cokernels in $\mathbf{Ab}_X$, regularity of monomorphisms and epimorphisms is inherited from $\mathbf{Ab}_X$ (and more generally, exactness can be checked at the level of underlying maps).

2.6.J. (a) The tensor product $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes \mathscr{G}$ should be universal with respect to sheaf-of-set morphisms out of $\mathscr{F} \times \mathscr{G}$ into an $\mathscr{O}_X$-module that are $\mathscr{O}_X(U)$-bilinear on every U.

On presheaves, we have such a tensor product by $(\mathscr{F} \otimes_{\mathrm{pre}} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$, with restriction maps $\big(\mathscr{F}(U) \to \mathscr{F}(V)\big) \otimes_{\mathscr{O}_X(U)} \big(\mathscr{G}(U) \to \mathscr{G}(V)\big)$, where the individual restriction maps are $\mathscr{O}_X(U)$-linear by the definition of an $\mathscr{O}_X$-module. Let $\mathscr{F} \otimes \mathscr{G} = (\mathscr{F} \otimes_{\mathrm{pre}} \mathscr{G})^{\mathrm{sh}}$, with the map $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes \mathscr{G}$ given by composition with the sheafification map, so $(f, g) \in (\mathscr{F} \times \mathscr{G})(U)$ maps to $\mathrm{sh}(f \otimes_{\mathrm{pre}} g)$. This is an $\mathscr{O}_X$-module, since (pre-)$\mathscr{O}_X$-module structures descend to sheafifications as in 2.6.I with the cokernel.

Suppose $\phi : \mathscr{F} \times \mathscr{G} \to \mathscr{H}$ is $\mathscr{O}_X(U)$-bilinear on each U. Then ϕ factors uniquely through an $\mathscr{O}_X$-linear map of presheaf $\mathscr{O}_X$-modules $\bar{\phi}_{\mathrm{pre}} : \mathscr{F} \otimes_{\mathrm{pre}} \mathscr{G} \to \mathscr{H}$. This then factors through a unique map of sheaves $\bar{\phi} : \mathscr{F} \otimes \mathscr{G} \to \mathscr{H}$ by the universal property of sheafification, which is $\mathscr{O}_X$-linear by the construction of the $\mathscr{O}_X$-module structure on sheafification. This proves the universal property.

(b) Since sheafification preserves stalks, we can consider the presheaf tensor product. Then for each $U \ni x$, we have a map $(\mathscr{F} \otimes_{\text{pre}} \mathscr{G})(U) = \mathscr{F}(U) \otimes \mathscr{G}(U) \to \mathscr{F}_x \otimes \mathscr{G}_x$ given by tensoring the maps $\mathscr{F}(U) \to \mathscr{F}_x$ and $\mathscr{G}(U) \to \mathscr{G}_x$. Since the latter commute with the individual restriction maps, the former commutes with the tensor product restriction maps, so we get an induced map $(\mathscr{F} \otimes_{\text{pre}} \mathscr{G})_x \to \mathscr{F}_x \otimes \mathscr{G}_x$, where elementary tensor stalks $(f \otimes g)_x$ map to $f_x \otimes g_x$.

Conversely, taking the morphism of sheaves of sets $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes_{\text{pre}} \mathscr{G}$ on stalks gives a morphism $\mathscr{F}_x \times \mathscr{G}_x = (\mathscr{F} \times \mathscr{G})_x \to (\mathscr{F} \otimes_{\text{pre}} \mathscr{G})_x$, where (f_x, g_x) maps to $(f \otimes g)_x$ after restricting f and g to the same domain. This is $(\mathscr{O}_X)_x$-bilinear, since the original map $\mathscr{F} \times \mathscr{G} \to \mathscr{F} \otimes_{\text{pre}} \mathscr{G}$ was $\mathscr{O}_X$-bilinear, so we get an induced map $\mathscr{F}_x \otimes \mathscr{G}_x \to (\mathscr{F} \otimes_{\text{pre}} \mathscr{G})_x$.

These two maps are inverse to each other, and so give an isomorphism $(\mathscr{F} \otimes_{\text{pre}} \mathscr{G})_x \cong \mathscr{F}_x \otimes \mathscr{G}_x$ of $(\mathscr{O}_X)_x$-modules.

2.7.A. For $U' \subseteq U$, the diagram of $\mathscr{G}(V)$ for $V \supseteq \pi(U)$ is a subcategory of that for $V \supseteq \pi(U')$, and so we get a natural morphism $\pi^{-1}\mathscr{G}^{\text{pre}}(U) \to \pi^{-1}\mathscr{G}^{\text{pre}}(U')$. Taking these to be the restriction maps gives a presheaf structure, where both functoriality conditions hold since the resulting morphism is uniquely determined by how it maps from the individual $\mathscr{G}(V)$'s.

It may not be a sheaf: take the continuous map $\pi : \{0, 1\} \to \{*\}$ on $\{0, 1\}$ with the discrete topology, and $\mathscr{G}(\{*\}) = \{a, b\}$ with $a \neq b$. Then $\pi^{-1}\mathscr{G}^{\text{pre}}(U) = \mathscr{G}(\{*\})$ for any non-empty $U \subseteq \{0, 1\}$, and hence we get two sections $a \in \pi^{-1}\mathscr{G}^{\text{pre}}(\{0\})$ and $b \in \pi^{-1}\mathscr{G}^{\text{pre}}(\{1\})$ which have no gluing to all of $\{0, 1\}$, since they are distinct and restriction is trivial.

2.7.B. Note that by plugging in $V = V'$ and $U = U'$ respectively, the squares (2.7.2.1) commute iff the following two triangles commute:

$$\begin{tikzcd} \mathscr{G}(V) \arrow[d, "\mathrm{res}_{V,V'}"'] \arrow[dr, "\phi_{VU'}"] & \\ \mathscr{G}(V') \arrow[r, "\phi_{V'U'}"'] & \mathscr{F}(U') \end{tikzcd} \qquad \begin{tikzcd} \mathscr{G}(V) \arrow[r, "\phi_{VU}"] \arrow[dr, "\phi_{VU'}"'] & \mathscr{F}(U) \arrow[d, "\mathrm{res}_{U,U'}"] \\ & \mathscr{F}(U') \end{tikzcd}$$

Now the triangle on the left is the condition for the ϕ_{VU}'s to give a map $\pi^{-1}\mathscr{G}^{\text{pre}}(U) \to \mathscr{F}(U)$ by the universal property of the colimit, and the triangle on the right is the condition for these maps to give a morphism $\pi^{-1}\mathscr{G}^{\text{pre}} \to \mathscr{F}$. Hence $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F})$ is in bijection with $\mathrm{Mor}_X(\pi^{-1}\mathscr{G}^{\text{pre}}, \mathscr{F})$, and further with $\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$ by the universal property of sheafification.

On the other hand, the triangle on the right is also equivalent to the maps ϕ_{VU} being simply given by restriction from $\phi_{V\pi^{-1}(V)}$, since $\pi^{-1}(V)$ is a maximal value for U. Then the triangle on the left is the condition for these maps to give a morphism $\mathscr{G} \to \pi_*\mathscr{F}$, so $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F})$ is also in bijection with $\mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$.

If $\psi : \mathscr{G}' \to \mathscr{G}$, then we get the induced map $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F}) \to \mathrm{Mor}_{YX}(\mathscr{G}', \mathscr{F})$ by $\phi_{VU} \mapsto \phi_{VU} \circ \psi(V)$, which clearly corresponds to composition with ψ in $\mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$, but also corresponds to composition with $\pi^{-1}\psi$ in $\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, since $\pi^{-1}\psi$ is induced directly from the maps $\psi(V)$.

Similarly, if $\psi : \mathscr{F} \to \mathscr{F}'$ we get $\mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F}) \to \mathrm{Mor}_{YX}(\mathscr{G}, \mathscr{F}')$ by $\phi_{VU} \mapsto \psi(U) \circ \phi_{VU}$, which clearly corresponds to composition with ψ in $\mathrm{Mor}_X(\pi^{-1}\mathscr{G}, \mathscr{F})$, but also corresponds to composition with $\pi_*\psi$ in $\mathrm{Mor}_Y(\mathscr{G}, \pi_*\mathscr{F})$ since $\pi_*\psi$ is given by the maps $\psi(\pi^{-1}(V))$.

2.7.C. Since sheafification preserves stalks, we can consider $\pi^{-1}\mathscr{G}^{\text{pre}}$. Then

$$(\pi^{-1}\mathscr{G}^{\text{pre}})_p = \varinjlim_{p \in U} \varinjlim_{\pi(U) \subseteq V} \mathscr{G}(V) \cong \varinjlim_{q \in V} \mathscr{G}(V) = \mathscr{G}_q,$$

since $V \supseteq \pi(U)$ for some $U \ni p$ iff $q \in V$, by taking $U = \pi^{-1}(V)$, and hence unwinding the universal property of the two colimits gives the same condition.

Alternatively, note that if $a : X \to Y$, $b : Y \to Z$ are continuous maps, then the functor $a^{-1} \circ b^{-1}$ is a left-adjoint for $b_* \circ a_* = (b \circ a)_*$, and so $a^{-1} \circ b^{-1}$ is naturally isomorphic to $(b \circ a)^{-1}$. Then letting $i : \{*\} \to X$ be the inclusion of p, we have $(\pi^{-1}\mathscr{G})_p$ given by $i^{-1}(\pi^{-1}\mathscr{G}) \cong (\pi \circ i)^{-1}\mathscr{G}$, which is $\mathscr{G}_q$ since $\pi \circ i$ is the inclusion of q.

2.7.D. If $W \subseteq U$ is open, then $i^{-1}\mathscr{G}(W) = \varinjlim_{W \subseteq V} \mathscr{G}(V) \cong \mathscr{G}(W)$ since W is final in the index category.

2.7.E. If $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact, then $\mathscr{F}_q \to \mathscr{G}_q \to \mathscr{H}_q$ is exact for each $q = \pi(p)$, and by 2.7.C we get that $(\pi^{-1}\mathscr{F})_p \to (\pi^{-1}\mathscr{G})_p \to (\pi^{-1}\mathscr{H})_p$ is exact. Hence $\pi^{-1}\mathscr{F} \to \pi^{-1}\mathscr{G} \to \pi^{-1}\mathscr{H}$ is exact, since exactness can be checked on stalks (see the solution to 2.6.E).

2.7.F. (a) If $q \notin Z$, then $q \in Y \setminus Z$ and $i^{-1}(Y \setminus Z) = \emptyset$, so $(i_*\mathscr{F})_q \cong \mathscr{F}(\emptyset)$. Otherwise, if U is an open neighbourhood of q in Y then $i^{-1}(U)$ is an open neighbourhood of q in Z, and any open neighbourhood of q in Z arises in this way by the definition of the subspace topology. Then $(i_*\mathscr{F})_q \cong \mathscr{F}_q$, both being given by the same diagram (up to duplicated objects).

(b) It is enough to check that the map is an isomorphism on stalks. From (a) and 2.7.D the stalk maps are isomorphisms for points in Z, and outside of Z both the source and target stalks are trivial (for the source this is because of $\operatorname{Supp}\mathscr{G} \subseteq Z$), so in either case we get an isomorphism.

2.7.G. skipped.

Part II

Schemes

Chapter 3

Toward schemes

3.1.A. Let $\mathscr{O}_X^c$ and $\mathscr{O}_Y^c$ be the sheaves of continuous functions on X and Y, so we have $\pi^\sharp : \mathscr{O}_Y^c \to \pi_*\mathscr{O}_X^c$. The condition for this to descend to the subsheafs of differentiable functions, and hence give a map $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$, is that $\pi^\sharp(f) \in \pi_*\mathscr{O}_X(U)$ whenever $f \in \mathscr{O}_Y(U)$. In other words whenever $f : U \to \mathbb{R}$ is differentiable, for $U \subseteq Y$ open, $f \circ \pi : \pi^{-1}(U) \to \mathbb{R}$ is also differentiable. If $U \subseteq Y$ is a coordinate patch, then π is differentiable on $\pi^{-1}(U)$, since its composites with each component of the coordinate system are differentiable. Hence π is differentiable on all of X, since the coordinate patches cover Y.

3.1.B. We get $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$, and hence $\pi_q^\sharp : \mathscr{O}_{Y,q} \to (\pi_*\mathscr{O}_X)_q$. The map $\mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$ is then obtained from composition with $(\pi_*\mathscr{O}_X)_q \to \mathscr{O}_{X,p}$ (see 2.2.I). If an element of $\mathfrak{m}_{Y,q}$ is represented by $f \in \mathscr{O}_Y(U)$, then f vanishes at q, and so $f \circ \pi$ vanishes at p. Now $\pi^\sharp(f_q) \in \mathscr{O}_{X,p}$ is simply the germ of $f \circ \pi$ at p, so it follows that $\pi^\sharp(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$.

3.2.A.
(a) The prime ideals of $k[\epsilon]/(\epsilon^2)$ are the prime ideals of $k[\epsilon]$ that contain ϵ^2. The only such prime is (ϵ), so $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ is the single point $[(\epsilon)]$.

(b) The prime ideals of $k[x]_{(x)}$ are the prime ideals of $k[x]$ contained in (x). The only such primes are (0) and (x), so $\operatorname{Spec} k[x]_{(x)}$ consists of the point $[(x)]$ and the point $[(0)]$.

3.2.B. By the automorphism $x \mapsto x - \frac{a}{2}$, the quotient is isomorphic to a ring of the form $\mathbb{R}[x]/(x^2 + r)$, with $r > 0$ by irreducibility. Then by the automorphism $x \mapsto x/\sqrt{r}$ this is isomorphic to $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

3.2.C. The primes of $\mathbb{Q}[x]$ are (0) and (f) for $f \in \mathbb{Q}[x]$ an irreducible polynomial. The latter correspond to algebraic numbers, up to equality of minimal polynomials, and so the points of $\mathbb{A}^1_{\mathbb{Q}}$ are $[(0)]$ together with the galois orbits over $\mathbb{Q}$ in the field $\mathbb{A}$ of algebraic numbers.

3.2.D. There are infinitely many monic irreducibles, since if $f_1, \ldots, f_r \in k[x]$ are monic irreducibles then $f_1 \cdots f_r + 1$ is not divisible by any of them, but must have some monic irreducible factor. Since distinct monic irreducibles generate distinct prime ideals, this shows that $\operatorname{Spec} k[x]$ is infinite.

3.2.E. Suppose $\mathfrak{p} \subseteq \mathbb{C}[x,y]$ is prime but not principal. Then there must exist two polynomials $f, g \in \mathfrak{p}$ not both generated by a single element of $\mathfrak{p}$. If d is a common factor of f and g, then $d \notin \mathfrak{p}$, so $f/d \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Hence we may divide out, and assume f and g have no common factor in $\mathbb{C}[x,y]$. Then they are also coprime in $\mathbb{C}(x)[y]$; if $r \in \mathbb{C}[x,y]$ is a common factor non-constant in y, then dividing out by polynomials in x preserves this, and results in factorizations in $\mathbb{C}[x,y]$. Hence 1 is a $\mathbb{C}(x)[y]$-linear combination of f and g, so that there is some denominator $h \in \mathbb{C}[x]$ with $h \in (f,g) \subseteq \mathfrak{p}$. Then h is non-constant, and one of its linear factors $x - a$ lies in $\mathfrak{p}$. By symmetry some $y - b \in \mathfrak{p}$, and hence $\mathfrak{p} = (x-a, y-b)$ by maximality.

3.2.F. If $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ is maximal, let $K = k[x_1, \ldots, x_n]/\mathfrak{m}$. By the Nullstellensatz K is a finite field extension of k, and since k is algebraically closed $K \cong k$. Hence we get a map of k-algebras $k[x_1, \ldots, x_n] \to k$, with kernel $\mathfrak{m}$. If $a_1, \ldots, a_n \in k$ are the images of $x_1, \ldots, x_n$ under this map, then $(x_1 - a_1, \ldots, x_n - a_n)$ is contained in the kernel $\mathfrak{m}$, and hence $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ by maximality.

3.2.G. If $x \in A\backslash 0$, then since A is a domain the multiplication map $A \xrightarrow{x} A$ is injective. Moreover it is k-linear, since A is a k-algebra. Since A is finite over k, the map must also be surjective, and in particular $1 \in A$ must be in the image. Therefore x is a unit, so A is a field.

3.2.H. For $(\pm\sqrt{2}, \pm\sqrt{2})$ the maximal ideal is $(x^2-2, x-y)$, whereas for $(\pm\sqrt{2}, \mp\sqrt{2})$ the maximal ideal is $(x^2-2, x+y)$. In both cases the residue field is isomorphic to $\mathbb{Q}[\sqrt{2}]$, but the possible induced maps $\mathbb{Q}[x,y] \to \mathbb{Q}[\sqrt{2}]$ differ.

3.2.I. (a) As in 3.2.E, the prime $\mathfrak{p} = \phi(\pi, \pi^2)$ is either principal or contains some non-constant $h(x) \in \mathbb{Q}[x]$. Since π is transcendental, $\mathfrak{p}$ is principal, and since $y - x^2 \in \mathfrak{p}$ is irreducible, $\mathfrak{p} = (y - x^2)$.

(b) Firstly, suppose $\mathfrak{p} \subseteq \mathbb{Q}[x,y]$ is a non-principal prime ideal. Then as in 2.3.E, we get irreducible polynomials $f(x), g(y) \in \mathfrak{p}$. Let $\alpha, \beta \in \mathbb{C}$ be roots of f and g respectively, so that $\mathfrak{p} \supseteq \phi(\alpha,\beta)$. Now $\mathbb{Q}[x,y]/\phi(\alpha,\beta) \cong \mathbb{Q}[\alpha,\beta]$ is a field, since α and β are algebraic over $\mathbb{Q}$ (so 3.2.G applies, since algebraic numbers over a field generate finite algebras). Hence $\phi(\alpha,\beta) = [\mathfrak{p}]$ by maximality.

Now suppose $\mathfrak{p} = (f(x,y))$, where $f(x,y) \in \mathbb{Q}[x,y]$ is irreducible. Without loss of generality, assume f is non-constant in x (otherwise it is non-constant in y). Then $f(x,\pi)$ is non-constant since π is transcendental, and so we have some $\alpha \in \mathbb{C}$ with $f(\alpha,\pi) = 0$. Now by the same argument as in (a), $\phi(\alpha,\pi) = [\mathfrak{p}]$.

Finally, since $\mathbb{C}$ is uncountable it has infinite transcendence degree over $\mathbb{Q}$. Hence we can find some $\alpha, \beta \in \mathbb{C}$ that are algebraically independent, so that $\phi(\alpha,\beta) = [(0)]$.

3.2.J. In general, if ϕ is any ring homomorphism then $\phi^{-1}(J)$ is an ideal whenever J is an ideal, and $\phi^{-1}(J)$ is prime whenever J is prime. Moreover $\phi(J)$ is an ideal in the subring $\operatorname{im}\phi$ whenever J is an ideal, and $\phi(J)$ is prime in $\operatorname{im}\phi$ whenever J is prime. Both maps preserve inclusions.

In the case of $\phi : A \to A/I$ the homomorphism is surjective, so $\operatorname{im}\phi$ is the whole codomain and $\phi(\phi^{-1}(J)) = J$. Hence we get an inclusion-preserving bijection between ideals (resp. prime ideals) of A/I, and ideals (resp. prime ideals) J of A such that $\phi^{-1}(\phi(J)) = J$. Since $\phi^{-1}(\phi(J)) = \ker\phi + J$, these are simply the ideals containing $\ker\phi = I$.

3.2.K. Let $\phi : A \to S^{-1}A$ be the natural map. If $I \subseteq A$ is an ideal, then $S^{-1}I$ is an ideal in $S^{-1}A$ since localization is left-exact. Moreover any ideal $J \subseteq S^{-1}A$ is of this form, since $J = S^{-1}\phi^{-1}(J)$; whenever $x/s \in J$ we get $x/1 = s(x/s) \in J$. This last observation even shows that we get a bijection (preserving inclusions) between ideals of $S^{-1}A$, and ideals $I \subseteq A$ such that $\phi^{-1}(S^{-1}I) = I$.

As in 3.2.J, prime ideals in $S^{-1}A$ correspond to prime ideals in A, and since $S^{-1}A/S^{-1}I \cong S^{-1}(A/I)$ is a domain whenever I is prime (being a subring of $\operatorname{Frac}(A/I)$), they correspond precisely to the prime ideals I in A such that $\phi^{-1}(S^{-1}I) = I$.

Now $\phi^{-1}(S^{-1}I) = \{x \in A : sx \in A \text{ for some } s \in S\}$, so if I is a prime ideal not meeting S then $\phi^{-1}(S^{-1}I) = I$, since $sx \in I$ implies $x \in I$. If instead I *does* meet S, then $S^{-1}I = (1)$ is trivial. Hence the primes in $S^{-1}A$ correspond precisely to those primes of A not meeting S.

(The fact that $S^{-1}A/S^{-1}I \cong S^{-1}(A/I)$ as A-modules follows from exactness of localization, and the isomorphism is in fact an isomorphism of rings if one checks what happens to multiplication.)

3.2.L. Consider the map $\mathbb{C}[x,y]/(xy) \to \mathbb{C}[x]$ given by $x \mapsto x$ and $y \mapsto 0$, which is well-defined since $(xy) \subseteq (y)$. This induces a map $(\mathbb{C}[x,y]/(xy))_x \to \mathbb{C}[x]_x$. Conversely, we have the composite map $\mathbb{C}[x] \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]/(xy) \to (\mathbb{C}[x,y]/(xy))_x$, which gives a map $\mathbb{C}[x]_x \to (\mathbb{C}[x,y]/(xy))_x$ by the universal property of localization.

The first map is given by $\overline{f(x,y)}/1 \mapsto f(x,0)/1$, and the second map is given by $f(x)/1 \mapsto \overline{f(x)}/1$. Since $\overline{xy} = 0$, $\overline{y}/1 = 0$ in $(\mathbb{C}[x,y]/(xy))_x$, and hence $\overline{f(x,y)}/1 = \overline{f(x,0)}/1$. This shows that the two maps give an isomorphism.

3.2.M. See 3.2.J; if $\phi(xy) = \phi(x)\phi(y) \in \mathfrak{p}$ then either $\phi(x) \in \mathfrak{p}$ or $\phi(y) \in \mathfrak{p}$.

3.2.N. (a) That was the definition of the inclusion in 3.2.J.

(b) That was the definition of the inclusion in 3.2.K.

3.2.O. Write $\phi : \mathbb{C}[y] \to \mathbb{C}[x]$ for the map $y \mapsto x^2$. A closed point $(x-a)$ in $\operatorname{Spec}\mathbb{C}[x]$ maps to

$$\begin{aligned}\{f(y) \in \mathbb{C}[y] : \phi(f(y)) \in (x-a)\} &= \{f(y) \in \mathbb{C}[y] : f(x^2) \in (x-a)\} \\ &= \{f(y) \in \mathbb{C}[y] : f(a^2) = 0\} \\ &= (y-a^2),\end{aligned}$$

so the fiber over a closed point $(y-b) \in \operatorname{Spec}\mathbb{C}[y]$ consists of those closed points $(x-a)$ such that $a^2 = b$, i.e. the points $(x \mp \sqrt{b})$.

3.2.P. (a) A ring homomorphism $\phi : A \to B$ such that $\phi(I) \subseteq J$ for ideals $I \subseteq A$, $J \subseteq B$ induces a homomorphism $A/I \to B/J$, because A/I is universal among A-algebras on which I acts trivially. This ring homomorphism induces a map $\operatorname{Spec}(B/J) \to \operatorname{Spec}(A/I)$ as usual.

(b) The map $\operatorname{Spec}(\phi) : \operatorname{Spec} k[x_1, \dots, x_m] \to \operatorname{Spec} k[y_1, \dots, y_n]$ sends a closed point $[(x_1 - a_1, \dots, x_m - a_m)]$ to the preimage

$$\begin{aligned}&\{F \in k[y_1, \dots, y_n] : \phi(F) \in (x_1 - a_1, \dots, x_m - a_m)\} \\ &\quad = \{F \in k[y_1, \dots, y_n] : F(f_1, \dots, f_n) \in (x_1 - a_1, \dots, x_m - a_m)\} \\ &\quad = \{F \in k[y_1, \dots, y_n] : F\big(f_1(a_1, \dots, a_m), \dots, f_n(a_1, \dots, a_m)\big) = 0\} \\ &\quad = \big[\big(y_1 - f_1(a_1, \dots, a_m), \dots, y_n - f_n(a_1, \dots, a_m)\big)\big].\end{aligned}$$

3.2.Q. The fiber $\pi^{-1}\big([(p)]\big)$ consists of primes containing (p), since $\pi(\mathfrak{p}) = \mathbb{Z} \cap \mathfrak{p}$, and moreover it consists of all such primes, since if $\mathbb{Z} \cap \mathfrak{p} \supseteq (p)$ then $\mathbb{Z} \cap \mathfrak{p} = (p)$ by maximality. Hence $\pi^{-1}\big([(p)]\big)$ is in bijection with $\operatorname{Spec}\mathbb{Z}[x_1, \dots, x_n]/(p) = \operatorname{Spec}\mathbb{F}_p[x_1, \dots, x_n] = \mathbb{A}^n_{\mathbb{F}_p}$.

The fiber over $[(0)]$ consists of those primes that don't meet $\mathbb{Z}$ away from 0, and hence is in bijection with the localization $\operatorname{Spec}\big((\mathbb{Z} \setminus 0)^{-1}\mathbb{Z}[x_1, \dots, x_n]\big) = \operatorname{Spec}\mathbb{Q}[x_1, \dots, x_n] = \mathbb{A}^n_{\mathbb{Q}}$.

3.2.R. (a) This amounts simply to showing that every prime ideal contains I. But every prime ideal contains every nilpotent, since every ideal contains zero.

(b) If $x^n = 0$, then $(rx)^n = 0$, and if $y^m = 0$, then $(x+y)^{n+m-1} = 0$ by binomial expansion.

3.2.S. By 3.2.R, $\mathfrak{N}(A)$ is contained in every prime ideal. Conversely, if $x \notin \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, then the localization $A_{\mathfrak{p}}$ is not the zero ring ($ux \neq 0$ for any $u \notin \mathfrak{p}$). Hence $A_{\mathfrak{p}}$ has some maximal ideal by Zorn's lemma, which by 3.2.K corresponds to a prime ideal of A not containing x.

3.2.T. The polynomial $f(x+\epsilon) - f(x)$ in $k[x, \epsilon]$ is divisible by ϵ by comparing powers of x, and hence we have

$$f(x+\epsilon) - f(x) = g(x,\epsilon)\epsilon$$

for some $g(x,\epsilon) \in k[x,\epsilon]$. Observing the coefficients of $f(x+\epsilon)$ with single powers of ϵ, we see that $g(x,0) = f'(x)$. Now in the quotient $k[x,\epsilon]/(\epsilon^2)$,

$$g(x,\epsilon)\epsilon = g(x,0)\epsilon,$$

so $f(x+\epsilon) = f(x) + f'(x)\epsilon$.

3.4.A. The x-axis corresponds to (y,z), and $(y,z) \supseteq (xy, yz)$. This shows that the generic point on the x-axis is in $V(xy,yz)$, and it follows that all the concrete points on the x-axis are as well, since as ideals they contain the generic point.

3.4.B. This is immediate from the fact that prime ideals are ideals, and the ideal generated by a set is minimal among ideals containing said set.

3.4.C. (a) We have $\emptyset = V(1)$, and $A = V(0) = V(\mathfrak{N})$.

(b) It is immediate that $\cap_i V(I_i) = V(\cup_i I_i)$, and this is equal to $V(\sum_i I_i)$ by 3.4.A.

(c) This follows from the fact that a prime $\mathfrak{p}$ contains an ideal product $I_1 I_2$ iff $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$, which one can show by considering $xy \in I_1 I_2$ for $x \in I_1 \setminus \mathfrak{p}$ and $y \in I_2$ (or vice versa).

3.4.D. The radical $\sqrt{I}$ corresponds to the nilradical $\mathfrak{N}(A/I)$, in the sense that $\sqrt{I} = \pi^{-1}(\mathfrak{N}(A/I))$ where $\pi : A \to A/I$ is the natural map. Hence it is an ideal as in 3.2.J, by 3.2.R.

That $V(\sqrt{I}) = V(I)$ follows from the inclusion-preserving association of 3.2.J, and the fact that any prime ideal in A/I contains the nilradical $\mathfrak{N}(A/I)$.

Since a power of a power is a power ($(x^n)^m = x^{nm}$), and any power is a power of a power ($x^n = (x^1)^n$), we have $\sqrt{\sqrt{I}} = \sqrt{I}$.

Finally, the association with $\mathfrak{N}(A/I)$ in fact shows that $\sqrt{I}$ is the intersection of all prime ideals containing I (using theorem 3.2.12), so in particular $\sqrt{\mathfrak{p}} = \mathfrak{p}$ for any prime $\mathfrak{p}$.

3.4.E. If a finite collection of powers lie in a finite collection of ideals, we can take the maximal such power, which will then lie in all of the ideals:

$$x^{n_i} \in I_i \text{ for } i = 1, \ldots, n \qquad \Longrightarrow \qquad x^{\max n_i} \in \bigcap_{i=1}^{n} I_i.$$

This shows $\cap_{i=1}^n \sqrt{I_i} \subseteq \sqrt{\cap_{i=1}^n I_i}$, and the reverse inclusion is clear from the fact that $\sqrt{\cdot}$ is order-preserving.

3.4.F. See 3.4.D.

3.4.G. The closed sets are $V(0) = \mathbb{A}^1_k$, $V(1) = \emptyset$, and intersections of sets of the form $V(f)$ for non-constant $f \in k[x]$. The latter consists of the closed points at which f vanishes, i.e. the irreducible factors of f. Hence the closed sets in $\mathbb{A}^1_k$ are the finite sets not containing $[(0)]$, and $\mathbb{A}^1_k$.

In other words, we have the cofinite topology on $\mathbb{A}^1_k \setminus [(0)] = \{\text{Galois orbits in } \overline{k}\}$, adjoined a dense point $[(0)]$.

