Discrete Mathematics for Computer Science

Lecture 16: Relation

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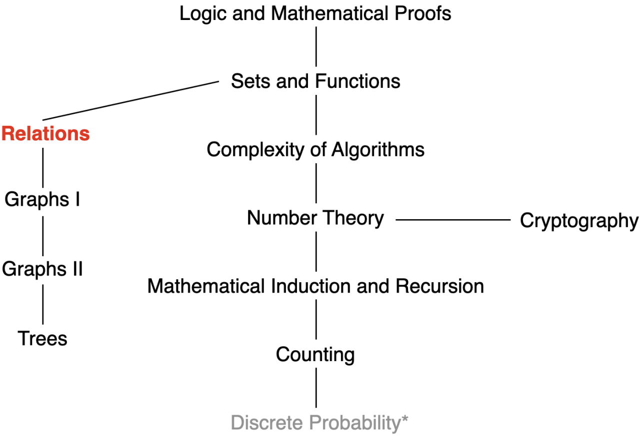
This Lecture



Relation, *n*-ary Relations, Representing Relations, Closures of Relations, ...

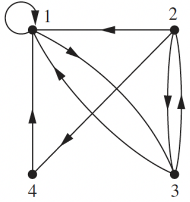
Directed Graph

A directed graph, or digraph, consists of a set V of vertices together with a set E of ordered pairs of elements of V called edges.

The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Example: Relation R is defined on $\{1, 2, 3, 4\}$:

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$



Closures of Relations

Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

Is this relation R reflexive?

No. $(2,2)$ and $(3,3)$ are not in R.

The question is what is the minimal relation $S \supseteq R$ that is reflexive?

How to make R reflexive by minimum number of additions?

Add $(2,2)$ and $(3,3)$

Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$.

The minimal set $S \supseteq R$ is called the reflexive closure of R.

Closures of Relations

The set S is called the reflexive closure of R if it:

- contains R
- is reflexive
- is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

Relations can have different properties:

- reflexive
- symmetric
- transitive

We define:

- reflexive closures
- symmetric closures
- transitive closures

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

S is the minimal set containing R satisfying the property P.

Example: $R = \{(1,2),(2,3),(2,2)\}$ on $A = \{1,2,3\}$. What is the symmetric closure S of R?

$S = \{(1,2),(2,3),(2,2),(2,1),(3,2)\}$.

What is the transitive closure S of R?

$S = \{(1,2),(2,2),(2,3),(1,3)\}$.

Overview

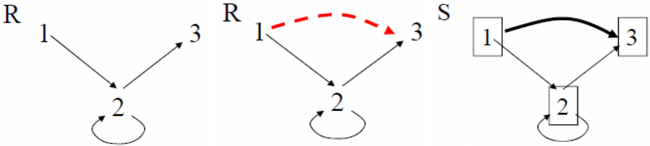
- Transitive Closure
- Path Length and Connectivity Relation
- Transitive Closure and Connectivity Relation

Transitive Closure

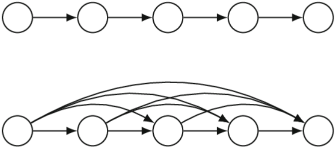
We can represent the relation on the graph.

Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Example: $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$. Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



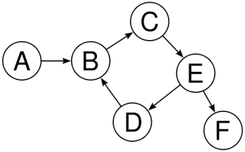
Example:



Overview

- Transitive Closure
- Path Length and Connectivity Relation
- Transitive Closure and Connectivity Relation

Paths in Directed Graphs



Definition: A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$.

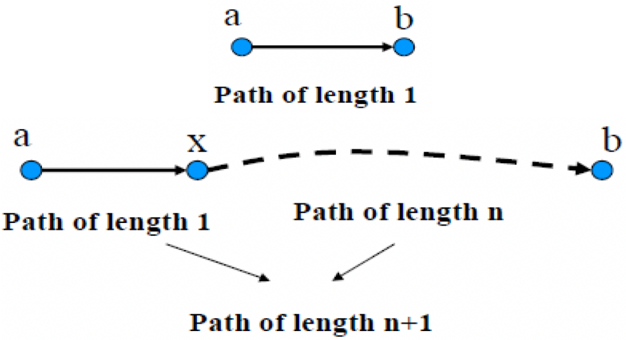
A path of length $n \geq 1$ that begins and ends at the same vertex is called a circuit or cycle.

Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Path Length

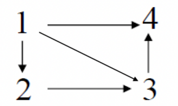
Theorem: Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Proof (by induction):



Recall that $R^{n+1} = R^n \circ R$

Path Length: Example



$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$

$R^2 = \{(1, 3), (2, 4), (1, 4)\}$

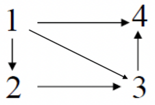
$R^3 = \{(1, 4)\}$

$R^4 = \emptyset$

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

$$R^* = \bigcup_{k=1}^{\infty} R^k$$



$A = \{1, 2, 3, 4\}$

$R = \{(1,2),(1,3),(1,4),(2,3),(3,4)\}$, $R^2 = \{(1,3),(2,4),(1,4)\}$

$R^3 = \{(1,4)\}$, $R^4 = \emptyset$

$R^* = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

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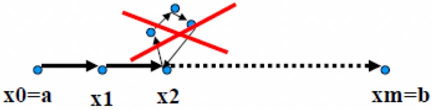
Connectivity

Lemma: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Proof (by intuition): There are at most n different elements we can visit on a path if the path does not have loops:

Loops may increase the length but the same node is visited more than once

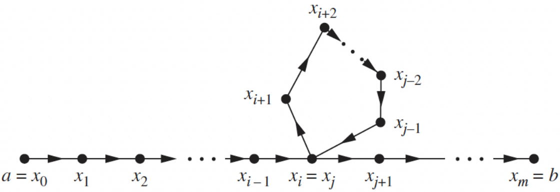


Connectivity

Lemma: Let A be a set with n elements, and R be a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

Proof: Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x1, x2, ..., x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a \neq b$ and that $m \geq n$. The $m + 1$ vertices are from n elements. According to the pigeonhole principle and $a \neq b$, at least two of the vertices $x_0, x_1, ..., x_{m-1}$ are equal.



There is a circuit that can be deleted until the length is $< n$.

Connectivity

Lemma: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Lemma: If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n.

Overview

- Transitive Closure
- Path Length and Connectivity Relation
- Transitive Closure and Connectivity Relation

Connectivity

Theorem: The transitive closure of a relation R equals the connectivity relation R^*, where $R^* = \bigcup_{k=1}^{\infty} R^k$.

Recall: Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path.

Recall: The connectivity relation R^* consists of all pairs (a, b) such that there is a path (of any length) between a and b in R:

Connectivity

Theorem: The transitive closure of a relation R equals the connectivity relation R^*, where $R^* = \bigcup_{k=1}^{\infty} R^k$.

R^* is a transitive closure of R:

- $R \subseteq R^*$
- R^* is transitive

  If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.
- $R^* \subseteq S$ whenever S is a transitive relation containing R
  - Suppose that S is a transitive relation containing R.
  - Transitive: $S^n \subseteq S$ for integer $n \geq 1$. (Recall S is transitive iff $S^n \subseteq S$). We have $S^* \subseteq S$.
  - $R \subseteq S$: then $R^* \subseteq S^*$, because any path in R is also a path in S.
  - Thus, $R^* \subseteq S^* \subseteq S$.

Find Transitive Closure

Recall that if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Thus,

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n.$$

Theorem: Let M_R be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots \vee M_R^{[n]},$$

where $M_R^{[n]} = \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{n \ M_R' s}$

Find Transitive Closure: Example

Find the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find Transitive Closure: Algorithm

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
    $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
    $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return B{**B** is the zero–one matrix for R^*}

- $n - 1$ Boolean products
- Each of these Boolean products use $n^2(2n-1)$ bit operations.
- $O(n^4)$ bit operations.

Roy-Warshall Algorithm

The transitive closure can be found by Warshall's algorithm using only $O(n^3)$ bit operations.

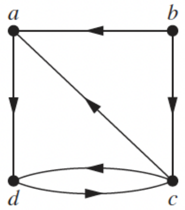
If $a, x_1, x_2, ..., x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, ..., x_{m-1}$.

Consider a list of vertices $v_1, v_2, ..., v_k, ..., v_n$. Define a zero-one matrix

$$\mathbf{W}_k = [w_{ij}^{(k)}],$$

where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, ..., v_k\}$ and is 0 otherwise.

Example of $\mathbf{W}_k$



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, and $v_4 = d$.

W_0 is the matrix of the relation.

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

W_1 has 1 as its (i, j)th entry if

- W_0 has 1 as its (i, j)th entry (i.e., no interior) or
- there is a path from v_i to v_j that has $v_1 = a$ as an interior vertex

Roy-Warshall Algorithm

Consider a list of vertices $v_1, v_2, ..., v_k, ..., v_n$. Define a zero-one matrix

$$\mathbf{W}_k = [w_{ij}^{(k)}],$$

where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, ..., v_k\}$ and is 0 otherwise.

Note that $\mathbf{W}_n = M_{R^*}$, because the (i, j)th entry of M_{R^*} is 1 if and only if there is a path from v_i to v_j with all interior vertices in the set $\{v_1, v_2, ..., v_n\}$.

