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Assignment 4

Q1. Using mathematical induction.

When $n=1$, $a^n-b^n \le na^{n-1}(a-b)$ is $a-b \le a-b$, obviously holds.

When $n=k$ holds, prove $n=k+1$ holds.

$$a^k-b^k \le k\,a^{k-1}(a-b)$$

$$\Leftrightarrow a^{k+1}-ab^k \le k a^k(a-b)$$

$$\Leftrightarrow a^{k+1}-ab^k+a^k(a-b) \le (k+1)a^k(a-b)$$

$$\Leftrightarrow a^{k+1}-b^{k+1}+a^k(a-b)-ab^k+b^{k+1} \le (k+1)a^k(a-b)$$

$$\Leftrightarrow a^{k+1}-b^{k+1}+(a^k-b^k)(a-b) \le (k+1)a^k(a-b)$$

$$\Rightarrow a^{k+1}-b^{k+1} \le (k+1)a^k(a-b)$$

The last step is because $0<b<a$, $a^k-b^k>0$, $a-b>0$, $(a^k-b^k)(a-b)>0$

By mathematical induction, $n=1$ holds, $n=k$ holds $\Rightarrow$ $n=k+1$ holds.

For any n, the inequation holds.

Q2. We can get 10\$, 20\$, 5k\$ ($k \ge 5$, $k \in \mathbb{Z}$).

Using strong induction. Suppose $P(k)$: 5k\$ can be get.

20\$ = 10\$ + 10\$, 25\$ = 25\$, hence $P(4)$, $P(5)$.

Now we prove, for any $k \ge 5$, $k \in \mathbb{Z}$, $P(4) \wedge P(5) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$.

Since when $P(4) \wedge P(5) \wedge \cdots \wedge P(k)$, $P(k-1)$. $5(k-1)$\$ can get, we add a 10\$, then $5(k+1)$\$ get, which means $P(k-1) \rightarrow P(k+1)$.

And we have $P(k-1)$, hence $P(k+1)$. That is, $P(4) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$ holds.

Then , since we know $P(4)$, $P(5)$, from strong induction, $\forall k \ge 4$, $P(k)$.

In addtion, 10\$ can directly get.

Above all, 10\$, 5k\$ ($k \ge 4$, $k \in \mathbb{Z}$) can be get.

Q3. $f(n) = f(4^k)$

$$= 5f(4^{k-1}) + 6\cdot 4^k$$
$$= 5(5f(4^{k-2}) + 6\cdot 4^{k-1}) + 6\cdot 4^k$$
$$= \cdots$$
$$= 5(5(\cdots(5f(1)+6\cdot 4^1)+6\cdot 4^2)+\cdots)+6\cdot 4^k$$
$$= 5^k + 6(5^{k-1}\cdot 4^1 + 5^{k-2}\cdot 4^2 + \cdots + 5^0\cdot 4^k)$$
$$= 5^k + 6\cdot 4^k\cdot \sum_{i=0}^{k-1}\left(\tfrac{5}{4}\right)^i$$
$$= 5^k + 6\cdot 4^k\cdot 4\left(\left(\tfrac{5}{4}\right)^k - 1\right)$$
$$= 25\cdot 5^k - 24\cdot 4^k$$
$$= 25\cdot 5^{\log_4 n} - 24n$$

Q4. (a) If $n=1$ or 2, there are 2 one-to-one function.

If $n>2$, 0 one-to-one function since $|A|>|B|$.

(b) $f(1)=f(n)=0$, but for $x=2,3,\cdots,n-1$, $f(x)=0$ or 1.

There 2^{n-2} functions.

(c) Only one 1 can assign to $1,2,\cdots,n-1$, others are 0.

There are $n-1$ functions.

Q5. First choose the rank of 2 pairs, $\binom{13}{2}$.

If we want to choose a pair in rank x, there are $\binom{4}{2}$.

