4.4

Problem

Solution

In pooling equilibria, both sender types play the same message.

1. Pooling on L :

- Sender strategy: Both t_1 and t_2 choose L.
  o Receiver strategy:
- Upon observing L, play d.
- Upon observing R (off-equilibrium), play d.
  o Beliefs:
- At L information set: $P(t_1) = 0.5, P(t_2) = 0.5$ (consistent with pooling).
- At R information set: $P(t_1) \leq 1/3$ (off-equilibrium, arbitrary but must justify playing d).
  o Equilibrium justification:
- On-path (L) : Receiver's expected payoff from u is $0.5 \times 1 + 0.5 \times 0 = 0.5$; from d is $0.5 \times 0 + 0.5 \times 1 = 0.5$. Receiver is indifferent, so playing d is optimal.
- Off-path (R) : With belief $P(t_1) \leq 1/3$, receiver's expected payoff from u is $2 \times P(t_1) \leq 2/3$, and from d is $1 - P(t_1) \geq 2/3$. Thus, playing d is optimal (strictly if $P(t_1) < 1/3$, weakly if $P(t_1) = 1/3$).
- Sender incentives:
- t_1 : Gets 2 from L (receiver plays d). Deviating to R : Receiver plays d, so payoff is 0 (worse than 2). No incentive to deviate.
- t_2 : Gets 1 from L (receiver plays d). Deviating to R : Receiver plays d, so payoff is 1 (same as equilibrium). Indifferent, so no strict incentive to deviate.

2. Pooling on R :

- Sender strategy: Both t_1 and t_2 choose R.
  o Receiver strategy:
- Upon observing R, play u.
- Upon observing L (off-equilibrium), play any action (e.g., u or d) depending on belief.
  o Beliefs:

- At R information set: $P\left(t_1\right)=0.5, P\left(t_2\right)=0.5$ (consistent with pooling).
- At L information set: Any belief $P\left(t_1\right)=q$ for $q \in[0,1]$, and receiver plays a best response to this belief.
- Equilibrium justification:
- On-path (R) : Receiver's expected payoff from u is $0.5 \times 2+0.5 \times 0=1$; from d is $0.5 \times 0+0.5 \times 1=0.5$. Playing u is strictly optimal ($1>0.5$).
- Off-path (L) : Receiver plays a best response to belief q. For example:
- If $q>0.5$, play u (since $q>1-q$).
- If $q<0.5$, play d (since $q<1-q$).
- If $q=0.5$, indifferent, can play either.
- Sender incentives:
- t_1 : Gets 2 from R (receiver plays u). Deviating to L : If receiver plays u, payoff is $1<2$; if receiver plays d, payoff is 2 (same as equilibrium). Thus, deviation not profitable.
- t_2 : Gets 1 from R (receiver plays u). Deviating to L : If receiver plays u, payoff is $0<1$; if receiver plays d, payoff is 1 (same as equilibrium). Thus, deviation not profitable.

In separating equilibria, sender types choose different messages.
3. Separating: t_1 plays L, t_2 plays R :

- Sender strategy: t_1 chooses L, t_2 chooses R.
  o Receiver strategy:
- Upon observing L, play u.
- Upon observing R, play d.
- Beliefs:
- At L information set: $P\left(t_1\right)=1, P\left(t_2\right)=0$ (consistent: only t_1 plays L).
- At R information set: $P\left(t_1\right)=0, P\left(t_2\right)=1$ (consistent: only t_2 plays R).
  o Equilibrium justification:
- Receiver best response:
- At L (Node_A1): u gives receiver $1, d$ gives 0 ; play u.
- At R (Node_B2): u gives receiver $0, d$ gives 1 ; play d.
- Sender incentives:
- t_1 : Plays L, receiver plays u, payoff is 1 . Deviating to R : Receiver plays d, payoff is $0<1$. No incentive.
- t_2 : Plays R, receiver plays d, payoff is 1 . Deviating to L : Receiver plays u, payoff is $0<1$. No incentive.

4. Separating: t_1 plays R, t_2 plays L :

- Sender strategy: t_1 chooses R, t_2 chooses L.
  o Receiver strategy:
- Upon observing R, play u.
- Upon observing L, play d.
  o Beliefs:
- At R information set: $P(t_1)=1, P(t_2)=0$ (consistent: only t_1 plays R).
- At L information set: $P(t_1)=0, P(t_2)=1$ (consistent: only t_2 plays L).
  o Equilibrium justification:
- Receiver best response:
- At R (Node_A2): u gives receiver $2, d$ gives 0; play u.
- At L (Node_B1): u gives receiver 0, d gives 1; play d.
- Sender incentives:
- t_1 : Plays R, receiver plays u, payoff is 2 . Deviating to L : Receiver plays d, payoff is 2 (same). Indifferent, but no strict incentive to deviate.
- t_2 : Plays L, receiver plays d, payoff is 1 . Deviating to R : Receiver plays u, payoff is 1 (same). Indifferent, but no strict incentive to deviate.

