CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #1
Due: 23:55 on Mar. 18th, 2024, please submit through Blackboard

Please answer questions in English. Using any other language will lead to a zero point.

Q. 1. Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You do not get 100 marks on the final.

(b) You get 100 marks on the final, but you do not get an A in this course.

(c) You will get an A in this course if you get 100 marks on the final.

(d) If you do not get 100 marks on the final, then you will not get an A in this course.

(e) Getting 100 marks on the final is sufficient for getting an A in this course.

(f) You get an A in this course, but you do not get 100 marks on the final.

(g) Whenever you get an A in this course, you got 100 marks on the final.

Solution:

(a) $\neg p$

(b) $p \wedge \neg q$

(c) $p \rightarrow q$

(d) $\neg p \rightarrow \neg q$

(e) $p \rightarrow q$

(f) $q \wedge \neg p$

(g) $q \to p$

$\square$

Q. 2. Construct a truth table for each of these compound propositions.

(a) $(p \oplus q) \to (p \wedge q)$

(b) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

Solution: The details are omitted. The final results are given as follows:

(a)

p	q	$(p \oplus q) \to (p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

p	q	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	T
T	F	T
F	T	T
F	F	T

$\square$

Q. 3. Suppose P and Q are predicates, and x and y are variables. Suppose all quantifiers we considered have the same nonempty domain.

(a) Prove or disprove that $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are logically equivalent.

(b) Prove or disprove that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to

$$\forall x \exists y(P(x) \wedge Q(y)).$$

Solution:

(a) They are not equivalent. For example, let $P(x)$ be a propositional function such that $P(x)$ is true for some x in the domain and false for the rest. Let $Q(x)$ be a propositional function that is always false for all x in the domain. Then, there exists an x_0 in the domain such that $P(x_0)$ is true and $Q(x_0)$ is false, i.e., $P(x_0) \to Q(x_0)$ is false. Thus, $\forall x(P(x) \to Q(x))$ is false. On the other hand, there exists an x_1 in the domain such that $P(x_1)$ is false. Thus, $\forall x P(x)$ is false, so $\forall x P(x) \to \forall x Q(x)$ is true.

(b) They are equivalent. We aim to show that they always lead to the same truth value, no matter what the predicates P and Q are.

- Suppose that $\forall x P(x) \wedge \exists x Q(x)$ is true. Then, $P(x)$ is true for all x in the domain and there is an element y in the domain such that $Q(y)$ is true. Let y_0 denote such an element that $Q(y_0)$ is true. Thus, $P(x) \wedge Q(y_0)$ is true for all x. This implies that for all x, there exists a y such that $P(x) \wedge Q(y)$ is true. That is, $\forall x \exists y(P(x) \wedge Q(y))$ is true.
- Suppose $\forall x \exists y(P(x) \wedge Q(y))$ is true. For all x in the domain, there exists a y such that $P(x) \wedge Q(y)$ is true. Thus, $P(x)$ must be true for all x, i.e., $\forall x P(x)$. In addition, there exists a y such that $Q(y)$ is true, i.e., $\exists x Q(x)$. As a result, $\forall x P(x) \wedge \exists x Q(x)$ is true.

Q. 4. Determine whether the following statements are correct or incorrect. Explain your answer. Assume that p, q and r are logical propositions, x and y are real numbers, and m and n are integers.

(1) $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is a tautology.

(2) $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are equivalent.

(3) Under the domain of all real numbers, the truth value of $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$ is T.

(4) Under the domain of all integers, the truth value of $\exists n \exists m(n^2+m^2 = 5)$ is T.

Solution:

(1) Incorrect. This can be proven using truth table.

p	q	$\neg p$	$p \rightarrow q$	$\neg p \wedge (p \rightarrow q)$	$\neg q$	$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$
T	T	F	T	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Since the truth value of $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is not always T, it is not a tautology. (Any proof is acceptable, as long as it explains that under some p and q, the truth value of $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$ is false.)

(2) Correct. This can be proven as follows:

$$\begin{aligned}
(p \vee q) \rightarrow r &\equiv \neg(p \vee q) \vee r && \text{(Useful Law)} \\
&\equiv (\neg p \wedge \neg q) \vee r && \text{(De Moegan's Law)} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) && \text{(Distributive Law)} \\
&\equiv (p \rightarrow r) \wedge (q \rightarrow r) && \text{(Useful Law)}
\end{aligned}$$

Any proof that shows the equivalence is acceptable.

