6

Problem

> In Example 5.3, if clerk % i % serves at an exponential rate % \lambda_{i}, i=1,2 %, prove that
>
> $$\mathrm{P}\{\text{ Smith is not the last one }\} = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$$

Proof

Given the independent increment, The probability is:

$$P(X_1 < X_2) \cdot P(X_1 < X_2) + P(X_1 > X_2) \cdot P(X_1 > X_2)$$

$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^2$$

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Problem

> Machine 1 is currently working, and Machine 2 will start working % t % time units from now. If the lifetime of machine % i % follows an exponential distribution with rate % \lambda_{i}(i=1,2) %, what is the probability that Machine 1 fails before Machine 2?

Solution

The probability is:

$$1 - P(X_1 > t)P(X_1 > X_2)$$

$$=1-e^{-\lambda_1 t}\frac{\lambda_2}{\lambda_1+\lambda_2}$$

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Problem

> There are three jobs to be processed. The processing time of job $i(i=1,2,3)$ is an exponential random variable with rate μ_{i}. There are two available processors, so two jobs can be processed immediately, and the third job starts when one of the initial two finishes.
> (a) Let T_{i} denote the completion time of job i. If the goal is to minimize $\mathrm{E}\left[T_{1}+T_{2}+T_{3}\right]$, which two jobs should be processed first when $\mu_{1}<\mu_{2}<\mu_{3}$?
> (b) Let M (called the makespan) be the total time until all three jobs are completed. Let S be the time when only one processor is working. Prove that:
>
> $$2\mathrm{E}[M]=\mathrm{E}[S]+\sum_{i=1}^{3}\frac{1}{\mu_i}$$
>
> For the following parts, assume $\mu_{1}=\mu_{2}=\mu, \mu_{3}=\lambda$. Let $P(\mu)$ denote the probability that the last job to finish is either job 1 or job 2, and let $P(\lambda)=1-P(\mu)$ denote the probability that the last job to finish is job 3.
> (c) Express $\mathrm{E}[S]$ in terms of $P(\mu)$ and $P(\lambda)$.
> Let $P_{i, j}(\mu)$ be the value of $P(\mu)$ when jobs i and j are processed first.
> (d) Prove that $P_{1,2}(\mu) \leq P_{1,3}(\mu)$.
> (e) If $\mu>\lambda$, show that $\mathrm{E}[M]$ is minimized when job 3 is one of the first two jobs to be processed.
> (f) If $\mu<\lambda$, show that $\mathrm{E}[M]$ is minimized when jobs 1 and 2 are processed first.

Solution (a)

$$\begin{aligned}&E[T_1+T_2+T_3]\\&=E[\min(X,Y)+\max(X,Y)+(\min(X,Y)+E[Z])]\\&=E[X+Y+Z]+E[\min(X,Y)]\end{aligned}$$

Therefore choose the job $2,3$

Solution (b)

Denote the time where each job finish as $T_{(1)},T_{(2)},T_{(3)}$, then, the total working time gives us:

$X+Y+Z=M+T_{(2)}=2M-S$

Then apply the expectation:

$2E[M]=E[S]+\sum_{i=1}^{3}\frac{1}{\mu_i}$

Solution (c)

$E[S]=P\left(M=T_1\right)\frac{1}{\mu}+P\left(M=T_2\right)\frac{1}{\mu}+P\left(M=T_3\right)\frac{1}{\lambda}=\frac{P\left(M=T_1\right)+P\left(M=T_2\right)}{\mu}+\frac{P\left(M=T_3\right)}{\lambda}=\frac{P(\mu)}{\mu}+\frac{P(\lambda)}{\lambda}.$

Solution (d)

$P_{1,2}(\mu)=P(Y>Z)=\frac{\lambda}{\lambda+\mu}$

$P_{1,3}(\mu)=1-\left(\frac{\mu}{\mu+\lambda}\right)^2$

Then:

$P_{1,3}(\mu)-P_{1,2}(\mu)=\frac{\lambda\mu}{(\lambda+\mu)^2}\geq 0$

Solution (e)

We need minimize $E[S]$, notice $\mu > \lambda$, then we need maximize $P(\mu)$ (from (c)), then $P_{1,3}(\mu)$ is larger.

