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Problem

> Consider a Markov chain with a state space consisting of integers $0, \pm 1, \pm 2, \ldots$, and transition probabilities:
>
> $$\mathrm{P}_{\mathrm{i,i+1}} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \pm 2, \ldots$$
>
> where $p \in (0,1)$. We can directly establish recurrence in the symmetric case and determine the probability of eventually returning to state 0 in the asymmetric case.
>
> (1) Let α denote the probability that the Markov chain eventually returns to state 0 given it is currently in state 1 . Show that $\alpha = 1 - p + p\alpha^2$. (Hint: Condition on the next state.)
>
> (2) Let $\beta = \mathrm{P}\{$ eventually return to 0$\}$. Show that:
>
> $\beta = \mathrm{P}\left\{\text{ eventually return to } 0 \mid X_1 = 1\right\} p + \mathrm{P}\left\{\text{ eventually return to } 0 \mid X_1 = -1\right\}(1-p)$
> .
>
> (3) Using the previous parts, show that if $p = 0.5$ (symmetric random walk), all states are recurrent.
>
> (4) Solve Problem 17 in Chapter 4 of the textbook.
>
> (5) Referring to part (4), show that when $p > 0.5, \mathrm{P}\left\{\text{ eventually return to } 0 \mid X_1 = -1\right\} = 1$.
>
> (6) Combine part (5) with the transience of all states to show that when $p > 0.5, \beta = 2(1-p)$.
>
> (7) Referring to part (4), show that when $p < 0.5, \alpha = 1$.
>
> (8) Combine part (7) with the transience of all states to show that when $p < 0.5, \beta = 2p$.
>
> (9) Combine parts (6) and (8) to derive the general form of β.

Solution (1)

When moves to 2, we need α^2 to return to 0, therefore:

We have:

$$\alpha = (1-p) + p\alpha^2$$

Solution (2)

Notice after first move, we can only be 1 or -1, then the question is well explainerd

Solution (3)

After calculation, we have:

$$\alpha = 1$$

Then:

$$\beta = p + 1 - p = 1$$

Also, it applies for every place.

Solution (4)

Denote X_n as the place at time n, Z_n as the n-th move, then:

$$E[Z_n] = p + (1-p)(-1) = 2p - 1$$

Then:

$$\frac{X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} Z_n = 2p - 1$$

Therefore:

$$X_n = n(2p-1)$$

When $n \to \infty$, we have $X_n \to \infty$, which means any place can be only visited finite times.

Solution (5)

$X_n \to +\infty$, then for any place less than 0, 0 is must visited

Solution (6)

We have:

$$\alpha = \frac{1-p}{p}$$

Then:

$$\beta = \alpha p + (1-p) = 2(1-p)$$

Solution (7)

Same as Solution 5

Solution (8)

Same as Solution 6

Solution (9)

$$\beta = 2\min(p, 1-p)$$

Chapter 4

Problem 14

> For each of the specified Markov chains below, classify the states and determine whether they are transient or recurrent.

$$\boldsymbol{P}_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad \boldsymbol{P}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{P}_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \boldsymbol{P}_4 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

P_1: all states communicates with each other.

All states are **recurrent**

P_2: $4 \to 3 \to 1 \text{ or } 2 \to 4$

All staes are **recurrent**

P_3: no other path goes to 2

Recurrent classes:

$\{1, 3\}, \{4, 5\}$

Transient classes:

$\{2\}$

P_4: no path goes to 5, no other path goes to 4

Recurrent classes:

$\{1, 2\}, \{3\}$

Transient classes:

$\{4\},\{5\}$

Problem 15

> Prove that if a Markov chain has M states and state j is reachable from state i, then it can be reached in at most M steps.

Solution

If there are more than M steps, we consider the index of the start and end, there must be a duplicated start index m

Then, m is revisited, therefore, any path in between can be eliminated, until the path is not longer than M

Problem 16

Solution

If $P_{ij} \neq 0$,$i \rightarrow j, j \nrightarrow i$, then i can't be recurrent, because there's a positive probability that $i \rightarrow j$ and never comes back, resulting contradiction.