3.4.H. We have

$$\begin{aligned}\pi^{-1}(V(I)) &= \{[\mathfrak{p}] \in \operatorname{Spec} A : \phi^{-1}(\mathfrak{p}) \supseteq I\} \\ &= \{[\mathfrak{p}] \in \operatorname{Spec} A : \mathfrak{p} \supseteq \phi(I)\} = V(\phi(I)),\end{aligned}$$

so π is continuous. Hence Spec takes morphisms of rings to morphisms of the topological spaces given by their spectra, by $\operatorname{Spec}(\phi)([\mathfrak{p}]) = [\phi^{-1}(\mathfrak{p})]$, which makes it a contravariant functor $\mathbf{CRing} \to \mathbf{Top}$.

3.4.I. (a) After inclusion, $\operatorname{Spec} B/I$ is simply the closed set $V(I)$ by 3.2.J. Similarly, $\operatorname{Spec} S^{-1}B$ is the open set $\operatorname{Spec} B \setminus V(f)$ for $S = \{1, f, f^2, \ldots\}$. For arbitrary S we may not get an open or closed set, since the point $[(0)] \in \operatorname{Spec} \mathbb{A}^1_k$ is neither closed nor open, but it is the image of $\operatorname{Spec} k(x)$.

(b) The inclusion maps are continuous, since they are induced by ring homomorphisms, and closed mappings onto their images, since the associations from 3.2.J/K are inclusion-preserving. Hence they are topological embeddings.

3.4.J. By definition f vanishes on $V(I)$ iff $f \in \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$, and $\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}$ by 3.4.F.

3.4.K. The closed sets are $\emptyset$, $\{[(0)]\}$, and $\{[(0)], [(x)]\}$. This is the Sierpiński topology on two points; one closed and one dense.

3.5.A. Any open set is the complement of some $V(S)$, and

$$\begin{aligned}\operatorname{Spec} A \setminus V(S) &= \operatorname{Spec} A \setminus \bigcap_{s \in S} V(s) \\ &= \bigcup_{s \in S} \big(\operatorname{Spec} A \setminus V(s)\big) \\ &= \bigcup_{s \in S} D(s).\end{aligned}$$

3.5.B. If $(f_i) = A$ then no prime can contain every f_i, so $\cup_{i\in J} D(f_i) = \operatorname{Spec} A$. If $(f_i) \neq A$, there is some maximal ideal containing (f_i), which is prime, so $\cup_{i\in J} D(f_i) \neq \operatorname{Spec} A$.

3.5.C. If $\cup_{i\in J} D(f_i) = \operatorname{Spec} A$, then we have $(f_i) = A$, so that 1 is a finite linear combination of f_i's, and hence a finite subset $J' \subseteq J$ satisfies $\cup_{i\in J'} D(f_i) = \operatorname{Spec} A$.

3.5.D. By the defining property of a prime ideal $\mathfrak{p}$, we have $fg \notin \mathfrak{p}$ iff $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.

3.5.E. We have $D(f) \subseteq D(g)$ iff $V(f) \supseteq V(g)$, and by 3.4.J this happens iff $f \in \sqrt{(g)}$. Now g is an invertible element of A_f iff we have $rg = f^n$ for some n, since if $g/1$ has inverse r/f^m then we get $f^k rg = f^{m+k}$ for some k. Again this is precisely the condition $f \in \sqrt{(g)}$.

3.5.F. We have $D(f) = \emptyset$ iff $V(f) = \operatorname{Spec} A$, i.e. iff $f \in \mathfrak{N}$ by 3.2.S.

3.6.A. If $[\mathfrak{p}] \in \operatorname{Spec} A_i$, then take its image in $\operatorname{Spec} A$ to be given by the prime

$$(1) \times \cdots \times \mathfrak{p} \times \cdots \times (1),$$

which is prime since its quotient ring is

$$0 \times \cdots \times A/\mathfrak{p} \times \cdots \times 0,$$

which is isomorphic to the domain $A/\mathfrak{p}$. If e_i is the image of the unit of A_i in A, then for any prime ideal $I = I_1 \times \cdots \times I_n$ of A not containing e_i, we have $e_j \in I$ for $j \neq i$, since $e_i e_j = 0$. Hence

$$I = (1) \times \cdots \times I_i \times \cdots \times (1),$$

and I_i is a prime ideal of A_i, since the quotient ring $A/I \cong A_i/I_i$ is a domain. This shows that the image of $\operatorname{Spec} A_i$ is $D(e_i)$. Since $(e_1, \ldots, e_n) = (1)$ we get surjectivity by 3.5.B, and injectivity is clear.

The bijection is a closed mapping, since the image of $V(I_i)$ for an ideal I_i in A_i is the closed set

$$V((1) \times \cdots \times I_i \times \cdots \times (1)),$$

and it is continuous, since the preimage of $V(I)$ for an ideal $I = I_1 \times \cdots \times I_n$ of A is the closed set

$$V(I_1) \amalg \cdots \amalg V(I_n),$$

since a prime $(1) \times \cdots \times \mathfrak{p} \times \cdots \times (1)$ from A_i contains I iff $\mathfrak{p} \supseteq I_i$.

3.6.B. (a) If $U \neq \emptyset$ is open in an irreducible space X, then any closed set $F \supseteq U$ has $F \cup (X \setminus U) = X$, and $X \setminus U \neq X$, so $F = X$.

(b) If $\overline{Z}$ is a union of two of its closed subsets, then one of them contains Z, by irreducibility of Z, and hence equals $\overline{Z}$, since Z is dense in $\overline{Z}$.

3.6.C. If A is a domain, then (0) is prime so we have the point $[(0)] \in \operatorname{Spec} A$. Since $V(0) = \operatorname{Spec} A$, this is a dense point, and hence $\operatorname{Spec} A$ is irreducible by 3.6.B(b).

3.6.D. If an irreducible space is a disjoint union of closed sets, then one of them must be the whole space by irreducibility, so the other must be empty. Hence it is connected (not a disjoint union of non-empty closed sets, which is equivalent to connectedness by taking complements).

3.6.E. Consider $A = \mathbb{C}[x, y]/(xy)$. We have $\operatorname{Spec} A = V(x) \cup V(y)$, while $V(x), V(y) \neq \operatorname{Spec} A$ because of the points $[(y)], [(x)]$. So $\operatorname{Spec} A$ is reducible, but $[(x, y)] \in V(x) \cap V(y)$ and both $V(x)$ and $V(y)$ are connected by 3.6.C and 3.6.D:

$$V(x) \cong \operatorname{Spec} A/(x) \cong \operatorname{Spec} k[y],$$
$$V(y) \cong \operatorname{Spec} A/(y) \cong \operatorname{Spec} k[x].$$

Hence $\operatorname{Spec} A$ is also connected.

3.6.F. (a) Let $\phi : k[w,x,y,z] \to k[a,b]$ be the homomorphism $w \mapsto b^3$, $x \mapsto ab^2$, $y \mapsto a^2b$, $z \mapsto a^3$. Then certainly $I \subseteq \ker\phi$, so we get an induced map $\bar{\phi} : k[w,x,y,z]/I \to k[a,b]$. Suppose $f(w,x,y,z) \in \ker\bar{\phi}$, where we may assume f is of the form

$$f(w,x,y,z) = p(w,z) + q(w,z)x + r(w,z)y$$

by applying the relations $x^2 = wy$, $y^2 = zx$ and $xy = wz$. Then

$$p(b^3,a^3) + q(b^3,a^3)ab^2 + r(b^3,a^3)a^2b = 0,$$

so by considering the monomials whose exponents in a modulo 3 are 0, 1, and 2 respectively, we see that $p = q = r = 0$, and hence $f = 0$. This shows that $\bar{\phi}$ is injective, so that $k[w,x,y,z]/I \cong \operatorname{im}\phi \subseteq k[a,b]$ is a domain.

(b) The corresponding ideal $I \subseteq k[x_1,\ldots,x_{n+1}]$ for $2 \times n$ matrices

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \end{pmatrix} \le 1$$

is generated by the minors

$$M_{ij} = \det\begin{pmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{pmatrix} = x_i x_{j+1} - x_{i+1}x_j$$

for $i \le j$. If we define $\phi : k[x_1,\ldots,x_{n+1}] \to k[a,b]$ by $x_i \mapsto a^{i-1}b^{n+1-i}$, then certainly $I \subseteq \ker\phi$, so again we have an induced map $\bar{\phi} : k[x_1,\ldots,x_{n+1}]/I \to k[a,b]$. Now any element of $\ker\bar{\phi}$ can be written in the form

$$p_1(x_1,x_{n+1}) + p_2(x_1,x_{n+1})x_2 + \cdots + p_n(x_1,x_{n+1})x_n,$$

since the relations give

$$x_i x_j = \begin{cases} x_1 x_{i+j-1} & \text{if } i+j-1 \le n+1 \\ x_{n+1}x_{i+j-n-1} & \text{if } i+j-1 \ge n+1. \end{cases}$$

Then

$$p_1(b^n,a^n) + p_2(b^n,a^n)b^{n-1}a + \cdots + p_n(b^n,a^n)ba^{n-1} = 0,$$

so considering the monomials whose exponents in a have given residues modulo n we get

$$p_1 = p_2 = \cdots = p_n = 0.$$

Hence $\bar{\phi}$ is injective, so $k[x_1,\ldots,x_{n+1}]/I \cong \operatorname{im}\phi \subseteq k[a,b]$ is a domain.

3.6.G. (a) If $\operatorname{Spec} A = \cup_{i\in I} U_i$, then we have

$$\operatorname{Spec} A = \bigcup_{i\in I}\ \bigcup_{D(f)\subseteq U_i} D(f),$$

and by 3.5.C there is a finite collection $f_1,\ldots,f_n$, where $D(f_k) \subseteq U_{i_k}$ for some $i_k \in I$, such that

$$\operatorname{Spec} A = \bigcup_{k=1}^{n} D(f_k) = \bigcup_{k=1}^{n} U_{i_k}.$$

Essentially this shows that quasicompactness can be "checked on a base", and 3.5.C already checked quasicompactness on the base of distinguished open sets.

(b) Consider $A = k[x_1, x_2, \ldots]$ with the maximal ideal $\mathfrak{m} = (x_1, x_2, \ldots)$. The open set $\operatorname{Spec} A \setminus V(\mathfrak{m})$ is a union of distinguished open sets (since they form a base), but it cannot be a finite union of distinguished open sets:

If $V(\mathfrak{m}) = V(f_1, \ldots, f_n)$, then $[\mathfrak{m}] \in V(f_1, \ldots, f_n)$, so $f_1, \ldots, f_n \in \mathfrak{m}$. Hence $f_1, \ldots, f_n \in (x_1, \ldots, x_N)$ for large enough N, which is a contradiction since then

$$[(x_1, \ldots, x_N)] \in V(f_1, \ldots, f_n) \setminus V(\mathfrak{m}).$$

3.6.H. (a) If $X = X_1 \cup \cdots \cup X_n$ with each X_i quasicompact, then an open covering of X has a finite subset covering each X_i, and the union of these finite sets gives a finite subcover.

(b) If X is quasicompact, and $F \subseteq X$ is closed, then an open covering of F gives an open covering of X after adjoining the open set $X \setminus F$, and since this has a finite subcover we get a finite subcover for the original cover of F.

3.6.I. The closure of $[\mathfrak{p}]$ is $V(\mathfrak{p})$, so $[\mathfrak{p}]$ is closed iff $V(\mathfrak{p}) = \{\mathfrak{p}\}$, which happens iff $\mathfrak{p}$ is maximal.

3.6.J. (a) Note that a prime $\mathfrak{p}$ of A is maximal iff $\operatorname{Frac}(A/\mathfrak{p})$ is finite over k, since the Nullstellensatz shows that $A/\mathfrak{m}$ is finite over k for any maximal $\mathfrak{m}$ (since A is finitely generated over k), and conversely if $\operatorname{Frac}(A/\mathfrak{p})$ is finite over k then so is $A/\mathfrak{p}$, and hence $A/\mathfrak{p}$ is a field by 3.2.G.

Now given $D(f) \neq \emptyset$ in $\operatorname{Spec} A$, there is a maximal ideal $\mathfrak{p}_f$ of A_f. Then $(A/\mathfrak{p})_f \cong A_f/\mathfrak{p}_f$ is a field, which is finite over k by the Nullstellensatz (adjoining $1/f$ to a finite generating set for A). Then $\operatorname{Frac}(A/\mathfrak{p}) = (A/\mathfrak{p})_f$ and we see that $[\mathfrak{p}]$ is a closed point from above. Hence $D(f)$ contains a closed point, so the closed points are dense since the distinguished open sets are a base.

(b) In the example of 3.4.K there is only one closed point, but the space has two points, so the closed points are not dense.

3.6.K. If $f \in A$ is zero on all the closed points, then by 3.6.J it is zero on all of the points, so $f \in \mathfrak{p}$ for every $[\mathfrak{p}] \in \operatorname{Spec} A$. It follows that $f \in \mathfrak{N}(A) = \{0\}$ by 3.2.12, so $f = 0$.

3.6.L. Note that as used in an earlier exercise, the closure of $[\mathfrak{p}]$ is $V(\mathfrak{p})$; any closed set $V(I)$ containing $\mathfrak{p}$ satisfies $\mathfrak{p} \supseteq I$, so $V(\mathfrak{p}) \subseteq V(I)$. It follows that $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ iff $\mathfrak{q} \in V(\mathfrak{p})$, i.e. $\mathfrak{p} \subseteq \mathfrak{q}$.

3.6.M. From 3.6.L, $V(y - x^2)$ is the closure of $[(y - x^2)]$.

3.6.N. Every open neighbourhood of q contains p, since the complement would otherwise be a closed set containing p but not q, so the closure K of p would not contain q. Hence every neighbourhood of q contains p. If $r \notin K$ then the complement of K is a neighbourhood of r not containing p.

3.6.O. The partially ordered set $\mathscr{I}$ of irreducible closed subsets of X containing x is non-empty, since the closure $\overline{\{x\}}$ is irreducible, so by Zorn's lemma there is a maximal totally ordered subset $\{Z_\alpha\}$ of $\mathscr{I}$. Then $\cup_\alpha Z_\alpha$ is irreducible; if it is contained in a union $F_1 \cup F_2$ of closed sets, then if $\cup_\alpha Z_\alpha \not\subseteq F_2$ every Z_α is contained in some $Z_\beta \not\subseteq F_2$ (otherwise by total ordering Z_α contains every such Z_β, so $Z_\alpha \not\subseteq F_2$). But $Z_\beta \subseteq F_1$ by irreducibility, so we get $\cup_\alpha Z_\alpha \subseteq F_1$. By construction $\cup_\alpha Z_\alpha$ is then an irreducible component of X, and contains x.

3.6.P. If every $Z_i = \mathbb{A}^2_{\mathbb{C}}$ then the sequence clearly stabilizes. Otherwise, take the first $Z_i \neq \mathbb{A}^2_{\mathbb{C}}$. It is a finite union of curves $V(f_1), \ldots, V(f_n)$ (with each f_j irreducible) and closed points $(a_1, b_1), \ldots, (a_m, b_m)$. If Z_{i+1} consists of the curves $V(g_1), \ldots, V(g_k)$ and a finite set of closed points, then since $V(g_i)$ is irreducible (being the closure of a point $[(g_1)]$) we have $V(g_1) \subseteq V(f_j)$ for some j, and in fact $V(g_1) = V(f_j)$ since f_j is irreducible. Hence Z_{i+1} is the union of some subset of the $V(f_j)$'s together with a finite set of closed points.

By finiteness, the subset of curves present must stabilize, so we may take Z_i such that $Z_{i+1}, Z_{i+2}, \ldots$ all consist of $V(f_1), \ldots, V(f_n)$ union finite sets of closed points (which may be assumed not to lie in any of the curves). Again by finiteness of the additional sets of points, the sequence must stabilize.

3.6.Q. By 3.6.N every point of a connected component is contained in an irreducible set, which by 3.6.D must be a subset of the connected component. Hence a connected component is the union of its irreducible subsets.

If V is open and closed, then any connected set lies either entirely in V or entirely in $X \setminus V$, since they form an open disjoint union. Hence the connected component of any point in V lies entirely in V, so V is the union of the connected components contained in it.

Suppose X is Noetherian, and $\{U_i\}$ are connected components of X. The closure of $\cup_i U_i$ is a finite union $Z_1 \cup \cdots \cup Z_n$ of irreducible sets, and each individual U_i is then a subunion $\cup_{j \in J_i} Z_j$ (connected components are closed). Then the whole union $\cup_i U_i = \cup_{j \in J} Z_j$ where $J = \cup_i J_i$ is finite, so $\cup_i U_i$ is closed. This shows that an arbitrary union of connected components in X is closed, and by taking complements it is also open (the complement of such a union is another such union, since the connected components form a partition).

3.6.R. If all ideals are finitely generated, then the ring is Noetherian, since the union of an ascending sequence of ideals is again an ideal, and its finite generating set must be picked up at some finite point in the sequence. Conversely, if a ring is Noetherian, any sequence of ideals of the form

$$(a_1) \subseteq (a_1, a_2) \subseteq (a_1, a_2, a_3) \subseteq \cdots$$

stabilizes, so every ideal has a finite generating set.

3.6.S. If $V(I_1) \supseteq V(I_2) \supseteq \cdots$, then $\sqrt{I_1} \supseteq \sqrt{I_2} \supseteq \cdots$. Since A is Noetherian, this sequence stabilizes, and hence the sequence of closed sets $V(I_n) = V(\sqrt{I_n})$ stabilizes.

The ring $A = k[x_1, x_2, \ldots]$ has $\operatorname{Spec} A$ non-Noetherian, since $V(x_1) \supseteq V(x_1, x_2) \supseteq \cdots$ never stabilizes (each $(x_1, \ldots, x_n)$ is prime).

3.6.T. In a Noetherian topological space any union of open sets is given by a finite sub-union, since there is a closed set minimal among the complements of finite sub-unions, which must be the complement of the whole union otherwise it could be made smaller.

3.6.U. This follows from applying the ascending chain condition to sequences of the form

$$(m_1) \subseteq (m_1, m_2) \subseteq (m_1, m_2, m_3) \subseteq \cdots .$$

In fact almost the same argument as in 3.6.Q shows that this is an equivalent definition.

3.6.V. Let $\sigma(M)$ denote the poset of submodules of M. Given a morphism $\phi : M \to N$ of modules, we get order-preserving maps $\phi_* : \sigma(M) \to \sigma(N)$ and $\phi^* : \sigma(N) \to \sigma(M)$ by taking images and pre-images. Moreover

$$(\phi^* \circ \phi_*)(L) = L + \ker \phi \qquad \text{and} \qquad (\phi_* \circ \phi^*)(K) = K \cap \operatorname{im} \phi,$$

so in particular $\phi^* \circ \phi_* = \mathrm{id}$ whenever ϕ is injective, and $\phi_* \circ \phi^* = \mathrm{id}$ whenever ϕ is surjective.

If $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is an exact sequence, we get the following strictly monotone mappings:

$$\begin{tikzcd}
\sigma(M') \arrow[dr, "i_*"] & & \\
& \sigma(M) \arrow[r, "{(i^*,\, p_*)}"] & \sigma(M') \times \sigma(M''), \\
\sigma(M'') \arrow[ur, "p^*"'] & &
\end{tikzcd}$$

where i_* and p^* are injective (and hence strictly monotone) since $i^* \circ i_* = \mathrm{id}$ and $p_* \circ p^* = \mathrm{id}$, and $(i^*,\ p_*)$ is also strictly monotone (though not generally injective!) since if $L \subseteq K \subseteq M$ and $x \in K \setminus L$, either $x \in N$, so $i^{-1}(x) \in i^*(K) \setminus i^*(L)$, or else $x \notin N$, so $p(x) \in p_*(K) \setminus p_*(L)$.

Now $\sigma(M')$ and $\sigma(M'')$ have the ascending chain condition iff $\sigma(M') \times \sigma(M'')$ does: the forward implication holds by the order-embeddings $N \mapsto (N, 0)$ and $K \mapsto (0, K)$, and the backward implication holds since a chain in the product stabilizes after its two projections stabilize. Moreover the ascending chain condition pulls back along strictly monotone maps, since a chain stabilizes whenever its image under a strictly monotone map stabilizes. Applying this to the diagram, we see that $\sigma(M)$ has the ascending chain condition iff both $\sigma(M')$ and $\sigma(M'')$ do.

3.6.W. The case $n = 1$ is clear, and if $A^{\oplus n}$ is Noetherian then both the submodule $A \subseteq A \oplus A^{\oplus n}$ and the quotient $A^{\oplus n} \cong (A \oplus A^{\oplus n})/A$ are Noetherian, so by 3.6.V the whole module $A \oplus A^{\oplus n}$ is Noetherian.

3.6.X. A finitely generated A-module is a quotient of $A^{\oplus n}$ for some n, and hence is Noetherian by 3.6.V and 3.6.W.

3.7.A. By definition $f \in I(S)$ iff f vanishes at $[(x)]$ and $[(x-1, y)]$, and since the y-axis is the closure of $[(x)]$, this is iff f vanishes on the y-axis and at the point $(1, 0)$. Now $(x) + (x - 1, y) = (1)$, so $I(S) = (x) \cap (x - 1, y) = (x) \cdot (x - 1, y) = (x^2 - x, xy)$.

3.7.B. Clearly $I(S) = (y, z) \cap (x, z) \cap (x, y)$. Now any element of (y, z) is of the form $f(x, y)y + g(x, y, z)z$, and if it is also an element of (x, z) we must have $f(x, y)y \in (x, z)$, so that x divides $f(x, y)$. This shows that $(y, z) \cap (x, z) = (xy, z)$. Now an element of (x, y) is of the form $a(x, z)x + b(y, z)y + c(x, y, z)xy$, and if it is also an element of (xy, z) then z divides $a(x, z)x + b(y, z)y$, and therefore z divides both $a(x, z)$ and $b(y, z)$. This shows that $I(S) = (yz, xz, xy)$.

3.7.C. The closure $\overline{S}$ is the smallest $V(J) \supseteq S$, which is given by the largest J such that $V(J) \supseteq S$. This is the largest J such that $\mathfrak{p} \supseteq J$ for every $[\mathfrak{p}] \in S$, which is precisely the intersection $I(S)$.

3.7.D. By definition $I(V(J))$ is the intersection of all the prime ideals containing J, which is the radical of J as seen in 3.4.F.

3.7.E. If $\mathfrak{p}$ is prime, then $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$ is irreducible. Conversely if S is irreducible, then whenever $fg \in I(S)$ we have $S \subseteq V(fg) = V(f) \cup V(g)$, so $S \subseteq V(f)$ or $S \subseteq V(g)$, i.e. $f \in I(S)$ or $g \in I(S)$. Hence we get a bijection $[\mathfrak{p}] \leftrightarrow \mathfrak{p} \leftrightarrow V(\mathfrak{p})$ between points of $\operatorname{Spec} A$ and closed irreducible subsets of $\operatorname{Spec} A$, which is given by taking the closure $V(\mathfrak{p})$ of a point.

3.7.F. A closed irreducible subset of $\operatorname{Spec} A$ is an irreducible component iff it is maximal with respect to inclusion among closed irreducible subsets, which is iff the prime it is associated to is minimal with respect to inclusion among primes. Note that all irreducible components arise in this way since irreducible components are closed.

3.7.G. The irreducible components of $\operatorname{Spec} k[x, y]/(xy)$ are the irreducible components of $V(xy) = V(x) \cup V(y)$ in $\mathbb{A}^2_k$, which are just $V(x)$ and $V(y)$ since (x) and (y) are prime. Hence the minimal primes of $k[x, y]/(xy)$ are (x) and (y) by 3.7.F.

Chapter 4

The structure sheaf

4.1.A. Let S be the multiplicative set that $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ is the localization of A at, so

$$S = \{g \in A : V(g) \subseteq V(f)\}.$$

Then $f \in S$, so we have a natural map $A_f \to A_S$ where $x/s \mapsto x/s$. Conversely, by 3.5.E any element of S is invertible in A_f, so we have a natural map $A_S \to A_f$ given by $x/s \mapsto (s/1)^{-1}(x/1)$. This gives an isomorphism $A_f \cong A_S$, since these two maps are clearly inverse to each other.

4.1.B. This is combined with the following exercise:

4.1.C. The inclusion $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is a topological embedding, with image $D(f)$. The distinguished open sets $D(x/s) = D(x/1)$ in $\operatorname{Spec} A_f$ are a base, and correspond to those $D(x) \subseteq D(f)$ in $\operatorname{Spec} A$. In this way the restriction of the structure sheaf from $\operatorname{Spec} A$ gives a sheaf on the base for $\operatorname{Spec} A_f$, which is in fact isomorphic to the genuine structure sheaf for $\operatorname{Spec} A_f$:

If $D(x) \subseteq D(f)$, then (by 3.5.E) f is invertible in A_x (equivalently, in any A-algebra where x is invertible). Hence $(A_f)_x \cong A_x$, since both satisfy the same universal property. This gives an open set-by-open set isomorphism of the sheaves, and it respects the restriction morphisms since in all cases such morphisms are the unique localizations of $A \to A$ to the given multiplicative sets.

Now suppose $D(f) = \cup_i D(f_i)$. Under the isomorphism, we have $\operatorname{Spec} A_f = \cup_i D(f_i/1)$, and it has been shown that identity and gluing both hold for this latter union. Since both properties are preserved by presheaf isomorphisms, it follows that identity and gluing both hold for the union $D(f) = \cup_i D(f_i)$ in $\operatorname{Spec} A$.

4.1.D. It clearly forms a presheaf on the base, since all the restriction morphisms are uniquely induced by the identity on M. Now let $S_f = \{g \in A : V(f) \subseteq V(g)\}$, so $\widetilde{M}(D(f)) = M_{S_f}$. Then M_{S_f} is an A_{S_f}-module in a unique way, and hence gives a presheaf $\widetilde{M_{S_f}}$ on $\operatorname{Spec} A_{S_f}$. As in 4.1.A, under the inclusion of $\operatorname{Spec} A_{S_f}$ as $D(f) \subseteq \operatorname{Spec} A$, this presheaf is isomorphic to the restriction sheaf $\widetilde{M}|_{D(f)}$, since $(M_{S_f})_{S_g} \cong M_{S_g}$ (uniquely, as extensions of M) for $S_g \supseteq S_f$. Hence it suffices to show identity and gluing in $\widetilde{M}$ only for open coverings $\operatorname{Spec} A = \cup_i D(f_i)$, where say $\operatorname{Spec} A = D(f_1) \cup \cdots \cup D(f_n)$ is a finite subcover. Then as previously we have $A = (f_1, \ldots, f_n)$.

Suppose $m \in M$ has $m_{f_i} = 0$ for each i. Then we have $f_i^N m = 0$ for $i \in \{1, \ldots, n\}$ for some N. Since $(f_1^N, \ldots, f_n^N) = A$, we have $1 = r_1 f_1^N + \cdots + r_n f_n^N$ for some $r_1, \ldots, r_n \in A$, and hence

$$m = (r_1 f_1^N + \cdots + r_n f_n^N)m = r_1 f_1^N m + \cdots + r_n f_n^N m = 0.$$

So identity holds.

For gluing, suppose we have $m_i/f_i^{k_i} \in M_{f_i}$ for each i that agree in $M_{f_i f_j}$. Since $D(f_i) = D(f_i^{k_i})$, we may assume $k_i = 1$. Then we have some N_{ij} with

$$(f_i f_j)^{N_{ij}}(f_j m_i - f_i m_j) = 0,$$

and in particular for $i, j \in \{1, \ldots, n\}$ we can uniformly choose N such that

$$(f_i f_j)^N (f_j m_i - f_i m_j) = 0.$$

Since $D(f_i) = D(f_i^{N+1})$ and $m_i/f_i = f_i^N m_i / f_i^{N+1}$ we can assume $N = 0$, so for $i, j \in \{1, \ldots, n\}$ we have $f_j m_i = f_i m_j$. Now since $(f_1, \ldots, f_n) = A$, we have $1 = r_1 f_1 + \cdots + r_n f_n$ for some $r_1, \ldots, r_n \in A$. Consider

$$m = r_1 m_1 + \cdots + r_n m_n.$$

For $i \in \{1, \ldots, n\}$, we have

$$f_i m = \sum_{j=1}^n f_i r_j m_j = \sum_{j=1}^n f_j r_j m_i = m_i,$$

so m agrees with m_i/f_i in $D(f_i)$. Now as with $\mathscr{O}_{\operatorname{Spec} A}$, we can repeat this argument with any other finite subcover, and by identity the resulting gluing must equal m. This shows that m agrees with each m_i/f_i on $D(f_i)$, so gluing holds.

Finally, each M_{S_f} is an A_{S_f}-module in a unique way extending the A-module structure, and this makes $\widetilde{M}$ an $\mathscr{O}_{\operatorname{Spec} A}$-module since in all cases the scalar multiplication is uniquely determined by that of M.

4.3.A. Ring isomorphisms $A' \to A$ certainly give isomorphisms $\operatorname{Spec} A \to \operatorname{Spec} A'$, since Spec is a contravariant functor. Conversely, an isomorphism $\operatorname{Spec} A \to \operatorname{Spec} A'$ contains an isomorphism of the global function rings $A' \to A$. By definition the former followed by the latter gives back the same map, so it remains only to prove that an isomorphism $\operatorname{Spec} A \to \operatorname{Spec} A'$ is induced by its isomorphism on the rings of global functions.

