Discrete Mathematics for Computer Science

Lecture 6: Complexity of Algorithms

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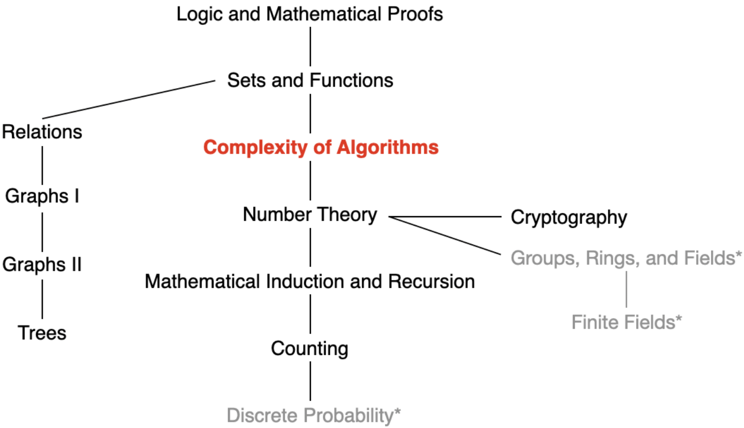
This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|,$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]

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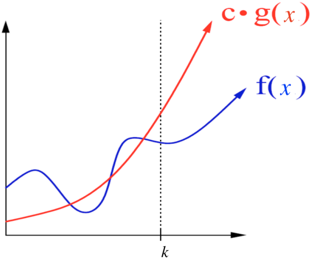

Big-O Estimates for Polynomials

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_n$ are real numbers. Then, $f(x) = O(x^n)$.

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Proof:

Assuming $x > 1$, we have

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\
&= x^n(|a_n| + |a_{n-1}|/x + \cdots + |a_1|/x^{n-1} + |a_0|/x^n) \\
&\leq x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|).
\end{aligned}$$

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&\leq x^n(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|).
\end{aligned}$$

The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-O Estimates for Some Functions

$1 + 2 + \cdots + n = O(n^2)$

$n! = O(n^n)$

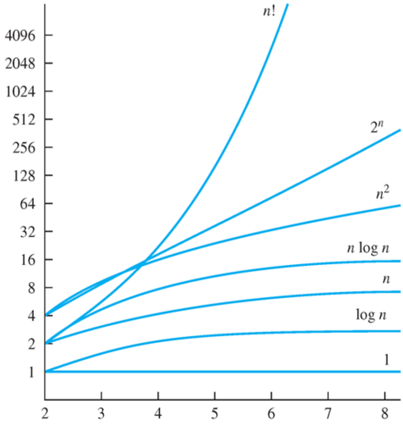
$\log n! = O(n \log n)$

$\log_a n = O(n)$ for an integer $a \geq 2$

$n^a = O(n^b)$ for integers $a \leq b$

$n^a = O(2^n)$ for an integer a

Big-O Estimates for Some Functions



Big-O Estimates for Some Functions

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Prove $\log_a n = O(n)$ for an integer $a \geq 2$.

Proof: We always have $\log_a n \leq n$ for $n \geq 1$. This can be proven using mathematical induction. ...

- $n = 1$: $log_a 1 = 0 < 1$
- Suppose $\log_a n \leq n$ for $n > 1$:

$$\log_a(n+1) \leq \log_a(an) = \log_a n + 1 \leq n + 1$$

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Thus, $n^a \leq 2^n$ for large enough n.

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Note: If f and g are functions such that

$$\lim_{n \to \infty} \frac{|f(x)|}{|g(x)|} = C < \infty,$$

then $f(x) \leq (C+1)g(x)$ for large enough x. So $f(n) = O(g(n))$. If that limit is ∞, then $f(n)$ is not $O(g(n))$.

Combinations of Functions

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x) = O(\max(|g_1(x)|, |g_2(x)|))$.

