2

(1)

Problem

> Are $\cup_{n=1}^{\infty} E_n$ and $\cap_{n=1}^{\infty} E_n$ events? If so, what do they represent?

Solution

Yes, they are events.

$\cup_{n=1}^{\infty} E_n$ is the event that one of them happens

$\cap_{n=1}^{\infty} E_n$ is the event that all of them happen

(2)

Problem

> Example: Three pairwise independent events that are not mutually independent.

Solution

Consider following setting:

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, E_1 = \{\omega_1, \omega_2\}, E_2 = \{\omega_1, \omega_3\}, E_3 = \{\omega_1, \omega_4\}$$

Then:

$$P(E_1 E_2) = \frac{1}{4}, P(E_1)P(E_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

same goes for $E_1 E_3, E_2 E_3$

However:

$$P(E_1 E_2 E_3) = \frac{1}{4}, P(E_1)P(E_2)P(E_3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

(3)

Problem

> Statement: If $F_1, \ldots, F_n$ are mutually exclusive and exhaustive $\left(\cup_{i=1}^n F_i = S\right)$, then: $P(E) = \sum_{i=1}^n P\left(E \mid F_i\right) P\left(F_i\right)$.

Proof

Notice:

$$P(E|F_i)P(F_i) = P(EF_i)$$

Since $F_i \cap F_j = \oslash, \forall i \neq j$

We can get:

$$EF_i \cap EF_j = E(F_i \cap F_j) = \oslash$$

Then:

$$P(E) = P(ES) = P(E(\cup_{i=1}^n F_i)) = \sum_{i=1}^n P(EF_i) = \sum_{i=1^n} P(E|F_i)P(F_i)$$

(4)

Problem

> Write Binomial Theorem

Solution

For any integers $n \geq 0$ and real numbers a, b :
$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$.

(5)

Problem

> Expectation of X if $Y = X + 1$ is Geometric

Solution

Since: $E[Y] = \frac{1}{p}$

Then: $E[X] = E[Y-1] = E[Y] - 1 = \frac{1}{p} - 1$

(6)

Problem

> Distribution of $X+Y$ for Independent Poisson Variables

Solution

Assuming: $p(x) = e^{-\lambda_1}\frac{\lambda_1^x}{x!}, p(y) = e^{-\lambda_2}\frac{\lambda_2^y}{y!},$

Then for $X+Y$

$$P(X+Y=z)$$
$$= \sum_{i=0}^{z} e^{-\lambda_1}\frac{\lambda_1^i}{i!}e^{-\lambda_2}\frac{\lambda_2^{z-i}}{(z-i)!}$$
$$= e^{-(\lambda_1+\lambda_2)}\sum_{i=0}^{z}\frac{\lambda_1^i\lambda_2^{z-i}}{i!(z-i)!}$$

Notice:

$$(\lambda_1+\lambda_2)^z = z!\sum_{i=0}^{z}\frac{\lambda_1^i\lambda_2^{z-i}}{i!(z-i)!}$$

Then:

$$P(X+Y=z) = e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^z}{z!}$$

Therefore:

$$X+Y \sim \mathrm{Poi}(\lambda_1+\lambda_2)$$

(7)

Problem

> Expected Number of Tosses Until First Head

Solution

Denote E as "First toss is Head",then:

$$
\begin{aligned}
&E(X) \\
&= E(X|E)P(E) + E(X|E^c)P(E^c) \\
&= 1 \cdot p + (\frac{1}{p} + 1) \cdot (1 - p) \\
&= \frac{1}{p}
\end{aligned}
$$

(8)

Problem

> Consider n independent trials, where each trial results in one of $1, \ldots, r$ with probabilities $p_1, \ldots, p_r$, respectively, such that $p_1 + \cdots + p_r = 1$. Let N_i denote the number of trials resulting in outcome i. The vector $(N_1, \ldots, N_r)$ follows a multinomial distribution. Using the conditional expectation formula, compute $\operatorname{Cov}(N_i, N_j)$