Solution (f)

Similarly, we need minimize $P(\mu)$, then $P_{1,2}(\mu)$ is smaller

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Problem

> A doctor has two scheduled appointments, one at 1:00 PM and another at 1:30 PM. The appointment durations are independent exponential random variables with a mean of 30 minutes. Assuming both patients arrive on time, find the expected time that the patient with the 1:30 PM appointment spends in the doctor's office.

Solution

$E[T] = P(X > 30)(E[X] + E[Y]) + P(X < 30)E[Y] = 60e^{-1} + 30(1 - e^{-1}) = 30 + 30e^{-1}$

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Problem

> Let $S(t)$ denote the price of a security at time t. A popular model for the process $\{S(t), t \geq 0\}$ assumes that the price remains constant until a "shock" occurs, at which point the price is multiplied by a random factor. If we let $N(t)$ denote the number of shocks up to time t, and X_i denote the multiplicative factor of the i-th shock, then this model assumes:

> $$S(t)=S(0)\prod_{i=1}^{N(t)} X_i$$
>
> where $\prod_{i=1}^{N(t)} X_i=1$ when $N(t)=0$. Assume that:
>
> - The X_i are independent exponential random variables with rate μ.
> - $\{N(t), t \geq 0\}$ is a Poisson process with rate λ.
> - $\{N(t), t \geq 0\}$ is independent of the X_i.
> - $S(0)=s$.
>   (a) Find $\mathrm{E}[S(t)]$.
>   (b) Find $\mathrm{E}\left[S^2(t)\right]$.

Solution (a)

$$\begin{aligned}
& E[S(t)] \\
& =S(0) E[\textstyle\prod_{i=1}^{N(t)} X_i] \\
& =S(0) E[E[\textstyle\prod_{i=1}^{N(t)} X_i] | N(t)] \\
& =S(0) E[\frac{1}{\mu^{N(t)}} | N(t)] \\
& == s \exp \left(-\lambda t\left(1-\frac{1}{\mu}\right)\right)
\end{aligned}$$

Solution (b)

Similarly:

$$E[S^2(t)]=s^2 \exp \left(-\lambda t\left(1-\frac{2}{\mu^2}\right)\right)$$

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Problem

> Let $\{M_i(t), t \geq 0\}$ $(i=1,2,3)$ be independent Poisson processes with rates $\lambda_i (i=1,2,3)$, and define:

> $$N_1(t) = M_1(t) + M_2(t), \quad N_2(t) = M_2(t) + M_3(t)$$

The stochastic process $\{(N_1(t), N_2(t)), t \geq 0\}$ is called a bivariate Poisson process.
(a) Find $\mathrm{P}\{N_1(t) = n, N_2(t) = m\}$.
(b) Find $\mathrm{Cov}(N_1(t), N_2(t))$.

Solution (a)

$$\begin{aligned}
&P(N_1(t) = n, N_2(t) = m) \\
&= \textstyle\sum_{i=0}^{\min(m,n)} P(M_1(t) = n-i, M_2(t) = i, M_3(t) = m-i) \\
&= \textstyle\sum_{i=0}^{\min(m,n)} \frac{(\lambda_1 t)^{n-i} e^{-\lambda_1 t}}{(n-i)!} \frac{(\lambda_2 t)^i e^{-\lambda_2 t}}{i!} \frac{(\lambda_3 t)^{m-i} e^{-\lambda_3 t}}{(m-i)!} \\
&= \textstyle\sum_{i=0}^{\min(m,n)} \frac{\lambda_1^n \lambda_3^n t^{m+n} e^{-(\lambda_1+\lambda_2+\lambda_3)t}}{n!m!} \frac{i!(m\cdots(m-i+1))(n\cdots(n-i+1))\lambda_2^i}{(\lambda_1\lambda_3 t)^i}
\end{aligned}$$

Solution (b)

$$\begin{aligned}
&Cov(N_1(t), N_2(t)) \\
&= Cov(M_1(t) + M_2(t), M_2(t) + M_3(t)) \\
&= E[M_2^2(t)] - (E[M_2(t)])^2 \\
&= Var[M_2^2(t)] \\
&= \lambda_2 t
\end{aligned}$$

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Problem

> Prove that if $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2, respectively, then $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$, where $N(t) = N_1(t) + N_2(t)$.

