No.

Date

HW1. $(G, \circ)$ be a group.

1. Can G have more than one identical element?
2. Can $x \in G$ have more than one inverse?
3. If $x_1, x_2, \cdots, x_m \in G$ have inverse $y_1, y_2, \cdots, y_m$ respectively, find the inverse of $x_1 \circ x_2 \circ \cdots \circ x_m$?

Sol. 1. No. Suppose e, e' are both identical element. $(e \neq e')$

G is a group so does a monoid, so $e \circ e' = e' \circ e$

Since e is identical element, $e' \circ e = e'$

Since e' is identical element, $e \circ e' = e$

So we get $e = e'$, identical element is unique.

2. No. Suppose x', x'' are both inverse of x. $(x' \neq x'')$

$$x' \circ x \circ x'' = (x' \circ x) \circ x'' \xlongequal{\text{(definition of inverse)}} e \circ x'' = x''$$

$$x' \circ x \circ x'' = x' \circ (x \circ x'') = x' \circ e = x'$$

So $x' = x''$, $x \in G$ cannot have more than one inverse.

3. Notice that

$$\begin{aligned}(x_1 \circ x_2 \circ \cdots \circ x_m) \circ (y_1 \circ y_2 \circ \cdots \circ y_m) &= x_1 \circ x_2 \circ \cdots \circ x_m \circ y_1 \circ y_2 \circ \cdots \circ y_m \\ &= (x_1 \circ y_1) \circ (x_2 \circ y_2) \circ \cdots \circ (x_m \circ y_m) \\ &= e \circ e \circ \cdots \circ e \\ &= e\end{aligned}$$

And since inverse is unique, the inverse of $x_1 \circ \cdots \circ x_m$ is $y_1 \circ y_2 \circ \cdots \circ y_m$.