2

Problem

> Assume that the interarrival time distribution of a renewal process is Poisson with mean μ. That is, assume
>
> $$\mathrm{P}\left\{X_n=k\right\}=\mathrm{e}^{-\mu} \frac{\mu^k}{k!}, \quad k=0,1, \cdots$$
>
> (a) Find the distribution of S_n.
> (b) Compute $\mathrm{P}\{N(t)=n\}$.

Solution (a)

Since $S_n=X_1+X_2+\cdots+X_n$ is the sum of n i.i.d. Poisson random variables, each with parameter μ, S_n follows a Poisson distribution with parameter $n\mu$. Therefore, the probability mass function (PMF) of S_n is:

$\mathrm{P}\left\{S_n=k\right\}=e^{-n\mu} \frac{(n\mu)^k}{k!}, \quad k=0,1,2, \ldots$

Solution (b)

The event $\{N(t)=n\}$ occurs if and only if the n-th renewal happens by time t and the $(n+1)$-th renewal happens after time t, i.e., $S_n \leq t$ and $S_{n+1}>t$. This can be expressed as:

$$\mathrm{P}\{N(t)=n\}=\mathrm{P}\left\{S_n \leq t\right\}-\mathrm{P}\left\{S_{n+1} \leq t\right\}$$

because $\left\{S_{n+1} \leq t\right\} \subseteq\left\{S_n \leq t\right\}$ (since $S_{n+1}=S_n+X_{n+1} \geq S_n$), and the difference gives the probability that $S_n \leq t$ but $S_{n+1}>t$.

- $S_n \sim \operatorname{Poisson}(n\mu)$, so $\mathrm{P}\left\{S_n \leq t\right\}=\sum_{k=0}^t e^{-n\mu} \frac{(n\mu)^k}{k!}$.
- $S_{n+1} \sim \operatorname{Poisson}((n+1)\mu)$, so $\mathrm{P}\left\{S_{n+1} \leq t\right\}=\sum_{k=0}^t e^{-(n+1)\mu} \frac{((n+1)\mu)^k}{k!}$.
  $\mathrm{P}\{N(t)=n\}=\sum_{k=0}^t e^{-n\mu} \frac{(n\mu)^k}{k!}-\sum_{k=0}^t e^{-(n+1)\mu} \frac{((n+1)\mu)^k}{k!}$

4

Problem

> Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$.
> (a) Are the interarrival times of $\{N(t), t \geq 0\}$ independent?
> (b) Are they identically distributed?
> (c) Is $\{N(t), t \geq 0\}$ a renewal process?

Solution (a)

No.

Solution (b)

No.

Solution (c)

No.

5

Problem

> Let $U_1, U_2, \cdots$ be independent uniform random variables on $(0, 1)$. Define N as
>
> $$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

> What is $\mathrm{E}[N]$?

Solution

$P\left(S_n \leq x\right) = \frac{1}{n!} \sum_{k=0}^{\lfloor x \rfloor} (-1)^k \binom{n}{k} (x-k)^n, \quad 0 \leq x \leq n$

$\mathrm{E}[N]=\sum_{k=0}^{\infty}\mathrm{P}(N>k)=\sum_{k=0}^{\infty}\frac{1}{k!}=e.$

，6， 8，10，12， 14，17，
18，19，22，26，27

6

Problem

> Consider a renewal process $\{N(t), t\geqslant 0\}$ with interarrival times following a $\Gamma(r,\lambda)$ distribution. That is, the interarrival density is
>
> $$f(x)=\frac{\lambda\mathrm{e}^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!},\quad x>0$$
>
> (a) Prove that
>
> $$\mathrm{P}\{N(t)\geqslant n\}=\sum_{i=nr}^{\infty}\frac{\mathrm{e}^{-\lambda t}(\lambda t)^i}{i!}$$
>
> (b) Prove that
>
> $$m(t)=\sum_{i=r}^{\infty}\left\lfloor\frac{i}{r}\right\rfloor\frac{\mathrm{e}^{-\lambda t}(\lambda t)^i}{i!}$$
>
> where $\lfloor i/r\rfloor$ is the greatest integer less than or equal to i/r.

Solution (a)

Recall that $N(t)\geqslant n$ if and only if the n-th arrival occurs by time t, i.e., $S_n\leqslant t$. Since S_n is the time of the nr-th event in the Poisson process $\{M(t)\}$, we have:

$$\mathrm{P}\{N(t)\geqslant n\}=\mathrm{P}\left\{S_n\leqslant t\right\}=\mathrm{P}\{M(t)\geqslant nr\}$$

This is because the nr-th event occurs by time t if and only if there are at least nr events in the Poisson process by time t.

The probability $\mathrm{P}\{M(t) \geqslant nr\}$ for a Poisson process with rate λ is given by the tail of the Poisson distribution:

$$\mathrm{P}\{M(t) \geqslant nr\} = \sum_{i=nr}^{\infty} \mathrm{P}\{M(t) = i\} = \sum_{i=nr}^{\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^i}{i!}$$

Therefore,

$$\mathrm{P}\{N(t) \geqslant n\} = \sum_{i=nr}^{\infty} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^i}{i!}$$

Solution (b)

The renewal function $m(t)$ is defined as the expected number of renewals by time t, i.e., $m(t) = \mathrm{E}[N(t)]$. From the relationship with the Poisson process, we have $N(t) = \left\lfloor \frac{M(t)}{r} \right\rfloor$. Thus,

$$m(t) = \mathrm{E}\left[\left\lfloor \frac{M(t)}{r} \right\rfloor\right].$$

Since $M(t)$ is a discrete random variable (Poisson with mean λt), the expectation can be computed as:

$$\mathrm{E}\left[\left\lfloor \frac{M(t)}{r} \right\rfloor\right] = \sum_{k=0}^{\infty} \left\lfloor \frac{k}{r} \right\rfloor \mathrm{P}\{M(t) = k\} = \sum_{k=0}^{\infty} \left\lfloor \frac{k}{r} \right\rfloor \frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!}.$$

