

1

Problem

Let C be convex

Let $\text{int}(C) \neq \emptyset$.

Then prove:

$$(i) \overline{\text{int}(C)} = \bar{C}$$

$$(ii) \text{int}(C) = \text{int}(\bar{C}) \text{ and then } \partial C = \partial \bar{C}.$$

Hint: $\text{int}(C) \cup \partial C = \bar{C} = \overline{\bar{C}} = \text{int}(\bar{C}) \cup \partial \bar{C}$ and $\text{int}(C) \cap \partial C = \emptyset$.

Solution (i)

First we show $\overline{\text{int}(C)} \subseteq \bar{C}$

by definition, $\text{int}(C) \subseteq C$, then $\overline{\text{int}(C)} \subseteq \bar{C}$

Then, we show $\bar{C} \subseteq \overline{\text{int}(C)}$

For $\vec{x} \in \bar{C}$, let $z \in \text{int}(C)$ by Lemma 1.21, we have:

$$w_n = \frac{1}{n}z + (1 - \frac{1}{n})x \in \text{int}(C)$$

Notice that: $w_n - x \rightarrow 0$

Then, x is a limit point of $\text{int}(C)$, by definition, $x \in \bar{C}$

□

Solution (ii)

- $\text{int}(C) = \text{int}(\bar{C})$

First, we prove $\text{int}(C) \subseteq \text{int}(\bar{C})$

by definition, $C \subseteq \bar{C}$, then we can get $\text{int}(C) \subseteq \text{int}(\bar{C})$

Then, we prove $\text{int}(\bar{C}) \subseteq \text{int}(C)$

Let $\vec{x} \in \text{int}(\bar{C})$

By definition, there exists $r > 0$ s.t. $B(\vec{x}, r) \subseteq \bar{C}$

Fix $\vec{z} \in \text{int}(C)$, consider $\vec{v} = \vec{x} - \vec{z}$

There exists some $\epsilon \in (0, 1)$ small enough s.t. $\vec{x} + \epsilon\vec{v} \in B(\vec{x}, r)$ (e.g. ϵ proportional to r)

Therefore, $\vec{x} + \epsilon\vec{v} \in \bar{C}$

Apply Lemma 1.21, we can get:

$$\frac{1}{1+\epsilon}(\vec{x} + \epsilon\vec{v}) + \frac{\epsilon}{1+\epsilon}\vec{z} = \vec{x} \in \text{int}(C)$$

□

2

Problem

Show that the following functions are convex:

1. $e^{x_1+x_2} + (x_1 - x_2)^2 + x_1^4$;
2. $e^{x_1} + e^{x_2} + (x_1 - 4x_2)^4 - 5$.

Solution 1

$$H = \begin{pmatrix} e^{x_1+x_2} + 2 + 12x_1^2 & e^{x_1+x_2} - 2 \\ e^{x_1+x_2} - 2 & e^{x_1+x_2} + 2 \end{pmatrix}$$

The trace is $2e^{x_1+x_2} + 4 + 12x_1^2 > 0$.

The determinant is $e^{x_1+x_2} (8 + 12x_1^2) + 24x_1^2 \geq 0$.

Thus, H is positive semi-definite everywhere, so f is convex.

Solution 2

Let $u = x_1 - 4x_2$. The Hessian matrix is

$$H = \begin{pmatrix} e^{x_1} + 12u^2 & -48u^2 \\ -48u^2 & e^{x_2} + 192u^2 \end{pmatrix}.$$

The trace is $e^{x_1} + e^{x_2} + 204u^2 > 0$.

The determinant is $e^{x_1}e^{x_2} + 192e^{x_1}u^2 + 12e^{x_2}u^2 \geq 0$.

Thus, H is positive semi-definite everywhere, so g is convex.

3

Problem

Consider the minimal-objective function of \mathbf{b} for fixed A and \mathbf{c} :

$$\begin{aligned} z(\mathbf{b}) &= \min \mathbf{c}^t \mathbf{x} \\ \text{s.t. } &A\mathbf{x} = \mathbf{b}, \\ &\mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Show that $z(\mathbf{b})$ as a function of \mathbf{b} is a convex function in \mathbf{b} for all feasible \mathbf{b} .

Solution

First, we show that the domain is convex

Let \vec{b}_1, \vec{b}_2 with corresponding \vec{x}_1, \vec{x}_2 s.t. $A\vec{x}_1 = \vec{b}_1, A\vec{x}_2 = \vec{b}_2$.

Then let $\vec{b}_\lambda = \lambda \vec{b}_1 + (1 - \lambda) \vec{b}_2$, we have:

$\vec{x}_\lambda = \lambda \vec{x}_1 + (1 - \lambda) \vec{x}_2 \geq \mathbf{0}$, then \vec{b}_λ is feasible

Therefore, the domain is convex.