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Proof:

By definition, there exist constants C_1, C_2, k_1, k_2 such that
$|f_1(x)| \leq C_1|g_1(x)|$ when $x > k_1$ and
$|f_2(x)| \leq C_2|g_2(x)|$ when $x > k_2$. Then

$$\begin{aligned}
|(f_1 + f_2)(x)| &= |f_1(x) + f_2(x)| \\
&\leq |f_1(x)| + |f_2(x)| \\
&\leq C_1|g_1(x)| + C_2|g_2(x)| \\
&\leq C_1|g(x)| + C_2|g(x)| \\
&= (C_1 + C_2)|g(x)| \\
&= C|g(x)|,
\end{aligned}$$

where $g(x) = \max(|g_1(x)|, |g_2(x)|)$ and $C = C_1 + C_2$.

$k = \max\{k_1, k_2\}$.

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Proof:

When k $> \max(k_1, k_2)$,

$$\begin{aligned} |(f_1 f_2)(x)| &= |f_1(x)||f_2(x)| \\ &\leq C_1|g_1(x)|C_2|g_2(x)| \\ &\leq C_1 C_2|(g_1 g_2)(x)| \\ &\leq C|(g_1 g_2)(x)|, \end{aligned}$$

where $C = C_1 C_2$.

Big-Omega Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that

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Big-Ω tells us that a function grows at least as fast as another.

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Note: $f(x)$ is $\Omega(g(x))$ if and only if $g(x)$ is $O(f(x))$.

Big-Theta Notation (Big-O & Big-Omega)

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers.We say that $f(x)$ is $\Theta(g(x))$ if

- $f(x)$ is $O(g(x))$ and
- $f(x)$ is $\Omega(g(x))$.

When $f(x)$ is $\Theta(g(x))$, we say that $f(x)$ is big-Theta of $g(x)$, that $f(x)$ is of order $g(x)$, and that $f(x)$ and $g(x)$ are of the same order.

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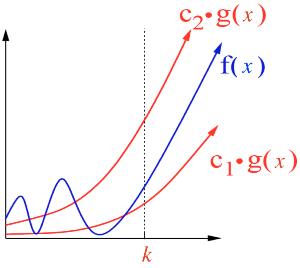

Big-Theta Notation

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is of order x^n.

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- $f(x) = O(x^n)$
- $f(x) = \Omega(x^n)$

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Note: If f and g are functions such that

$$\lim_{n\to\infty} \frac{|f(x)|}{|g(x)|} = C < \infty,$$

and

$$C \neq 0,$$

then $f(n) = \Theta(g(n))$.

Big-Theta Notation: Examples

$3n^2 + 4n = \Theta(n)$?

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$3n^2 + 4n = \Theta(n^3)$?

$n/5 + 10n \log n = \Theta(n^2)$?

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$n^2/5 + 10n \log n = \Theta(n^2)$?

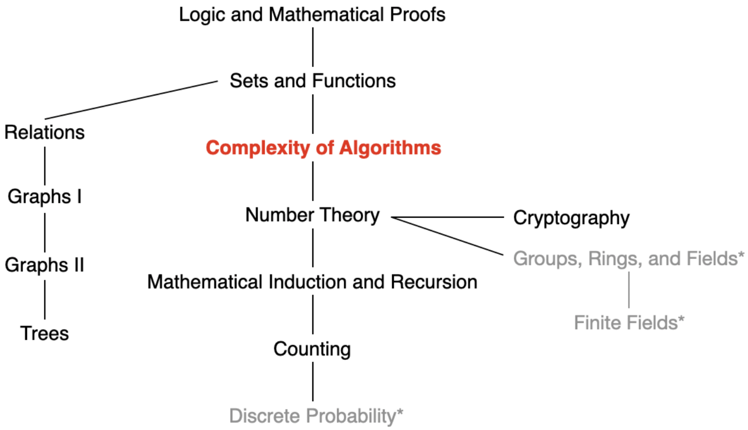
Big-Theta Notation: Examples

$3n^2 + 4n = \Theta(n)$?	No
$3n^2 + 4n = \Theta(n^2)$?	Yes
$3n^2 + 4n = \Theta(n^3)$?	No, but $O(n^3)$
$n/5 + 10n \log n = \Theta(n^2)$?	No, but $O(n^2)$
$n^2/5 + 10n \log n = \Theta(n \log n)$?	No
$n^2/5 + 10n \log n = \Theta(n^2)$?	Yes

The Growth of Functions

- Big-O notation, e.g., $O(n^2)$
  - Upper bound
- Big-Omega notation, e.g., $\Omega(n^2)$
  - Lower bound
- Big-Theta notation, e.g., $\Theta(n^2)$
  - Of the same order

This Lecture



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Example (Computational Problem and Algorithm):

- Computational Problem: Input n numbers $a_1, a_2, ..., a_n$; Output the sum of the n numbers.