Problem 20

> A transition probability matrix $\boldsymbol{P}$ is called doubly stochastic if the sum of each column is 1 , i.e., for all j,
>
> $\sum_i P_{ij} = 1.$
>
> If such a chain is irreducible and aperiodic with $M+1$ states $0,1,\ldots,M$, prove that the long-run proportion is
>
> $\pi_j = \frac{1}{M+1}, \quad j = 0,1,\ldots,M$

Solution

We have:

$\pi_j = \sum_i \pi_i P_{ij} = (\vec{\pi}, \vec{P_j})$

Therefore:

$$\vec{\pi}^T = \vec{\pi}^T P$$

Then we get:

$$\lambda = 1$$

With corresponding vector:

$$\vec{v} = (\frac{1}{M+1}, \ldots, \frac{1}{M+1})^T$$

Then we get the answer:

$$\pi_j = \frac{1}{M+1}$$

Problem 21

> A standard model for mutations at a specific DNA nucleotide site is a Markov chain where a nucleotide remains unchanged with probability $1 - 3\alpha \left(0 < \alpha < \frac{1}{3}\right)$ or changes to one of the other three nucleotides with equal probability α.
>
> (a) Show that $P^n_{1,1} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$.
>
> (b) What is the long-run proportion of time the chain spends in each state?

Solution (a)

The matrix P is:

$$\begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix}$$

When $n = 1$, $P^n_{1,1} = 1 - 3\alpha$ holds.

Assume it holds for $n - 1$, then:

$$
\begin{aligned}
&P_{1,1}^{n} \\
&= (1-3\alpha)P_{1,1}^{n-1} + \alpha(1-P_{1,1}^{n-1}) \\
&= (1-3\alpha)(\frac{1}{4}+\frac{3}{4}(1-4\alpha)^{n-1}) + \alpha(1-\frac{1}{4}-\frac{3}{4}(1-4\alpha)^{n-1}) \\
&= \frac{1}{4}+\frac{3}{4}(1-4\alpha)^{n}
\end{aligned}
$$

Solution (b)

By symmetric, we can easily get:

$$\pi_j = \frac{1}{4}, j = 1,2,3,4$$

Problem 22

> Let Y_n be the sum of n independent rolls of a fair six-sided die. Find $\lim_{n\to\infty} \mathrm{P}\left\{Y_n \text{ is a multiple of 13}\right\}$.

Solution

Notice the chain is irreducible and aperiodic, then:

$$\lim_{n\to\infty} \mathrm{P}\left\{Y_n \text{ is a multiple of 13}\right\} = \frac{1}{13}$$

Problem 23

> In good weather years, the number of storms follows a Poisson distribution with mean 1 ; in bad weather years, the number of storms follows a Poisson distribution with mean 3 . The weather condition of any year depends only on the previous year's weather. After a good weather year, the next year is equally likely to be good or bad; after a bad weather year, the next year is twice as likely to be bad as good. Assume year 0 (last year) was a good weather year.
>
> (a) Find the expected total number of storms in the next two years (year 1 and year 2).
>
> (b) Find the probability that year 3 has no storms.
>
> (c) Find the long-run average number of storms per year.

Solution (a)

Year 1:

$$E[X_1] = \frac{1}{2} + \frac{1}{2} \cdot 3 = 2$$

Year 2:

$$E[X_2] = \frac{5}{12} + \frac{7}{12} \cdot 3 = \frac{13}{6}$$

Solution (b)

$$P(G_3) = \frac{29}{72}, P(B_3) = \frac{43}{72}$$

Solution (c)

Probability transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Then solve the π:

$$\pi = (\frac{2}{5}, \frac{3}{5})^T$$

Then:

$$\frac{2}{5} \cdot 1 + \frac{3}{5} \cdot 3 = \frac{11}{5}$$

Problem 26

> Prove: repeatedly moving a randomly selected card to the top eventually results in a uniform distribution over all n ! permutations.

