Discrete Martingale

1

Problem

> If $\{Z_n, n \geq 1\}$ is a martingale, prove that for $1 \leq k < n$,
>
> $\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right] = Z_k$.
>
> Similarly, if $\{Z_n, n \geq 1\}$ is a submartingale, prove that
>
> $\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right] \geq Z_k$

Proof (a)

Notice:

$$\begin{aligned}&\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right]\\&= E[E[Z_n|Z_{n-1}, \ldots, Z_1]|Z_k, \ldots, Z_1]\\&= E[Z_{n-1}|Z_k, \ldots, Z_1]\end{aligned}$$

By induction, we can get:

$\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right] = Z_k$

Proof (b)

Similarly, we have:

$$\begin{aligned}&\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right]\\&= E[E[Z_n|Z_{n-1}, \ldots, Z_1]|Z_k, \ldots, Z_1]\\&\geq E[Z_{n-1}|Z_k, \ldots, Z_1]\end{aligned}$$

By induction, we can get:

$\mathbb{E}\left[Z_n \mid Z_k, \ldots, Z_1\right] \geq Z_k$

2

Problem

> Let $S_0 = 0, S_n = \epsilon_1 + \cdots + \epsilon_n$ for $n \geq 1$, where $\{\epsilon_i\}$ are i.i.d. exponential random variables with $\lambda = 1$. Show that
>
> $$X_n = 2^n \exp\left(-S_n\right), \quad n \geq 1$$
>
> is a martingale.

Solution

First:

$$E[|X_n|] \\ = E[2^n e^{-n}] \\ < \infty$$

Then:

$$E[X_{n+1}|X_1, \ldots, X_n] \\ = E[2^{n+1} e^{-S_n} e^{-\epsilon_{n+1}}] \\ = 2^{n+1} e^{-S_n} E[e^{-\epsilon_{n+1}}] \\ = 2^{n+1} e^{-S_n} \int_0^\infty e^{-x} e^{-x} dx \\ = 2^n e^{-S_n} \\ = X_n$$

3

Problem

> If $X_i (i \geq 1)$ are i.i.d. with $\mathbb{E}[|X|] < \infty$, and N is a stopping time for $\{X_i\}$ with $\mathbb{P}(N < \infty) = 1$ and $\mathbb{E}[N] < \infty$, show that
>
> $\mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}[N]\mathbb{E}[X]$.

Solution

$$
\begin{aligned}
&E[\textstyle\sum_{i=1}^{N} X_i] \\
&= E[\textstyle\sum_{i=1}^{\infty} X_i I_{i\le N}] \\
&= \textstyle\sum_{i=1}^{\infty} E[X]P(N \ge i) \\
&= \textstyle\sum_{i=1}^{\infty} E[X] \textstyle\sum_{j=i}^{\infty} P(N = j) \\
&= \textstyle\sum_{j=1}^{\infty} E[X] \textstyle\sum_{i=1}^{j} P(N = j) \\
&= \textstyle\sum_{j=1}^{\infty} E[X] jP(N = j) \\
&= E[N]E[X]
\end{aligned}
$$

4

Problem

> Consider a process $\{X_n, n \ge 0\}$ where X_0 is a positive integer. If $X_n = 0$, then $X_{n+1} = 0$. If $X_n > 0$,
>
> $$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } 0.5 \\ X_n - 1, & \text{with probability } 0.5 \end{cases}$$
>
> (a) Show that X_n is a non-negative martingale.
>
> (b) For $X_0 = i > 0$, use Kolmogorov's inequality for submartingales to bound $\mathbb{P}(\exists n \ge 0, X_n \ge N \mid X_0 = i)$

Solution (a)

Notice $X_n \ge 0$, and:

$$
\begin{aligned}
&E[X_{n+1}|X_1, \ldots, X_n] \\
&= E[X_{n+1}|X_n] \\
&= \begin{cases} 0.5(X_n + 1) + 0.5(X_n - 1) & X_n > 0 \\ X_n & X_n = 0 \end{cases} \\
&= X_n
\end{aligned}
$$

Then:

$$
\begin{aligned}
&E[|X_n|] \\
&= E[X_n] \\
&= E[X_0] \\
&= X_0 < \infty
\end{aligned}
$$

Solution (b)

$$
\begin{aligned}
&P(\max(X_0, \ldots, X_n) > N | X_0 = i) \\
&\leq \frac{E[X_n | X_0 = i]}{N} \\
&= \frac{i}{N}
\end{aligned}
$$

which is the upper bound.

5

Problem

> Prove Kolmogorov's Inequality: Let $X_1, X_2, \ldots$ be independent random variables with mean 0 . Define $S_k = X_1 + \cdots + X_k$. For any $a > 0$, show that:
>
> $P\left\{\max_{1 \leq k \leq n} |S_k| \geq a\right\} \leq \frac{\mathrm{Var}(S_n)}{a^2}$

Proof (1)

> (1) Show that $\{S_k, k = 1, 2, \ldots\}$ is a martingale with mean 0 .