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A computational problem is a specification of the desired input-output relationship.

Example (Computational Problem and Algorithm):

- Computational Problem: Input n numbers $a_1, a_2, ..., a_n$; Output the sum of the n numbers.
- Algorithm: the following procedures

  Step 1: set $S = 0$

  Step 2: for $i = 1$ to n, replace S by $S + a_i$

  Step 3: output S

Instance

An instance of a problem is a realization of all the inputs needed to compute a solution to the problem.

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A correct algorithm halts with the correct output for every input instance. We can then say that the algorithm solves the problem.

Time and Space Complexity

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Example (Algorithm)

Step 1: set $S = 0$

Step 2: for $i = 1$ to n, replace S by $S + a_i$

Step 3: output S

Time Complexity:

- Steps 1 and 3 take one operation.
- Step 2 takes $2n$ operations.

Therefore, altogether this algorithm takes $1 + 2n + 1$ operations. The time complexity is $O(n)$.

Horner's Algorithm and Its Complexity

Example: Consider the evaluation of $f(x) = 1 + 2x + 3x^2 + 4x^3$. Direct computation takes 3 additions and 6 multiplications.

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Horner's algorithm for computing
$f(x) = a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n = a_0 + x(a_1 + ... + x(a_{n-1} + a_nx))$
at a particular x:

Step 1: set $S = a_n$

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Step 1: set $S = a_n$

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Step 3: output S

The number of operations needed in this algorithm is $1 + 3n + 1 = 3n + 2$. So the time complexity of this algorithm is $O(n)$.

Note: Operations: addition, multiplication, comparison, replacement, etc.

Time Complexity: Example

Determine the time complexity of the following algorithm:

for $i := 1$ to n

  for $j := 1$ to n

    $a := 2 * n + i * j;$

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- For each i, it takes $4n$ operations to complete the second loop.
- Thus, this algorithm takes $n \times 4n = 4n^2$ operations to complete the two loops. The time complexity of this algorithm is $O(n^2)$.

Time Complexity: Example

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end for

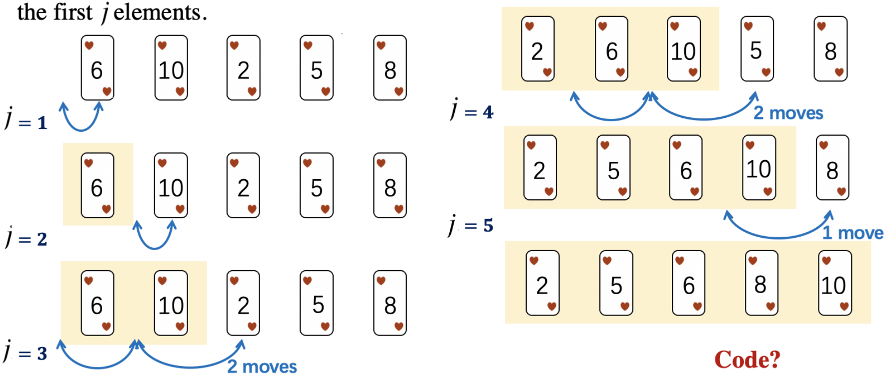
- Computing $S := S + i \times j$ takes 3 operations.
- For each i , completing the second loop takes $3i$ operations.
- Thus, this algorithm takes

$$1 + \sum_{i=1}^{n} 3i = 1 + 3\frac{n(n+1)}{2}$$

So the complexity of this algorithm is $O(n^2)$.

Time Complexity: Example - Insertion Sort

In iteration j, we move the j-th element left until its correct place is found among the first j elements.



Code?

