CS201: Discrete Math for Computer Science
2024 Spring Semester Written Assignment #4
Due: May 3rd, 2024

The assignment needs to be written in English. Assignments in any other language will get zero point. Any plagiarism behavior will lead to zero point.

Q. 1. Suppose that a and b are real numbers with $0 < b < a$. Use mathematical induction to prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.

Solution: It turns out to be easier to think about the given statement as $na^{n-1}(a-b) \geq a^n - b^n$. The basic step ($n = 1$) is true since $a - b \geq a - b$. Assume that the inductive hypothesis, that $ka^{k-1}(a-b) \geq a^k - b^k$; we must show that $(k+1)a^k(a-b) \geq a^{k+1} - b^{k+1}$. We have

$$\begin{aligned}(k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b)\\ &\geq a(a^k - b^k) + a^k(a-b)\\ &= a^{k+1} - ab^k + a^{k+1} - ba^k.\end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \geq a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \geq 0$, which factors into $(a^k - b^k)(a-b) \geq 0$, and this is true, because we are given that $a > b$.

$\square$

Q. 2. A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Solution:

By checking the first few values $10, 20, 25, 30, 35, 40, 45, 50, \ldots$, we guess that we can make \$$n$ in amount of money, where

$$n \in \{10\} \cup \{5m : \ m \geq 4 \text{ and } m \in \mathbb{Z}^+\}.$$

Let $P(n)$ be the statement "we can make \$$5m$ in gift certificate in amount of \$10 and \$25."

- Basic step: $m = 4, 5$, we can make \$20 and \$25 in gift certificate.
- Inductive step: Suppose \$$5k$ for $4 \leq k < m$. We now prove $P(m)$ for $m \geq 6$. Note that $5m = 10 + 5(m-2)$. Since $4 \leq m-2 < m$, $P(m-2)$ is true. So we can make \$$5(m-2)$ in gift certificate. It then follows that we can \$$5m$ in gift certificate by adding an extra \$10 certificate.

□

Q. 3. Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Solution: $f(n) = 25n^{\log_4 5} - 24n$.

□

Q. 4. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Solution:

(a) 2 if $n = 1$, 2 if $n = 2$, and 0 if $n \geq 3$.

(b) 2^{n-2} for $n > 1$; 1 if $n = 1$.

(c) $2(n-1)$.

□

Q. 5. How many 6-card poker hands consist of exactly 2 pairs? That is two of one rank of card, two of another rank of card, one of a third rank, and one of a fourth rank of card? Recall that a deck of cards consists of 4 suits each with one card of each of the 13 ranks.

You should leave your answer as an equation.

Solution: First, we choose the ranks of the 2 pairs, noting that the order we pick these two ranks does not matter, so there are $\binom{13}{2}$ options here. Next we pick the 2 suits for the first pair, $\binom{4}{2}$ and the suits for the second pair $\binom{4}{2}$. Then we decide which 2 ranks of the remaining 11 to use for the other cards, $\binom{11}{2}$, and finally choose each of their suits $\binom{4}{1}\binom{4}{1}$. Altogether, by the product rule, this gives $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{2}\binom{4}{1}\binom{4}{1}$ hands.

$\square$

Q. 6. Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Solution:

Since $\gcd(2, 121) = 1$, it suffices to prove that $2|\binom{240}{120}$ and $121|\binom{240}{120}$. We prove these two divisibilities in general, i.e.,

$$2\Big|\binom{2n}{n}, \text{ and } (n+1)\Big|\binom{2n}{n}.$$

Since

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n!n!} \\
&= \frac{2n \cdot (2n-1)!}{n!n!} \\
&= \frac{2 \cdot (2n-1)!}{(n-1)!n!} \\
&= 2 \cdot \binom{2n-1}{n},
\end{aligned}$$

we have 2 divides $\binom{2n}{n}$. Since

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\
&= \frac{(2n)!}{(n+1)!n!} \\
&= \frac{1}{n+1}\binom{2n}{n},
\end{aligned}$$

which is an integer, we have $n+1$ divides $\binom{2n}{n}$. This completes the proof.

$\square$

Q. 7. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \bmod 5 = a_2 \bmod 5$ and $b_1 \bmod 5 = b_2 \bmod 5$.

Solution: Working modulo 5 there are 25 pairs: $(0,0), (0,1), \ldots, (4,4)$. Thus, we could have 25 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle, however, guarantees that if we have 26 such pairs, then at least two of them will have the same coordinates, modulo 5.

$\square$

Q. 8. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution:

Let $K(x)$ be the number of other people at the party that person x knows. The possible values for $K(x)$ are $0, 1, \ldots, n-1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ to be in the range of K, since if one person knows everybody else, then nobody can know no one else (we assume that "knowing" is symmetric). Therefore, the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

$\square$

Q. 9. Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of <u>non-empty</u> subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Solution: We may split S_n into 3 cases :

> Case (1): item 1 is not in the subset. We must now choose a non-empty subset of $\{2, \ldots, n\}$. There are a_{n-1} ways to do this.

Case (2): item 1 is in the subset, and there are more elements. We must now choose a non-empty subset of $\{3, \ldots, n\}$. There are a_{n-2} ways to do this.

Case (3): item 1 is in the subset, and no other elements are. There is 1 way to do this.

Thus, we have the recurrence relation as: $a_n = a_{n-1} + a_{n-2} + 1$.

$\square$

Q. 10. Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [Hint: Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

Solution:

First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned}
\sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=1}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\
&= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r.
\end{aligned}$$

Thus,

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n, r)$ must equal $C(n-1, r) + C(n-1, r-1)$ for $1 \le r \le n-1$, as desired.

$\square$

Q. 11. Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

Solution: The CE is

$$r^3 - 2r^2 - r + 2 = (r+1)(r-1)(r-2).$$

The roots are $r = -1$, $r = 1$ and $r = 2$. Hence, the solutions to this recurrence are of the form

$$a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n.$$

To find the constants α_1, α_2 and α_3, we use the initial conditions. Plugging in $n = 0, n = 1$, and $n = 2$, we have

$$a_0 = 1 = \alpha_1 + \alpha_2 + \alpha_3 a_1 = 0 = -\alpha_1 + \alpha_2 + 2\alpha_3 a_2 = 7 = \alpha_1 + \alpha_2 + 4\alpha_3.$$

We then have $\alpha_1 = 3/2$, $\alpha_2 = -5/2$, and $\alpha_3 = 2$. Hence,

$$a_n = 3/2 \cdot (-1)^n - 5/2 \cdot 1^n + 2 \cdot 2^n = 2^{n+1} + (-1)^n \cdot 3/2 - 5/2.$$

$\square$