

# Lecture 2 - Evolutionary Search Operators

Yuhui Shi  
CSE, SUSTech

# Summary of the Previous Lecture

- Why Natural Computation?
- What is Evolutionary Computation?
- Different Types of Evolutionary Algorithms
- Major Areas in Evolutionary Computation

# Recall: Main Steps of Evolutionary Algorithms

1. Initialise the population at random
  2. REPEAT
    - a) Evaluate fitness of individuals in the population  
( $\mu$  is the population size)
    - b) Compute the selection probability for each individual
    - c) REPEAT
      - I. Select two individuals as parents according to the probabilities in Step 2.b
      - II. Crossover the two individuals in Step 2.c.I with a crossover rate  
/\* After this step, we have two individuals \*/
      - III. Mutate the two individuals in Step 2.c.II with a mutation rateUNTIL we have obtained  $\mu$  new individuals
    - d) Use the  $\mu$  new individuals to replace the previous populationUNTIL stopping criteria are met
- /\* Output the best individual in the population \*/

In this lecture, we will focus on the crossover and mutation operators.

# Outline of This Lecture

- Recombination/Crossover Operators
  - ✓ Recombination/Crossover Operators for Discrete Representation
  - ✓ Recombination/Crossover Operators for Real-valued Representation
- Mutation Operators
  - ✓ Mutation Operators for Discrete Representation
  - ✓ Mutation Operators for Real-valued Representation
- Summary of this Lecture

# Outline of This Lecture

- **Recombination/Crossover Operators**
  - ✓ **Recombination/Crossover Operators for Discrete Representation**
  - ✓ **Recombination/Crossover Operators for Real-valued Representation**
- **Mutation Operators**
  - ✓ Mutation Operators for Discrete Representation
  - ✓ Mutation Operators for Real-valued Representation
- **Summary of this Lecture**

# Recombination/Crossover

- Pick up two parents to generate two offspring.
- Crossover rate: The probability of applying crossover.
- Aim to keep and recombine good building blocks of the parents.  
(But how do we know which part is good?)

Illustrative example of Lecture 1:

**Parent 1**

1	1	0	0	0
---	---	---	---	---

**Parent 2**

0	1	1	0	1
---	---	---	---	---

**Offspring 1**

1	1	0	0	1
---	---	---	---	---

**Offspring 2**

0	1	1	0	0
---	---	---	---	---

# Recombination/Crossover Operators

- Recombination for **discrete** representation
  - ✓ One-point crossover / k-point crossover ( $k > 1$ )
  - ✓ Uniform crossover
  - ✓ ...
- Recombination for **real-valued** representation (mainly 2 categories):
  - ✓ Discrete recombination does not change actual (gene) values.  
Very similar to the crossover operators on binary strings.,
  - ✓ Intermediate recombination does change actual (gene) values.  
Usually based on some kind of average/mixture among multiple parents.
  - ✓ ...

# One-point Crossover

Example: choose a **random** crossover point at 3

**Parent 1**

0	1	0	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---

**Parent 2**

0	1	1	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

**Offspring 1**

0	1	0	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

**Offspring 2**

0	1	1	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---



# k-point Crossover ( $k > 1$ )

- Example:  $k = 2$ , choose **two** random crossover points at 3 and 6

Parent 1 

0	1	0	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---

Parent 2 

0	1	1	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

Offspring 1 

0	1	0	1	0	1	1	0	0	0
---	---	---	---	---	---	---	---	---	---

Offspring 2 

0	1	1	1	0	0	1	0	1	1
---	---	---	---	---	---	---	---	---	---

- Example:  $k = 3$ , choose **three** random crossover points at 3, 6, 8

Parent 1 

0	1	0	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---

Parent 2 

0	1	1	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

Offspring 1 

0	1	0	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

Offspring 2 

0	1	1	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---

# Uniform Crossover

Example: at every point, generate a random number  $\in [0, 1]$  with crossover rate 0.5

Gene Index	1	2	3	4	5	6	7	8	9	10
Random	0.31	0.63	0.58	0.07	0.29	0.42	0.59	0.63	0.13	0.73

Parent 1

0	1	0	1	0	0	1	0	0	0
---	---	---	---	---	---	---	---	---	---

Parent 2

0	1	1	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

Offspring 1

0	1	0	1	0	1	1	0	1	0
---	---	---	---	---	---	---	---	---	---

Offspring 2

0	1	1	1	0	0	1	0	0	1
---	---	---	---	---	---	---	---	---	---

# Multi-point Discrete Recombination

- Similar to that for the binary representation.
- [Example] two parents with a random crossover point at 1:

