Discrete Mathematics for Computer Science

Lecture 20: Graph

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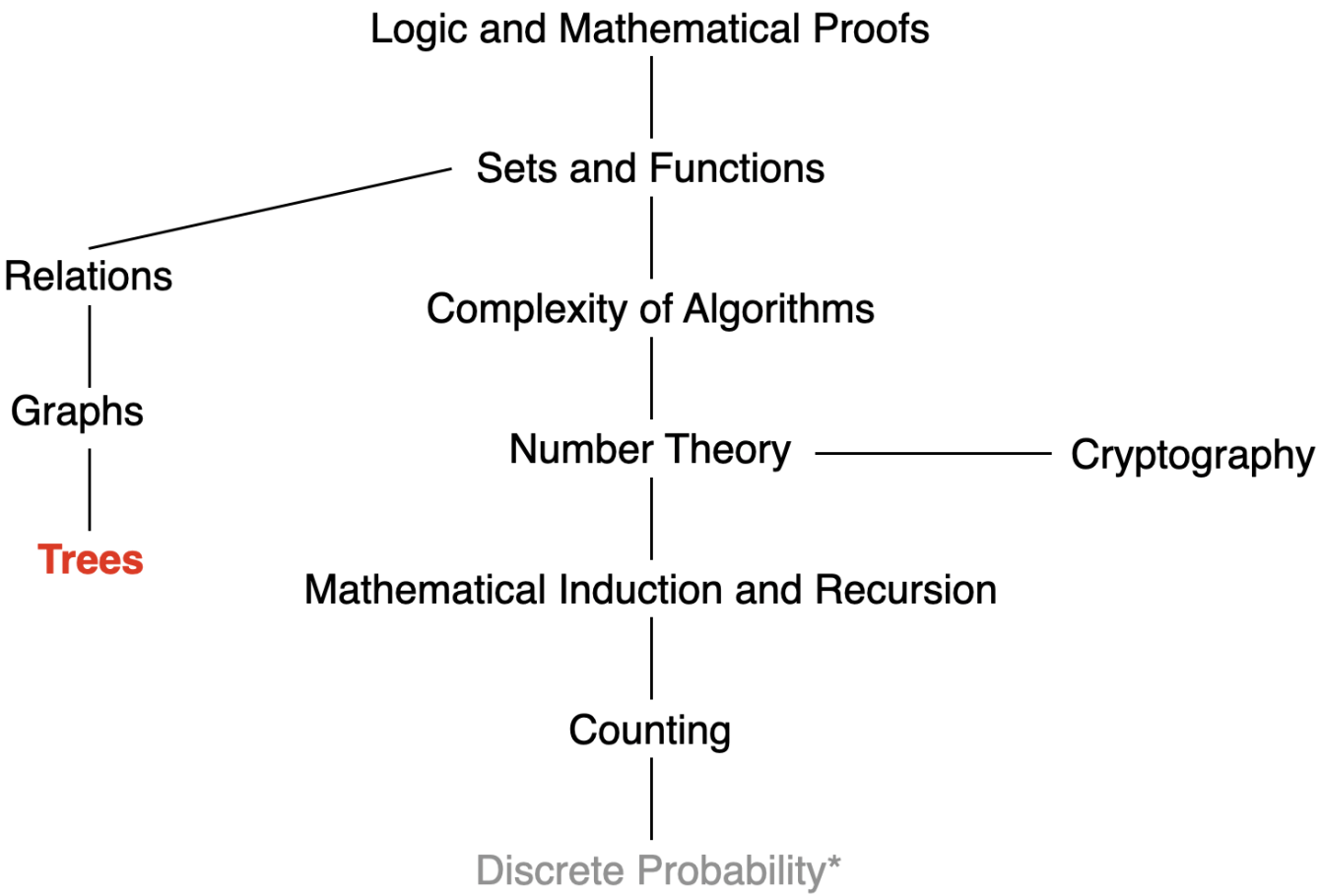
This Lecture



..., Euler and Hamiliton path, shortest-path problem, planar graphs, graph coloring, ...

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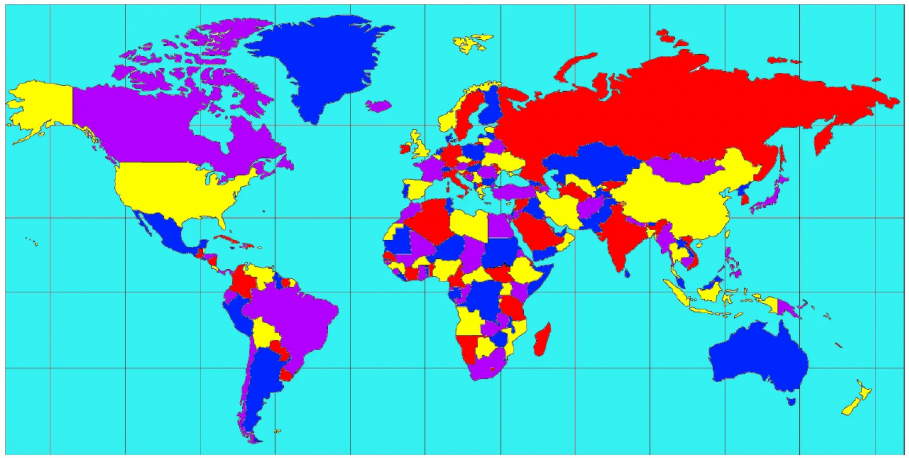
Graph Coloring

Four-color theorem: Given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



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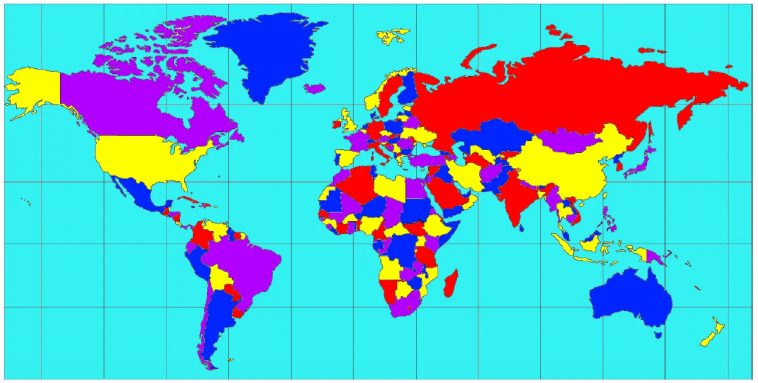
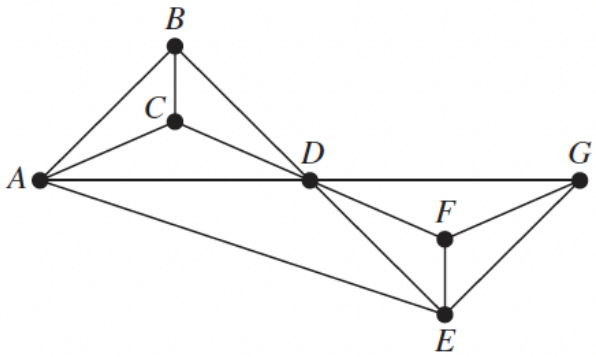
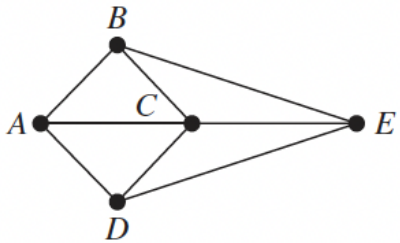
Graph Coloring







- A map can be represented as a planar graph
- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

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Graph Coloring

The chromatic number of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.

Theorem (Four Color Theorem): The chromatic number of a planar graph is no greater than four.