Note that for $k < r$, $\left\lfloor \frac{k}{r} \right\rfloor = 0$ because $0 \leq k/r < 1$. Therefore, the sum can start from $k = r$:

$$\mathrm{E}\left[\left\lfloor \frac{M(t)}{r} \right\rfloor\right] = \sum_{k=r}^{\infty} \left\lfloor \frac{k}{r} \right\rfloor \frac{\mathrm{e}^{-\lambda t}(\lambda t)^k}{k!}$$

Changing the index to i for consistency with the given expression, we have:

$$m(t) = \sum_{i=r}^{\infty} \left\lfloor \frac{i}{r} \right\rfloor \frac{\mathrm{e}^{-\lambda t}(\lambda t)^i}{i!}.$$

8

Problem

> Replace a machine when it fails or has been in use for T years. If the successive lifetimes of the machines are independent, with a common distribution F having density function f, prove that

> (a) The long-run rate at which machines are replaced is
>
> $$\left[\int_0^T xf(x)dx + T(1-F(T))\right]^{-1}$$
>
> (b) The long-run rate at which machines fail is
>
> $$\frac{F(T)}{\int_0^T xf(x)dx + T[1-F(T)]}$$

Solution (a)

We compute $E[Y] = E[\min(X,T)]$. For a non-negative random variable, the expectation can be expressed as:

$$E[Y] = \int_0^\infty P(Y>y)dy$$

The survival function $P(Y>y)$ is:

- $P(Y>y) = P(\min(X,T)>y) = P(X>y)$ for $0 \le y < T$, since $\min(X,T)>y$ iff $X>y$ when $y<T$.
- $P(Y>y) = 0$ for $y \ge T$, because $Y \le T$.

Thus,

$$P(Y>y) = \begin{cases} 1-F(y) & \text{if } 0 \le y < T \\ 0 & \text{if } y \ge T \end{cases}$$

and

$$E[Y] = \int_0^T (1-F(y))dy$$

We now show that this equals the expression inside the inverse in part (a). Consider:

$$\int_0^T xf(x)dx + T(1-F(T))$$

Using integration by parts on $\int_0^T xf(x)dx$, let $u=x$ and $dv = f(x)dx$. Then $du = dx$ and $v = F(x)$ (since $F'(x) = f(x)$). Assuming $F(0)=0$ (as lifetime is positive), we have:

$$\int_0^T xf(x)dx = [xF(x)]_0^T - \int_0^T F(x)dx = TF(T) - \int_0^T F(x)dx$$

Substitute this into the expression:

$$= TF(T) - \int_0^T F(x)dx + T - TF(T) = T - \int_0^T F(x)dx = \int_0^T (1 - F(x))dx$$

Thus,

$$E[Y] = \int_0^T (1 - F(y))dy = \int_0^T xf(x)dx + T(1 - F(T))$$

Therefore, the long-run replacement rate is:

$$\frac{1}{E[Y]} = \left[\int_0^T xf(x)dx + T(1 - F(T))\right]^{-1}$$

Solution (b)

- $R = 1$ if the replacement is due to failure (i.e., $X < T$).
- $R = 0$ if the replacement is due to reaching age T (i.e., $X \geq T$).

The expected reward per cycle is:

$$E[R] = P(\text{ failure }) = P(X < T) = F(T)$$

The expected cycle length is $E[Y]$, as computed in part (a). By the renewal reward theorem, the long-run failure rate (reward per unit time) is:

$$\frac{E[R]}{E[Y]} = \frac{F(T)}{E[Y]} = \frac{F(T)}{\int_0^T xf(x)dx + T[1 - F(T)]}$$

10

Problem

> Consider a renewal process with mean interarrival time μ. Suppose each event of this process is counted with probability p. Let $N_C(t)$ denote the number of counted events up to time $t(t > 0)$.
> (a) Is $\{N_C(t), t \geqslant 0\}$ a renewal process?
> (b) What is $\lim_{t\to\infty} \frac{N_G(t)}{t}$?

Solution (a)

Yes.

- Let M_k be the number of original events between the $(k-1)$-th and k-th counted event (including the k-th counted event). Since each event is counted independently with probability p, M_k follows a geometric distribution with success probability p, i.e., $\mathbb{P}\left(M_k=m\right)=(1-p)^{m-1} p$ for $m=1,2,3, \ldots$.
- Then $S_k=\sum_{i=1}^{M_k} T_i^{(k)}$, where $\left\{T_i^{(k)}\right\}$ are i.i.d. copies of T_i.

Since:

- The sequence $\left\{M_k\right\}_{k=1}^{\infty}$ is i.i.d. (geometric with parameter p),
- The interarrival times $\left\{T_i^{(k)}\right\}$ are i.i.d. for each k and independent of $\left\{M_k\right\}$,
- And for each k, S_k depends only on M_k and the corresponding $T_i^{(k)}$,

Solution (b)

From part (a), $N_C(t)$ is a renewal process with i.i.d. interarrival times S_k, where $S_k=\sum_{i=1}^{M_k} T_i$ and $M_k \sim \operatorname{Geometric}(p)$. The expected interarrival time is:

$$
\mathbb{E}\left[S_k\right]=\mathbb{E}\left[\sum_{i=1}^{M_k} T_i\right] .
$$

Using the law of total expectation and the independence of M_k and $\left\{T_i\right\}$:

$$
\mathbb{E}\left[S_k\right]=\mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{M_k} T_i \mid M_k\right]\right]=\mathbb{E}\left[\sum_{i=1}^{M_k} \mathbb{E}\left[T_i\right]\right]=\mathbb{E}\left[M_k \cdot \mu\right]=\mu \cdot \mathbb{E}\left[M_k\right] .
$$

