CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #2
Due: 23:55 on Apr. 1th, 2024, please submit through Blackboard

Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. Consider sets A and B. Prove or disprove the following.

(1) $\mathcal{P}(A \times B) = \mathcal{P}(B \times A)$.

(2) $(A \oplus B) \oplus B = A$, where $A \oplus B$ denotes the set containing those elements in either A or B, but not both.

(3) For any function $f : A \to B$, $f(S \cap T) = f(S) \cap f(T)$, for any two sets $S, T \subseteq A$.

(4) For function $f : A \to B$, suppose its inverse function f^{-1} exists. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, for any $S, T \subseteq B$.

Q. 2. Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Q. 3. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Q. 4. Let $f_1 : \mathbf{R} \to \mathbf{R}^+$ and $f_2 : \mathbf{R} \to \mathbf{R}^+$. Let $g : \mathbf{R} \to \mathbf{R}$, and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

(a) Prove that $f_1(x) + f_2(x)$ is $\Theta(g(x))$.

(b) Suppose we change the range of functions $f_1(x)$ and $f_2(x)$ to the set of real numbers, i.e., $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$. Prove or disprove that $f_1(x) + f_2(x)$ is always $\Theta(g(x))$.

Q. 5. Let $f_1 : \mathbf{Z}^+ \to \mathbf{R}^+$, and $f_2 : \mathbf{Z}^+ \to \mathbf{R}^+$. Let $g : \mathbf{Z}^+ \to \mathbf{R}$, and suppose $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

(a) Prove or disprove that $(f_1 - f_2)(x)$ is $\Theta(g(x))$.

(b) Prove or disprove that $(f_1 f_2)(x)$ is $\Theta(g^2(x))$, where $g^2(x) = (g(x))^2$.

Q. 6. Prove or disprove that there exists an infinite set A such that $|A| < |\mathbf{Z}^+|$.

Q. 7. Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q. 8. Derive the formula for $\sum_{k=1}^{n} k^3$.

Q. 9. For each set defined below, determine whether the set is countable or uncountable. Explain your answers. Recall that $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b) | a,\ b \in \mathbf{N}\}$

(c) $\{(a, b) | a \in \mathbf{N},\ b \in \mathbf{R}\}$

Q. 10. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m+n-2)(m+n-1)/2+m$ is one-to-one and onto.

Q. 11. Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Q. 12. (5 points) Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

Q. 13. Consider **Horner's method**. This pseudocode shows how to use this method to find the value of $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$.

Algorithm 1 Horner ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
y := a_n
for i := 1 to n do
    y := y * c + a_{n-i}
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0}
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Exactly how many multiplications and additions are used by this algorithm to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable.)