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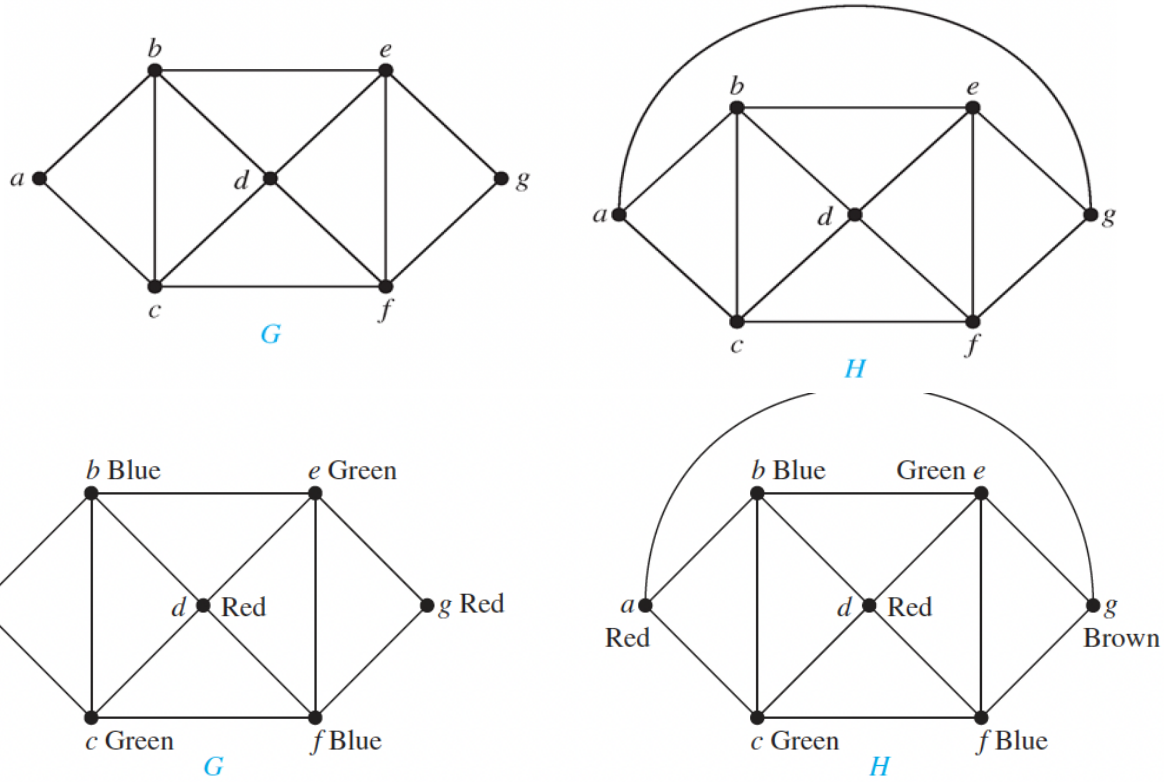
Graph Coloring: Example

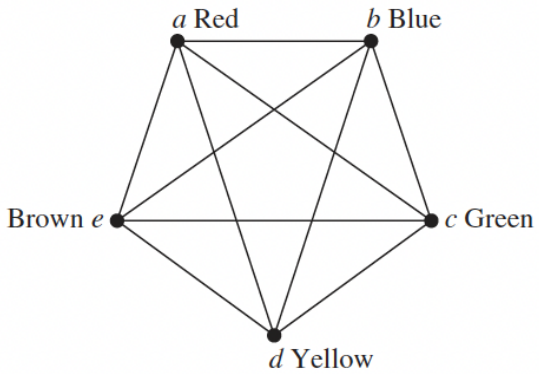


The chromatic number of G is at least three, because the vertices a, b, and c must be assigned different colors.

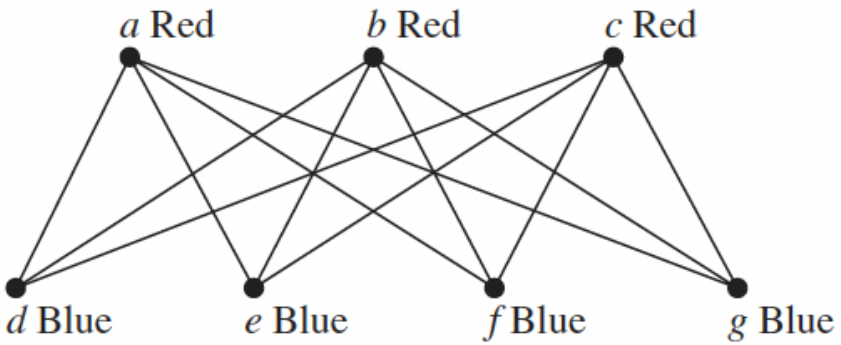
- Assign red to a, blue to b, and green to c;
- d can (and must) be colored red because it is adjacent to b and c;
- e can (and must) be colored green because it is adjacent only to vertices colored red and blue;

Graph Coloring: Example

What is the chromatic number of K_n? $\chi(K_n) = n$



What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers? $\chi(K_{m,n}) = 2$



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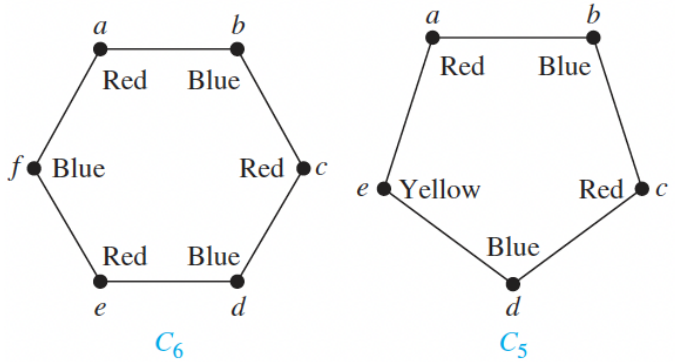
Graph Coloring: Example

What is the chromatic number of the graph C_n, where $n \geq 3$? (Recall that C_n is the cycle with n vertices.)



- When n is even, $\chi(C_n) = 2$
- When n is odd and $n > 1$, $\chi(C_n) = 3$

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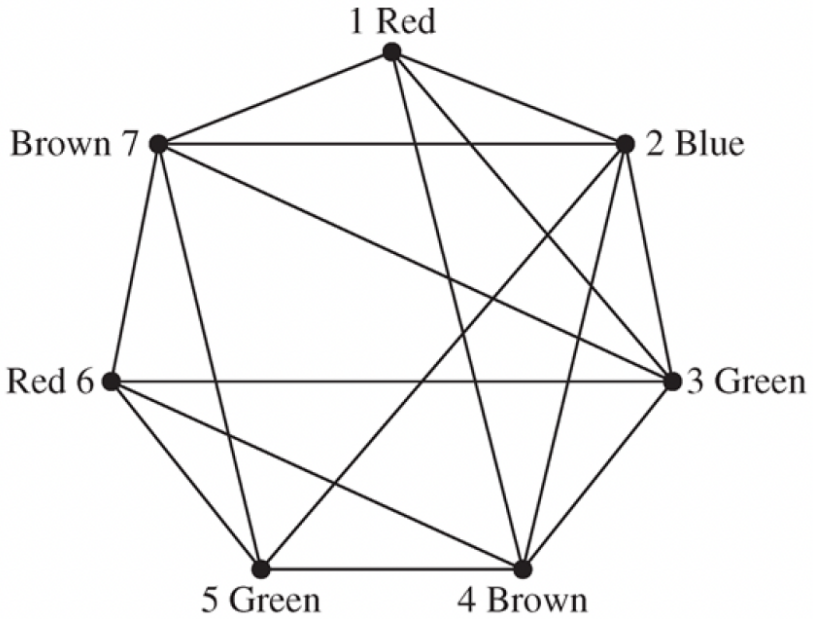
Applications of Graph Coloring

Scheduling Final Exams: Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

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Applications of Graph Coloring

Channel Assignments: Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Graph Coloring $\in$ NP-Complete

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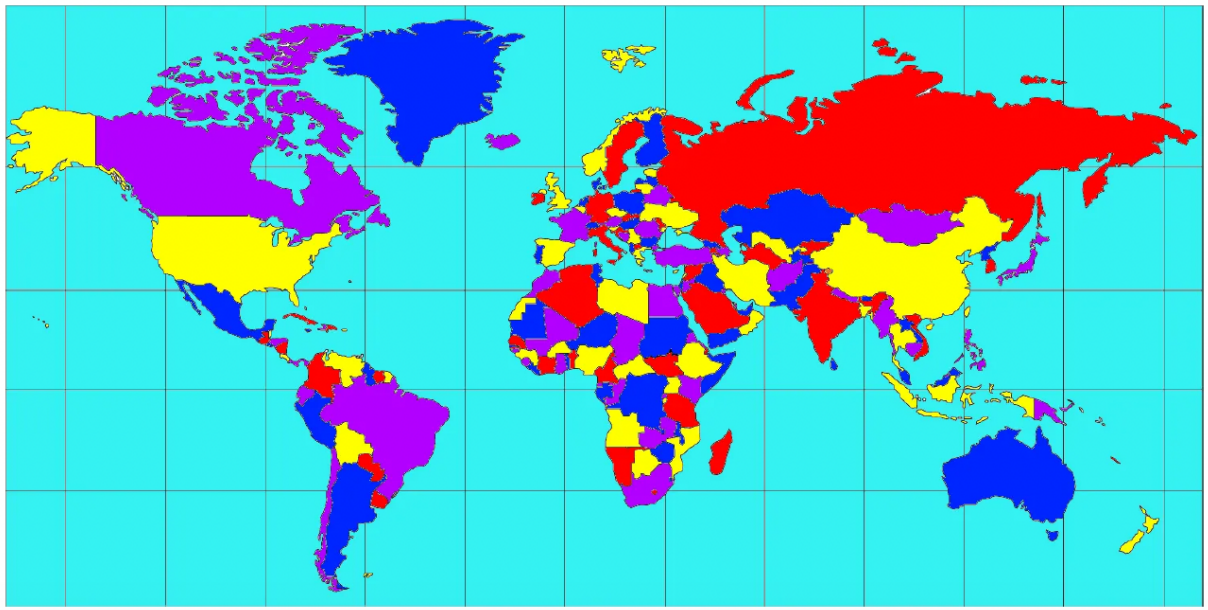
Graph Coloring

Theorem (Four Color Theorem): The chromatic number of a planar graph is no greater than four.



