Discrete Mathematics for Computer Science

Lecture 7: Number Theory

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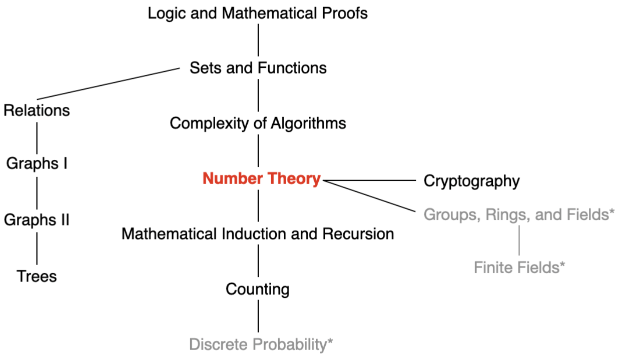
Number Theory



Number Theory: divisibility and modular arithmetic, integer representations, primes, greatest common divisors, ...

Number Theory

Number theory is a branch of mathematics that explores integers and their properties, is the basis of cryptography, coding theory, computer security, e-commerce, etc.

Division

If a and b are integers with $a \neq 0$,

- we say that a divides b if there is an integer c such that $b = ac$, or equivalently b/a is an integer.
- b is divisible/divided by a

In this case, we say that a is a factor or divisor of b, and b is a multiple of a. (We use the notations $a|b$, $a \nmid b$)

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Example:

- $4|24$
- $4 \nmid 5$

Divisibility

All integers divisible by $d > 0$ can be enumerated as:

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Question: Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

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Question: Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

Answer: Count the number of integers such that $0 < kd \leq n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.

Divisibility: Properties

Let a, b, c be integers. Then the following hold:

(i) if $a|b$ and $a|c$, then $a|(b+c)$

(ii) if $a|b$ then $a|bc$ for all integers c

(iii) if $a|b$ and $b|c$, then $a|c$

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Proof: Suppose that $a|b$ and $a|c$. Then, from the definition of divisibility, it follows that there are integers s and t with $b = as$ and $c = at$. Hence,

$$b + c = as + at = a(s + t).$$

Therefore, a divides $b + c$.

Divisibility

Corollary: If a, b, c are integers, where $a \neq 0$, such that $a|b$ and $a|c$, then $a|(mb + nc)$ whenever m and n are integers.

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Proof: By part (ii) and part (i) of Properties.

The Division Algorithm

If a is an integer and d a positive integer, then there are unique integers q and r, with $0 \leq r < d$, such that

$$a = dq + r.$$

In this case, d is called the divisor, a is called the dividend, q is called the quotient, and r is called the remainder.

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Example: The quotient and remainder when 101 is divided by 11?

$$101 = 11 \times 9 + 2$$

Hence, the quotient is $9 = 101 \textbf{ div } 11$, and the remainder is $2 = 101 \textbf{ mod } 11$.

Congruence Relation

If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$, denoted by $a \equiv b \pmod{m}$. This is called congruence and m is its modulus.

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- $-1 \equiv 11 \pmod{6}$

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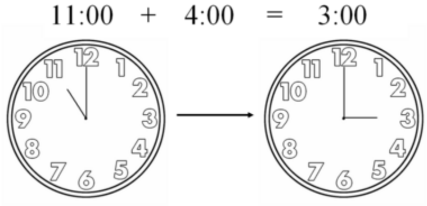

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Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that

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Proof:

- If part: If there is an integer k such that $a = b + km$, then $km = a - b$. Hence, m divides $a - b$, so that $a \equiv b \pmod{m}$.
- Only if part: If $a \equiv b \pmod{m}$, by the definition of congruence, we know that $m|(a-b)$. This means that there is an integer k such that $a - b = km$, so that $a = b + km$.

$(\mathbf{mod}\ m)$ and $\mathbf{mod}\ m$ Notations

Notations $a \equiv b \pmod{m}$ and $a \bmod m$ are different.

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Let a and b be integers, and let m be a positive integer. Then, $a \equiv b \pmod{m}$ if and only if

$$a \bmod m = b \bmod m.$$

Congruence: Properties

Theorem: Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m}$$

$$ac \equiv bd \pmod{m}$$

Proof:

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Proof: We use a direct proof. Since $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, there are integers s and t with $a = b + sm$ and $c = d + tm$. Hence,

$$b + d = (a - sm) + (c - tm) = (a + c) + m(-s - t)$$

$$bd = (a - sm)(c - tm) = ac + m(-at - cs + stm)$$

Hence, $a + c \equiv b + d \pmod{m}$, $ac \equiv bd \pmod{m}$.

