HW3

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PROBLEM 1: Let X be a nonempty convex set in $\mathbb{R}^n$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{g} : \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{\mathbf{m}}$ be convex and let $\mathbf{h}$ be affine, i.e. $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$. If *System 1* below has no solution $\mathbf{x}$, then *System 2* has a solution $(\lambda_0, \boldsymbol{\lambda}, \boldsymbol{\mu})$. The converse holds if $\lambda_0 > 0$.

$$\textit{System 1}\ \alpha(\mathbf{x}) < 0, \quad \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \quad \mathbf{h}(\mathbf{x}) = \mathbf{0} \text{ for some } \mathbf{x} \in X$$

$$\textit{System 2}\ \lambda_0\alpha(\mathbf{x}) + \boldsymbol{\lambda}^t\mathbf{g}(\mathbf{x}) + \boldsymbol{\mu}^{\mathbf{t}}\mathbf{h}(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in X$$

$$(\lambda_0, \boldsymbol{\lambda}) \geq \mathbf{0}, \quad (\lambda_{\mathbf{0}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \neq \mathbf{0}.$$

Hint: Consider the set $\{(z_1, \mathbf{z}_2, \mathbf{z}_3) : \text{there exists } \mathbf{x} \in X \text{ such that } \alpha(\mathbf{x}) < z_1, \mathbf{g}(\mathbf{x}) \leq \mathbf{z}_2, \mathbf{h}(\mathbf{x}) = \mathbf{z}_3\}$.

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PROBLEM 2: Let $E = \{i : g_i(\mathbf{x}^*) = 0\} = \{1, \ldots, r\}$ and the vectors

$$\nabla g_1(\mathbf{x}^*), \ldots, \nabla g_r(\mathbf{x}^*), \nabla h_1(\mathbf{x}^*), \ldots, \nabla h_k(\mathbf{x}^*)$$

are linearly independent. Then the system

$$\nabla \mathbf{g}_E(\mathbf{x}^*)\mathbf{z} < \mathbf{0}, \qquad \nabla \mathbf{h}(\mathbf{x}^*)\mathbf{z} = \mathbf{0},$$

has a solution $\mathbf{z}$ in $\mathbb{R}^n$.

Hint: Use the separation theorem for a point and a convex set.

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PROBLEM 3: Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ be an m vector. Then exactly one of the following two systems has a solution:

$$\textit{System } 1 \quad \mathbf{A}\mathbf{x} = \mathbf{b} \qquad \text{for some } \mathbf{x} \in \mathbb{R}^n$$

$$\textit{System } 2 \quad \mathbf{A}^{\mathbf{t}}\mathbf{y} = \mathbf{0}, \mathbf{b}^{\mathbf{t}}\mathbf{y} = 1 \qquad \text{for some } \mathbf{y} \in \mathbb{R}^m$$

Hint: Consider the closed convex set $\{\mathbf{y} : \mathbf{y} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{R}^n\}$.

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PROBLEM 4: Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b}$ be an m vector. Then exactly one of the following two systems has a solution:

$$\textit{System } 1 \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \qquad \text{for some } \mathbf{x} \in \mathbb{R}^n$$

System 2 $\mathbf{A^t y} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{b^t y} < 0$ for some $\mathbf{y} \in \mathbb{R}^m$

Hint: Let $\mathbf{x} = \mathbf{w} - \mathbf{v}, \mathbf{w}, \mathbf{v} \geq \mathbf{0}$. Or consider a new system $\mathbf{Ax} \leq t\mathbf{b}, t > 0$. Or consider the system $\mathbf{Ax} \leq \mathbf{0}, -\mathbf{Ax} \leq \mathbf{0}, -\mathbf{y} \leq \mathbf{0}, -\mathbf{b^t y} > 0$.

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PROBLEM 5: $\min f(\mathbf{x}) = (x_1 - 2)^4 + (x_1 - 2x_2)^2$.

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PROBLEM 6: (Linear Regression) In the linear regression problem n points $(x_1, y_1), (x_2, y_2), \ldots,$ (x_n, y_n) are given in the xy-plane and it is required to "fit" a straight line $y = ax + b$ to these points in such a way that the sum of the squares of the vertical distances of the given points from the line is minimized. That is, a and b are to be chosen so that

$$f(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

is minimized. The resulting line is called the linear regression line for the given points. Show that the coefficients a and b of the linear regression line are given by

$$b = \bar{y} - a\bar{x}, \qquad a = \frac{n\bar{x}\bar{y} - \sum_{i=1}^{n} x_i y_i}{n(\bar{x})^2 - \sum_{i=1}^{n} x_i^2},$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \qquad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

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PROBLEM 7: Maximize $f(\mathbf{x}) = x_1^2 + x_1 x_2 + x_2^2$ subject to

$$-3x_1 - 2x_2 + 6 \leq 0, \qquad -x_1 + x_2 - 3 \leq 0, \qquad x_1 - 2 \leq 0.$$

1. Sketch the feasible set.
2. Show that a solution exists.
3. Find the solution.

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