Summary of all pure-strategy PBEs:

- Pooling on L : Both types play L; receiver plays d on L and d on R with belief $P(t_1) \leq 1/3$ at R
- Pooling on R : Both types play R; receiver plays u on R and any action on L with any belief.
- Separating ($t_1 : L, t_2 : R$): Receiver plays u on L, d on R.
- Separating ($t_1 : R, t_2 : L$): Receiver plays u on R, d on L.

1. Pooling on L :

- Sender strategy: Both t_1 and t_2 send L.
- Receiver beliefs:
- After $L : P(t_1)=0.5, P(t_2)=0.5$ (consistent with pooling).
- After R (off-equilibrium): $P(t_1) \leq \frac{2}{3}$ (arbitrary but must support equilibrium).
- Receiver strategy:
- After L : Plays u (expected payoff $0.5 \times 0+0.5 \times 3=1.5>0.5 \times 1+0.5 \times 1=1$).
- After R : Plays u (optimal if $P(t_1) \leq \frac{2}{3}$ since $2(1-P(t_1)) \geq P(t_1)$).
- Incentive compatibility:
- t_1 : Gets 3 (plays L, receiver plays u). Deviating to R : Receiver plays u, payoff 0 (worse).
- t_2 : Gets 3 (plays L, receiver plays u). Deviating to R : Receiver plays u, payoff 1 (worse).

- Conclusion: This is a PBE if off-equilibrium belief satisfies $P\left(t_1 \mid R\right) \leq \frac{2}{3}$.

2. Pooling on R :

- Sender strategy: Both t_1 and t_2 send R.
- Receiver beliefs:

◦ After $R: P\left(t_1\right)=0.5, P\left(t_2\right)=0.5$ (consistent).

- After L (off-equilibrium): Any $P\left(t_1\right)=q \in[0,1]$.
- Receiver strategy:
- After R : Plays u (expected payoff $0.5 \times 0+0.5 \times 2=1>0.5 \times 1+0.5 \times 0=0.5$).
- After L : Plays best response to q :
- If $q<\frac{2}{3}$, plays u (since $3(1-q)>1$).
- If $q>\frac{2}{3}$, plays d (since $3(1-q)<1$).
- If $q=\frac{2}{3}$, indifferent.
- Incentive compatibility fails:
- t_1 : In equilibrium, gets 0 (plays R, receiver plays u). Deviating to L :
- If receiver plays u, gets $3>0$.
- If receiver plays d, gets $1>0$.
- Always profitable to deviate.
- Conclusion: No pooling equilibrium on R.

1. Separating: t_1 sends L, t_2 sends R

- Sender strategy: $t_1 \rightarrow L, t_2 \rightarrow R$.
- Receiver beliefs:
- After $L: P\left(t_1\right)=1, P\left(t_2\right)=0$ (consistent).

◦ After $R: P\left(t_1\right)=0, P\left(t_2\right)=1$ (consistent).

- Receiver strategy:
o After L : Plays d (payoff $1>0$ for u at A_1).
- After R : Plays u (payoff $2>0$ for d at B_2).
- Incentive compatibility:
- t_1 : Gets 1 (plays L, receiver plays d). Deviating to R : Receiver plays u, payoff 0 (worse).
- t_2 : Gets 1 (plays R, receiver plays u). Deviating to L : Receiver plays d, payoff 0 (worse).
- Conclusion: This is a PBE.

2. Separating: t_1 sends R, t_2 sends L

- Sender strategy: $t_1 \rightarrow R, t_2 \rightarrow L$.

- Receiver beliefs:
  o After $L: P(t_1)=0, P(t_2)=1$ (consistent).
- After $R: P(t_1)=1, P(t_2)=0$ (consistent).
- Receiver strategy:
- After L : Plays u (payoff $3>1$ for d at B_1).
- After R : Plays d (payoff $1>0$ for u at A_2).
- Incentive compatibility:
- t_1 : Gets 4 (plays R, receiver plays d). Deviating to L : Receiver plays u, payoff 3 (worse).
- t_2 : Gets 3 (plays L, receiver plays u). Deviating to R : Receiver plays d, payoff 2 (worse).
- Conclusion: This is a PBE.

Summary of All Pure-Strategy PBEs

1. Pooling on L :

- Both types send L.
- Receiver plays u after L and u after R.
- Belief after $R: P(t_1 \mid R) \leq \frac{2}{3}$.

2. Separating Equilibrium 1:

- t_1 sends L, t_2 sends R.
- Receiver plays d after L, u after R.
- Beliefs: Certainty at each information set.

3. Separating Equilibrium 2:

- t_1 sends R, t_2 sends L.
- Receiver plays u after L, d after R.
- Beliefs: Certainty at each information set.