**Parent 1**

1.0	2.4	0.8	7.6
-----	-----	-----	-----

**Parent 2**

0.2	1.1	1.6	3.4
-----	-----	-----	-----

**Offspring 1**

1.0	1.1	1.6	3.4
-----	-----	-----	-----

**Offspring 2**

0.2	2.4	0.8	7.6
-----	-----	-----	-----

# Global Discrete Recombination

- Similar to uniform crossover for the binary representation.
- On each dimension, randomly select a parent from the population and take its gene of the dimension.
- Selection on each dimension is independent.

**Parent 1**

1.0	2.4	0.8	7.6
-----	-----	-----	-----

**Parent 2**

0.2	1.1	1.6	3.4
-----	-----	-----	-----

**Offspring**

0.2	2.4	0.8	3.4
-----	-----	-----	-----

# Intermediate Recombination

- Discrete Recombination does not change actual (gene) values.  
→ Create new values using **intermediate recombination** (usually based on some kind of average/mixture among multiple parents).

# Arithmetic Recombination

## *-A Commonly Used Intermediate Recombination Operator*

- Given
  - ✓  $d$  is the individual/chromosome length,
  - ✓ two parents  $x_1$  and  $x_2$ , and
  - ✓ a weight  $\alpha \in [0, 1]$ :

$$\text{Offspring 1 : } x'_i = \alpha x_{1i} + (1 - \alpha)x_{2i}, \quad i \in \{1, \dots, d\}$$

$$\text{Offspring 2 : } x''_i = \alpha x_{2i} + (1 - \alpha)x_{1i}, \quad i \in \{1, \dots, d\}$$

- If  $\alpha = 0.5$ , the two offspring/children are identical.

# Arithmetic Recombination

[Example] two parents with  $\alpha = 0.4$

**Parent 1**

1.0	2.4	0.8	7.6
-----	-----	-----	-----

**Parent 2**

0.2	1.1	1.6	3.4
-----	-----	-----	-----

$$\begin{array}{ll} 0.4 * 1.0 + (1 - 0.4) * 0.2 = 0.52, & 0.4 * 0.2 + (1 - 0.4) * 1.0 = 0.68 \\ 0.4 * 2.4 + (1 - 0.4) * 1.1 = 1.62, & 0.4 * 1.1 + (1 - 0.4) * 2.4 = 1.88 \\ 0.4 * 0.8 + (1 - 0.4) * 1.6 = 1.28, & 0.4 * 1.6 + (1 - 0.4) * 0.8 = 1.12 \\ 0.4 * 7.6 + (1 - 0.4) * 3.4 = 5.08, & 0.4 * 3.4 + (1 - 0.4) * 7.6 = 5.92 \end{array}$$

**Offspring 1**

0.52	1.62	1.28	5.08
------	------	------	------

**Offspring 2**

0.68	1.88	1.12	5.92
------	------	------	------

# Simple Arithmetic Recombination

## - *Combining Arithmetic Recombination with One-Point Recombination*

- Given

- ✓ two parents  $x_1$  and  $x_2$ ,
- ✓ a weight  $\alpha \in [0, 1]$ , and
- ✓ a randomly selected point  $k$ :

$$\text{Offspring 1 : } x'_i = x_{1i}, \quad i \in \{1, \dots, k\}$$

$$x'_i = \alpha x_{1i} + (1 - \alpha)x_{2i}, \quad i \in \{k + 1, \dots, d\}$$

$$\text{Offspring 2 : } x''_i = x_{2i}, \quad i \in \{1, \dots, k\}$$

$$x''_i = \alpha x_{2i} + (1 - \alpha)x_{1i}, \quad i \in \{k + 1, \dots, d\}$$

- If  $\alpha = 0.5$ , the part after the  $k^{\text{th}}$  gene of the two offspring are identical.



