现代信号处理: Homework 4

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要求:latex
DDL:2025/12/14 下午 23: 59 分前提交 pdf 电子版
电子版以”homework4-姓名-学号” 形式发送到 12432643@mail.sustech.edu.cn 邮箱

Problem 1

Consider the continuous-time signal $x(t)$ defined as

$$x(t) = \frac{\sin 2\pi t}{\pi t}$$

Prove that $x(t)$ is square integrable, but is not absolutely integrable.

平方可积性要求:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty.$$

已知:

$$x(t) = \frac{\sin(2\pi t)}{\pi t}.$$

其傅里叶变换为（设 $W = 1$）:

$$X(f) = \begin{cases} 1, & |f| < 1, \\ 0, & |f| > 1. \end{cases}$$

由 Parseval 定理:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \int_{-1}^{1} 1 \, df = 2 < \infty.$$

所以 $x(t)$ 平方可积。

绝对可积要求:

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty.$$

由于 $|x(t)| = \frac{|\sin(2\pi t)|}{\pi |t|}$，考虑 $t \geq 1$ 的部分:

$$\int_{1}^{\infty} \frac{|\sin(2\pi t)|}{\pi t} \, dt.$$

在区间 $[n, n+1]$ 上，有:

$$\int_{n}^{n+1} \frac{|\sin(2\pi t)|}{\pi t} \, dt \geq \frac{1}{\pi(n+1)} \int_{n}^{n+1} |\sin(2\pi t)| \, dt.$$

而 $\int_{n}^{n+1} |\sin(2\pi t)| \, dt = \frac{2}{\pi}$，所以:

$$\int_{n}^{n+1} |x(t)| \, dt \geq \frac{2}{\pi^2(n+1)}.$$

于是:

$$\int_{1}^{\infty} |x(t)| \, dt \geq \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

同理，负半轴也发散。因此 $x(t)$ 不是绝对可积的。

Problem 2

We wish to estimate the amplitudes of exponential signals in noise. The observed data is given by

$$x[n] = \sum_{i=1}^{p} A_i r_i^n + w[n], \quad n = 0, 1, \ldots, N-1$$

where $w[n]$ is white Gaussian noise (WGN) with variance σ^2. Find the MVU estimator of the amplitudes and their covariance. Evaluate your results for the specific case where $p = 2$, $r_1 = 1$, $r_2 = -1$, and N is even.

模型为:

$$\mathbf{x} = \mathbf{H}\mathbf{A} + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

其中

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_p \\ r_1^2 & r_2^2 & \cdots & r_p^2 \\ \vdots & \vdots & & \vdots \\ r_1^{N-1} & r_2^{N-1} & \cdots & r_p^{N-1} \end{bmatrix}.$$

MVU（最小方差无偏）估计量及其协方差矩阵为:

$$\hat{\mathbf{A}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}, \quad \mathrm{Cov}(\hat{\mathbf{A}}) = \sigma^2(\mathbf{H}^T\mathbf{H})^{-1}.$$

$p = 2, r_1 = 1, r_2 = -1, N$ 为偶数

此时:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}^T\mathbf{H} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}.$$

于是:

$$\hat{A}_1 = \frac{1}{N}\sum_{n=0}^{N-1} x[n], \quad \hat{A}_2 = \frac{1}{N}\sum_{n=0}^{N-1}(-1)^n x[n],$$

$$\mathrm{Var}(\hat{A}_1) = \mathrm{Var}(\hat{A}_2) = \frac{\sigma^2}{N}, \quad \mathrm{Cov}(\hat{A}_1, \hat{A}_2) = 0.$$

Problem 3

Consider the observation matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}$$

where ϵ is small. Compute $(\mathbf{H}^T\mathbf{H})^{-1}$, and examine what happens as $\epsilon \to 0$. If the observation vector is $\mathbf{x} = [2 \;\; 2 \;\; 2]^T$, find the MVU estimator. Describe what happens as $\epsilon \to 0$.