Roy-Warshall Algorithm

Warshall's algorithm computes M_{R^*} by efficiently computing $\mathbf{W}_0 = M_R, W_1, W_2, ..., \mathbf{W}_n = M_{R^*}$.

Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

The transitive closure can be found by Warshall's algorithm using only $O(n^3)$ bit operations.

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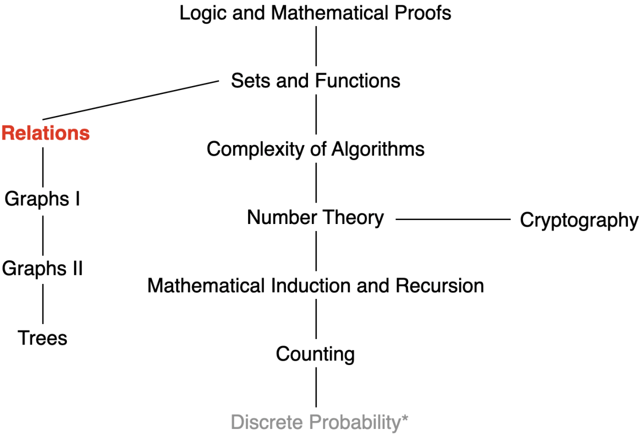


Relation, *n*-ary Relations, Representing Relations, Closures of Relations, Relation Equivalence, ...

Equivalence Relation

Definition: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

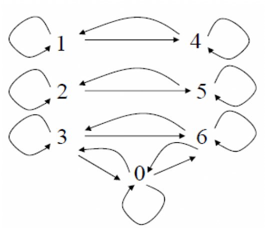
Example:
$A = \{0, 1, 2, 3, 4, 5, 6\}$ $R = \{(a, b) : a \equiv b \mod 3\}$
R has the following pairs:

- (0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)
- (1, 1), (1, 4), (4, 1), (4, 4)
- (2, 2), (2, 5), (5, 2), (5, 5)

Equivalence Relation

Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

- (0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)
- (1, 1), (1, 4), (4, 1), (4, 4)
- (2, 2), (2, 5), (5, 2), (5, 5)



Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes

R is an equivalence relation.

Examples of Equivalence Relations

- "Strings a and b have the same length."
- "Integers a and b have the same absolute value."
- "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

Equivalence Class

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

Example: $A = \{0, 1, 2, 3, 4, 5, 6\}$
$R = \{(a, b) : a \equiv b \mod 3\}$

$[0] = [3] = [6] = \{0, 3, 6\}$
$[1] = [4] = \{1, 4\}$
$[2] = [5] = \{2, 5\}$

Examples of Equivalence Classes

"Strings a and b have the same length."
$[a] =$ the set of all strings of the same length as a.

"Integers a and b have the same absolute value."
$[a] =$ the set $\{a, -a\}$

"Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."
$[a] =$ the set $\{..., a-2, a-1, a, a+1, a+2, ...\}$

Equivalence Class

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Proof:

- (i) $\rightarrow$ (ii): prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$
  Suppose $c \in [a]$. Then, aRc.
  Because aRb and R is symmetric, we know that bRa.
  Since R is transitive and bRa and aRc, it follows that bRc.
  Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$.

Equivalence Class

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

(i)aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

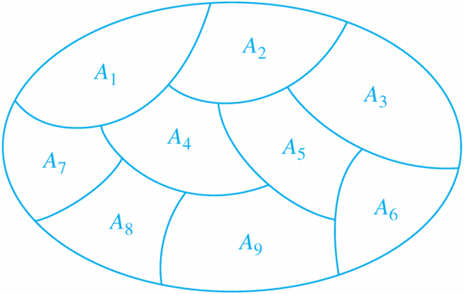
Proof:

- (i) $\to$ (ii): prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$
- (ii) $\to$ (iii): Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ as R is reflexive).
- (iii) $\to$ (i): Suppose that $[a] \cap [b] \neq \emptyset$. There exists a c such that $c \in [a]$ and $c \in [b]$, i.e., aRc and bRc.
  By the symmetric property, cRb.
  Then by transitivity, because aRc and cRb, we have aRb.

Partition of a Set S

Definition: Let S be a set. A collection of nonempty subsets of S, i.e A_1, A_2, . . . , A_k, is called a partition of S if:

$$A_i \cap A_j = \emptyset, i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$



Example: $A = \{0, 1, 2, 3, 4, 5, 6\}$
$A_1 = \{0, 3, 6\}$, $A_2 = \{1, 4\}$, $A_3 = \{2, 5\}$
Is A_1, A_2, A_3 a partition of S?

Equivalence Classes and Partitions

Theorem: Let R be an equivalence relation on a set A. Then, union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A} [a]_R$$

Theorem: The equivalence classes form a partition of A.