Solution

$N(0) = 0$

$N(t_i) - N(t_{i-1}) = [N_1(t_i) - N_1(t_{i-1})] + [N_2(t_i) - N_2(t_{i-1})]$ is independent

$N(t+s)-N(s)=[N_1(t+s)-N_1(s)]+[N_2(t+s)-N_2(s)]$ is poission distributed.

Therefore, $N(t+s)-N(s) \sim$ Poisson $((\lambda_1+\lambda_2)\,t)$.

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Problem

> Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the n-th event. Find:
> (a) $\mathrm{E}\left[S_4\right]$,
> (b) $\mathrm{E}\left[S_4 \mid N(1)=2\right]$,
> (c) $\mathrm{E}[N(4)-N(2) \mid N(1)=3]$.

Solution (a)

$$E[S_4]=\frac{4}{\lambda}$$

Solution (b)

$$1+\frac{2}{\lambda}$$

Solution (c)

$$2\lambda$$

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Problem

> Events occur according to a Poisson process with rate λ. At each event occurrence time, we must decide whether to continue or stop, with the goal of stopping at the last event time before a

> fixed time T, where $T > 1/\lambda$. Specifically:
>
> - If an event occurs at time $t(0 \leq t \leq T)$ and we decide to stop, we win if no additional events occur in $(t, T]$, otherwise we lose.
> - If we choose not to stop at an event and no additional events occur before T, we lose.
> - If no events occur before T, we lose.

> Consider the strategy of stopping at the first event that occurs after a fixed time $s(0 \leq s \leq T)$.
> (a) What is the probability of winning when using this strategy?
> (b) What value of s maximizes the winning probability?
> (c) Show that when using the optimal s from (b), the winning probability is $1/e$.

Solution (a)

Only one event occurs in $(s, T]$, so the probability of winning is:

$\lambda(T-s)e^{-\lambda(T-S)}$

Solution (b)

$$\lambda(T-s)=1 \Longrightarrow s=T-\frac{1}{\lambda}.$$

Solution (c)

$$P(\text{win})=\lambda\left(T-\left(T-\frac{1}{\lambda}\right)\right)e^{-\lambda\left(T-\left(T-\frac{1}{\lambda}\right)\right)}=\lambda\left(\frac{1}{\lambda}\right)e^{-\lambda\left(\frac{1}{\lambda}\right)}=1\cdot e^{-1}=\frac{1}{e}.$$

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Problem

> The time between consecutive train arrivals is uniformly distributed over $(0, 1)$ hours.
> Passengers arrive according to a Poisson process with a rate of 7 per hour. Suppose a train has just departed.
> Let X be the number of passengers who board the next train. Find:

> (a) $\mathrm{E}[X]$,
> (b) $\mathrm{Var}(X)$.

Solution (a)

$$\begin{aligned}&E[X]\\&=E[E[X|T]]\\&=E[\lambda T]=\frac{\lambda}{2}\\&=3.5\end{aligned}$$

Solution (b)

$$\begin{aligned}&Var(X)\\&=Var(E(X|T))+E(Var(X|T))\\&=Var(\lambda T)+E(\lambda T)\\&=\frac{\lambda^2}{12}+\frac{7}{2}\\&=\frac{91}{12}\end{aligned}$$

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Problem

> Team 1 and Team 2 are playing a match. The teams score according to independent Poisson processes with rates λ_1 and λ_2, respectively. The match stops when one team leads by k points. Find the probability that Team 1 wins.

Solution

$$P_0=\frac{\left(\frac{\lambda_1}{\lambda_2}\right)^k}{1+\left(\frac{\lambda_1}{\lambda_2}\right)^k}=\frac{\frac{\lambda_1^k}{\lambda_2^k}}{1+\frac{\lambda_1^k}{\lambda_2^k}}=\frac{\lambda_1^k}{\lambda_1^k+\lambda_2^k}$$

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Problem

> An insurance company has two types of claims. Let $N_i(t)$ denote the number of type i claims up to time t, and assume that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The successive claim amounts for type 1 are independent exponential random variables with a mean of \$1000, while those for type 2 are independent exponential random variables with a mean of $\$5000$. Given that a claim of $\$4000$ has just arrived, what is the probability that it is a type 1 claim?