(3) Incorrect. This proposition means that there is a real number x for which $y \neq 0 \rightarrow xy = 1$ for every real number y. Consider an arbitrary x. Suppose $y_1 \neq 0$ and $xy_1 = 1$. Let $y_2 = 2y_1$. Then, $xy_2 = 2$, i.e., $y \neq 0 \rightarrow xy = 1$ does not hold for every y.

(4) Correct. When $n = 1$ and $m = 2$, $n^2 + m^2 = 5$.

Q. 5. For each of the following argument, determine whether it is valid or invalid. Explain using the validity of its argument form.

(1) Premise 1: If you did not finish your homework, then you cannot answer this question.

Premise 2: You finished your homework.

Conclusion: You can answer this question.

(2) Premise 1: If all students in this class has submitted their homework, then all students can get 100 in the final exam.

Premise 2: There is a student who did not submit his or her homework.

Conclusion: It is not the case that all student can get 100 in the final exam.

Solution:

(1) Invalid. Let p denote "you finished your homework". Let q denote "you can answer this question". Thus, premises 1 and 2 can be represented as $\neg p \rightarrow \neg q$ and p, respectively. Conclusion can be represented as q. This argument form is not valid, since $((\neg p \rightarrow \neg q) \wedge p) \rightarrow q$ is not a tautology. This is because when p is T and q is F, the truth value of $((\neg p \rightarrow \neg q) \wedge p) \rightarrow q$ is F.

(2) Invalid. Consider the domain of this class. Let $P(x)$ denote "student x has submitted his or her homework". Let $Q(x)$ denote "student x can get 100 in the final exam". Premises 1 and 2 can be represented as $\forall x P(x) \to \forall x Q(x)$ and $\exists x(\neg P(x))$, respectively. The conclusion can be represented as $\neg \forall x Q(x)$. This argument form is not valid, since $((\forall x P(x) \to \forall x Q(x)) \wedge \exists x(\neg P(x)) \to \neg \forall x Q(x)$ is not a tautology. Consider the case where both $\exists x(\neg P(x))$ and $\forall x Q(x)$ are T. Thus, $\forall x P(x)$ is F, since $\neg \forall x P(x) \equiv \exists x(\neg P(x))$ is T. Hence, $((\forall x P(x) \to \forall x Q(x)) \wedge \exists x(\neg P(x))$ is T. However, since $\neg \forall x Q(x)$ is F, the entire proposition is F.

Q. 6. Suppose that p, q, r, s are all logical propositions. You are given the following statement

$$(\neg r \vee (p \wedge \neg q)) \to (r \wedge p \wedge \neg q)$$

Prove that this implies $r \vee s$ using logical equivalences and rules of inference.

Solution:

$$\begin{array}{lll}
 & (\neg r \vee (p \wedge \neg q)) \to (r \wedge p \wedge \neg q) & \\
\equiv & \neg(\neg r \vee (p \wedge \neg q)) \vee (r \wedge p \wedge \neg q) & \text{Useful} \\
\equiv & (r \wedge \neg(p \wedge \neg q)) \vee (r \wedge p \wedge \neg q) & \text{De Morgan's} \\
\equiv & (r \wedge (\neg p \vee q)) \vee (r \wedge p \wedge \neg q) & \text{De Morgan's} \\
\equiv & (r \wedge (\neg p \vee q)) \vee (r \wedge (p \wedge \neg q)) & \text{Associative} \\
\equiv & r \wedge ((\neg p \vee q) \vee (p \wedge \neg q)) & \text{Distributive} \\
\equiv & r \wedge ((\neg p \vee q) \vee \neg(\neg(p \wedge \neg q))) & \text{Double negation} \\
\equiv & r \wedge ((\neg p \vee q) \vee \neg(\neg p \vee q))) & \text{De Morgan's} \\
\equiv & r \wedge T & \text{Negation} \\
\equiv & r & \text{Identity} \\
\to & r \vee s & \text{Addition}
\end{array}$$

Q. 7. Use logical equivalences to prove the following statements.

(a) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

(b) $\neg(p \to q) \to \neg q$ is a tautology.

(c) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$\begin{aligned}
&\neg(p \oplus q) \\
&\quad\equiv \neg((p \wedge \neg q) \vee (\neg p \wedge q)) \quad \text{Definition} \\
&\quad\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q) \quad \text{De Morgan} \\
&\quad\equiv (\neg p \vee q) \wedge (p \vee \neg q) \quad \text{De Morgan} \\
&\quad\equiv (p \to q) \wedge (q \to p) \quad \text{Useful} \\
&\quad\equiv p \leftrightarrow q \quad \text{Definition}
\end{aligned}$$

Thus, they are equivalent.

