CSCE 222-200, Discrete Structures for Computing, Honors
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Homework 3

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Instructions:

- The exercises are from the textbook. MAKE SURE YOU HAVE THE CORRECT EDITION! You are encouraged to work extra problems to aid in your learning; remember, the solutions to the odd-numbered problems are in the back of the book.
- Each exercise is worth 5 points.
- Grading will be based on correctness, clarity, and whether your solution is of the appropriate length.
- Always justify your answers.
- Don't forget to acknowledge all sources of assistance on the cover sheet, and write up your solutions on your own.
- *Turn in your pdf file on Canvas by 3:00 PM, Wednesday, October 6.*

LaTeX hints: Read this .tex file for some explanations that are in the comments.

Math formulas are enclosed in $ signs, e.g., `$x + y = z$` becomes $x + y = z$.

Logical operators: $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$.

Here is a truth table using the "tabular" environment:

p	$\neg p$
T	F
F	T

**** Delete the instructions and the LaTeX hints in your solution. ****

Exercises for Section 2.4 (pp. 177–179):

32(c): 21215

36): The result is $1 - \frac{1}{n+1}$

Exercises for Section 2.5 (pp. 186–187):

Clarification for 2(e) and 4(e): To exhibit a one-to-one correspondence, it's sufficient to give a diagram of the pattern; if you provide the actual function and prove it is a bijection, you can get extra credit.

2(e): The result is countably infinite. Lining upt the elements of the set like so:

$$1 : (2, 1), 2 : (3, 1), 3 : (2, 2), 4 : (3, 2), 5 : (2, 3), 6 : (3, 3)...$$

We can create a function from each integer that creates an element of the cartesian product.

$$f(n) = (2 + (n + 1) \mod 2, n - (n + 1) \mod 2)$$

The function is one-to-one. $(f(a) = f(b)) \to a = b$. If $a - (a + 1) \mod 2 = b - (b + 1) \mod 2$, then $a = b$. Even if a and b have the same odd-even parity, they have to be equal. The function can be proven that it is onto. $\forall x \in S \exists n(f(n) = x)$.

4(a): The function should follow a pattern of $f(1) = 1, f(2) = 2, f(3) = 4, f(4) = 5, f(5) = 7, f(6) = 8$

(A): Combining two countable sets, even if they are countably infinite sets, should always result in a countable set.

(B): If there exists a one-to-one function from set A to set B and set B to set A, then A and B must have the same cardinality.

10a: A is the set of all real numbers, B is the set of all real numbers as well.
10b: A is the set of real numbers less than or equal to 1, B is the set of all real numbers less than 1.
10c: A is the set of real numbers, B is the set of real numbers bigger than 5.

34a: Showing that $f(x) = \frac{2x-1}{2x(1-x)}$ is one-to-one:
The derivative is $\frac{2x^2-2x+1}{2x^2(1-x^2)}$
The derivate is always positive for x in the domain $(0, 1)$, Hence the function is always increasing and should be one-to-one.

34b: $f(x)$ is a function that reaches positive and negtive infinity in the domain $0 \leq x \leq 1$ as the denominator approaches 0 when x approaches 0 and 1.

38: Each element in the domain $(0, 1)$ can be written in the form $0.d_1d_2d_n$ and has one-to-one corrseponence to a function in the set of functions that map a positive integer to the set of integers 0 to 9. The set of real numbers in the domain $(0, 1)$ is uncountable and hence the set of functions must also be uncountable.

Exercises for Section 3.1 (pp. 213–216):

28:

$arr \leftarrow [a_1, a_2, ..., a_n]$
$left \leftarrow 1, right \leftarrow n$
$m2 \leftarrow 0$
while $left < right$ **do**
  $m2 \leftarrow \left\lfloor \frac{l+r}{2} \right\rfloor$
  $m1 \leftarrow \left\lfloor \frac{l+m2}{2} \right\rfloor$
  $m3 \leftarrow \left\lfloor \frac{m2+r}{2} \right\rfloor$
  if $val == arr[m1]$ **then**
    return m1
  end if
  if $val == arr[m2]$ **then**
    return m2
  end if
  if $val == arr[m3]$ **then**
    return m3
  end if
  if $val < arr[m1]$ **then**
    $right \leftarrow m1$
  end if
  if $val < arr[m2]$ **then**
    $right \leftarrow m2$
    $left \leftarrow m1$
  end if
  if $val < arr[m3]$ **then**
    $right \leftarrow m3$
    $left \leftarrow m2$

end if
if $val < arr[right]$ **then**
$left \leftarrow m3$
end if
end while
return -1

Exercises for Section 3.2 (pp. 228–231):

28a: $f(x) = 10$ is not $\Omega(x)$ as there is no such C, k pair where $10 \geq Cx$ for all $x > k$. Because 10 is a constant, x eventually exceeds it.
28b: $f(x) = 3x + 7$ is $\Omega(x)$ as the function is greater for C = 1, and k = 1. Since it is also $O(x)$, the function is $\Theta(x)$.
28c: $f(x) = x^2 + x + 1$ is $\Omega(x)$, as the function's leading term has a higher exponent. It is actually $\omega(x)$. However, the function is not $O(x)$, so therefore, it is not $\Theta(x)$.
28d: $f(x) = 5logx$ is not $\Omega(x)$ as it's a smaller function since it's a logarithm. Therefore it's not $\Theta(x)$.
28e: $f(x) = \lfloor x \rfloor$ is not $\Omega(x)$ as no matter what C and k values are chosen, $f(x)$ will not be greater than or equal to x since decimal values would be floored. Hence, it is also not $\Theta(x)$
28f: $f(x) = \left\lfloor \frac{x}{2} \right\rfloor$ is also not $\Omega(x)$ as no matter what C and k values are chosen, $f(x)$ will not be greater than or equal to x since decimal values would be floored. Hence, it is also not $\Theta(x)$

50: The function $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is $O(g(x))$ where $g(x) = x^n$. There exists at least one C, k pair such that $f(x) \leq C * g(x)$ for $x > k$. This is intuitive as we can say that $C \geq a_n$, and that it should also account for the remaining n terms in $f(x)$. Since $g(x)$ is function that increases quicker than a function with a leading exponenet less than n, we know that we can make C and k appropriately high enough such that $g(x) > (f(x) - a_n x^n)$.