Problem 3 continued on next page...

已知:

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} 3 & 3+\epsilon \\ 3+\epsilon & 4+2\epsilon+\epsilon^2 \end{bmatrix}$$

行列式:

$$\Delta = 3(4+2\epsilon+\epsilon^2) - (3+\epsilon)^2 = 3+2\epsilon^2$$

逆矩阵:

$$(\mathbf{H}^T\mathbf{H})^{-1} = \frac{1}{3+2\epsilon^2} \begin{bmatrix} 4+2\epsilon+\epsilon^2 & -(3+\epsilon) \\ -(3+\epsilon) & 3 \end{bmatrix}$$

MVU 估计量

$$\mathbf{H}^T\mathbf{x} = \begin{bmatrix} 6 \\ 6+2\epsilon \end{bmatrix}$$

$$\hat{\mathbf{A}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} = \frac{1}{3+2\epsilon^2} \begin{bmatrix} (4+2\epsilon+\epsilon^2)\cdot 6 - (3+\epsilon)(6+2\epsilon) \\ -(3+\epsilon)\cdot 6 + 3(6+2\epsilon) \end{bmatrix}$$

计算得:

$$\hat{A}_1 = \frac{6+4\epsilon^2}{3+2\epsilon^2}, \quad \hat{A}_2 = 0$$

当 $\epsilon \to 0$ 时

$$\hat{A}_1 \to 2, \quad \hat{A}_2 \to 0$$

当 $\epsilon = 0$ 时,$\mathbf{H}$ 的列线性相关,参数不可唯一辨识。

Problem 4

In practice, we sometimes encounter the linear model $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$, where $\mathbf{H}$ is composed of random variables. Suppose we ignore this difference and use the usual estimator

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$$

where we assume the specific realization of $\mathbf{H}$ is known. Prove that if $\mathbf{H}$ and $\mathbf{w}$ are independent, then the mean and covariance of $\hat{\boldsymbol{\theta}}$ are

$$\begin{aligned} E(\hat{\boldsymbol{\theta}}) &= \boldsymbol{\theta} \\ \mathbf{C}_{\hat{\theta}} &= \sigma^2 E_H[(\mathbf{H}^T\mathbf{H})^{-1}] \end{aligned}$$

模型:$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$,其中 $\mathbf{H}$ 与 $\mathbf{w}$ 独立,$\mathbf{w} \sim (\mathbf{0}, \sigma^2\mathbf{I})$。
使用估计量:

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}.$$

代入 $\mathbf{x}$：

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T(\mathbf{H}\boldsymbol{\theta} + \mathbf{w}) = \boldsymbol{\theta} + (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{w}.$$

$$E[\hat{\boldsymbol{\theta}}] = \boldsymbol{\theta} + E\left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{w}\right].$$

由于 $\mathbf{H}$ 与 $\mathbf{w}$ 独立且 $E[\mathbf{w}] = \mathbf{0}$，有条件期望：

$$E[\mathbf{w} \mid \mathbf{H}] = \mathbf{0} \quad \Rightarrow \quad E\left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{w}\right] = \mathbf{0}.$$

因此：

$$E[\hat{\boldsymbol{\theta}}] = \boldsymbol{\theta}.$$

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{w},$$

$$\mathbf{C}_{\hat{\theta}} = E\left[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\right] = E\left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{w}\mathbf{w}^T\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\right].$$

对 $\mathbf{w}$ 取条件期望：

$$E[\mathbf{w}\mathbf{w}^T \mid \mathbf{H}] = \sigma^2\mathbf{I},$$

所以：

$$\mathbf{C}_{\hat{\theta}} = E_{\mathbf{H}}\left[(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T(\sigma^2\mathbf{I})\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\right] = \sigma^2\, E_{\mathbf{H}}\left[(\mathbf{H}^T\mathbf{H})^{-1}\right].$$

