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Assignment 1

Q1. (a) $\neg p$ (b) $p \wedge \neg q$ (c) $p \to q$ (d) $\neg p \to \neg q$

(e) $p \to q$ (f) $q \wedge \neg p$ (g) $q \to p$

Q2. (a)

p	q	$p \oplus q$	$p \wedge q$	$(p \oplus q) \to (p \wedge q)$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(b)

p	q	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	T	F	T

Q3. (a) Counter example: $P(x)$: $0 < x < 4$ $Q(x)$: $0 < x < 2$

domain: real number x

$\forall x\,(P(x) \to Q(x))$ is F, $\forall x\, P(x) \to \forall x\, Q(x) = F \to F = T$

(b) Prove it.

First, if $\forall x\, P(x) \wedge \exists x\, Q(x) = T$, $\forall x\, P(x) = T$, $\exists x\, Q(x) = T$

Suppose y_0 s.t. $Q(y_0) = T$

Then for $\forall x$, $P(x) \wedge Q(y_0) = T \wedge T = T$, that is $\forall x \exists y (P(x) \wedge Q(y)) = T$.

Second, if $\forall x \exists y\,(P(x) \wedge Q(y)) = T$, that is, for any x_0, there exist y_0, such that $P(x_0) \wedge Q(y_0) = T$, which means $P(x_0) = T$, $Q(y_0) = T$.

So $\forall x\, P(x) = T$, $\exists y\, Q(y) = T$, $\forall x\, P(x) \wedge \exists x\, Q(x)$.

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Q4. (1) $(\neg p \wedge (p \to q)) \to \neg q = \neg(\neg p \wedge (p \to q)) \vee \neg q$ (useful)

$= p \vee \neg(p \to q) \vee \neg q$ (De Morgan)

$= p \vee \neg(\neg p \vee q) \vee \neg q$ (useful)

$= p \vee (p \wedge \neg q) \vee \neg q$ (De Morgan)

$= (p \vee p \vee \neg q) \wedge (p \vee \neg q \vee \neg q)$ (distributive)

$= (p \vee \neg q) \wedge (p \vee \neg q)$ (idempotent)

$= p \vee \neg q$ not a tautology.

(2) $(p \vee q) \to r = \neg(p \vee q) \vee r$ (useful)

$= (\neg p \wedge \neg q) \vee r$ (De Morgan)

$= (\neg p \vee r) \wedge (\neg q \vee r)$ (distributive)

$(p \to r) \wedge (q \to r) = (\neg p \vee r) \wedge (\neg q \vee r)$

Hence the two are equivalent.

(3) It's false. ~~Proved~~ by contradiction.

If there exist x_0, such that for $\forall y$, if $y \neq 0$, $xy = 1$

Then we can find y_1, y_2, ($y_1 \neq y_2$), both are nonzero,

satisfied $x_0 y_1 = 1$, $x_0 y_2 = 1$, so $y_1 = y_2 = \frac{1}{x_0}$,

contradict to $y_1 \neq y_2$.

(4) It's true. $m=1, n=2$ satisfies $n^2+m^2=5$, which proved existence.

Q5. (1) $P(x)$: x finished homework.

$Q(x)$: x can answer this question.

domain: all students

Premise 1: $\neg P(you) \to \neg Q(you) = T$ Premise 2: $P(you) = T$

From these we can not get the conclusion $Q(you) = T$.

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(2) $P(x)$: x submitted homework.

$Q(x)$: x get 100 in the final exam.

domain: all student in this class

Premise 1: $\forall x\, P(x) \to \forall x\, Q(x)$ Premise 2: $\exists x\, \neg P(x)$

Conclusion: $\exists x\, \neg Q(x)$

$\exists x\, \neg P(x)$, so $\forall x\, P(x) = F$, $F \to \forall x\, Q(x) = T$, we don't know $\forall x\, Q(x) = T$ or F. so we can not get the conclusion.