# Simple Arithmetic Recombination

- [Example 1] two parents with  $\alpha = 0.4$  and a random point 1

Parent 1 

1.0	2.4	0.8	7.6
-----	-----	-----	-----

Parent 2 

0.2	1.1	1.6	3.4
-----	-----	-----	-----

Offspring 1 

1.0	1.62	1.28	5.08
-----	------	------	------

Offspring 2 

0.2	1.88	1.12	5.92
-----	------	------	------

- [Example 2] two parents with  $\alpha = 0.5$  and a random point 2

Parent 1 

1.0	2.4	0.8	7.6
-----	-----	-----	-----

Parent 2 

0.2	1.1	1.6	3.4
-----	-----	-----	-----

Offspring 1 

1.0	2.4	1.2	5.5
-----	-----	-----	-----

Offspring 2 

0.2	1.1	1.2	5.5
-----	-----	-----	-----

# Single Arithmetic Recombination

- Given
  - ✓ two parents  $x_1$  and  $x_2$ ,
  - ✓ a weight  $\alpha \in [0, 1]$ , and
  - ✓ a randomly selected point  $k$ :

$$\text{Offspring 1 : } x'_i = x_{1i}, \quad i \in \{1, \dots, d\} \textbf{ and } i \neq k$$

$$x'_k = \alpha x_{1k} + (1 - \alpha)x_{2k}$$

$$\text{Offspring 2 : } x''_i = x_{2i}, \quad i \in \{1, \dots, d\} \textbf{ and } i \neq k$$

$$x''_k = \alpha x_{2k} + (1 - \alpha)x_{1k}$$

# Single Arithmetic Recombination

[Example] two parents with  $\alpha = 0.4$  and a random point 2

<b>Parent 1</b>	1.0	2.4	0.8	7.6
<b>Parent 2</b>	0.2	1.1	1.6	3.4
<b>Offspring 1</b>	1.0	1.62	0.8	7.6
<b>Offspring 2</b>	0.2	1.88	1.6	3.4

# Other Recombination Operators for Real-valued Representation I

## Heuristic Recombination

Assume  $x_2$  is no worse than  $x_1$ .

$$x' = \alpha(x_2 - x_1) + x_2,$$

where  $\alpha$  is a uniformly distributed random number  $\in [0, 1]$ .

# Other Recombination Operators for Real-valued Representation II

## Simplex Recombination

Randomly select a group ( $> 2$ ) of parents. Assume  $x_b$  is the best individual and  $x_w$  is the worst in the group. Compute the centroid,  $c$ , of the group without  $x_w$ . Let the following  $x'$  replace  $x_w$ .

$$x' = c + (x_b - x_w).$$

# Other Recombination Operators for Real-valued Representation III

## Geometric Recombination

Can be generalised to multiple parents.

$$\mathbf{x}' = (\sqrt{x_{11}x_{21}}, \sqrt{x_{12}x_{22}}, \dots)$$

# Other Recombination Operators for Real-valued Representation IV

## Quadratic Recombination

Let  $x_{i,j}$  be the  $j$ -th component of the vectors  $\mathbf{x}_i$ ,

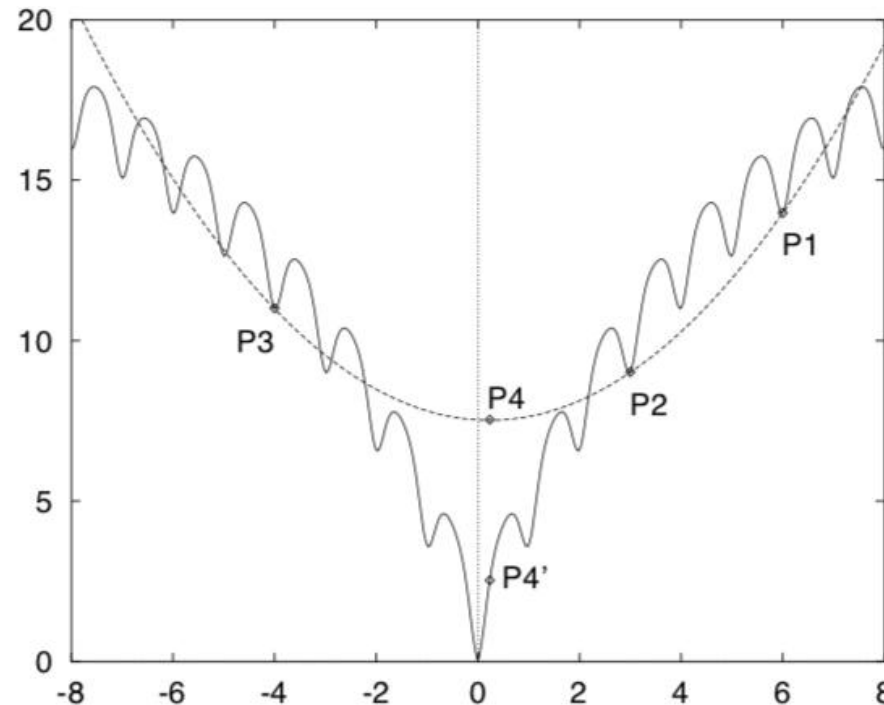
$\forall i \in \{1, 2, 3\}, j \in \{1, \dots, d\}$ , where  $d$  is the dimensionality. We approximate the position of  $P_4$  using the quadratic interpolation method as follow:

$$x_{4,j} = \frac{1}{2} \cdot \frac{(x_{2,j}^2 - x_{3,j}^2)f(\mathbf{x}_1) + (x_{3,j}^2 - x_{1,j}^2)f(\mathbf{x}_2) + (x_{1,j}^2 - x_{2,j}^2)f(\mathbf{x}_3)}{(x_{2,j} - x_{3,j})f(\mathbf{x}_1) + (x_{3,j} - x_{1,j})f(\mathbf{x}_2) + (x_{1,j} - x_{2,j})f(\mathbf{x}_3)}.$$

# What Does Quadratic Recombination Mean?

$$x_{4,j} = \frac{1}{2} \cdot \frac{(x_{2,j}^2 - x_{3,j}^2)f(\mathbf{x}_1) + (x_{3,j}^2 - x_{1,j}^2)f(\mathbf{x}_2) + (x_{1,j}^2 - x_{2,j}^2)f(\mathbf{x}_3)}{(x_{2,j} - x_{3,j})f(\mathbf{x}_1) + (x_{3,j} - x_{1,j})f(\mathbf{x}_2) + (x_{1,j} - x_{2,j})f(\mathbf{x}_3)}.$$

Note that we are minimising “fitness” here.





# Outline of This Lecture

- Recombination/Crossover Operators
  - ✓ Recombination/Crossover Operators for Discrete Representation
  - ✓ Recombination/Crossover Operators for Real-valued Representation
- **Mutation Operators**
  - ✓ **Mutation Operators for Discrete Representation**
  - ✓ **Mutation Operators for Real-valued Representation**
- Summary of this Lecture

# Mutation

- Change values of gene(s) at random.
- Mutation rate: Note the difference between per bit (gene) and per chromosome (individual) mutation rates.
- Be careful with the randomised part when implementing.

Illustrative example of Lecture 1:

(After crossover, before mutation)

Offspring 1 

1	1	0	0	1
---	---	---	---	---

Offspring 2 

0	1	1	0	0
---	---	---	---	---

(After mutation)

Offspring 1 

1	1	0	0	1
---	---	---	---	---

Offspring 2 

0	1	1	0	1
---	---	---	---	---

# Mutation Operators

- For **discrete** representation:
  - ✓ Bit-flipping
  - ✓ Random bit assignment
  - ✓ Swap mutation
  - ✓ Inverse mutation
  - ✓ ...
- For **real-valued** representation (mainly two categories):
  - ✓ Uniform mutation
  - ✓ Nonuniform mutation

# Bit-flipping

- One-bit flipping / One-bit mutation:

Flip one of the bits uniformly at random, e.g.,

Gene index	1	2	3	4	5	6	7	8	9	10
Random index	3									

**Before mutation**

0	1	0	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

**After mutation**

0	1	1	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

- Multi-bit flipping:

Select multiple bits and flip their values, e.g.,

Gene index	1	2	3	4	5	6	7	8	9	10
Random indices	3,	6,	7	(e.g. 3-bit flipping)						

**Before mutation**

0	1	0	1	0	1	1	0	1	1
---	---	---	---	---	---	---	---	---	---

**After mutation**

0	1	1	1	0	0	0	0	1	1
---	---	---	---	---	---	---	---	---	---

# Bitwise mutation

- At every point, generate a random number  $\in [0, 1]$  with mutation rate  $p$  (usually  $p = 1/d$ )

Gene	1	2	3	4	5	6	7	8	9	10
Random	0.03	0.31	0.07	0.58	0.79	0.52	0.19	0.13	0.93	0.23
Before mutation	0	1	0	1	0	1	1	0	1	1
After mutation	1	1	1	1	0	1	1	0	1	1

# Random mutation

- Extension of bit-flipping:  
binary representation  $\rightarrow$  integer representation.
- Example:
  - ✓ An integer representation with values  $\in \{0, 1, 2, 3, 4\}$
  - ✓ At every point, generate a random number  $\in [0, 1]$  with mutation rate 0.1, mutate to another possible value uniformly at random.