Then, we show that $z(\vec{b})$ is convex

Let $\vec{c}'\vec{x}_1^* = z(\vec{b}_1)$, $\vec{c}'\vec{x}_2^* = z(\vec{b}_2)$, then:

\vec{b}_λ is feasible for \vec{x}_λ

Therefore: $z(\vec{b}_\lambda) \leq \lambda z(\vec{b}_1) + (1 - \lambda)z(\vec{b}_2)$

4

Problem

Let X be a nonempty compact set and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Denote $\theta(\boldsymbol{\lambda}) = \inf\{f(\mathbf{x}) + \langle \boldsymbol{\lambda}, \mathbf{g}(\mathbf{x}) \rangle : \mathbf{x} \in X\}$. Prove that $\theta(\boldsymbol{\lambda})$ is concave over \mathbb{R}^m .

Solution

Let $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^m$ and $t \in [0, 1]$.

Define $\boldsymbol{\mu} = t\boldsymbol{\lambda}_1 + (1 - t)\boldsymbol{\lambda}_2$.

Then,

$$\theta(\boldsymbol{\mu}) = \inf_{\mathbf{x} \in X} \{f(\mathbf{x}) + \langle \boldsymbol{\mu}, \mathbf{g}(\mathbf{x}) \rangle\} = \inf_{\mathbf{x} \in X} \{f(\mathbf{x}) + t \langle \boldsymbol{\lambda}_1, \mathbf{g}(\mathbf{x}) \rangle + (1 - t) \langle \boldsymbol{\lambda}_2, \mathbf{g}(\mathbf{x}) \rangle\}.$$

Denote $A(\mathbf{x}) = f(\mathbf{x}) + \langle \boldsymbol{\lambda}_1, \mathbf{g}(\mathbf{x}) \rangle$ and $B(\mathbf{x}) = f(\mathbf{x}) + \langle \boldsymbol{\lambda}_2, \mathbf{g}(\mathbf{x}) \rangle$.

The expression becomes:

$$\theta(\boldsymbol{\mu}) = \inf_{\mathbf{x} \in X} \{tA(\mathbf{x}) + (1 - t)B(\mathbf{x})\}$$

In general, for any functions u and v , we have:

$$\inf(u + v) \geq \inf u + \inf v.$$

Applying this to $u(\mathbf{x}) = tA(\mathbf{x})$ and $v(\mathbf{x}) = (1 - t)B(\mathbf{x})$,

and notice that $\inf(tA) = t \inf A$ and $\inf((1 - t)B) = (1 - t) \inf B$ (since $t, 1 - t \geq 0$), we have:

$$\theta(\boldsymbol{\mu}) \geq t \inf_{\mathbf{x} \in X} A(\mathbf{x}) + (1 - t) \inf_{\mathbf{x} \in X} B(\mathbf{x}) = t\theta(\boldsymbol{\lambda}_1) + (1 - t)\theta(\boldsymbol{\lambda}_2).$$

Thus, $\theta(\boldsymbol{\lambda})$ is concave.

5

Problem

Prove that every local solution of the following problem is a global solution as well:

$$\min_{x_1, x_2, x_3 \in \mathbb{R}} e^{x_1 - 2x_2 + x_3} + (x_1 - 5x_2 + 6x_3)^2 + (-x_1 + 2x_2 + 3x_3)^6$$

$$\text{s.t. } x_1 + x_2 - 7x_3 = 1$$

$$x_1^2 + x_2^2 + e^{x_1 - 2x_2 - x_3} \leq 2$$

$$x_1 \geq 0$$

$$x_3 \geq 0$$

Solution

First, we show that the function is convex

$$e^{x_1 - 2x_2 + x_3}:$$

- $x_1 - 2x_2 + x_3$ is linear, e^x is convex, so it's convex (Lemma 1.41)

$$(x_1 - 5x_2 + 6x_3)^2:$$

- $x_1 - 5x_2 + 6x_3$ is linear, x^2 is convex, so it's convex (Lemma 1.41)

Similarly, $(-x_1 + 2x_2 + 3x_3)^6$ is convex

Then the whole function is convex (Lemma 1.40)

Then, we show that the feasible set is convex:

$x_1 + x_2 - 7x_3 = 1$: This is a linear function, defines a convex set. (Lemma 1.38)

$x_1^2 + x_2^2 + e^{x_1 - 2x_2 - x_3} \leq 2$: Similarly, this function is convex, so this inequality is defines a convex set. (Lemma 1.38)

$x_1 \geq 0, x_3 \geq 0$: These are linear inequality, defines a convex set. (Lemma 1.38)

Therefore, the intersection of convex sets is also convex (Lemma 1.39)

□

6

Problem

Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & 2x^2 - x^3 \\ \text{s.t.} \quad & x \in \{-2, -1, 0, 1, 2\} \end{aligned}$$

1. Convert the above problem to an optimization problem with a linear objective.
2. Draw the feasible set of the reformulated problem.
3. Convexify the reformulated problem and draw the feasible set of the resulting convex problem.