Q6. $(\neg r \vee (p \wedge \neg q)) \to (r \wedge p \wedge \neg q)$

$= \neg(\neg r \vee (p \wedge \neg q)) \vee (r \wedge p \wedge \neg q)$ (useful)

$= (r \wedge \neg(p \wedge \neg q)) \vee (r \wedge p \wedge \neg q)$ (De Morgan)

$= (r \wedge (\neg p \vee q)) \vee (r \wedge p \wedge \neg q)$ (De Morgan)

$= (r \wedge \neg p) \vee (r \wedge q) \vee (r \wedge p \wedge \neg q)$ (distributive)

$= r \wedge (r \vee p) \wedge (r \vee \neg q) \wedge (r \vee q) \wedge (r \vee p \vee q) \wedge T \wedge (r \vee \neg p) \wedge (r \vee \neg p \vee \neg q) \wedge T$
$\wedge (r \vee \neg p \vee q) \wedge T \wedge T$ (distributive)

$= r$ (absorption, domination)

From $r = T$, we get $r \vee s = T$.

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Q7. (a) $\neg(p \oplus q)$

$$
\begin{array}{ll}
= \neg((p \wedge \neg q) \vee (\neg p \wedge q)) & \text{(definition)} \\
= \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) & \text{(De Morgan)} \\
= (\neg p \vee q) \wedge (p \vee \neg q) & \text{(De Morgan)} \\
= (\neg p \wedge p) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg q) & \text{(distributive)} \\
= F \vee (\neg p \wedge \neg q) \vee (p \wedge q) \vee F & \text{(negation)} \\
= (\neg p \wedge \neg q) \vee (p \wedge q) & \text{(identity)} \\
= p \leftrightarrow q & \text{(definition)}
\end{array}
$$

(b) $\neg(p \to q) \to \neg q$

$$
\begin{array}{ll}
= \neg(\neg(p \to q)) \vee \neg q & \text{(useful)} \\
= (p \to q) \vee \neg q & \text{(double negation)} \\
= (\neg p \vee q) \vee \neg q & \text{(useful)} \\
= \neg p \vee (q \vee \neg q) & \text{(associative)} \\
= \neg p \vee T & \text{(negation)} \\
= T & \text{(domination)}
\end{array}
$$

Hence, $\neg(p \to q) \to \neg q$ is a tautology.

(c) $(p \to q) \to ((r \to p) \to (r \to q))$

$$
\begin{array}{ll}
= \neg(p \to q) \vee ((r \to p) \to (r \to q)) & \text{(useful)} \\
= \neg(\neg p \vee q) \vee ((\neg r \vee p) \to (\neg r \vee q)) & \text{(useful)} \\
= (p \wedge \neg q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) & \text{(useful)} \\
= (p \wedge \neg q) \vee ((r \wedge \neg p) \vee \neg r \vee q) & \text{(De Morgan)} \\
= (p \vee (r \wedge \neg p) \vee \neg r \vee q) \wedge (\neg q \vee (r \wedge \neg p) \vee \neg r \vee q) & \text{(distributive)} \\
= (p \vee r \vee \neg r \vee q) \wedge (p \vee \neg p \vee \neg r \vee q) \wedge (\neg q \vee q \vee (r \wedge \neg p) \vee \neg r) & \text{(distributive)} \\
= (p \vee T \vee q) \wedge (T \vee \neg r \vee q) \wedge (T \vee (r \wedge \neg p) \vee \neg r) & \text{(negation)} \\
= T \wedge T \wedge T & \text{(domination)} \\
= T & \text{(domination)}
\end{array}
$$

Hence, $(p \to q) \to ((r \to p) \to (r \to q))$ is tautology.

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Q8. (a) $\exists x\,(C(x) \land D(x) \land F(x))$

(b) $\forall x\,(C(x) \lor D(x) \lor F(x))$

(c) $\exists x\,(C(x) \land F(x) \land \neg D(x))$

(d) $\neg \exists x\,(C(x) \land D(x) \land F(x))$

(e) $(\exists x\, C(x)) \land (\exists x\, D(x)) \land (\exists x\, F(x))$

Q9.
① $p \land q$ premise
② p from ①, $p \land q = T$ then $p = T$
③ $p \to \neg(q \land r)$ premise
④ $\neg(q \land r)$ from ②③.
⑤ q from ①, $p \land q = T$ then $q = T$
⑥ $\neg r$ from ④⑤
⑦ $s \to r$ premise
⑧ $\neg s$ from ⑥⑦

Q10. (a) Suppose $f(x)$: x is in this class, $g(x)$: x enjoys whale watching, $h(x)$: x cares about ocean pollution.