So suppose $\pi : \operatorname{Spec} A \to \operatorname{Spec} A'$ is an isomorphism. Then for $[\mathfrak{p}] \in \operatorname{Spec} A$, $[\mathfrak{q}] = \pi([\mathfrak{p}])$, we have an isomorphism of stalks induced by $\pi^\sharp : \mathscr{O}_{\operatorname{Spec} A'} \to \pi_* \mathscr{O}_{\operatorname{Spec} A}$.

$$\begin{tikzcd}
\Gamma(\mathscr{O}_{\operatorname{Spec} A'}, D(1)) \arrow[r, equal] & A' \arrow[r, "\pi^\sharp(D(1))"] \arrow[d] & A \arrow[r, equal] \arrow[d] & \Gamma(\mathscr{O}_{\operatorname{Spec} A}, D(1)) \\
& \mathscr{O}_{\operatorname{Spec} A', [\mathfrak{q}]} \arrow[r, "\pi^\sharp_{[\mathfrak{q}]}"] & \mathscr{O}_{\operatorname{Spec} A, [\mathfrak{p}]} &
\end{tikzcd}$$

Now from 1.4.E (the limit works in the category of rings, since having commuting maps from each A_s automatically gives an A-algebra structure that they respect) we have

$$A_\mathfrak{p} = \varinjlim_{f \notin \mathfrak{p}} A_f = \mathscr{O}_{\operatorname{Spec} A, [\mathfrak{p}]},$$

since stalks can be computed on the base. Hence the commuting square can be viewed as

$$\begin{tikzcd}
A' \arrow[r, "\pi^\sharp(D(1))"] \arrow[d] & A \arrow[d] \\
A'_\mathfrak{q} \arrow[r, "\pi^\sharp_{[\mathfrak{q}]}"] & A_\mathfrak{p}.
\end{tikzcd}$$

Since the horizontal maps are isomorphisms, the maximal ideal in $A_\mathfrak{p}$ pulls back to the maximal ideal in $A'_\mathfrak{q}$, and hence to $\mathfrak{q}$ in A'. But it also pulls back to $\mathfrak{p}$ in A, so from the square we see that $\mathfrak{q}$ is the pull-back of $\mathfrak{p}$ along $\pi^\sharp(D(1))$. This shows that on points (and hence topologically), π is the same as the map $\operatorname{Spec} \pi^\sharp(D(1))$.

Moreover, the above square shows that $\pi^\sharp_{[\mathfrak{q}]}$ is uniquely determined by $\pi^\sharp(D(1))$, by the universal property of localization. This is true for any morphism of sheaves $\mathscr{O}_{\operatorname{Spec} A'} \to \pi_* \mathscr{O}_{\operatorname{Spec} A}$, and hence shows that π and $\operatorname{Spec} \pi^\sharp(D(1))$ are the same as maps of ringed spaces, since they induce the same maps on stalks.

4.3.B. See 4.1.C for a description of an isomorphism of sheaves on the base

$$\mathscr{O}_{\operatorname{Spec} A}|_{D(f)} \cong \pi_* \mathscr{O}_{\operatorname{Spec} A_f},$$

where $\pi : \operatorname{Spec} A_f \to D(f)$ is the topological embedding of $\operatorname{Spec} A_f$ into $\operatorname{Spec} A$. This makes π into an isomorphism of ringed spaces:

$$(\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}) \cong (D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}).$$

4.3.C. Given an open neighbourhood of a point in an affine scheme $[\mathfrak{p}] \in U \subseteq \operatorname{Spec} A$, there is some distinguished open set $[\mathfrak{p}] \in D(f) \subseteq U$, and from 4.3.B we have $(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}) \cong \operatorname{Spec} A_f$. This shows that $(U, \mathscr{O}_{\operatorname{Spec} A}|_U)$ is a scheme, and it follows that $(U, \mathscr{O}_X|_U)$ is a scheme for arbitrary schemes X, since it is locally of the former form by intersecting with affine neighbourhoods.

4.3.D. If $U \subseteq X$ is open, then since $(U, \mathscr{O}_X|_U)$ is a scheme we have a covering of U by affine open subsets (take an affine neighbourhood of every point in U).

4.3.E. (a) The disjoint union $X = \operatorname{Spec} A_1 \amalg \cdots \amalg \operatorname{Spec} A_n$ is homeomorphic to $\operatorname{Spec} A$, where $A = A_1 \times \cdots \times A_n$, by 3.6.A. The bases of distinguished open sets for each $\operatorname{Spec} A_i$ combine to give a base for X, and hence also for $\operatorname{Spec} A$, and we have an isomorphism of sheaves on this base; if $f \in A_i$, then

$$\begin{aligned}\mathscr{O}_{\operatorname{Spec} A}(D(f)) &= A_f \\ &= (A_i)_f \\ &= \mathscr{O}_{\operatorname{Spec} A_i}(D(f)) \\ &= \mathscr{O}_X(D(f)),\end{aligned}$$

where $A_f = (A_i)_f$ since $fe_j = 0$ for $j \neq i$.

(b) An infinite disjoint union of non-empty spaces is not quasicompact, since the defining union has no subcover. Hence an infinite disjoint union of non-empty affine schemes is not affine, since affine schemes are quasicompact.

4.3.F. See 4.3.A.

4.3.G. (a) Suppose $f \in \mathscr{O}_X(U)$ for a ringed space $(X, \mathscr{O}_X)$. If the germ f_p at $p \in U$ is invertible, then there is some neighbourhood $p \in V \subseteq U$ and a $g \in \mathscr{O}_X(V)$ with $g \cdot \operatorname{res}_{U,V}(f) = 1$. Then for any $q \in V$, we have $g_q f_q = 1$ so f_q is invertible. This shows that the subset of U where the germ of f is invertible is open.

In a locally ringed space, the germ is invertible iff it doesn't lie in the maximal ideal (the complement of the maximal ideal of a local ring consists of units), and hence this is just the non-vanishing locus of f.

(b) Multiplication by f and its restrictions gives an endomorphism of sheaves of rings, and it is an isomorphism on stalks since non-vanishing implies invertibility in the stalk. It is therefore an isomorphism globally, so f is invertible.

4.4.A. Define the topological space X to be the quotient of the disjoint union $\amalg_{i \in I} X_i$ by the equivalence relation identifying $x_i \in X_i$ and $x_j \in X_j$ iff $f_{ij}(x_i) = x_j$. Then we have an open cover $X = \cup_i X_i$, such that the embeddings induce the homeomorphisms f_{ij} on each intersection. If Y is any other topological space, with an open covering $Y = \cup_i Y_i$ and homeomorphisms $Y_i \cong X_i$ that induce the homeomorphisms f_{ij} on the intersections $Y_i \cap Y_j$, then we have a unique continuous map $\amalg_i X_i \to Y$ respecting the homeomorphisms $Y_i \cong X_i$ by the universal property of the coproduct, and this descends (uniquely) to a homeomorphism with the quotient space X, because of the condition on intersections.

Now on this space X with the covering $X = \cup_i X_i$, we have sheaves $\mathscr{O}_{X_i}$ on each X_i which satisfy the conditions of 2.5.D. Hence there is a sheaf of rings $\mathscr{O}_X$ on X such that $\mathscr{O}_X|_{X_i} \cong \mathscr{O}_{X_i}$, with the induced isomorphisms

$$\mathscr{O}_{X_i}|_{X_{ij}} \cong \mathscr{O}_X|_{X_i \cap X_j} \cong \mathscr{O}_{X_j}|_{X_{ji}}$$

given by f_{ij}. Moreover $\mathscr{O}_X$ is unique among sheaves with these conditions, up to unique isomorphisms respecting the isomorphisms $\mathscr{O}_X|_{X_i} \cong \mathscr{O}_{X_i}$.

Combining the uniqueness properties on the topological space and the sheaf of rings, we see that this ringed space $(X, \mathscr{O}_X)$ is unique among ringed spaces that have open coverings with each restriction isomorphic to $(X_i, \mathscr{O}_{X_i})$ and inducing the isomorphisms f_{ij} on intersections, up to unique isomorphisms respecting the inclusions $(X_i, \mathscr{O}_{X_i}) \to (X, \mathscr{O}_X)$.

Any such ringed space is also a scheme, since its restriction to each open set in the covering is a scheme.

4.4.B. The sheaf property implies that the global section ring is the fibered product:

$$\begin{array}{ccc} A & \longrightarrow & k[t] \\ \downarrow & & \downarrow \\ k[u] & \longrightarrow & k[t,1/t] \cong k[u,1/u] \end{array}$$

Now the given isomorphism $k[t,1/t] \cong k[u,1/u]$ is induced from the obvious isomorphism $k[t] \cong k[u]$, so this fibered product is equivalent to

$$\begin{array}{ccc} A & \longrightarrow & k[t] \\ \downarrow & & \downarrow \\ k[u] \cong k[t] & \longrightarrow & k[t,1/t] \end{array}$$

Then clearly $A \cong k[t]$, with the restriction maps to X and Y given by the identity and the obvious isomorphism $k[t] \cong k[u]$.

Now if this scheme were affine, the closed set $V(x)$ of points where $x \in k[x]$ vanishes (see 4.3.G) would be a single point, since (x) is a maximal ideal. But $V(x)$ consists of the two points $[(t)] \in X$ and $[(u)] \in Y$, which are the only unidentified points.

4.4.C. Define the affine plane with doubled origin by gluing two copies of $\mathbb{A}^2$, say $\operatorname{Spec} k[x,y]$ and $\operatorname{Spec} k[u,v]$, along the open sets $D(x) \cup D(y)$ and $D(u) \cup D(v)$, by the isomorphism of open subschemes induced by the natural isomorphism $\operatorname{Spec} k[x,y] \cong \operatorname{Spec} k[u,v]$. The images of the two copies of $\mathbb{A}^2$ give open affine subschemes, which cover the space, but their intersection is isomorphic to the scheme $\mathbb{A}^2 \setminus \{(0,0)\}$, which has been shown to be non-affine.

4.4.D. The isomorphisms on $U_i \cap U_j$ and $U_j \cap U_k$

$$\operatorname{Spec} k[x_{0/i}, \dots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong \operatorname{Spec} k[x_{0/j}, \dots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$
$$\operatorname{Spec} k[x_{0/j}, \dots, x_{n/j}, 1/x_{k/j}]/(x_{j/j} - 1) \cong \operatorname{Spec} k[x_{0/k}, \dots, x_{n/k}, 1/x_{j/k}]/(x_{k/k} - 1)$$

descend to isomorphisms on the triple overlap $U_i \cap U_j \cap U_k$

$$\operatorname{Spec} k[x_{0/i}, \dots, x_{n/i}, 1/x_{j/i}, 1/x_{k/i}]/(x_{i/i} - 1) \cong \operatorname{Spec} k[x_{0/j}, \dots, x_{n/j}, 1/x_{i/j}, 1/x_{k/j}]/(x_{j/j} - 1)$$
$$\operatorname{Spec} k[x_{0/j}, \dots, x_{n/j}, 1/x_{k/j}, 1/x_{i/j}]/(x_{j/j} - 1) \cong \operatorname{Spec} k[x_{0/k}, \dots, x_{n/k}, 1/x_{j/k}, 1/x_{i/k}]/(x_{k/k} - 1)$$

since $x_{k/i} \mapsto x_{k/j}/x_{i/j}$, which is a unit multiple of $x_{k/j}$, and similar for the second isomorphism. This gives a composite isomorphism

$$\begin{aligned} &\operatorname{Spec} k[x_{0/i}, \dots, x_{n/i}, 1/x_{j/i}, 1/x_{k/i}]/(x_{i/i} - 1) \\ &\qquad \cong \operatorname{Spec} k[x_{0/k}, \dots, x_{n/k}, 1/x_{j/k}, 1/x_{i/k}]/(x_{k/k} - 1), \end{aligned}$$

where

$$\begin{aligned} x_{m/i} &\mapsto x_{m/j}/x_{i/j} \\ &\mapsto (x_{i/k}/x_{j/k})^{-1}(x_{m/k}/x_{j/k}) \\ &= x_{m/k}/x_{i/k}, \end{aligned}$$

which is the same as the map induced directly from $U_i \cap U_k$.

4.4.E. Suppose we have a function on $U_i \cup U_j$ for $i \neq j$, given by

$$f\big(x_{0/i}, \ldots, x_{(i-1)/i}, 1, x_{(i+1)/i}, \ldots, x_{n/i}\big) \quad \text{on } U_i,$$

and

$$g\big(x_{0/j}, \ldots, x_{(j-1)/j}, 1, x_{(j+1)/j}, \ldots, x_{n/j}\big) \quad \text{on } U_j.$$

Agreement on $U_i \cap U_j$ implies that in $k[x_{0/j}, \ldots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$ we have

$$\begin{aligned} &f\big(x_{0/j}/x_{i/j}, \ldots, x_{(i-1)/j}/x_{i/j}, 1, x_{(i+1)/j}/x_{i/j}, \ldots, x_{n/j}/x_{i/j}\big) \\ &\quad = g\big(x_{0/j}, \ldots, x_{(j-1)/j}, 1, x_{(j+1)/j}, \ldots, x_{n/j}\big). \end{aligned}$$

If A denotes the domain $k[x_{0/j}, \ldots, x_{n/j}]/(x_{j/j} - 1)$, then we have an element of $A[x_{i/j}]$ equalling an element of $A[1/x_{i/j}]$ in $\operatorname{Frac}(A)$. This is only possible if it is in fact an element of A, which can only happen if f is a constant. Hence $\Gamma(U_i \cup U_j, \mathscr{O}) \cong k$, and by gluing $\Gamma(\mathbb{P}^n_k, \mathscr{O}) \cong k$. It follows that $\mathbb{P}^n_k$ cannot be affine, since it is not the single point space $\operatorname{Spec} k$.

4.4.F. We can view U_i as the closed subset $V(x_i - 1) \subseteq \mathbb{A}^{n+1}_k$. Then U_i is glued to U_j along the subsets $D(x_j) \cap V(x_i - 1)$ and $D(x_i) \cap V(x_j - 1)$ respectively, where a closed point $(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$ in $D(x_j) \cap V(x_i - 1)$ is identified with the closed point $(a_0/a_j, \ldots, a_{j-1}/a_j, 1, a_{j+1}/a_j, \ldots, a_n/a_j)$ in $D(x_i) \cap V(x_j - 1)$. Now given a scale-invariant point $[a_0, \ldots, a_n]$ with some $a_i \neq 0$, we get a well-defined closed point $(a_0/a_i, \ldots, a_{i-1}/a_i, 1, a_{i+1}/a_i, \ldots, a_n/a_i)$ in U_i. The condition for this to give a well-defined closed point of $\mathbb{P}^n_k$ is that whenever a_j is also non-zero, this closed point in U_i is identified with the closed point $(a_0/a_j, \ldots, a_{j-1}/a_j, 1, a_{j+1}/a_j, \ldots, a_n/a_j)$ in U_j. But this is clear from the previous description of the identifications. Hence the scale-invariant points $[a_0, \ldots, a_n]$ are in bijection with the closed points of $\mathbb{P}^n_k$, since every closed point $(a_0, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$ of U_i arises in this way.

4.5.A. The isomorphisms from the definition of $\mathbb{P}^2_k$ descend to isomorphisms

$$\begin{aligned} &\operatorname{Spec} k[x_{0/i}, x_{1/i}, x_{2/i}, 1/x_{j/i}]/(x_{i/i} - 1, x_{0/i}^2 + x_{1/i}^2 - x_{2/i}^2) \\ &\quad \cong \operatorname{Spec} k[x_{0/j}, x_{1/j}, x_{2/j}, 1/x_{i/j}]/(x_{j/j} - 1, x_{0/j}^2 + x_{1/j}^2 - x_{2/j}^2), \end{aligned}$$

since the defining equation $x_0^2 + x_1^2 - x_2^2 = 0$ is homogeneous; we get

$$\begin{aligned} x_{0/i}^2 + x_{1/i}^2 - x_{2/i}^2 &\mapsto (x_{0/j}/x_{i/j})^2 + (x_{1/j}/x_{i/j})^2 - (x_{2/j}/x_{i/j})^2 \\ &= (1/x_{i/j})^2 \cdot (x_{0/j}^2 + x_{1/j}^2 - x_{2/j}^2), \end{aligned}$$

a unit multiple of $x_{0/j}^2 + x_{1/j}^2 - x_{2/j}^2$. These quotient isomorphisms inherit the agreement on triple overlaps, so we can glue these schemes together to get the scheme for $x_0^2 + x_1^2 - x_2^2 = 0$ "in $\mathbb{P}^2_k$".

4.5.B. Let I be the ideal generated by the homogeneous polynomials f_i. As in 4.5.A, the isomorphisms from the definition of $\mathbb{P}^n_A$ descend to isomorphisms

$$\operatorname{Spec} k[x_0, \ldots, x_n, 1/x_k]/(x_j - 1, I) \cong \operatorname{Spec} k[x_0, \ldots, x_n, 1/x_j]/(x_k - 1, I),$$

since each $f_i(x_0, \ldots, x_n)$ maps to a unit multiple

$$f_i(x_0/x_j, \ldots, x_n/x_j) = (1/x_j)^{\deg f_i} f_i(x_0, \ldots, x_n).$$

Moreover these isomorphisms agree on triple overlaps as inherited from the isomorphisms they are induced by. Hence we can glue the schemes $\operatorname{Spec} k[x_0, \ldots, x_n]/(x_j - 1, I)$ along these isomorphisms of open sets, to get what could be called the scheme for the vanishing of I in $\mathbb{P}^n_A$.

4.5.C. (a) If I is homogeneous, then every element $f \in I$ is a linear combination

$$f = s_1 f_1 + \cdots + s_k f_k,$$

where $s_1, \ldots, s_n \in S_\bullet$ and $f_1, \ldots, f_n \in I$ are homogeneous. If $[f]_n$ denotes the degree n piece of f, then we have

$$\begin{aligned} [f]_n &= [s_1 f_1]_n + \cdots + [s_k f_k]_n \\ &= [s_1]_{n-\deg f_1} \cdot f_1 + \cdots + [s_k]_{n-\deg f_k} \cdot f_k \\ &\in I. \end{aligned}$$

Conversely, if I contains the homogeneous components of all of its elements, then it is generated by those homogeneous components (even additively), and hence is homogeneous.

(b) If I_i is a homogeneous ideal for each $i \in \mathscr{I}$, then any element of the sum $I = \sum_{i \in \mathscr{I}} I_i$

$$f = f_{i_1} + \cdots + f_{i_k},$$

where each $f_{i_j} \in I_{i_j}$, satisifes

$$[f]_n = [f_{i_1}]_n + \cdots + [f_{i_k}]_n \in I$$

because $[f_{i_j}]_n \in I_{i_j}$ for each j, since I_{i_j} is homogeneous. Hence I is homogeneous. Similarly, any $f \in \cap_{i \in \mathscr{I}} I_i$ has $[f]_n \in I_i$ for each i since I_i is homogeneous, so therefore $[f]_n \in \cap_{i \in \mathscr{I}} I_i$, and hence $\cap_{i \in \mathscr{I}} I_i$ is homogeneous. Now if I and J are homogeneous ideals, then any element of their product IJ is of the form

$$f = f_1 g_1 + \cdots + f_k g_k,$$

where each $f_i \in I$ and $g_i \in J$. Then

$$\begin{aligned} [f]_n &= [f_1 g_1]_n + \cdots + [f_k g_k]_n \\ &= \sum_j [f_1]_j [g_1]_{n-j} + \cdots + \sum_j [f_k]_j [g_k]_j \\ &= \sum_{ij} [f_i]_j [g_i]_{n-j} \\ &\in IJ, \end{aligned}$$

because each $[f_i]_j \in I$ and $[g_i]_{n-j} \in J$, since I and J are homogeneous. Hence IJ is homogeneous, and by induction any product of (necessarily finitely many) homogeneous ideals is homogeneous. Finally, suppose I is a homogeneous ideal and $f \in \sqrt{I}$, say $f^N \in I$. Then if m is minimal such that $[f]_m \neq 0$, we have

$$[f]_m^N = [f^N]_{Nm} \in I$$

since I is homogeneous, and so $[f]_m \in \sqrt{I}$. By induction on m, replacing f by $f - [f]_m$, we get $[f]_n \in \sqrt{I}$ for all $n \geq m$ (and trivially for $n < m$), so $\sqrt{I}$ is homogeneous.

(c) Suppose I satisfies the given property, and $fg \in I$ for arbitrary $f, g \in S_\bullet$. Suppose n, m are minimal such that $[f]_n \neq 0$ and $[g]_m \neq 0$. Then we have

$$[f]_n [g]_m = [fg]_{nm} \in I$$

since I is homogeneous, so therefore either $[f]_n \in I$ or $[g]_m \in I$ by assumption. Without loss of generality say $[f]_n \in I$, so $(f - [f]_n) g \in I$. Then by induction on the number of non-zero homogeneous components of f and g, we get $g \in I$ or $f - [f]_n \in I$ (in which case $f \in I$). Hence I is prime.

4.5.D. (a) If $S_\bullet$ is generated as an A-algebra by homogeneous elements $s_1, \dots, s_n$ of positive degree, then S_+ is generated as an ideal by $s_1, \dots, s_n$: any $s \in S_+$ is a polynomial $s \in A[s_1, \dots, s_n]$, with no constant coefficient since s has no degree zero term, so s is an $S_\bullet$-linear combination of $s_1, \dots, s_n$.

Conversely, suppose S_+ is generated as an ideal by $s_1, \dots, s_n \in S_+$ (homogeneous without loss of generality, since S_+ is homogeneous). If $f \in S_k$, we have

$$f = f_1 s_1 + \cdots + f_n s_n$$

for some $f_1, \dots, f_n \in S_\bullet$, and hence

$$\begin{aligned} f &= [f_1 s_1 + \cdots + f_n s_n]_k \\ &= \sum_{\deg s_i \leq k} [f_i]_{k - \deg s_i} \cdot s_i. \end{aligned}$$

Now $k - \deg s_i < k$ since $s_i \in S_+$, so $f \in A[s_1, \dots, s_n]$ since by induction each $[f_i]_{k-\deg s_i} \in A[s_1, \dots, s_n]$. Hence $S_\bullet$ is generated by $s_1, \dots, s_n$ as an A-algebra, since it is generated additively by the homogeneous elements.

(b) This follows from 3.6.V because of the exact sequence of S-modules

$$0 \to S_+ \to S_\bullet \to S/S_+ \to 0,$$

since $A \cong S/S_+$ is a Noetherian ring iff it is a Noetherian S-module (the S-module structure is just an extension by zero of the A-module structure).

4.5.E. (a) Taking the intersection with A_0, a homogeneous prime of A gives a prime of A_0. Conversely, suppose we have a prime $\mathfrak{p}$ in A_0. Define Q_i to be the set of those $a \in A_i$ such that $a^{\deg f}/f^i \in \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, we have $A_i \cdot Q_j \subseteq Q_{i+j}$, and since $\mathfrak{p}$ is prime, if $a \in A_i$, $b \in A_j$ satisfy $ab \in Q_{i+j}$ then $a \in Q_i$ or $b \in Q_j$. Now if $a, b \in Q_i$, we get $a^2, ab, b^2 \in Q_{2i}$, so $(a+b)(a+b) \in Q_{2i}$ and hence $a + b \in Q_i$. This shows that $\oplus_i Q_i$ is a homogeneous ideal, and that it is prime (by 4.5.C).

Now certainly $Q_0 = \mathfrak{p}$, and if $\mathfrak{q}$ is a homogeneous prime of A then for $a \in A_i$ we have $a \in \mathfrak{q}$ iff $a^{\deg f} \in \mathfrak{q}$ (as $\mathfrak{q}$ is prime) iff $a^{\deg f}/f^i \in \mathfrak{q}$ (as f is a unit), so this construction gives a bijection between primes of A_0 and homogeneous primes of A.

(b) The prime ideals of $((S_\bullet)_f)_0$ are in bijection with the homogeneous primes of $(S_\bullet)_f$, which are in bijection with the homogeneous primes of $S_\bullet$ not containing f. (A prime in $S_\bullet$ not containing f is homogeneous iff the corresponding prime in $(S_\bullet)_f$ is homogeneous, since the homogeneous components of a/f^n are the homogeneous components of a over f^n.) Note that from the construction in (a) we see that such primes do not contain A_+, otherwise for every $a \in A_0$ we would have $fa \in A_1 = Q_1$, so $a = fa/f \in \mathfrak{p}$, which contradicts $\mathfrak{p} \neq A_0$.

4.5.F. By definition $D(f)$ is the set of homogeneous prime ideal in $S_\bullet$ containing f but not S_+. From 4.5.E these correspond bijectively (and in an inclusion-preserving manner) with $\operatorname{Spec}((S_\bullet)_f)_0$.

4.5.G. Any open set is of the form $\operatorname{Proj} S_\bullet \setminus V(T)$ for some set $T \subseteq S_+$ of homogeneous elements, which is $\cup_{f \in T} D(f)$.

4.5.H. (a) Replacing $S_\bullet$ by $(S/I)_\bullet$, which is still a $\mathbb{Z}_{\geq 0}$-graded ring, we may assume $I = 0$ since $\operatorname{Proj}(S/I)_\bullet$ corresponds to $V(I)$ in $\operatorname{Proj} S_\bullet$ (a prime of $(S/I)_\bullet$ is homogeneous iff the associated prime of S is, and $(S/I)_+ = S_+/I$).

Then the claim is that $f \in \mathfrak{N}(S_\bullet)$ iff $V(f) = \operatorname{Proj} S_\bullet$. If $f \in \mathfrak{N}(S_\bullet)$, then f is certainly contained in every prime ideal of $S_\bullet$, so $V(f) = \operatorname{Proj} S_\bullet$. Conversely, if $f \notin \mathfrak{N}(S_\bullet)$ then $((S_\bullet)_f)_0$ is a non-zero ring. Then it has a maximal ideal, which by 4.5.E corresponds to an element of $\operatorname{Proj} S_\bullet$ not containing f, so $V(f) \neq \operatorname{Proj} S_\bullet$.

(b) Let

$$I(Z) = S_+ \cap \bigcap_{[\mathfrak{p}] \in Z} \mathfrak{p},$$

which is a homogeneous ideal by 4.5.C. We have $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$ as a basic property of intersections of sets.

(c) Any closed set containing Z is of the form $V(J)$ for some homogeneous ideal $J \subseteq S_+$. Then since $V(J) \supseteq Z$, we have $I(Z) \supseteq S_+ \cap J = J$, so $V(I(Z)) \subseteq V(J)$. This shows that the closed set $V(I(Z))$ is the closure of Z.

4.5.I.

(a) $\iff$ (b): We have $\cup_i D(f_i) = \operatorname{Proj} S_\bullet \setminus \cap_i V(f_i) = \operatorname{Proj} S_\bullet \setminus V(I)$.

(a) $\implies$ (c): If $f \in S_+$ is homogeneous, then $f \in \sqrt{I}$ by 4.5.H.

(c) $\implies$ (a): Since $V(I) = V(\sqrt{I})$, we get $V(I) = \emptyset$ because no element of $\operatorname{Proj} S_\bullet$ contains $\sqrt{I} \supseteq S_+$.

4.5.J. The injection $\operatorname{Spec}((S_\bullet)_f)_0 \hookrightarrow \operatorname{Proj} S_\bullet$ preserves inclusions in both directions, and hence is a homeomorphism onto its image.

4.5.K. The element $g^{\deg f}/f^{\deg g}$ is invertible in $((S_\bullet)_{fg})_0$, and is a multiple of g, so $((S_\bullet)_{fg})_0$ satisfies the universal property of $(((S_\bullet)_f)_0)_{g^{\deg f}/f^{\deg g}}$ as an $((S_\bullet)_f)_0$-algebra. Since $D(g^{\deg f}/f^{\deg g})$ is isomorphic to the Spec of the latter, it is uniquely isomorphic to the Spec of the former. Note that when included into $\operatorname{Proj} S_\bullet$, this distinguished open set is the intersection $D(f) \cap D(g)$; a prime in $((S_\bullet)_f)_0$ contains $g^{\deg f}/f^{\deg g}$ iff its associated homogeneous prime in $S_\bullet$ contains g.

4.5.L. From 4.5.J we have a structure sheaf on the distinguished open set $D(f) \subseteq \operatorname{Proj} S_\bullet$ for each homogeneous f of positive degree, given by $\mathscr{F}_f = \pi_f^{-1} \operatorname{Spec}((S_\bullet)_f)_0$, where $\pi_f : D(f) \to \operatorname{Spec}((S_\bullet)_f)_0$ is a homeomorphism. From 4.5.K we have isomorphisms on the overlaps $\mathscr{F}_f|_{D(fg)} \cong \mathscr{F}_{fg}$, and for these to behave well on triple overlaps we need the composite

$$\mathscr{F}_f|_{D(fgh)} \cong \mathscr{F}_{fg}|_{D(fgh)} \cong \mathscr{F}_{fgh}$$

to equal the direct isomorphism

$$\mathscr{F}_f|_{D(fgh)} \cong \mathscr{F}_{fgh}.$$

But this holds, since the morphisms involved are all unique $((S_\bullet)_f)_0$-algebra morphisms (conflating the isomorphisms of affine schemes with their associated isomorphisms of rings). Hence we can glue these sheaves to get a structure sheaf on $\operatorname{Proj} S_\bullet$, which is a scheme since each $\operatorname{Spec}((S_\bullet)_f)_0$ was a scheme.

4.5.M. The stalks of $\operatorname{Proj} S_\bullet$ are given by localizations $(((S_\bullet)_f)_0)_\mathfrak{p}$ for non-zero homogeneous $f \in S_+$ and primes $\mathfrak{p}$ of $((S_\bullet)_f)_0$. Now if $\mathfrak{q}$ is the homogeneous prime of $(S_\bullet)_f$ associated to $\mathfrak{p}$ from 4.5.E, then by its construction we have $(((S_\bullet)_f)_0)_\mathfrak{p} \cong (((S_\bullet)_f)_\mathfrak{q})_0$. Since $f \notin \mathfrak{q}$, we can replace $\mathfrak{q}$ by its image in $\operatorname{Proj} S_\bullet$ and get $(((S_\bullet)_f)_0)_\mathfrak{p} \cong ((S_\bullet)_\mathfrak{q})_0$. This characterizes the stalks of $\operatorname{Proj} S_\bullet$ as the degree-zero parts of the localizations of $S_\bullet$ at homogeneous prime ideals not containing S_+.

Now if $s_\mathfrak{q} \in ((S_\bullet)_\mathfrak{q})_0$ is a collection of germs for each $[\mathfrak{q}] \in U \subseteq \operatorname{Proj} S_\bullet$ for some open set U, then they form a compatible collection of germs iff they are locally given by sections on some basic open sets $D(f)$. In other words, for every $[\mathfrak{q}] \in U$ we have some $D(f) \subseteq U$ containing $[\mathfrak{q}]$, and an element $s \in ((S_\bullet)_f)_0$ such that each $s_{\mathfrak{q}'}$ for $\mathfrak{q}' \in D(f)$ is the germ of s (i.e. the image of s under the map $((S_\bullet)_f)_0 \to ((S_\bullet)_{\mathfrak{q}'})_0$).

