HW 4

1. Prove that a norm in linear space satisfying the parallelogram law $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ is induced by an inner product.

Proof. $\|x+y\|^2+\|x-y\|^2=\langle x+y,x+y\rangle+\langle x-y,x-y\rangle$

$$=(\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle)+(\langle x,x\rangle+\langle x,-y\rangle+\langle -y,x\rangle+\langle y,y\rangle)$$

$$=2\langle x,x\rangle+2\langle y,y\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle x,-y\rangle+\langle -y,x\rangle$$

$$=2\|x\|^2+2\|y\|^2+\langle x,y\rangle+\langle y,x\rangle+\overline{\langle -y,x\rangle}-\langle y,x\rangle$$

$$=2\|x\|^2+2\|y\|^2+\langle x,y\rangle-\overline{\langle y,x\rangle}$$

$$=2\|x\|^2+2\|y\|^2+\langle x,y\rangle-\langle x,y\rangle$$

$$=2\|x\|^2+2\|y\|^2$$

2. Verify that $\langle f,g\rangle=\int_a^b f(t)\overline{g(t)}\,dt$ is a inner product on $C[a,b]$ and that the L_2-norm is derived from it.

Sol. 1) $\langle f,f\rangle=\int_a^b f(t)\overline{f(t)}\,dt=\int_a^b |f(t)|^2dt\geq 0$ nonnegativity

1a) $\langle f,f\rangle=\int_a^b |f(t)|^2dt=0 \Leftrightarrow |f(t)|^2=0 \Leftrightarrow f=0$ positivity

2) $\langle f+g,h\rangle=\int_a^b (f(t)+g(t))\overline{h(t)}\,dt=\int_a^b\left[f(t)\overline{h(t)}+g(t)\overline{h(t)}\right]dt$

$=\int_a^b f(t)\overline{h(t)}\,dt+\int_a^b g(t)\overline{h(t)}\,dt=\langle f,h\rangle+\langle g,h\rangle$ additivity

3) $\langle cf,g\rangle=\int_a^b cf(t)\overline{g(t)}\,dt=c\int_a^b f(t)\overline{g(t)}\,dt=c\langle f,g\rangle$ homogenety

4) $\langle f,g\rangle=\int_a^b f(t)\overline{g(t)}\,dt=\int_a^b \overline{\overline{f(t)}g(t)}\,dt=\overline{\left(\int_a^b g(t)\overline{f(t)}\,dt\right)}$

$=\overline{\langle g,f\rangle}$ Hermitian property

Hence, $\langle f,g\rangle=\int_a^b f(t)\overline{g(t)}\,dt$ is a inner product.

$\langle f,f\rangle=\int_a^b|f(t)|^2dt=\|f\|_2^2$ by definition, so $\|f\|_2=\langle f,f\rangle^{\frac{1}{2}}$

The L_2-norm is derived from it.

HW#

The maximum column sum $|||\cdot|||_1$ is defined on M_n by $|||A|||= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$.
Prove directly from the definition that $|||\cdot|||_1$ is a matrix norm.

Proof. 1) $\forall i,j$, $|a_{ij}|\ge 0$, $|||A||| = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \ge 0$ nonnegativity

~~1a) Suppose when $j=k$, $\sum_{i=1}^{k} |a_{ij}|$ reach its max value.~~

~~If $|||A|||=0$, $\sum_{i=1}^{k} |a_{il}|=0$,~~

1a) $|||A|||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = 0$, for $\forall j$, $\sum_{i=1}^{n} |a_{ij}| \le 0$

And we have $\sum_{i=1}^{n} |a_{ij}| \ge 0$, so $\forall j$, $\sum_{i=1}^{n} |a_{ij}| = 0$

So $\forall i,j$, $|a_{ij}|=0$ which means $A=0$ positivity

2) $|||cA|||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |ca_{ij}| = \max_{1\le j\le n} \sum_{i=1}^{n} |c||a_{ij}| = |c| \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| = |c|\, |||A|||_1$

homogenety

3) $|||A+B|||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}+b_{ij}| \xlongequal{\text{suppose } j=l \text{ get max}} \sum_{i=1}^{n} |a_{il}+b_{il}| \le \sum_{i=1}^{n} |a_{il}| + \sum_{i=1}^{n} |b_{il}|$

$\le \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| + \max_{1\le j\le n} \sum_{i=1}^{n} |b_{ij}| = |||A|||_1 + |||B|||_1$ triangle inequality

4) $|||AB|||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right| \le \max_{1\le j\le n} \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}||b_{kj}| \xlongequal{\text{suppose } j=l \text{ get max}} \sum_{i=1}^{n}\sum_{k=1}^{n} |a_{ik}||b_{kl}|$

$\le \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \cdot \max_{1\le j\le n} \sum_{i=1}^{n} |b_{ij}| = |||A|||_1 \cdot |||B|||_1$ submultiplicativity

Hence $|||\cdot|||_1$ is a matrix norm.