Solution

Notice if we do it reversely, that is `remove the top card and insert into position $i$`, every move has possibility $\frac{1}{n}$ to tansit to another state, so it is double stochastic.

Problem 27

> Any individual in a population of N may be active or inactive in each time period. If an individual is active in a given time period, then independently of all other individuals, the probability that they are also active in the next time period is α. Similarly, if an individual is inactive in a given time period, then independently of all other individuals, the probability that they remain inactive in the next time period is β. Let X_n denote the number of active individuals in time period n.
>
> (a) Prove that $\{X_n, n \geqslant 0\}$ is a Markov chain.
>
> (b) Find $\mathrm{E}\left[X_n \mid X_0 = i\right]$.
>
> (c) Derive an expression for the transition probabilities.
>
> (d) Find the long-run proportion of time that exactly j individuals are active.

Solution (a)

Notice state $n+1$ only depends on state n

Solution (b)

$$E[X_{n+1}|X_n] = \alpha X_n + (N - X_n)(1-\beta)$$

With induction:

$$E\left[X_n \mid X_0 = i\right] = \left(i - \frac{N(1-\beta)}{2-\alpha-\beta}\right)(\alpha+\beta-1)^n + \frac{N(1-\beta)}{2-\alpha-\beta}$$

Solution (c)

$P(X_{n+1} = j | X_n = k) = \sum_{m=\max(0,j-(N-k))}^{\min(k,j)} \binom{k}{m} \alpha^m (1-\alpha)^{k-m} \cdot \binom{N-k}{j-m} (1-\beta)^{j-m} \beta^{N-k-(j-m)}$

Solution (d)

For $N = 1$, the stationary distribution π satisfies:
$\pi_1 = \dfrac{1-\beta}{2-\alpha-\beta}, \quad \pi_0 = \frac{1-\alpha}{2-\alpha-\beta}$

For general N, individuals act independently. The stationary distribution is binomial with parameters N and $p = \dfrac{1-\beta}{2-\alpha-\beta}$:
$\pi_j = \binom{N}{j} \left(\dfrac{1-\beta}{2-\alpha-\beta}\right)^j \left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{N-j}$

Problem 36

> A process changes its state daily according to a two-state Markov chain. If the process is in state i on one day, then the next day it is in state j with probability $P_{i,j}$, where
> $P_{0,0} = 0.4, \quad P_{0,1} = 0.6, \quad P_{1,0} = 0.2, \quad P_{1,1} = 0.8$

Each day, a message is sent. If the Markov chain is in state i on that day, the probability that the message sent is a good message is p_i, and the probability that it is a bad message is $q_i = 1 - p_i$, for $i = 0, 1$.

> (a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
>
> (b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
>
> (c) In the long run, what is the proportion of messages that are good?
>
> (d) If a good message is sent on day n, let $Y_n = 1$; otherwise, let $Y_n = 2$. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If yes, provide its transition probability matrix. If no, briefly explain why not.

Solution (a)

$0.4p_0 + 0.6p_1$

Solution (b)

We have:

$$P^4 = \begin{bmatrix} 0.2512 & 0.7488 \\ 0.2496 & 0.7504 \end{bmatrix}$$

Then it's:

$$0.2512p_0 + 0.7488p_1$$

Solution (c)

Solve $\pi^T = \pi^T P$ we get:

$$\pi = (\frac{1}{4}, \frac{3}{4})^T$$

Then:

$$0.25p_0 + 0.75p_1$$

Solution (d)

No.

Y_{n+1} is determined by X_n, instead of Y_n

Problem 37

> Prove that the stationary probabilities of a Markov chain with transition probabilities $P_{i,j}$ are also the stationary probabilities of the Markov chain defined by the transition probabilities
> $Q_{i,j} = P_{i,j}^k$
> for a specific positive integer k.