Hence we get an equivalent definition of the structure sheaf on $\operatorname{Proj} S_\bullet$, by saying that the stalk at $[\mathfrak{q}] \in \operatorname{Proj} S_\bullet$ is $((S_\bullet)_\mathfrak{q})_0$, and that a collection of germs $s_\mathfrak{q}$ is compatible on the open set $D(f)$ iff there is some $s \in ((S_\bullet)_f)_0$ such that $s_\mathfrak{q}$ is the image of s in $((S_\bullet)_\mathfrak{q})_0$ for each $[\mathfrak{q}] \in D(f)$, induced by the natural graded map $(S_\bullet)_f \to (S_\bullet)_\mathfrak{q}$.

4.5.N. The earlier construction is isomorphic to the gluing of the subschemes $D(x_i) \subseteq \operatorname{Proj} A[x_0, \ldots, x_n]$ as they are in $\operatorname{Proj} A[x_0, \ldots, x_n]$, given that

$$A[x_0, \ldots, x_n]/(x_i - 1) \cong A[x_0/x_i, \ldots, x_n/x_i] = A[x_0, \ldots, x_n, 1/x_i]_0.$$

Now the open sets $D(x_i)$ cover $\operatorname{Proj} A[x_0, \ldots, x_n]$ by 4.5.I, since the irrelevant ideal is $(x_1, \ldots, x_n)$. Hence the result of this gluing is simply isomorphic to $\operatorname{Proj} A[x_0, \ldots, x_n]$.

4.5.O. The point $[a_0, \ldots, a_n]$, when $a_i \neq 0$, corresponds to the maximal ideal $(x_{0/i} - a_{0/i}, \ldots, x_{n/i} - a_{n/i})$ of $k[x_0, \ldots, x_n, 1/x_i]_0 \cong k[x_{0/i}, \ldots, x_{n/i}]/(x_{i/i} - 1)$. Applying the construction of 4.5.E, this corresponds to the homogeneous prime ideal of $k[x_0, \ldots, x_n, 1/x_i]$ generated by the homogeneous elements $f(x_0, \ldots, x_n)/x_i^k$ such that

$$f(x_0, \ldots, x_n)/x_i^{\deg f} \in (x_{0/i} - a_{0/i}, \ldots, x_{n/i} - a_{n/i});$$

i.e. those that vanish at the point $[a_0, \ldots, a_n]$. This then pulls back to the homogeneous prime ideal of $k[x_0, \ldots, x_n]$ generated by the homogeneous polynomials vanishing at $[a_0, \ldots, a_n]$.

4.5.P. For a homogeneous ideal I, the quotient $S_\bullet/I$ is naturally graded, and the association between ideals of $S_\bullet/I$ and ideals of $S_\bullet$ containing I gives a topological embedding of $\operatorname{Proj} S_\bullet/I$ into $\operatorname{Proj} S_\bullet$. The image of this embedding is $V(I)$, so in this way we get a structure sheaf making $V(I)$ a scheme (strictly depending on I).

4.5.Q. Given a basis $f_0, \ldots, f_n$ for $V^\vee$, we have an isomorphism of graded k-algebras $\operatorname{Sym}^\bullet V^\vee \cong k[x_0, \ldots, x_n]$, and hence a bijection between the closed points of $\mathbb{P}V$ and the one-dimensional subspaces of k^n from 4.4.F. Now using the dual basis of $f_0, \ldots, f_n$ for V we have an isomorphism $k^n \cong V$, which gives a bijection between the one-dimensional subspaces of k^n and V.

Composing these, the bijection between closed points of $\mathbb{P}V$ and one-dimensional subspaces of V is given concretely as follows, using 4.5.O:

$$\begin{aligned}(v) \subseteq V &\leftrightarrow [f_0(v), \ldots, f_n(v)] \\ &\leftrightarrow \mathfrak{p} = (P \in k[x_0, \ldots, x_n] : P \text{ homogeneous, } P(f_0(v), \ldots, f_n(v)) = 0)) \\ &\leftrightarrow \mathfrak{p} = (f \in \operatorname{Sym}^\bullet V^\vee : f \text{ homogeneous, } f(v) = 0).\end{aligned}$$

Here elements of $\operatorname{Sym}^\bullet V^\vee$ naturally evaluate at vectors $v \in V$ by $(f \otimes g)(v) = f(v) \cdot g(v)$, and we see that the composite bijection is natural.

Chapter 5

Some properties of schemes

5.1.A. The closure of the point $[(0)] \in \operatorname{Proj} k[x_0, \ldots, x_n]$ is $V(0) = \mathbb{P}_k^n$. Since points are irreducible, and irreducibility transfers to the closure, $\mathbb{P}_k^n$ is irreducible.

5.1.B. Suppose $Z \subseteq X$ is an irreducible closed subset of a scheme X. For each open affine subscheme $U_i \subseteq X$ intersecting Z, we have a generic point z_i for the irreducible closed subset $Z \cap U_i$ of U_i by 3.7.E. Let Z_i be the closure of $\{z_i\}$ in X, so we have $Z \cap U_i \subseteq Z_i \subseteq Z$ and $Z = \cup_i Z_i$. By irreducibility of Z we get $Z = Z_i$ for some i, and hence Z is the closure of $\{z_i\}$. Any other point whose closure is Z lies in some U_j, and hence equals z_j by 3.7.E. Then the closed set $Z \setminus U_j$ cannot contain z_i, since it doesn't contain the closure Z of z_i, so we must have $z_i \in U_j$ and therefore $z_i = z_j$ by 3.7.E.

5.1.C. If X is a topological space, let $\sigma(X)$ denote the poset of closed sets in X. If $X = \cup_i U_i$, taking intersections gives a monotone map $\sigma(X) \to \sigma(U_i)$ for each i, so we get a monotone map $\psi : \sigma(X) \to \prod_i \sigma(U_i)$, which is injective since the U_i's cover X.

If each U_i is a Noetherian space, and the union is finite, then since the descending chain condition is preserved under finite products and pulls back along strictly monotone maps (see the solution to 3.6.V), we get that X is Noetherian.

5.1.D. If it is quasicompact, the cover by affine open subschemes that exists since it is a scheme has a finite subcover. Conversely, if it is a finite union of affine open subschemes then it is quasicompact, since affine schemes are quasicompact and a finite union of quasicompact spaces is quasicompact (a finite union of finite subcovers is a finite subcover).

5.1.E. If X is quasicompact, then any non-empty $S \subseteq X$ totally ordered by specialization of points has

$$\bigcap_{s \in S} \overline{\{s\}} \neq \emptyset,$$

otherwise the complements would be an open cover with no finite subcover. Now specialization of points is a pre-order on X, and if X is also a scheme then it is a partial order by 5.1.B. Hence by Zorn's lemma we have extremal specializations of any point in X, which are precisely the closed points of its closure. Then if $Z \subseteq X$ is closed and $z \in Z$, we get a closed point w in $\overline{\{z\}} \subseteq Z$.

5.1.F. This is necessary by 5.1.D, since affine schemes are quasicompact, and is sufficient since by 5.1.D intersections of quasicompact open subschemes reduce to intersections of affine open subschemes.

5.1.G. If $U, V \subseteq X$ are quasicompact open subschemes of an affine scheme X, then we have finite covers $U = \cup_{i=1}^n D(f_i)$, $V = \cup_{j=1}^m D(g_j)$ by distinguished open sets in X. Then $U \cap V = X \setminus V(I)$, where I is the finitely generated ideal

$$I = (f_1, \ldots, f_n) \cdot (g_1, \ldots, g_m) = (f_i g_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m),$$

and hence $U \cap V = \cup_{i,j} D(f_i g_j)$ is a finite union of distinguished open sets, which are quasicompact.

5.1.H. By 5.1.D and 5.1.F this is necessary, and sufficient for quasicompactness. Let $X = \cup_{i=1}^n A_i$ be the given cover, and suppose $U, V \subseteq X$ are affine open subschemes. Since U and V are (finite by quasicompactness) unions of affine open subschemes of the A_i's, we can assume $U \subseteq A_i$ and $V \subseteq A_j$ for some i and j. Let $A_i \cap A_j = \cup_{k=1}^m B_k$ be the given cover. By 5.1.G in A_i and A_j, the $U \cap B_k$'s and $V \cap B_k$'s are quasicompact, and then by 5.1.G in each B_k the $U \cap V \cap B_k$'s are quasicompact. Hence $U \cap V$ is quasicompact, so X is quasiseparated.

5.1.I. Suppose a $\mathbb{Z}_{\geq 0}$-graded algebra $S_\bullet$ is generated over $S_0 = A$ by $f_1, \ldots, f_n$. Then $S_+ = (f_1, \ldots, f_n)$ by 4.5.D, so $\operatorname{Proj} S_\bullet = D(f_1) \cup \cdots D(f_n)$ by 4.5.I. Moreover each intersection $D(f_i) \cap D(f_j) = D(f_i f_j)$ is affine, so by 5.1.H we get that $\operatorname{Proj} S_\bullet$ is quasicompact and quasiseparated.

5.1.J. The two open subsets isomorphic to X are quasicompact, but their intersection which is isomorphic to U is not by 3.6.G.

5.2.A. Existence of a non-zero nilpotent section implies existence of a non-zero nilpotent germ, since non-zero sections must have some non-zero germ. Conversely, taking a representative of any non-zero nilpotent germ gives a non-zero nilpotent section.

If a function f on a reduced scheme X vanishes at every point, then on affine neighbourhoods, and hence in every stalk, f is nilpotent. By reducedness we then get that f is zero on stalks, so $f = 0$.

5.2.B. If A is reduced, then $A_\mathfrak{p}$ is reduced for every prime $\mathfrak{p}$: if $(a/1)^n = 0/1$ in $A_\mathfrak{p}$ then $xa^n = 0$ for some $x \notin \mathfrak{p}$, so $(xa)^n = 0$ in A and hence $xa = 0$, so that $a/1 = 0/1$. Both $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are covered by spectra of polynomial rings over k, which are reduced, so $\mathbb{A}_k^n$ and $\mathbb{P}_k^n$ are reduced.

5.2.C. The natural map $k[x]_x \to \big(k[x,y]/(y^2, xy)\big)_x$ is surjective since $y/1 = xy/x = 0/1$ in the image. It is injective, since localization is flat and $k[x] \cap (y^2, xy) = 0$. Hence $\big(k[x,y]/(y^2, xy)\big)_x \cong k[x]$ is reduced.

As in 5.2.B, it follows that any localization of $\big(k[x,y]/(y^2, xy)\big)_x$ is reduced, so the only non-reduced stalks $\big(k[x,y]/(y^2, xy)\big)_\mathfrak{p}$ must have $x \in \mathfrak{p}$. The only such prime in $k[x,y]/(y^2, xy)$ is $\mathfrak{p} = (x, y)$, which does in fact have a non-reduced stalk since y is non-zero in $\big(k[x,y]/(y^2, xy)\big)_\mathfrak{p}$; multiplication by y in $k[x,y]/(y^2, xy)$ determines the constant term.

5.2.D. If a stalk $\mathscr{O}_{X,p}$ is non-reduced for some $p \in X$, then in fact $\mathscr{O}_{X,q}$ is non-reduced for every $q \in \overline{\{p\}}$, since we have a natural map $\mathscr{O}_{X,q} \to \mathscr{O}_{X,p}$ (every neighbourhood of q contains p), and . Hence X is non-reduced iff it is non-reduced at every closed point by 5.1.E.

5.2.E. The restriction of f to affine open subschemes is nilpotent, and since a finite number of affine open subschemes cover X it follows that f is nilpotent. In the non-quasicompact case, consider the scheme

$$X = \coprod_{n=1}^{\infty} \operatorname{Spec} k[\epsilon_n]/(\epsilon_n^n).$$

The coordinate ring is given by

$$\Gamma(X, \mathscr{O}_X) = \prod_{n=1}^{\infty} k[\epsilon_n]/(\epsilon_n^n),$$

which contains the non-nilpotent element $(\epsilon_1, \epsilon_2, \ldots)$. This function vanishes at every point of X, since each ϵ_n vanishes at the only point of $\operatorname{Spec} k[\epsilon_n]/(\epsilon_n^n)$.

5.2.F. Since integral domains are reduced, reducedness is necessary. Moreover irreducibility is necessary, since if open subsets U, V are disjoint then $\mathscr{O}_X(U \cup V) \cong \mathscr{O}_X(U) \times \mathscr{O}_X(V)$, so one of them must have zero sections, and hence is empty (schemes have non-zero stalks).

Conversely suppose X is irreducible and reduced, with sections $f, g \in \mathscr{O}_X(U)$ having $fg = 0$. Let $F, G \subseteq U$ be the (open by 4.3.G) sets on which f and g respectively don't vanish. Then by 4.3.G(b) the restrictions of g and f to F and G are zero, so F and G are disjoint, and by irreducibility one is empty. Then by reducedness either $f = 0$ or $g = 0$, since a section that vanishes everywhere is zero.

5.2.G. If A is an integral domain then $\operatorname{Spec} A$ is reduced, since each stalk is integral and certainly reduced. Moreover $\operatorname{Spec} A$ is irreducible by 3.6.C, so $\operatorname{Spec} A$ is integral. The converse is clear.

5.2.H. Note that A is an integral domain. Since η is dense in any open subset, we see that it must be given by the unique generic point $[(0)]$ of $\operatorname{Spec} A$. The stalk is hence the localization $A_{(0)} = K(A)$.

5.2.I. If a section is zero on some non-empty open set, then it is zero on any affine open subscheme intersecting that open set; the non-zero restriction maps on integral affine schemes are injective (being inclusions of subrings in the fraction field). Since an integral scheme is irreducible this is every non-empty affine open subscheme, and hence the section is zero. Then the map $\mathscr{O}_X(U) \to \mathscr{O}_{X,\eta}$ is an inclusion, since by definition a germ is zero iff it is given by the zero section of some non-empty open set.

5.3.A. By the affine communication lemma, any affine open subscheme of a locally Noetherian scheme is Noetherian. An intersection of affine open subschemes is then an open subscheme of a Noetherian scheme, and hence is quasicompact. This shows quasiseparatedness for locally Noetherian schemes.

5.3.B. By 3.6.15 and 5.1.C, a Noetherian scheme can be written as a finite union of closed irreducible subsets, none contained in any other. Any irreducible component must be contained in one by irreducibility, and must equal it by maximality, so there are finitely many irreducible components. Since connected components are unions of irreducible components (3.6.P), this shows that there can only be finitely many connected components, all given by finite unions of the finitely many irreducible components.

5.3.C. An integral scheme is certainly non-empty and connected (connectedness is required for irreducibility), and the stalks are integral domains as inherited from the rings of sections.

Conversely if every stalk of a non-empty and connected Noetherian scheme X is an integral domain, then X is reduced by 5.2.A. Moreover the irreducible components of X are disjoint: a point in an affine scheme whose stalk is an integral domain has a unique irreducible component through it, since an integral domain has a unique minimal prime, and hence points in X have unique irreducible components through them (the generic point of any such component lies in any affine neighbourhood of the point). By 5.1.B there are finitely many irreducible components of X, which must be the connected components since they are closed and disjoint, so X is irreducible because it is connected. Hence X is an integral scheme.

5.3.D. (a) In a projective A-scheme $\operatorname{Proj} S_\bullet$, the A-algebras $((S_\bullet)_f)_0$ are finitely generated: given a finite generating set of homogeneous elements $g_1, \ldots, g_n \in S_+$ for $S_\bullet$ over A, the A-module $((S_\bullet)_f)_0$ is generated by

$$\left\{ \left(\prod g_i^{e_i} \right) / f^k : \sum e_i \deg g_i = k \deg f \right\},$$

and to generate $((S_\bullet)_f)_0$ as an A-algebra we can factor out terms with $e_i \geq \deg f$, reducing to just the generators with $e_i < \deg f$ for each i (of which there are finitely many), by adjoining the finite set

$$\left\{ g_i^{\deg f} / f^{\deg g_i} : i = 1, \ldots, n \right\}.$$

Since the distinguished open sets $D(f)$ are a base for the topology on $\operatorname{Proj} S_\bullet$, this shows that any open subset of $\operatorname{Proj} S_\bullet$ is locally of finite type, so any quasiprojective A-scheme is of finite type. If A is Noetherian, then any A-scheme of finite type is Noetherian by the Hilbert Basis Theorem.

(b) If $U \subseteq X$ is an open subscheme of a projective A-scheme X, then U is locally of finite type as was shown in (a). If A is Noetherian, then X is a Noetherian scheme by (a), since X is trivially quasiprojective. By 3.6.T, we get that U is quasicompact. This may fail for non-Noetherian rings, since any affine scheme is projective (4.5.11), but open sets of an affine scheme may not be quasicompact (3.6.G).

5.3.E. (a) Since finite-generation and reducedness are affine-local, an affine k-scheme $\operatorname{Spec} A$ is an affine k-variety iff A is finitely-generated over k and reduced, i.e. isomorphic to some $k[x_1, \ldots, x_n]/I$ with I radical.

(b) Since I is radical, $k[x_0,\dots,x_n]/I$ is reduced. Hence each $\mathrm{Spec}((k[x_0,\dots,x_n]/I)_f)_0$ is reduced, since reducedness is affine-local and preserved in subrings. Therefore $\mathrm{Proj}\, k[x_0,\dots,x_n]/I$ is a projective k-variety.

5.3.F. Note that since we have a cover by affine open sets, a point is closed iff it is closed in every affine open neighbourhood. Now 3.6.J(a) gives

$$\begin{aligned} p \text{ closed in an affine open neighbourhood} &\iff \kappa(p) \text{ finite over } k \\ &\iff p \text{ closed in every affine open neighbourhood,} \end{aligned}$$

so the closed points are precisely those with residue field finite over k. Hence closedness is actually a local property in this scenario. From 3.6.J(a) we have that the closed points are dense in each affine open set, and it follows that they are dense globally.

5.3.G. $\star\star$ skipped.

5.3.H. We have a chain in the product $\prod A_{f_i}$ given by $\prod_{i=1}^n I_{i,1} \subseteq \prod_{i=1}^n I_{i,2} \subseteq \cdots$, which stabilizes since the projected chains $I_{i,1} \subseteq I_{i,2} \subseteq \cdots$ all do (and the product is finite). The preimage of this chain under the map $A \to \prod A_{f_i}$ is the original chain of I_i's, since that is the preimage of each projected chain, and since this map is injective the I_i's must stabilize.

5.3.I. Given $r \in A$, we have a polynomial in the $r_{ij}/f_i^{k_j}$ giving $r/1 \in A_{f_i}$, and hence $r/1 = p_i/f_i^N$ for some $p_i \in A[f_i, \{r_{ij}\}_j]$, where we can take N large enough so that $f_i^N r = p_i$, and large enough to work for each i. Raising $1 = \sum_i c_i f_i$ to the power of N, we get $1 = \sum_i q_i f_i^N$ with $q_i \in A[\{f_i\}_i, \{c_i\}_i]$, and can consider

$$b = \sum_i q_i p_i \in A[\{f_i\}_i, \{c_i\}_i, \{r_{ij}\}_{ij}].$$

Since

$$f_i^N p_j = f_i^N f_j^N r = f_j^N p_i,$$

we have

$$f_i^N b = \sum_j f_i^N q_j p_j = \sum_j f_j^N q_j p_i = p_i,$$

and hence $b/1 = p_i/f_i^N = r/1$ in A_{f_i}. Therefore $b = a$ in A, so $a \in A[\{f_i\}_i, \{c_i\}_i, \{r_{ij}\}_{ij}]$.

5.4.A. If $x \in K(A)$ satisfies

$$x^n + (a_1/s_1)x^{n-1} + \cdots + (a_n/s_n) = 0,$$

we may assume $s_1 = \cdots = s_n = s$ by multiplying each a_i by $\prod_{j \neq i} s_j$, and then

$$(sx)^n + a_1(sx)^{n-1} + sa_2(sx)^{n-2} + \cdots + s^{n-1}a_n = 0,$$

so $sx \in A$ since A is normal, and hence $x \in S^{-1}A$.

5.4.B. If a Noetherian scheme is normal, then all its stalks are integral domains, and by 5.3.C each of its finitely many connected components (which are clopen) are integral subschemes. Hence it is a finite disjoint union of integral Noetherian normal schemes, since normality and Noetherianness are affine-local. Any such disjoint union is normal, since normality is affine-local.

5.4.C. Suppose $x \in \cap A_{\mathfrak{m}}$. Then the ideal of denominators $I = \{r \in A : rx \in A\}$ is not contained in any maximal ideal, and hence must be the trivial ideal (1). Hence $x \in A$.

5.4.D. Note that $wz - xy$ is irreducible, since it is the homogenization of the irreducible polynomial $z - xy$, so A is indeed integral. Now clearly (y, z) is contained in the ideal I of denominators for $w/y = x/z$, and $I = (y, z)$ since x is not a zerodivisor in $A/(y, z) \cong k[w, x]$; if $a \in A$ is a denominator for x/z then ax is a multiple of z, and hence vanishes in $A/(y, z)$.

5.4.E. Suppose A is a UFD, and $S^{-1}A$ a non-zero localization. The ideals of $S^{-1}A$ are in an order-preserving bijection with a subset of the ideals of A, and a principal ideal (a/s) corresponds to the principal ideal (a), so $S^{-1}A$ inherits the ACCP from A. Hence it suffices to show that irreducibles in $S^{-1}A$ are prime.

Suppose $a \in A$ is irreducible in $S^{-1}A$ (the general irreducible element is a unit multiple of such). Write $a = uf_1 \cdots f_n$ with u a unit in $S^{-1}A$, and $f_1, \ldots, f_n$ non-units that are irreducible elements of A. Since a is irreducible in $S^{-1}A$ we have $n = 1$, so $(a) = (f_1)$. Since prime ideals in A correspond to prime ideals in $S^{-1}A$, and f_1 is prime in A, we get that a is prime in $S^{-1}A$.

5.4.F. Suppose $x/y \in K(A) \setminus A$ satisfies

$$(x/y)^n + a_1(x/y)^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n \in A$. Then multiplying out we get $x \mid y^n$ and $y \mid x^n$, so since y is not a unit in A (otherwise $x/y \in A$) there is an irreducible factor of y, which is also an irreducible factor of x. But there can only be finitely many such factors, and we can divide it out and start again, so reductio ad absurdum gives a contradiction.

5.4.G. If A is a UFD, then $A[x]$ is also a UFD:

- If $f \in A[x]$ is irreducible, then f is irreducible in $K(A)[x]$ by Gauss's lemma, and hence prime since $K(A)[x]$ is a PID. Then f is prime in $A[x]$, again by Gauss's lemma.
- If $(f_1) \subseteq (f_2) \subseteq \cdots$ is an ascending chain of principal ideals in $A[x]$, then eventually the degrees $\deg f_i$ stabilize (since they are decreasing), and since the leading coefficients have finitely many irreducible factors the chain itself also stabilizes.

Since $\mathbb{A}^n_A$ and $\mathbb{P}^n_A$ are covered by spectra of localizations of polynomial rings over A, we see that they are factorial whenever A is a UFD. Note that fields k are UFD's, and $\mathbb{Z}$ is a UFD, so in particular $\mathbb{A}^n_k$, $\mathbb{P}^n_k$, and $\operatorname{Spec}\mathbb{Z}$ are factorial.

Gauss's lemma: If A is a UFD, and $f \in K(A)[x]$ is given by P/a, where $P \in A[x]$ and $a \in A$, define the *content* $C(f)$ of f to be g_P/a, where $g_P \in A$ is the GCD in A of the coefficients of P.

- $C(f)$ is defined up to multiplication by a unit of A: If $Q/b = P/a$ then $aQ = bP$ and hence $ag_Q \sim bg_P$, where $\sim$ denotes equivalence up to units of A.
- $C(fg) \sim C(f)C(g)$: Say $P = \sum_{i=0}^n p_i x^i$ and $Q = \sum_{j=0}^m q_j x^j$ are in $A[x]$. Since $g_P g_Q$ divides each $p_i q_j$, we have $g_P g_Q \mid g_{PQ}$. Conversely, any irreducible common factor $r \in A$ of the coefficients of PQ is prime in A (since A is a UFD), and hence prime in $A[x]$ (constants divide term-by-term). Hence $r \mid P$ or $r \mid Q$, so by dividing out each such r we see that $g_{PQ} \mid g_P g_Q$.
- $C(f)$ is equivalent to an element of A iff $f \in A[x]$. If $C(f) \sim 1$ we say f is *primitive*. Any non-constant irreducible in $A[x]$ is primitive, since otherwise the content is a non-unit constant factor.
- If f is primitive, and $f = gh$ in $K(A)[x]$, then $f = (g/C(g))(h/C(h))$ is a factorization in $A[x]$. Hence if a primitive f is irreducible in $K(A)[x]$, it is irreducible in $A[x]$.
- If f is primitive, and $fg = h$ in $K(A)[x]$, then $C(g) \sim C(h)$, so $g \in A[x]$ iff $h \in A[x]$. Hence if a primitive f is prime in $K(A)[x]$, it is prime in $A[x]$.

5.4.H. (a) Since $z^2 - f$ is irreducible, and $A[z]$ is a UFD, we get that $B = A[z]/(z^2 - f)$ is an integral domain. Suppose we have a monic $F(x) \in B[x]$ with a root $\alpha \in K(B) \setminus B$. Replacing $F(x)$ by $\overline{F}(x)F(x)$ we can assume $F(x) \in A[x]$ (where $\overline{F}(x)$ has coefficients given by the endomorphism $z \mapsto -z$). Say $\alpha = g + hz$, where $g, h \in K(A)$ (anything in $K(B)$ is of this form since we can rationalize the denominator). Then $\alpha^2 - 2g\alpha + g^2 - fh^2 = 0$, so $Q(x) = x^2 - 2gx + g^2 - fh^2 \in K(A)[x]$ is the minimal polynomial of α over $K(A)$, as otherwise $\alpha \in K(A)$ and then $\alpha \in A \subseteq B$ since A is normal and $F(x) \in A[x]$. Hence $Q(x)$ divides $F(x)$ in $K(A)[x]$, and by Gauss's lemma $Q(x)$ is primitive (since everything is monic), so $2g \in A$ and $g^2 - fh^2 \in A$. Hence $g \in A$ and $fh^2 \in A$, so $h \in A$ since f has no repeated prime factors. Then $\alpha \in B$, which is a contradiction.

(b) Suppose $r \in A$ is a repeated prime factor of f. Then $(z/r)^2 = f/r^2 \in A$ so $z/r \in K(B)$ is integral over B, but $z/r \notin B$ since r is not a unit in B (otherwise quotienting out z we get that r is a unit in A).

5.4.I. (a) If $x \in \mathbb{Q}$ is integral over $\mathbb{Z}[\sqrt{n}]$, then by multiplying the monic polynomial with its conjugate we see that x is also integral over $\mathbb{Z}$, and hence $x \in \mathbb{Z}$.

Now suppose $(a+b\sqrt{n})/c \in \mathbb{Q}(\sqrt{n}) \setminus \mathbb{Q}$ is integral over $\mathbb{Z}[\sqrt{n}]$. By Gauss' lemma, its minimal polynomial over $\mathbb{Q}$ is primitive, and hence has integer coefficients. Since this polynomial is $x^2 - 2(a/c)x + (a^2 - nb^2)/c^2$, we get $c \mid 2a$ and $c^2 \mid a^2 - nb^2$. Then we may assume c is even, as otherwise $c \mid a$, so $c^2 \mid nb^2$ and $c \mid b$ since n is square-free. Then

$$0 \equiv c^2 \equiv a^2 - nb^2 \equiv a^2 + b^2 \mod 4,$$

so a and b are both even. Reducing to the case where there are no common factors, we get a contradiction.

(b) Note that in 5.4.H(a), $z^2 - f$ is automatically irreducible whenever f is a non-unit, since f then has no square root. Taking $A = k[x_2, \ldots, x_n]$ and $f = -(x_2^2 + \cdots + x_m^2)$, it then suffices to show that f has no repeated prime factors. Since the degree is 2, this is equivalent to f not being a square. But a potential linear square root has to be of the form $\pm\sqrt{-1}x_2 \pm \cdots \pm \sqrt{-1}x_m$, whose square is not f whatever signs are chosen. Hence $A[z]/(z^2 - f) \cong k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_m^2)$ is normal.

(c) We have

$$k[w,x,y,z]/(wz - xy) \cong k[x_1, x_2, x_3, x_4]/(x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

by

$$x_1 \mapsto (w+z)/2,\ x_2 \mapsto (x-y)/2,\ x_3 \mapsto (w-z)/2,\ x_4 \mapsto (x+y)/2,$$

and $k[x_1, x_2, x_3, x_4]/(x_1^2 + x_2^2 - x_3^2 - x_4^2)$ is normal by the same argument as in (b); $-x_2^2 + x_3^2 + x_4^2$ is not a square.

5.4.J. (a) Let $x_1, \ldots, x_n$ be the variables. If the form contains a term λx_1^2 with $\lambda \neq 0$, then we can complete the square:

$$\begin{aligned} &\lambda x_1^2 + x_1(a_2x_2 + \cdots + a_nx_n) + P(x_2, \ldots, x_n) \\ &= \left(\sqrt{\lambda}x_1 + \frac{a_2x_2 + \cdots + a_nx_n}{2\sqrt{\lambda}}\right)^2 + Q(x_2, \ldots, x_n), \end{aligned}$$

which gives an invertible linear change of coordinates

$$y = \sqrt{\lambda}x_1 + \frac{a_2x_2 + \cdots + a_nx_n}{2\sqrt{\lambda}}$$

after which the form is given by $y^2 + Q(x_2, \ldots, x_n)$. Otherwise if there are no square terms, but we have a non-zero product λx_1x_2, take the invertible linear change of coordinates $u = x_1 + x_2$, $v = x_1 - x_2$. We get

$$\lambda x_1x_2 + R(x_1, \ldots, x_n) = \frac{\lambda}{4}(u^2 - v^2) + S(u, v, x_3, \ldots, x_n),$$

where R has no x_1x_2 term so S has no u^2 term. Hence we can reduce to the first case.

By induction on the number of variables, we can repeat these steps until the remainder is zero, and as a composite we get an invertible linear change of coordinates after which the form is given by $x_1^2 + \cdots + x_m^2$ for some $m \leq n$.