Then choose the rank of 2 single, $\binom{11}{2}$, with each there are $\binom{4}{1}$ choose of suits.

Total, $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ choices.

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Q6. $\binom{240}{120} = \frac{240!}{(120!)^2}$

First we calculate how many factor 2 and 11 in 120!

\# of factor 11 $= \lfloor\frac{120}{11}\rfloor + \lfloor\frac{120}{11^2}\rfloor = 10+0 = 10$

\# of factor 2 $= \lfloor\frac{120}{2}\rfloor + \lfloor\frac{120}{2^2}\rfloor + \lfloor\frac{120}{2^3}\rfloor + \cdots + \lfloor\frac{120}{2^6}\rfloor + \lfloor\frac{120}{2^7}\rfloor$

$= 60+30+15+7+3+1+0 = 116$

Then, similarly, calculate factor 2 and 11 in 240!

\# of factor 11 $= \lfloor\frac{240}{11}\rfloor + \lfloor\frac{240}{11^2}\rfloor = 21+1 = 22$

\# of factor 2 $= \sum_{i=1}^{8} \lfloor\frac{240}{2^i}\rfloor = 120+60+30+15+7+3+1+0 = 236$

Hence $\binom{240}{120} = \frac{240!}{(120!)^2} = \frac{2^{236}\times 11^{22}\times\cdots}{(2^{116}\times 11^{10}\times\cdots)^2} = 2^4\times 11^2\times\cdots$

and $242 = 2\times 11^2$, so $242 \mid \binom{240}{120}$

Q7. For (a,b), $a \bmod 5$ and $b \bmod 5$ both can be $0,1,2,3,4$

There're 25 values of $(a \bmod 5, b \bmod 5)$

So we need 26 pair to guarantee.

Q8. If there're n people in this party, a person can know $0,1,\cdots,n-1$.

Discuss by cases.

① Suppose each person knows at least 1 others, a person can know $1,2,\cdots,n-1$ others, but there're n people. From pigeonhole principle, at least 2 people know the same number of others.

② There exist person who know only 0 people. Then no one can know all $n-1$ people, so a person can know $0,1,\cdots,n-2$ people. Similarly, from pigeonhole priciple we can get the conclusion.

~~About~~ Above all, there're 2 people who know the same number of other.

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Q9. For $\{1, 2, \cdots, n\}$, if we don't choose n, the number of subset in the question is a_{n-1}. If we choose n, then $n-1$ can't be choosen, the number of subset described in question is a_{n-2}.

So $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$

Q10. $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$

By binomial coefficient,

$$\sum_{r=0}^{n} C(n,r)x^r = \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1}$$

$$= 1 + \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r$$

$$= 1 + \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n-1} C(n-1,r-1)x^r + x^n$$

And we have $\sum_{r=0}^{n} C(n,r)x^r = 1 + \sum_{r=1}^{n-1} C(n,r)x^r + x^n$

Hence, $\sum_{r=1}^{n-1} C(n,r)x^r = \sum_{r=1}^{n-1} \big(C(n-1,r) + C(n-1,r-1)\big)x^r$

That is, $C(n,r) = C(n-1,r) + C(n-1,r-1)$

Q11. Characteristic equation is $r^3 = 2r^2 + r - 2$.

$r_1 = -1$, $r_2 = 1$, $r_3 = 2$. Then $a_n = a(-1)^n + b \cdot 1^n + c \cdot 2^n$

$$\begin{cases} a_0 = a+b+c = 1 \\ a_1 = -a+b+2c = 0 \\ a_2 = a+b+4c = 7 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{2} \\ b = -\frac{5}{2} \\ c = 2 \end{cases}$$

$a_n = \frac{3}{2}(-1)^n + -\frac{5}{2} \cdot 1^n + 2 \cdot 2^n = \frac{3}{2}(-1)^n + 2^{n+1} - \frac{5}{2}$