结果为：

$$E(\hat{\boldsymbol{\theta}}) = \boldsymbol{\theta}, \quad \mathbf{C}_{\hat{\theta}} = \sigma^2\, E_{\mathbf{H}}\left[(\mathbf{H}^T\mathbf{H})^{-1}\right]$$

Problem 5

Suppose we observe a fading signal in noise. We view this fading signal as being derived from another ”on” or ”off” signal. For example, consider a DC level in WGN, i.e., $x[n] = A + w[n]$, for $n = 0, 1, \ldots, N-1$. When the signal fades, the data model becomes

$$x[n] = \begin{cases} A + w[n] & n = 0, 1, \ldots, M-1 \\ w[n] & n = M, M+1, \ldots, N-1 \end{cases}$$

where the probability of fading is ϵ. Assume we know when the signal has undergone fading. Use the results of Problem 4 to determine the estimator of A and its variance, and compare this result with the case of no fading.

模型：

$$x[n] = \begin{cases} A + w[n], & n = 0, 1, \ldots, M-1 \\ w[n], & n = M, M+1, \ldots, N-1 \end{cases}$$

其中 $w[n] \sim \mathcal{N}(0, \sigma^2)$，衰落概率 ϵ，已知 M。

$$\mathbf{x} = \mathbf{H}A + \mathbf{w}, \quad \mathbf{H} = [\underbrace{1, 1, \ldots, 1}_{M}, \underbrace{0, 0, \ldots, 0}_{N-M}]^T$$

$$\mathbf{H}^T\mathbf{H} = M, \quad (\mathbf{H}^T\mathbf{H})^{-1} = \frac{1}{M}$$

$$\hat{A} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} = \frac{1}{M}\sum_{n=0}^{M-1} x[n]$$

由 Problem 4，$\mathbf{H}$ 随机时：

$$\mathrm{Var}(\hat{A}) = \sigma^2 E\left[(\mathbf{H}^T\mathbf{H})^{-1}\right] = \sigma^2 E\left[\frac{1}{M}\right]$$

$M \sim \mathrm{Binomial}(N, 1-\epsilon)$，所以：

$$E\left[\frac{1}{M}\right] = \sum_{m=1}^{N} \frac{1}{m}\binom{N}{m}(1-\epsilon)^m \epsilon^{N-m}$$

因此：

$$\mathrm{Var}(\hat{A}) = \sigma^2 \sum_{m=1}^{N} \frac{1}{m}\binom{N}{m}(1-\epsilon)^m \epsilon^{N-m}$$

无衰落 ($\epsilon = 0$)：$M = N$，方差 σ^2/N。
有衰落 ($\epsilon > 0$)：方差更大，因为 $E[1/M] > 1/N$。

Problem 6

The IID observations $x[n]$ for $n = 0, 1, \ldots, N-1$ have the exponential PDF

$$p(x[n];\lambda) = \begin{cases} \lambda \exp(-\lambda x[n]) & x[n] > 0 \\ 0 & x[n] < 0 \end{cases}$$

Find a sufficient statistic for λ.

已知 $x[n] \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(\lambda)$，PDF 为：

$$p(x[n];\lambda) = \begin{cases} \lambda \exp(-\lambda x[n]), & x[n] > 0 \\ 0, & x[n] \le 0 \end{cases}$$

$$p(\mathbf{x};\lambda) = \prod_{n=0}^{N-1} \lambda \exp(-\lambda x[n]) = \lambda^N \exp\left(-\lambda \sum_{n=0}^{N-1} x[n]\right), \quad x[n] > 0$$

Neyman–Fisher 因子分解定理：
令

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]$$

则：

$$p(\mathbf{x};\lambda) = \underbrace{\lambda^N \exp\left(-\lambda T(\mathbf{x})\right)}_{g(T(\mathbf{x}),\lambda)} \cdot \underbrace{1}_{h(\mathbf{x})}$$

其中 $h(\mathbf{x})$ 不依赖于 λ。

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]$$

是参数 λ 的充分统计量。

Problem 7

Assume $x[n]$ is the result of a Bernoulli trial (coin toss), with

$$\Pr\{x[n]=1\}=\theta$$
$$\Pr\{x[n]=0\}=1-\theta$$

and N IID observations are made. Assuming the Neyman-Fisher Factorization Theorem holds for discrete random variables, find a sufficient statistic for θ. Then, assuming completeness, find the MVU estimator of θ.

