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Assignment 2

Q1. (1) Disprove. Suppose $A=\{0\}$, $B=\{0,1\}$, $A\times B=\{(0,0),(0,1)\}$,

$B\times A=\{(0,0),(1,0)\}$

$P(A\times B)=\{\emptyset,\{(0,0)\},\{(0,1)\},\{(0,0),(0,1)\}\}$

$P(B\times A)=\{\emptyset,\{(0,0)\},\{(1,0)\},\{(0,0),(1,0)\}\}$

$P(A\times B)\neq P(B\times A)$

(2) $A\oplus B=(A-B)\cup(B-A)$ (definition of $\oplus$)

$=(A\cap\overline{B})\cup(B\cap\overline{A})$ (definition of A-B)

$(A\oplus B)\oplus B=((A\cap\overline{B})\cup(B\cap\overline{A}))\oplus B$ (definition)

$=(((A\cap\overline{B})\cup(B\cap\overline{A}))\cap\overline{B})\cup(B\cap\overline{(A\cap\overline{B})\cup(B\cap\overline{A})})$ (definition)

$=(A\cap\overline{B}\cap\overline{B})\cup(B\cap\overline{A}\cap\overline{B})\cup(B\cap\overline{A\cap\overline{B}}\cap\overline{B\cap\overline{A}})$ (distributive, De Morgan)

$=\emptyset\cup(\overline{B}\cap\overline{A})\cup(B\cap(\overline{A}\cup B)\cap(\overline{B}\cup A))$ (complement, idempotent)

$=(\overline{B}\cap\overline{A})\cup(B\cap(\overline{B}\cup A))$ (identity, absorbtion)

$=(\overline{B}\cap\overline{A})\cup((B\cap\overline{B})\cup(B\cap A))$ (distributive)

$=(\overline{B}\cap\overline{A})\cup\emptyset\cup(B\cap A)$ (complement)

$=(\overline{B}\cap\overline{A})\cup(B\cap A)$ (identity)

$=\cancel{B\cap(\overline{A}\cup A)}\ A\cap(\overline{B}\cup B)$ (distributive)

$=A\cap U$ (complement)

$=A$ (domination)

Hence $(A\oplus B)\oplus B=A$ is true.

(3) Disprove. $A=\{1,2,3,4\}$, $B=\{a,b,c\}$, $S=\{1,2\}$, $T=\{3,4\}$, $S\cap T=\emptyset$

$f: 1\to a,\ 2\to b,\ 3\to b,\ 4\to c$; so $f(S)=\{a,b\}$, $f(T)=\{b,c\}$

$f(S)\cap f(T)=\{b\}$, $f(S\cap T)=f(\emptyset)=\emptyset$

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(4) $\forall y \in S\cap T,\ y\in S,\ y\in T \Rightarrow f^{-1}(y)\in f^{-1}(S),\ f^{-1}(y)\in f^{-1}(T)$

$\Rightarrow f^{-1}(y)\in f^{-1}(S)\cap f^{-1}(T)$

From arbitrariness of y, $f^{-1}(S\cap T)\subseteq f^{-1}(S)\cap f^{-1}(T)$ ①

$\forall x\in f^{-1}(S)\cap f^{-1}(T),\ x\in f^{-1}(S),\ x\in f^{-1}(T) \Rightarrow f(x)\in S,\ f(x)\in T$

$\Rightarrow f(x)\in S\cap T \Rightarrow x\in f^{-1}(S\cap T)$

From arbitrariness of x, $f^{-1}(S)\cap f^{-1}(T)\subseteq f^{-1}(S\cap T)$ ②

From ①②, $f^{-1}(S\cap T) = f^{-1}(S)\cap f^{-1}(T)$

Q2. (1) $(B-A)\cup(C-A) = (B\cap\bar{A})\cup(C\cap\bar{A})$ (definition)

$\cancel{= (B\cup C)\cap(B\cup\bar{A})\cap(C\cup\bar{A})\cap(\bar{A}\cup\bar{A})}$

$= (B\cup C)\cap\bar{A}$ (distributive)

$= (B\cup C)-A$ (definition)

(2) $(A\cap B)\cap(\overline{B\cap C})\cap(A\cap C)$

$= (A\cap B)\cap(\bar{B}\cup\bar{C})\cap(A\cap C)$ (De Morgan)

$= A\cap B\cap A\cap C\cap(\bar{B}\cup\bar{C})$ (commutative, associative)