Gene index	1	2	3	4	5	6	7	8	9	10
Random for selecting gene	0.03	0.31	0.07	0.58	0.79	0.52	0.19	0.13	0.93	0.23
Random for selecting value	0.21	-	0.72	-	-	-	-	-	-	-

**Before mutation**

0	1	3	2	0	1	2	2	4	1
---	---	---	---	---	---	---	---	---	---

**After mutation**

1	1	2	2	0	1	2	2	4	1
---	---	---	---	---	---	---	---	---	---

## [Exercises]

1. Search by yourself to find out what is **random bit assignment**?
2. Implement one-bit flipping and bitwise mutation, then compare them using the objective function of Lab 1.

# Swap mutation

- Swap the values of two genes selected uniformly at random.
- Example:

**Before mutation**

1	2	3	4	5	6	7
---	---	---	---	---	---	---

**After mutation**

1	2	6	4	5	3	7
---	---	---	---	---	---	---



# Inversion mutation

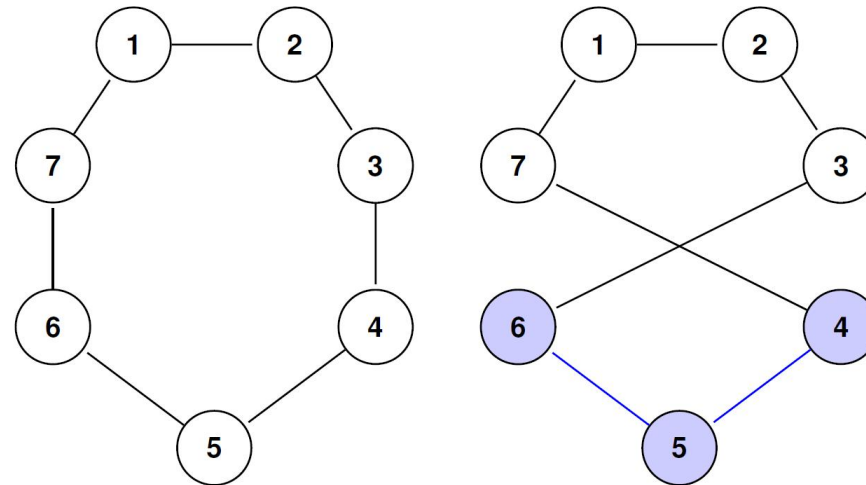
- Invert the order of a subset of string.
- Example:

**Before mutation**

1	2	3	4	5	6	7
---	---	---	---	---	---	---

**After mutation**

1	2	3	6	5	4	7
---	---	---	---	---	---	---



# Uniform Mutation

- Similar to uniform mutation for the binary representation:  
$$x'_i = \text{UniformRandom}(\text{LowerBound}_i, \text{UpperBound}_i), \quad \forall i \in \{1, \dots, d\}$$
- Potential problem: possibility of diverging too much from an already good solution.
- *[Example] One parent with mutation rate 0.1 and random numbers generated as follows:*

Gene	1	2	3	4	
Interval	[0.0, 1.0]	[1.0, 4.0]	[0.5, 2.3]	[2.0, 30]	
Random	0.02	0.63	0.58	0.07	// mutate if random value < 0.1
Random	0.30	-	-	4.5	// randomly generate new value

Before mutation

1.0	2.4	0.8	7.6
-----	-----	-----	-----

After mutation

0.3	2.4	0.8	4.5
-----	-----	-----	-----

# Nonuniform Mutation

Usually, new value = current value + **random perturbation** ( $\Delta$ ), i.e.,

$$x'_i = x_i + \Delta, \quad \forall i \in \{1, \dots, d\} \text{ and } LB_i \leq x'_i \leq UB_i,$$

where  $\Delta$  is sampled from a distribution with 0 mean, and a given standard deviation ( $\sigma$ ), also called **mutation step-size**.