Exercises for Section 3.3 (pp. 241–244):

4: *Justify your answer.*
$4log_2 n + 2$. Each iteration of the while loop has 4 operations, two arithmetic and two assignments. It gets executed log_2 times and there are two initial assignments.

14a: The loop can be expanded to show all the calculations.
For i = 1, $y = 3 * 2 + 1$
For i = 2, $y = 7 * 3 + 1$
For i = 3, $y = 22 * 3 + 0$

The result is $y = 66$.
14b: There are n multiplication and n addition operations.

26: There are 4 preliminary assignment operations. The loop will iterate approximately $log_4 n$ times. Each iteration does 26 opertions. Each comparison of the loop is an operation. There are 12 operations to calculate the indices of the middle values. There are 6 comparison that use 12 operations due to the indexing Hence the complexity is $4 + 26log_n$.

Exercises for Section 5.1 (pp. 350–354):

4a: $P(1)$ is $1^3 = (\frac{1(1+1)}{2})^2$ which is true. **4b:** $1^3 = (\frac{1(1+1)}{2})^2$ simplifies to $1 = 1$ which is a true statement. This proved that the base case is true.
4c: The inductive hypothesis is that $P(k)$ is true. This is the statement

$$\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$$

4d: It should be shown that $P(k) \to P(k+1)$
4e: The following statement must be shown.

$$1^3 + 2^3 + ... + k^3 + (k+1)^3 = (\frac{(k+1)(k+2)}{2})^2$$

The inductive hypothesis can be used to replace the first k terms. This yields the statement:

$$(\frac{k(k+1)}{2})^2 + (k+1)^3 = (\frac{(k+1)(k+2)}{2})^2$$

We can distribute the squares into their expressions.

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

We then multiply by four and divide by $(k+1)^2$.

$$k^2 + 4(k+1) = (k+2)^2$$

Expanding the both sides reveals that they are equal.

$$k^2 + 4k + 4 = k^2 + 4k + 4$$

Hence, we have proved that $P(k) \to P(k+1)$, so the statement $\forall n P(n)$ is true.
4f: Mathematical induction relies on the idea that a statement being true for a value k implies that is true for k + 1, the next integer. If we prove the statement for a base case such as n = 1, the smallest positive integer and that each case implies the next case is true, this effectively proves the statement for all integers. $((P(1)) \wedge (P(k) \to P(k+1)) \to \forall n P(n))$ where n is a positive integrer.

14: The goal is to prove the statement

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$$

Showing that the statement is true for the base case $n = 1$:

$$1(2^1) = (1-1)2^{1+1} + 2$$

Constructing the inductive hypothesis:

$$\sum_{j=1}^{k} j2^j = (k-1)2^{k+1} + 2$$

Attempting to prove $P(k) \to P(k+1)$

$$1 * 2^1 + 2 * 2^2 + 3 * 2^3 + ... + k * 2^k + (k+1) * 2^{k+1} = (k) * 2^{k+2} + 2$$

The inductive hypothesis can be used to replace the first k terms.

$$(k-1)2^{k+1} + 2 + (k+1) * 2^{k+1} = (k) * 2^{k+2} + 2$$

Subtracting the constants and expanding the left side reveals the follwing statement:

$$(k)2^{k+1} + (k) * 2^{k+1} + 2^{k+1} - 2^{k+1} = (k) * 2^{k+2}$$

Cancelling terms and simplifying the statements proves their equality:

$$2 * (k)2^{k+1} = (k) * 2^{k+2}$$

Hence, since $P(1)$ is true and $P(k) \rightarrow P(k+1)$, the statement $\forall n P(n)$ where n is a positive integer is true. **20:** The goal is to prove the statement

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$$

Showing that the statement is true for the base case $n = 7$:

$$3^7 < 7! \text{ is } 2187 < 5040$$

Constructing the inductive hypothesis:

$$3^k < k!$$

Attempting to prove $P(k) \rightarrow P(k+1)$

$$3^{k+1} < (k+1)!$$

We can split the left and right sides and use the inductive hypothesis:

$$3^k 3^1 < (k+1)(k)!$$

Since we know from the inductive hypothesis that $3^k < k!$ and that $3^1 < k+1$ since k is an integer greater than 6, the statement is true.
Hence, since $P(1)$ is true and $P(k) \rightarrow P(k+1)$, the statement $\forall n P(n)$ where n is an integer greater than 6 is true.
40: Showing that $(A_1 \cap A_2 \cap ... \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap ... \cap (A_n \cup B)$ is true for all n. We can see this relationship in a smaller case: $(A \cap B) \cup C$. If $k \epsilon (A \cap B) \cup C$ then k is an element of C or k is an element of $(A \cap B)$ or both. So we can say that k should be a member of A or C and B or C. This evaluates to $k \epsilon ((A \cap C) \cup (B \cap C))$. This is a proof of the distributive law and can be applied to any number of sets.