CS201: Discrete Math for Computer Science
Quiz 2, Spring 2024

The quiz needs to be written in English. Quiz in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1 (Mathematical Induction, 30 points)**.** Consider the "proof" of the following clearly false claim: *Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. Prove that $P(n)$ is true for all positive integers $n \geq 2$.*

Proof:

BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.

INDUCTIVE STEP:

(S1) Inductive hypothesis: $P(k)$ is true for the positive integer $k \geq 2$. That is, every set of k lines in the plane, no two of which are parallel, meet in a common point.

(S2) Consider a set of $k+1$ distinct lines in the plane. By the inductive hypothesis, the first k of these lines meet in a common point $p1$. Moreover, by the inductive hypothesis, the last k of these lines meet in a common point $p2$.

(S3) We will show that $p1$ and $p2$ must be the same point. This is because if $p1$ and $p2$ were different points, all lines containing both of them must be the same line because two points determine a line. Contradiction. We conclude that the point $p1 = p2$ lies on all $k+1$ lines.

(S4) We have shown that $P(k+1)$ is true assuming that $P(k)$ is true.

Answer the following questions:

(a) Is the BASIS STEP correct? ________________ [Yes/No]

(b) Which of the step in INDUCTIVE STEP is incorrect? ________ [S1/S2/S3/S4]. Please choose one of them. If you think multiple steps have errors, please choose the one with the major error.

(c) Explain the error you refer to in b). Explain the reason.

Solution: a) Yes. b) S3. c) The following statement is incorrect: $p1$ and $p2$ must be the same point. Consider $k = 2$ and $k+1 = 3$. The first two lines meet in $p1$. The last two lines meet at $p2$. However, $p1$ and $p2$ do not need to be the same point. $p1$ and $p2$ can be any two points along the second line.

Q. 2 (Counting, 20 points)**.** Distribute six indistinguishable balls into nine distinguishable bins.

(a) How many ways to give these ball to the bins? ________________

(b) The answer to the above question is the coefficient of term ____________ of generating function ________________________________.

Solution:

(a) $C(14, 6)$

(b) x^6; $(1 + x^1 + x^2 + x^3 + \cdots)^9$ or $(1 + x^1 + x^2 + x^3 + \cdots + x^6)^9$ (any of these functions is ok)

Q. 3 (Counting, 20 points). Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers. (Hint: Write each of the $n+1$ integers a_1, a_2, . . . , a_{n+1} as a power of 2 times an odd integer, i.e., let $a_j = 2^{k_j} q_j$ for $j = 1, 2, ..., n+1$, where k_j is a nonnegative integer and q_j is odd.)

Solution: Write each of the $n+1$ integers a_1, a_2, . . . , a_{n+1} as a power of 2 times an odd integer, i.e., let $a_j = 2^{k_j} q_j$ for $j = 1, 2, ..., n+1$, where k_j is a nonnegative integer and q_j is odd. The integers q_1, q_2, . . . , q_{n+1} are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers q_1, q_2, . . . , q_{n+1} must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i.

Q. 4 (Linear Recurrence Relation, 30 points). Let $l(n)$ denote the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time. For example, if there are five stairs, the possibilities include $(1,1,1,1,1)$, $(1,1,1,2)$, $(1,1,2,1)$, ... $(1,2,2)$, ...

(a) What are the initial conditions $l(1)$ and $l(2)$? ____________________

(b) What is the recursive function of $l(n)$ for $n \geq 3$?____________________

(c) Derive the closed-form of $l(n)$ using the general approach we have learned for solving linear recurrence relation. Please provide the derivation details. Do NOT use mathematical induction.

Solution:

(a) $l(1) = 1$; $l(2) = 2$;

(b) $l(n) = l(n-1) + l(n-2)$;

(c) The characteristic equation is $r^2 - r - 1 = 0$. Thus,

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Let $l(n) = \alpha_1 r_1^n + \alpha_2 r_2^n$. Construct another initial condition $l(0) = 1$. Substituting the initial conditions $l(0)$ and $l(1)$:

$$\alpha_1 + \alpha_2 = 1, \quad \alpha_1 \frac{1+\sqrt{5}}{2} + \alpha_2 \frac{1-\sqrt{5}}{2} = 1.$$

Thus, we can determine $\alpha_1 = 1/2 + 1/(2\sqrt{5})$, $\alpha_2 = 1/2 - 1/(2\sqrt{5})$. To sum up,

$$l(n) = \left(\frac{1}{2} + \frac{1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} - \frac{1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$