Discrete Mathematics for Computer Science

Lecture 14: Counting

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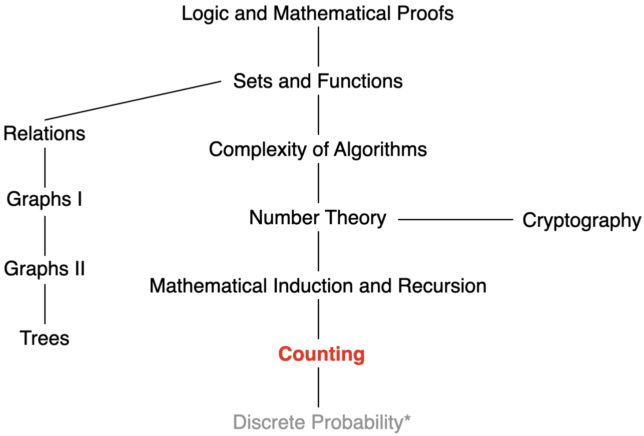
This Lecture



Counting basis, Permutations and Combinations, Binomial Coefficients, The Birthday Paradox, Generalized Permutations and Combinations, Generating Function, Solving Linear Recurrence Relations , ...

Solving Linear Recurrence Relations

Solve the closed-form:

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Solving Linear Recurrence Relations

One important class of recurrence relations can be explicitly solved in a systematic way.

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

- linear: it is a linear combination of previous terms
- homogeneous: all terms are multiples of a_j's
- degree k: a_n is expressed by the previous k terms
- constant coefficients: coefficients are constants

Example:

- $a_n = a_{n-1} + (a_{n-2})^2$: not linear.
- $H_n = 2H_{n-1} + 1$: not homogeneous.
- $B_n = nB_{n-1}$: not constant coefficients.
- $P_n = 1.11 \cdot P_{n-1}$: Yes; of degree 1
- $f_n = f_{n-1} + f_{n-2}$: Yes; of degree 2

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Solving Linear Recurrence Relations

By induction, such a recurrence relation is uniquely determined by this recurrence relation and k initial conditions $a_0, a_1, \ldots, a_{k-1}$.

Example: Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2},$$

Which of the following are solutions?

- $a_n = 3n$ Yes
- $a_n = 2^n$ No
- $a_n = 5$ Yes

Question: Why not unique?

Question: Any systematic way?

Solving Linear Recurrence Relations

Definition: A linear homogeneous relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k},$$

where c_1, c_2, . . . , c_k are real numbers, and $c_k \neq 0$.

Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + ... + c_k r^{n-k}.$$

Divide both sides by r^{n-k},

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_{k-1} r - c_k = 0.$$

Characteristic equation of the recurrence relation.

Solving Linear Recurrence Relations

Why of the form of $a_n = r^n$?

Short answer 1: we know that exponentials satisfy linear recurrences. So we try to fit the linear recurrence to an exponential. We check the result to justify our work a posteriori.

Short answer 2: someone has worked this method out in the past, so we use it.

Medium answer: we can write a linear recurrence relation as a matrix equation

$$v_{n+1} = Av_n$$

where v_n is a vector whose components are consecutive terms of the linear recurrence. This is easy to solve:

$$v_n = A^n v_0$$

If A is diagonalizable, we can obtain a closed form for v_n by diagonalizing and computing the exponential.

Long answer: there is a theory of *difference equations* that is quite analogous to the theory of differential equations. Given a function f, we define the (forward) difference Δf to be the function $(\Delta f)(n) = f(n+1) - f(n)$. Many things in the theory of differential calculus have analogs in this difference calculus.

https://math.stackexchange.com/questions/926112/
what-is-the-intuitive-idea-behind-looking-for-a-solution-of-the-form-an-rn-for

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Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

The sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ with the initial condition $a_0 = C_0$ and $a_1 = C_1$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, ...,$$

where α_1 and α_2 are constants.

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation for any arbitrary constants α_1 and α_2.
- For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Since the solution is unique, the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation.

Since r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. Suppose $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}$$

Solving Linear Recurrence Relations: Degree Two

For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Suppose that $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold.

$$a_0 = C_0 = \alpha_1 + \alpha_2, \qquad a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Thus,

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \qquad \alpha_2 = C_0 - \alpha_1 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

α_1 and α_2 exist since $r_1 \neq r_2$.

Solving Linear Recurrence Relations: Degree Two

Proof: $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is the solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and satisfy the initial conditions.

- $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ is a solution of the recurrence relation.
- For every recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, there exist α_1 and α_2 that satisfy the initial conditions.

Note that there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, so the if and only if statement holds.

Solving Linear Recurrence Relations: Degree Two

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \text{ for } n = 0, 1, 2, ...,$$

where α_1 and α_2 are constants.