Solution

We have:

$$E[N_i] = np_i$$

Notice: $(N_i|N_j = y) \sim \mathrm{B}(n - y, \frac{p_i}{1 - p_j})$

Then:

$$E[N_i N_j | N_j = y] = y(n - y)\frac{p_i}{1 - p_j}$$

$$
\begin{aligned}
&E[N_iN_j] \\
&= E[E[N_iN_j|N_j]] \\
&= \sum_{y=0}^{n} y(n-y)\frac{p_i}{1-p_j} \cdot \binom{n}{y} p_j^y (1-p_j)^{n-y}
\end{aligned}
$$

Notice:

$\sum_{y=0}^{n} y\binom{n}{y} p_j^y (1-p_j)^{n-y} = E[N_j] = np_j$

$\sum_{y=0}^{n} y^2\binom{n}{y} p_j^y (1-p_j)^{n-y} = E[N_j^2] = (E[N_j])^2 + Var[N_j] = n^2p_j^2 + np_j(1-p_j)$

Then:

$$
\begin{aligned}
&E[N_iN_j] \\
&= \frac{p_i}{1-p_j}(n^2p_j - n^2p_j^2 - np_j(1-p_j)) \\
&= p_i(n^2p_j - np_j) \\
&= n(n-1)p_ip_j
\end{aligned}
$$

Then:

$$
\begin{aligned}
&Cov(N_j, N_j) \\
&= E[N_iN_j] - E[N_i]E[N_j] \\
&= n(n-1)p_ip_j - np_i \cdot np_j \\
&= -np_ip_j
\end{aligned}
$$

(9)

Problem

> 9. Let $X_1, X_2, \ldots$ be i.i.d. random variables, and N be a non-negative integer-valued random variable independent of the X-sequence. Define $S_0 = 0$ and $S_N = \sum_{i=1}^{N} X_i$ (a compound random variable). Assume N follows a Poisson distribution with mean λ.Assume X_i is interger random variable, $P\{X_1 = j\} = a_j, j > 0$. Moreover, $\mathrm{P}\{M = n\} = \frac{n\mathrm{P}\{N=n\}}{\mathrm{E}[N]}$. Find an expression for $P_n = P\left\{S_N = n\right\}, n = 0, 1, \ldots$

Solution

We have:

$$
\begin{aligned}
&P_n \\
&= \sum_{k=0}^{\infty} P(S_k = n) \cdot P(N = k) \\
&= \sum_{k=0}^{n} P(S_k = n) \cdot P(N = k) \\
&= \sum_{k=0}^{n} P(S_k = n) \cdot e^{-\lambda} \frac{\lambda^k}{k!}
\end{aligned}
$$

With MGF:

$$\phi(t) = E[e^{tS_N}] = \sum_{n=0}^{\infty} P_n e^{tn}$$

$$\phi'(t) = \sum_{n=0}^{\infty} nP_n e^{tn}$$

On the other hand:

$$
\begin{aligned}
&E[e^{tS_N}] \\
&= E[E[e^{tS_N}|N]] \\
&= E[(E[e^{tX_1}])^N] \\
&= e^{\lambda(E[e^{tX_1}]-1)} \\
&= e^{\lambda(\sum_{j=1}^{\infty}(a_j e^{tj})-1)}
\end{aligned}
$$

Then:

$$\phi'(t) = \lambda \sum_{j=1}^{\infty} (ja_j e^{tj}) \phi(t)$$

Replace $\phi(t)$ and $\phi'(t)$, we get:

$$\sum_{n=0}^{\infty} nP_n e^{tn} = \lambda \sum_{j=1}^{\infty} (ja_j e^{tj}) \sum_{n=0}^{\infty} P_n e^{tn}$$

To match the power of e, we get:

$$nP_n = \lambda \sum_{j=1}^{n} ja_j P_{n-j}$$

That is:

$$P_n = \begin{cases} e^{-\lambda} & \text{if } n = 0, \\ \frac{\lambda}{n} \sum_{j=1}^{n} j a_j P_{n-j} & \text{if } n \geq 1. \end{cases}$$

(10)

Problem

> 10. If N follows a binomial distribution with parameters r and p, find $P_n = P\left\{S_N = n\right\}, n = 0, 1, \ldots$.