Algebraic Manipulation of Congruence

Question: If $ca \equiv cb \pmod{m}$, then $a \equiv b \pmod{m}$?

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Question: If $a \equiv b \pmod{m}$ and c is an integer, then

- $ca \equiv cb \pmod{m}$? Yes
- $c + a \equiv c + b \pmod{m}$? Yes
- $a/c \equiv b/c \pmod{m}$? No

Computing the mod Function

Corollary: Let m be a positive integer and let a and b be integers. Then,

$$(a+b) \textbf{ mod } m = ((a \textbf{ mod } m) + (b \textbf{ mod } m)) \textbf{ mod } m$$

$$ab \textbf{ mod } m = ((a \textbf{ mod } m)(b \textbf{ mod } m)) \textbf{ mod } m$$

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Proof: By the definitions of **mod** m and of congruence modulo m, we know that $a \equiv (a \mathbf{\ mod\ } m)(\mathbf{mod\ } m)$ and $b \equiv (b \mathbf{\ mod\ } m)(\mathbf{mod\ } m)$. Hence,

$$a+b \equiv (a \mathbf{\ mod\ } m) + (b \mathbf{\ mod\ } m)(\mathbf{mod\ } m)$$

$$ab \equiv (a \mathbf{\ mod\ } m)(b \mathbf{\ mod\ } m)(\mathbf{mod\ } m).$$

According to the theorem that $a \equiv b \ (\mathbf{mod\ } m)$ if and only if $a \mathbf{\ mod\ } m = b \mathbf{\ mod\ } m$, we obtain the above equalities.

Arithmetic Modulo m

Let $\mathbf{Z}_m$ be the set of nonnegative integers less than m: $\{0, 1, ..., m-1\}$.

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- $7 \cdot_{11} 9 =?$ 8

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Associativity: If a, b, and c belong to $\mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$.

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Additive inverses: If $a \neq 0$ and $a \in \mathbf{Z}_m$, then $m - a$ is an additive inverse of a modulo m. That is, $a +_m (m - a) = 0$ and $0 +_m 0 = 0$.

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Commutativity: If $a, b \in \mathbf{Z}_m$, then $a +_m b = b +_m a$.

Distributivity: If $a, b, c \in \mathbf{Z}_m$, then

$$a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$$

$$(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$$

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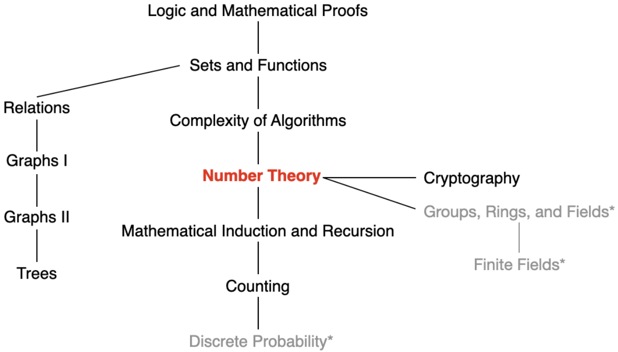

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Representations of Integers

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Let $b > 1$ be an integer. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \quad + a_1 b + a_0,$$

where k is nonnegative, a_k's are nonnegative integers less than b. The representation of n is called the base-b expansion of n and is denoted by $(a_k a_{k-1} ... a_1 a_0)_b$.

Base-b Expansions

From binary, octal, hexadecimal expansions to the decimal expansion:

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Example

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- ◇ $(7016)_8 = 7 \cdot 8^3 + 1 \cdot 8 + 6 = 3598$

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Conversions between binary and octal (or hexadecimal) expansions:

Example

- $(101011111)_2 = (\underline{101}\,\overline{011}\,\underline{111}) = (537)_8$
- $(7016)_8 = (\underline{111}\,\overline{000}\,\underline{001}\,\overline{110})_2$
  $= (\underline{1110}\,\overline{0000}\,\underline{1110})_2 = (E0E)_{16}$

