CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #2
Due: 23:55 on Apr. 1th, 2024, please submit through Blackboard

Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. Consider sets A and B. Prove or disprove the following.

(1) $\mathcal{P}(A \times B) = \mathcal{P}(B \times A)$.

(2) $(A \oplus B) \oplus B = A$, where $A \oplus B$ denotes the set containing those elements in either A or B, but not both.

(3) For any function $f : A \to B$, $f(S \cap T) = f(S) \cap f(T)$, for any two sets $S, T \subseteq A$.

(4) For function $f : A \to B$, suppose its inverse function f^{-1} exists. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, for any $S, T \subseteq B$.

Solution:

(1) This statement is false. Consider a counterexample $A = \{1\}$ and $B = \{2\}$. Thus, $A \times B = \{(1, 2)\}$ and $B \times A = \{(2, 1)\}$. Hence, $\mathcal{P}(A \times B) = \{\emptyset, (1, 2)\}$ and $\mathcal{P}(B \times A) = \{\emptyset, (2, 1)\}$. Since $(1, 2) \in \mathcal{P}(A \times B)$ and $(1, 2) \notin \mathcal{P}(B \times A)$, $\mathcal{P}(A \times B) = \mathcal{P}(B \times A)$ does not hold.

(2) This statement is true. This can be proven using membership table.

A	B	$A \oplus B$	$(A \oplus B) \oplus B$
1	1	0	1
1	0	1	1
0	1	1	0
0	0	0	0

(3) The statement is false. A counterexample is: $f(n) = n^2$ for $A = \mathbf{R}$ and $B = \mathbf{R}^+$. Consider $S = \{1\}$ and $T = \{-1\}$. Then, $f(S) = \{1\}$ and $f(T) = \{1\}$. Thus, $f(S) \cap f(T) = \{1\}$. However, $S \cap T = \emptyset$ and hence $f(S \cap T) = \emptyset$.

(4) This statement is true. Since the inverse function f^{-1} exists, f must be a bijection, i.e., it is both one-to-one and onto. Thus, f^{-1} is also a bijection. We complete the proof by showing that $f^{-1}(S \cap T)$ and $f^{-1}(S) \cap f^{-1}(T)$ are subsets of each other:

- $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$: For any $x \in f^{-1}(S \cap T)$, there exists a y such that $y \in S \cap T$ and $f^{-1}(y) = x$. Thus, we have $y \in S$ and $y \in T$. This implies that $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$. Hence, $x \in f^{-1}(S) \cap f^{-1}(T)$.
- To prove $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$: If $x \in f^{-1}(S) \cap f^{-1}(T)$, then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$. Then, there exists $y_1 \in S$ and $y_2 \in T$ such that $f^{-1}(y_1) = x$ and $f^{-1}(y_2) = x$. Since f^{-1} is a bijection, it is one-to-one. Thus, $y_1 = y_2 \in S \cap T$. This implies that $x \in f^{-1}(S \cup T)$.

Q. 2. Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Solution:

(1) We have

$$\begin{aligned} (B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) && \text{by definition} \\ &= \overline{A} \cap (B \cup C) && \text{ditributive law} \\ &= (B \cup C) - A && \text{by definition} \end{aligned}$$

(2) We have

$$\begin{aligned} &(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) \\ &= (A \cap B) \cap (A \cap C) \cap \overline{(B \cap C)} && \text{commutative law} \\ &= (A \cap B \cap C) \cap \overline{(B \cap C)} && \text{associative law} \\ &= (A \cap B \cap C) \cap (\overline{B} \cup \overline{C}) && \text{De Morgan} \\ &= ((A \cap B \cap C) \cap \overline{B}) \cup ((A \cap B \cap C) \cap \overline{C}) && \text{distributive law} \\ &= \emptyset \cup \emptyset && \text{Complement} \\ &= \emptyset. \end{aligned}$$

$\square$

Q. 3. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution: For the "if" part, given $A \subseteq B$, we want to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., if $C \subseteq A$, then $C \subseteq B$. Since $A \subseteq B$, $A \subseteq C$ directly follows.

For the "only if" part, given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show that $A \subseteq B$. Suppose that $a \in A$. Then $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$, which means that $\{a\} \subseteq B$. This implies that $a \in B$, and completes the proof.

$\square$

Q. 4. Let $f_1 : \mathbf{R} \to \mathbf{R}^+$ and $f_2 : \mathbf{R} \to \mathbf{R}^+$. Let $g : \mathbf{R} \to \mathbf{R}$, and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

(a) Prove that $f_1(x) + f_2(x)$ is $\Theta(g(x))$.

(b) Suppose we change the range of functions $f_1(x)$ and $f_2(x)$ to the set of real numbers, i.e., $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$. Prove or disprove that $f_1(x) + f_2(x)$ is always $\Theta(g(x))$.

