

HW 3

For any $m \times n$ matrix A , and any matrices P and Q such that PA and AQ are defined, we have

1. $N(A) \subseteq N(PA)$ with equality if P^{-1} exist.
2. $RS(PA) \subseteq RS(A)$ with equality if P^{-1} exist.
3. $CS(AQ) \subseteq CS(A)$ with equality if Q^{-1} exist.

Sol. 1. For $\forall x \in N(A)$, we have $Ax = 0$

$$\cancel{P} PAx = P(Ax) = P \cdot 0 = 0 \Rightarrow x \in N(PA) \Rightarrow N(A) \subseteq N(PA)$$

If P^{-1} exist, for $\forall x \in N(PA)$, $PAx = 0$

$$P Ax = P^{-1} PAx = P^{-1} \cdot 0 = 0 \Rightarrow x \in N(A) \Rightarrow N(PA) \subseteq N(A)$$

2. For $\forall x \in RS(PA)$, $\exists y$ s.t. $x = yPA$

Let $z = yP$, then $\exists z$ s.t. $x = za$, so $x \in RS(A)$

Hence $RS(PA) \subseteq RS(A)$

If P^{-1} exist, for $\forall x \in RS(A)$, $\exists y$ s.t. $x = ya$

$$x = ya = (yP^{-1})PA \Rightarrow x \in RS(PA) \Rightarrow RS(A) \subseteq RS(PA)$$

3. For $\forall y \in CS(AQ)$, $\exists x$ s.t. $y = AQx$

$$y = AQx = A(Qx) \Rightarrow y \in CS(A) \Rightarrow CS(AQ) \subseteq CS(A)$$

If Q^{-1} exist, for $\forall y \in CS(A)$, $\exists x$ s.t. $y = Ax$

$$y = Ax = A(Q^{-1}x) \Rightarrow y \in CS(AQ) \Rightarrow CS(A) \subseteq CS(AQ)$$

If $\|\cdot\|$ is a vector seminorm on a real or complex vector space V ,
then $\|x - y\| \leq \|x - y\|$ for all $x, y \in V$.

Sol. From definition of seminorm's triangle inequality.

$$\|(x-y)+y\| \leq \|x-y\| + \|y\| \text{ that is } \|x\| \leq \|x-y\| + \|y\|$$

$$\text{Then } \|x\| - \|y\| \leq \|x-y\|$$

$$\text{Similarly, } \|y-x\| \leq \|y-x\|$$

$$\text{By homogeneity, } \|y-x\| = \|(-1)(x-y)\| = |-1| \|x-y\| = \|x-y\|$$

$$\text{So } \|y-x\| - \|x\| \leq \|x-y\|$$

$$\text{Combine } \|x\| - \|y\| \leq \|x-y\|, \cancel{\|y-x\|}$$

$$\|x\| - \|y\| \leq \|x-y\|$$