$= A\cap B\cap C\cap(\bar{B}\cup\bar{C})$ (idemponent)

$= (A\cap B\cap C\cap\bar{B})\cup(A\cap B\cap C\cap\bar{C})$ (distributive)

$= (A\cap C\cap\emptyset)\cup(A\cap B\cap\emptyset)$ (commutative, complement)

$= \emptyset\cup\emptyset$ (domination)

$= \emptyset$ (identity)

Q3. $A\subseteq B \Rightarrow \mathcal{P}(A)\subseteq\mathcal{P}(B)$. $\forall C\in\mathcal{P}(A)$, $C\subseteq A$, since $A\subseteq B$, $C\subseteq B$, hence $C\in\mathcal{P}(B)$. From arbitrariness of C, $\mathcal{P}(A)\subseteq\mathcal{P}(B)$.

$\mathcal{P}(A)\subseteq\mathcal{P}(B) \Rightarrow A\subseteq B$. $\forall x\in A$, $\{x\}\in\mathcal{P}(A)$, since $\mathcal{P}(A)\subseteq\mathcal{P}(B)$, $\{x\}\in\mathcal{P}(B)$, hence $x\in B$. From arbitrariness of x, $A\subseteq B$.

Above all, $A\subseteq B \Leftrightarrow \mathcal{P}(A)\subseteq\mathcal{P}(B)$.

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Q4. (a) $f_1(x)$ are $\Theta(g(x)) \Rightarrow \exists\, c_1, c_2$, $c_1|g(x)| \le |f_1(x)| \le c_2|g(x)|$ when $x > N_1$

$f_2(x)$ are $\Theta(g(x)) \Rightarrow \exists\, c_3, c_4$, $c_3|g(x)| \le |f_2(x)| \le c_4|g(x)|$ when $x > N_2$

Take $N = \max(N_1, N_2)$, when $x > N$,

$c_1|g(x)| \le |f_1(x)| \le c_2|g(x)|$, $c_3|g(x)| \le |f_2(x)| \le c_4|g(x)|$

Since $f_1: \mathbb{R} \to \mathbb{R}^+$, $f_2: \mathbb{R} \to \mathbb{R}^+$

$(c_1+c_2)|g(x)| \le |f_1(x)| + |f_2(x)| = f_1(x) + f_2(x) = |(f_1+f_2)(x)| \le (c_3+c_4)|g(x)|$

Hence $\exists\, C_1 = c_1+c_2$, $C_2 = c_3+c_4$, s.t. when $x > N$, $C_1|g(x)| \le |(f_1+f_2)(x)| \le C_2|g(x)|$

f_1+f_2 is $\Theta(g(x))$.

(b) ~~Disapr~~ Disprove. $f_1(x) = x$, $f_2(x) = -x$, $g(x) = x$.

f_1, f_2 are both $\Theta(g(x))$. $f_1+f_2 = 0$ is not $\Theta(g(x))$.

Q5. (a) Disprove. $f_1(x) = x$, $f_2(x) = x$, $g(x) = x$.

f_1, f_2 are both $\Theta(g)$, $(f_1 - f_2)(x) = 0$ is not $\Theta(g(x))$

(b) $f_1(x)$ is $\Theta(g(x)) \Rightarrow \exists\, c_1, c_2 > 0$, s.t. when $x > N_1$, $c_1|g(x)| \le |f_1(x)| \le c_2|g(x)|$.

$f_2(x)$ is $\Theta(g(x)) \Rightarrow \exists\, c_3, c_4 > 0$, s.t. when $x > N_2$, $c_3|g(x)| \le |f_2(x)| \le c_4|g(x)|$.

Then $c_1c_3|g^2(x)| \le |(f_1f_2)(x)| \le c_2c_4|g^2(x)|$.

Hence, $\exists\, C_1 = c_1c_3$, $C_2 = c_2c_4 > 0$, s.t. when $x > N = \max(N_1, N_2)$,

$C_1|g^2(x)| \le |(f_1f_2)(x)| \le C_2|g^2(x)|$.

$(f_1f_2)(x)$ is $\Theta(g^2(x))$.

Q6. Disprove. If there exist infinite set A, $|A| < |\mathbb{Z}^+|$. We can ~~count~~ list elements of A one by one, this cause a mapping from $\mathbb{Z}^+$ to A and the map is bijective. Hence $|A| = |\mathbb{Z}^+|$, which means it ~~don't~~ doesn't exist such a set.