Solution

we have:

$$\pi^T = \pi^T P$$

Therefore:

$$\pi^T = \pi^T P = \pi^T = \pi^T P^2 = \pi^T = \pi^T P^k = \pi^T = \pi^T Q$$

Problem 42

> Let A be a set of states, and A^c be the set of remaining states.
> (a) What does $\sum_{i\in A}\sum_{j\in A^c}\pi_i P_{ij}$ represent?
> (b) What does $\sum_{i\in A^c}\sum_{j\in A}\pi_i P_{ij}$ represent?
> (c) Explain the identity
> $\sum_{i\in A}\sum_{j\in A^c}\pi_i P_{ij} = \sum_{i\in A^c}\sum_{j\in A}\pi_i P_{ij}$

Solution (a)

Probability of A flows to A^C

Solution (b)

Probability of A^C flows to A

Solution (c)

Two flows must be equal

Problem 45

> Consider an irreducible finite Markov chain with states $0, 1, \cdots, N$.
>
> (a) Starting from state i, what is the probability that the process eventually visits state j ? Provide an explanation.
>
> (b) Let $x_i = \mathrm{P}\{$ visiting state N before visiting state $0 \mid$ starting at $i\}$. Compute the system of linear equations satisfied by x_i, for $i = 0, 1, \cdots, N$.
>
> (c) If for $i = 1, \cdots, N-1$, it holds that $\sum_j jP_{ij} = i$, prove that $x_i = i/N$ is a solution to the equations in (b).

Solution (a)

1

finite and irreducible, therefore every state is **recurrent**

Solution (b)

We have:

$$\left\{\begin{array}{l}x_i=\sum_{j=0}^N P_{ij} x_j \quad \text{ for } i=1,2,\ldots,N-1 \\ x_0=0 \\ x_N=1\end{array}\right.$$

Solution (c)

$$\begin{aligned}
& \sum_{j=0}^N P_{ij} x_j \\
& =\sum_{j=0}^N P_{ij} \cdot \frac{j}{N} \\
& =\frac{1}{N} \sum_{j=0}^N j P_{ij} \\
& =\frac{i}{N} \\
& =x_i
\end{aligned}$$

Problem 47

> Consider an ergodic Markov chain $\{X_n, n \geqslant 0\}$ with limiting probabilities π_i. Define the process $\{Y_n, n \geqslant 1\}$ by $Y_n=(X_{n-1}, X_n)$. That is, Y_n tracks the last two states of the original chain. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, determine its transition probabilities and find $\lim_{n \rightarrow \infty} \mathrm{P}\{Y_n=(i, j)\}$

Solution

$$P\left(Y_{n+1}=(j, l) \mid Y_n=(i, j)\right)=P\left(X_{n+1}=l \mid X_n=j, X_{n-1}=i\right)=P_{jl}.$$

Therefore, Y_n is a Markov chain

$$P\left(Y_{n+1}=(k, l) \mid Y_n=(i, j)\right)=\left\{\begin{array}{ll}P_{jl} & \text{ if } k=j, \\ 0 & \text{ if } k \neq j.\end{array}\right.$$

$\lim_{n\to\infty} \mathrm{P}\{Y_n = (i,j)\} = \pi_i P_{ij}$.

Problem 57

> A particle moves between $n+1$ vertices located on a circle in the following manner: at each step, it moves one step clockwise with probability p, or one step counterclockwise with probability $q = 1-p$. Starting from a special state 0 , let T be the first time it returns to state 0 . Find the probability that all states have been visited before T.

Solution

if first move is p and goes to 1, and we want to visit $n+1$ before 0, we have:

We have:

$$\begin{cases} x_i = px_{i-1} + (1-p)x_{i+1} \quad \text{for } i = 1,2,\ldots,N \\ x_0 = 0 \\ x_{N+1} = 1 \end{cases}$$

When $p \neq \dfrac{1}{2}$ We can get:

$$x_1 = \frac{1-\frac{p}{q}}{1-\frac{p^n}{q^n}}$$

Similarly we can get the answer:

$$P = px_1 + qy_1 = p\frac{1-\frac{p}{q}}{1-\frac{p^n}{q^n}} + q\frac{1-\frac{q}{p}}{1-\frac{q^n}{p^n}}$$