Solution

Notice:

$(M-1) \sim \mathrm{B}(r-1, p)$

With the corollary:

$\mathrm{P}\left\{S_N = k\right\} = \frac{\mathrm{E}[N]}{k} \cdot \sum_{j=1}^{k} j \alpha_j \cdot \mathrm{P}\left\{S_{M-1} = k - j\right\}, k > 0$

We denote $P_r(k) = P(S_{N_{(r)}} = k)$, where $N(r) \sim \mathrm{B}(r, p)$, then:

$P_r(0) = (1-p)^r$

$P_r(k) = \dfrac{rp}{k} \sum_{j=1}^{k} j a_j P_{r-1}(k-j), k > 0$

Chapter 1

10

Problem

> Prove Bool's Inequality: $\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \sum_{i=1}^{n} \mathrm{P}\left(E_i\right)$

Solution

We prove by induction:

When $n = 1$, it is obvious that:

$P(E_1) \leq P(E_1)$

For $n - 1$, assume that:

$\mathrm{P}\left(\bigcup_{i=1}^{n-1} E_i\right) \leqslant \sum_{i=1}^{n-1} \mathrm{P}\left(E_i\right)$

Then:

$$
\begin{aligned}
& \mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) \\
& = \mathrm{P}\left(\bigcup_{i=1}^{n-1} E_i\right) + \mathrm{P}\left(E_n\right) - \mathrm{P}\left(\bigcup_{i=1}^{n-1} E_i \cap E_n\right) \\
& \leq \mathrm{P}\left(\bigcup_{i=1}^{n-1} E_i\right) + \mathrm{P}\left(E_n\right) \\
& \leq \sum_{i=1}^{n-1} \mathrm{P}\left(E_i\right) + \mathrm{P}\left(E_n\right) \\
& = \sum_{i=1}^{n} \mathrm{P}\left(E_i\right)
\end{aligned}
$$

12

Problem

> Let E and F be mutually exclusive events in the sample space of a certain experiment. Suppose the experiment is repeated until either E or F occurs. What is the sample space of this super experiment? Prove that the probability of event E occurring before event F is $\frac{\mathrm{P}(E)}{\mathrm{P}(E)+\mathrm{P}(F)}$

Solution

For the sample space:

$\Omega = \cup_{k=0}^{\infty} \{(E \cup F)^c\}^k \times \{E, F\}$

Denote G_E as "E happens at first experiment",G_F as "F happens at first experiment", H as the needed event. Notice that:

$$
\begin{aligned}
&P(H|(G_E \cup G_F)^c) = P(H) \\
&P(H) \\
&= P(H|G_E)P(G_E) + P(H|G_F)P(G_F) + P(H|(G_E \cup G_F)^c)P((G_E \cup G_F)^c) \\
&= 1 \cdot P(E) + 0 + P(H) \cdot (1 - P(E) - P(H))
\end{aligned}
$$

Then we get:

$$P(H) = \frac{P(E)}{P(E) + P(F)}$$

23

Problem

> For events $E_1, E_2, \cdots, E_n$ Prove:
>
> $$\mathrm{P}\left(E_1 E_2 \cdots E_n\right) = \mathrm{P}\left(E_1\right) \mathrm{P}\left(E_2 \mid E_1\right) \mathrm{P}\left(E_3 \mid E_1 E_2\right) \cdots \mathrm{P}\left(E_n \mid E_1 \cdots E_{n-1}\right)$$

Solution

We prove by induction:

For $n - 1$, it's obvious that:

$$P(E_1) = P(E_1)$$

Then, assume case $n - 1$ holds, that is:

$$P(E_1 E_2 \cdots E_{n-1}) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \cdots P(E_{n-1}|E_1 \cdots E_{n-2})$$

Notice:

$$P(E_1 E_2 \cdots E_n) = P(E_n|E_1 \cdots E_{n-1})P(E_1 \cdots E_{n-1})$$

Then:

$$
\begin{aligned}
&P(E_1 E_2 \cdots E_n) \\
&= P(E_n|E_1 \cdots E_{n-1})P(E_1 \cdots E_{n-1}) \\
&= P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \cdots P(E_n|E_1 \cdots E_{n-1})
\end{aligned}
$$

28

Problem

> If the occurrence of B makes A more likely to occur, does the occurrence of A make B more likely to occur?