We have premise $\exists x\,(f(x) \land g(x))$, $\forall x\,(g(x) \to h(x))$, we want to get $\exists x\,(f(x) \land h(x))$

① $\exists x\,(f(x) \land g(x))$ premise
② $f(y) \land g(y)$ using existness
③ $g(y)$ from ②, $f(y) \land g(y) = T$ then $g(y) = T$
④ $\forall x\,(g(x) \to h(x))$ premise
⑤ $g(y) \to h(y)$ using arbitrariness
⑥ $h(y)$ from ③⑤
⑦ $f(y)$ from ②
⑧ $f(y) \land h(y)$ from ⑥⑦
⑨ $\exists x\,(f(x) \land h(x))$ using existness

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(b) $f(x)$: x in this class, $g(x)$: x owns a personal computer,
$h(x)$: x can use a word processing program
We have the following premises. $\forall x\,(f(x)\to g(x))$, $\forall x\,(g(x)\to h(x))$,
$f(\text{Zeke})$. We want to get $h(\text{Zeke})$.

① $\forall x\,(f(x)\to g(x))$ premise
② $f(\text{Zeke})\to g(\text{Zeke})$ using existness
③ $f(\text{Zeke})$ premise
④ $g(\text{Zeke})$ from ②③
⑤ $\forall x\,(g(x)\to h(x))$ premise
⑥ $g(\text{Zeke})\to h(\text{Zeke})$ using existness
⑦ $h(\text{Zeke})$ from ④⑥

(c) $f(x)$: x lives in this room (x is one of ABCDE)
$g(x)$: x has taken a course in ~~de~~ discrete math.
$h(x)$: x can take a course in algorithms.
We have $\forall x\,(f(x)\to g(x))$, $\forall x\,(g(x)\to h(x))$, we want to
get $\forall x\,(f(x)\to h(x))$.

① $\forall x\,(f(x)\to g(x))$ premise
② $f(y)\to g(y)$ from ①, y is an arbitrary student
③ $\forall x\,(g(x)\to h(x))$ premise
④ $g(y)\to h(y)$ from ③
⑤ $f(y)\to h(y)$ from ②④
⑥ $\forall x\,(f(x)\to h(x))$ using arbitrariness of y

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Q11. (a) $\exists n \in \mathbb{N} ((n^3+6n+5 \text{ is odd}) \land (n \text{ is not even}))$

(b) The original statement is true.

If n^3+6n+5 is odd, then n^3+5 is odd, because $6n$ is even.

From n^3+5 is odd, we get n^3 is even, then n is even.

Hence, $\forall n \in \mathbb{N} (n^3+6n+5 \text{ is odd} \Rightarrow n \text{ is even})$

Q12. If a^2+b^2 is even, then a^2+b^2+2ab is even, which means $(a+b)^2$ is even, so $a+b$ is even.

Q13. Disprove. $a=2$, $b=\frac{1}{2}$ are both rational, but $a^b = 2^{\frac{1}{2}} = \sqrt{2}$ is not.

Q14. Prove by contradiction.

Suppose $\sqrt[3]{2}$ is irrational, it can be written as $\sqrt[3]{2} = \frac{n}{m}$ ($\gcd(m,n)=1$)

so $\frac{n^3}{m^3} = 2$, $n^3 = 2m^3$ is even, so n is even.

Suppose $n=2k$, $n^3 = 8k^3 = 2m^3$, $m^3 = 4k^3$ is even, so m is even.

Contradict to $\gcd(m,n)=1$.

Q15. Prove by contradiction.

By the theorem, $\sqrt{2}$ and $\sqrt{3}$ are not rational.

Suppose $\sqrt{2}+\sqrt{3}$ is rational, $\sqrt{2}+\sqrt{3} = x$.

$x^2 = (\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$, then $\sqrt{6} = \frac{x^2-5}{2}$

x is rational, $\frac{x^2-5}{2}$ is rational, but by theorem, $\sqrt{6}$ is irrational, which cause a contradiction.

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Q16. Suppose $\max\{b_1, b_2, \cdots, b_{n-1}\} = b$, then discuss by cases.

Case 1, $b < v_n$. When $b_n \geq b$, get payoff $v_n - b$, otherwise 0. Set $b_n = v_n$ the payoff is no smaller than others.

Case 2, $b = v_n$. This case, payoff is always 0. The payoff is no smaller than ~~$b_n = v_n$~~ others when $b_n = v_n$.

Case 3, $b > v_n$. When $b_n > b$, payoff is $v_n - b < 0$, otherwise 0. Set $b_n = v_n$ also get the smallest lost.

Above all, bid $b_n = v_n$ always lead to a payoff that is no smaller than $b_n \neq v_n$.