Since M_k is geometric with success probability $p, \mathbb{E}\left[M_k\right]=\frac{1}{p}$. Thus,

$$
\mathbb{E}\left[S_k\right]=\mu \cdot \frac{1}{p}=\frac{\mu}{p}
$$

By the elementary renewal theorem, for a renewal process with finite mean interarrival time $\mathbb{E}\left[S_k\right]$, the long-run rate is:

$$
\lim _{t \rightarrow \infty} \frac{N_C(t)}{t}=\frac{1}{\mathbb{E}\left[S_k\right]}=\frac{1}{\mu / p}=\frac{p}{\mu} .
$$

12

Problem

> Events occur according to a Poisson process with rate λ. An event that occurs within time d after the event immediately preceding it is called a " d-event". For example, if $d=1$, and events occur at times $2, 2.8, 4, 6, 6.6, \ldots$, then the events at times 2.8 and 6.6 are d-events.
> (a) What is the rate at which d-events occur?
> (b) What is the proportion of d-events among all events?

Solution (a)

The overall event occurrence rate is λ. Since each event (after the first) is a d-event with probability $p = 1-e^{-\lambda d}$ independently, the long-run rate of d-events is:

$$\lambda \times p = \lambda\left(1-e^{-\lambda d}\right)$$

This can also be derived by noting that the process of d-events forms a renewal process where the interarrival times between consecutive d-events have a specific distribution. Specifically, the time between consecutive d-events is the sum of a geometric number of exponential interarrival times:

- Let N be the number of events until the next d-event, including the d-event itself. Then N follows a geometric distribution with success probability $p=1-e^{-\lambda d}$, so $P(N=k)=(1-p)^{k-1} p$ for $k=1,2,3,\ldots$, and $E[N]=1/p$.
- The interarrival time between consecutive d-events is $W=\sum_{i=1}^{N} S_i$, where $S_i \sim \operatorname{Exp}(\lambda)$. By independence, $E[W]=E[N] \cdot E\left[S_i\right]=(1/p) \cdot(1/\lambda)=1/(\lambda p)$.
- The long-run rate of d-events is $1/E[W]=\lambda p=\lambda\left(1-e^{-\lambda d}\right)$.

Thus, the rate at which d-events occur is $\lambda\left(1-e^{-\lambda d}\right)$.

Solution (b)

Let $N(t)$ be the total number of events up to time t, and $N_d(t)$ be the number of d-events up to time t. The proportion of d-events is $\lim_{t \rightarrow \infty} \frac{N_d(t)}{N(t)}$.

Excluding the first event (which cannot be a d-event and becomes negligible as $t \rightarrow \infty$), we have:

$$N_d(t)=\sum_{i=2}^{N(t)} I_i, \quad \text{where} \quad I_i=\mathbf{1}_{\{S_i \leq d\}}.$$

The indicators I_i are i.i.d. Bernoulli random variables with success probability $p = 1 - e^{-\lambda d}$. As $t \to \infty, N(t) \to \infty$ almost surely. By the strong law of large numbers:

$$\frac{1}{N(t)-1} \sum_{i=2}^{N(t)} I_i \to p \quad \text{almost surely.}$$

Since $\frac{N(t)-1}{N(t)} \to 1$ as $t \to \infty$, it follows that:

$$\frac{N_d(t)}{N(t)} = \frac{\sum_{i=2}^{N(t)} I_i}{N(t)} = \frac{\sum_{i=2}^{N(t)} I_i}{N(t)-1} \cdot \frac{N(t)-1}{N(t)} \to p \cdot 1 = 1 - e^{-\lambda d}.$$

Thus, the long-run proportion of d-events among all events is $1 - e^{-\lambda d}$.

14

Problem

Consider the gambler's ruin problem where in each game, the gambler wins 1 yuan with probability p and loses 1 yuan with probability $1-p$. The gambler continues playing until their fortune reaches either $N - i$ yuan or $-i$ yuan. (That is, the gambler starts with i yuan and stops when their fortune reaches 0 or N.) Let T be the number of games played before the gambler stops. Using Wald's equation and the known probability that the gambler ends up with a fortune of N (i.e., a net gain of $N-i$), find $\mathrm{E}[T]$.

Solution

$$\mathrm{E}\left[X_j\right] = p \cdot 1 + (1-p) \cdot (-1) = 2p - 1$$

The gambler starts with an initial fortune of i yuan. The process stops when the fortune reaches 0 or N, so T is a stopping time. The total winnings up to time T is $\sum_{j=1}^{T} X_j = S_T - S_0$, where $S_0 = i$ is the initial fortune, and S_T is the fortune at stopping time T. Therefore:

$$\sum_{j=1}^{T} X_j = S_T - i$$

The possible values of S_T are 0 or N, so the possible values of $\sum_{j=1}^{T} X_j$ are:

- $0 - i = -i$ (if ruin occurs, fortune reaches 0),
- $N - i$ (if success occurs, fortune reaches N).