Solution

$$P(\text{ Type } 1 \mid X = 4000) = \frac{P(\text{ Type } 1) \cdot f_{X_1}(4000)}{P(\text{ Type } 1) \cdot f_{X_1}(4000) + P(\text{ Type } 2) \cdot f_{X_2}(4000)}$$

$$= \frac{\frac{10}{11000} e^{-4}}{\frac{10}{11000} e^{-4} + \frac{1}{55000} e^{-0.8}} = \frac{e^{-4}}{e^{-4} + \frac{1}{5} e^{-0.8}}$$

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Problem

> Customers enter a bank according to a Poisson process with rate λ. Suppose two customers arrive in the first hour. What are the probabilities of the following events?
> (a) Both customers arrive in the first 20 minutes.
> (b) At least one customer arrives in the first 20 minutes.

Solution (a)

$$P\left(T_1 \leq \frac{1}{3}, T_2 \leq \frac{1}{3}\right) = P\left(T_1 \leq \frac{1}{3}\right) \cdot P\left(T_2 \leq \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Solution (b)

$P(\text{ At least one }) = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}.$

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Problem

> Assume people arrive at a bus stop according to a Poisson process with rate λ. The bus departs at time t. Let X denote the total waiting time of all people boarding the bus by time t. We want to determine $\operatorname{Var}(X)$. Let $N(t)$ be the number of people who arrived by time t.
> (a) What is $\mathrm{E}[X \mid N(t)]$?
> (b) Argue that $\operatorname{Var}(X \mid N(t)) = N(t) t^2 / 12$.
> (c) What is $\operatorname{Var}(X)$?

Solution (a)

$\mathrm{E}[X \mid N(t) = n] = \sum_{i=1}^n \mathrm{E}\left[t - T_i\right] = n\left(t - \mathrm{E}\left[T_i\right]\right) = n\left(t - \frac{t}{2}\right) = \frac{nt}{2}.$

Solution (b)

$\operatorname{Var}(X \mid N(t) = n) = \sum_{i=1}^n \operatorname{Var}\left(t - T_i\right) = n \cdot \operatorname{Var}\left(T_i\right) = n \cdot \frac{t^2}{12} = \frac{nt^2}{12}.$

Solution (c)

$\operatorname{Var}(X) = \mathrm{E}[\operatorname{Var}(X \mid N(t))] + \operatorname{Var}(\mathrm{E}[X \mid N(t)])$

Then:

$$\begin{aligned}
\mathrm{E}[\operatorname{Var}(X \mid N(t))] = \mathrm{E}\left[\frac{N(t) t^2}{12}\right] &= \frac{t^2}{12} \mathrm{E}[N(t)] = \frac{t^2}{12} \lambda t = \frac{\lambda t^3}{12}. \\
\operatorname{Var}(\mathrm{E}[X \mid N(t)]) = \operatorname{Var}\left(\frac{N(t) t}{2}\right) &= \frac{t^2}{4} \operatorname{Var}(N(t)) = \frac{t^2}{4} \lambda t = \frac{\lambda t^3}{4}.
\end{aligned}$$

Then:

$$\mathrm{Var}(X) = \frac{\lambda t^3}{12} + \frac{\lambda t^3}{4} = \frac{\lambda t^3}{3}.$$

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Problem

> Satellites are launched into space according to a Poisson process with rate λ. Each satellite remains in space for a random time (with distribution G) before landing. Find the probability that at time t, there are no satellites in space that were launched before time s, where $s < t$.

Solution

$P(L > t-u) = 1 - G(t-u).$

$\lambda \int_0^s P(L > t-u) du = \lambda \int_0^s (1 - G(t-u)) du$

$P(\text{ None survive }) = \exp\left(-\lambda \int_{t-s}^t (1 - G(v)) dv\right)$

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Problem

> A store opens at 8:00 AM. Customers arrive according to the following Poisson process rates:
>
> - From 8:00 AM to 10:00 AM: 4 customers per hour.
> - From 10:00 AM to 12:00 PM: 8 customers per hour.
> - From 12:00 PM to 2:00 PM: The arrival rate increases linearly from 8 to 10 customers per hour.
> - From 2:00 PM to 5:00 PM: The arrival rate decreases linearly from 10 to 4 customers per hour.
> Determine the distribution of the total number of customers entering the store on a given day.

Solution

$N \sim$ Poisson $(8+16+18+21=63)$.

