

### HW 3

For any  $m \times n$  matrix  $A$ , and any matrices  $P$  and  $Q$  such that  $PA$  and  $AQ$  are defined, we have

1.  $N(A) \subseteq N(PA)$  with equality if  $P^{-1}$  exist.
2.  $RS(PA) \subseteq RS(A)$  with equality if  $P^{-1}$  exist.
3.  $CS(AQ) \subseteq CS(A)$  with equality if  $Q^{-1}$  exist.

Sol. 1. For  $\forall x \in N(A)$ , we have  $Ax = 0$

$$\Rightarrow PAx = P(Ax) = P \cdot 0 = 0 \Rightarrow x \in N(PA) \Rightarrow N(A) \subseteq N(PA)$$

If  $P^{-1}$  exist, for  $\forall x \in N(PA)$ ,  $PAx = 0$

$$\Rightarrow Ax = P^{-1}PAx = P^{-1} \cdot 0 = 0 \Rightarrow x \in N(A) \Rightarrow N(PA) \subseteq N(A)$$

2. For  $\forall x \in RS(PA)$ ,  $\exists y$  s.t.  $x = yPA$

Let  $z = yP$ , then  $\exists z$  s.t.  $x = zA$ , so  $x \in RS(A)$

Hence  $RS(PA) \subseteq RS(A)$

If  $P^{-1}$  exist, for  $\forall x \in RS(A)$ ,  $\exists y$  s.t.  $x = yA$

$$x = yA = (yP^{-1})PA \Rightarrow x \in RS(PA) \Rightarrow RS(A) \subseteq RS(PA)$$

3. For  $\forall y \in CS(AQ)$ ,  $\exists x$  s.t.  $y = AQx$

$$y = AQx = A(Qx) \Rightarrow y \in CS(A) \Rightarrow CS(AQ) \subseteq CS(A)$$

If  $Q^{-1}$  exist, for  $\forall y \in CS(A)$ ,  $\exists x$  s.t.  $y = Ax$

$$y = Ax = A(QQ^{-1}x) \Rightarrow y \in CS(AQ) \Rightarrow CS(A) \subseteq CS(AQ)$$

If  $\|\cdot\|$  is a vector seminorm on a real or complex vector space  $V$ , then  $|\|x\| - \|y\|| \leq \|x - y\|$  for all  $x, y \in V$ .

Sol. From definition of seminorm's triangle inequality.

$$\|(x-y) + y\| \leq \|x-y\| + \|y\| \text{ that is } \|x\| \leq \|x-y\| + \|y\|$$

$$\text{Then } \|x\| - \|y\| \leq \|x-y\|$$

$$\text{Similarly, } \|y\| - \|x\| \leq \|y-x\|$$

$$\text{By homogeneity, } \|y-x\| = \|(-1)(x-y)\| = |-1| \|x-y\| = \|x-y\|$$

$$\text{So } \|y\| - \|x\| \leq \|x-y\|$$

$$\text{Combine } \|x\| - \|y\| \leq \|x-y\|, \text{ and } \|y\| - \|x\| \leq \|x-y\|$$

$$|\|x\| - \|y\|| \leq \|x-y\|$$