Solution

Yes.

We translate the problem into:

Given: $P(A|B) > P(A)$,

Prove or not: $P(B|A) > P(B)$

Notice:

$$P(A|B)P(B) = P(AB) = P(B|A)P(A)$$

Rearrange it:

$$\frac{P(A|B)}{P(A)} = \frac{P(B|A)}{P(B)}$$

Given $P(A|B) > P(A)$

We get:

$$P(B|A) > P(B)$$

39

Problem

> Assume that stores A, B, and C have 50, 75, and 100 employees, respectively. Among them, 50%, 60%, and 70% are female. The probability of resignation is the same for all employees, regardless of gender. Now, an employee has resigned, and she is female. What is the probability that she worked at store C?

Solution

Denote E as "Employee is female", F as "Employee works in C, then:

$$
\begin{aligned}
&P(F|E) \\
&= \frac{P(EF)}{P(E)} \\
&= \frac{100 \times 70\%}{50 \times 50\% + 75 \times 60\% + 100 \times 70\%} \\
&= \frac{1}{2}
\end{aligned}
$$

Chapter 2

7

Problem

> Suppose a coin with a probability of 0.7 for landing heads is tossed 3 times. Let X denote the number of heads that appear in these 3 tosses. Determine the probability mass function of X.

Solution

Notice:

$$X \sim \mathrm{B}(3, 0.7)$$

Then:

$$
P(X = k) = \begin{cases} 0.027, & \text{if } k = 0, \\ 0.189, & \text{if } k = 1, \\ 0.441, & \text{if } k = 2, \\ 0.343, & \text{if } k = 3 \end{cases}
$$

23

Problem

> Negative binomial distribution: Continuously toss a coin with probability p of landing heads until r heads appear. Derive the probability that the number of tosses required, X, is n (where $n \geqslant r$):
>
> $\mathrm{P}\{X=n\}=\binom{n-1}{r-1} p^r(1-p)^{n-r}, \quad n \geqslant r$

Solution

Notice that we need $r-1$ heads in $n-1$ times, then:

$P\{X=n\}=\binom{n-1}{r-1} p^r(1-p)^{n-r}$

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Problem (a)

> If X is a non-negative integer-valued random variable, prove that: $\mathrm{E}[X]=\sum_{n=1}^{\infty} \mathrm{P}\{X \geqslant n\}=\sum_{n=0}^{\infty} \mathrm{P}\{X>n\}$.

Solution (a)

Notice:

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \mathrm{P}\{X \geqslant n\} \\
& =\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mathrm{P}\{X=k\} \\
& =\sum_{k=1}^{\infty} \sum_{n=1}^{k} \mathrm{P}\{X=k\} \\
& =\sum_{k=1}^{\infty} k P\{X=k\} \\
& =E[X]
\end{aligned}
$$

Also:

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \mathrm{P}\{X \geqslant n\} \\
&= \sum_{n=1}^{\infty} \mathrm{P}\{X > n-1\} \\
&= \sum_{n=0}^{\infty} \mathrm{P}\{X > n\}
\end{aligned}
$$

Problem (b)

> If X and Y are both non-negative integer-valued random variables, prove that:
>
> $\mathrm{E}[XY] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathrm{P}(X \geqslant n, Y \geqslant m)$.

