CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #4
Due: May 3rd, 2024

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. Suppose that a and b are real numbers with $0 < b < a$. Use mathematical induction to prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

Q. 2. A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Q. 3. Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Q. 4. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Q. 5. How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks.

<u>You should leave your answer as an equation</u>.

Q. 6. Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Q. 7. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Q. 8. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Q. 9. Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of non-empty subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Q. 10. Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Q. 11. Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.