The expected value of the sum is:

$$\mathrm{E}\left[\sum_{j=1}^{T} X_j\right] = \mathrm{E}\left[S_T - i\right] = \mathrm{E}\left[S_T\right] - i$$

Let P_i be the probability that the gambler reaches fortune N (i.e., ends with a net gain of $N - i$). Then:

$$\mathrm{E}\left[S_T\right] = P_i \cdot N + (1 - P_i) \cdot 0 = P_i N$$

so:

$$\mathrm{E}\left[\sum^{T} X_j\right] = P_i N - i$$

Solving for $\mathrm{E}[T]$:

$$\mathrm{E}[T] = \frac{P_i N - i}{2p - 1}$$

This expression is valid for $p \neq \frac{1}{2}$. When $p = \frac{1}{2}, \mathrm{E}\left[X_j\right] = 2 \cdot \frac{1}{2} - 1 = 0$, and Wald's equation gives:

$$\mathrm{E}\left[\sum_{j=1}^{T} X_j\right] = 0 \cdot \mathrm{E}[T] = 0$$

which is consistent since $P_i = \frac{i}{N}$ (as derived below) and $P_i N - i = \frac{i}{N} \cdot N - i = 0$. However, the expression $\frac{P_i N - i}{2p - 1}$ is undefined when $p = \frac{1}{2}$. In this case, the expected time is known to be $\mathrm{E}[T] = i(N - i)$.

Solving for $\mathrm{E}[T]$:

$$\mathrm{E}[T] = \frac{P_i N - i}{2p - 1}$$

This expression is valid for $p \neq \frac{1}{2}$. When $p = \frac{1}{2}, \mathrm{E}\left[X_j\right] = 2 \cdot \frac{1}{2} - 1 = 0$, and Wald's equation gives:

$$\mathrm{E}\left[\sum_{j=1}^{T} X_j\right] = 0 \cdot \mathrm{E}[T] = 0$$

The probability P_i that the gambler reaches N before 0 , starting from i, is a standard result in gambler's ruin:

- If $p \neq \frac{1}{2}$, let $q = 1 - p$. Then:

$$P_i = \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N}$$

- If $p = \frac{1}{2}$, then:

$$P_i = \frac{i}{N}$$

Final Expression for $\mathrm{E}[T]$

- For $p \neq \frac{1}{2}$:

$$\mathrm{E}[T] = \frac{P_i N - i}{2p - 1} = \frac{\left(\frac{1 - \binom{p}{p}^i}{1 - \binom{p}{p}^N}\right) N - i}{2p - 1}$$

- For $p = \frac{1}{2}$:

$$\mathrm{E}[T] = i(N - i)$$

17

Problem

In Example 7.6, assume that potential customers arrive according to a renewal process with interarrival distribution F. Does the number of events up to time t form a renewal process (possibly delayed) if an event corresponds to a customer who:
(a) enters the bank?
(b) leaves the bank?

What if F is exponential?

Solution (a)

Yes. Yes.

Solution (b)

No. Yes.

18

Problem

> Calculate the renewal function when the interarrival distribution F satisfies $1 - F(t) = pe^{-\mu_1 t} + (1-p)e^{-\mu_2 t}$.

Solution

The survival function is given by:

$$1 - F(t) = pe^{-\mu_1 t} + (1-p)e^{-\mu_2 t}, \quad t \geq 0$$

The cumulative distribution function (CDF) is:

$$F(t) = 1 - pe^{-\mu_1 t} - (1-p)e^{-\mu_2 t}$$

The probability density function (PDF) is the derivative of the CDF:

$$f(t) = \frac{d}{dt}F(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}$$

This is a mixture of two exponential distributions.
The renewal function $M(t)$ is defined as the expected number of renewals (events) up to time t. To find $M(t)$, we use the Laplace transform approach. The Laplace-Stieltjes transform of the renewal function satisfies:

$$\mathcal{L}_{dM}(s) = \int_0^\infty e^{-st} dM(t) = \frac{\mathcal{L}_f(s)}{1 - \mathcal{L}_f(s)}$$

where $\mathcal{L}_f(s)$ is the Laplace transform of the interarrival density $f(t)$.

First, compute $\mathcal{L}_f(s)$:

$$\mathcal{L}_f(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st} \left[p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t} \right] dt$$

This splits into two integrals:

$$\mathcal{L}_f(s) = p\mu_1 \int_0^\infty e^{-(s+\mu_1)t} dt + (1-p)\mu_2 \int_0^\infty e^{-(s+\mu_2)t} dt = p\mu_1 \frac{1}{s+\mu_1} + (1-p)\mu_2 \frac{1}{s+\mu_2}$$

Next, compute $1 - \mathcal{L}_f(s)$:

$$1-\mathcal{L}_f(s)=1-\left(\frac{p\mu_1}{s+\mu_1}+\frac{(1-p)\mu_2}{s+\mu_2}\right)$$

Combining over a common denominator $(s+\mu_1)(s+\mu_2)$:

$$\mathcal{L}_f(s)=\frac{p\mu_1(s+\mu_2)+(1-p)\mu_2(s+\mu_1)}{(s+\mu_1)(s+\mu_2)}=\frac{s(p\mu_1+(1-p)\mu_2)+\mu_1\mu_2}{(s+\mu_1)(s+\mu_2)}.$$

Thus,

$$1-\mathcal{L}_f(s)=\frac{(s+\mu_1)(s+\mu_2)-[s(p\mu_1+(1-p)\mu_2)+\mu_1\mu_2]}{(s+\mu_1)(s+\mu_2)}=\frac{s^2+s(p\mu_2+(1-p)\mu_1)}{(s+\mu_1)(s+\mu_2)}.$$

Now,

$$\mathcal{L}_{dM}(s)=\frac{\mathcal{L}_f(s)}{1-\mathcal{L}_f(s)}=\frac{\frac{s(p\mu_1+(1-p)\mu_2)+\mu_1\mu_2}{(s+\mu_1)(s+\mu_2)}}{\frac{s^2+s(p\mu_2+(1-p)\mu_1)}{(s+\mu_1)(s+\mu_2)}}=\frac{s(p\mu_1+(1-p)\mu_2)+\mu_1\mu_2}{s^2+s(p\mu_2+(1-p)\mu_1)}$$