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Problem

> Suppose events occur according to a nonhomogeneous Poisson process with intensity function $\lambda(t)$, $t>0$. Further assume that an event occurring at time s is a type 1 event with probability $p(s)$, $s>0$. If $N_1(t)$ is the number of type 1 events that occur by time t, what type of process is $\{N_1(t), t \geq 0\}$?

Solution

Independent Increments

$N_1(t)-N_1(s)$ is poission with mean: $\int_s^t p(u)\lambda(u)du$

Then: it's a nonhomogeneous poission process

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Problem

> Suppose $\{N_0(t), t \geq 0\}$ is a Poisson process with rate $\lambda=1$. Let $\lambda(t)$ be a nonnegative function of t, and define:
>
> $$m(t)=\int_0^t \lambda(s)ds$$
>
> Define $N(t)=N_0(m(t))$. Show that $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t)(t \geq 0)$.

Solution

$N(0)=N_0(m(0))=N_0(0)=0$.

$N\left(t_i\right)-N\left(t_{i-1}\right)=N_0\left(m\left(t_i\right)\right)-N_0\left(m\left(t_{i-1}\right)\right)$ are independent.

$N_0(m(t))-N_0(m(s)) \sim$ Poisson $(m(t)-m(s))$ =Poisson$\left(\int_s^t \lambda(u) du\right)$

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> In good years, storms occur according to a Poisson process with rate 3 per unit time, while in other years, they occur at rate 5 per unit time. Suppose the probability that next year is a good year is 0.3 . Let $N(t)$ denote the number of storms in the first t units of time next year.
> (a) Find $\mathrm{P}\{N(t)=n\}$.
> (b) Is $\{N(t), t \geq 0\}$ a Poisson process?
> (c) Does $\{N(t), t \geq 0\}$ have stationary increments? Why?
> (d) Does it have independent increments? Why?
> (e) If there are 3 storms by $t=1$, what is the conditional probability that it is a good year?

Solution (a)

$$\mathrm{P}\{N(t)=n\}=0.3 \cdot \frac{(3 t)^n e^{-3 t}}{n!}+0.7 \cdot \frac{(5 t)^n e^{-5 t}}{n!}$$

Solution (b)

No, there's no stationary increment.

Solution (c)

No, depends on year type

Solution (d)

No, it depends on the year type.

Solution (e)

$$\mathrm{P}(\text{Good} \mid N(1)=3)=\frac{\mathrm{P}(N(1)=3 \mid \text{ Good }) \cdot \mathrm{P}(\text{ Good })}{\mathrm{P}(N(1)=3)}=$$
$$\frac{0.3 \cdot \frac{27 e^{-3}}{6}}{0.3 \cdot \frac{27 e^{-3}}{6}+0.7 \cdot \frac{125 e^{-5}}{6}}=\frac{8.1 e^{-3}}{8.1 e^{-3}+87.5 e^{-5}}.$$

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Problem

> When $\{X(t), t \geq 0\}$ is a compound Poisson process, determine:
>
> $$\operatorname{Cov}(X(t), X(t+s))$$

Solution

$\operatorname{Cov}(X(t), X(t+s))=\operatorname{Cov}(X(t), X(t)+\Delta X)=\operatorname{Var}(X(t))+\operatorname{Cov}(X(t), \Delta X)=$
$\operatorname{Var}(X(t))=\lambda t \mathrm{E}\left[Y^2\right]$

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Problem

> Customers arrive at an ATM according to a Poisson process with a rate of 12 per hour. The amount withdrawn per transaction is a random variable with a mean of $30 and a standard deviation of $50 (negative withdrawals represent deposits). The ATM operates for 15 hours each day. Find the approximate probability that the total daily withdrawals are less than $\$6000$.

Solution

$\mathrm{E}[X(t)]=\lambda t \mathrm{E}[Y]=12 \times 15 \times 30=5400$

$$\operatorname{Var}(X(t)) = \lambda t \mathrm{E}\left[Y^2\right] = \lambda t\left(\sigma_Y^2 + (\mathrm{E}[Y])^2\right) = 12 \times 15 \times \left(50^2 + 30^2\right) = 612,000$$

$$P(X(15) < 6000) = P\left(Z < \frac{6000 - 5400}{782.30}\right) = P(Z < 0.767) \approx 0.778$$