(b) We have

$$\begin{aligned}
&\neg(p \to q) \to \neg q \\
&\quad\equiv \neg\neg(p \to q) \vee \neg q \quad \text{Useful} \\
&\quad\equiv (p \to q) \vee \neg q \quad \text{Double negation} \\
&\quad\equiv (\neg p \vee q) \vee \neg q \quad \text{Useful} \\
&\quad\equiv \neg p \vee (q \vee \neg q) \quad \text{Associative} \\
&\quad\equiv T \quad \text{Domination}
\end{aligned}$$

Therefore, it is a tautology.

(c) We have

$$\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\quad\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\quad\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\quad\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\quad\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\quad\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\quad\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\quad\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\quad\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\quad\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\quad\equiv T \vee \neg r \quad \text{Complement} \\
&\quad\equiv T \quad \text{Identity.}
\end{aligned}$$

Thus, it is a tautology.

$\square$

Q. 8. Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret." Express each of these sentences in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution:

(a) $\exists x(C(x) \land D(x) \land F(x))$

(b) $\forall x(C(x) \lor D(x) \lor F(x))$

(c) $\exists x(C(x) \land F(x) \land \neg D(x))$

(d) $\neg \exists x(C(x) \land D(x) \land F(x))$

(e) $(\exists x C(x)) \land (\exists x D(x)) \land (\exists x F(x))$

$\square$

Q. 9. Prove that if $p \land q$, $p \to \neg(q \land r)$, $s \to r$, then $\neg s$.

Solution:

(1)	$p \wedge q$	Premise
(2)	p	Simplication of (1)
(3)	$p \to \neg(q \wedge r)$	Premise
(4)	$\neg(q \wedge r)$	Modens ponens (2) (3)
(5)	$\neg q \vee \neg r$	De Morgan
(6)	q	Simplication of (1)
(7)	$\neg r$	Disjunctive syllogism (6) (7)
(8)	$s \to r$	Premise
(9)	$\neg s$	Modus tollens (7) (8)

□

Q. 10. For each of these arguments, explain which rules of inference are used for each step.

(a) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

(b) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

(c) "Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Solution:

(a) Let $c(x)$ denote "x is in this class", $w(x)$ denote "x enjoys whale watching", and $p(x)$ denote "x cares about ocean pollution." The premises are $\exists x(c(x) \wedge w(x))$ and $\forall x(w(x) \to p(x))$. From the first premise, $c(y) \wedge w(y)$ for a particular person y. Using simplification, $w(y)$ follows. Using the second premise and universal instantiation, $w(y) \to p(y)$ follows. Using modus ponens, $p(y)$ follows, and by conjunction, $c(y) \wedge p(y)$

follows. Finally, by existential generalization, the desired conclusion, $\exists x(c(x) \wedge p(x))$ follows.

(b) Let $c(x)$ be "x is in this class," $p(x)$ be "x owns a PC", and $w(x)$ be "x can use a word-processing program". The premises are $c(Zeke)$, $\forall x(c(x) \to p(x))$, and $\forall x(p(x) \to w(x))$. Using the second premise and universal instantiation and modus ponens, $c(Zeke) \to p(Zeke)$ follows. Using the first premise and modus ponens, $p(Zeke)$ follows. Using the third premise and universal instantiation, $p(Zeke) \to w(Zeke)$ follows. Finally, using modus ponens, $w(Zeke)$, the desired conclusion follows.

(c) Let $r(x)$ be "x is one of the five roommates listed", let $d(x)$ be "x has taken a course in discrete mathematics", and let $a(x)$ be "x can take a course in algorithms". We are given premises $\forall x(r(x) \to d(x)$, $\forall x(d(x) \to a(x)$, and we want to conclude $\forall x(r(x) \wedge a(x))$.

Step	**Reason**
1. $\forall x(r(x) \to d(x))$	Hypothesis
2. $r(y) \to d(y)$	Universal Instantiation using 1.
3. $\forall x(d(x) \to a(x))$	Hypothesis
4. $d(y) \to a(y)$	Universal instantiation using 3.
5. $r(y) \to a(y)$	Hypothetical syllogism using 2. and 4.
6. $\forall x(r(x) \to a(x))$	Universal generalization using 5.

$\square$

Q. 11. (a) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Solution:

(a) The negation is

$$\exists x \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \wedge n \text{ is odd}).$$

(b) If n is odd then $n^3 + 6n + 5$ is even because n^3 is then odd and $6n$ is then even. Therefore, the original statement is true.

