VE215 RC3

Erdao Liang, Chongye Yang

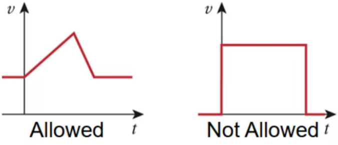
UM-SJTU JI

June 13, 2023

Overview

Capacitors

1. **Open Circuit Property** When the voltage across a capacitor is not changing with time (**DC steady state**), **the capacitor could be treated as an open circuit**.
2. **Continuity property** The voltage on a capacitor must be continuous.



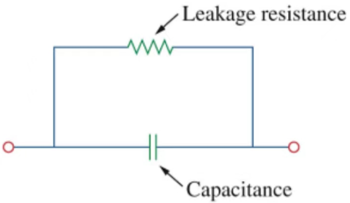
3. **Capacitors IV relationship**

$$i = C\frac{dv}{dt}$$

property 2 can be intuitively shown be property 3. If the voltage across the capacitor is not continuous, say $\frac{dv}{dt} = \infty$, which will cause i to be infinity.

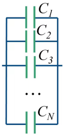
Capacitors

1. **An ideal capacitor will not dissipate energy.** It takes power from the circuit when storing energy in its electric field and returns previously stored energy when delivering power to the circuit.
2. **A real capacitor has a large leakage resistance**



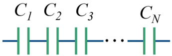
Capacitors in parallel & in series

- **capacitors in parallel**



$$C_{eq} = C_1 + C_2 + C_3 + ... + C_N$$

- **capacitors in series**



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4} + ... + \frac{1}{C_N}$$

Energy stored in Capacitors

The instantaneous power delivered to the capacitor is

$$p = vi = v(C\frac{dv}{dt})$$

Therefore, **the total energy** stored in the capacitor is

$$w = \frac{1}{2}CV^2$$

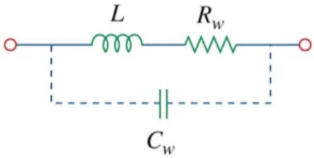
Inductors

1. **Short Circuit Property** When the current through an inductor is not changing with time (**DC steady state**), **the inductor could be treated as a short circuit in the circuit**.
2. **Continuity property** The current through a capacitor must be continuous.
3. **Inductor IV relationship**

$$v = L\frac{di}{dt}$$

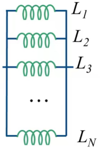
Inductors

1. **An ideal inductor will not dissipate energy.** It takes power from the circuit when storing energy in its magnetic field and returns previously stored energy when delivering power to the circuit.
2. **A real inductor has a significant winding resistance and a small winding capacitance**



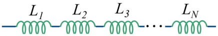
Inductors in parallel & in series

- **inductors in parallel**



$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4} + ... + \frac{1}{L_N}$$

- **inductors in series**



$$L_{eq} = L_1 + L_2 + L_3 + L_4 + ... + L_N$$

Energy stored in Inductors

The instantaneous power delivered to the inductor is

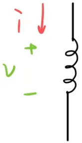
$$p = vi = (L\frac{di}{dt})i$$

Therefore, **the total energy** stored in the inductor is

$$w = \frac{1}{2}Li^2$$

Summary of Capacitors and Inductors

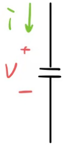
	Capacitor	Inductor
Electric/magnetic	q	ψ
	q=Cv	ψ=Li
i-v (or v-i) relation	i=C×dv/dt	v=L×di/dt
energy	1/2Cv^2	1/2Li^2



$v = L\frac{di}{dt}$

i increase

→ charging the inductor



$i = C\frac{dv}{dt}$

v increase

→ charging the capacitor

Overview

Capacitors and Inductors

First-Order Circuit

Second-Order Circuit

Duality

Introduction

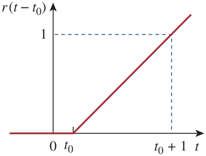
Definition: a first-order circuit is a circuit that contains **only ONE** capacitor/inductor after circuit simplification.
Motivation: we want to investigate how the circuit responses if we

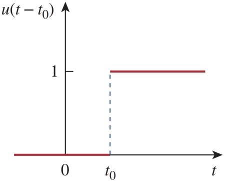
- Store energy to capacitor/inductor
- Let the capacitor/inductor releases energy