Main differences compared to the uniform mutation presented previously:

- ✓ Nonuniform.
- ✓ + A random perturbation, instead of generating a totally new value.

$$x'_i = x_i + \Delta, \quad \forall i \in \{1, \dots, d\} \text{ and } LB_i \leq x'_i \leq UB_i,$$

## Question

*Why the perturbation is sampled from a distribution with 0 mean not a mean  $< 0$  or  $> 0$ ?*

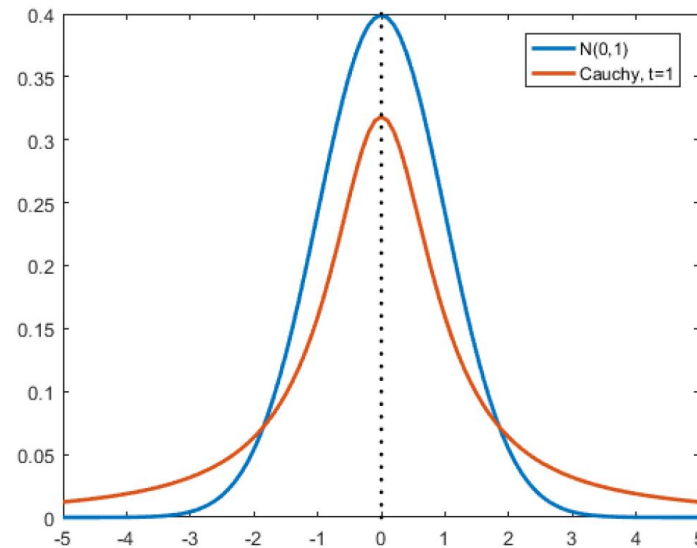
# Nonuniform Mutation (continued)

- ✓ Perturbation applied with probability 1 per gene.
- ✓ Different perturbation techniques:
  - Differ in the distribution:
    - ❑ **Gaussian** distribution:  $\Delta \sim N(0, \sigma^2)$
    - ❑ **Cauchy** distribution:  $\Delta \sim C(0, t)$ 
      - A “fatter” tail. → higher probability of generating larger values.
  - Differ in how  $\sigma$  is updated:
    - ❑ **Self-adaptive**: demonstrated to be successful for real-valued, binary and integer search space.
      - **More in the next lecture.**
- ✓ Curtailing the resulting value to locate in  $[LB_i, UB_i]$ , if not.

# Normal Distribution and Cauchy Distribution

**PDF of Gaussian:**  $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

**PDF of Cauchy:**  $f(x; x_0, t) = \frac{1}{\pi t \left[ 1 + \left( \frac{x-x_0}{t} \right)^2 \right]} = \frac{1}{\pi t} \left[ \frac{t^2}{(x-x_0)^2 + t^2} \right]$ , where  $t$  is the scale parameter which specifies the half-width at half-maximum.



# Outline of This Lecture

- Recombination/Crossover Operators
  - ✓ Recombination/Crossover Operators for Discrete Representation
  - ✓ Recombination/Crossover Operators for Real-valued Representation
- Mutation Operators
  - ✓ Mutation Operators for Discrete Representation
  - ✓ Mutation Operators for Real-valued Representation
- **Summary of this Lecture**

# Summary

1. Evolutionary algorithms vary in the
  - representation,
  - search operators (crossover operators and mutation operators)
  - and selection schemes.
2. (THIS LECTURE) crossover operators and mutation operators for discrete and real-valued representations.
3. Different problems require different search operators and selection schemes. There is no universally best one.

We will learn selection schemes in the next lecture.

# Essential Reading for This Lecture

1. X. Yao, "Evolutionary computation: A gentle introduction," In Evolutionary Optimization, R. Sarker, M. Mohammadian and X. Yao (eds.), Chapter 2, pp.27-53, Kluwer Academic Publishers, Boston, 2002. (ISBN 0-7923-7654-4)  
[https://link.springer.com/chapter/10.1007/0-306-48041-7\\_2](https://link.springer.com/chapter/10.1007/0-306-48041-7_2)
2. H. G.Beyer & H. P. Schwefel, "Evolution strategies-A comprehensive introduction" . Natural Computing, 1(1), 3-52, 2002.



# Essential Reading for Next Lecture

1. T. Bäck, D. B. Fogel, and Z. Michalewicz (eds.), Handbook of Evolutionary Computation, IOP Publ. Co. & Oxford University Press, 1997. Part C. (The part of selection schemes.)
2. X. Yao, Y. Liu and G. Lin, "Evolutionary programming made faster," IEEE Transactions on Evolutionary Computation, 3(2):82-102, July 1999.  
[https://www.cs.bham.ac.uk/~xin/papers/published\\_tec\\_jul99.pdf](https://www.cs.bham.ac.uk/~xin/papers/published_tec_jul99.pdf)  
(You can also visit IEEEExplore and download the pdf through VPN of our university)