(b) If the form is given by

$$\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

then for an invertible linear change of coordinates

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

the form is given by

$$\begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} P^T M P \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Now since P and P^T are invertible, $\operatorname{rank}(P^T M P) = \operatorname{rank}(M)$. Hence the rank of M is invariant, and for $x_1^2 + \cdots + x_m^2$ the rank is clearly m.

5.4.K. It is integrally closed by 5.4.I(a). Now $2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5})$, and the norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$ is multiplicative. Hence

- Each number $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is irreducible, since 2 and 3 have prime norms, and $1 \pm \sqrt{-5}$ with norm 6 could only have 2 or 3 as non-unit factors (and it doesn't).
- The irreducibles on the right cannot be unit multiples of the irreducibles on the left, since the norms are distinct.

From this we see that $\mathbb{Z}[\sqrt{-5}]$ cannot be a UFD.

5.4.L. (a) This was proved in 4.5.I(c).

(b) If A is a $\mathbb{Z}$-graded integral domain, and $f \in A$ is homogeneous and non-zero, suppose $f = gh$ with $g, h \in A$. Then g, h are homogeneous: if g_n, h_m are the least degree parts, then $f = g_n h_m$ and hence $(g - g_n)(h - h_m) = 0$. Then without loss of generality $g = g_n$, so $f = gh_m$, and hence $g(h - h_m) = 0$. Therefore $h = h_m$ so g and h are homogeneous.

Now $A = k[w, x, y, z]/(wz - xy)$ is a $\mathbb{Z}_{\geq 0}$-graded integral domain, and so we see that the homogeneous elements w, x, y, z can only have homogeneous factorizations. Since they have degree 1, they are irreducible. Moreover they are not unit multiples of each other, since units have degree 0 and for $\lambda \in k$ we have $w - \lambda x$, e.t.c. not in $(wz - xy)$. Hence A is not a UFD, because $wz = xy$ are distinct irreducible factorizations.

5.4.M. The basis gives an isomorphism $l \cong k^{\oplus d}$ of k-modules, and hence an isomorphism

$$A \otimes_k l \cong A \otimes_k k^{\oplus d} \cong (A \otimes_k k)^{\oplus d} \cong A^{\oplus d}$$

of A-modules, where the generators of $A^{\oplus d}$ are the images of $1 \otimes b_1, \ldots, 1 \otimes b_d$. Then $A \to A \otimes_k l$; $a \mapsto a \otimes 1$ is the injection $a \mapsto (a, 0, \ldots, 0)$ since $b_1 = 1$, so A is an integral domain and we have the canonical injection $A \to K(A)$. Similarly $K(A)$ injects into $K(A) \otimes_k l$, and tensoring $A \to K(A)$ with l gives a map $A \otimes_k l \to K(A) \otimes_k l$. This gives a commutative square since both composites are $a \mapsto a \otimes 1$, and the bottom map is injective since it is $(A \to K(A))^{\oplus d}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & K(A) \arrow[d] \\ A \otimes_k l \arrow[r] & K(A) \otimes_k l \end{tikzcd}$$

Now $(A \otimes_k l) \cap K(A) = A$, since if $a/b \in K(A)$ and $\sum a_i \otimes b_i \in A \otimes_k l$ have the same image

$$(a/b) \otimes 1 = \sum a_i \otimes b_i \in K(A) \otimes_k l,$$

then $a/b = a_1$ and $a_i = 0$ for $i > 1$ because $\{1 \otimes b_i\}$ is a $K(A)$-basis and $b_1 = 1$. In other words a/b and $\sum a_i \otimes b_i$ are the images of $a_1 \in A$.

Also the composite $A \to A \otimes_k l \to K(A \otimes_k l)$ factors through $K(A)$, since its target is a field, and $K(A) \to K(A \otimes_k l)$ factors through $K(A) \otimes_k l$, since its target is an l-algebra respecting l/k. Moreover the factor map $K(A) \otimes_k l \to K(A \otimes_k l)$ is an injection, since its kernel is prime and any element of $K(A) \otimes_k l$ is a product of something from $K(A)$ with something from $A \otimes_k l$ by taking a common denominator, both of which inject into $K(A \otimes_k l)$. Hence we have the following diagram of injections, and everything is an integral domain.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & K(A) \arrow[d] \arrow[ddl] \\
A \otimes_k l \arrow[r] \arrow[d] & K(A) \otimes_k l \arrow[dl] \\
K(A \otimes_k l) &
\end{tikzcd}$$

[In fact $K(A) \otimes_k l$ is a field, since it is a domain finite-dimensional over $K(A)$, so $K(A) \otimes_k l \to K(A \otimes_k l)$ is an isomorphism. This can fail in the infinite-dimensional case though, and isn't necessary for the argument.]

Finally, if an element of $K(A)$ is integral over A, then it is integral over $A \otimes_k l$ in $K(A) \otimes_k l$, and hence lies in $A \otimes_k l$ since $A \otimes_k l$ is normal. But then it lies in $(A \otimes_k l) \cap K(A) = A$, and we see that A is normal.

5.4.N. Since $\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 1$, they generate the unit ideal in A, and hence they give distinguished open sets covering $\operatorname{Spec} A$. Now $A = \mathbb{Q}[\frac{x^2}{x^2+y^2}, \frac{xy}{x^2+y^2}, \frac{y^2}{x^2+y^2}]$, so

$$\begin{aligned}
A_{\frac{x^2}{x^2+y^2}} &= \mathbb{Q}\left[\frac{x^2}{x^2+y^2}, \frac{xy}{x^2+y^2}, \frac{y^2}{x^2+y^2}, \frac{x^2+y^2}{x^2}\right] \\
&= \mathbb{Q}\left[\frac{1}{1+(y/x)^2}, \frac{(y/x)}{1+(y/x)^2}, \frac{(y/x)^2}{1+(y/x)^2}, 1+(y/x)^2\right] \\
&= \mathbb{Q}[y/x]_{1+(y/x)^2},
\end{aligned}$$

which is a UFD since $\mathbb{Q}[y/x] \cong \mathbb{Q}[t]$ is. By symmetry $A_{\frac{y^2}{x^2+y^2}}$ is also a UFD. However, A is not: if

$$\frac{P(x,y)}{x^2+y^2} = \frac{Q(x,y)}{(x^2+y^2)^n} \cdot \frac{R(x,y)}{(x^2+y^2)^m}$$

in A, then since x^2+y^2 is prime in $\mathbb{Q}[x,y]$ we can divide out and assume $n=0$ or $m=0$, so one of the factors is a unit. Hence $\frac{P(x,y)}{x^2+y^2}$ is irreducible (or a unit), so

$$\left(\frac{xy}{x^2+y^2}\right)\left(\frac{xy}{x^2+y^2}\right) = \left(\frac{x^2}{x^2+y^2}\right)\left(\frac{y^2}{x^2+y^2}\right)$$

are two irreducible factorizations. But the only units in A are constant (again since x^2+y^2 is prime), so these factorizations are not equivalent.

5.5.A. If $f = p(x) + q(x)y$ with $f(x) \neq 0$, then f is supported everywhere since the subring $k[x]$ is preserved in localizations. For a point $[(x-a, y)]$ with $a \in k^*$ the stalk is $k[x]_{(x-a)}$, so f is supported at $[(x-a,y)]$ iff $p(x) \neq 0$. Hence f is either supported everywhere, or supported only on a subset of $\{[(x,y)], [(y)]\}$. The only closed such subsets are the empty set and the origin, as required. Both are possible, since 0 has no support and y is supported at the origin.

5.5.B. A non-zero $f \in A$ is supported everywhere, since $A \to A_{\mathfrak{p}} \to K(A)$ is an injection. Hence the only associated point is the generic point.

5.5.C. Suppose $f \in A$ and let $U = \operatorname{Spec} A \setminus \overline{D(f)}$. Then f vanishes everywhere in U, so $f|_U$ is nilpotent, and $f|_U = 0$ since A is reduced. Hence $\operatorname{Supp} f \subseteq \overline{D(f)}$, and the reverse inclusion is clear so $\operatorname{Supp} f = \overline{D(f)}$. Now the irreducible components of $\overline{D(f)}$ give irreducible components of the open subset $D(f)$, whose generic points are minimal primes of A_f, i.e. the generic points of irreducible components from $\operatorname{Spec} A$ that meet $D(f)$. Hence $\operatorname{Supp} f = \overline{D(f)}$ is a union of irreducible components from $\operatorname{Spec} A$.

5.5.D. From (A) we have that $\operatorname{Supp}(m)$ is the closure of some collection of associated points of M, so it is certainly the closure of all such points that it contains.

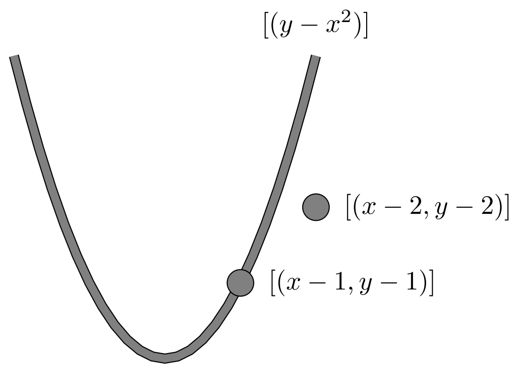
5.5.E. If $\mathfrak{q} \subseteq \mathfrak{p}$ and $A_\mathfrak{p}$ is reduced, then $A_\mathfrak{q} \cong (A_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ is reduced. Hence reducedness is preserved under generization, so non-reducedness is preserved under specialization. It remains to show that any non-reduced point is the specialization of a non-reduced associated point. But if $A_\mathfrak{p}$ is non-reduced, then there is some non-zero nilpotent $f/1 \in A_\mathfrak{p}$. Then $[\mathfrak{p}] \in \operatorname{Supp}(f)$ is a specialization of some associated point $[\mathfrak{q}] \in \operatorname{Supp}(f)$, and $f/1$ is nilpotent in $A_\mathfrak{q} \cong (A_\mathfrak{p})_{\mathfrak{q}_\mathfrak{p}}$ so $A_\mathfrak{q}$ is non-reduced.

5.5.F. Note that for $m/s \in S^{-1}M$, we have $\operatorname{Ann}_{S^{-1}A}(m/s) = \operatorname{Ann}_{S^{-1}A}(m/1)$ since $1/s$ is a unit, and $\operatorname{Ann}_{S^{-1}A}(m/1) = S^{-1}\operatorname{Ann}_A(m)$ since localization is left-exact.

Now $\operatorname{Supp} m = V(\operatorname{Ann} m)$, so the associated primes of M are those primes in A that are minimal over some $\operatorname{Ann} m$. Hence the associated primes of $S^{-1}M$ are those primes of $S^{-1}A$ that are minimal over some $S^{-1}\operatorname{Ann} m$. The corresponding primes in A are minimal over $\operatorname{Ann} m$, since smaller primes remain disjoint from S, and so the associated primes of $S^{-1}M$ correspond precisely to the associated primes of M not meeting S.

5.5.G. If $U = \cup U_i$ where each $U_i = \operatorname{Spec} A_i$ is an affine open subscheme, then $\mathscr{O}_X(U) \to \prod \mathscr{O}_X(U_i)$ is an injection, so it suffices to consider affine U. But the affine case was shown in 5.5.3, so we are done.

5.5.H. (a)



(b) The associated point $[(x-1, y-1)]$ is embedded in $[(y-x^2)]$, so it is non-reduced. The non-embedded points $[(y-x^2)]$ and $[(x-2, y-2)]$ may or may not be reduced.

(c) The zerodivisors are precisely those functions that vanish at some associated point, i.e. those that vanish at either $(1,1)$ or $(2,2)$. Hence $x+y-2$ and $y-x^2$ are zerodivisors, but $x+y-3$ is not.

5.5.I. If $g \in k[x_1, \ldots, x_n]$ gives a non-trivial zerodivisor in A, then we have $gh = rf$ for some $h, r \in k[x_1, \ldots, x_n]$, and $g, h \notin (f)$. Hence every prime factor of f divides either g or h, and not all of them divide h (counting with multiplicity) so one of them divides g. Say this prime factor is p, so we have $f, g \in (p)$ and (p) prime. Then (p) is a minimal prime over (f): if $\mathfrak{p} \subsetneq (p)$ then $\mathfrak{p} = (p) \cdot \mathfrak{p}$, so $\mathfrak{p} = 0$ since nothing else is infinitely divisible by p. Hence (p) is the generic point for an irreducible component of $\operatorname{Spec} A$, and since $g \in (p)$ we get that g vanishes at a non-embedded point.

Hence if $[\mathfrak{q}] \in \operatorname{Spec} A$ is an associated point, any function vanishing at $[\mathfrak{q}]$ also vanishes at some non-embedded associated point. In other words,

$$\mathfrak{q} \subseteq \bigcup_{\text{non-embedded } \mathfrak{p}} \mathfrak{p}.$$

This is a finite union because of (B), so by prime avoidance $\mathfrak{q} \subseteq \mathfrak{p}$ for some non-embedded $\mathfrak{p}$. But $\mathfrak{p}$ is minimal, so $\mathfrak{q} = \mathfrak{p}$ is non-embedded.

5.5.J. Suppose $I = \operatorname{Ann}(m)$ and $fg \in I$. Then $\operatorname{Ann}(gm) \supseteq I$, so either $gm = 0$ or $\operatorname{Ann}(gm) = I$. In the first case $g \in I$, and in the second case $f \in \operatorname{Ann}(gm) = I$.

5.5.K. If $m \neq 0$ then $\operatorname{Ann}(m)$ is proper, and there is a proper ideal containing $\operatorname{Ann}(m)$ maximal among those that are annhilators. By 5.5.J this is an associated prime $\mathfrak{p} \in \operatorname{Ass} M$, and $m_{\mathfrak{p}} \neq 0$ since $\mathfrak{p} \supseteq \operatorname{Ann}(m)$.

5.5.L. Since $M' \subseteq M$, we have $\operatorname{Ass} M' \subseteq \operatorname{Ass} M$. Now suppose $m \in M$ has $\operatorname{Ann}(m) = \mathfrak{p} \in \operatorname{Ass} M$, and let m'' be the image of m in M''. We have a short exact sequence

$$0 \to (A \cdot m) \cap M' \to A \cdot m \to A \cdot m'' \to 0,$$

so if $(A \cdot m) \cap M' = 0$ we get $\mathfrak{p} = \operatorname{Ann}(m'') \in \operatorname{Ass} M''$. Otherwise there is some non-zero $x \in (A \cdot m) \cap M'$, and $\operatorname{Ann}(x) = \mathfrak{p}$ since $A \cdot m \cong A/\mathfrak{p}$ is a domain, so $\mathfrak{p} \in \operatorname{Ass} M'$.

5.5.M. (a) If no such sequence exists, then $\operatorname{Ass} M$ is non-empty because $M \neq M_0$, so there is a suitable M_1, and $\operatorname{Ass} M/M_1$ is non-empty because $M \neq M_1$, so there is a suitable M_2, and by induction we get an ascending chain

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots,$$

contradicting the fact that M is Noetherian.

(b) By 5.5.L we have

$$\begin{aligned} \operatorname{Ass} M &\subseteq \operatorname{Ass} M_{n-1} \cup \operatorname{Ass} A/\mathfrak{p}_{n-1} \\ &\subseteq \operatorname{Ass} M_{n-2} \cup \operatorname{Ass} A/\mathfrak{p}_{n-2} \cup \operatorname{Ass} A/\mathfrak{p}_{n-1} \\ &\subseteq \cdots \\ &\subseteq \operatorname{Ass} A/\mathfrak{p}_0 \cup \cdots \cup \operatorname{Ass} A/\mathfrak{p}_{n-1} = \{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\}. \end{aligned}$$

(c) We have

$$(M_{i+1})_{\mathfrak{p}_i}/(M_i)_{\mathfrak{p}_i} = (M_{i+1}/M_i)_{\mathfrak{p}_i} = K(A/\mathfrak{p}_i) \neq 0,$$

so $(M_{i+1})_{\mathfrak{p}_i} \neq 0$ and hence $[\mathfrak{p}_i] \in \operatorname{Supp} M$. Now since M has a finite generating set $m_1, \ldots, m_n$, we get that $\operatorname{Supp} M = \cup \operatorname{Supp} m_i$ is closed. Therefore $\operatorname{Supp} A/\mathfrak{p}_i = \overline{\{[\mathfrak{p}_i]\}} \subseteq \operatorname{Supp} M$.

5.5.N. As in 5.5.F we have $\operatorname{Ann}_{S^{-1}A}(m/s) = S^{-1} \operatorname{Ann}_A(m)$. Hence

$$\begin{aligned} \operatorname{Ass}_{S^{-1}A} S^{-1}M &= \{\text{primes of the form } S^{-1} \operatorname{Ann}_A(m)\} \\ &\leftrightarrow \{\text{primes of the form } \operatorname{Ann}_A(m) \text{ not meeting } S\} \\ &= \operatorname{Ass}_A M \cap \operatorname{Spec} S^{-1}A. \end{aligned}$$

5.5.O. Given $[\mathfrak{p}] \in \operatorname{Supp} m = V(\operatorname{Ann} m)$, by 5.5.J there is an associated prime of $M_{\mathfrak{p}}$ containing $(\operatorname{Ann} m)_{\mathfrak{p}}$, and by 5.5.N this gives an associated prime $\mathfrak{q} \in \operatorname{Ass} M$ with $\operatorname{Ann} m \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. Hence $\operatorname{Supp} m$ is the closure of $\operatorname{Ass} M \cap \operatorname{Supp} m$.

Conversely, any closure of a subset of $\operatorname{Ass} M$ is a finite union $\cup_i V(\mathfrak{p}_i)$ of $\mathfrak{p}_i = \operatorname{Ann} m_i$, and we may assume $V(\mathfrak{p}_i) \not\subseteq V(\mathfrak{p}_j)$ for $i \neq j$. If $m = \sum_i m_i$, then $\operatorname{Supp} m \subseteq \cup_i \operatorname{Supp} m_i = \cup_i V(\mathfrak{p}_i)$, and since $[\mathfrak{p}_i] \notin V(\mathfrak{p}_j) = \operatorname{Supp} m_j$ we have $m_{\mathfrak{p}_j} = (m_j)_{\mathfrak{p}_j} \neq 0$ so $\mathfrak{p}_j \in \operatorname{Supp} m$. Hence $\operatorname{Supp} m = \cup_i V(\mathfrak{p}_i)$.

5.5.P. (a) Every associated point $\mathfrak{p} = \operatorname{Ann} m$ is the generic point of $\operatorname{Supp} m = V(\operatorname{Ann} m)$.

(b) This was the first paragraph in the solution to 5.5.O.

5.5.Q. If X is an integral scheme, and $\operatorname{Spec} A \subseteq X$ an affine open subscheme, then A is an integral domain. Hence any $f \in A$ has $\operatorname{Supp} f = V(\operatorname{Ann} f) = V(0)$, so the only associated point of A is the generic point $[(0)]$. Then the only associated point of X is the generic point of X, so (B) is immediate. Also (C) follows, since the only zerodivisor is zero, which is the only function vanishing at the generic point.

5.5.R. The irreducible components of X are $V(y-x^2)$ and $V(x-2, y-2)$, so these give the non-embedded associated points. Then the only possible embedded points are $[(x-a, y-a^2)]$ for $a \in \mathbb{C}$. Since $x-1$ is a zerodivisor, and $[(x-1, y-1)]$ is the only possible associated point at which it vanishes, we see that $[(x-1, y-1)]$ is an embedded associated point. Now for $a \notin \{1, 2\}$ suppose $(x-a)f(x, y) \in I$. Since $y - x^2 \nmid x - a$, we have $f \in (y-x^2)^3$, and since $(x-a) \notin (x-2, y-2)$, we have $f \in (x-2, y-2)$. Also since $(x-a) \notin (x-1, y-1)$, the vanishing order of f at $(1, 1)$ must be at least 15. Hence $f(x, y) \in I$, so $x-a$ is not a zerodivisor and $[(x-a, y-a^2)]$ is not an associated point. Finally, the same argument shows that $y-4$ is not a zerodivisor, so $[(x-1, y-1)]$ is the only embedded associated point.

Part III

Morphisms

Chapter 6

Morphisms of schemes

6.2.A. There is a unique morphism of topological spaces $\pi : X \to Y$ as described, by 2.2.F. Now $\pi_i^{-1}\mathscr{O}_Y = (\pi^{-1}\mathscr{O}_Y)|_{U_i}$, since if $V \subseteq U_i$ we have $\pi_i(V) = \pi(V)$ because $\pi_i = \pi|_{U_i}$. Hence the pullback map for π_i is equivalent to a morphism of sheaves $(\pi^{-1}\mathscr{O}_Y)|_{U_i} \to \mathscr{O}_X|_{U_i}$. These morphisms agree when restricted to $U_i \cap U_j$ by assumption, and so since $\mathscr{H}om(\pi^{-1}\mathscr{O}_Y, \mathscr{O}_X)$ is a sheaf there is a unique morphism of sheaves $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ that restricts to the pullback $\pi_i^{-1}\mathscr{O}_Y \to \mathscr{O}_X|_{U_i}$ on U_i.

6.2.B. The pullback map makes $\mathscr{O}_X(\pi^{-1}(U))$ an $\mathscr{O}_Y(U)$-algebra, and hence $\mathscr{O}_X(\pi^{-1}(U))$-modules become $\mathscr{O}_Y(U)$-modules in a way that preserves morphisms, and this commutes with restriction since the pullback map is a sheaf morphism. In this way an $\mathscr{O}_X$-module $\widetilde{M}$ gives an $\mathscr{O}_Y$-module $\pi_*\widetilde{M}$ by $(\pi_*\widetilde{M})(U) = \widetilde{M}(\pi^{-1}(U))$, and morphisms are preserved so this is a functor $\mathbf{Mod}_{\mathscr{O}_X} \to \mathbf{Mod}_{\mathscr{O}_Y}$.

6.2.C. From 2.2.I and 2.3.A we have maps $(\pi_*\mathscr{O}_X)_q \to (\mathscr{O}_X)_p$ and $(\mathscr{O}_Y)_q \to (\pi_*\mathscr{O}_X)_q$.

6.2.D. Firstly note that $\pi^{-1}D(g) = D(\pi^\sharp g)$ because $g \in (\pi^\sharp)^{-1}(\mathfrak{p})$ iff $\pi^\sharp g \in \mathfrak{p}$. Now $B_g \to A_{\pi^\sharp g}$ is the localization of $\pi^\sharp$ at g as a map of B-algebras, so $A_{\pi^\sharp g} \cong B_g \otimes_B A$. Hence if $D(g) = D(h)$, the unique B-algebra isomorphism $B_g \cong B_h$ gives an A-algebra isomorphism $A_{\pi^\sharp g} \cong A_{\pi^\sharp h}$ by tensoring with A.

Moreover this gives a morphism of sheaves on the base, since if $D(f) \subseteq D(g)$ the following square commutes; the vertical map on the right is unique among A-algebra morphisms and $(\cdot) \otimes_B A$ is functorial.

$$\begin{tikzcd} B_g \arrow[r] \arrow[d] & A_{\pi^\sharp g} \arrow[d] \\ B_f \arrow[r] & A_{\pi^\sharp f} \end{tikzcd}$$

6.2.E. As described, we can take the continuous map $\pi : \operatorname{Spec} k(x) \to \operatorname{Spec} k[y]_{(y)}$ with image $[(y)]$, and define a pullback $\mathscr{O}_{\operatorname{Spec} k[y]_{(y)}} \to \pi_*\mathscr{O}_{\operatorname{Spec} k(x)}$ by the obvious map $k[y]_{(y)} \to k(x)$ on global sections, which determines a sheaf morphism since $\pi_*\mathscr{O}_{\operatorname{Spec} k(x)}$ has zero non-global sections, making π a morphism of ringed spaces. If this were to come from a ring map $k[y]_{(y)} \to k(x)$ as in 6.2.D, then that ring map must be given by the map on global sections. But the obvious map $k[y]_{(y)} \to k(x)$ does not pull (0) back to (y), so π cannot arise in this way.

6.3.A. Using 6.2.A it suffices to show that a morphism of ringed spaces preserves vanishing iff its restrictions to an open cover all preserve vanishing. But this is clear, because restricting to an open neighbourhood doesn't affect the map on stalks.

6.3.B. (a) See 4.3.F.

(b) An element $f/s \in B_\mathfrak{q}$ vanishes at $[\mathfrak{q}]$ iff $f \in \mathfrak{q}$. If $[\mathfrak{q}] = \pi([\mathfrak{p}]) = [(\pi^\sharp)^{-1}(\mathfrak{p})]$, then $\pi^\sharp(f) \in \mathfrak{p}$, so the pullback $\pi^\sharp(f)/\pi^\sharp(s) \in A_\mathfrak{p}$ vanishes at $[\mathfrak{p}]$.

6.3.C. Suppose $\pi : X \to Y$ is a morphism of schemes, and we have affine open subschemes $\operatorname{Spec} B \subseteq Y$, $\operatorname{Spec} A \subseteq \pi^{-1}(\operatorname{Spec} B)$.

Note that if $f : Z \to W$ is a morphism of (locally) ringed spaces, and we have open sets $U \subseteq Z$, $f(Z) \subseteq V \subseteq W$, then

- There is an inclusion morphism $i : U \to Z$, with pullback given by the identity on $i^{-1}\mathscr{O}_Z = \mathscr{O}_Z|_U$.
- Since $f^{-1}\mathscr{O}_W = f^{-1}(\mathscr{O}_W|_V)$, f is also a morphism $Z \to V$, and hence factors through $V \to W$.

Applying this to our situation, we can compose π with the inclusion $\operatorname{Spec} A \to X$ to get $\pi|_{\operatorname{Spec} A} : \operatorname{Spec} A \to Y$, and this factors through $\operatorname{Spec} B \to Y$ so in fact we have a morphism $\pi|_{\operatorname{Spec} A} : \operatorname{Spec} A \to \operatorname{Spec} B$. From 6.3.2 this morphism of schemes must be induced by the map $B \to A$ of global sections, and hence π has the desired property.

Conversely, if π looks locally like morphisms of affine schemes, either on all affine open subschemes, or just on a fixed cover $X = \cup_i \operatorname{Spec} A_i$, $Y = \cup_i \operatorname{Spec} B_i$ with $\pi(\operatorname{Spec} A_i) \subseteq \operatorname{Spec} B_i$, then π will be a morphism of locally ringed spaces by 6.3.B(b) and 6.3.A.

6.3.D. Clearly Spec gives a functor from the category of rings to the opposite category of affine schemes. Conversely, let Γ denote the functor in the reverse direction given by taking global function rings, which sends a morphism $\pi : X \to Y$ to the pullback map $\pi^\sharp : \Gamma(Y) \to \Gamma(Y, \pi_*\mathscr{O}_X) = \Gamma(X)$. Clearly we have a canonical isomorphism $\Gamma \circ \operatorname{Spec} \cong \mathrm{id}$. Conversely, suppose $\phi_X : X \xrightarrow{\sim} \operatorname{Spec} A_X$ for each affine scheme X. Then $\phi_X^{-1} \circ \operatorname{Spec}(\phi_X^\sharp) : \operatorname{Spec}\Gamma(X) \xrightarrow{\sim} X$. Given a morphism $\pi : X \to Y$, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\pi"] & Y \\ \operatorname{Spec}(\Gamma(X)) \arrow[u, "\phi_X^{-1}\circ\operatorname{Spec}(\phi_X^\sharp)"] \arrow[r, "\operatorname{Spec}(\pi^\sharp)"] & \operatorname{Spec}(\Gamma(Y)) \arrow[u, "\phi_Y^{-1}\circ\operatorname{Spec}(\phi_Y^\sharp)"'] \end{tikzcd}$$

since

$$\operatorname{Spec}(\phi_Y^\sharp)^{-1} \circ \phi_Y \circ \pi \circ \phi_X^{-1} \circ \operatorname{Spec}(\phi_X^\sharp) = \operatorname{Spec}(\pi^\sharp)$$

by 6.3.2, because the pullback map for the LHS is

$$\phi_X^\sharp \circ (\phi_X^\sharp)^{-1} \circ \pi^\sharp \circ \phi_Y^\sharp \circ (\phi_Y^\sharp)^{-1} = \pi^\sharp.$$

Hence this gives a natural isomorphism $\operatorname{Spec} \circ \Gamma \cong \mathrm{id}$. See 6.3.4 for a canonical version.

6.3.E. We can replace k with a general ring B. As well as the cover

$$\mathbb{P}^n_B = D_{\mathbb{P}^n_B}(x_0) \cup \cdots \cup D_{\mathbb{P}^n_B}(x_n),$$

we also have

$$\mathbb{A}^{n+1}_B \setminus \{\vec{0}\} = \mathbb{A}^{n+1}_B \setminus V(x_0, \ldots, x_n) = D_{\mathbb{A}^{n+1}_B}(x_0) \cup \cdots \cup D_{\mathbb{A}^{n+1}_B}(x_n).$$

These affine open subschemes are given by

$$D_{\mathbb{A}^{n+1}_B}(x_i) = \operatorname{Spec} B[x_0, \ldots, x_n, 1/x_i]$$
$$D_{\mathbb{P}^n_B}(x_i) = \operatorname{Spec} B[x_0/x_i, \ldots, x_n/x_i],$$

so $B[x_0/x_i, \ldots, x_n/x_i] \subseteq B[x_0, \ldots, x_n, 1/x_i]$ gives a morphism of schemes $D_{\mathbb{A}^{n+1}_B}(x_i) \to D_{\mathbb{P}^n_B}(x_i)$.