Solution (1)

Let $y = (y_{-2}, y_{-1}, y_0, y_1, y_2)$ be binary variables such that:

$$y_i = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\begin{aligned} \min_y \quad & 16y_{-2} + 3y_{-1} + 0 \cdot y_0 + 1 \cdot y_1 + 0 \cdot y_2 \\ \text{s.t.} \quad & y_{-2} + y_{-1} + y_0 + y_1 + y_2 = 1 \\ & y_i \in \{0, 1\}, \quad i \in \{-2, -1, 0, 1, 2\} \end{aligned}$$

Solution (2)

$$\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$$

Solution (3)

Relax the variables:

$$\begin{aligned} \min_y \quad & 16y_{-2} + 3y_{-1} + 0 \cdot y_0 + 1 \cdot y_1 + 0 \cdot y_2 \\ \text{s.t.} \quad & y_{-2} + y_{-1} + y_0 + y_1 + y_2 = 1 \\ & y_i \geq 0 \end{aligned}$$

Then the answer:

$$\{y \in \mathbb{R}^5 : y_i \geq 0, \sum y_i = 1\}$$

7

Problem

Employ AI to evaluate the benefits of studying convex programming and to provide a concrete example illustrating how convex programming can be applied to solve practical problems.

Solution

Studying convex programming is highly beneficial because it provides a powerful framework for solving a wide range of optimization problems reliably and efficiently. Its value is evident in both theoretical research and practical applications across various industries.

The Key Benefits of Convex Programming

The primary advantages of convex programming stem from the mathematical properties of convex functions and convex sets.

- **Guaranteed Global Optimality:** Perhaps the most significant benefit is that for a convex problem, any local solution you find is guaranteed to be a global solution. This eliminates the uncertainty of whether a better solution exists, which is a major challenge in non-convex optimization.
- **Efficient and Reliable Algorithms:** Because of their well-behaved structure, convex problems can be solved using highly efficient algorithms. There are well-established methods for various types of convex problems, such as Quadratic Programming (QP) and Second-Order Cone Programming (SOCP), which are both reliable and can handle large-scale problems.
- **The Foundation for Complex Problems:** Convex optimization serves as a cornerstone for tackling more complex, non-convex problems. Techniques like **Sequential Convex Programming (SCP)** approximate a non-convex problem as a sequence of convex sub-problems, solving them iteratively. Similarly, **convex relaxations** (e.g., using Second-Order Cones) can transform intractable non-convex constraints into solvable convex ones, providing useful bounds and approximations.

A Concrete Example: Pollution Control in Wastewater Networks

A great example of convex programming in action is the management of sewer networks and wastewater treatment plants, a problem tackled by researchers using a Model-Predictive Control (MPC) framework.

- **The Practical Problem:** The goal is to control actuators (like gates and pumps) in a sewer system to minimize flooding and the release of untreated pollutants, especially during heavy rain. At its core, this involves balancing the flow of wastewater to treatment plants over time.
- **The Optimization Challenge:** The natural mathematical model of this system is **non-convex**, primarily due to two factors:
 - i. **Microbial growth kinetics** in treatment plants, which follow complex, non-linear rules.
 - ii. **Bilinear terms** arising from the product of flow rates and pollutant concentrations.
Solving non-convex optimization problems in real-time for a dynamic system like this is computationally very difficult.
- **The Convex Programming Solution:** To make the problem tractable for real-time control, the researchers applied convexification techniques:
 - They used a **second-order cone (SOC) relaxation** to convexify the complex microbial growth kinetics.
 - They introduced a novel approximation that separated the modeling of flow rates in the pipes from concentrations in the plants, effectively **eliminating the nonlinearities**.
- **The Outcome:** After these transformations, the complex, non-convex trajectory optimization problem in each control step becomes a **Second-Order Cone Program (SOCP)**. This convex formulation can be solved quickly and reliably. In simulations, this convex optimization-based controller achieved a **15% reduction in released pollutant mass** compared to conventional methods while treating nearly the same volume of wastewater.

How to Deepen Your Understanding

To further your study of convex programming, you can explore its intersections with other cutting-edge fields:

- **Integration with Machine Learning:** Concepts from convex optimization are being deeply integrated into machine learning, leading to approaches like **Decision-Focused Learning**, where ML models are tailored to improve the outcomes of downstream convex optimization solvers.
- **Convex Optimization in Autonomous Systems:** Beyond wastewater management, convex programming is crucial in safety verification for autonomous vehicles, helping to ensure that a car's steering decisions are safe and reliable.

I hope this overview provides you with a clear picture of the value and application of convex programming. If you are interested in a specific industry or type of problem, feel free to ask for more detailed examples.

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