Solution:

(a) By the definition of Θ, since $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, there exist real numbers C_1, C_1', C_2, and C_2' and positive real numbers k_1 and k_2 such that

$$C_1|g(x)| \leq |f_1(x)| \leq C_1'|g(x)|, \ x > k_1,$$

$$C_2|g(x)| \leq |f_2(x)| \leq C_2'|g(x)|, \ x > k_2.$$

Thus, let $k = \max\{k_1, k_2\}$. Then, since $f_1(x) > 0$ and $f_2(x) > 0$, we have

$$(C_1 + C_2)|g(x)| \leq |f_1(x) + f_2(x)| \leq (C_1' + C_2')|g(x)|, \ x > k.$$

Thus, $f_1(x) + f_2(x)$ is $\Theta(g(x))$.

(b) The statement $f_1(x) + f_2(x)$ is $\Theta(g(x))$ does not always hold. Suppose $f_1(x) = x^2$ and $f_2(x) = -x^2$. We have $f_1(x)$ and $f_2(x)$ are $\Theta(x^2)$. However, $f_1(x) + f_2(x) = 0$, which is no longer $\Theta(x^2)$.

Q. 5. Let $f_1 : \mathbf{Z}^+ \to \mathbf{R}^+$, and $f_2 : \mathbf{Z}^+ \to \mathbf{R}^+$. Let $g : \mathbf{Z}^+ \to \mathbf{R}$, and suppose $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

(a) Prove or disprove that $(f_1 - f_2)(x)$ is $\Theta(g(x))$.

(b) Prove or disprove that $(f_1 f_2)(x)$ is $\Theta(g^2(x))$, where $g^2(x) = (g(x))^2$.

Solution:

(a) This is false. Consider a counterexample. Let $f_1(x) = x^2 + 2$, $f_2(x) = x^2 + 1$, and $g(x) = x^2$. Thus, $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$. Note that $(f_1 - f_2)(x) = 1$, which is not $\Theta(g(x))$.

(b) It is true that $(f_1 f_2)(x)$ is $\Theta(g^2(x))$. By the definition of Θ, since $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, there exist real numbers C_1, C_1', C_2, and C_2' and positive real numbers k_1 and k_2 such that

$$C_1|g(x)| \le |f_1(x)| \le C_1'|g(x)|, \ x > k_1,$$

$$C_2|g(x)| \le |f_2(x)| \le C_2'|g(x)|, \ x > k_2.$$

Thus, let $k = \max\{k_1, k_2\}$, $C = C_1 C_2$, and $C' = C_1' C_2'$. Then, since $f_1(x) > 0$ and $f_2(x) > 0$, we have

$$C(|g(x)|)^2 \le |(f_1 f_2)(x)| \le C'(|g(x)|)^2, \ x > k.$$

That is, $C|(g(x))^2| \le |(f_1 f_2)(x)| \le |C'(g(x))^2|, \ x > k$. Thus, $(f_1 f_2)(x)$ is $\Theta(g^2(x))$.

Q. 6. Prove or disprove that there exists an infinite set A such that $|A| < |\mathbf{Z}^+|$.

Solution: This statement is false. Suppose there exists an infinite set A such that $|A| < |\mathbf{Z}^+|$. This means that $|A| \le |\mathbf{Z}^+|$ and $|A| \ne |\mathbf{Z}^+|$.

- Since $|A| \ne |\mathbf{Z}^+|$, there does not exist any one-to-one correspondence that maps from A to $\mathbf{Z}^+$. Thus, A cannot be countable infinite.

- Since $|A| \le |\mathbf{Z}^+|$, there exists a one-to-one function maps from A to $\mathbf{Z}^+$. There is a subset $S \subset \mathbf{Z}^+$ such that there exists a one-to-one correspondence that maps from A to S. Since the subset of a countable set is also countable, S is countable. Thus, S is either finite or there exists a one-to-one correspondence from S to $\mathbf{Z}^+$. This leads to the fact that A is either finite or countable infinite.

Thus, contradiction occurs. This complete the disprove.

Q. 7. Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Solution:

Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed $n = \lfloor x \rfloor$.

- if $x \in [n, n + \frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n + 1)$, so $\lfloor 3x \rfloor = 3n$. Moreover in this case $x + \frac{1}{3}$ is still less than $n + 1$, and $x + \frac{2}{3}$ is still less than $n + 1$. So, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + n = 3n$ as well.
- if $x \in [n + \frac{1}{3}, n + \frac{2}{3})$, then $3x \in [3n + 1, 3n + 2)$, so $\lfloor 3x \rfloor = 3n + 1$. Moreover in this case $x+\frac{1}{3}$ is in $[n+\frac{2}{3}, n+1)$, and $x+\frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + (n + 1) = 3n + 1$ as well.
- if $x \in [n + \frac{2}{3}, n + 1)$, similar and both sides equal $3n + 2$.