Time Complexity: Example - Insertion Sort

Input: $A[1 \dots n]$ is an array of numbers

for $j := 2$ to n

  $key = A[j]$;

  $i = j - 1$;

  while $i \geq 1$ and $A[i] > key$ do

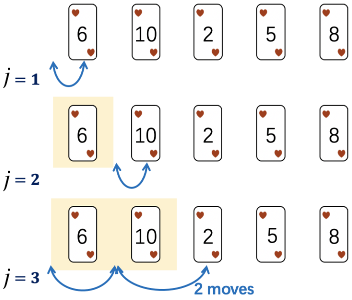
    $A[i+1] = A[i]$;

    $i - -$;

  end while

  $A[i+1] = key$;

end for



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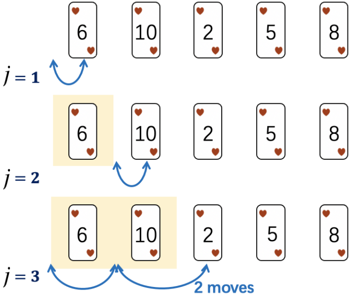

The time complexity depends on the input array $A[1, ..., n]$.

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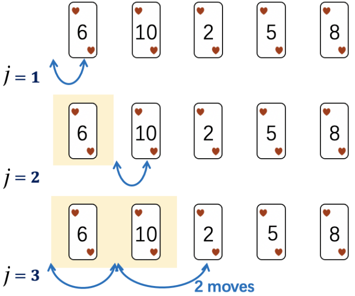

The time complexity depends on the input array $A[1, \ldots, n]$.

Consider only the number of comparisons

Three Cases of Analysis: Best-Case

Best-Case Complexity: The smallest number of operations needed to solve the given problem using this algorithm on input of specified size.

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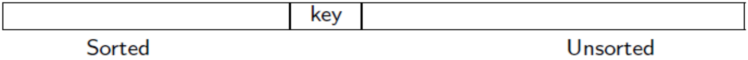
Example: (Insertion Sort)

$$A[1] \leq A[2] \leq A[3] \leq \cdots \leq A[n]$$

The number of comparisons needed is

$$\underbrace{1 + 1 + 1 + \cdots + 1}_{n-1} = n - 1 = \Theta(n)$$



"key" is compared to only the element right before it.

Three Cases of Analysis: Worst-Case

Worst-Case Complexity: The largest number of operations needed to solve the given problem using this algorithm on input of specified size.

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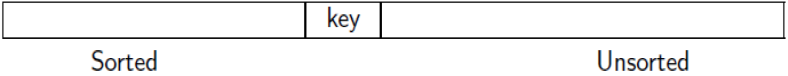
Example: (Insertion Sort)

$$A[1] \geq A[2] \geq A[3] \geq \cdots \geq A[n]$$

The number of comparisons needed is

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \Theta(n^2)$$



"key" is compared to everything element before it.

Three Cases of Analysis: Average-Case

Average-Case Complexity: The average number of operations used to solve the problem over all possible inputs of a given size is found in this type of analysis.

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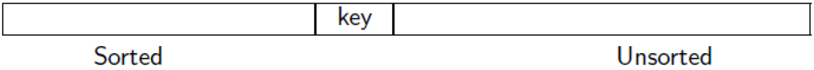
Example: (Insertion Sort)

$\Theta(n^2)$ assuming that each of the $n!$ instances are equally likely



On average, “key” is compared to half of the elements before it.

Three Cases of Analysis: Average-Case

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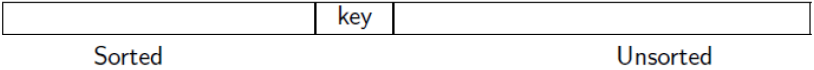

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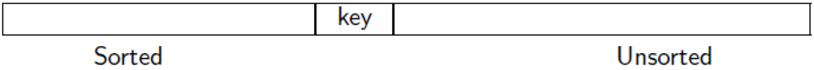

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Average-case complexity is usually difficult to compute.

Some Thoughts on Algorithm Design

Algorithm Design is mainly about designing algorithms that have small Big-O running time.