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Q7. Suppose $x = \lfloor x \rfloor + y$, $0 \le y < 1$. Then discuss by cases.

① $0 \le y < \frac{1}{3}$

$LHS = \lfloor 3x \rfloor = \lfloor 3\lfloor x \rfloor + 3y \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor$

$RHS = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = 3\lfloor x \rfloor = LHS$

② $\frac{1}{3} \le y < \frac{2}{3}$

$LHS = \lfloor 3x \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor + 1$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1$

$RHS = \lfloor x \rfloor + \lfloor x \rfloor + (\lfloor x \rfloor + 1) = 3\lfloor x \rfloor + 1 = LHS$

③ $\frac{2}{3} \le y < 1$

$LHS = \lfloor 3x \rfloor = 3\lfloor x \rfloor + \lfloor 3y \rfloor = 3\lfloor x \rfloor + 2$

$\lfloor x + \frac{1}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{1}{3} \rfloor = \lfloor x \rfloor + 1$

$\lfloor x + \frac{2}{3} \rfloor = \lfloor x \rfloor + \lfloor y + \frac{2}{3} \rfloor = \lfloor x \rfloor + 1$

$RHS = \lfloor x \rfloor + (\lfloor x \rfloor + 1) + (\lfloor x \rfloor + 1) = 3\lfloor x \rfloor + 2 = LHS$

Hence, the equation holds.

Q8. $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$

$$\sum_{k=1}^{n} [k^4 - (k-1)^4] = 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$n^4 = 4\sum_{k=1}^{n} k^3 - 6\frac{2n^3+3n^2+n}{6} + 4 \cdot \frac{n^2+n}{2} - n$$

$$\therefore 4\sum_{k=1}^{n} k^3 = n^4 + 6 \cdot \frac{2n^3+3n^2+n}{6} - 4\frac{n^2+n}{2} + n$$

$$= n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n$$

$$= n^4 + 2n^3 + n^2$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2$$

Q9. (a) Countable. Number of students in CS201 is finite, its power set is also finite.

(b) Countable. We can list them as follows. Hence is countable.

$$\begin{array}{ccc} (0,0) \to & (0,1) \to & (0,2) \quad \cdots \\ (1,0) & (1,1) & (1,2) \\ (2,0) & (2,1) & \\ \vdots & & \end{array}$$

(c) Uncountable. $A = \{(1,b) \mid b \in \mathbb{R}\}$, $A \subseteq \{(a,b) \mid a \in \mathbb{N}, b \in \mathbb{R}\}$,

so $|A| \le |\{(a,b) \mid a \in \mathbb{N}, b \in \mathbb{R}\}|$

$f: A \to \mathbb{R}$, $f(1,b) = b$ is bijective, so $|A| = |\mathbb{R}|$

Then $|\mathbb{R}| \le |\{(a,b) \mid a \in \mathbb{N}, b \in \mathbb{R}\}|$, it is uncountable.

Q10. If we list (m,n) with increasing of the sum of m,n,

we get $(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), \cdots$

And $f(m,n)$ would be $1, 2, 3, 4, 5, 6, \cdots$

Hence $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ is one-to-one and onto.

Q11. $|A| = |B|$ means there exist function $f: A \to B$ is bijective.

$|B| = |C| \Rightarrow \exists\, g: B \to C$ is bijective.

$g \circ f: A \to C$ is bijective, so $|A| = |C|$.

Q12. $f(x)$ is $\Theta(g(x)) \Rightarrow \exists\, c_1, c_2 > 0$, s.t. when $x > N_1$, $c_1|g(x)| \le |f(x)| \le c_2|g(x)|$

$g(x)$ is $\Theta(h(x)) \Rightarrow \exists\, c_3, c_4 > 0$, s.t. when $x > N_2$, $c_3|h(x)| \le |g(x)| \le c_4|h(x)|$

When $x > N = \max(N_1, N_2)$, $c_1c_3|h(x)| \le c_1|g(x)| \le |f(x)| \le c_2|g(x)| \le c_2c_4|h(x)|$

Hence, $\exists\, C_1 = c_1c_3$, $C_2 = c_2c_4$, s.t. when $x > N$, $C_1|h(x)| \le |f(x)| \le C_2|h(x)|$,

which means $f(x)$ is $\Theta(h(x))$

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Q13. In each loop, multiplication and addtion are both once.
Since this loop is n times, the multiplications and additions are both n times.