Let $a=p\mu_1+(1-p)\mu_2$ and $b=p\mu_2+(1-p)\mu_1$, so:

$$\mathcal{L}_{dM}(s)=\frac{as+\mu_1\mu_2}{s(s+b)}$$

The Laplace transform of $M(t)$, denoted $\tilde{M}(s)=\int_0^\infty e^{-st}M(t)dt$, satisfies $\mathcal{L}_{dM}(s)=s\tilde{M}(s)$ since $M(0)=0$. Thus:

$$s\tilde{M}(s)=\frac{as+\mu_1\mu_2}{s(s+b)}\Longrightarrow\tilde{M}(s)=\frac{as+\mu_1\mu_2}{s^2(s+b)}$$

To find $M(t)$, we invert this Laplace transform. Perform partial fraction decomposition:

$$\frac{as+\mu_1\mu_2}{s^2(s+b)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+b}$$

Solving for the constants:

$$as+\mu_1\mu_2=As(s+b)+B(s+b)+Cs^2.$$

Equate coefficients:

- $s^2: A+C=0$
- $s^1: Ab+B=a$
- $s^0: Bb=\mu_1\mu_2$

From $Bb=\mu_1\mu_2$, we have $B=\frac{\mu_1\mu_2}{b}$.
From $A+C=0$, we have $C=-A$.
Substitute into the s^1 equation:

$$Ab+\frac{\mu_1\mu_2}{b}=a \Longrightarrow A=\frac{a}{b}-\frac{\mu_1\mu_2}{b^2}.$$

Thus,

$$C=-\left(\frac{a}{b}-\frac{\mu_1\mu_2}{b^2}\right)$$

The mean interarrival time μ is:

$$\mu=\int_0^{\infty}(1-F(t))dt=\int_0^{\infty}\left(pe^{-\mu_1 t}+(1-p)e^{-\mu_2 t}\right)dt=\frac{p}{\mu_1}+\frac{1-p}{\mu_2}$$

Note that $\frac{1}{\mu}=\frac{\mu_1\mu_2}{b}$, so $B=\frac{1}{\mu}$.

Compute $ab-\mu_1\mu_2$:

$$ab=(p\mu_1+(1-p)\mu_2)(p\mu_2+(1-p)\mu_1)=p(1-p)\left(\mu_1^2+\mu_2^2\right)+p^2\mu_1\mu_2+(1-p)^2\mu_1\mu_2$$
$$ab-\mu_1\mu_2=p(1-p)\left(\mu_1^2-2\mu_1\mu_2+\mu_2^2\right)=p(1-p)\left(\mu_1-\mu_2\right)^2$$

Thus,

$$A=\frac{p(1-p)\left(\mu_1-\mu_2\right)^2}{b^2}$$

The inverse Laplace transform gives:

$$M(t)=\mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+b}\right\}=A+Bt+Ce^{-bt}$$

Substituting $C=-A$:

$$M(t)=A+Bt-Ae^{-bt}=Bt+A\left(1-e^{-bt}\right)$$

Substitute the expressions for A and B :

$$M(t)=\frac{t}{\mu}+\frac{p(1-p)\left(\mu_1-\mu_2\right)^2}{b^2}\left(1-e^{-bt}\right)$$

where $b=p\mu_2+(1-p)\mu_1$ and $\mu=\frac{p}{\mu_1}+\frac{1-p}{\mu_2}$.

$M(t)=\frac{t}{\mu}+\frac{p(1-p)(\mu_1-\mu_2)^2}{[p\mu_2+(1-p)\mu_1]^2}\left(1-e^{-[p\mu_2+(1-p)\mu_1]t}\right)$,

19

Problem

> Consider a renewal process with interarrival times uniformly distributed on (0,1). Determine the expected time from $t = 1$ until the next renewal.

Solution

Let U be the time of the last renewal before or at time $t = 1$. The next renewal occurs at time $U + Y$, where Y is the interarrival time following U, and $Y \sim \text{Uniform}(0, 1)$, independent of U. The time from $t = 1$ to the next renewal is $B(1) = (U + Y) - 1$. Since there are no renewals in $(U, 1]$, it follows that $U + Y > 1$ almost surely.

The cumulative distribution function (CDF) of U is derived as follows. For $u \in [0, 1], P(U \leq u) = P$ (no renewals in $(u, 1]$). This probability is given by:

$$P(U \leq u) = 1 + e^u(u - 1).$$

The probability density function (PDF) of U is:

$$f_U(u) = \frac{d}{du}\left[1 + e^u(u - 1)\right] = ue^u, \quad u \in [0, 1].$$

The expected value of U is:

$$E[U] = \int_0^1 u \cdot ue^u du = \int_0^1 u^2 e^u du$$

Using integration by parts:

- Let $v = u^2, dw = e^u du$, so $dv = 2udu, w = e^u$.
- Then:

$$\int u^2 e^u du = u^2 e^u - \int 2ue^u du$$

- Now, $\int ue^u du = ue^u - \int e^u du = ue^u - e^u$.
- Substituting back:

$$\int u^2 e^u du = u^2 e^u - 2\left(ue^u - e^u\right) = u^2 e^u - 2ue^u + 2e^u$$

- Evaluating from 0 to 1 :

$$\left[u^2e^u - 2ue^u + 2e^u\right]_0^1 = (1 \cdot e - 2 \cdot e + 2e) - (0 - 0 + 2) = e - 2.$$

Thus, $E[U] = e - 2$.