Solve Linear Recurrence Relations:

- Solve r_1 and r_2 with $r^2 - c_1 r - c_2 = 0$.
- Solve α_1 and α_2 with the initial conditions.

Example 1: Fibonacci number

Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

What is the closed-form expression of F_n?

To solve r_1 and r_2, consider $r^n = r^{n-1} + r^{n-2}$, i.e., $r^2 - r - 1 = 0$. There are two different roots:

$$r_1 = \frac{1+\sqrt{5}}{2}, \qquad r_2 = \frac{1+\sqrt{5}}{2}$$

Consider the form of $F_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. To solve α_1 and α_2, by $F_0 = 0$ and $F_1 = 1$, we have $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 r_1 + \alpha_1 r_2 = 1$.

Thus, $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -\alpha_1$. Hence,

$$F_n = \alpha_1 r_1^n + \alpha_2 r_2^n = \frac{r_1^n - r_2^n}{\sqrt{5}}.$$

Example 2

$a_n = a_{n-1} + 2a_{n-2}$, with $a_0 = 2$, $a_1 = 7$.

The characteristic equation is

$$r^2 - r - 2 = 0.$$

The two roots are 2 and -1. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \qquad a_1 = 2\alpha_1 - \alpha_2 = 7.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Example 3

$a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2$, $a_1 = 1$

The characteristic equation is

$$r^2 - 7r + 10 = 0.$$

Two roots are 2 and 5. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 5^n.$$

By the two initial conditions, we have

$$a_0 = \alpha_1 + \alpha_2 = 2, \qquad a_1 = 2\alpha_1 + 5\alpha_2 = 1.$$

We get $\alpha_1 = 3$ and $\alpha_2 = -1$. Thus,

$$a_n = 3 \cdot 2^n - 5^n.$$

Solving Linear Recurrence Relations of Degree k

Consider an arbitrary linear homogeneous relation of degree k with constant coefficients:

$$a_n = \sum_{i-1}^{k} c_i a_{n-i}.$$

The characteristic equation (CE) is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0.$$

Theorem: If this CE has k distinct roots r_i, then the solutions to the recurrence are of the form

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the α_i's are constants.

Example

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

The characteristic equation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. So, assume that

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

By the three initial conditions, we have $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$, $\quad a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$, $\quad a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$.

We get $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Thus,

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

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The Case of Degenerate Roots: Degree Two

Theorem: If the $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0, then

$$a_n = (\alpha_1 + \alpha_2 n) r_0^n,$$

for all $n \geq 0$ and two constants α_1 and α_2.

Example

$a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

The only root is 2. So, assume that

$$a_n = (\alpha_1 + \alpha_2 n)2^n.$$

By the three initial conditions, we have

$$a_0 = \alpha_1 = 1, \qquad a_1 = 2 \cdot (\alpha_1 + \alpha_2) = 0.$$

We get $\alpha_1 = 1$, $\alpha_2 = -1$. Thus,

$$a_n = (1 - n)2^n.$$

The Case of Degenerate Roots: Degree k

Theorem: Suppose that there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then,

$$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$ and constants $\alpha_{i,j}$.

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

Example

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, $a_2 = -1$.

The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

There is a single root $r = -1$ of multiplicity three of the characteristic equation. Thus, assume that

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n.$$

By the three initial conditions, we have ...

We get $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, $\alpha_{1,2} = -2$. Thus, $a_n = (1 + 3n - 2n^2)(-1)^n$.

Solving Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Linear Nonhomogeneous Recurrence Relations

Linear Nonhomogeneous Recurrence Relations

Definition: A linear nonhomogeneous relation with constant coefficients may contain some terms $F(n)$ that depend only on n

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n).$$

The recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ is called the associated homogeneous recurrence relation.

Example:

- $a_n = a_{n-1} + 2^n$ $\quad a_n = a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ $\quad a_n = a_{n-1} + a_{n-2}$
- $a_n = 3a_{n-1} + n3^n$ $\quad a_n = 3a_{n-1}$
- $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ $\quad a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Linear Nonhomogeneous Recurrence Relations

Every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.

Note: $a_n^{(p)}$ does not need to satisfy the initial conditions.

Linear Nonhomogeneous Recurrence Relations

Proof: Suppose $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \ldots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation,

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + \ldots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$.

Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n.

Linear Nonhomogeneous Recurrence Relations

Theorem: If $\{a_n^{(p)}\}$ is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $\{a_n^{(h)}\}$ is any solution to the associated homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$.

The key is to find the particular solution to the linear nonhomogeneous relation. However, there is no general method for finding such a solution.

Example 1

There are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants.