已知 $x[n] \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$，即

$$\Pr\{x[n]=1\}=\theta, \quad \Pr\{x[n]=0\}=1-\theta.$$

$$p(\mathbf{x};\theta)=\prod_{n=0}^{N-1}\theta^{x[n]}(1-\theta)^{1-x[n]}=\theta^{\sum_{n=0}^{N-1}x[n]}(1-\theta)^{N-\sum_{n=0}^{N-1}x[n]}.$$

令 $T(\mathbf{x})=\sum_{n=0}^{N-1}x[n]$，则

$$p(\mathbf{x};\theta)=\underbrace{\theta^{T(\mathbf{x})}(1-\theta)^{N-T(\mathbf{x})}}_{g(T(\mathbf{x}),\theta)}\cdot\underbrace{1}_{h(\mathbf{x})}.$$

由 Neyman–Fisher 定理，$T(\mathbf{x})$ 是 θ 的充分统计量。
考虑估计量

$$\hat{\theta}=\frac{T(\mathbf{x})}{N}=\frac{1}{N}\sum_{n=0}^{N-1}x[n].$$

它是无偏的：$E[\hat{\theta}]=\theta$。
Bernoulli 分布是指数族，参数空间 $(0,1)$ 包含开区间，故 $T(\mathbf{x})$ 是完全的充分统计量。
由 Lehmann–Scheffé 定理，$\hat{\theta}$ 是 MVU 估计量。
结论

$$T(\mathbf{x})=\sum_{n=0}^{N-1}x[n], \quad \hat{\theta}_{\text{MVU}}=\frac{1}{N}\sum_{n=0}^{N-1}x[n]$$

Problem 8

Consider a sinusoidal signal of known frequency in WGN, i.e.,

$$x[n]=A\cos 2\pi f_0 n+w[n] \quad n=0,1,\ldots,N-1$$

where $w[n]$ is WGN with variance σ^2. Find the MVU estimators for the following parameters:

1. Amplitude A, assuming σ^2 is known;
2. Amplitude A and noise variance σ^2.

You may assume the sufficient statistic is complete.

模型：

$$x[n] = A\cos(2\pi f_0 n) + w[n], \quad w[n] \sim \mathcal{N}(0, \sigma^2), \quad n = 0, 1, \ldots, N-1$$

f_0 已知。
(1) 已知 σ^2，估计 A

线性模型：$\mathbf{x} = \mathbf{H}A + \mathbf{w}$，其中

$$\mathbf{H} = [\cos(2\pi f_0 \cdot 0), \cos(2\pi f_0 \cdot 1), \ldots, \cos(2\pi f_0 \cdot (N-1))]^T.$$

MVU 估计量：

$$\hat{A} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} = \frac{\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)}{\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n)}.$$

方差：

$$\mathrm{Var}(\hat{A}) = \sigma^2(\mathbf{H}^T\mathbf{H})^{-1} = \frac{\sigma^2}{\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n)}.$$

(2) 同时估计 A 和 σ^2

充分统计量：

$$T_1(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n), \quad T_2(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]^2.$$

A 的 MVU 估计量同上：

$$\hat{A} = \frac{T_1(\mathbf{x})}{\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n)}.$$

σ^2 的 MVU 估计量：

$$\hat{\sigma}^2 = \frac{\sum_{n=0}^{N-1}(x[n] - \hat{A}\cos(2\pi f_0 n))^2}{N-1}.$$

结果：

$$\hat{A} = \frac{\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)}{\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n)}, \quad \hat{\sigma}^2 = \frac{\sum_{n=0}^{N-1}(x[n] - \hat{A}\cos(2\pi f_0 n))^2}{N-1}$$