Solution

Similarly:

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathrm{P}(X \geqslant n, Y \geqslant m) \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{x=n}^{\infty} \sum_{y=m}^{\infty} \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \sum_{n=1}^{x} \sum_{m=1}^{y} \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} xy \mathrm{P}(X = x, Y = y) \\
&= E[XY]
\end{aligned}
$$

48

Problem

> If X is a non-negative random variable and g is a differentiable function with $g(0) = 0$, then:
>
> $\mathrm{E}[g(X)] = \int_0^{\infty} \mathrm{P}(X > t) g'(t) \mathrm{d}t$
>
> Prove the above result when X is a continuous random variable.

Solution

We have:

$$
\begin{aligned}
&\int_0^{\infty} \mathrm{P}(X > t) g'(t) \mathrm{d}t \\
&= \int_0^{\infty} \int_t^{\infty} P(X = x) g'(t) \mathrm{d}x \mathrm{d}t \\
&= \int_0^{\infty} \int_0^{x} P(X = x) g'(t) \mathrm{d}t \mathrm{d}x \\
&= \int_0^{\infty} g(x) P(X = x) \mathrm{d}x \\
&= E[g(x)]
\end{aligned}
$$

71

Problem

> Prove that the sum of independent and identically distributed (i.i.d.) exponential random variables has a gamma distribution.

Solution

Notice for exponential distribution:

$M_X(t) = \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$

Then:

$M_X(t) = \lambda \cdot \frac{1}{\lambda - t} = \frac{\lambda}{\lambda - t} \quad (t < \lambda)$

Also, for Gamma distribution:

$M_X(t) = \int_0^{\infty} e^{tx} \cdot \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x} dx = \frac{\lambda^{\alpha}}{(\lambda - t)^{\alpha}} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha)}$

Then:

$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$

Therefore, for i.i.d. exponential distributions:

$M_{\sum_{i=1}^{\alpha} X_i}(t) = (\frac{\lambda}{\lambda - t})^{\alpha}$

Therefore:

$\sum_{i=1}^{\alpha} X_i \sim \text{Gamma}(\alpha)$

Chapter 3

7

Problem

> Suppose the joint probability mass function $p(x, y, z)$ of X, Y, and Z is given by:
>
> $$\begin{array}{llll} p(1,1,1) = \frac{1}{8}, & p(2,1,1) = \frac{1}{4}, & p(1,1,2) = \frac{1}{8}, & p(2,1,2) = \frac{3}{16} \\ p(1,2,1) = \frac{1}{16}, & p(2,2,1) = 0, & p(1,2,2) = 0, & p(2,2,2) = \frac{1}{4} \end{array}$$
>
> What is $\mathrm{E}[X \mid Y = 2]$? What is $\mathrm{E}[X \mid Y = 2, Z = 1]$?

Solution

We have:

$P(Y = 2) = \sum_{x,z} p(x, 2, z) = \frac{5}{16}$

With conditional probability:

$P(X = 1|Y = 2) = \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \frac{1}{5}$

$P(X = 2|Y = 2) = \frac{P(X = 2, Y = 2)}{P(Y = 2)} = \frac{4}{5}$

Then:

$$\begin{aligned} & E[X|Y = 2] \\ & = \sum_x x \cdot P(X = x|Y = 2) \\ & = \frac{9}{5} \end{aligned}$$

Similarly:

$$
\begin{aligned}
&E[X|Y=2,Z=1] \\
&=\sum_x x\cdot P(X=x|Y=2,Z=1) \\
&=1
\end{aligned}
$$

14

Problem

> Let X be a uniform random variable on $(0,1)$. Find $\mathrm{E}[X \mid X<1/2]$

Solution

We have:

$p(x)=1,x\in(0,1)$

Then:

$$
\begin{aligned}
&E[X|X<\frac{1}{2}] \\
&=\int_0^1 xP(X=x|X<\frac{1}{2})dx \\
&=\int_0^{\frac{1}{2}} x\frac{P(X=x)}{\frac{1}{2}}dx \\
&=\frac{1}{4}
\end{aligned}
$$

27

Problem

> Continuously toss a coin with probability p of landing heads until the pattern $\mathrm{T},\mathrm{T},\mathrm{H}$ appears. (That is, you stop tossing when the most recent toss is heads, and the two tosses immediately before it are tails.) Let X be the number of tosses. Find $\mathrm{E}[X]$.