When $p = \dfrac{1}{2}$:

$$x_1 = \frac{1}{2}$$

$$P = px_1 + qy_1 = \frac{1}{2}$$

Problem 59

> For the gambler's ruin model in Section 4.5.1, it is known that a gambler starts with wealth i (where $i = 0, 1, \ldots, N$), and M_i denotes the average number of bets required until the gambler either goes bankrupt (wealth reaches 0) or achieves wealth N. Prove that M_i satisfies the following system of equations:
> $M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad \text{for } i = 1, \ldots, N-1$

> Solve the above system of equations to obtain:
> $M_i = \begin{cases} i(N-i), & \text{if } p = \frac{1}{2} \\ \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2} \end{cases}$

Solution

It is trivial that:$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \cdots, N-1$

When $p = \dfrac{1}{2}$:

$$p(M_{i+1} - M_i) - q(M_i - M_{i-1}) = -1$$

Then:

$$M_N - M_{N-1} = -\frac{1}{p} - \frac{q}{p^2} - \cdots - \frac{q^{N-2}}{p^{N-1}} + \frac{q^{N-1}}{p^{N-1}}M_1 = -\frac{1 - \frac{q^{N-1}}{p^{N-1}}}{p-q} + \frac{q^{N-1}}{p^{N-1}}M_1$$

Then:

$$M_N = \frac{N}{p-q} - \frac{1 - \frac{q^N}{p^N}}{(p-q)(1-\frac{q}{p})} + \frac{1 - \frac{q^N}{p^N}}{1 - \frac{q}{p}}M_1 = 0$$

Then:

$$M_1 = \frac{1}{p-q} - \frac{N}{p(1 - \frac{q^N}{p^N})}$$

Therefore:

$$M_i = \tfrac{i}{q-P} - \tfrac{N}{q-P}\tfrac{1-(q/p)^i}{1-(q/p)^N}$$

When $p = \dfrac{1}{2}$:

$$M_N - M_{N-1} = -2(N-1) + M_1$$

Then:

$$M_N = -N(N-1) + (N-1)M_1 = 0$$

Then:

$$M_1 = \frac{1}{N}$$

Then:

$$M_i = i(N-i)$$

Therefore:

$$M_i = \begin{cases} i(N-i), & \text{if } P = \frac{1}{2} \\ \frac{i}{q-P} - \frac{N}{q-P} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } P \neq \frac{1}{2} \end{cases}$$

Problem 61

> Suppose in the gambler's ruin problem, the probability of winning a round depends on the gambler's current wealth. Specifically, let α_i be the probability that the gambler wins a round when his wealth is i. Given that the gambler's initial wealth is i, let $P(i)$ denote the probability that the gambler's wealth reaches N before reaching 0 .
>
> (a) Derive a formula relating $P(i)$ to $P(i-1)$ and $P(i+1)$.
>
> (b) Using the same method as in the gambler's ruin problem, solve the equation for $P(i)$ from part (a).
>
> (c) Suppose there are initially i balls in jar 1 and $N-i$ balls in jar 2, and each time a ball is randomly selected from the N balls and moved to the other jar. Find the probability that jar 1 becomes empty before jar 2.

Solution (a)

$$P(i) = \alpha_i P(i+1) + (1-\alpha_i)\, P(i-1)$$

Solution (b)

We omit the steps:

$$P(i) = \frac{\sum_{j=1}^{i}(-1)^{j-1}\prod_{k=1}^{j-1}\frac{1-\alpha_k}{\alpha_k}}{\sum_{j=1}^{N}(-1)^{j-1}\prod_{k=1}^{j-1}\frac{1-\alpha_k}{\alpha_k}}$$

Solution (c)

$\alpha_i = \dfrac{i}{N}$, then:

$$\frac{\sum_{j=i+1}^{N}(j-1)!(N-j)!}{\sum_{j=1}^{N}(j-1)!(N-j)!}$$

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