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Graph Coloring

Theorem (Six Color Theorem): The chromatic number of a planar graph is no greater than six.

Proof (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

- Basic step: For one single vertex, pick an arbitrary color.
- Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.
- Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

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Graph Coloring

Theorem (Five Color Theorem): The chromatic number of a planar graph is no greater than five.

Proof (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

- Basic step: For one single vertex, pick an arbitrary color.
- Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.
- Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in.

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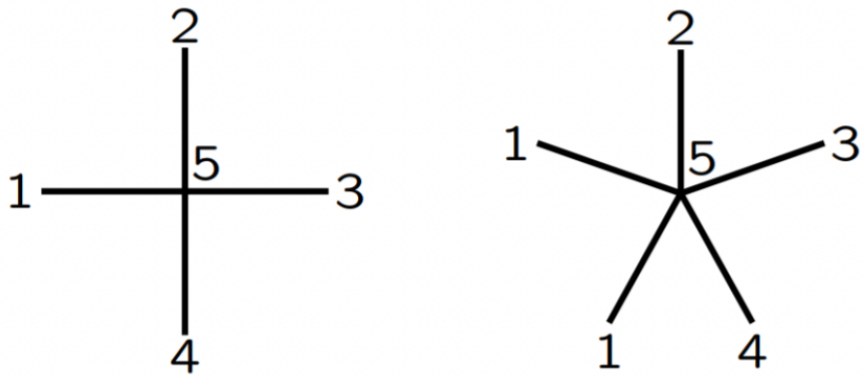
Graph Coloring

Theorem (Five Color Theorem): The chromatic number of a planar graph is no greater than five.

Proof (by induction on the number of vertices): W.l.o.g., assume that the graph is connected.

Case 1: If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we can pick an available color for it.



Case 2: If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the "special" vertex (degree 5) 1 to 5 (in order).

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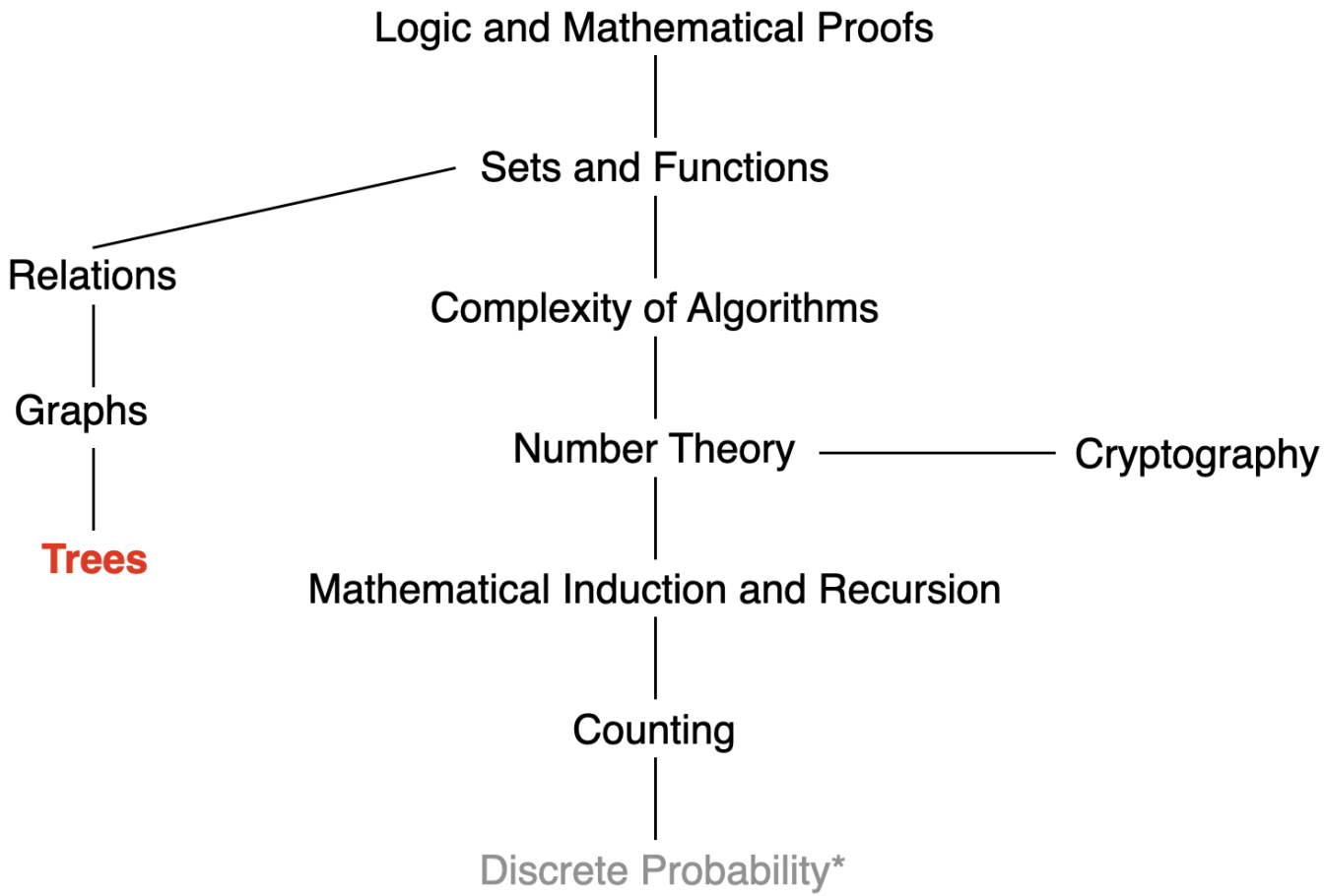