The intersections are

$$D_{\mathbb{A}^{n+1}_B}(x_i) \cap D_{\mathbb{A}^{n+1}_B}(x_j) = D_{\mathbb{A}^{n+1}_B}(x_ix_j) = \operatorname{Spec} B[x_0, \ldots, x_n, 1/x_i, 1/x_j]$$
$$D_{\mathbb{P}^n_B}(x_i) \cap D_{\mathbb{P}^n_B}(x_j) = D_{\mathbb{P}^n_B}(x_ix_j) = \operatorname{Spec} B[x_0/x_i, \ldots, x_n/x_i, x_0/x_j, \ldots, x_n/x_j],$$

which include into $D_{\mathbb{A}_B^{n+1}}(x_i)$ / $D_{\mathbb{P}_B^n}(x_i)$ by

$$B[x_0, \dots, x_n, 1/x_i] \subseteq B[x_0, \dots, x_n, 1/x_i, 1/x_j]$$
$$B[x_0/x_i, \dots, x_n/x_i] \subseteq B[x_0/x_i, \dots, x_n/x_i, x_0/x_j, \dots, x_n/x_j].$$

Hence the restriction of $D_{\mathbb{A}_B^{n+1}}(x_i) \to D_{\mathbb{P}_B^n}(x_i)$ to $D_{\mathbb{A}_B^{n+1}}(x_i) \cap D_{\mathbb{A}_B^{n+1}}(x_j)$ is given by

$$B[x_0/x_i, \dots, x_n/x_i] \subseteq B[x_0, \dots, x_n, 1/x_i, 1/x_j],$$

which factors through the morphism $D_{\mathbb{A}_B^{n+1}}(x_i) \cap D_{\mathbb{A}_B^{n+1}}(x_j) \to D_{\mathbb{P}_B^n}(x_i) \cap D_{\mathbb{P}_B^n}(x_j)$ given by

$$B[x_0/x_i, \dots, x_n/x_i, x_0/x_j, \dots, x_n/x_j] \subseteq B[x_0, \dots, x_n, 1/x_i, 1/x_j].$$

Since this final morphism is symmetric with respect to i and j, we see that $D_{\mathbb{A}_B^{n+1}}(x_i) \to \mathbb{P}_B^n$ and $D_{\mathbb{A}_B^{n+1}}(x_j) \to \mathbb{P}_B^n$ restrict to the same morphism on $D_{\mathbb{A}_B^{n+1}}(x_i) \cap D_{\mathbb{A}_B^{n+1}}(x_j)$. By gluing, we then get a morphism $\mathbb{A}_B^{n+1} \setminus \{\vec{0}\} \to \mathbb{P}_B^n$. All maps considered were inclusions of B-algebras, so this is a morphism of B-schemes.

6.3.F. Let $\phi_i : \operatorname{Spec} B_i \to U_i \subseteq X$ be open immersions covering X. We may assume the cover is closed under intersections, by using distinguished open sets.

Given $f : A \to \Gamma(X)$, we have morphisms $\operatorname{Spec}(\phi_i^\sharp \circ \operatorname{res}_{X,U_i} \circ f) \circ \phi_i^{-1} : U_i \to \operatorname{Spec} A$. If $U_j \subseteq U_i$, then

$$\big(\operatorname{Spec}(\phi_i^\sharp \circ \operatorname{res}_{X,U_i} \circ f) \circ \phi_i^{-1}\big)|_{U_j} = \operatorname{Spec}(\phi_j^\sharp \circ \operatorname{res}_{X,U_j} \circ f) \circ \phi_j^{-1},$$

because

$$\big(\operatorname{Spec}(\phi_i^\sharp \circ \operatorname{res}_{X,U_i} \circ f) \circ \phi_i^{-1}\big)|_{U_j} \circ \phi_j = \operatorname{Spec}(\phi_j^\sharp \circ \operatorname{res}_{X,U_j} \circ f)$$

by 6.3.2, since the pullback on global sections for the LHS is

$$\phi_j^\sharp \circ \operatorname{res}_{U_i,U_j} \circ (\phi_i^\sharp)^{-1} \circ (\phi_i^\sharp \circ \operatorname{res}_{X,U_i} \circ f) = \phi_j^\sharp \circ \operatorname{res}_{X,U_j} \circ f.$$

Hence these morphisms agree on overlaps, and by gluing we get a morphism $\pi : X \to \operatorname{Spec} A$. By construction we have $\operatorname{res}_{X,U_i} \circ \pi^\sharp = (\pi|_{U_i})^\sharp = \operatorname{res}_{X,U_i} \circ f$, so $\pi^\sharp = f$. By 6.3.2, the restrictions $\pi|_{U_i}$ are uniquely determined by this property, and it follows that π is unique.

This shows that the functor Γ composed with the obvious natural isomorphism $\Gamma \circ \operatorname{Spec} \cong \operatorname{id}$ gives a bijection $\operatorname{Mor}(X, \operatorname{Spec} A) \to \operatorname{Mor}(A, \Gamma(X))$, and this is natural in both arguments since Γ is a functor and $\Gamma \circ \operatorname{Spec} \cong \operatorname{id}$ is natural. Hence Γ and Spec, which were an equivalence of categories when restricted to affine schemes, are an adjoint pair on schemes in general.

Suppose X is a locally ringed spaced, and we have a map $A \to \Gamma(X)$. For a point $p \in X$, the maximal ideal $\mathfrak{m}_p \subseteq \mathscr{O}_{X,p}$ has a prime preimage $\mathfrak{q}_p \subseteq \Gamma(X)$, which has a prime preimage $\mathfrak{p}_p \subseteq A$. Hence we have a map of sets $X \to \operatorname{Spec} A$ by $p \mapsto [\mathfrak{p}_p]$. The closed set $V(I)$ pulls back to the vanishing set of an ideal in $\Gamma(X)$, which is closed by 4.3.G. Hence this is a continuous map.

Now for $p \in X$, we have the composite $A \to \Gamma(X) \to \mathscr{O}_{X,p}$, and since $\mathfrak{p}_p$ is the preimage of $\mathfrak{m}_p$, the multiplicative set $A \setminus \mathfrak{p}_p$ maps into $\mathscr{O}_{X,p} \setminus \mathfrak{m}_p$. Hence we get a map of stalks $A_{\mathfrak{p}_p} \to \mathscr{O}_{X,p}$. Suppose we have a section $a/f^n \in A_f$, where $[\mathfrak{p}_p] \in D(f)$. The germs of this section have images $b_q/g_q^n \in \mathscr{O}_{X,q}$ where b_q and g_q are the images of a and f, and g_q is invertible for q in some neighbourhood U of p by 4.3.G. But these are just the germs of the section $b/g^n \in \mathscr{O}_X(U)$, where b and g are the images of a and f, and $g \in \mathscr{O}_X(U)$ is invertible since all its germs are (using 2.4.E). Therefore these maps of stalks preserve compatability of germs, and so give a sheaf morphism from the pushforward of $\mathscr{O}_{\operatorname{Spec} A}$ to $\mathscr{O}_X$.

By construction the maximal ideal $\mathfrak{m}_p$ pulls back to the maximal ideal in $A_\mathfrak{p}$, and hence we have a morphism of locally ringed spaces $X \to \operatorname{Spec} A$. The pullback on global sections is the original map $A \to \Gamma(X)$, by applying the definition to $D(1)$. Moreover the morphism $X \to \operatorname{Spec} A$ is unique with this property, since all choices of images in the previous paragraphs were determined by the map $A \to \Gamma(X)$ and the properties of morphisms (note that the action on points is forced, since a morphism of locally ringed spaces must preserve the ideal vanishing at a point).

This shows that the functor Γ composed with the obvious natural isomorphism $\Gamma \circ \operatorname{Spec} \cong \mathrm{id}$ gives a bijection $\operatorname{Mor}(X, \operatorname{Spec} A) \to \operatorname{Mor}(A, \Gamma(X))$, since we have constructed an inverse for it. Since F is a functor and $\Gamma \circ \operatorname{Spec} \cong \mathrm{id}$ is natural, this is a natural isomorphism. Hence Γ and Spec, which were an equivalence of categories when restricted to affine schemes, are an adjoint pair on locally ringed spaces in general.

6.3.G. Note that having an A-algebra structure on $\Gamma(U, \mathscr{O}_X)$ for each U in a way that commutes with restriction is equivalent to having an A-algebra structure on $\Gamma(X, \mathscr{O}_X)$, since $\mathrm{res}_{X,U}$ makes $\Gamma(U, \mathscr{O}_X)$ a $\Gamma(X, \mathscr{O}_X)$-algebra. Hence A-schemes in the old sense are equivalent to A-schemes in the new sense (as objects) by 6.3.F.

Now a morphism $X \to Y$ respects the A-algebra structure on all section rings iff it does on the global section rings, since both the morphism and the A-algebra structure commute with restriction. Hence $X \to Y$ is a morphism of A-schemes in the old sense iff $\Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is a map of A-algebras, i.e. it factors $A \to \Gamma(X, \mathscr{O}_X)$ through $A \to \Gamma(X, \mathscr{O}_X)$. By 6.3.F, this is iff $X \to Y$ factors $X \to \operatorname{Spec} A$ through $Y \to \operatorname{Spec} A$, which is the definition of a morphism of A-schemes in the new sense.

6.3.H. The maps $A \to ((S_\bullet)_f)_0$ for homogeneous $f \in S_+$ agree on intersections $((S_\bullet)_{fg})_0$, as they are just given by composing $A \to S_\bullet$ with the localization maps. Hence we can glue them to get a canonical morphism $\operatorname{Proj} S_\bullet \to \operatorname{Spec} A$.

6.3.I. Clearly $\operatorname{Spec} \mathbb{Z}$ is final in the category of affine schemes, by equivalence with rings. Hence there is a unique morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ for any affine open subscheme $\operatorname{Spec} A \subseteq X$. By uniqueness these agree on overlaps, so by gluing we get a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$.

In the category of A-schemes $\operatorname{Spec} A$ is final essentially by definition. This applies when $A = k$.

6.3.J. (a) For an affine scheme X, we can compose the natural map $\operatorname{Spec} \mathscr{O}_{X,p} \to \operatorname{Spec} \Gamma(X)$ with the canonical isomorphism $X \cong \operatorname{Spec} \Gamma(X)$ to get $\operatorname{Spec} \mathscr{O}_{X,p} \to X$. This gives a natural transformation from Spec of the stalk functor to the identity in the category of affine schemes with basepoint.

From this naturality, affine open neighbourhoods $W \subseteq U \cap V$ of a point p in a general scheme X make the following diagram commute.

$$\begin{tikzcd} & U \\ \operatorname{Spec} \mathscr{O}_{X,p} \arrow[ur] \arrow[r] \arrow[dr] & W \arrow[u, hook] \arrow[d, hook] \\ & V \end{tikzcd}$$

Hence the map $\operatorname{Spec} \mathscr{O}_{X,p} \to X$ is independent of choice of neighbourhood.

(b) We can compose the map from (a) with Spec of the quotient map $\mathscr{O}_{X,p} \to \mathscr{O}_{X,p}/\mathfrak{m}_p = \kappa(p)$.

6.3.K. An open set containing p pulls back to an open set containing $[\mathfrak{m}]$, and every point in $\operatorname{Spec} A$ is a generization of $[\mathfrak{m}]$ so the only such open set is the whole of $\operatorname{Spec} A$.

Hence the morphisms $\operatorname{Spec} A \to X$ sending $[\mathfrak{m}]$ to p are in bijection with the morphisms $\operatorname{Spec} A \to \operatorname{Spec} B$ sending $[\mathfrak{m}]$ to $p = [\mathfrak{p}]$ for an affine open neighbourhood $\operatorname{Spec} B$ of p, i.e. homomorphisms $\phi : B \to A$ with $\phi^{-1}(\mathfrak{m}) = \mathfrak{p}$. Since A is local, we get $\phi(B \setminus \mathfrak{p}) \subseteq A \setminus \mathfrak{m} \subseteq U(A)$, so these correspond to local homomorphisms $B_\mathfrak{p} \to A$, and of course $B_\mathfrak{p} \cong \mathscr{O}_{X,p}$.

In particular, that $B \to A$ factors through $B \to B_\mathfrak{p}$ means every morphism $\operatorname{Spec} A \to X$ sending $[\mathfrak{m}]$ to p factors through the morphism $\operatorname{Spec} \mathscr{O}_{X,p} \to X$ from 6.3.J.

6.3.L. (a) Simply form the composite $Z \to X \to Y$.

(b) There are multiple morphisms $\operatorname{Spec} k \to \operatorname{Spec} k$ when $\operatorname{Aut} k$ is non-trivial (e.g. complex conjugation on $\mathbb{C}$), but the topological space underlying $\operatorname{Spec} k$ has only one point.

Now there is a canonical X-valued point in $X(X)$ given by the identity, and its image in $Y(X)$ is just the original morphism $X \to Y$, so the map $X(X) \to Y(X)$ determines the morphism $X \to Y$.

6.3.M. (a) We have a morphism of B-schemes $X \to \mathbb{A}^{n+1}_B$ given by the B-algebra map $B[x_0, \ldots, x_n] \to \Gamma(X)$; $x_i \mapsto f_i$. The preimage of $\vec{0} = V(x_0, \ldots, x_n)$ is $V(f_0, \ldots, f_n)$, which is empty by assumption. Hence this gives a morphism of B-schemes $X \to \mathbb{A}^{n+1}_B \setminus \{\vec{0}\}$, and by 6.3.E we get a morphism of B-schemes $X \to \mathbb{P}^n_B$.

(b) The restricted morphism $D(f_i) \to D_{\mathbb{A}^{n+1}_B}(x_i) \to D_{\mathbb{P}^n_B}(x_i)$ is given by

$$B[x_0/x_i, \ldots, x_n/x_i] \subseteq B[x_0, \ldots, x_n, 1/x_i] \to B[f_0, \ldots, f_n, 1/f_i] \subseteq \Gamma(D(f_i), X),$$

and the restricted morphism $D(gf_i) \to D_{\mathbb{A}^{n+1}_B}(x_i) \to D_{\mathbb{P}^n_B}(x_i)$ is given by

$$B[x_0/x_i, \ldots, x_n/x_i] \subseteq B[x_0, \ldots, x_n, 1/x_i] \to B[gf_0, \ldots, gf_n, 1/(gf_i)] \subseteq \Gamma(D(gf_i), X).$$

Both are the same composite map, since $(gf_j)/(gf_i) = f_j/f_i$.

6.3.N. $\star\star$ skipped.

6.4.A. For $f \in S_+$ homogeneous of positive degree we have a morphism $D(\phi(f)) \to D(f)$, since ϕ localizes to a graded map $(S_\bullet)_f \to (R_\bullet)_{\phi(f)}$. The restriction to $D(\phi(fg))$ is just the further localization $(S_\bullet)_{fg} \to (R_\bullet)_{\phi(fg)}$, so we get agreement on overlaps. By gluing we then get a morphism $\operatorname{Proj} R_\bullet \setminus V(\phi(S_+)) \to \operatorname{Proj} S_\bullet$, since $V(\phi(S_+)) = \cap V(\phi(f))$.

6.4.B. This is immediate from 4.5.I.

6.4.C. For any $\mathbb{Z}_{\geq 0}$-graded ring $S_\bullet$ and any homogeneous ideal $I \subseteq S_\bullet$, if $f \in S_+ \setminus I$ is homogeneous of positive degree we get

$$(S_\bullet/(S_+ \cdot I))_f \cong (S_\bullet/I)_f,$$

since S_+ generates the unit ideal in $(S_\bullet)_f$. Hence the obvious quotient map gives an isomorphism $\operatorname{Proj} S_\bullet/(S_+ \cdot I) \cong \operatorname{Proj} S_\bullet/I$.

For $R_\bullet = k[x, y, z]/(xz, yz, z^2)$ we get $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} k[x, y, z]/(z) \cong \mathbb{P}^1_k$, and since $(x, y, z) \mapsto (x, y, 0)$ on $R_\bullet$ induces the identity on the quotient $k[x, y, z]/(z)$, it induces the identity on $\operatorname{Proj} R_\bullet$.

6.4.D. If $f \in S_+$ is homogeneous of positive degree divisible by n, then $((S_\bullet)_f)_0 = ((S_{n\bullet})_f)_0$ since the numerator of an element of the former must have degree divisible by n. Now $D(g) = D(g^n)$ for any homogeneous $g \in S_+$, so the open sets $D(f)$ as above cover $\operatorname{Proj} S_\bullet$, and hence $\operatorname{Proj} S_\bullet = \operatorname{Proj} S_{n\bullet}$.

6.4.E. Any $f \in S_{nm}$ is given by a polynomial in $A[S_1]$, and we can assume all the monomials have degree nm so f is an A-linear combination from $\{\prod_{i=1}^{nm} g_i : g_i \in S_1\}$. Now $\prod_{i=1}^{nm} g_i$ is a product of m elements from S_n, so we see that $f \in A[S_n]$. Hence $S_{n\bullet}$ is generated in degree 1. Note that if we had finitely many generators in S_1 then this would give a finite generating set for $S_{n\bullet}$ in degree 1 (their n-fold products).

6.4.F. We can interpret the condition as saying we have a graded map $R_\bullet \to S_\bullet$ which restricts to an isomorphism of abelian groups on all but finitely many R_n's. Then for a large value of N, it induces an isomorphism $R_{N\bullet} \to S_{N\bullet}$. Hence $\operatorname{Proj} R_\bullet \cong \operatorname{Proj} S_\bullet$ by 6.4.D.

6.4.G. We can assume $f_1, \ldots, f_n$ are homogeneous of positive degree. Let $d = \prod d_i$ be the product of their degrees $d_i = \deg f_i$. Suppose a product $f_1^{a_1} \cdots f_n^{a_n}$ has degree $a_1 d_1 + \cdots + a_n d_n \geq nd$. Then we must have $a_i \geq d/d_i$ for some i, and hence can pull out the factor f_i^{d/d_i} of degree d. By induction, $f_1^{a_1} \cdots f_n^{a_n}$ is a monomial in the f_i^{d/d_i}'s multiplied by a homogeneous element of degree less than nd. If $a_1 d_1 + \cdots + a_n d_n$ is a multiple of nd, then this latter element has degree divisible by nd, and hence lies in S_0. Moreover the number of factors of the form f_i^{d/d_i} must be divisible by n, so it follows that $S_{nd\bullet}$ is generated over S_0 by the finite set of n-fold products from $\{f_i^{d/d_i}\}$.

6.4.H. By 6.4.G, we have $S_{N\bullet}$ finitely generated in degree 1 for some N. Then $S_{nN\bullet}$ is also finitely generated in degree 1 by 6.4.E. Let $f_1, \ldots, f_r \in S_+$ be homogeneous generators with $d_i = \deg f_i$. For $0 \leq j < N$, any element of $S_{nN\bullet+nj}$ is an S_0-linear combination of monomials $\prod_i f_i^{a_i}$ with $\sum_i a_i d_i = nj$ modulo nN. We can assume each d_i divides nN, so by the same argument as in 6.4.G we can pull out factors f_i^{nN/d_i} until $\sum_i a_i d_i = nj$. Hence $S_{nN\bullet+nj}$ is finitely-generated as an $S_{nN\bullet}$-module by $\{\prod_i f_i^{a_i} : \sum_i a_i d_i = nj\}$. Then $S_{n\bullet} = \oplus_j S_{nN\bullet+nj}$ is finite over $S_{nN\bullet}$, and hence finitely-generated over S_0.

6.5.A. If π is defined on a dense open set U, then the generic point p of X lies in U, and $\pi(U) \subseteq \overline{\{\pi(p)\}}$ since $U \subseteq \overline{\{p\}}$. Hence $\pi(U)$ is dense if $\pi(p)$ is the generic point of Y, and the converse is clear since $\pi(p) \in \pi(U)$.

6.5.B. If $p \in X$, $q \in Y$ are the generic points, then from 6.5.A we have a map of stalks $K(Y) = \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p} = K(X)$.

6.5.C. If $a_1, \ldots, a_n \in K$ are generators over k, then consider the map $k[x_1, \ldots, x_n] \to K$ given by $x_i \mapsto a_i$. This gives an injection $k[x_1, \ldots, x_n]/\mathfrak{p} \to K$ (note that K is an integral domain so the kernel must have been prime), and the image generates K as a field so K is the fraction field of $k[x_1, \ldots, x_n]/\mathfrak{p}$. Hence $\operatorname{Spec} k[x_1, \ldots, x_n]/\mathfrak{p}$ gives an integral affine k-variety with function field K.

6.5.D. By 6.5.B, taking function fields gives a functor K from (a) to (c). Now if we choose a generating set for everything in (c), then by 6.5.C and 6.5.7 we get a functor F from (c) to the full subcategory (b) of (a). Clearly $K \circ F \cong \mathrm{id}$, and applying 6.5.7 to the identity gives $F \circ K \cong \mathrm{id}$, since a morphism inducing the identity on $K(X)$ must be id_X by considering an affine cover. This shows that both (a) and (b) are equivalent to (c).

6.5.E. Apart from p, every rational point q on C is given by $x = (m^2-1)/(m^2+1)$, $y = -2m/(m^2+1)$ for some $m \in \mathbb{Q}$. Multiplying out denominators this is given by $x = (a^2-b^2)/(a^2+b^2)$, $y = -2ab/(a^2+b^2)$ for $a, b \in \mathbb{Z}$ not both zero, where the case $b = 0$ has added in p. Hence every pythagorean triple is of the form $(a^2-b^2, -2ab, a^2+b^2)$ for $a, b \in \mathbb{Z}$, where the case $a = b = 0$ has added in the one triple $(0,0,0)$ which does not project onto the circle.

6.5.F. In this and later exercises we are using 6.4.A and 6.3.M to define morphisms. The equivalent fact to 6.3.M(b) for morphisms defined as in 6.4.A can be checked. Recall that $V(I) = \emptyset$ if I contains a power of the irrelevant ideal.

Write $C = \operatorname{Proj} k[x,y,z]/(x^2+y^2-z^2)$. We have a map $\mathbb{P}^1_k \to C$ given by $[x,y] \mapsto [x^2-y^2, -2xy, x^2+y^2]$, since $(x^2-y^2, -2xy, x^2+y^2) = (x,y)^2$. The preimage of $D(x-z)$ is $D(-2y^2) = D(y)$, and the preimage of $D(x+z)$ is $D(2x^2) = D(x)$. Moreover, we have inverses $D(x-z) \to D(y)$ and $D(x+z) \to D(x)$ given by $[x,y,z] \mapsto [y, x-z]$ and $[x,y,z] \mapsto [x+z, -y]$; the composites are

$$[x,y] \mapsto [x^2-y^2, -2xy, x^2+y^2] \mapsto [-2xy, -2y^2] = [x,y]$$
$$[x,y,z] \mapsto [y, x-z] \mapsto [2(z-x)x, 2(z-x)y, 2(z-x)z] = [x,y,z]$$

and

$$[x,y] \mapsto [x^2-y^2, -2xy, x^2+y^2] \mapsto [2x^2, 2xy] = [x,y]$$
$$[x,y,z] \mapsto [x+z, -y] \mapsto [2(x+z)x, 2(x+z)y, 2(x+z)z] = [x,y,z].$$

Hence this gives an isomorphism $C \cong \mathbb{P}^1_k$.

6.5.G. If $C = \operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x^3 - x^2)$, then $(x,y) \mapsto y/x$ gives a rational map $C \dashrightarrow \mathbb{A}^1_{\mathbb{Q}}$ defined on $D(x) = C \setminus \{(0,0)\}$. Since the line with slope m through $(0,0)$ meets C where $m^2x^2 = x^3 + x^2$, the only other intersection is (m^2-1, m^3-m). Hence the morphism $\mathbb{A}^1_{\mathbb{Q}} \to C$ given by $m \mapsto (m^2-1, m^3-m)$ is an inverse to the rational map $C \dashrightarrow \mathbb{A}^1_{\mathbb{Q}}$. In fact the two extraneous points ± 1 map to $(0,0)$, so every rational point on C is given by (m^2-1, m^3-m) for some $m \in \mathbb{Q}$.

6.5.H. We have a map $f : \mathbb{P}^2_{\mathbb{Q}} \dashrightarrow Q$ given by intersecting the line through $[1,0,0,1]$ and $[x,y,z,0]$:

$$[x,y,z] \mapsto [y^2 - z^2 - x^2, -2yx, -2zx, y^2 - z^2 + x^2],$$

defined on $D(x) \cup D(y^2 - z^2)$. The inverse is defined on $Q \setminus \{[1,0,0,1]\}$ by $[x,y,z,w] \mapsto [x-w, y, z]$, so the map is birational. Now the preimage of $V(x, y^2 - z^2)$ in $Q \setminus \{[1,0,0,1]\}$ is given by the two lines $V(x-w, y-z) \cup V(x-w, y+z)$, so Q is covered by the images of three maps:

$$\begin{aligned} \mathbb{P}^2_{\mathbb{Q}} \supseteq D(x) \cup D(y^2-z^2) \to Q; &\qquad [x,y,z] \mapsto [y^2 - z^2 - x^2, -2yx, -2zx, y^2 - z^2 + x^2], \\ \mathbb{P}^1_{\mathbb{Q}} \to Q; &\qquad [x,y] \mapsto [x,y,y,x], \\ \mathbb{P}^1_{\mathbb{Q}} \to Q; &\qquad [x,y] \mapsto [x,y,-y,x]. \end{aligned}$$

6.5.I. On $D(xyz)$ we get
$$[1/x, 1/y, 1/z] = [yz, xz, xy],$$
so it can be extended to
$$D(yz) \cup D(xz) \cup D(xy) = \mathbb{P}^2_k \setminus \{[1,0,0],[0,1,0],[0,0,1]\}.$$

6.5.J. Write $C = \operatorname{Proj}\mathbb{C}[x,y,z]/(x^n+y^n-z^n)$. Since y^n-1 is divisible by $y-1$ exactly once, the polynomial x^n+y^n-1 is irreducible (if it is a product of f and g, they can't both be non-constant in $\mathbb{C}[x,y]/(y-1)$). Hence the homogenization $x^n+y^n-z^n$ is irreducible, so C is an integral $\mathbb{C}$-variety.

To show that there is no dominant rational map of $\mathbb{C}$-schemes $\mathbb{P}^1_{\mathbb{C}} \dashrightarrow C$, we reduce to showing that there are no maps of $\mathbb{C}$-extensions $K(C) \to K(\mathbb{P}^1_{\mathbb{C}})$ by 6.5.B. Since $K(C)$ is the stalk at the generic point, we can reduce to an affine neighbourhood, and by the universal property of the fraction field we want to show non-existence of an injective $\mathbb{C}$-algebra map $\mathbb{C}[x,y]/(x^n+y^n-1) \hookrightarrow K(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$. Considering the images of x and y, it suffices to show that no non-constant functions $f,g \in \mathbb{C}(t)$ satisfy $f^n+g^n=1$. Note that this fails when $n=2$, since $\left(\frac{t^2-1}{t^2+1}\right)^2 + \left(\frac{2t}{t^2+1}\right)^2 = 1$.

Bringing f and g to a common denominator, we have an equation of polynomials $p(t)^n + q(t)^n = r(t)^n$ with $p,q,r \in \mathbb{C}[t]$ having no common factor, and we want to show that they are constant. Now if ζ is a primitive nth root of unity, we get
$$\prod_{k=0}^{n-1}(r-\zeta^k q) = p^n.$$
For $\zeta^k \neq \zeta^{k'}$, since q and r are linear combinations of $r-\zeta^k q$ and $r-\zeta^{k'}q$, we get that no irreducible factor of p can divide both. Moreover every irreducible factor of $r-\zeta^k q$ divides p^n, and hence p, so the irreducible factorization of $r-\zeta^k q$ is the nth power of a part of the irreducible factorization of p. Then since $1,\zeta,\zeta^2$ are distinct (as $n>2$), we get an alternative solution with relatively prime polynomials given by
$$\left(\sqrt[n]{r-q}\right)^n + \left(\sqrt[n]{\alpha(r-\zeta q)}\right)^n = \left(\sqrt[n]{\beta(r-\zeta^2 q)}\right)^n$$
whenever $\alpha,\beta \in \mathbb{C}$ satisfy $1+\alpha=\beta$ and $1+\zeta\alpha = \zeta^2\beta$. Such a choice of α and β is possible, since $(1,-1)$ and $(\zeta^2,-\zeta)$ are linearly independent. If $\max\{\deg r, \deg q\} > 0$, we have
$$\max\{\deg \sqrt[n]{r-q}, \deg \sqrt[n]{\alpha(r-\zeta q)}\} \le \frac{1}{n}\max\{\deg q, \deg r\} < \max\{\deg q, \deg r\},$$
so by induction $r-q$ and $r-\zeta q$ are constant. Hence r and q are constant, so p is too.

6.5.K. Since any nonempty open subset of an irreducible space is dense, the curves C and $\mathbb{P}^1_{\mathbb{C}}$ from 6.5.J cannot have nonempty isomorphic open subsets.

6.5.L. (a) We have distinct $[\alpha_1,\beta_1],\dots,[\alpha_4,\beta_4] \in \mathbb{P}^2_{\mathbb{C}}$ with each $L_i = \alpha_i P + \beta_i Q$ a perfect square. We can replace P and Q with linear combinations that have the same span, since such combinations remain relatively prime. Applying this to L_1 and L_2 we can assume $[\alpha_1,\beta_1]=[1,0]$ and $[\alpha_2,\beta_2]=[0,1]$, so $[\alpha_3,\beta_3]=[1,-\mu]$ and $[\alpha_4,\beta_4]=[1,-\lambda]$ for some $\lambda,\mu \in \mathbb{C}^*$. Then replacing Q by μQ we can assume $\mu=1$, and still $[\alpha_1,\beta_1]=[1,0]$, $[\alpha_2,\beta_2]=[0,1]$. So P, Q, $P-Q$, and $P-\lambda Q$ are perfect squares, with $\lambda \in \mathbb{C}\setminus\{0,1\}$.

Now P and Q are coprime, so $u=\sqrt{P}$ and $v=\sqrt{Q}$ are coprime. Then $u \pm v$ and $u \pm \sqrt{\lambda}v$ are coprime pairs, so since $(u+v)(u-v) = P-Q$ and $(u+\sqrt{\lambda}v)(u-\sqrt{\lambda}v) = P-\lambda Q$ are squares we get that $u\pm v$ and $u \pm \sqrt{\lambda}v$ are squares. This is a non-degenerate solution since $\lambda \notin \{0,1\}$, and if $\max\{\deg P, \deg Q\} > 0$ we get
$$\max\{\deg u, \deg v\} = \frac{1}{2}\max\{\deg P, \deg Q\} < \max\{\deg P, \deg Q\},$$
so by induction u and v are constant. Hence P and Q are constant.