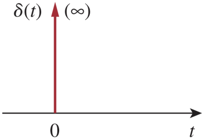
	Only one Capacitor	Only one inductor
Store energy	Step input RC	Step input RL
Release energy	Source free RC	Source free RL

Singularity Functions

Unit ramp	**Unit step**	**Unit impulse**
$r(t) = \begin{cases} 0, t \leq 0 \\ t, t > 0 \end{cases}$	$u(t) = \begin{cases} 0, t \leq 0 \\ 1, t > 0 \end{cases}$	$\delta(t) = \begin{cases} 0, t \neq 0 \\ \text{Undef.}, t = 0 \end{cases}$





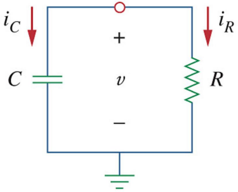


Give a nice way to represent "Switch on/off" of the sources/part of circuits.

$$\delta(t) \xrightarrow{\int} u(t) \xrightarrow{\int} r(t)$$

Source-Free Circuits (I) Response

Source-free RC

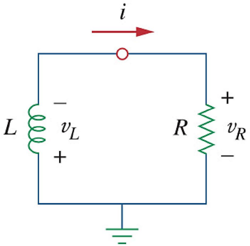


Voltage: $v = v_0 e^{-t/RC}$

Time constant: $\tau = RC$

Current: $i_R = \frac{v}{R} = \frac{v_0}{R} e^{-t/\tau}$
Power: $p = vi_R = \frac{v_0^2}{R} e^{-2t/\tau}$
Energy: $w_R = \int_0^t p dt = \frac{1}{2} C v_0^2$

Source-free RL

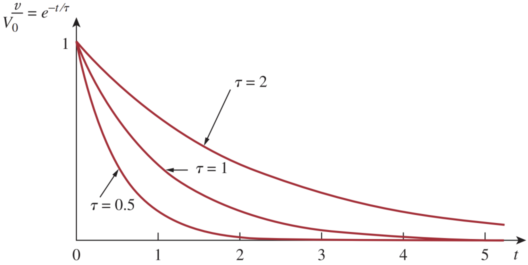


Current: $i = i_0 e^{-t/(L/R)}$

Time constant: $\tau = L/R$

Voltage: $v_R = iR = \frac{i_0}{R} e^{-t/\tau}$
Power: $p = v_R i = i_0^2 R e^{-2t/\tau}$
Energy: $w_R = \int_0^t p dt = \frac{1}{2} L i_0^2$

Source-Free Circuits (II) Time Constant

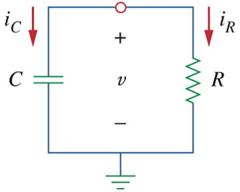


	Source-free RC	Source-free RL
Time constant	$\tau = RC$	$\tau = L/R$
Relation to initial decay rate	$\frac{d}{dt}(\frac{v}{v_0}) = -1/\tau$	$\frac{d}{dt}(\frac{i}{i_0}) = -1/\tau$

- Time required for the response to decay to a factor of $1/e$ or 36.8% of its initial value
- Indicates the initial decaying rate
- Assume complete decay after 5τ

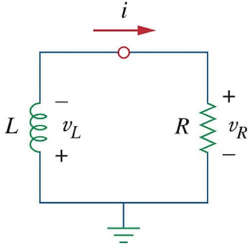
Source-Free Circuits (III) General Steps

Source-free RC



(1) Find initial voltage v_0
(2) Find time constant $\tau = RC$
(3) Obtain v_c, then i_c, v_R, i_R

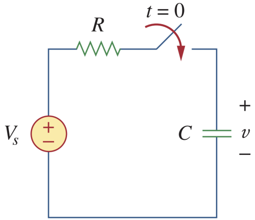
Source-free RL



(1) Find initial voltage i_0
(2) Find time constant $\tau = L/R$
(3) Obtain i_L, then v_L, v_R, i_R

Circuits with Step Input (I) Response

Step-input RC

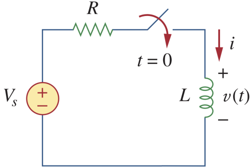


Initial condition:
$v(0^+) = v(0^-) = V_0$

Equation:
(KVL) $(C\frac{dv}{dt}R + v = V_s)$

Response:
$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}$

Step-input RL



Initial condition:
$i(0^+) = i(0^-) = I_0$

Equation:
(KCL) $iR + L\frac{di}{dt} = V_s$

Response:
$i(t) = \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-t/\tau}$