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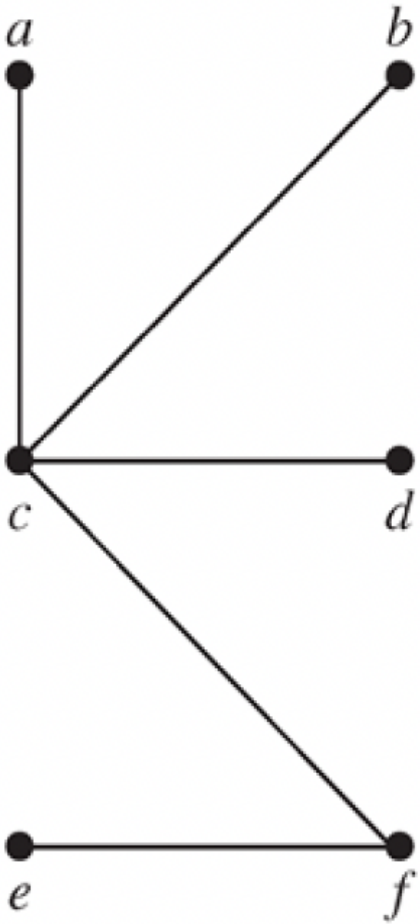
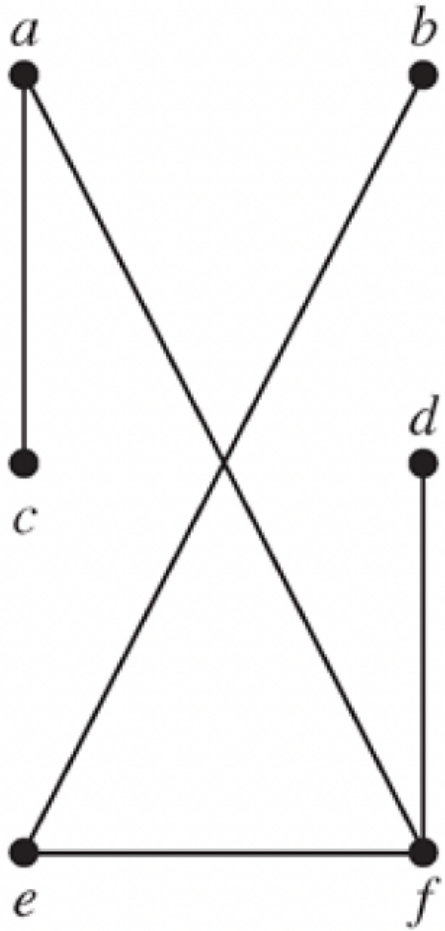
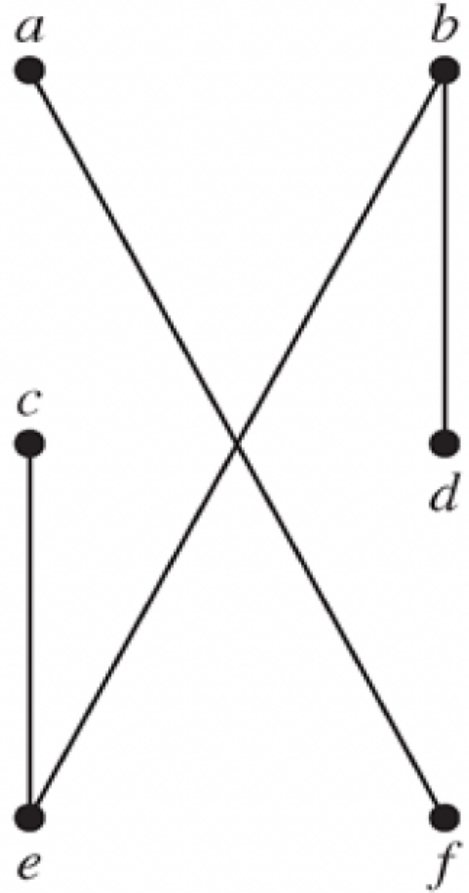
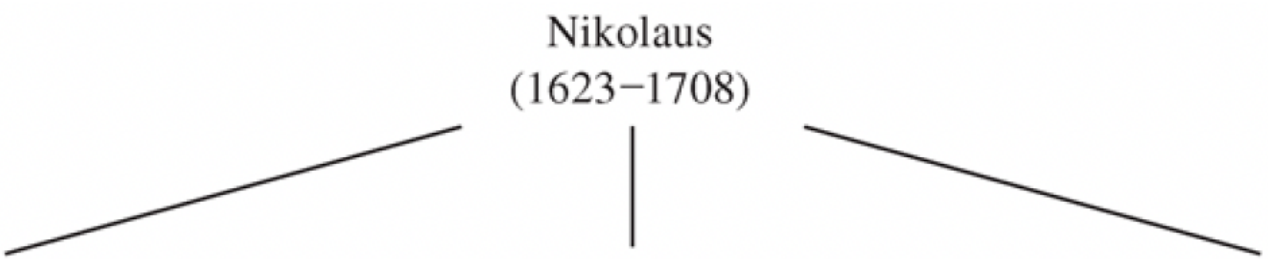
Trees

Definition: A tree is a connected undirected graph with no simple circuits.









Trees

Theorem: An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Proof: Two properties of tree: connected, no circuit

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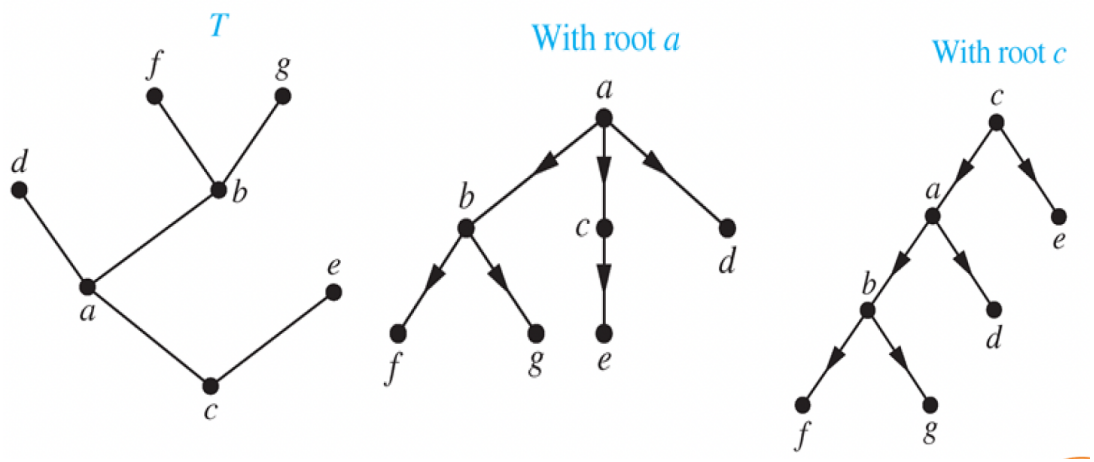
Rooted Trees

Definition: A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.



We can change an unrooted tree into a rooted tree by choosing any vertex as the root.

The arrows indicating the directions of the edges in a rooted tree can be omitted, because the choice of root determines the directions of the edges.

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Rooted Trees

The parent of v is the unique vertex u such that there is a directed edge from u to v.

When u is the parent of v, v is called a child of u.

Vertices with the same parent are called siblings.

The ancestors of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.

The descendants of a vertex v are those vertices that have v as an ancestor.

A vertex of a rooted tree is called a leaf if it has no children.

Vertices that have children are called internal vertices.

Subtree with a as its root: consists of a and its descendants and all edges incident to these descendants.

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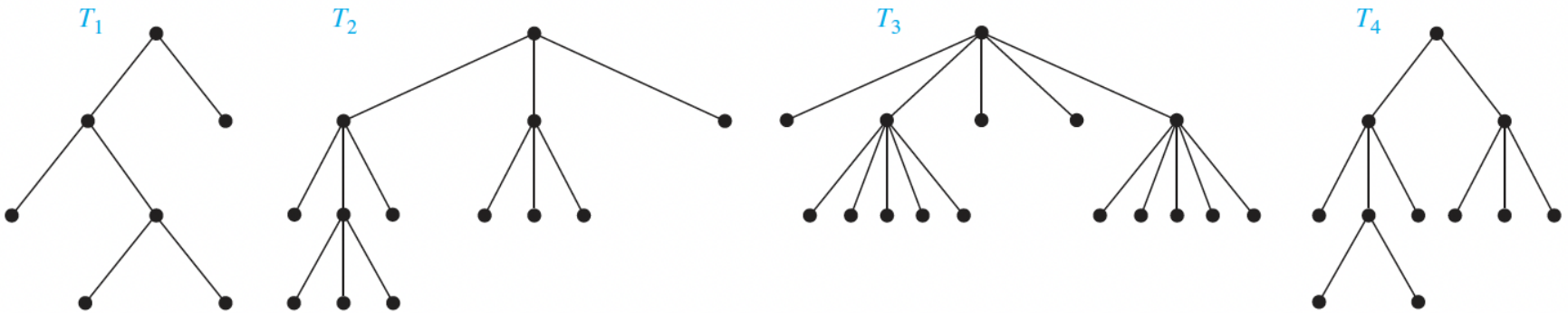
m-Ary Trees

Definition: A rooted tree is called an m-ary tree if every internal vertex has no more than m children. The tree is called a full m-ary tree if every internal vertex has exactly m children. In particular, an m-ary tree with $m = 2$ is called a binary tree.



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Ordered Rooted Tree

Definition: An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered. Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right.

In an ordered binary tree (usually called just a binary tree), if an internal vertex has two children, the first child is called the left child and the second child is called the right child.

The tree rooted at the left child of a vertex is called the left subtree of this vertex, and the tree rooted at the right child of a vertex is called the right subtree of the vertex.

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Counting Vertices in a Full m-Ary Trees

Theorem: A tree with n vertices has $n-1$ edges.

Theorem: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices.