$\square$

Q. 12. Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.

Solution: Observe that $a^2+b^2 = (a+b)^2 - 2ab$. Thus, $(a+b)^2$ has the same parity as $a^2 + b^2$. So $(a + b)^2$ is even. Then $a + b$ is also even.

$\square$

Q. 13. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

Solution: Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and this number is not rational.

$\square$

Q. 14. Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q. 15. Suppose that we have a theorem: "$\sqrt{n}$ is irrational whenever n is a positive integer that is <u>not</u> a perfect square." Use this theorem to prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: We give a proof by contradiction. If $\sqrt{2}+\sqrt{3}$ is a rational number, then its square is also rational, which is $5+2\sqrt{6}$. Subtracting 5 and dividing by 2, we have $\sqrt{6}$ is also rational. However, this contradicts the theorem.

$\square$

Q. 16. (Second-Price Auction) Please read the following description carefully and answer questions. In an auction, an auctioneer is responsible for selling a product, and bidders bid for the product. The winner of the auction wins the product and pays for it. We consider a single object sealed second-price auction. The detailed settings are as follows:

- There is one product to be sold.
- There are N bidders, denoted by $\mathcal{N} = \{1, 2, ..., N\}$. Bidder $n \in \mathcal{N}$ has a valuation over the product of v_n.
- Every bidder submits his or her bid in a sealed envelope, so other bidders do not know his or her bid. Bidder $n \in \mathcal{N}$ submits a bid of b_n.
- After receiving the bids from all bidders, the auctioneer announces the winner and payment. The winner is the bidder who submits the highest bid. The payment of this winner is the second highest bid. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 4$, $b_3 = 5$. Then, the winner is bidder 3, and the payment is the second highest bid 4.
- If multiple bidders have the same bid, then they draw a lottery. Each of them has equally probability of winning. In this case, the payment is equal to their bids. For example, consider three bidders. Suppose $b_1 = 2$, $b_2 = 5$, $b_3 = 5$. The winner is either 2 or 3 with equal probability. The payment is 5.
- After the auction, the payoffs of the bidders are as follows:
  - If bidder n loses, his or her payoff is zero.
  - If bidder n wins, his or her payoff is equal to its valuation v_n minus the payment.

  For bidder n, the higher payoff, the better.

Now, suppose you are a bidder in this auction, e.g., bidder n, and you do not know any other bidders' valuations and bids. You know your valuation v_n. You can choose your bid b_n to maximize your payoff. Prove that for an arbitrary bidder $n \in \mathcal{N}$, submitting a bid $b_n = v_n$ will always lead to a payoff that is no smaller than submitting a bid with $b_n \neq v_n$.

(Note: This second-price auction is commonly used, due to the property that bidders are willing to submit their valuation as their bid.)

(Hint: Use proof by cases; consider the highest bid of the others, and compare it with your valuation v_n; enumerate all possibilities.)

Solution: Let b^* be the highest bid of all bidders except bidder n, i.e., $b^* = \max_{\mathcal{N}\setminus\{n\}} b_n$. Note that if bidder n wins, then b^* is his or her payment.

Consider an arbitrary bidder $n \in \mathcal{N}$. There are three cases:

- $v_n < b^*$:
  - Submitting $b_n = v_n$: loses; payoff is zero
  - Submitting $b_n < v_n$: loses; payoff is zero
  - Submitting $b^* > b_n > v_n$: loses; payoff is zero
  - Submitting $b_n = b^*$: may win; if wins, payoff is negative, as $v_n - b^* < 0$; if loses, payoff is zero
  - Submitting $b_n > b^*$: wins; payoff is negative, as $v_n - b^* < 0$
- $v_n = b^*$
  - Submitting $b_n = v_n$: may win; if wins, payoff is $v_n - b^* = 0$; if loses, payoff is zero
  - Submitting $b_n > v_n$: wins; payoff is $v_n - b^* = 0$
  - Submitting $b_n < v_n$: loses; payoff is zero
- $v_n > b^*$
  - Submitting $b_n = v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b_n > v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b^* < b_n < v_n$: wins; payoff is $v_n - b^* > 0$
  - Submitting $b_n = b^*$: may win; if wins, payoff is $v_n - b^* > 0$; if loses, payoff is zero
  - Submitting $b_n < b^*$: loses; payoff is zero

For all these three cases, submitting bid $b_n = v_n$ will always lead to a payoff that is no smaller than submitting bid $b_n \neq v_n$.