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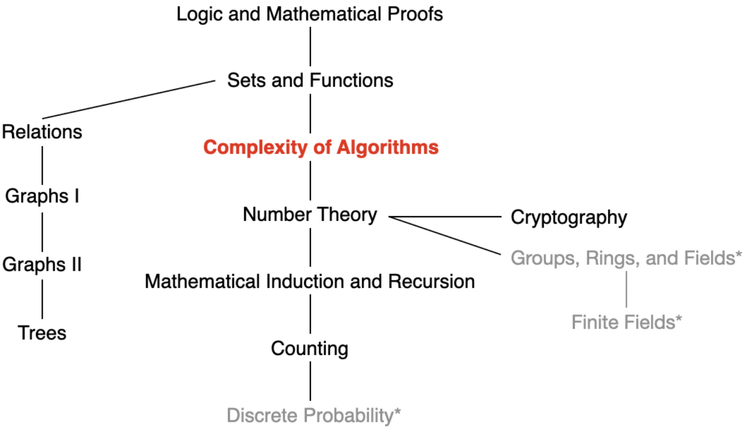
- The most straightforward manner based on the statement of the problem and the definitions of terms

A few hours of abstract thought devoted to algorithm design could speed up the solution substantially and simplified it!

This Lecture



The growth of functions, complexity of algorithm, P and NP problem, ...

Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Blame yourself.

I couldn't find a polynomial-time algorithm.
I guess I am too dumb.

Dealing with Hard Problems

What happens if you cannot find an efficient algorithm for a given problem?

Show that no-efficient algorithm exists.

I couldn't find a polynomial-time algorithm,
because no such algorithm exists.

Dealing with Hard Problems

Showing that a problem has an efficient algorithm is, relatively easy:

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We will now learn about NP-Complete problems, which provides us with a way to approach this question.

NP-Complete

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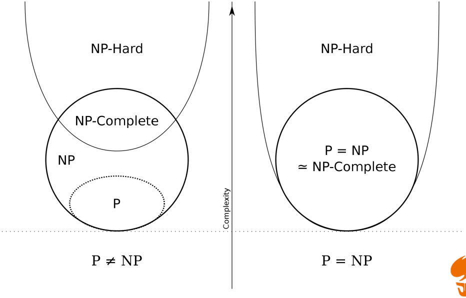

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NP-Complete and NP-Hard problems are very likely to be hard.

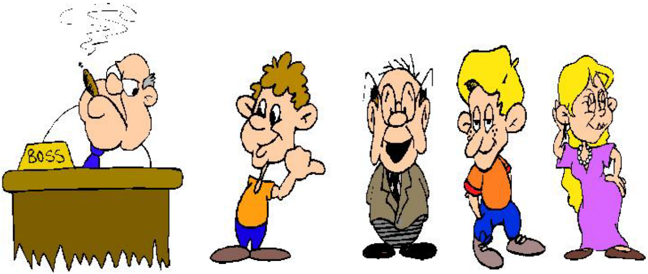
Thus, to proving that no efficient algorithm exists for a particular problem?

Prove that your problem is NP-Complete or even NP-Hard:

- Show that your problem can be reduced to a typical (well-known) NP-Complete or NP-Hard problem.

NP-Complete

What do you actually do:



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Details for P and NP

Encoding the Inputs of Problems

Complexity of a problem is measure with respect to the size of input:

- E.g., for insertion sort, $\Theta(n^2)$ is the average-case complexity, where n is the length of the array.

In order to formally discuss how hard a problem is, we need to be much more formal than before about the input size of a problem.

The Input Size of Problems

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For most problems, it is sufficient to choose some natural and (usually) simple encoding and use the size s of this encoding.

- E.g., 5 can be encoded as 101.

Input Size Example: Composite

Example: Input a positive integer n; output if there are integers $j, k > 1$ such that $n = jk$? (i.e., is n a composite number?)

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Thus, a natural measure of input size is $\lceil \log_2(n+1) \rceil$ (or just $\log_2 n$)

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Note: Back to our earlier discussions for complexity, when we use fixed length encoding regardless of a_i for $i = 1, 2, \ldots, n$, the value of m becomes a constant. Thus, we can omit the constant m.

Complexity in terms of Input Size

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This makes $\Theta(n)$ comparisons, so it might seem linear and very efficient.

But, the input size of this problem is $\log_2 n$ instead of n. The number of comparisons performed is actually $\Theta(n)$, which can be represented as $\Theta(2^{(\log_2 n)})$. It is exponential with respect to the input size.

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Example:
The standard multiplication algorithm has time $O(m_1 m_2)$, where m_1 and m_2 denote the number of digits in the two integers, respectively.

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- The algorithm is nonpolynomial!