Theorem: A full m-ary tree with

(i) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n + 1]/m$ leaves,

(ii) i internal vertices has $n = mi + 1$ vertices and $l = (m-1)i + 1$ leaves,

(iii) l leaves has $n = (ml - 1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Using $n = mi + 1$ and $n = i + l$

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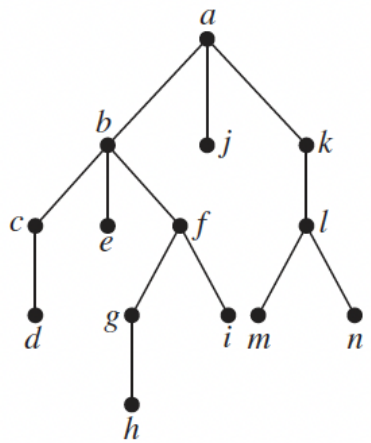
Level and Height

The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.

The height of a rooted tree is the maximum of the levels of the vertices.



The root a is at level 0. Vertices b, j, and k are at level 1. ... Finally, vertex h is at level 4. Because the largest level of any vertex is 4, this tree has height 4.

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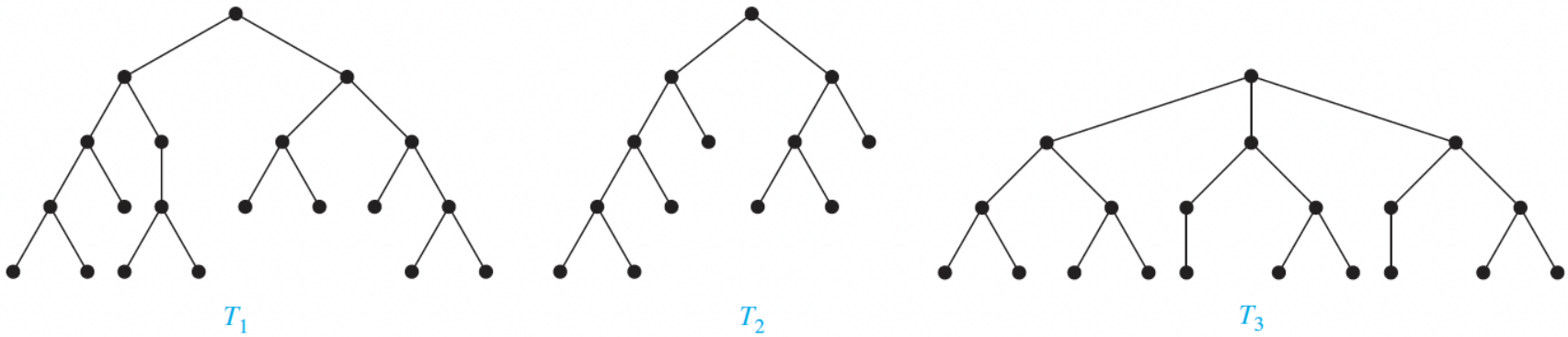
Level and Height

A rooted m-ary tree of height h is balanced if all leaves are at levels h or $h-1$.



T_1 is balanced, because all its leaves are at levels 3 and 4.
T_2 is not balanced, because it has leaves at levels 2, 3, and 4.
T_3 is balanced, because all its leaves are at level 3.

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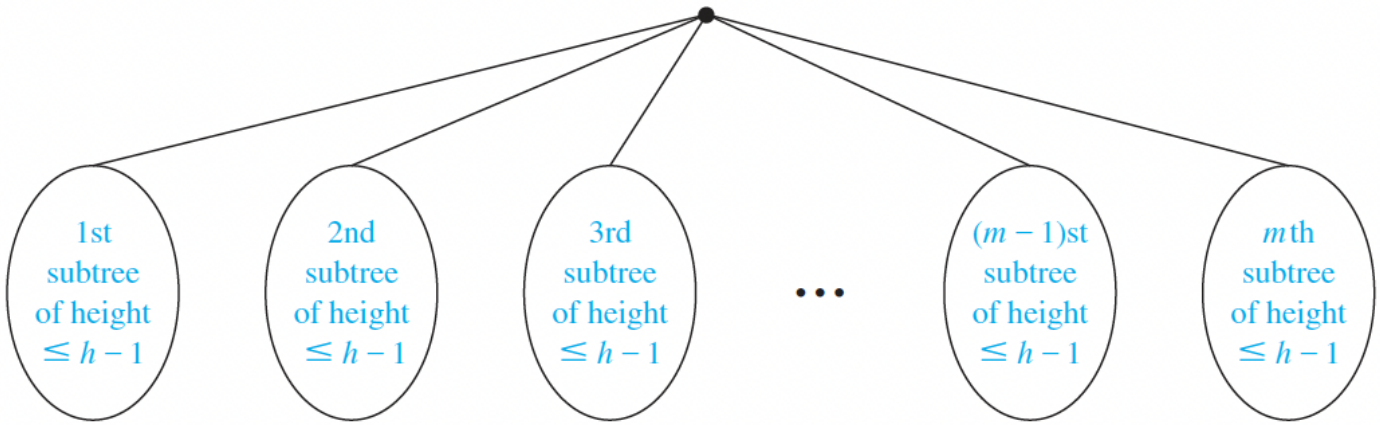
The Number of Leave

Theorem: There are at most m^h leaves in an m-ary tree of height h.

Proof (by induction)

- Basic Step: $h = 1$, at most m leaves
- Inductive Step: Now assume that the result is true for all m-ary trees of height less than h; Consider an m-ary tree of height h:



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The Number of Leave

Corollary: If an m-ary tree of height h has l leaves, then $h \geq \lceil \log_m l \rceil$. If the m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$.

Proof:

- We know that $l \leq m^h$. Taking logarithms to the base m shows that $\log_m l \leq h$. Because h is an integer,we have $h \geq \lceil \log_m l \rceil$.
- Now suppose that the tree is balanced. Then, each leaf is at level h or $h-1$, and because the height is h, there is at least one leaf at level h. Since the tree is full, there must be more than m^{h-1} leaves.
- Because $l \leq m^h$, we have $m^{h-1} < l \leq m^h$. Taking logarithms to the base m in this inequality gives $h - 1 < \log_m l \leq h$. Hence, $h = \lceil \log_m l \rceil$.

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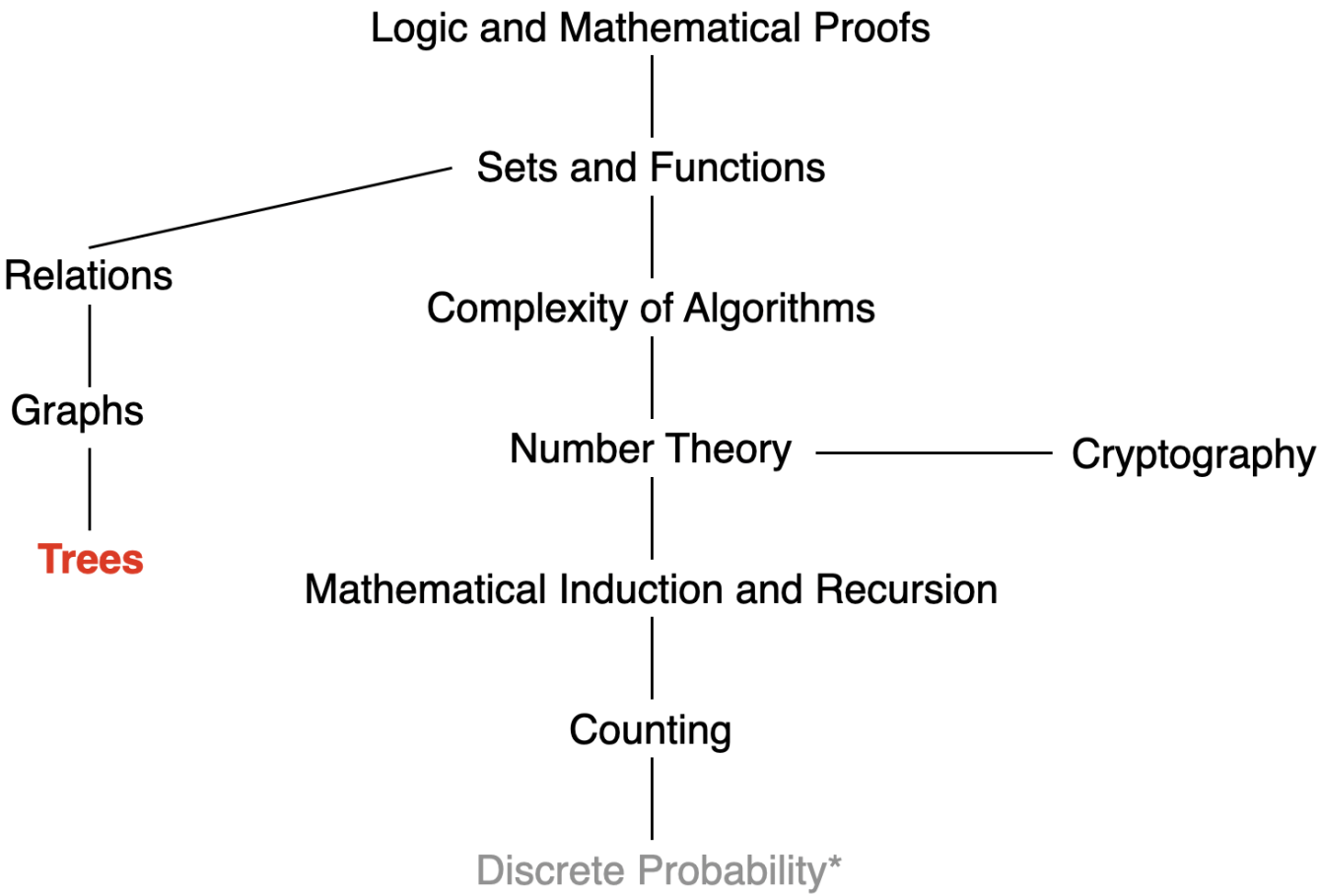

Tree, Tree Traversal, ...