$\square$

Q. 8. Derive the formula for $\sum_{k=1}^{n} k^3$.

Solution: Again, we use "telescoping" to derive this formula. Since $k^4 - (k - 1)^4 = k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1) = 4k^3 - 6k^2 + 4k - 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n}[k^4 - (k-1)^4] &= 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
&= 4\sum_{k=1}^{n} k^3 - 6n(n+1)(2n+1)/6 + 4n(n+1)/2 - n \\
&= 4\sum_{k=1}^{n} k^3 - n(n+1)(2n+1) + 2n(n+1) - n \\
&= n^4.
\end{aligned}$$

Thus, it then follows that

$$\begin{aligned}
4\sum_{k=1}^{n} k^3 &= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\
&= n^2(n+1)^2.
\end{aligned}$$

Then we get the formula $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$.

□

Q. 9. For each set defined below, determine whether the set is countable or uncountable. Explain your answers. Recall that $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b)|a,\ b \in \mathbf{N}\}$

(c) $\{(a, b)|a \in \mathbf{N},\ b \in \mathbf{R}\}$

Solution:

(a) Countable. The number of students in CS201 is finite, so the size of its power set is also finite. All finite sets are countable.

(b) Countable. The set is the same as $\mathbf{N} \times \mathbf{N}$. We now show that these elements can be listed in a sequence:

$$(0, 0), (1, 0), (1, 1), (0, 1), (2, 0), (2, 1), (2, 2), (1, 2), (0, 2), \dots.$$

That is, we start with $a = 0$, list $(0, 0)$. Then, we work on $a = 1$, list $(1, 0)$, $(1, 1)$, $(0, 1)$. Subsequently, for any $a = i$, we list $(i, 0)$, $(i, 1)$, ..., (i, i), $(i-1, i)$, ...$(0, i)$. Then, we set $a = i+1$ and continue the process. It can be easily checked that all elements in set $\{(a, b)|a,\ b \in \mathbf{N}\}$ are in this sequence. (Note: as long as students can show there is a sequence that can list all the elements or there is a one-to-one corresponds from the set of positive integers to this set, then it is correct.)

(c) Uncountable. We will prove by contradiction. Suppose $\{(a, b) \mid a \in \mathbf{N},\ b \in \mathbf{R}\}$ is countable. Then, it's subset $\{(a, b) \mid a = 1,\ b \in \mathbf{R}, 0 < b < 1\}$ is also countable. Thus, we can list all the elements in this set in a sequence. Let $(1, r_1)$, $(1, r_2)$, $(1, r_3)$ ... be the elements in the sequence, where

- $r_1 = 0.d_{11}d_{12}d_{13}...$
- $r_2 = 0.d_{21}d_{22}d_{23}...$
- $r_3 = 0.d_{31}d_{32}d_{33}...$
- ...

Now, we aim to construct a tuple $(1, r)$ that is not in this sequence. Let $r = d_1 d_2 d_3 ....$ Set $d_i = 3$ if $d_{ii} \neq 3$, and $d_i = 2$ if $d_{ii} = 3$. It can be seen that r is different from any element in the sequence. Thus, this leads to a contradiction.

$\square$

Q. 10. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m+n-2)(m+n-1)/2+m$ is one-to-one and onto.

Solution: It is clear from the formula that the range of values the function takes on for a fixed value of $m+n$, say $m+n = x$, is $(x-2)(x-1)/2+1$ through $(x-2)(x-1)/2+(x-1)$, because m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m+n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x+1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1, 1) + 1 = f(1, x)$. We have $f(x-1, 1) + 1 = (x-2)(x-1)/2 + (x-1) + 1 = (x^2 - x + 2)/2 = (x-1)x/2 + 1 = f(1, x)$.

$\square$

Q. 11. Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

By definition, we have one-to-one and onto functions $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a one-to-one and onto function from A to C, so we have $|A| = |C|$.

$\square$

Q. 12. (5 points) Suppose that $f(x), g(x)$ and $h(x)$ are functions such that $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$. Show that $f(x)$ is $\Theta(h(x))$.

Solution: The definition of "$f(x)$ is $\Theta(g(x))$" is that $f(x)$ if both $O(g(x))$ and $\Omega(g(x))$. This means that there are positive constants C_1, k_1, C_2, and k_2 such that $|f(x)| \leq C_2|g(x)|$ for all $x > k_2$ and $|f(x) \geq C_1|g(x)|$ for all $x > k_1$. Similarly, we have that there are positive constants C'_1, k'_1, C'_2, and k'_2 such that $|g(x)| \leq C'_2|h(x)|$ for all $x > k'_2$ and $|g(x)| \geq C'_1|h(x)|$ for all $x > k'_1$. We can combine these inequalities to obtain $|f(x)| \leq C_2C'_2|h(x)|$ for all $x > \max(k_2, k'_2)$ and $|f(x)| \geq C_1C'_1|h(x)|$ for all $x > \max(k_1, k'_1)$. This means that $f(x)$ is $\Theta(h(x))$.

□

Q. 13. Consider **Horner's method**. This pseudocode shows how to use this method to find the value of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$.

Algorithm 1 Horner ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
y := a_n
for i := 1 to n do
    y := y * c + a_{n-i}
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ... + a_1 c + a_0}
```

Exactly how many multiplications and additions are used by this algorithm to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable.)

Solution:

n multiplications and n additions.

□