We have:

$E[|S_n|] \leq \sqrt{E[S_n^2]} = Var[S_n] < \infty$

Also:

$$
\begin{aligned}
&E[S_{n+1} | S_1, \ldots, S_n] \\
&= E[\textstyle\sum_{i=1}^{n+1} X_i | X_1, \ldots, X_n] \\
&= E[X_1 + \cdots + X_n | X_1, \ldots, X_n] + E[X_{n+1}] \\
&= S_n
\end{aligned}
$$

Proof (2)

> (2) Define $\{Z_k\}$:
>
> $Z_{k+1} = \begin{cases} S_{k+1}, & \text{if } \max_{1 \leq i \leq k} |S_i| < a \\ Z_k, & \text{if } \max_{1 \leq i \leq k} |S_i| \geq a \end{cases}$
>
> with $Z_0 = 0$. Show $\{Z_k\}$ is a martingale.

Notice $E[|Z_{n+1}|] < \infty$, and:

$$
\begin{aligned}
&E[Z_{n+1}|Z_1, \ldots, Z_n] \\
&= E[S_{n+1}|Z_1, \ldots, Z_n]P(\max_{1\leq i\leq n}|S_i| < a) + E[Z_n|Z_1, \ldots, Z_n]P(\max_{1\leq i\leq n}|S_i| \geq a) \\
&= E[S_{n+1}|S_1, \ldots, S_n]P(\max_{1\leq i\leq n}|S_i| < a) + Z_nP(\max_{1\leq i\leq n}|S_i| \geq a) \\
&= S_nP(\max_{1\leq i\leq n}|S_i| < a) + Z_nP(\max_{1\leq i\leq n}|S_i| \geq a) \\
&= Z_nP(\max_{1\leq i\leq n}|S_i| < a) + Z_nP(\max_{1\leq i\leq n}|S_i| \geq a) \\
&= Z_n
\end{aligned}
$$

Proof (3)

> (3) For a martingale $\{M_k\}$ with $M_0 = 0$, show:
>
> $\sum_{i=1}^n E\left[(M_i - M_{i-1})^2\right] = E\left[M_n^2\right].$

We have:

$$
\begin{aligned}
&\sum_{i=1}^n E[(M_i - M_{i-1})^2] \\
&= \sum_{i=1}^n E[E[M_i^2 - 2M_iM_{i-1} + M_{i-1}^2]|M_{i-1}] \\
&= \sum_{i=1}^n E[E[M_i^2] - M_{i-1}^2] \\
&= \sum_{i=1}^n E[M_i^2] - E[M_{i-1}^2] \\
&= E[M_n^2] - E[M_0^2] \\
&= E[M_n^2]
\end{aligned}
$$

Proof (4)

> (4) Use Chebyshev's inequality to show:
>
> $P\left\{\max_{1\leq k\leq n}|S_k| \geq a\right\} \leq \frac{E\left[S_n^2\right]}{a^2} = \frac{\mathrm{Var}(S_n)}{a^2}.$

We have:

$$
\begin{aligned}
&P\left\{\max_{1\leq k\leq n}|S_k| \geq a\right\} \\
&= P\{|Z_n| \geq a\} \\
&\leq \frac{E[Z_n^2]}{a^2} \\
&\leq \frac{E[S_n^2]}{a^2} \\
&= \frac{Var(S_n)}{a^2}
\end{aligned}
$$

Answer (5)

> (5) Does this proof apply to all $\{S_k\}$ that are mean-0 martingales?

Answer: No

The proof relies on the independent increments of $\{S_k\}$.

Specifically:

- Step (3) assumes $\{M_k\}$ has orthogonal increments (true for martingales).
- Step (4) uses the fact that $Z_n = S_n$ if no stopping occurs, which implicitly requires S_n to be a sum of independent variables.

For general martingales, Kolmogorov's inequality does not hold; instead, Doob's inequality is used, which requires different techniques. Thus, independence is essential here.

Chapter 4

2

Problem

> Suppose whether it rains today depends on the weather conditions of the previous three days. Explain how to analyze this system using a Markov chain. How many states must there be?

Solution

8, each day has 2 states, and there are:

$$2^3 = 8$$

states.