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Tree Traversal

The procedures for systematically visiting every vertex of an ordered tree are called traversals.

The three most commonly used traversals are preorder traversal, inorder traversal, postorder traversal.

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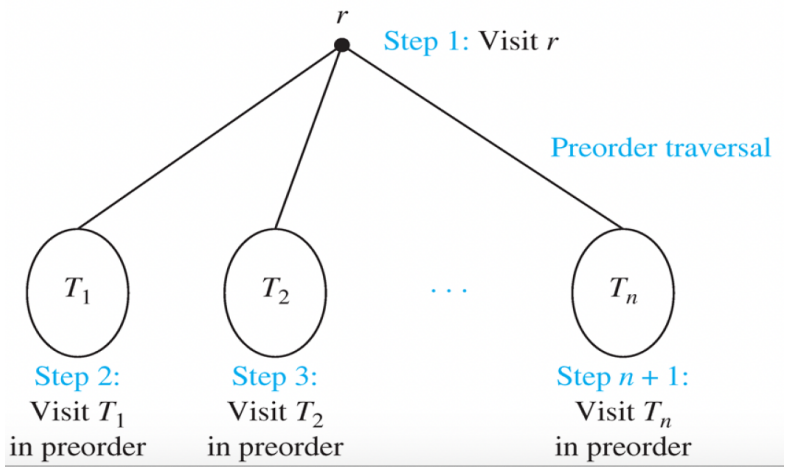
Preorder Traversal

Definition Let T be an ordered rooted tree with root r. If T consists only of r, then r is the preorder traversal of T.

Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The preorder traversal begins by visiting r, and continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.



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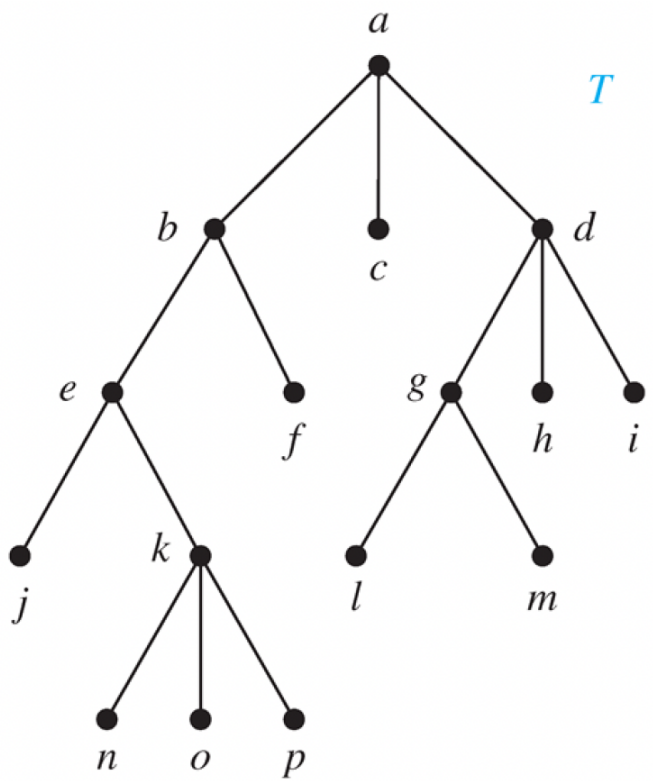
Preorder Traversal: Example



a, b, e, j, k, n, o, p, f, c, d, g, l, m, h, i

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Preorder Traversal: Algorithm

procedure *preorder* (*T*: ordered rooted tree)
r := root of *T*
list *r*
for each child *c* of *r* from left to right
    T(*c*) := subtree with *c* as root
    preorder(*T*(*c*))

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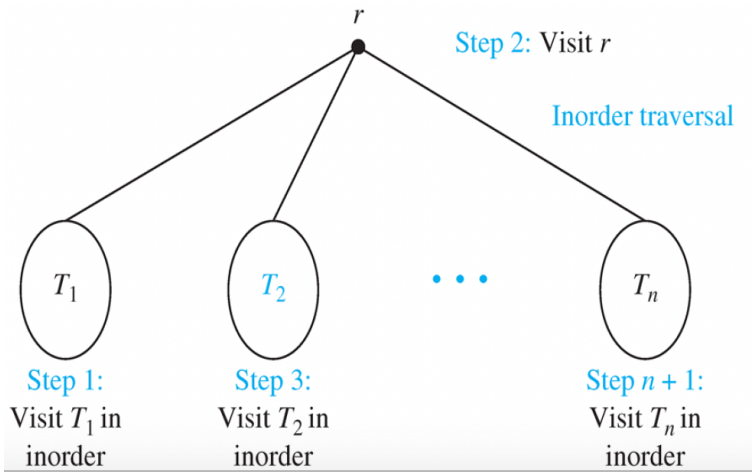
Inorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the inorder traversal of T.

Otherwise, suppose that $T_1, T_2, \ldots, T_n$ are the subtrees of r from left to right in T. The inorder traversal begins by traversing T_1 in inorder, then visiting r, and continues by traversing T_2 in inorder, and so on, until T_n is traversed in inorder.



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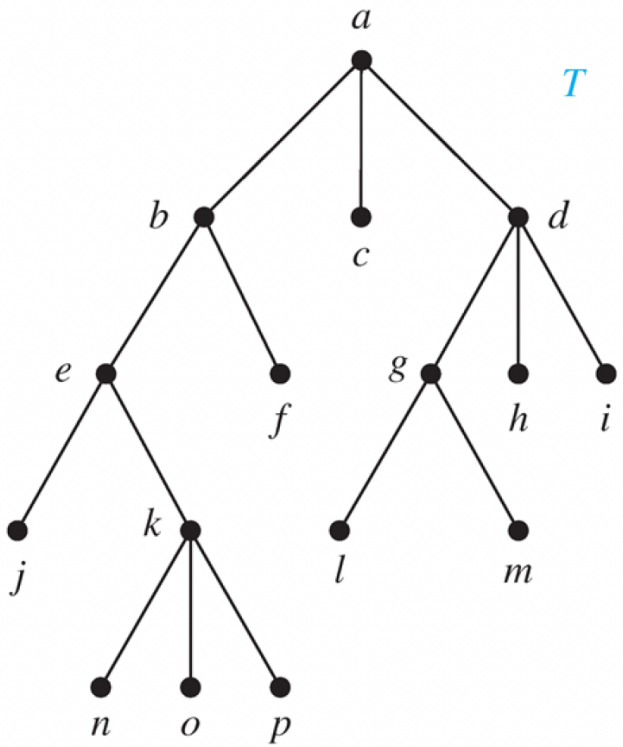
Inorder Traversal: Example



j, e, n, k, o, p, b, f, a, c, l, g, m, d, h, i

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Inorder Traversal: Algorithm

```
procedure inorder (T: ordered rooted tree)
r := root of T
if r is a leaf then list r
else
    l := first child of r from left to right
    T(l) := subtree with l as its root
    inorder(T(l))
    list(r)
    for each child c of r from left to right
        T(c) := subtree with c as root
        inorder(T(c))
```