Theorem: Let $\{A_1, A_2, ..., A_i, ...\}$ be a partition of S. Then, there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Equivalence Classes and Partitions: Example

List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$.

Solution: The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition:

- (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class;
- (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class;
- (6, 6) belongs to R because 6 is an equivalence class.

Equivalence Classes and Partitions: Example

What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: There are four congruence classes, corresponding to $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$. They are the sets

- $[0]_4 = \{..., -8, -4, 0, 4, 8, ...\}$
- $[1]_4 = \{..., -7, -3, 1, 5, 9, ...\}$
- $[2]_4 = \{..., -6, -2, 2, 6, 10, ...\}$
- $[3]_4 = \{..., -5, -1, 3, 7, 11, ...\}$

These congruence classes are disjoint, and every integer is in exactly one of them.

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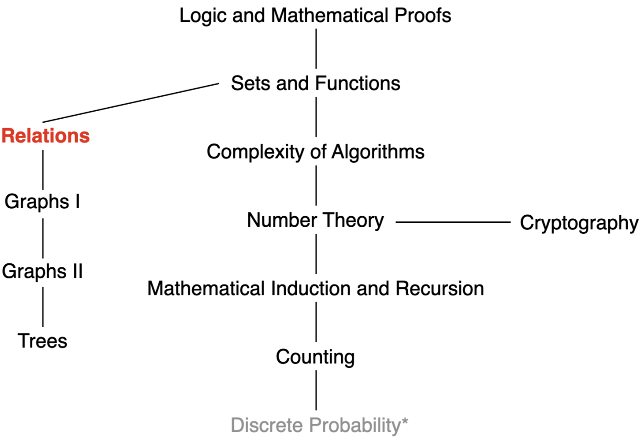

Relation, *n*-ary Relations, Representing Relations, Closures of Relations, Relation Equivalence, Partial Ordering, ...

Partial Ordering

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive.

A set S together with a partial ordering R is called a partially ordered set, or poset, denoted by (S, R).

Example: $S = \{1, 2, 3, 4, 5\}$, R denotes the "$\geq$" relation:

- Is R reflexive? Yes
- Is R antisymmetric? Yes
- Is R transitive? Yes

R is a partial ordering

Partial Ordering: Example

$S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$|$" relation

- Is R reflexive? Yes
- Is R antisymmetric? Yes
- Is R transitive? Yes

R is a partial ordering

Comparability

The notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R).

The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$.

Definition: The elements a and b of a poset $(S, \preccurlyeq)$ are comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are called incomparable.

Example: $S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$|$" relation.
$2, 4$ are comparable, $3, 5$ are incomparable.

Total Ordering

Definition: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Example: $S = \{1, 2, 3, 4, 5, 6\}$, R denotes the "$\geq$" relation S is a chain.

Well-Ordered Set

$(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Example: The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2), (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set.

The set Z, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Suppose x_0 is the least element of a well ordered set. Then $P(x)$ is true for all $x \in S$, if

Basic Step: $P(x_0)$ is true.

Inductive Step: For every $y \in S \setminus \{x_0\}$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Or equivalently,
Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

The Principle of Well-Ordered Induction

The Principle of Well-Ordered Induction: Suppose that $(S, \preccurlyeq)$ is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

Inductive Step: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false.

Consequently, the set $A = \{x \in S | P(x) \text{is false}\}$ is nonempty. Because S is well ordered, A has a least element a.

By the choice of a as a least element of A, we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. By the inductive step, $P(a)$ is true.

This contradiction shows that $P(x)$ must be true for all $x \in S$.

Questions from Section 5 (Induction)

The Well-Ordering Property: Every nonempty set of nonnegative integers has a least element.

The principle of mathematical induction follows from the well-ordering property.

Question from students: Consider the set of negative integers. Although it does not has a least element, it has a greatest element. Can we solve it using mathematical induction?

Yes. We can solve it using the principle of well-ordered induction if we can find a relation $\preccurlyeq$ such that $(S, \preccurlyeq)$ is a well-ordered set.

Questions from Section 5 (Induction)

(i) The principle of mathematical induction, (ii) strong induction, and (iii) well-ordering property are all equivalent principles.

That is, the validity of each can be proved from either of the other two. (See Section 5.2 Exercise 41, 42, 43)

- (i) $\rightarrow$ (ii): The inductive hypothesis of a proof by mathematical induction is part of the inductive hypothesis in a proof by strong induction.
- (ii) $\rightarrow$ (iii) Use strong induction to show that the set of nonnegative integers has a least element.
- (iii) $\rightarrow$ (i) The principle of mathematical induction follows from the well-ordering property.