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- Compute $a_n^{(h)}$
- Compute $a_n^{(p)}$
- Initial condition

Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$:

The characteristic equation is

$$r^2 - 3r = 0.$$

The roots are $r_1 = 3$ and $r_2 = 0$. By So, assume that

$$a_n^{(h)} = \alpha 3^n.$$

To compute $a_n^{(p)}$: Try $a_n^{(p)} = cn + d$. Thus,

$$cn + d = 3(c(n-1) + d) + 2n.$$

We get $c = -1$ and $d = -3/2$. Thus, $a_n^{(p)} = -n - 3/2$.

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Example 1

Example 1: $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

To compute $a_n^{(h)}$: $a_n^{(h)} = \alpha 3^n$.

To compute $a_n^{(p)}$: $a_n^{(p)} = -n - 3/2$.

Initial condition:

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha 3^n - n - 3/2.$$

Base on the initial condition $a_1 = 3$. We have $3 = -1 - 3/2 + 3\alpha$, which implies $\alpha = 11/6$. Thus, $a_n = -n - 3/2 + (11/6)3^n$.

Example 2

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$. (Since we do not provide the initial conditions, obtain the general form would be sufficient.)

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- Try $a_n^{(p)} = C \cdot 7^n$:

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$$

Thus, $C = 49/20$, and $a_n^{(p)} = (49/20)7^n$.

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Linear Nonhomogeneous Recurrence Relations

For the previous two examples, we made a guess that there are solutions of a particular form. This was not an accident.

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Taken care when $s = 1$!

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$:
The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

This characteristic equation has a single root $r = 3$ of multiplicity $m = 2$.

$$a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n.$$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = n^2 2^n$:

Since $s = 2$ is not a root of the characteristic equation, we have

$$a_n^{(p)} = (p_2 n^2 + p_1 n + p_0)2^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

$$\begin{aligned}(p_2 n^2 + p_1 n + p_0)2^n = 6(&p_2(n-1)^2 + p_1(n-1) + p_0)2^{n-1} \\ &- 9(p_2(n-2)^2 + p_1(n-2) + p_0)2^{n-2} + n^2 2^n.\end{aligned}$$

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + (p_2 n^2 + p_1 n + p_0)2^n.$

Example 1

$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ with $F(n) = n^2 2^n$ and $F(n) = (n^2+1)3^n$.

To compute $a_n^{(h)}$: $a_n^{(h)} = (\alpha_1 + \alpha_2 n)3^n$.

To compute $a_n^{(p)}$ of $F(n) = (n^2+1)3^n$:

Since $s = 3$ is a root of the characteristic equation
with multiplicity $m = 2$, we have

$$a_n^{(p)} = n^2(p_2 n^2 + p_1 n + p_0)3^n.$$

Substituting $a_n^{(p)}$ into $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ to derive p_2, p_1, and p_0:

...

$a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)3^n + n^2(p_2 n^2 + p_1 n + p_0)3^n.$

Example 2: The Term n^m

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n$$

Solution:

- $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$
- $a_n^{(p)}$ should be in the form of $np_0 2^n$.
- Try $a_n^{(p)} = p_0 \cdot 2^n$ instead:

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2} + 2^n.$$

Since $s = 2$ is a root of the characteristic equation,

$$p_0 \cdot 2^n = 5p_0 \cdot 2^{n-1} - 6p_0 \cdot 2^{n-2}$$

always holds. Thus, we obtain $0 = 4$. Contradiction.

Generating Function

Generating function and recurrent relation ...

Useful Generating Functions

$$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$$

$$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. We aim to first derive the formulation of $G(x)$.

$$
\begin{aligned}
G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\
&= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1})x^k \\
&= 2,
\end{aligned}
$$

Thus, $G(x) - 3xG(x) = (1-3x)G(x) = 2$:

$$G(x) = \frac{2}{(1-3x)}.$$

Example 1

Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Solution: We aim to first derive the formulation of $G(x)$.

$$G(x) = \frac{2}{(1-3x)}.$$

Then, derive a_k using the identity $1/(1-ax) = \sum_{k=0}^{\infty} a^k x^k$. That is,

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$$

Consequently, $a_k = 2 \cdot 3^k$.

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: We extend this sequence by setting $a_0 = 1$. We have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned}
G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\
&= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8xG(x) + x/(1 - 10x),
\end{aligned}$$

Example 2

Consider the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1},$$

and the initial condition $a_1 = 9$.

Solution: Thus,

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)} = G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Thus, $a_n = \frac{1}{2}(8^n + 10^n)$.

Generating function to solve recurrence relations

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Based on the recurrence relations, derive the formulation of $G(x)$.
- Using identities (or the useful facts of generating functions), derive sequence $\{a_k\}$.

Next Lecture