Polynomial- vs. Nonpolynomial-Time

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In reality, an $O(n^{20})$ algorithm is not really practical.

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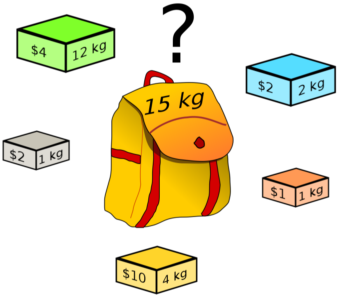
Examples:
Knapsack vs. Decision Knapsack (DKnapsack)

Knapsack V.S. DKnapsack

We have a knapsack of capacity W (a positive integer) and N objects with weights $w_1, \ldots, w_N$ and values $v_1, \ldots, v_N$, where v_n and w_n are positive integers.



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The optimization problem is at least as hard as the decision problem.

Decision Problems and Optimization Problem

Given a subroutine for solving the optimization problem, solving the corresponding decision problem is usually trivial.

- First, solve the optimization problem
- Then, check the decision problem.

Thus, if we prove that a given decision problem is hard to solve efficiently, then it is obvious that the optimization problem must be (at least as) hard.

Complexity Classes

Theory of Complexity deals with

1. the classification of certain "decision problems" into several classes:
   - the class of "easy" problems
   - the class of "hard" problems
   - the class of "hardest" problems
2. relations among the three classes
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P problem, NP problem, ...

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The Class P

Definition: A problem is solvable in polynomial time (or more simply, the problem is in polynomial time) if there exists an algorithm which solves the problem in polynomial time

- This problem is called tractable.

Definition (The Class P): The class P consists of all decision problems that are solvable in polynomial time. That is, there exists an algorithm that will decide in polynomial time if any given input is a yes-input or a no-input.

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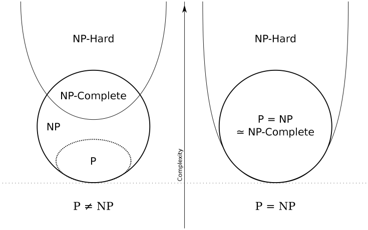
- Some other definitions for potentially harder problems ....



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Proposition: The problem **LongPath(G,k)** is in **NP**.
Proof: (PARTIAL!)

1. Note that **LongPath(G,k)** is a decision problem, as the definition of NP requires!
2. Here's my notion of certificate: A certificate is a list of vertices comprising a path of length at least k
3. Here's my algorithm for verifying a certificate:

Verify(G,k,C)

```
1. Read G, k, store graph G in an adjacency matrix
2. Read certificate C into an array
3. if m < k, where m is the length of C, return FALSE
4. for i = 1 to m - 1 do
     if G has no edge from vertex C[i-1] to C[i] return FALSE
5. for i = 0 to m - 1 do
     for j = i + 1 to m - 1 do
       if C[i] == C[j] return FALSE
6. return TRUE
```

The Class NP

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DKnapsack is an NP problem.

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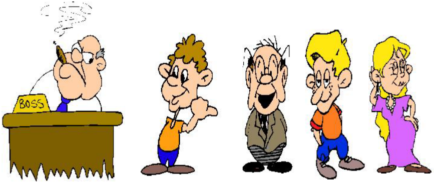

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- NP-Hard: informally "at least as hard as the hardest problems in NP"
- NP-Complete: If the problem is NP and all other NP problems are polynomial-time reducible to it.

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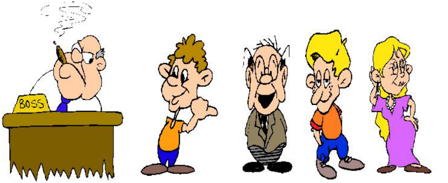
- Intuitively, $NP \subseteq P$ is doubtful.

What do you actually do:



I couldn't find a polynomial-time algorithm,
but neither could all these other smart people!

- NP-Hard: informally "at least as hard as the hardest problems in NP"
- NP-Complete: If the problem is NP and all other NP problems are polynomial-time reducible to it.

However, we are still no closer to solving it.

What We Covered

- Decision problem and optimization
- Polynomial-time algorithms
- P problem and NP problem

We will not cover the concept of P and NP problems and the related proofs in homework or exam. If you decide to do research, these concepts and proofs are important.