1

Problem

> 1. Consider the following simultaneous moving game.

		L_2	$Player2$ M_2	R_2
	L_1	$1,1$	$6,0$	$0,0$
$Player1$	M_1	$0,6$	$4,4$	$0,0$
	R_1	$0,0$	$0,0$	$2,2$

(a) Identify a mixed strategy for Player 1 that strictly dominates the strategy M_1.
(b) Compute the set of all Nash equilibria, including those associated with mixed strategies. It is important to note that in any mixed strategy Nash equilibrium, strictly dominated strategies receive a probability of zero.
(c) Now consider a two-stage game where the stage game is the simultaneous move game and the discount factor is set to $\delta = 1$. Is it possible to identify a subgame perfect Nash equilibrium in which strategies M_1 and M_2 are employed in some stage? If so, please specify the subgame perfect Nash equilibrium; if not, please provide an explanation for why this is the case.
(d) Referring to the two-stage game discussed in part (c), what is the maximum payoff achievable for each of the two players when they engage in rational play?
(e) Consider the infinitely repeated game, using the simultaneous move game as the stage game, and let the discount factor be δ. What is the minimum value of δ under which the two players implement grim trigger strategies and M_1 and M_2 are employed in every stage? Additionally, please outline the grim trigger strategy for each player.

Solution (a)

$$(\frac{3}{4}, 0, \frac{1}{4})$$

Solution (b)

Pure: $(L_1, L_2), (R_1, R_2)$
Mixed: $((\frac{2}{3}, 0, 13), (\frac{2}{3}, 0, 13))$

Solution (c)

- Stage 2: Must be an NE- $(L_1, L_2), (R_1, R_2)$, or mixed-none include M_1 or M_2.
- Stage 1: Playing (M_1, M_2) (payoff 4,4) deviates to L_1 (6 against M_2), breaking NE unless enforced, but stage 2 constraints prevent it.

No subgame-perfect NE includes M_1 or M_2 due to domination and NE requirements.
Answer for (c): Not possible, as M_1 is strictly dominated and excluded from NE, including stage 2 .

Solution (d)

$2+2=4$

Solution (e)

Grim trigger: Play (M_1, M_2) $(4,4)$ until deviation, then revert to NE (e.g., (L_1, L_2), payoff 1,1) forever.

- Cooperation: $4+4\delta+4\delta^2+\cdots=\frac{4}{1-\delta}$
- Deviate: Player 1 plays L_1 vs. $M_2: 6$, then 1 forever: $6+\frac{\delta}{1-\delta}$
- Incentive: $\frac{4}{1-\delta} \geq 6+\frac{\delta}{1-\delta}$
- Solve: $4 \geq 6(1-\delta)+\delta \rightarrow 4 \geq 6-6\delta+\delta \rightarrow 4 \geq 6-5\delta \rightarrow 5\delta \geq 2 \rightarrow \delta \geq \frac{2}{5}$

Answer for (e): Minimum $\delta=\frac{2}{5}$. Grim trigger: Play M_1, M_2 unless deviation, then L_1, L_2 forever.

2.10

Problem

> 2.10. The accompanying simultaneous-move game is played twice, with the outcome of the first stage observed before the second stage begins. There is no discounting. The variable x is greater than 4 , so that $(4,4)$ is not an equilibrium payoff in the one-shot game. For what values of x is the following strategy (played by both players) a subgame-perfect Nash equilibrium?

Play Q_i in the first stage. If the first-stage outcome is (Q_1, Q_2), play P_i in the second stage. If the first-stage outcome is (y, Q_2) where $y \neq Q_1$, play R_i in the second stage. If the first-stage outcome is (Q_1, z) where $z \neq Q_2$, play S_i in the second stage. If the first-stage outcome is (y, z) where $y \neq Q_1$ and $z \neq Q_2$, play P_i in the second stage.

	P_2	Q_2	R_2	S_2
P_1	$2,2$	$x,0$	$-1,0$	$0,0$
Q_1	$0,x$	$4,4$	$-1,0$	$0,0$
R_1	$0,0$	$0,0$	$0,2$	$0,0$
S_1	$0,-1$	$0,-1$	$-1,-1$	$2,0$

Solution

- Both follow the strategy: Play (Q_1, Q_2).
  - First stage: (Q_1, Q_2) gives $(4,4)$.
  - Second stage: (P_1, P_2) gives $(2,2)$.
  - Total payoff: $(4+2, 4+2) = (6,6)$.
- Player 1 deviates to P_1, Player 2 plays Q_2 :
  - First stage: (P_1, Q_2) gives $(\mathrm{x}, 0)$.
  - Second stage: Since $y = P_1 \neq Q_1$, play $(\mathrm{R}_1, \mathrm{R}_2)$ gives $(0,2)$.
  - Total for Player 1: $\mathrm{x} + 0 = \mathrm{x}$.
  - Total for Player 2: $0 + 2 = 2$.
- Player 1's incentive: Compare total payoffs:
  - Sticking to $Q_1 : 6$.
  - Deviating to $P_1 : x$.
  - No deviation if $6 \geq x$ or $x \leq 6$.
- Player 2 deviates to P_2, Player 1 plays Q_1 :
  - First stage: (Q_1, P_2) gives $(0, \mathrm{x})$.

- Second stage: Since $z = P_2 \neq Q_2$, play (S_1, S_2) gives $(2, 0)$.
- Total for Player 2: $x + 0 = x$.
- Total for Player 1: $0 + 2 = 2$.
- Player 2's incentive: 6 (sticking) vs. x (deviating).
- No deviation if $x \leq 6$.

Therefore if $x < 6$ it's subgame-perfect NE

2.11

Problem

2.11. The simultaneous-move game (below) is played twice, with the outcome of the first stage observed before the second stage begins. There is no discounting. Can the payoff $(4, 4)$ be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium? If so, give strategies that do so. If not, prove why not.

	L	C	R
T	$3,1$	$0,0$	$5,0$
M	$2,1$	$1,2$	$3,1$
B	$1,2$	$0,1$	$4,4$

Solution

- Both play (B, R): First stage (4,4), second stage (T, L) with $(3, 1)$, total $(7, 5)$.
- Player 1 deviates to T, Player 2 plays R:
  - First stage: (T, R) with $(5, 0)$.
  - Second stage: Since $(T, R) \neq (B, R)$, play (M, C) with $(1, 2)$.
  - Total: Player 1: $5 + 1 = 6 < 7$, Player 2: $0 + 2 = 2 < 5$.
- Player 1 deviates to M, Player 2 plays R:
  - First stage: (M, R) with $(3, 1)$.
  - Second stage: (M, C) with $(1, 2)$.
  - Total: Player 1: $3 + 1 = 4 < 7$, Player 2: $1 + 2 = 3 < 5$.
- Player 2 deviates to L, Player 1 plays B:
  - First stage: (B, L) with $(1, 2)$.
  - Second stage: (M, C) with $(1, 2)$.
  - Total: Player 1: $1 + 1 = 2 < 7$, Player 2: $2 + 2 = 4 < 5$.

- Player 2 deviates to C, Player 1 plays B:
  - First stage: (B, C) with $(0, 1)$.
  - Second stage: (M, C) with $(1, 2)$.
  - Total: Player 1: $0 + 1 = 1 < 7$, Player 2: $1 + 2 = 3 < 5$.

No unilateral deviation improves either player's total payoff:

- Player 1: Sticking with B gives 7, deviating gives 6 (T), 4 (M).
- Player 2: Sticking with R gives 5, deviating gives 4 (L), 3 (C).

Thus, (B,R) is a NE in the first stage given the second-stage strategies.