3

Problem

> In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if there was no rain in the past three days, then it will rain today with probability 0.2; and in other cases, today's weather will be the same as yesterday's with probability 0.6. Determine the transition probability matrix P

Solution

We denote R as rain and N as no rain, then:

	R R R	R R N	R N R	R N N	N R R	N N R	N N R
R R R	0.8	0.2	0	0	0	0	0
R R N	0	0	0.4	0.6	0	0	0
R N R	0	0	0	0	0.6	0	0
R N N	0	0	0	0	0	0.4	0.6
N R R	0.6	0.4	0	0	0	0	0
N R N	0	0	0.4	0.6	0	0	0
N N R	0	0	0	0	0.6	0.2	0.8
N N N	0	0	0	0	0	0	0

4

Problem

> Consider a process $\{X_n, n \geq 0\}$ taking values 0,1 , or 2 . Suppose
>
> $$\mathrm{P}\left\{X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0\right\}=\begin{cases} P_{ij}^{\mathrm{I}}, & \text{if } n \text{ is even} \\ P_{ij}^{\mathrm{II}}, & \text{if } n \text{ is odd.} \end{cases}$$
>
> Is $\{X_n, n \geq 0\}$ a Markov chain? If not, explain how to enlarge the state space to make it a Markov chain.

Solution

No, it depends on if n is odd.

We can extend to (i,k), where $k = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

5

Problem

> A Markov chain $\{X_n, n \geq 0\}$ with states $0, 1, 2$ has the transition probability matrix:
>
> $$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$
>
> Given $\mathrm{P}(X_0 = 0) = \mathrm{P}(X_0 = 1) = \frac{1}{4}$, find $\mathrm{E}[X_3]$.

Solution

Notice:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^3 = \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \dfrac{2}{9} & \frac{13}{36} \end{bmatrix}$$

Then:

$$\begin{aligned} &\left[\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\right] \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \dfrac{2}{9} & \frac{13}{36} \end{bmatrix} \\ &= \left[\tfrac{59}{144}, \tfrac{43}{216}, \tfrac{169}{432}\right] \end{aligned}$$

Therefore:

$$\mathrm{E}[X_3] = 0 \cdot \frac{59}{144} + 1 \cdot \frac{43}{216} + 2 \cdot \frac{169}{432} = \frac{43}{216} + \frac{338}{432} = \frac{53}{54}$$

8

Problem

> Suppose Coin 1 lands heads with probability 0.7, and Coin 2 lands heads with probability 0.6. If the coin tossed today lands heads, we choose Coin 1 to toss tomorrow; if it lands tails, we choose Coin 2 to toss tomorrow. Initially, we toss either Coin 1 or Coin 2 with equal probability.
>
> (a) What is the probability that Coin 1 is tossed on the third day after starting?
>
> (b) If the coin tossed on Monday lands heads, what is the probability that the coin tossed on Friday of the same week also lands heads?

Solution (a)

The probability transition matrix:

	1	2
1	0.7	0.3
2	0.6	0.4

Then for the third day:

$$[0.5, 0.5]\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}^2 = [0.665, 0.335]$$

So the probability is 0.665

Solution (b)

For Thursday:

$$[1, 0]\begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}^3 = [0.6667, 0.3333]$$

Then for Friday:

$$P(H) = 0.6667 * 0.7 + 0.3333 * 0.6 = 0.6667$$

11

Problem

> In Example 4.3, Gary was in glum 4 days ago. Given that he has not felt cheerful for a week, what is the probability that he is in glum today?
>
> The transition probability matrix of (C, S, G) (cheerful, so-so, glum) is: $\boldsymbol{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$

Solution

Notice:

$$[\frac{7}{15}, \frac{8}{15}] \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix} = [\frac{7}{15}, \frac{8}{15}]$$

Therefore, we can approximate that:

$$[0, 1] \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}^4 \approx \frac{8}{15}$$

12

Problem

> For a Markov chain $\{X_n, n \geq 0\}$ with transition probabilities $P_{i,j}$, consider the conditional probability $\mathrm{P}\left(X_n = m \mid X_0 = i, X_k \neq r \text{ for } k = 1, \ldots, n\right)$. Is this equal to the n-step transition probability $Q_{i,m}^n$ of a chain with state space excluding r and adjusted transitions $Q_{i,j} = \frac{P_{i,j}}{1-P_{i,r}}$? Prove or provide a counterexample.

Solution

Consider a Markov chain with states $\{0, 1, 2\}$ where $r = 2$.

Transition probabilities from state 0: $P_{0,0} = 1/3, P_{0,1} = 1/3, P_{0,2} = 1/3$.

Transition probabilities from state $1: P_{1,0} = 1/2, P_{1,1} = 1/4, P_{1,2} = 1/4$.

Then:

$Q_{0,0}^2 = (1/2)(1/2) + (1/2)(2/3) = 1/4 + 1/3 = 7/12$

$$\begin{aligned} &\mathrm{P}\left(X_2 = 0 \mid X_0 = 0, X_1 \neq 2, X_2 \neq 2\right) \\ &= \frac{1/9 + 1/6}{1/9 + 1/9 + 1/6 + 1/12} \\ &= 10/17 \end{aligned}$$

Answer is No.

13

Problem

> Prove that if $\boldsymbol{P}^r$ has all positive entries for some r, then $\boldsymbol{P}^n$ has all positive entries for $n \geq r$

Solution

We have:

$P^n(i,j) = \sum_k P^r(i,k) P^{n-r}(k,j) > 0$