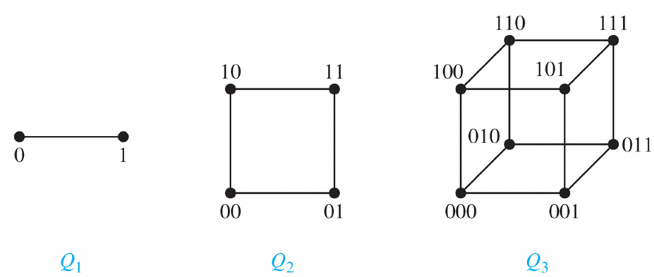
CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #5
Due: May 17th, 2024

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. An n-dimensional hypercube, or n-cube, Q_n is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position. Let $l(n)$ denote the number of edges of Q_n.



(a) What is the initial condition of $l(n)$?

(b) What is the recursive function of $l(n)$?

(c) Derive the closed-form of $l(n)$ using the general approach we have learned for solving linear recurrence relation. Please provide the derivation details. Please do NOT use mathematical induction.

Q. 2. Consider 10 identical balloons (i.e., non-distinguishable balloons). We aim to give these balloons to four children, and each child should receive at least one balloons.

(a) How many ways to give these balloons to the children? Explain the reason.

(b) The answer to the above question is the coefficient of term ____________ of generating function ________________.

Q. 3. How many relations are there on a set with n elements that are

a) antisymmetric?

b) irreflexive?

c) neither reflexive nor irreflexive?

d) symmetric, antisymmetric and transitive?

Please explain your answer.

Q. 4. Prove or disprove the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Q. 5. Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Q. 6. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also be symmetric? Explain your answer.

Q. 7. Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Q. 8. Suppose that the relation R is symmetric. Show that R^* is symmetric.

Q. 9. Which of the following are equivalence relations on the set of all people?

(1) $\{(x, y)|x$ and y have the same sign of the zodiac$\}$

(2) $\{(x, y)|x$ and y were born in the same year$\}$

(3) $\{(x, y)|x$ and y have been in the same city$\}$

Q. 10. Show that $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Q. 11. Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Q. 12. Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both equivalence relations, then T is also an equivalence relation.

Q. 13. Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

a) Prove that $\preceq$ is a partial ordering.

b) Prove or disprove: $\preceq$ is a total ordering.

Q. 14. We consider partially ordered sets whose elements are sets of natural numbers, and for which the ordering is given by $\subseteq$. For each such partially ordered set, we can ask if it has a minimal or maximal element. For example, the set $\{\{0\}, \{0, 1\}, \{2\}\}$, has minimal elements $\{0\}, \{2\}$, and maximal elements $\{0, 1\}, \{2\}$.

(a) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no maximal element.

(b) Prove or disprove: there exists a nonempty $R \subseteq \mathcal{P}(\mathbb{N})$ with no minimal element.

(c) Prove or disprove: there exists a nonempty $T \subseteq \mathcal{P}(\mathbb{N})$ that has neither minimal nor maximal elements.