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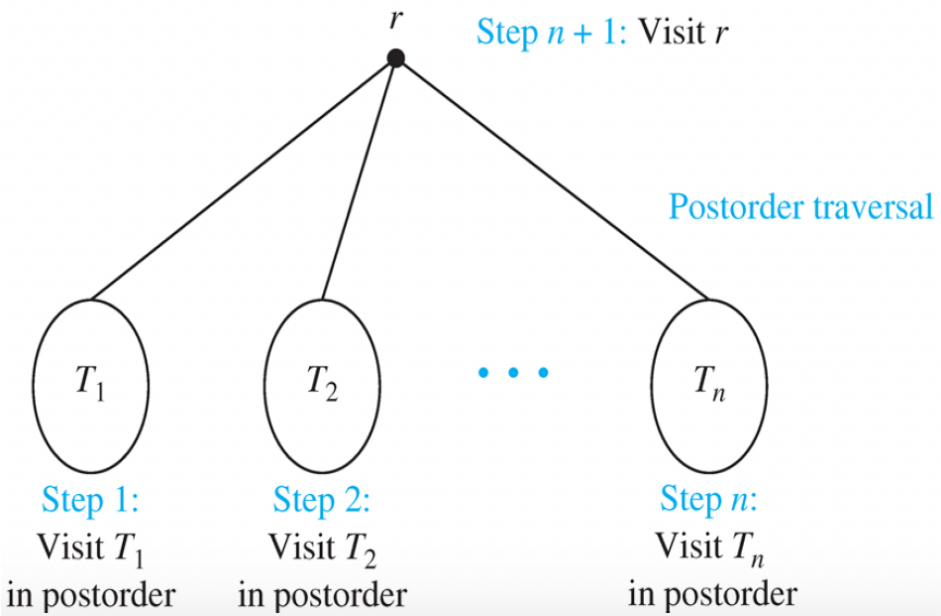
Postorder Traversal

Definition: Let T be an ordered rooted tree with root r. If T consists only of r, then r is the postorder traversal of T.

Otherwise, suppose that $T_1, T_2, ..., T_n$ are the subtrees of r from left to right in T. The postorder traversal begins by traversing T_1 in postorder, then T_2 in postorder, and so on, after T_n is traversed in postorder, r is visited.



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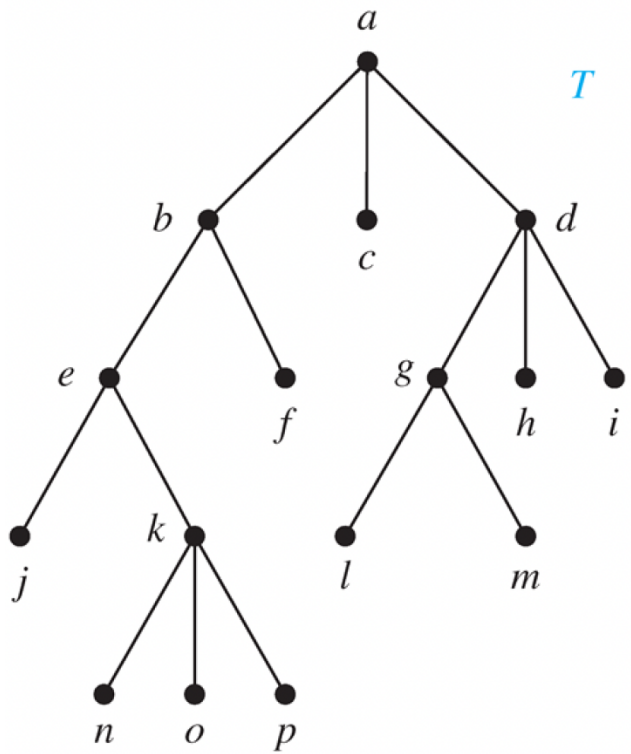
Postorder Traversal



j, n, o, p, k, e, f, b, c, l, m, g, h, i, d, a

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Postorder Traversal: Algorithm

procedure *postordered* (T: ordered rooted tree)
r := root of T
for each child c of r from left to right
    $T(c)$:= subtree with c as root
    postorder($T(c)$)
list r

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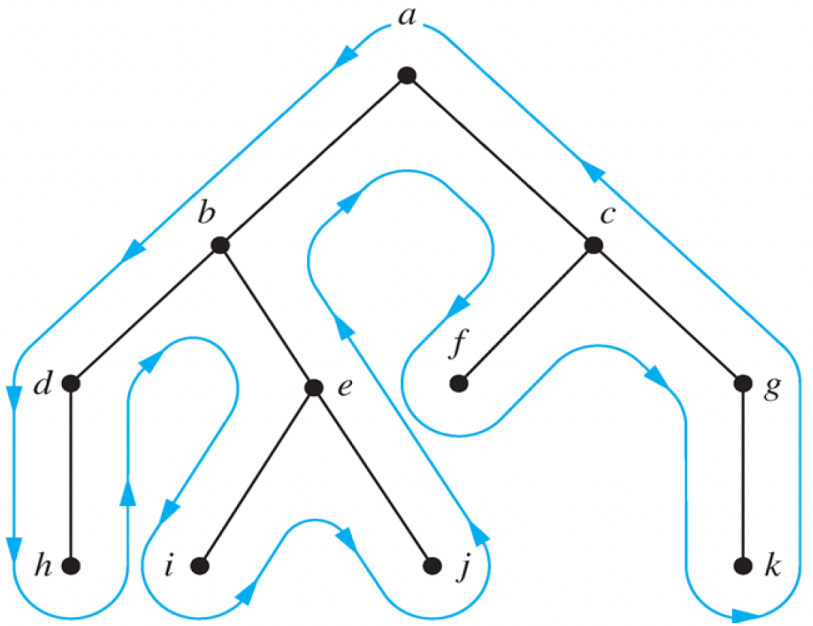
Preorder, Inorder, Postorder Traversal



- in preorder: listing each vertex the first time this curve passes it.
  a, b, d, h, e, i, j , c, f , g, k
- in inorder: listing a leaf the first time the curve passes it and listing each internal vertex the second time the curve passes it.
  h, d, b, i, e, j, a, f , c, k, g
- in postorder: listing a vertex the last time it is passed on the way back up to its parent.
  h, d, i, j , e, b, f , k, g, c, a

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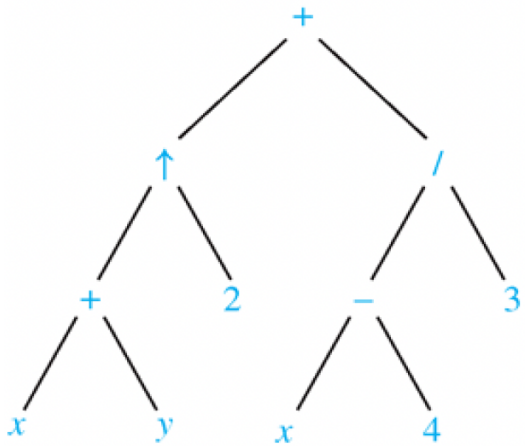
Expression Trees

Complex expressions can be represented using ordered rooted trees.

- the internal vertices represent operations
- the leaves represent the variables or numbers

Example: Consider the expression $((x + y) \uparrow 2) + ((x - 4)/3)$



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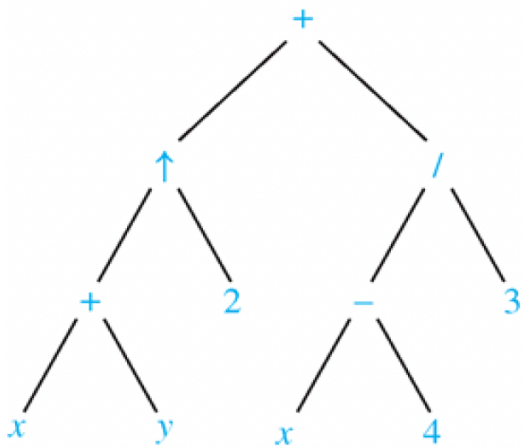
Infix Notation

An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operation. The fully parenthesized expression obtained in this way is said to be in infix form.