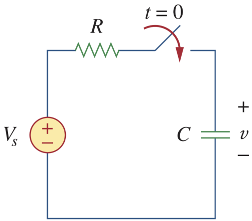
Circuits with Step Input (II) Interpretation of Response

There are three ways to look at the result.
Take $v(t) = V_s + (V_0 - V_s)e^{-t/\tau}$ as example,

Interpretation	First component	Second component
$v(t) = v_n(t) + v_f(t)$	$v_n(t) = (V_o - V_s)e^{-t/\tau}$ Natural response	$v_f(t) = V_s$ Forced response
$v(t) = v_t(t) + v_{ss}(t)$	$v_t(t) = (V_o - V_s)e^{-t/\tau}$ Temporary response	$v_{ss}(t) = V_s$ Steady-state response
$v(t) = v_{zp}(t) + v_{zs}(t)$	$v_{zp}(t) = V_0 e^{t/\tau}$ Zero-input response	$v_{zs}(t) = (1 - e^{-t/\tau})V_s$ Zero-state response

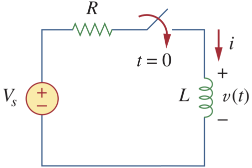
Circuits with Step Input (III) General Steps

Step-input RC



(1) Find initial voltage $v(0^+)$
(2) Find final voltage $v(\infty)$
(3) Find time constant

Step-input RL



(1) Find initial current $i(0^+)$
(2) Find find current $i(\infty)$
(3) Find time constant

General Formula for First-Order Circuits

General formula for RC:

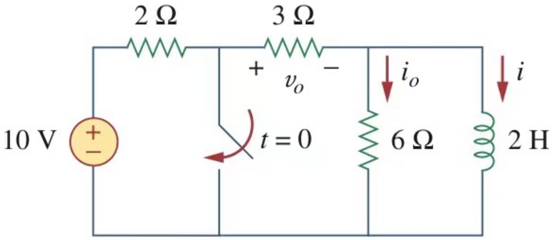
$$v(t) = v(\infty) + \left[v(0^+) - v(\infty)\right] e^{-t/\tau}$$

General formula for RL:

$$i(t) = i(\infty) + \left[i(0^+) - i(\infty)\right] e^{-t/\tau}$$

Exercise

Find i_0, v_o and i for all time, assuming that the switch was open for a long time.



Overview

Introduction

Definition: a second-order circuit is a circuit that consists of resistors and **TWO** capacitors/inductors after circuit simplification.

Workflow to solve 2nd-order circuits:

Find initial and final values and its derivative

$\downarrow$

List differential equation and find its solution

$\downarrow$

Use initial and final values to determine the coefficients in the solution

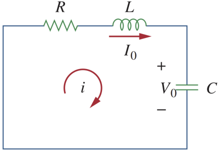
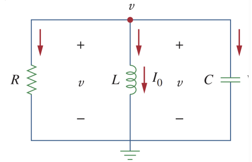
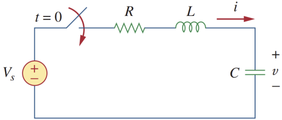
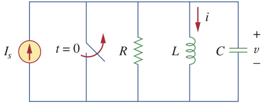
Initial and Final Values

- $v(0)$,$i(0)$,$v(\infty)$,$i(\infty)$: same method as the first-order circuit.
- $dv(0)/dt = I_C/C$ (Trick here is to use the property that the current across an inductor cannot change abruptly.)
- $di(0)/dt = V_L/L$ (Trick here is to use the property that the voltage of a capacitor cannot change abruptly.)

Caution

Please do care about the polarity when calculating the initial derivatives! You should always remember that current flows from high voltage to low voltage.

