2.4

Problem

> Two partners would like to complete a project. Each partner receives the payoff V when the project is completed but neither receives any payoff before completion. The cost remaining before the project can be completed is R. Neither partner can commit to making a future contribution towards completing the project, so they decide to play the following two-period game: In period one partner 1 chooses to contribute c_1 towards completion. If this contribution is sufficient to complete the project then the game ends and each partner receives V. If this contribution is not sufficient to complete the project (i.e., $c_1 < R$) then in period two partner 2 chooses to contribute c_2 towards completion. If the (undiscounted) sum of the two contributions is sufficient to complete the project then the game ends and each partner receives V. If this sum is not sufficient to complete the project then the game ends and both partners receive zero.

> Each partner must generate the funds for a contribution by taking money away from other profitable activities. The optimal way to do this is to take money away from the least profitable alternatives first. The resulting (opportunity) cost of a contribution is thus convex in the size of the contribution. Suppose that the cost of a contribution c is c^2 for each partner. Assume that partner 1 discounts second-period benefits by the discount factor δ. Compute the unique backwards-induction outcome of this twoperiod contribution game for each triple of parameters $\{V, R, \delta\}$; see Admati and Perry (1991) for the infinite-horizon case.

Solution

Partner 2's optimal c_2 is:

- If contributing $R - c_1$ gives non-negative payoff: $V - (R - c_1)^2 \geq 0$.
  - Then $c_2 = R - c_1$.
- Else, $c_2 = 0$ (no contribution, project not completed).

Partner 1 will choose c_1 to maximize their payoff among these options.
Option 1: $c_1 \geq R$

- Best is to set $c_1 = R$ (since higher c_1 increases cost without benefit).
- Payoff: $V - R^2$.

Option 2: $R-\sqrt{V} \leq c_1 < R$

- To maximize $\delta V - c_1^2$, set c_1 as small as possible within this range, i.e., $c_1 = R - \sqrt{V}$.
- Payoff: $\delta V - (R-\sqrt{V})^2 = \delta V - \left(R^2 - 2R\sqrt{V} + V\right)$.

Option 3: $c_1 < R - \sqrt{V}$

- Best is $c_1 = 0$ (no cost, no payoff).
- Payoff: 0 .

Case 1: $R \leq \sqrt{V}$

- Then $R - \sqrt{V} \leq 0$, so minimal c_1 in Option 2 is 0 .
- Option 2 becomes contributing $0 \leq c_1 < R$.
- Best in Option 2 is $c_1 = 0$, payoff $\delta V - 0 = \delta V$.
- Option 1: $c_1 = R$, payoff $V - R^2$.
- Compare δV and $V - R^2$.
- $\delta V \geq V - R^2$ if $R^2 \geq (1-\delta)V$.
- Otherwise, $V - R^2 > \delta V$.

Case 2: $R > \sqrt{V}$

- Then $R - \sqrt{V} > 0$.
- Option 2: $c_1 = R - \sqrt{V}$, payoff $\delta V - (R - \sqrt{V})^2$.
- Option 1: $c_1 = R$, payoff $V - R^2$.
- Compare as earlier.

Then:

1. Partner 1 contributes enough to complete the project in Period $1\ (c_1 \geq R)$:

- Optimal is $c_1 = R$.
- Payoff: $V - R^2$.
- This is better than other options if $V - R^2 \geq \delta V - (R - \sqrt{V})^2$ and $V - R^2 \geq 0$.
- The first inequality simplifies to $(2-\delta)\sqrt{V} \geq 2R$.

2. Partner 1 contributes $c_1 = R - \sqrt{V}$ (leading to completion in Period 2):

- This is optimal if $(2-\delta)\sqrt{V} < 2R$ and $\delta V - (R - \sqrt{V})^2 \geq 0$.
- The second inequality is $(R - \sqrt{V})^2 \leq \delta V$.

3. Partner 1 contributes $c_1 = 0$ (project not completed):
   o If $\delta V - (R - \sqrt{V})^2 < 0$ and $V - R^2 < 0$.

Final answer:

1. If $V \geq R^2$ and $(2-\delta)\sqrt{V} \geq 2R$:

- Partner 1 contributes $c_1 = R$ in Period 1, completing the project.
- Partner 2 does not need to contribute.

2. If $(R - \sqrt{V})^2 \leq \delta V$ and $(2-\delta)\sqrt{V} < 2R$:

- Partner 1 contributes $c_1 = R - \sqrt{V}$ in Period 1 .
- Partner 2 contributes $c_2 = \sqrt{V}$ in Period 2, completing the project.

3. Else:

- Partner 1 contributes $c_1 = 0$.
- Partner 2 contributes $c_2 = 0$.
- Project is not completed.