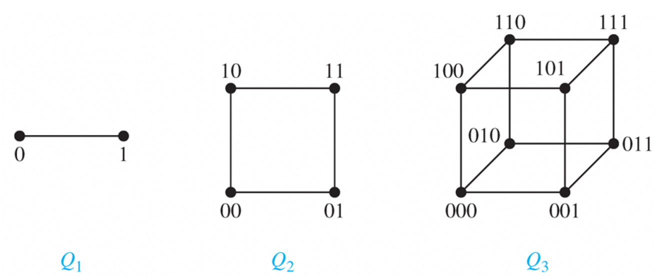
CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #5
Due: May 17th, 2024

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position. Let $l(n)$ denote the number of edges of Q_n.



(a) What is the initial condition of $l(n)$?

(b) What is the recursive function of $l(n)$?

(c) Derive the closed-form of $l(n)$ using the general approach we have learned for solving linear recurrence relation. Please provide the derivation details. Please do NOT use mathematical induction.

Solution:

(a) $l(1) = 1$;

(b) $l(n) = 2l(n-1) + 2^{n-1}$;

(c) First, compute the solution to the associated homogeneous recurrence relation $l^{(h)}(n)$. Since the characteristic equation is $r - 2 = 0$. Thus, $l^{(h)}(n) = \alpha 2^n$. Second, compute the particular solution $l^{(p)}(n)$. According to the formulation of $l(n)$, the particular solution is in the following form: $l^{(p)}(n) = pn2^n$. Substituting $l^{(p)}(n)$ into the recurrence relation of $l(n)$, we have

$$pn2^n = 2p(n-1)2^{n-1} + 2^{n-1}.$$

Thus, $p = 1/2$. Thus, the closed-form solution

$$l(n) = l^{(h)}(n) + l^{(p)}(n) = \alpha 2^n + \frac{1}{2} n 2^n.$$

By substituting the initial condition,

$$l(1) = 2\alpha + 1 = 1.$$

Thus, $\alpha = 0$. As a result $l(n) = n2^{n-1}$.

Q. 2. Consider 10 identical balloons (i.e., non-distinguishable balloons). We aim to give these balloons to four children, and each child should receive at least one balloons.

(a) How many ways to give these balloons to the children? Explain the reason.

(b) The answer to the above question is the coefficient of term ____________ of generating function ________________.

Solution:

(a) $C(9, 6)$

(b) x^{10}; $(x^1 + x^2 + x^3 + \cdots)^4$ or $(x^1 + x^2 + x^3 + \cdots + x^{10})^4$ or $(x^1 + x^2 + x^3 + \cdots + x^7)^4$ (any of these functions is ok)

Or alternatively, x^6; $(1 + x^1 + \cdots)^4$ or $(1 + x^1 + \cdots + x^6)^4$ (any of these functions is ok)

Q. 3. How many relations are there on a set with n elements that are

a) antisymmetric?

b) irreflexive?

c) neither reflexive nor irreflexive?

d) symmetric, antisymmetric and transitive?

Please explain your answer.

Solution:

a) $2^n 3^{n(n-1)/2}$

b) $2^{n(n-1)}$

c) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

d) 2^n

$\square$

Q. 4. Prove or disprove the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Solution: Counterexample: Consider $A = \{1, 2, 3\}$ and

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}.$$

Then R is symmetric and reflexive, but not transitive.

$\square$

Q. 5. Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Solution: Suppose that $(a, b) \in R$. Because $(b, b) \in R$, it then follows that $(a, b) \in R^2$. Thus, R is a subset of R^2.

$\square$

Q. 6. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also be symmetric? Explain your answer.

Solution: Yes. Yes. For both R_1 and R_2, the corresponding 0-1 matrices are both symmetric. Thus. the two matrices representing $R_1 \cap R_2$ and $R_1 \cup R_2$ are also symmetric.