Example:



Infix form: $((x + y) \uparrow 2) + ((x - 4)/3)$

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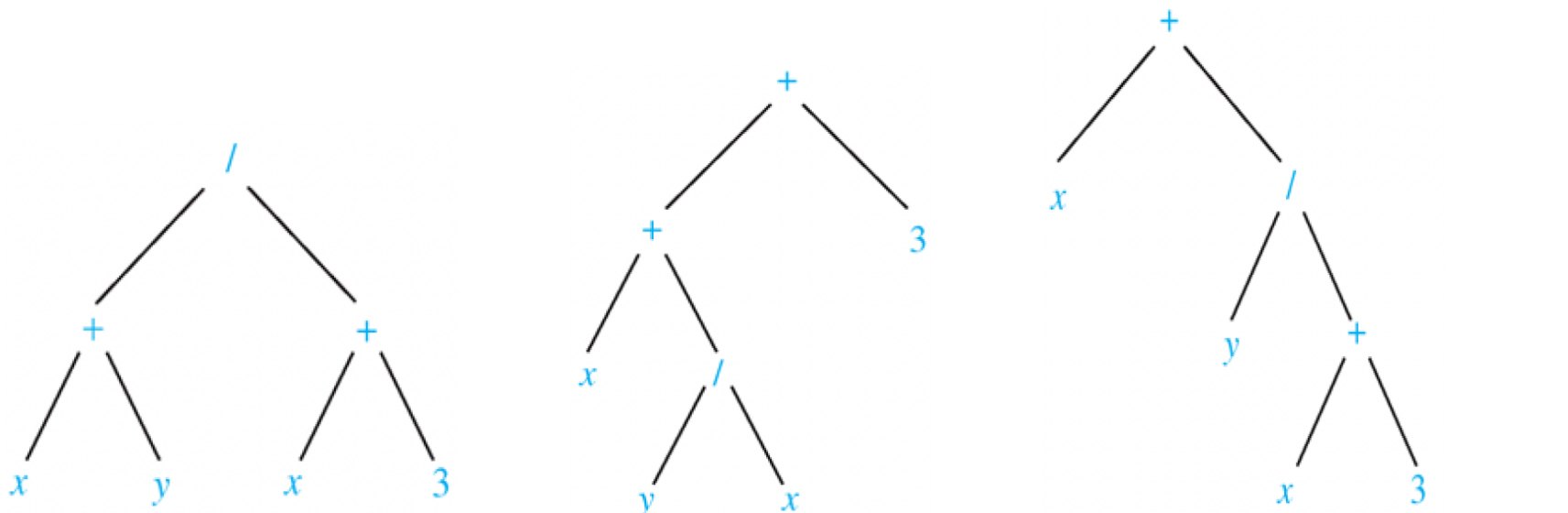
Infix Notation

An inorder traversal of the tree representing an expression produces the original expression when parentheses are included except for unary operation. The fully parenthesized expression obtained in this way is said to be in infix form.

Why parentheses are needed?

$(x + y)/(x + 3)$, $(x + (y/x)) + 3$, $x + (y/(x + 3))$



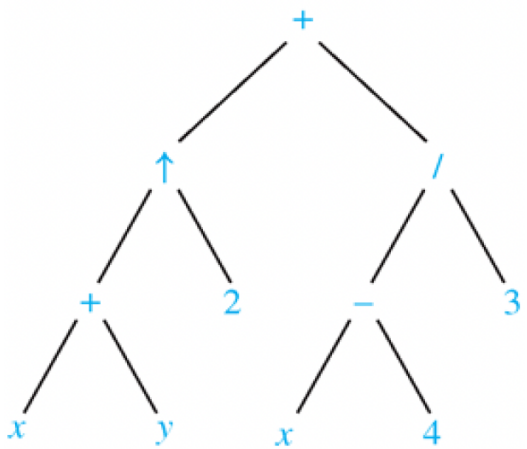
Infix form: $x + y/x + 3$

Prefix Notation

The preorder traversal of expression trees leads to the prefix form of the expression (Polish notation).

An expression in prefix notation, is unambiguous, so no parentheses are needed in such an expression.

Example: What is the prefix form for $((x + y) \uparrow 2) + ((x - 4)/3)$?



Prefix form: $+ \uparrow + x\ y\ 2\ /\ -\ x\ 4\ 3$.

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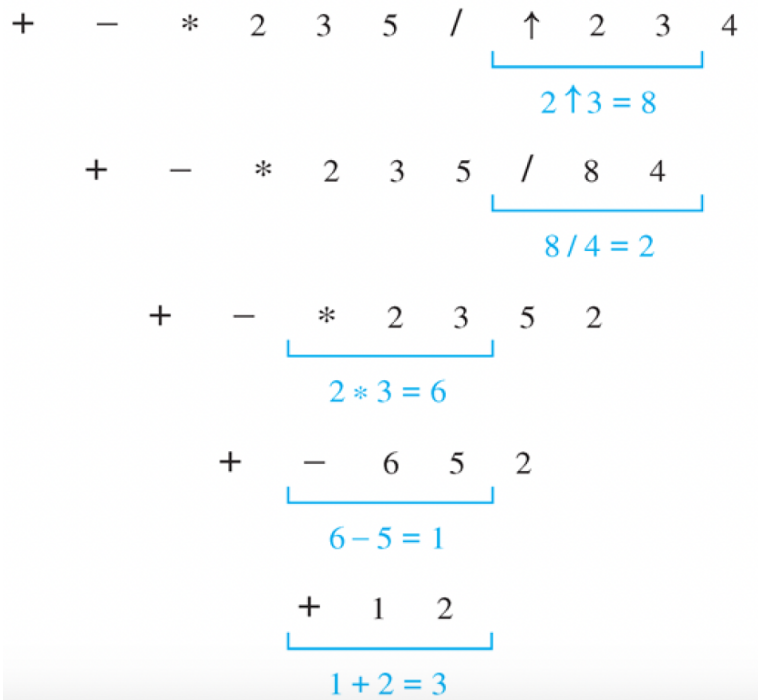
Prefix Notation

Prefix expressions are evaluated by working from right to left. When we encounter an operator, we perform the operation with the two operands to the right.

Example: $+ \; - \; * \; 2 \; 3 \; 5 \; / \; \uparrow \; 2 \; 3 \; 4$



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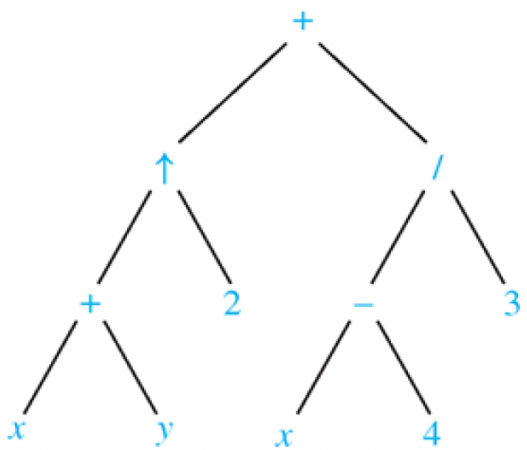
Postfix Notation

The postorder traversal of expression trees leads to the postfix form of the expression (reverse Polish notation).

Expressions in reverse Polish notation are unambiguous, so parentheses are not needed.

Example: What is the postfix form of the expression $((x + y) \uparrow 2) + ((x - 4)/3)$?



Postfix form: $x\ y\ +\ 2\ \uparrow\ x\ 4\ -\ 3\ /\ +$.

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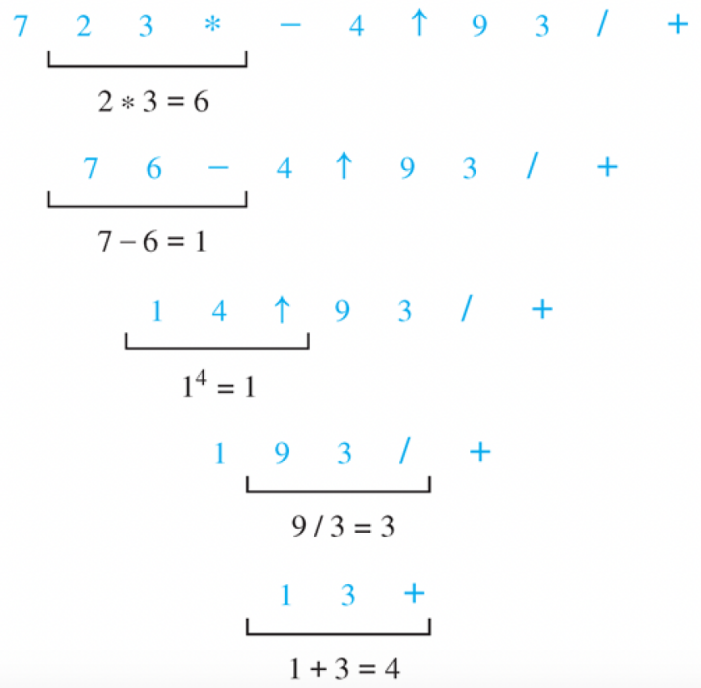
Postfix Notation

Postfix expressions are evaluated by working from left to right. When we encounter an operator, we perform the operation with the two operands to the left.

Example: $7\ 2\ 3\ *\ -\ 4\ \uparrow\ 9\ 3\ /\ +$



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