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$$
\begin{aligned}
n &= a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \cdots + a_2 b^2 + a_1 b + a_0 \\
&= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \cdots + a_2 b + a_1) + a_0 \\
&= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \cdots + a_2) + a_1) + a_0 \\
&= \cdots
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- Divide n by b to obtain $n = bq_0 + a_0$, with $0 \leq a0 < b$
- The remainder a_0 is the rightmost digit in the base-b; expansion of n. Then divide q_0 by b to get $q_0 = bq_1 + a_1$ with $0 \leq a1 < b$;

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- The remainder a_0 is the rightmost digit in the base-b; expansion of n. Then divide q_0 by b to get $q_0 = bq_1 + a_1$ with $0 \leq a1 < b$;
- a_1 is the second digit from the right; continue by successively dividing the quotients by b until the quotient is 0

Base-b Expansions

procedure *base b expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return$(a_{k-1}, \dots, a_1, a_0)$ $\{(a_{k-1} \dots a_1 a_0)_b$ is base b expansion of $n\}$

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Base-b Expansions

Example: Find the hexadecimal expansion of $(177130)_{10}$.

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Solution: First divide 177130 by 16 to obtain

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Successively dividing quotients by 16 gives

$$11070 = 16 \cdot 691 + 14,$$

$$691 = 16 \cdot 43 + 3,$$

$$43 = 16 \cdot 2 + 11,$$

$$2 = 16 \cdot 0 + 2.$$

The successive remainders that we have found, 10, 14, 3, 11, 2. It follows that $(177130)_{10} = (2B3EA)_{16}$.

Binary Addition of Integers

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$$a = (a_{n-1}a_{n-2}\ldots a_1a_0)_2,\ b = (b_{n-1}b_{n-2}\ldots b_1b_0)_2$$

procedure *add*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1},a_{n-2},\ldots,a_0)_2$ and $(b_{n-1},b_{n-2},\ldots,b_0)_2$, respectively}
$c := 0$
for $j := 0$ to $n-1$
  $d := \lfloor (a_j + b_j + c)/2 \rfloor$
  $s_j := a_j + b_j + c - 2d$
  $c := d$
$s_n := c$
return($s_0, s_1, \ldots, s_n$){the binary expansion of the sum is $(s_n, s_{n-1}, \ldots, s_0)_2$}

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    $s_j := a_j + b_j + c - 2d$
    $c := d$
$s_n := c$
return$(s_0, s_1, ..., s_n)${the binary expansion of the sum is $(s_n, s_{n-1}, ..., s_0)_2$}

$O(n)$ bit additions

Algorithm: Binary Multiplication of Integers

$a = (a_{n-1}a_{n-2}...a_1 a_0)_2$, $b = (b_{n-1}b_{n-2}...b_1 b_0)_2$

$ab = a(b_0 2^0 + b_1 2^1 + \ + b_{n-1}2^{n-1}) = a(b_0 2^0) + a(b_1 2^1) + \ + a(b_{n-1}2^{n-1})$

Algorithm: Binary Multiplication of Integers

$a = (a_{n-1}a_{n-2}...a_1a_0)_2,\ b = (b_{n-1}b_{n-2}...b_1b_0)_2$

$ab = a(b_0 2^0 + b_1 2^1 + \ + b_{n-1}2^{n-1}) = a(b_0 2^0) + a(b_1 2^1) + \ + a(b_{n-1}2^{n-1})$

procedure *multiply*(a, b: positive integers)
{the binary expansions of a and b are $(a_{n-1}, a_{n-2}, ..., a_0)_2$ and $(b_{n-1}, b_{n-2}, ..., b_0)_2$, respectively}
for $j := 0$ to $n-1$
    if $b_j = 1$ **then** $c_j = a$ shifted j places
    else $c_j := 0$
{$c_0, c_1, ..., c_{n-1}$ are the partial products}
$p := 0$
for $j := 0$ to $n-1$
  $p := p + c_j$
return p {p is the value of ab}

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$O(n^2)$ shifts and $O(n^2)$ bit additions

Algorithm: Computing div and mod

Compute $q = a$ **div** d and $r = a$ **mod** d:

procedure *division algorithm* (a: integer, d: positive integer)
$q := 0$
$r := |a|$
while $r \geq d$
    $r := r - d$
    $q := q + 1$
if $a < 0$ and $r > 0$ **then**
    $r := d - r$
    $q := -(q+1)$
return (q, r) {$q = a$ **div** d *is the quotient*, $r = a$ **mod** d *is the remainder* }