Solution

We give the solution by states:

We denote:

S_0: the most recent toss is H or nothing

S_1: the most recent toss is T

S_2: the most recent tosses are T, T

We denote:

E_i as the tosses needed for T, T, H in state i

Then:

$E_0 = (1-p)E_1 + pE_0 + 1$

$E_1 = (1-p)E_2 + pE_0 + 1$

$E_2 = p \cdot 0 + (1-p)E_2 + 1$

By solving this, we get:

$E_2 = \frac{1}{p}$

$E_0 = \frac{1/p}{1-2p+p^2}$

Therefore:

$E[X] = \frac{1}{p(1-p)^2}$

57

Problem

> The number of storms in the next rainy season follows a Poisson distribution, but its parameter is uniformly distributed over $(0, 5)$. That is, Λ is uniformly distributed over $(0, 5)$, and given $\Lambda = \lambda$,

> the number of storms is a Poisson random variable with mean λ. Find the probability that there are at least three storms in this rainy season.

Solution

Denote X as the number of storms in this rainy season.

Notice:

$$p(\lambda) = \frac{1}{5}$$

$$P(X = x|\Lambda = \lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$$

Then:

$$\begin{aligned}
&P(X \geq 3) \\
&= \int_0^5 P(X \geq 3|\Lambda = \lambda)P(\Lambda = \lambda)d\lambda \\
&= \frac{1}{5}\int_0^5 (1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2}{2}e^{-\lambda})d\lambda \\
&= \frac{1}{5}[\lambda + e^{-\lambda} + (\lambda+1)e^{-\lambda} + (\frac{\lambda^2}{2} + \lambda + 1)e^{-\lambda}]_0^5 \\
&= \frac{1}{5}[\lambda + (\frac{\lambda^2}{2} + 2\lambda + 3)e^{-\lambda}]_0^5 \\
&= \frac{2}{5} + \frac{51}{10}e^{-5}
\end{aligned}$$

59

Problem

> Assume that each new coupon collected is independent of the past, and the probability of collecting a coupon of type i is p_i. A total of n coupons are collected. Let A_i denote the event "at least one of the n coupons is of type i." For $i \neq j$, compute $P\left(A_i A_j\right)$ using the following methods:
> (a) Condition on the number of type i coupons N_i among the n coupons;
> (b) Condition on the time F_i when the first type i coupon is collected;
> (c) Use the identity $\mathrm{P}\left(A_i \cup A_j\right) = \mathrm{P}\left(A_i\right) + \mathrm{P}\left(A_j\right) - \mathrm{P}\left(A_i A_j\right)$.

Solution (a)

We have:

$$
\begin{aligned}
&P(A_iA_j) \\
&= \sum_{k=1}^{n} P(A_j|N_i = k)P(N_i = k) \\
&= \sum_{k=1}^{n}(1 - (1 - \frac{p_j}{1-p_i})^{n-k})\binom{n}{k}p_i^k(1-p_i)^{n-k}
\end{aligned}
$$

Solution (b)

Similarly, we have:

$$
\begin{aligned}
&P(A_iA_j) \\
&= \sum_{k=1}^{n} P(A_j|F_i = k)P(F_i = k) \\
&= \sum_{m=1}^{n}\left[1 - \left(1 - \frac{p_j}{1-p_i}\right)^{m-1}(1-p_j)^{n-m}\right](1-p_i)^{m-1}\,p_i
\end{aligned}
$$

Solution (c)

We have:

$$
\begin{aligned}
&P(A_iA_j) \\
&= P(A_i) + P(A_j) - P(A_i \cup A_j) \\
&= (1 - (1-p_i)^n) + (1 - (1-p_j)^n) - (1 - (1 - p_i - p_j)^n) \\
&= 1 - (1-p_i)^n - (1-p_j)^n + (1 - p_i - p_j)^n
\end{aligned}
$$