Basic RLC Circuits

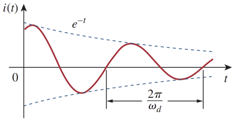
	Series connection	Parallel connection
Source-free	(By KVL) $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$  	(By KCL) $\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} v = 0$ 
Step input	(By KVL) $\frac{d^2 v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC} v = V_s$  	(By KCL) $\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{1}{LC} i = I_s$ 

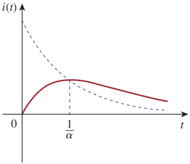
Solving 2nd-Order Differential Equations

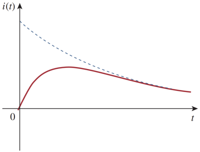
Through analysis we will obtain the differential equation for x,

- For source-free circuits, $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ (homogeneous, 2nd-order, const coefficient)
- For step input circuits, $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = X_0$ (non-homogeneous, 2nd-order, const coefficient)

Underdamped	Critically damped	Overdamped
$b^2 - 4ac < 0$	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$
No root	One root s	Two roots s_1, s_2







Solving 2nd-Order Differential Equations

For $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ (homogeneous),

- No root:

$$r = -\frac{b}{2a}, \omega = \frac{\sqrt{4ac - b^2}}{2a}$$

$$x(t) = e^{rt}(C_1 \sin \omega t + C_2 \cos \omega t)$$

- One root s:

$$x(t) = (C_1 + C_2 t)e^{st}$$

- To roots s_1, s_2:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

For $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = X_0$ (non-homogeneous), adopt

$$x(t) = x_{homogeneous}(t) + x_{particular}(t)$$

General Steps for Second-Order Circuits

What we want: $v(t)$ or $i(t)$ of some part of the circuit

- For source-free circuit:
  1. Draw the circuit for $t < 0$, obtain $i(0^+)$ and $v(0^+)$
  2. Draw the circuit for $t > 0$, list equations to obtain $\frac{di(0^+)}{dt}$ or $\frac{dv(0^+)}{dt}$
  3. Get the differential equation for the variable we want to study
     $\longrightarrow$ Judge how many roots in its characteristic equation
     $\longrightarrow$ Select the form of solution (with two coefficients unsolved)
  4. Use the initial conditions to solve the two coefficients C_1, C_2
- For step-input circuit:
  1. Follow the same steps to obtain the differential equation
  2. **Solve its corresponding homogeneous differential equation** $y_{homogeneous}$ (with two coefficients unsolved)
  3. Find a **constant** particular solution, i.e. $y_{particular} = C$ such that $a \times 0 + b \times 0 + c \times C = X_0 \Rightarrow C = X_0/c$
  4. General solution $x = x_{homogeneous} + x_{particular}$
  5. Use the initial conditions to solve the two coefficients C_1, C_2

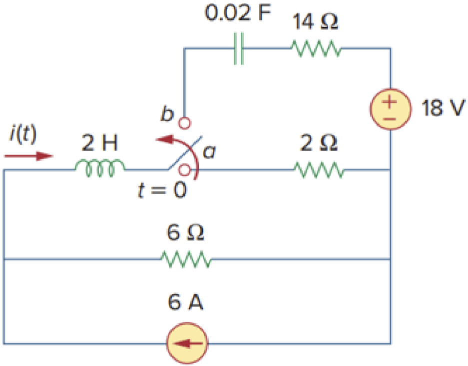
General Steps for Second-Order Circuits

Tips for finding the differential equation:

- Start from KCL or KVL?
  Better to try KVL in a scenario similar to series connection, and to try KCL in a scenario similar to parallel connection
- Take advantage of the property of capacitor & inductor:
  - There might be both v and i in the original equation, but only one of them is desired.
  - So consider using $i = C\frac{dv}{dt}$ and $v = L\frac{di}{dt}$ to "kill" one of them

Exercise

The switch in the circuit has been in position for a long time. At $t = 0$ the switch move instantaneously to position b. Find $i(t)$.



Overview

Dual Pairs

Resistance R	$\longleftrightarrow$	Conductance G
Inductance L	$\longleftrightarrow$	Capacitance C
Voltage v	$\longleftrightarrow$	Current i
Node	$\longleftrightarrow$	Mesh
Serie path	$\longleftrightarrow$	Parallel path
Open circuit	$\longleftrightarrow$	Short circuit
KVL	$\longleftrightarrow$	KCL
Thevenin	$\longleftrightarrow$	Norton

Steps to Draw Dual Circuits

- Place a node at the center of each mesh of the given circuit. Place the reference node (the ground) of the dual circuit outside the given circuit.
- Draw lines between the nodes such that each line crosses an element. Replace that element by its dual.
- To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the non-reference node.

References

1. 2023 Summer VE215 slides, Rui Yang
2. Fundamentals of Electric Circuits, 5th e, Sadiku, Matthew
3. 2022 Fall RC4, Zhiyu Zhou
4. 2022 Fall Mid RC, Zhiyu Zhou, Yifei Cai, Yuxuan Peng

Thank you!