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$O(q \log a)$ bit operations. But there exist more efficient algorithms with complextiy $O(n^2)$, where $n = \max(\log a, \log d)$

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$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}} \cdots b^{a_1\cdot 2} \cdot b^{a_0}$$

Recall that

$$ab \textbf{ mod } m = ((a \textbf{ mod } m)(b \textbf{ mod } m)) \textbf{mod } m.$$

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Successively finds b **mod** m, b^2 **mod** m, b^4 **mod** m, . . . , $b^{2^{k-1}}$ **mod** m, and multiplies together the terms b^{2^j}, where $a_j = 1$.

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procedure *modular exponentiation*(b: integer, $n = (a_{k-1}a_{k-2} \ldots a_1 a_0)_2$, m: positive integers)
$x := 1$
$power := b$ **mod** m
for $i := 0$ to $k - 1$
  if $a_i = 1$ **then** $x := (x \cdot power)$ **mod** m
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return x {x equals b^n **mod** m }

Algorithm: Binary Modular Exponentiation

Use the algorithm to find 3^{644} mod 645:

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The algorithm initially sets $x = 1$ and $power = 3 \textbf{ mod } 645 = 3$. The binary expansion of 644 is $(1010000100)_2$. Here are the steps used:

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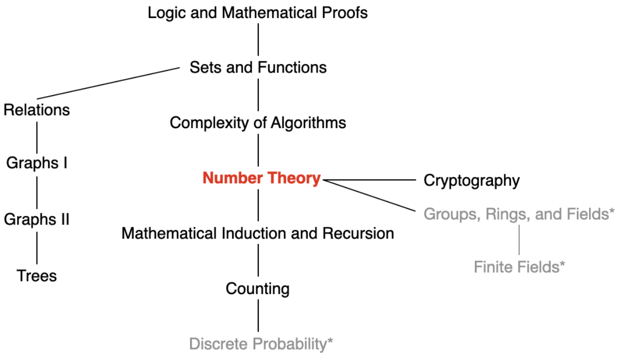
The algorithm initially sets $x = 1$ and $power = 3$ **mod** $645 = 3$. The binary expansion of 644 is $(1010000100)_2$. Here are the steps used:

$i = 0$: Because $a_0 = 0$, we have $x = 1$ and $power = 3^2 \textbf{ mod } 645 = 9 \textbf{ mod } 645 = 9$;
$i = 1$: Because $a_1 = 0$, we have $x = 1$ and $power = 9^2 \textbf{ mod } 645 = 81 \textbf{ mod } 645 = 81$;
$i = 2$: Because $a_2 = 1$, we have $x = 1 \cdot 81 \textbf{ mod } 645 = 81$ and $power = 81^2 \textbf{ mod } 645 = 6561 \textbf{ mod } 645 = 111$;
$i = 3$: Because $a_3 = 0$, we have $x = 81$ and $power = 111^2 \textbf{ mod } 645 = 12{,}321 \textbf{ mod } 645 = 66$;
$i = 4$: Because $a_4 = 0$, we have $x = 81$ and $power = 66^2 \textbf{ mod } 645 = 4356 \textbf{ mod } 645 = 486$;
$i = 5$: Because $a_5 = 0$, we have $x = 81$ and $power = 486^2 \textbf{ mod } 645 = 236{,}196 \textbf{ mod } 645 = 126$;
$i = 6$: Because $a_6 = 0$, we have $x = 81$ and $power = 126^2 \textbf{ mod } 645 = 15{,}876 \textbf{ mod } 645 = 396$;
$i = 7$: Because $a_7 = 1$, we find that $x = (81 \cdot 396) \textbf{ mod } 645 = 471$ and $power = 396^2 \textbf{ mod } 645 = 156{,}816 \textbf{ mod } 645 = 81$;
$i = 8$: Because $a_8 = 0$, we have $x = 471$ and $power = 81^2 \textbf{ mod } 645 = 6561 \textbf{ mod } 645 = 111$;
$i = 9$: Because $a_9 = 1$, we find that $x = (471 \cdot 111) \textbf{ mod } 645 = 36$.

Next Lecture



Number Theory: divisibility and modular arithmetic, integer representations, primes, greatest common divisors, ...