(b) Suppose $(x, y) = (p/q, r/s)$ is a solution to 6.5.10.1, with p/q and r/s in lowest terms. Then

$$r^2q^3 = s^2(p - aq)(p - bq)(p - cq),$$

where q is coprime to $(p - aq)(p - bq)(p - cq)$ since p and q are coprime, and s is coprime to r^2 since r and s are coprime. Hence $q^3 \mid s^2$ and $s^2 \mid q^3$, so $s^2 = \delta q^3$ for some $\delta \in \mathbb{C}^*$ and therefore

$$r^2 = \delta(p - aq)(p - bq)(p - cq).$$

Then $p - aq$, $p - bq$, $p - cq$ are squares, since they are coprime, and $q = (s/\sqrt{\delta}q)^2$ so by (a) we get that p and q are constant, since a, b, c are distinct. Then r and s are constant, since r^2 and s^2 are given in terms p and q.

6.6.A. By 6.3.F there is a natural bijection between $\mathbb{C}$-scheme morphisms $X \to \mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{C}$-algebra morphisms $\mathbb{C}[x] \to \Gamma(X)$. Now there is a natural transformation from $\mathbb{C}$-algebra morphisms $\mathbb{C}[x] \to \Gamma(X)$ to elements of $\Gamma(X)$, given by evaluation at x. This is injective, because x generates $\mathbb{C}[x]$ as a $\mathbb{C}$-algebra, and surjective, because given any $f \in \Gamma(X)$ there is the evaluation map $\mathbb{C}[x] \to \Gamma(X)$ given by $p(x) \mapsto p(f)$.

6.6.B. By taking an affine neighbourhood $\operatorname{Spec} A$ of the generic point, contained in the domain of definition of a rational map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$, we have a map $\mathbb{Z}[x] \to A$ (i.e. an element of $A \subseteq K(X)$), and this uniquely determines the rational map since $\operatorname{Spec} A$ is dense (contains the generic point). Since any element of $K(X)$ is defined on some such neighbourhood, this shows that maps to $\mathbb{A}^1_{\mathbb{Z}}$ correspond to rational functions.

6.6.C. Natural transformations $h_Y \to h_Z$ are in bijection with morphisms $Y \to Z$ by 1.3.Z(b), and this bijection respects composition since it is just given by evaluating at $\mathrm{id}_Y \in h_Y(Y)$. Hence a natural isomorphism $h_Y \cong h_Z$ induces an isomorphism $Y \cong Z$.

6.6.D. Since {Grothendieck, A.} is a singleton set, it suffices to find a Y such that $\operatorname{Mor}(X, Y)$ is a singleton set for each X. In other words, it suffices to find a final object Y in **Sch**. This is $\operatorname{Spec} \mathbb{Z}$.

6.6.E. (a) Note that products in the category of functors $\mathscr{C} \to$ **Set** can be taken pointwise. Now if $X, Y, Z \in \mathscr{C}$ have a natural isomorphism $h_Z \cong h_X \times h_Y$, then by definition a morphism $A \to Z$ corresponds naturally to morphisms $A \to X$ and $A \to Y$. In other words Z satisfies the universal property of $X \times Y$, where the maps $Z \to X$ and $Z \to Y$ come from applying the natural isomorphism to the identity map $Z \to Z$.

Since Spec is adjoint to Γ, we want to show that $\mathbb{Z}[x_1, \ldots, x_n]$ represents the functor $R \mapsto R^n$ on commutative rings. This follows from the fact that maps $\mathbb{Z}[x_1, \ldots, x_n] \to R$ are uniquely and freely determined by the choice of images for $x_1, \ldots, x_n$.

Since the functors being represented are n-fold products of Γ, the comment from above shows that $\mathbb{A}^n_{\mathbb{Z}}$ is an n-fold product of $\mathbb{A}^1_{\mathbb{Z}}$ in **Sch**, or equivalently $\mathbb{Z}[x_1, \ldots, x_n]$ is the n-fold coproduct in **CRing** (i.e. tensor product over $\mathbb{Z}$) of $\mathbb{Z}[x_1]$. The maps $\mathbb{A}^2_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ come from the inclusions $\mathbb{Z}[x_i] \subseteq \mathbb{Z}[x_1, x_2]$.

(b) Since Spec is adjoint to Γ, it suffices to show that the functor $R \mapsto U(R)$ on commutative rings is represented by $\mathbb{Z}[t, t^{-1}]$. But maps from $\mathbb{Z}[t, t^{-1}]$ correspond to choices of image for t, which must be units since t is a unit, and can be arbitrary units $u \in U(R)$ since evaluation $p(t, t^{-1}) \mapsto p(u, u^{-1})$ is then a homomorphism.

6.6.F. This is the adjoint relationship between Spec and Γ for locally ringed spaces, which was shown in 6.3.F.

6.6.G. The group operation $\mathbb{A}^2_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ is given by $(x, y) \mapsto x + y$; i.e. the map $\mathbb{Z}[t] \to \mathbb{Z}[x, y]$ where $t \mapsto x + y$. Inversion $\mathbb{A}^1_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ is $x \mapsto -x$, and identity $\operatorname{Spec} \mathbb{Z} \to \mathbb{A}^1_{\mathbb{Z}}$ is the element $0 = [(x - 0)]$, i.e. the map $\mathbb{Z}[x] \to \mathbb{Z}$ given by $x \mapsto 0$. Associativity holds, since on rings the two composites $\mathbb{Z}[t] \to \mathbb{Z}[x, y, z]$ are $t \mapsto (x + y) + z$ and $t \mapsto x + (y + z)$, which are equal. Identity holds, since on rings the two composites $\mathbb{Z}[x] \to \mathbb{Z}[x]$ are $x \mapsto 0 + x$ and $x \mapsto x + 0$, and are both the identity. Inversion works, since both composites $\mathbb{Z}[x] \to \mathbb{Z}[x]$ are $x \mapsto (-x) + x$ and $x \mapsto x + (-x)$, which are the same as $\mathbb{Z}[x] \to \mathbb{Z} \subseteq \mathbb{Z}[x]$ where $x \mapsto 0$.

6.6.H. Since $\mathrm{Mor}(X, \cdot)$ is a covariant functor, applying it to the structural morphisms on G make $\mathrm{Mor}(X,G)$ a group object in **Set**. But group objects in **Set** are just groups. Concretely, if $m_G : G \times G \to G$ is the multiplication morphism, then the group operation on $\mathrm{Mor}(X,G)$ is $(f,g) \mapsto m_G \circ (f,g)$. If $\phi : X \to Y$, then the pullback $\mathrm{Mor}(Y,G) \to \mathrm{Mor}(X,G)$ is a group homomorphism; if $f,g : Y \to G$ then $m_G \circ (f,g) \circ \phi = m_G \circ (f \circ \phi, g \circ \phi)$.

6.6.I. Given $f, g \in \Gamma(X)$, the associated maps $X \to \mathbb{A}^1_{\mathbb{Z}}$ are given by $\mathbb{Z}[t] \to \Gamma(X)$ with $t \mapsto f$ and $t \mapsto g$. Then the product $X \times X \to \mathbb{A}^2_{\mathbb{Z}}$ is given by $\mathbb{Z}[x,y] \to \Gamma(X)$ where $x \mapsto f$, $y \mapsto g$. Composing with $m : \mathbb{A}^2_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$ means composing with $\mathbb{Z}[t] \to \mathbb{Z}[x,y]$; $t \mapsto x+y$, so we get $\mathbb{Z}[t] \to \Gamma(X)$ with $t \mapsto f+g$. Hence the group operation is $(f,g) \mapsto f+g$.

6.6.J. If we remove the inverse map $i : X \to X$ and the axioms involving it from the definition of a group object, we get the definition of a monoid object. We say that a monoid object is *commutative* if the multiplication map $m : X \times X \to X$ satisfies $m \circ S = m$, where $S : X \times X \to X \times X$ is the swap morphism (π_2, π_1) with π_1, π_2 the projections $X \times X \to X$.

An abelian group object is a group object such that the underlying monoid object is commutative. A ring object is an abelian group object X, say with $a : X \times X \to X$, $n : X \to X$, $z : Z \to X$, along with an additional commutative monoid structure $m : X \times X \to X$, $i : Z \to X$ satisfying *distributivity*:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mathrm{id}\times a"] \arrow[d, "F"] & X \times X \arrow[d, "m"] \\ X \times X \arrow[r, "a"] & X \end{tikzcd}$$

commutes, where F is given by

$$X \times X \times X \xrightarrow{(\pi_1,\pi_2,\pi_1,\pi_3)} X \times X \times X \times X \xrightarrow{m \times m} X \times X.$$

6.6.K. The group structure on $F(X)$ gives maps

- $\mathrm{Mor}(X, Y \times Y) = \mathrm{Mor}(X,Y) \times \mathrm{Mor}(X,Y) \to \mathrm{Mor}(X,Y)$
- $\mathrm{Mor}(X,Y) \to \mathrm{Mor}(X,Y)$
- $\mathrm{Mor}(X,Z) = \{*\} \to \mathrm{Mor}(X,Y)$

that satisfy the group axioms, and are natural in X since F is a functor. By Yoneda's lemma, specifically 1.3.Z(a), these are induced by morphisms $Y \times Y \to Y$, $Y \to Y$, $Z \to Y$ respectively, which again satisfy the group axioms. In other words, Y has a group object structure inducing the group structure on $\mathrm{Mor}(X,Y)$ we originally had from $F(X)$.

6.6.L. The group structure on $\Gamma(X)^n$ gives m, i and e for $\mathbb{A}^n_{\mathbb{Z}}$ as follows:

- $m : \mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}} \to \mathbb{A}^n_{\mathbb{Z}}$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}}) & & \mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}}) \ni m \\ \| & & \| \\ \Gamma(\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}})^n \times \Gamma(\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}})^n & \xrightarrow{\quad + \quad} & \Gamma(\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^n_{\mathbb{Z}})^n \\ \Psi & & \Psi \\ \big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) & & (x_1 + y_1, \dots, x_n + y_n) \end{array}$$

i.e. Spec of $\mathbb{Z}[t_1, \dots, t_n] \to \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$ where $t_i \mapsto x_i + y_i$.

- $i : \mathbb{A}^n_{\mathbb{Z}} \to \mathbb{A}^n_{\mathbb{Z}}$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}}) & & \mathrm{Mor}(\mathbb{A}^n_{\mathbb{Z}}, \mathbb{A}^n_{\mathbb{Z}}) \ni i \\ \| & & \| \\ (x_1, \dots, x_n) \in \Gamma(\mathbb{A}^n_{\mathbb{Z}})^n & \xrightarrow{-} & \Gamma(\mathbb{A}^n_{\mathbb{Z}})^n \ni (-x_1, \dots, -x_n) \end{array}$$

  i.e. Spec of $\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}[x_1, \dots, x_n]$ where $x_i \mapsto -x_i$.
- $e : \operatorname{Spec} \mathbb{Z} \to \mathbb{A}^n_{\mathbb{Z}}$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\operatorname{Spec} \mathbb{Z}, \operatorname{Spec} \mathbb{Z}) & & \mathrm{Mor}(\operatorname{Spec} \mathbb{Z}, \mathbb{A}^n_{\mathbb{Z}}) \ni e \\ \| & & \| \\ * \in \{*\} & \xrightarrow{0} & \Gamma(\operatorname{Spec} \mathbb{Z})^n \ni (0, \dots, 0) \end{array}$$

  i.e. Spec of $\mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}$ where $x_i \mapsto 0$.

The group structure on $U(\Gamma(X))$ gives m, i and e for $\mathbb{G}_m$ as follows:

- $m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m \times \mathbb{G}_m) & & \mathrm{Mor}(\mathbb{G}_m \times \mathbb{G}_m, \mathbb{G}_m) \ni m \\ \| & & \| \\ (s,t) \in U(\Gamma(\mathbb{G}_m \times \mathbb{G}_m)) \times U(\Gamma(\mathbb{G}_m \times \mathbb{G}_m)) & \xrightarrow{\times} & U(\Gamma(\mathbb{G}_m \times \mathbb{G}_m)) \ni st \end{array}$$

  i.e. Spec of $\mathbb{Z}[u, u^{-1}] \to \mathbb{Z}[t, t^{-1}, s, s^{-1}]$ where $u \mapsto st$.
- $i : \mathbb{G}_m \to \mathbb{G}_m$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\mathbb{G}_m, \mathbb{G}_m) & & \mathrm{Mor}(\mathbb{G}_m, \mathbb{G}_m) \ni i \\ \| & & \| \\ t \in U(\Gamma(\mathbb{G}_m)) & \xrightarrow{(\cdot)^{-1}} & U(\Gamma(\mathbb{G}_m)) \ni t^{-1} \end{array}$$

  i.e. Spec of $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]$ where $t \mapsto t^{-1}$.
- $e : \operatorname{Spec} \mathbb{Z} \to \mathbb{G}_m$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\operatorname{Spec} \mathbb{Z}, \operatorname{Spec} \mathbb{Z}) & & \mathrm{Mor}(\operatorname{Spec} \mathbb{Z}, \mathbb{G}_m) \ni e \\ \| & & \| \\ * \in \{*\} & \xrightarrow{1} & U(\Gamma(\operatorname{Spec} \mathbb{Z})) \ni 1 \end{array}$$

  i.e. Spec of $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}$ where $t \mapsto 1$.

6.6.M. If X and Y are group objects, representing contravariant functors $h_X, h_Y : \mathscr{C} \to \mathbf{Grp}$, then the product $X \times Y$ represents $h_{X \times Y} \cong h_X \times h_Y$, which is naturally a functor to **Grp** since products (and more generally, limits) are functorial. Hence $X \times Y$ gets a natural group object structure.

Concretely, the multiplication map $(X \times Y) \times (X \times Y) \to X \times Y$ is given by

$$X \times Y \times X \times Y \xrightarrow{(\pi_1, \pi_3, \pi_2, \pi_4)} X \times X \times Y \times Y \xrightarrow{m_X \times m_Y} X \times Y.$$

6.6.N. (a) A group scheme X gives a contravariant functor $\mathrm{Mor}(\cdot, X)$ to **Grp**. Define a morphism of group schemes $X \to Y$ to be a natural transformation $\mathrm{Mor}(\cdot, X) \to \mathrm{Mor}(\cdot, Y)$. By Yoneda's lemma 1.3.Z(a), such transformations in **Set** correspond to morphisms $\phi : X \to Y$, and the condition for the natural transformation to consist of group homomorphisms is that ϕ commutes with the structural maps making X and Y group schemes, i.e. $\phi \circ m_X = m_Y \circ (\phi \times \phi)$. This implies $\phi \circ i_X = i_Y \circ \phi$ and $\phi \circ e_X = e_Y$, by the same argument that shows group homomorphisms (defined as preserving the group operation) preserve identity and inverses.

(b) The covariant functor $\mathbf{Alg}_A \to \mathbf{Grp}$ given by $S \mapsto \mathrm{GL}_n(S)$ is represented by $A[(x_{i,j}), 1/\det(x_{i,j})]$, so the contravariant functor $\mathbf{Sch}_A \to \mathbf{Grp}$ given by $X \mapsto \mathrm{GL}_n(\Gamma(X))$ is represented by the group scheme $\mathrm{GL}_n := \operatorname{Spec} A[(x_{i,j}), 1/\det(x_{i,j})]$. In other words, the distinguished open set of $\mathbb{A}_A^{n\times n}$ given by $\det \in \Gamma(\mathbb{A}_A^{n\times n})$. Then det is an invertible function on GL_n, and hence corresponds to a morphism $\mathrm{GL}_n \to \mathbb{G}_m$. Concretely, this is Spec of $A[t, t^{-1}] \to A[(x_{i,j}), 1/\det(x_{i,j})]$ where $t \mapsto \det(x_{i,j})$.

(c) We have a morphism of schemes $\mathbb{G}_m \to \mathbb{G}_m$ from the A-algebra map $t \mapsto t^n$ on $A[t, t^{-1}]$. Moreover it is a morphism of group schemes, since the composites with the multiplication morphism would come from $u \mapsto (st)^n$ and $u \mapsto s^n t^n$, which are the same.

6.6.O. Note that the zero object in the category of functors to **Grp** is just the constant functor whose value is the trivial group. Note also that this functor is represented by any final object, e.g. $\operatorname{Spec} A$ for $\mathbf{Sch}_A$. Moreover a functor to **Grp** has a kernel in the category of such functors, given by computing kernels pointwise. This follows from the same argument as in 2.3.F.

The category of group schemes is equivalent to the full subcategory of representable functors in the category of functors to **Grp**. Hence the claim follows from a general fact: if a category $\mathscr{C}$ has a zero object, with a full subcategory $\mathscr{C}'$ containing this zero object, and if a morphism $f : A \to B$ in $\mathscr{C}'$ has a kernel $g : K \to A$ in $\mathscr{C}$ such that $K \in \mathscr{C}'$, then g is a kernel for f in $\mathscr{C}'$. This is immediate because being a kernel for f in $\mathscr{C}$ is a strictly stronger condition than being a kernel for f in $\mathscr{C}'$, where the only difference between the two conditions is quantification.

6.6.P. The functor in question is $X \mapsto \{u \in U(\Gamma(X)) : u^n = 1\}$. This is representable by the closed subscheme $\operatorname{Spec} A[t, t^{-1}]/(t^n - 1)$ of $\mathbb{G}_m$. Over a field k of characteristic p dividing n we have $t^n - 1 = (t^{n/p} - 1)^p$, so $t^{n/p} - 1$ gives a non-trivial nilpotent function on this group scheme.

6.6.Q. The functor in question is $X \mapsto \{M \in \mathrm{GL}_n(\Gamma(X)) : \det(M) = 1\} = \mathrm{SL}_n(\Gamma(X))$. This is representable by the closed subscheme $\operatorname{Spec} A[(x_{i,j}), 1/\det(x_{i,j})]/(\det(x_{i,j}) - 1)$ of GL_n, or equivalently the closed subscheme $\operatorname{Spec} A[(x_{i,j})]/(\det(x_{i,j}) - 1)$ of $\mathbb{A}_A^{n\times n}$.

6.6.R. We have a contravariant functor $\mathbf{Sch}_A \to \mathbf{CRing}$ by $X \mapsto \Gamma(X)$, whose composition with the forgetful functor to **Set** is represented by $\mathbb{A}_A^1$. The ring structure on $\Gamma(X)$ hence makes $\mathbb{A}_A^1$ a ring A-scheme in a natural way: the additive structure has been covered already, and the multiplication map $\mathbb{A}_A^1 \to \mathbb{A}_A^1$ comes from

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{Mor}(\mathbb{A}_A^1 \times \mathbb{A}_A^1, \mathbb{A}_A^1 \times \mathbb{A}_A^1) & & \mathrm{Mor}(\mathbb{A}_A^1 \times \mathbb{A}_A^1, \mathbb{A}_A^1) \ni m \\ \| & & \| \\ (x, y) \in \Gamma(\mathbb{A}_A^1 \times \mathbb{A}_A^1) \times \Gamma(\mathbb{A}_A^1 \times \mathbb{A}_A^1) & \xrightarrow{\times} & \Gamma(\mathbb{A}_A^1 \times \mathbb{A}_A^1) \ni xy \end{array}$$

i.e. Spec of $A[t] \to A[x, y]$ where $t \mapsto xy$. The ring axioms are all satisfied since they are satisfied pointwise in $\Gamma(X)$.

6.6.S. (a) A left action of a group scheme X on a scheme Y is a morphism $\sigma : X \times Y \to Y$ such that

$$\begin{array}{ccc} X \times X \times Y & \xrightarrow{\mathrm{id}\times\sigma} & X \times Y \\ \downarrow{\scriptstyle m\times\mathrm{id}} & & \downarrow{\scriptstyle\sigma} \\ X \times Y & \xrightarrow{\sigma} & Y \end{array} \tag{G}$$

commutes and the composite

$$Y \xrightarrow{(Z,\mathrm{id})} Z \times Y \xrightarrow{(i,\mathrm{id})} X \times Y \xrightarrow{\sigma} Y \tag{I}$$

is the identity. Then we have a traditional action of the group $\mathrm{Mor}(F, X)$ on the set $\mathrm{Mor}(F, Y)$ for any scheme F.

(b) Note that $A \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] = A[t, t^{-1}]$, so a group scheme action of $\mathbb{G}_m$ on $\operatorname{Spec} A$ corresponds to a map $f : A \to A[t, t^{-1}]$ such that if $f(a) = \sum_n a_n t^n$ we have $\sum_n a_n (st)^n = \sum_n f(a_n) s^n$ for (G) and $a = \sum_n a_n$ for (I). In other words $a = \sum_n a_n$ and $f(a_n) = a_n t^n$. Now evaluation at 1 is a left inverse for f, so $\ker f = 0$, and hence the natural $\mathbb{Z}$-grading on $A[t, t^{-1}]$ pulls back to a grading on A:

$$A = \bigoplus_{n \in \mathbb{Z}} f^{-1}(t^n A).$$

Moreover any $a_n \in f^{-1}(t^n A)$ satisfies $f(a_n) = a_n t^n$ from (G), so the map f is entirely determined by this grading. Hence it suffices to show that such an f exists for any given $\mathbb{Z}$-grading on A. Certainly defining $f(a_n) = a_n t^n$ for homogeneous a_n of degree n gives a map of abelian groups, and it is a ring homomorphism since if b_m is homogeneous of degree m then $f(a_n b_m) = a_n b_m t^{n+m} = (a_n t^n)(b_m t^m) = f(a_n) f(b_m)$. But it satisfies (G) and (I) by construction, so we are done.

6.6.T. The group operation on $\mathbb{A}^1_{\mathbb{Z}}$ was given by Spec of $\mathbb{Z}[t] \to \mathbb{Z}[x, y]$ where $t \mapsto x + y$. This is using the identification $\mathbb{A}^2_{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$, which from the point of view of rings is $\mathbb{Z}[x, y] = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[y]$. Hence the comultiplication map $\mathbb{Z}[t] \to \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[y]$ is given by $t \mapsto x \otimes 1 + 1 \otimes y$.

6.7.A. Any map $A^{\oplus n} \to A^{\oplus n}$ is uniquely represented by a matrix in $M_n(A)$ with respect to the standard basis. If such a map is an isomorphism, then the corresponding matrix has an inverse (the matrix for the inverse map), and hence is invertible. If $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ are bases for $A^{\oplus n}$, then we have a composite isomorphism $A^{\oplus n} \to A^{\oplus n}$ given by

$$a_1 w_1 + \cdots + a_n w_n \mapsto (a_1, \ldots, a_n) \mapsto a_1 v_1 + \cdots + a_n v_n$$

which is hence represented by an invertible matrix $P \in \mathrm{GL}_n(A)$. (Note that $P \in M_n(A)$ has an inverse iff $\det(P) \in U(A)$, because of the adjugate matrix.) Then $v_i = P w_i$, and P is unique with this property.

6.7.B. Let $P \in \mathrm{GL}_n(A)$ be such that $v = Pw$. A plane from U_v is the columnspan with respect to v of a matrix

$$M(X) = \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ x_{1(k+1)} & x_{2(k+1)} & \cdots & x_{k(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix},$$

and the columnspan with respect to w of $P \cdot M(X)$. This lies in U_w iff the reduced column echelon form of $P \cdot M(X)$ is some $M(Y)$, i.e. iff the pivots of $P \cdot M(X)$ are the first k rows. Hence if $N_P(X)$ denotes the first k rows of $P \cdot M(X)$, the intersection $U_v \cap U_w$ under $U_v \cong \mathbb{A}_A^{k \times (n-k)}$ is the distinguished open set $D(\det N_P(X))$. We then glue the two schemes $D(\det N_P(X))$ and $D(\det N_{P^{-1}}(Y))$ representing $U_v \cap U_w$ along the isomorphism

$$A[(y_{ji}), 1/\det N_{P^{-1}}(Y)] \to A[(x_{ji}), 1/\det N_P(X)]$$

given by

$$M(Y) \mapsto P \cdot M(X) N_P(X)^{-1},$$

since right multiplication by $N_P(X)^{-1}$ puts $P \cdot M(X)$ into reduced column echelon form. Note that when $P = I$ this map is the identity on U_v, so the next exercise actually proves that it is an isomorphism.

6.7.C. Suppose we have a third basis u, with $w = Qu$. The triple overlap is given by the three rings

$$\begin{aligned} R_v &= A[(x_{ji}), 1/\det N_P(X), 1/\det N_{QP}(X)] \\ R_w &= A[(y_{ji}), 1/\det N_{P^{-1}}(Y), 1/\det N_Q(Y)] \\ R_u &= A[(z_{ji}), 1/\det N_{P^{-1}Q^{-1}}(Z), 1/\det N_{Q^{-1}}(Z)], \end{aligned}$$

with maps

$$\begin{aligned} R_w &\to R_v; & M(Y) &\mapsto P \cdot M(X) N_P(X)^{-1} \\ R_u &\to R_w; & M(Z) &\mapsto Q \cdot M(Y) N_Q(Y)^{-1} \\ R_u &\to R_v; & M(Z) &\mapsto QP \cdot M(X) N_{QP}(X)^{-1}. \end{aligned}$$

Hence it suffices to show

$$QP \cdot M(X) N_{QP}(X)^{-1} = Q \cdot M(Y) N_Q(Y)^{-1}$$

given $M(Y) = P \cdot M(X) N_P(X)^{-1}$. By substitution, this is equivalent to $N_{QP}(X) = N_Q(Y) N_P(X)$. Decompose P with a $k \times k$ top left corner as follows:

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right].$$

Then $N_P(X) = P_{11} + P_{12} X$ and $Y = (P_{21} + P_{22} X) N_P(X)^{-1}$, so

$$\begin{aligned} N_Q(Y) N_P(X) &= \left[Q_{11} + Q_{12}(P_{21} + P_{22} X) N_P(X)^{-1} \right] N_P(X) \\ &= Q_{11}(P_{11} + P_{12} X) + Q_{12}(P_{21} + P_{22} X) \\ &= (Q_{11} P_{11} + Q_{12} P_{21}) + (Q_{11} P_{12} + Q_{12} P_{22}) X \\ &= (QP)_{11} + (QP)_{12} X \\ &= N_{QP}(X) \end{aligned}$$

as required. This verifies the cocycle condition, so the maps are indeed isomorphisms and we can glue the U_v to get the structure of an A-scheme on $G(k, n)$.

Chapter 7

Useful classes of morphisms

7.1.A. Suppose $\pi = \tau \circ \rho$ is an open embedding onto $U \subseteq Y$. If $V \subseteq Y$ is open, and $W = \pi^{-1}(V)$, then $\pi|_W = \tau|_{U \cap V} \circ \rho|_W$. Now the restriction of the isomorphism ρ to W is again an isomorphism from W to $\rho(W) = U \cap V$, and the restriction of the inclusion τ to $U \cap V$ is again an inclusion $U \cap V \to Y$, so $\pi|_W$ is again an open embedding.

Now suppose $\pi' = \tau' \circ \rho'$ is an open embedding from Y onto $V \subseteq Z$, so we have

$$(X, \mathscr{O}_X) \xrightarrow{\rho} (U, \mathscr{O}_Y|_U) \xhookrightarrow{\tau} (Y, \mathscr{O}_Y) \xrightarrow{\rho'} (V, \mathscr{O}_Z|_V) \xhookrightarrow{\tau'} (Z, \mathscr{O}_Z).$$

It is a basic property of inclusions that $(\cdot) \circ \tau = (\cdot)|_U$, and $\rho'|_U$ is an isomorphism onto $\rho'(U)$ followed by the inclusion of $\rho'(U)$ into V, so we have

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (U, \mathscr{O}_Y|_Y) \xrightarrow{\sim} (\rho'(U), \mathscr{O}_Z|_{\rho'(U)}) \hookrightarrow (V, \mathscr{O}_Z|_V) \hookrightarrow (Z, \mathscr{O}_Z).$$

Since isomorphisms and inclusions are closed under composition, it follows that $\pi' \circ \pi$ is an open embedding.

7.1.B. If $i : U \to Z$ is an open embedding, and $\rho : Y \to Z$ an arbitrary morphism, consider the open subscheme $W = \rho^{-1}(U) \subseteq Y$. By definition the following squares commutes:

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\rho|_W"] & Y \arrow[d, "\rho"] \\ U \arrow[r, hook, "i"] & Z \end{tikzcd}$$

Now suppose we have morphisms $\alpha : A \to U$ and $\beta : A \to Y$ such that $i \circ \alpha = \rho \circ \beta$. Then $A = \alpha^{-1}(U) = \beta^{-1}(W)$, so β factors through W and α is the composition with $\rho|_W$. This verifies the universal property of the fibered product, and property (iii) holds since the open embedding i was lifted to an open inclusion $W \subseteq Y$.

7.1.C. If $U \subseteq X$ is an affine open subscheme, then $\pi|_U$ gives an isomorphism of it onto the open subscheme $\pi(U) \subseteq Y$. Then $U \cong \pi(U)$ is affine, and hence Noetherian if Y is locally Noetherian. Therefore X is locally Noetherian whenever Y is. If Y is Noetherian, then the open subset $\pi(X)$ is quasicompact, and hence $X \cong \pi(X)$ is quasicompact. Therefore X is Noetherian whenever Y is.

By 3.6.G(b), it is possible for Y to be quasicompact (even affine) while X is not quasicompact.

7.2.A. Since $(b_1, \ldots, b_n) = 1$ in B, $(\phi(b_1), \ldots, \phi(b_n)) = 1$ in $\phi(B)$. Also the image of $B_{b_i} \to A_{\phi(b_i)}$ is $\phi(B)_{\phi(b_i)} \subseteq A_{\phi(b_i)}$, since $B_{b_i} \to \phi(B)_{\phi(b_i)}$ is surjective (as $B \to \phi(B)$ was). By definition ϕ is integral iff $\phi(B) \to A$ is, so we are reduced to the case of $B \subseteq A$.

Now $a \in A$ satisfies integral equations in A_{b_i} over B_{b_i} for each i, and multiplying these equations by powers of a we can assume that they are all of degree m. Say

$$(a/1)^m = \sum_{j=0}^{m-1} (c_{ij}/b_i^t)(a/1)^j$$

where $c_{ij} \in B$, and t is chosen large enough to work for all i and j. Then by increasing t even further, we can guarantee
$$b_i^t a^m = \sum_{j=0}^{m-1} c_{ij} a^j.$$
Now $(b_1^t, \ldots, b_n^t) = 1$ in B, so we can write $1 = \sum f_i b_i^t$ with $f_i \in B$. Then a is integral over B, since
$$a^m = \sum_{i=1}^{n} f_i b_i^t a^m = \sum_{i=1}^{n} \sum_{j=0}^{m-1} f_i c_{ij} a^j.$$

7.2.B. (a) If $a \in A$ satisfies a monic polynomial $P(x) \in B[x]$, then $a/1 \in \phi(T)^{-1}A$ also satisfies $P(x)/1 \in T^{-1}B[x]$, which is again a monic polynomial if $0 \notin T$. Also $a + \phi(J)A \in A/\phi(J)A$ satisfies $P(x) + J[x] \in (B/J)[x]$, which is again a monic polynomial if $1 \notin J$. Moreover $a + I \in A/I$ still satisfies $P(x)$. Now the identity $\mathbb{Z} \to \mathbb{Z}$ is clearly integral, but if we localize the codomain to get $\mathbb{Z} \hookrightarrow \mathbb{Q}$ it is no longer integral, since $\mathbb{Z}$ is integrally closed.