Thus, $E[U] = e - 2$.
Given U, the conditional distribution of Y is uniform on ($1 - U, 1$), because $Y > 1 - U$ must hold. The conditional expectation of Y given U is:

$$E[Y \mid U] = E[Y \mid Y > 1 - U] = \frac{(1-U)+1}{2} = 1 - \frac{U}{2}.$$

The conditional expectation of $B(1)$ given U is:

$$E[B(1) \mid U] = E[(U + Y - 1) \mid U] = U + E[Y \mid U] - 1 = U + \left(1 - \frac{U}{2}\right) - 1 = \frac{U}{2}.$$

The unconditional expectation is:

$$E[B(1)] = E[E[B(1) \mid U]] = E\left[\frac{U}{2}\right] = \frac{1}{2}E[U] = \frac{1}{2}(e-2).$$

Alternatively, using the age $A(1) = 1 - U$, the distribution of $A(1)$ has PDF:

$$f_A(a) = f_U(1-a)\left|\frac{du}{da}\right| = (1-a)e^{1-a}, \quad a \in [0,1].$$

Given $A(1) = a$, the remaining life $B(1)$ is uniformly distributed on $(0, 1-a)$, so:

$$E[B(1) \mid A(1) = a] = \frac{1-a}{2}.$$

Then:

$$E[B(1)] = E[E[B(1) \mid A(1)]] = E\left[\frac{1-A(1)}{2}\right] = \frac{1}{2}E[1 - A(1)] = \frac{1}{2}E[U] = \frac{1}{2}(e-2),$$

22

Problem

> J's strategy for buying cars is as follows: For the first T units of time after acquiring a new car, all failures are repaired. After the car reaches an age of T, upon the first failure, it is sent to the junkyard

> and a new car is purchased. Assume that the time to first failure for a new car is an exponential random variable with rate λ, and the time to next failure for a repaired car is an exponential random variable with rate μ.
> (a) What is the rate at which J buys new cars?
> (b) Suppose a new car costs C, and each repair costs r. What is J's long-run average cost per unit time?

Solution(a)

To find $\mathbb{E}[L]$, define the failure times in a cycle. Let Z_1 be the time to the first failure, $Z_1 \sim \operatorname{Exp}(\lambda)$. For $k \geq 2$, let Z_k be the time to the next failure after a repair, $Z_k \sim \operatorname{Exp}(\mu)$, and all Z_k are independent. The cumulative time to the k-th failure is $S_k = \sum_{i=1}^{k} Z_i$. The cycle ends at the smallest N such that $S_N > T$, so $L = S_N$.

By Wald's equation and the properties of stopping times,

$$\mathbb{E}[L] = \mathbb{E}\left[S_N\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[Z_i\right] \mathbb{P}(N \geq i)$$

Here, $\mathbb{E}\left[Z_1\right] = \frac{1}{\lambda}, \mathbb{E}\left[Z_i\right] = \frac{1}{\mu}$ for $i \geq 2$, and $\mathbb{P}(N \geq i) = \mathbb{P}\left(S_{i-1} \leq T\right)$ for $i \geq 2$ (with $S_0 = 0$, so $\mathbb{P}(N \geq 1) = 1$). Thus,

$$\mathbb{E}[L] = \frac{1}{\lambda} \cdot 1 + \sum_{i=2}^{\infty} \frac{1}{\mu} \mathbb{P}\left(S_{i-1} \leq T\right) = \frac{1}{\lambda} + \frac{1}{\mu} \sum_{k=1}^{\infty} \mathbb{P}\left(S_k \leq T\right),$$

where $k = i - 1$.

The sum $\sum_{k=1}^{\infty} \mathbb{P}\left(S_k \leq T\right) = \mathbb{E}[N(T)]$, where $N(T)$ is the number of failures up to time T in a delayed renewal process. The first interarrival time is $\operatorname{Exp}(\lambda)$, and subsequent interarrival times are $\operatorname{Exp}(\mu)$. The renewal function $m(t) = \mathbb{E}[N(t)]$ satisfies

$$m(t) = F_1(t) + \int_0^t m_2(t-s) f_1(s) ds$$

where $F_1(t) = 1 - e^{-\lambda t}$ is the CDF of $\operatorname{Exp}(\lambda)$, $f_1(t) = \lambda e^{-\lambda t}$ is the PDF, and $m_2(t) = \mu t$ is the renewal function for the ordinary renewal process with interarrival $\operatorname{Exp}(\mu)$. Thus,

$$m(t) = \left(1 - e^{-\lambda t}\right) + \int_0^t (\mu(t-s)) \lambda e^{-\lambda s} ds$$

The integral is

$$\int_0^t \mu(t-s)\lambda e^{-\lambda s} ds = \mu\lambda \left[\frac{t}{\lambda} + \frac{1}{\lambda^2}e^{-\lambda t} - \frac{1}{\lambda^2}\right] = \mu t + \frac{\mu}{\lambda}e^{-\lambda t} - \frac{\mu}{\lambda}$$

so

$$m(t) = 1 - e^{-\lambda t} + \mu t + \frac{\mu}{\lambda}e^{-\lambda t} - \frac{\mu}{\lambda} = \mu t + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda t}\right)$$

Therefore, $\mathbb{E}[N(T)] = m(T) = \mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right)$, and

$$\mathbb{E}[L] = \frac{1}{\lambda} + \frac{1}{\mu}\left[\mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right)\right] = T + \frac{1}{\mu} + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)e^{-\lambda T}.$$

The rate of buying new cars is

$$\frac{1}{\mathbb{E}[L]} = \left[T + \frac{1}{\mu} + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)e^{-\lambda T}\right]^{-1}.$$

Solution (b)