$\square$

Q. 7. Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Solution: Suppose that (a, b) is in the symmetric closure of the transitive closure of R. We must show that (a, b) is in the transitive closure of the symmetric closure of R. We know that at least one of (a, b) and (b, a) is in the transitive closure of R. Hence, there is either a path from a to b in R or a path from b to a in R (or both). In the former case, there is a path from a to b in the symmetric closure of R. In the latter case, we can form a path from a to b in the symmetric closure of R by reversing the directions of all the edges in a path from b to a, going backward.

Hence, (a, b) is in the transitive closure of the symmetric closure of R.

□

Q. 8. Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution: The result follows from

$$(R^*)^{-1} = (\cup_{n=1}^{\infty} R^n)^{-1} = \cup_{n=1}^{\infty} (R^n)^{-1} = \cup_{n=1}^{\infty} R^n = R^*.$$

□

Q. 9. Which of the following are equivalence relations on the set of all people?

(1) $\{(x, y) | x$ and y have the same sign of the zodiac$\}$

(2) $\{(x, y) | x$ and y were born in the same year$\}$

(3) $\{(x, y) | x$ and y have been in the same city$\}$

Solution:

(1) This is an equivalence relation.

(2) This is an equivalence relation.

(3) This is not an equivalence relation, since it is not transitive.

□

Q. 10. Show that $\{(x, y) | x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Solution: This relation is reflexive, since $x - x = 0 \in \mathbb{Q}$. To see that it is symmetric, suppose that $x-y \in \mathbb{Q}$. Then $y-x = -(x-y)$ is again a rational number. For transitivity, if $x-y \in \mathbb{Q}$ and $y-z \in \mathbb{Q}$, then their sum, namely $x-z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of 1/2 are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number, in other words, $\{\pi + r | r \in \mathbb{Q}\}$.

$\square$

Q. 11. Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Solution: The subset relation is a partial ordering on any collection of sets, because it is reflexive, antisymmetric, and transitive. Here the collection of sets is $\mathbf{R}(S)$.

$\square$

Q. 12. Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both equivalence relations, then T is also an equivalence relation.

Solution: We need to show that T is reflexive, symmetric, and transitive.

Reflexive: For any x, we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.

Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.

Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

$\square$

Q. 13. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

a) Prove that $\preceq$ is a partial ordering.

b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \le f(x)$, so $f \preceq f$.

   Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \le g(x) \le f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

   Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \le g(x) \le h(x)$, giving $f(x) \le h(x)$. So, $f \preceq h$.

b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\le -1 = g(1)$ and $g(-1) = 1 \not\le -1 = f(-1)$. So it is not the case that for all x, $f(x) \le g(x)$, and it is not the case that for all x, $g(x) \le f(x)$. That is, these two functions are incomparable.

$\square$

Q. 14. We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0,1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0,1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.

Solution:

(a) There are many choices here. One is to let $R = \{A_0, A_1, A_2, \ldots\}$ where $A_i = \{j \in \mathbb{N} | j < i\}$. Then R has no maximal element, because for any $A_i \in R$, we have $A_i \not\subseteq A_{i+1} \in R$.

(b) For this we will do the same thing as above in reverse. Let $S = \{B_0, B_1, B_2, \ldots\}$ where $B_i = \{j \in \mathbb{N} | j \ge i\}$. Then S has no minimal element, because for any $B_i \in S$, we have $B_i \not\supseteq B_{i+1}$.

(c) Here we can combine the previous two results. Let $T = \{C_{ij} | i \in \mathbb{N},\ j \in \mathbb{N}\}$ where each $x \in \mathbb{N}$ is in C_{ij} if and only if $x = 2k$ and $k < i$, or $x = 2k + 1$ and $K \geq j$. Now T has no minimal or maximal elements, because for any $C_{ij} \in T$, $C_{i,j+1} \subsetneq C_{ij} \subsetneq C_{i+1,j}$.

$\square$