(b) By (a) localization of B preserves integrality, and since localization is exact it preserves inclusion. Again by (a) localization of A does not preserve integrality in general, and of course quotienting A does not preserve inclusion in general (e.g. $\mathbb{Z} \subseteq \mathbb{Z}$ but $\mathbb{Z} \nsubseteq \mathbb{Z}/(2)$).

(c) The kernel of $B \to A \to A/I$ is precisely the intersection of I with B, which is J. Hence $B/J \to A/JA \to A/I$ is an injection, so $B/J \to A/JA$ must be an injection. By (a) it is also integral.

7.2.C. Firstly, if $B \to A$ and $C \to B$ are finite algebras, then $C \to A$ is a finite algebra; finite generating sets X and Y for A and B give a finite generating set $X \cdot Y = \{xy : x \in X, y \in Y\}$ for A over C.

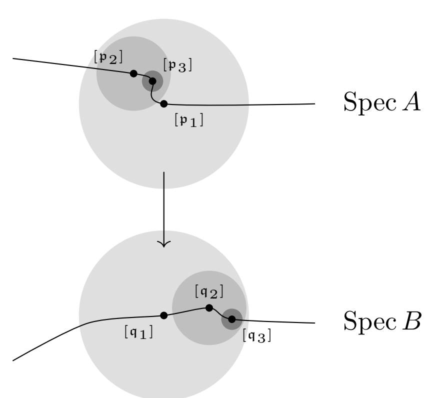
Now if $a \in A$ satisfies $a^{m+1} = \sum_{j=0}^{m} b_j a^j$, then $C[b_0]$ is finite over C since b_0 is integral over C, and $C[b_0, b_1]$ is finite over $C[b_0]$ since b_1 is integral over $C[b_0]$. Continuing inductively we get that $C[b_0, \ldots, b_m]$ is finite over C. But $C[b_0, \ldots, b_m, a]$ is finite over $C[b_0, \ldots, b_m]$ with generators $1, a, \ldots, a^m$, so $C[b_0, \ldots, b_m, a]$ is finite over C. Hence a is integral over C.

7.2.D. If $a_1, \ldots, a_n \in A$ are integral over B, then the same argument as in 7.2.C shows that $B[a_1, \ldots, a_n]$ is finite over B. Hence any polynomial over B in the a_i's is integral over B.

7.2.E. Note that we are considering the case $B \subseteq A$ already (integral *extension*). When A is a field, we must have $\mathfrak{p} = 0$, so the theorem statement is that B has no non-zero prime ideals, i.e. B is a field. We prove this as follows: Suppose $b \in B$ is non-zero, so $1/b \in A$. Then $(1/b)^m = \sum_{j=0}^{m-1} c_j (1/b)^j$ for some $c_j \in B$, so $1/b = \sum_{j=0}^{m-1} c_j b^{m-1-j} \in B$.

7.2.F. (a) It suffices to show the case $n = m + 1$, and then inductively grow the chain one prime at a time. Moreover the primes before $\mathfrak{p}_m$ and $\mathfrak{q}_m$ are irrelevant, so we can assume $m = 1$, $n = 2$. Replacing A and B with $A/\mathfrak{p}_1$ and $B/\mathfrak{q}_1$ lets us assume $\mathfrak{p}_1 = \mathfrak{q}_1 = 0$, preserving integrality by 7.2.B. Then the fact that $\mathfrak{p}_1$ lies over $\mathfrak{q}_1$ means $A \to B$ is injective, and the existence of a prime lying over $\mathfrak{q}_2$ is the content of the Lying Over Theorem.

(b)



7.2.G. The map $N \to M/IM$ is surjective iff $N + IM = M$, i.e. $I(M/N) = M/N$. But then $M/N = 0$ by Nakayama's lemma, since M/N is finitely-generated.

7.2.H. Let $N = (f_1, \ldots, f_n) \subseteq M$. The assumption is that $N \to M/\mathfrak{m}M$ is surjective, so $M = N$ by 7.2.G.

7.2.I. Let $m_1, \ldots, m_n$ be generators for M as an S-module. Then $rm_i = \sum_j s_{ij} m_j$ for some $s_{ij} \in S$, and the same argument as in Nakayama's lemma shows that $p(r)M = 0$, where $p(x)$ is the characteristic polynomial of (s_{ij}). But $p(x) \in S[x]$ is monic, and $p(r) = 0$ since M is faithful, so r is integral over S.

7.2.J. Suppose $x \in K(A)$ satisfies an equation $x^m = \sum_{j=0}^{m-1} a_j x^j$ where $a_j \in \widetilde{A}$. Since each a_j is integral over A, we get that $A[a_0, \ldots, a_{m-1}]$ is finite over A. But x is integral over $A[a_0, \ldots, a_{m-1}]$, so $A[a_0, \ldots, a_{m-1}, x]$ is also finite over A and hence $x \in \widetilde{A}$.

7.3.A. If $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ are quasicompact, and $U \subseteq C$ is a quasicompact open set, then $\psi^{-1}(U)$ is a quasicompact open set, so $\phi^{-1}(\psi^{-1}(U))$ is a quasicompact open set. Hence $\psi \circ \phi$ is quasicompact.

7.3.B. (a) Open subsets of Noetherian spaces are quasicompact.

(b) Open subsets of quasiseparated spaces are quasiseparated.

7.3.C. (a) We check that the condition of having quasicompact preimage under π is affine-local.

Suppose $U \subseteq Y$ is an affine open set with $\pi^{-1}(U)$ quasicompact, and suppose $D(f) \subseteq U$ is a distinguished open set. Take a finite cover of $\pi^{-1}(U)$ by affine open sets V_j. The preimage $\pi^{-1}(D(f))$ is the non-vanishing locus of $\pi^\sharp f \in \Gamma(\pi^{-1}(U_i), \mathscr{O}_X)$, and hence $\pi^{-1}(D(f)) \cap V_j = D((\pi^\sharp f)|_{V_j})$ is distinguished in V_j. Then $\pi^{-1}(D(f))$ is a finite union of affine open sets, and therefore is quasicompact.

If instead U has a finite cover by distinguished open sets whose preimages are quasicompact, then $\pi^{-1}(U)$ is a finite union of quasicompact open subsets, and hence is also quasicompact.

Hence π is quasicompact by the Affine Communication Lemma.

(b) Note that open subsets of quasiseparated spaces are again quasiseparated. Moreover, anything with a finite cover by open quasiseparated subsets is again quasiseparated. Hence π is quasiseparated by the Affine Communication Lemma.

7.3.D. Affine schemes are quasicompact and quasiseparated.

7.3.E. Restriction gives a map $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_s, \mathscr{O}_X)$, and the image of s is invertible since s is invertible in every stalk of X_s.

7.3.F. We can assume the cover is given by distinguished open sets, and so the complement of Z in each covering set is also a distinguished open set. Then the intersection of Y with these affine open sets is affine, so applying 7.3.4 to the inclusion $Y \hookrightarrow \operatorname{Spec} A$ shows that Y is affine.

7.3.G. We check that "$\pi^{-1}(\operatorname{Spec} A)$ is the spectrum of a finite A-algebra" is affine-local; suppose $(f_1, \ldots, f_n) = 1$ in A. If $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ with B finite over A, then $\pi^{-1}(\operatorname{Spec} A_{f_i}) = \operatorname{Spec} B_{g_i}$ where g_i is the image of f_i in B. Moreover B_{g_i} is finite over A_{f_i}, by the same generating set as for A over B. Conversely, suppose $\pi^{-1}(\operatorname{Spec} A_{f_i})$ is the spectrum of a finite A_{f_i}-algebra. By 7.3.4, we get that $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ is affine. Then $\pi^{-1}(\operatorname{Spec} A_{f_i}) = \operatorname{Spec} B_{g_i}$ where g_i is the image of f_i in B, and by assumption B_{g_i} is finite over A_{f_i}. We may assume we have a finite generating set of the form $\{b_{ij}/1\}_j$ for B_{g_i} as an A_{f_i}-module. Then any $b \in B$ has

$$g_i^N b = \sum_j a_j b_{ij}$$

for each i, with $a_j \in A$, by taking N large enough. Then since $1 = \sum_i e_i f_i^N$ with $e_i \in A$, we get

$$b = \sum_{i,j} (e_i a_j) b_{ij}.$$

Hence $\{b_{ij}\}$ gives a finite generating set for B as an A-module.

7.3.H. It suffices to prove the affine case, as then X has an open cover by discrete sets, so X is discrete, and discrete schemes are affine. Now if A is a finite k-algebra, then every prime in A is maximal by 3.2.G. But $\operatorname{Spec} A$ is a finite union of irreducible components, which are the closures of generic points. Since all the points are closed, this shows that $\operatorname{Spec} A$ is a finite union of closed points, and hence is a finite discrete set. The residue field at a point $[\mathfrak{m}]$ is $A/\mathfrak{m}$, which is finite over k since A is.

7.3.I. This holds since a composite of finite algebras is finite (see the first sentence in the solution to 7.2.C).

7.3.J. The multiplication maps $S_n \otimes S_m \to S_{n+m}$ are just the maps $R \otimes R \to R$, $A \otimes R \to R$, $A \otimes A \to A$ given by the ring and algebra structures. If x denotes the element of S_1 given by $1 \in R$, then $(x) \supseteq S_{\geq 2}$ since $S_{n+1} = xS_n$ for $n \geq 1$, and therefore $\sqrt{(x)} \supseteq S_+$. Hence $\operatorname{Proj} S_\bullet \cong \operatorname{Spec}((S_\bullet)_x)_0$, and it suffices to show that $((S_\bullet)_x)_0 \cong R$. But

$$\begin{aligned} ((S_\bullet)_x)_0 &= \sum_{n\geq 0} S_n/x^n \\ &= \sum_{n\geq 1} S_n/x^n && \text{since } S_0 = xS_0/x \subseteq S_1/x \\ &= S_1/x && \text{since } S_n = x^{n-1}S_1, \end{aligned}$$

so the obvious injection $R \to ((S_\bullet)_x)_0$ given by $r \mapsto r/x$ is surjective.

Now if $r_1, \ldots, r_n$ generate R as an A-module, then they generate S_1 as an S_0-module, and hence they generate all of $S_\bullet$ as an A-algebra by multiplication with x as above. So in this case $S_\bullet$ is a finitely generated graded A-algebra, and $\operatorname{Spec} R \cong \operatorname{Proj} S_\bullet$ is projective over A.

7.3.K. Since π is affine, we can restrict to the preimage of an affine neighbourhood of $q \in Y$, and assume $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then we can restrict to the preimage of the closure of $q = [\mathfrak{q}]$, which is the underlying set map for the induced morphism $\operatorname{Spec} A/I \to \operatorname{Spec} B/\mathfrak{q}$ where $I = A\mathfrak{q}$, which is again finite using the images of generators for A over B. The localized map $\operatorname{Spec} S^{-1}(A/I) \to \operatorname{Spec} S^{-1}(B/\mathfrak{q})$ where $S = (B/\mathfrak{q})^*$ corresponds to restricting to $\{[\mathfrak{q}]\}$ and its fibre for the underlying sets, and remains finite by again using images of generators for A over B. But $S^{-1}(B/\mathfrak{q}) = K(B/\mathfrak{q})$ is a field, so by 7.3.H we get that $\operatorname{Spec} S^{-1}(A/I)$ has finitely many points as required.

7.3.L. It is injective on points, so the fibres are certainly finite. It is affine, since the preimage $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} = D(x)$ of $\mathbb{A}^1_{\mathbb{C}}$ is affine. However the pullback map $\mathbb{C}[x] \subseteq \mathbb{C}[x, 1/x]$ is not a finite extension, since a finite set cannot generate elements with arbitrarily large degree in $1/x$.

7.3.M. It suffices to check locally, so consider an integral morphism $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ and a closed set $V(I) \subseteq \operatorname{Spec} A$. Then if $J = (\pi^\sharp)^{-1}(I)$ we have $\pi(V(I)) \subseteq V(J)$, and the map $B/J \to A/I$ is injective by construction, and integral by 7.2.B. Hence $\operatorname{Spec} A/I \to \operatorname{Spec} B/J$ is surjective by Lying Over, so $\pi(V(I)) = V(J)$ is closed.

7.3.N. Since the elements integral over C form a C-subalgebra of $A \otimes_B C$, it suffices to consider elementary tensors $a \otimes c$. Now a is integral over B, so we have $a^m = \sum_{j=0}^{m-1} b_j a^j$ for some $b_j \in B$. Then

$$(a \otimes c)^m = c^m \cdot (a \otimes 1)^m = c^m \sum_{j=0}^{m-1} b_j (a \otimes 1)^j = \sum_{j=0}^{m-1} b_j c^{m-j} (a \otimes c)^j,$$

so $a \otimes c$ is integral over C as required.

7.3.O. The property that $\pi^{-1}(\operatorname{Spec} B)$ is locally finite type over B is affine-local:

- If $\pi^{-1}(\operatorname{Spec} B) = \cup_i \operatorname{Spec} A_i$ with A_i finitely generated over B, and $f \in B$, then $\pi^{-1}(\operatorname{Spec} B_f) = \cup_i \operatorname{Spec} A_{f_i}$ where $f_i \in A_i$ is the image of f, and $(A_i)_{f_i}$ is finitely generated over B_f by the same generators as for A_i over B.

- If $(f_1, \ldots, f_n) = 1$ in B, and $\pi^{-1}(\operatorname{Spec} B_{f_i}) = \cup_j \operatorname{Spec} A_{ij}$ with A_{ij} finitely generated over B_{f_i}, then $\pi^{-1}(\operatorname{Spec} B) = \cup_{ij} \operatorname{Spec} A_{ij}$, and A_{ij} is finitely generated over B by adjoining $1/f_i$ to a generating set over B_{f_i}.

7.3.P. (a) For a finite morphism, the preimage of $\operatorname{Spec} B$ is affine and finite over B, which certainly counts as a union of affines finitely generated over B.

(b) From (a) and 7.2.2, it suffices to show that an integral morphism of finite type is finite. Since integral morphisms are affine, it suffices to show that if B is a finitely generated integral A-algebra, then B is finite over A. But this is implicit in the solution 7.2.D; if $b_1, \ldots, b_n$ generate B as an A-algebra, then $A[b_1, \ldots, b_{i+1}]$ is finite over $A[b_1, \ldots, b_i]$ by the proof of 7.2.1, and hence B is finite over A by 7.2.C.

7.3.Q. (a) An open subset of $\operatorname{Spec} A$ is covered by affine open sets $\operatorname{Spec} A_f$, and $A_f = A[1/f]$ is finitely generated over A. Hence open embeddings are locally of finite type. If A is Noetherian, then there is a finite subcover by 3.6.T. Hence open embeddings into locally Noetherian schemes are of finite type.

(b) This is immediate from 7.2.C.

(c) We have $Y = \cup_i \operatorname{Spec} A_i$ with A_i Noetherian, and $\pi^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$ with B_i finitely generated over A_i, so B_i is Noetherian by the Hilbert Basis Theorem, and hence $X = \cup_i \operatorname{Spec} B_i$ is locally Noetherian. If finitely many A_i's suffice, then finitely many B_i's suffice, so X is Noetherian whenever Y is.

7.3.R. (a) Let $\psi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ be given by $f \mapsto f^p$. This is a ring homomorphism, since the characteristic is p, and $\operatorname{Spec} \psi$ is the identity on points since $f^p \in \mathfrak{p}$ iff $f \in \mathfrak{p}$. Now $\psi^r = \phi^r$, since k is the splitting field of $x^{p^r} - x$ over $\mathbb{F}_p$, so $F^r = \operatorname{Spec} \psi^r$ is also the identity on points. However F^r is not the identity morphism, since ϕ^r is not the identity.

(b) Since F^r is the identity on points, F is a bijection on points, but it cannot be an isomorphism since ϕ is not surjective.

(c) Since $K = \varinjlim \mathbb{F}_{p^n}$ with the inclusions $\mathbb{F}_{p^n} \subseteq \mathbb{F}_{p^{nk}}$, any $f \in K[x_1, \ldots, x_n]$ has coefficients coming from $\mathbb{F}^{p^{N_f}}$ for some large N_f. If $\phi = F^\sharp$ is the pullback on global functions, then $\phi^{N_f}(f) = f^{p^{N_f}}$, so F is bijective:

- it is injective since $\mathfrak{p} = \{f : f^{N_f} \in \mathfrak{p}\} = \{f : \phi^{N_f - 1}(f) \in \phi^{-1}(\mathfrak{p})\}$.
- it is surjective by Lying Over, since $f^{p^{N_f}} - \phi^{N_f}(f) = 0$ makes f integral over $\operatorname{im} \phi$.

However no power of F is the identity on points, since F^N does not fix the closed points with coordinates in $\mathbb{F}_{p^{2N}} \setminus \mathbb{F}_{p^N}$.

7.3.S. The restriction to a distinguished open set $D(g) \subseteq \operatorname{Spec} A$ is the map $A \to A_g$ given by $f \mapsto f^p/1$, which is the same as the composite of $f \mapsto f^p$ on A_g with the canonical map $A \to A_g$. By 5.3.1, this is enough to conclude that the morphisms glue.

7.3.T. Suppose $\operatorname{Spec} B = \cup_f \operatorname{Spec} B_f$. If we have $\pi^{-1}(\operatorname{Spec} B) = \cup_i \operatorname{Spec} A_i$ with A_i finitely presented over B, then $\pi^{-1}(\operatorname{Spec} B_f) = \cup_i \operatorname{Spec}(A_i)_f$, and $(A_i)_f$ is finitely presented over B_f by exactness of localization:

$$\begin{aligned}\big(B[x_1, \ldots, x_n]/(r_1, \ldots, r_m)\big)_f &\cong B[x_1, \ldots, x_n]_f/(r_1, \ldots, r_m)_f \\ &\cong B_f[x_1, \ldots, x_n]/(r_1/1, \ldots, r_m/1).\end{aligned}$$

(Here $(A_i)_f$ denotes localization at the image of f in A_i.)

Conversely if $\pi^{-1}(\operatorname{Spec} B_f) = \cup_i \operatorname{Spec} A_i^{(f)}$ with $A_i^{(f)}$ finitely presented over B_f, then $\pi^{-1}(\operatorname{Spec} B) = \cup_{i,f} \operatorname{Spec} A_i^{(f)}$, and $A_i^{(f)}$ is finitely presented over B; we can assume the relations over B_f have coefficients in B since f is a unit, and then we get

$$\begin{aligned}A_i^{(f)} &\cong B_f[x_1, \ldots, x_n]/(r_1, \ldots, r_m) \\ &\cong B[x_1, \ldots, x_n, z]/(r_1, \ldots, r_m, zf - 1).\end{aligned}$$

7.3.U. We show that being locally of finite presentation is affine-local on the source by showing that finite-presentation of B-algebras is affine-local; suppose $(f_1, \dots, f_r) = 1$ in a B-algebra A.

- If A is finitely presented, then A_{f_i} is finitely presented by adjoining a generator z with the relation $zs_i = 1$ where s_i is a polynomial expressing $f_i \in A$.
- If each A_{f_i} is finitely presented, then A is finitely generated by 5.3.3. Say $a_1, \dots, a_n \in A$ generate A over B. Adjoining $1/f_i$ gives a generating set for A_{f_i}, so from the lemma below we get a finite presentation
$$A_{f_i} \cong B[x_1, \dots, x_n, z]/(r_1^{(i)}, \dots, r_m^{(i)})$$
where $x_j \mapsto a_j/1$ and $z \mapsto 1/f_i$. Taking a polynomial expression $f_i = s_i(a_1, \dots, a_n)$, we have $zs_i - 1 \in (r_1^{(i)}, \dots, r_m^{(i)})$, so
$$\begin{aligned}(r_1^{(i)}, \dots, r_m^{(i)}) &\supseteq (s_i^N r_1^{(i)}, \dots, s_i^N r_m^{(i)}, zs_i - 1),\\ &\supseteq (z^N s_i^N r_1^{(i)}, \dots, z^N s_i^N r_m^{(i)}, zs_i - 1)\\ &= (r_1^{(i)}, \dots, r_m^{(i)}, zs_i - 1) = (r_1^{(i)}, \dots, r_m^{(i)}).\end{aligned}$$
Hence $(r_1^{(i)}, \dots, r_m^{(i)}) = (s_i^N r_1^{(i)}, \dots, s_i^N r_m^{(i)}, zs_i - 1)$, and by elimination we can get relations of the form $(r_1^{(i)}, \dots, r_m^{(i)}, zs_i - 1)$ with $r_k^{(i)} \in B[x_1, \dots, x_n]$. Using an even larger value for N, we can ensure $r_k^{(i)}(a_1, \dots, a_n) = 0$ in A, since the relation in A_{f_i} becomes an equality after multiplication by some power of f_i. Moreover we can get this to hold uniformly in i, so
$$\begin{aligned}A_{f_i} &\cong B[x_1, \dots, x_n, z]/(\{r_k^{(i)}\}_k, zs_i - 1)\\ &\cong B[x_1, \dots, x_n, z]/(\{r_k^{(j)}\}_{jk}, zs_i - 1)\\ &\cong \big(B[x_1, \dots, x_n]/(\{r_k^{(j)}\}_{jk})\big)[z]/(zs_i - 1)\\ &\cong \big(B[x_1, \dots, x_n]/(\{r_k^{(j)}\}_{jk})\big)_{s_i}.\end{aligned}$$
If $S = B[x_1, \dots, x_n]/(\{r_k^{(j)}\}_{jk})$, this means the surjection $S \to A$ becomes an isomorphism when localizing at each $s_i \in S$. Since $(f_1, \dots, f_r) = 1$ in A, we can add a relation ensuring $(s_1, \dots, s_n) = 1$ in S. Then $S \to A$ is an isomorphism, since it is an isomorphism locally, so A is finitely presented.

Lemma: If $B \to A$ is a finitely presented algebra, and $\phi : B[x_1, \dots, x_n] \to A$ is a surjection of B-algebras, then $\ker \phi$ is finitely generated.

Proof: Suppose $A = B[y_1, \dots, y_m]/(r_1, \dots, r_k)$. By surjectivity there is some $p_i \in \phi^{-1}(y_i)$, and then $r_1(p_1, \dots, p_m), \dots, r_n(p_1, \dots, p_m)$ generate $\ker \phi$.

7.3.V. Since the notion is affine-local on the target, and distinguished open sets form a base, it suffices to observe that $A_f \cong A[z]/(zf - 1)$ is finitely presented over A.

7.3.W. It suffices to show that a composition of two finitely presented algebras is finitely presented. Say $B = A[x_1, \dots, x_n]/(r_1, \dots, r_j)$ and $C = B[y_1, \dots, y_m]/(s_1, \dots, s_k)$. Then $s_i(y_1, \dots, y_m)$ is given by some $S_i \in A[x_1, \dots, x_n, y_1, \dots, y_m]$ by expanding the coefficients using the presentation of B, and hence
$$C = A[x_1, \dots, x_n, y_1, \dots, y_m]/(r_1, \dots, r_j, S_1, \dots, S_k)$$
is finitely presented.

7.4.A. Firstly, note that (i) and (iii) ensure the presence of open and closed sets, so (ii) ensures the presence of locally closed sets. Then since finite unions are the complements of finite intersections of complements, we must have every finite union of locally closed sets. In fact these give the whole family:

- Open sets are locally closed, so (i) is satisfied.

- A finite intersection of finite unions of locally closed sets is a finite union of finite intersections of locally closed sets, and each intersection is locally closed since openness and closedness are preserved by finite intersections, so (ii) is satisfied.
- The complement of a finite union of locally closed sets is a finite union of locally closed sets:

$$\left(\bigcup_{i=1}^{n} U_i \cap F_i\right)^c = \bigcup_{I \subseteq \{1,\ldots,n\}} \left[\bigcap_{i \in I} U_i^c \cap \bigcap_{i \notin I} F_i^c\right].$$

Moreover any finite union of locally closed sets is also a finite disjoint union of locally closed sets:

$$\bigcup_{i=1}^{n}(U_i \cap F_i) = \coprod_{\substack{I,J \subseteq \{1,\ldots,n\} \\ I \cap J \neq \emptyset}} \left[\bigcap_{i \in I} U_i \cap \bigcap_{j \in J} F_j \cap \bigcap_{i \notin I} U_i^c \cap \bigcap_{j \notin J} F_j^c\right].$$

Hence we have

$$\begin{aligned}\{\text{constructible sets}\} &= \{\text{finite unions of locally closed sets}\} \\ &= \{\text{finite disjoint unions of locally closed sets}\}.\end{aligned}$$

7.4.B. By 7.4.A, the generic point forms a constructible set iff it forms a locally closed set. Since it is dense, this is iff it forms an open set. But its complement is not closed, since no non-zero ideal is contained in every non-zero prime (no non-zero polynomial has infinitely many irreducible factors).

7.4.C. (a) Clearly closed sets are stable under specialization. Suppose $Z \subseteq X$ is a constructible set in a Noetherian space X, and Z is stable under specialization. By 7.4.A we have a finite union $Z = \cup_i (U_i \cap F_i)$ with U_i open and F_i closed. Since X is Noetherian we may assume F_i is irreducible, and since X is a scheme there is a generic point $g_i \in F_i$. Moreover we can assume $U_i \cap F_i \neq \emptyset$, so $g_i \in U_i$. Then $F_i \subseteq Z$ since Z is stable under specialization, so $Z = \cup_i F_i$ is closed.

(b) Its complement is again constructible, so by (a) it is open iff its complement is stable under specialization. This is equivalent to being stable under generization by contraposition.

7.4.D. For the affine case $X = \operatorname{Spec} A$, it suffices to show that all the generators are integral. If not, we get an inclusion $k[x] \to A$, i.e. a dominant map $\operatorname{Spec} A \to \mathbb{A}^1_k$. Since $\operatorname{Spec} A$ is finite, the image of this map is a finite dense set. Removing closed points preserves constructibility, but must result in the non-constructible set $\{[0]\}$, contradicting Chevalley's Theorem.

For general X, we have an open cover by affines, which are then finite over k, and hence discrete by 7.3.H. Then X is discrete, so X is already affine.

7.4.E. By 5.1.E fibres over closed points contain closed points, so surjectivity implies surjectivity on closed points. For the converse, it suffices to show that non-empty constructible sets have closed points, by applying Chevalley's Theorem to the complement of the image. We can restrict to locally closed sets, and further to closed subsets of affine (in particular, quasicompact) open subschemes. These have closed points by 5.1.E, which are closed in the original scheme by 5.3.F.

7.4.F. Suppose M is a finitely-generated B module. If $m_1, \ldots, m_n$ is a generating set, we can take $m_1, \ldots, m_r$ to be a basis for $M_{(0)}$ over $K(B)$. Then let f be the product of the denominators used to express $m_{r+1}, \ldots, m_n$ as linear combinations of $m_1, \ldots, m_r$. By construction $m_1, \ldots, m_r$ generate M_f over B_f, and they are linearly independent so M_f is a free B_f-module.

7.4.G. Firstly, if A satisfies (†) then so does A/I for any ideal I, since A/I-modules are just A-modules annhilated by I. Now any finitely-generated B-algebra is of the form $B[x_1, \ldots, x_n]/I$, and if we prove that (†) transfers from A to $A[T]$ we would get that $B[x_1, \ldots, x_n]$ satisfies (†) from 7.4.A and induction, and hence the quotient $B[x_1, \ldots, x_n]/I$ also satisfies (†).

7.4.H. We have $TM_{n-1} \subseteq M_n$ so this map is well-defined, and it is a surjection since $M_{n+1} = M_n + TM_n$.

7.4.I. We have an ascending chain of submodules $\ker\psi_1 \subseteq \ker(\psi_2\psi_1) \subseteq \cdots \subseteq M_1$, which stabilizes since M_1 is finitely-generated over A. Hence for $n \gg 0$ we get $\ker(\psi_n\cdots\psi_1) = \ker(\psi_{n-1}\cdots\psi_1)$, so ψ_n is injective since $\psi_{n-1}\cdots\psi_1$ is surjective.

7.4.J. By assumption there is an f_i with $(M_{i+1}/M_i)_{f_i}$ free over B_{f_i}, and if $f_i \mid f$ then $(M_{i+1}/M_i)_f$ is free over B_f by exactness of localization. Hence we can take f to be the product of the f_i's for the values of i before ψ_i becomes an isomorphism.

7.4.K. We can take ϕ_1 to be the identity on M_1, choose isomorphisms $p_n : M_n \oplus M_{n+1}/M_n \to M_{n+1}$ extending $M_n \hookrightarrow M_{n+1}$, and define $\phi_{n+1} = p_n \circ (\phi_n \oplus \mathrm{id})$. Then the ϕ_n's give a commuting diagram,

$$\begin{tikzcd}
\cdots \arrow[r, hook] & \oplus_{i=0}^{n-1} M_{i+1}/M_i \arrow[r, hook] \arrow[d, "\phi_n"] & \oplus_{i=0}^{n} M_{i+1}/M_i \arrow[r, hook] \arrow[d, "\phi_{n+1}"] & \cdots \\
\cdots \arrow[r, hook] & M_n \arrow[r, hook] & M_{n+1} \arrow[r, hook] & \cdots
\end{tikzcd}$$

and hence we get a map $\phi = \varinjlim \phi_n$ from $\varinjlim \oplus_{i=0}^{n-1} M_{i+1}/M_i = \oplus_{i=0}^{\infty} M_{i+1}/M_i$ to $\varinjlim M_n = M$. Moreover ϕ is an isomorphism, since filtered colimits are exact and each ϕ_n is an isomorphism.

7.4.L.