The total cost per cycle is $C + r(N-1)$. The expected cost per cycle is

$$\mathbb{E}[C + r(N-1)] = C + r(\mathbb{E}[N] - 1)$$

From part (a), $\mathbb{E}[N] = 1 + \mathbb{E}[N(T)] = 1 + m(T) = 1 + \mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right)$, so

$$\mathbb{E}[N] - 1 = \mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right).$$

Thus, the expected cost per cycle is

$$C + r\left[\mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right)\right]$$

The long-run average cost per unit time is the expected cost per cycle divided by the expected cycle length:

$$\frac{C + r\left[\mu T + \left(1 - \frac{\mu}{\lambda}\right)\left(1 - e^{-\lambda T}\right)\right]}{T + \frac{1}{\mu} + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right)e^{-\lambda T}}.$$

26

Problem

> Consider a train station where passengers arrive according to a Poisson process with rate λ. Whenever there are N passengers waiting at the station, a train is dispatched. However, it takes K units of time for the train to arrive at the station, and it carries all waiting passengers upon arrival. Assume that the station incurs a cost at a rate of $n \cdot c$ per unit time when there are n passengers waiting, where c is the cost rate per passenger per unit time. Find the long-run average cost.

Solution

Since passenger arrivals follow a Poisson process with rate λ, T_1 is the time of the N-th arrival, which follows an Erlang distribution with shape parameter N and rate λ. Thus,

$$\mathbb{E}\left[T_1\right]=\frac{N}{\lambda}$$

The expected cycle length is

$$\mathbb{E}[T]=\mathbb{E}\left[T_1+K\right]=\frac{N}{\lambda}+K .$$

The cost per cycle is $C_{\text{cycle}}=c \int_0^T n(s) d s$, where $n(s)$ is the number of waiting passengers at time s. We split the integral into two intervals: $\left[0, T_1\right]$ and $\left[T_1, T_1+K\right]$ (since $T=T_1+K$).

- Interval $\left[0, T_1\right]$: Passengers arrive, and $n(s)$ is the number of arrivals by time s. At $s=T_1, n(s)=N$. The integral $\int_0^{T_1} n(s) d s$ can be expressed using the arrival times. Let T_i be the arrival time of the i-th passenger $(i=1,2, \ldots, N)$. Then,

  $$\int_0^{T_1} n(s) d s=\sum_{i=1}^N\left(T_1-T_i\right)$$

  since passenger i contributes waiting time from T_i to T_1. The expectation is

  $$\mathbb{E}\left[\sum_{i=1}^N\left(T_1-T_i\right)\right]=\mathbb{E}\left[N T_1-\sum_{i=1}^N T_i\right]$$

For a Poisson process, $\mathbb{E}\left[T_i\right]=\frac{i}{\lambda}$ and $\mathbb{E}\left[T_1\right]=\mathbb{E}\left[T_N\right]=\frac{N}{\lambda}$. Thus,

$$\mathbb{E}\left[\sum_{i=1}^{N} T_i\right] = \sum_{i=1}^{N} \mathbb{E}\left[T_i\right] = \sum_{i=1}^{N} \frac{i}{\lambda} = \frac{1}{\lambda} \cdot \frac{N(N+1)}{2},$$
$$\mathbb{E}\left[N T_1\right] = N \cdot \frac{N}{\lambda} = \frac{N^2}{\lambda},$$

so

$$\mathbb{E}\left[\sum_{i=1}^{N}\left(T_1 - T_i\right)\right] = \frac{N^2}{\lambda} - \frac{1}{\lambda} \cdot \frac{N(N+1)}{2} = \frac{N(N-1)}{2\lambda}.$$

- Interval $\left[T_1, T_1+K\right]$: At $s=T_1$, there are N passengers. During $\left[T_1, T_1+K\right]$, additional passengers arrive according to a Poisson process with rate λ, independent of the past. Let $N'(u)$ be the number of arrivals by time u in this interval, where $u=s-T_1$. Then $n(s)=N+N'\left(s-T_1\right)$, and

$$\int_{T_1}^{T_1+K} n(s) ds = \int_0^K \left[N+N'(u)\right] du$$

Combining both intervals, the expected value of the integral is

$$\mathbb{E}\left[\int_0^T n(s) ds\right] = \frac{N(N-1)}{2\lambda} + NK + \frac{\lambda K^2}{2}.$$

Thus, the expected cost per cycle is

$$\mathbb{E}\left[C_{\text{cycle}}\right] = c\left(\frac{N(N-1)}{2\lambda} + NK + \frac{\lambda K^2}{2}\right).$$

By the renewal reward theorem,

$$\text{Long-run average cost } = \frac{\mathbb{E}\left[C_{\text{cycle}}\right]}{\mathbb{E}[T]} = \frac{c\left(\frac{N(N-1)}{2\lambda} + NK + \frac{\lambda K^2}{2}\right)}{\frac{N}{\lambda} + K}.$$

Simplifying the expression:

$$= c \cdot \frac{\frac{N(N-1)}{2\lambda} + NK + \frac{\lambda K^2}{2}}{\frac{N}{\lambda} + K} = c \cdot \frac{\frac{N(N-1)+2NK\lambda+\lambda^2 K^2}{2\lambda}}{\frac{N+K\lambda}{\lambda}} = c \cdot \frac{N(N-1)+2NK\lambda+\lambda^2 K^2}{2(N+\lambda K)}.$$

Let $Q=\lambda K$. Then the numerator is $N(N-1)+2NQ+Q^2$, and

$$N(N-1)+2NQ+Q^2 = \left(N^2+2NQ+Q^2\right) - N = (N+Q)^2 - N.$$

Thus,

The long-run average cost is

$$\frac{c}{2}\left(N+\lambda K-\frac{N}{N+\lambda K}\right)$$

27

Problem

> Consider a machine consisting of two independent components, where the i-th component operates for an exponential time with rate λ_i. The machine functions as long as at least one component is working (i.e., it fails only when both components fail). When a machine fails, a new machine with both components working is immediately put into use. Each machine failure incurs a cost K, and while the machine is in use, it incurs an operating cost at a rate of c_i per unit time when there are i working components $(i=1,2)$. Find the long-run average cost per unit time.

Solution

The system can be modeled as a renewal process, where each renewal cycle begins when a new machine (both components working) is put into use and ends when the machine fails (both components failed). The long-run average cost per unit time is given by the renewal reward theorem:

$$\text{Long-run average cost } = \frac{\mathbb{E}[\text{ Cost per cycle }]}{\mathbb{E}[\text{ Cycle length }]}.$$

We define the states of the machine within a cycle:

- State 2: Both components working.
- State 1A: Component 1 working, component 2 failed.
- State 1B: Component 2 working, component 1 failed.
- State 0: Both components failed (machine failure).

The transition rates are:

- From state 2 , the machine transitions to state 1 B at rate λ_1 (if component 1 fails) or to state 1 A at rate λ_2 (if component 2 fails). The total transition rate out of state 2 is $\lambda_1+\lambda_2$.
- From state 1 A , the machine transitions to state 0 at rate λ_1 (component 1 fails).
- From state 1 B , the machine transitions to state 0 at rate λ_2 (component 2 fails).

Step 1: Expected Cycle Length
Let T be the cycle length (time from state 2 to state 0). Define:

- E_2 : Expected time to state 0 starting from state 2.
- E_{1A} : Expected time to state 0 starting from state 1A.
- E_{1B} : Expected time to state 0 starting from state 1 B .

From state 1A or 1B, the time to failure is exponential:

$$E_{1A} = \frac{1}{\lambda_1}, \quad E_{1B} = \frac{1}{\lambda_2}.$$

From state 2, the probability of transitioning to state 1A is $P(\text{ to 1 A}) = \frac{\lambda_2}{\lambda_1+\lambda_2}$ (if component 2 fails), and to state 1 B is $P(\text{ to 1 B}) = \frac{\lambda_1}{\lambda_1+\lambda_2}$ (if component 1 fails). The expected time spent in state 2 is $\frac{1}{\lambda_1+\lambda_2}$. Thus:

$$E_2 = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2}\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2}\right) = \frac{1}{\lambda_1+\lambda_2}\left(1 + \frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}\right).$$

Further simplification:

$$1 + \frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2} = \frac{\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}$$

so:

$$E_2 = \frac{\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2(\lambda_1+\lambda_2)} = \frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1\lambda_2(\lambda_1+\lambda_2)}$$

Alternatively, using algebraic manipulation:

$$E_2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}.$$

Thus, the expected cycle length is:

$$\mathbb{E}[T] = E_2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}.$$

The cost per cycle consists of:

- A fixed cost K incurred when the machine fails (upon entering state 0).
- Operating costs: At rate c_2 per unit time in state 2 , and c_1 per unit time in state 1 (either 1A or 1B).

Let T_2 be the time spent in state 2 per cycle, and T_1 be the total time spent in state 1 per cycle. The expected cost per cycle is:

$$\mathbb{E}[\text{ Cost per cycle }] = K + c_2\mathbb{E}[T_2] + c_1\mathbb{E}[T_1].$$

- Expected time in state 2 : The time in state 2 is exponential with rate $\lambda_1+\lambda_2$, so:

$$\mathbb{E}\left[T_2\right]=\frac{1}{\lambda_1+\lambda_2}.$$

- Expected time in state 1: The time in state 1 depends on which substate is entered:
- If entered state 1A (probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$), the time is exponential with rate λ_1, so expected time is $\frac{1}{\lambda_1}$.
- If entered state 1 B (probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$), the time is exponential with rate λ_2, so expected time is $\frac{1}{\lambda_2}$. Thus:

$$\mathbb{E}\left[T_1\right]=\mathbb{E}[T]-\mathbb{E}\left[T_2\right]=\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}\right)-\frac{1}{\lambda_1+\lambda_2}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{2}{\lambda_1+\lambda_2}=$$

By the renewal reward theorem:

$$\text{Long-run average cost }=\frac{\mathbb{E}[\text{ Cost per cycle }]}{\mathbb{E}[T]}=\frac{K+c_2 \cdot \frac{1}{\lambda_1+\lambda_2}+c_1 \cdot \frac{\lambda_1^2+\lambda_2^2}{\lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)}}{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{1}{\lambda_1+\lambda_2}}.$$

Substitute $\mathbb{E}[T]=\frac{\lambda_1^2+\lambda_1 \lambda_2+\lambda_2^2}{\lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)}$ and write the numerator with a common denominator:

$$\mathbb{E}[\text{ Cost per cycle }]=K+\frac{c_2 \lambda_1 \lambda_2+c_1\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)}.$$

Thus:

$$\text{Long-run average cost }=\frac{K+\frac{c_2 \lambda_1 \lambda_2+c_1\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)}}{\frac{\lambda_1^2+\lambda_1 \lambda_2+\lambda_2^2}{\lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)}}=\frac{K \lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)+c_2 \lambda_1 \lambda_2+c_1\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_1^2+\lambda_1 \lambda_2+\lambda_2^2}.$$

The denominator is $\lambda_1^2+\lambda_1 \lambda_2+\lambda_2^2$, and the numerator is $K \lambda_1 \lambda_2\left(\lambda_1+\lambda_2\right)+c_1\left(\lambda_1^2+\lambda_2^2\right)+c_2 \lambda_1 \lambda_2$