现代信号处理

Lecture 16

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3.4 Cramer-Rao Lower Bound

Theorem 3.1 (Cramer-Rao Lower Bound - Scalar Parameter) *It is assumed that the PDF $p(x;\theta)$ satisfies the "regularity" condition*

$$E\left[\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] = 0 \qquad \text{for all } \theta$$

where the expectation is taken with respect to $p(x;\theta)$. *Then, the variance of any unbiased estimator $\hat{\theta}$ must satisfy*

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]}$$

where the derivative is evaluated at the true value of θ and the expectation is taken with respect to $p(x;\theta)$. Furthermore, an unbiased estimator may be found that attains the bound for all θ if and only if

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)(g(x) - \theta)$$

for some functions g and I. That estimator, which is the MVU estimator, is $\hat{\theta} = g(x)$, and the minimum variance is $1/I(\theta)$.

Example – CRLB for the PDF Dependence on Unknown Parameter example

If a single sample is observed as

$$x[0] = A + w[0]$$

where $w[0] \sim N(0,\sigma^2)$, and it is desired to estimate A

$$p(x[0];A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x[0]-A)^2\right]$$

$$\frac{\partial \ln p(x[0];A)}{\partial A} = \frac{1}{\sigma^2}(x[0]-A)$$

$$-\frac{\partial^2 \ln p(x[0];A)}{\partial A^2} = \frac{1}{\sigma^2}$$

$$E\left[\frac{\partial \ln p(x[0];A)}{\partial A}\right] = E\left[\frac{1}{\sigma^2}(x[0]-A)\right]$$

$$= \frac{1}{\sigma^2}E(x[0]) - \frac{1}{\sigma^2}A$$

$$= \frac{1}{\sigma^2}A - \frac{1}{\sigma^2}A = 0$$

$p(x;\theta)$ *satisfies the "regularity" condition*

$$\mathrm{var}(\hat{A}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;A)}{\partial A^2}\right]} = \sigma^2 \quad \text{for all } A$$

Thus, no unbiased estimator can exist whose variance is lower than σ^2 for even a single value of A. If $\hat{A} = x[0]$, then $\mathrm{var}(\hat{A}) = \sigma^2$. Since $x[0]$ is unbiased and attains the CRLB, it must therefore be the MVU estimator.

$$\frac{\partial \ln p(x[0];A)}{\partial A} = \frac{1}{\sigma^2}(x[0]-A) \qquad \frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)(g(x)-\theta)$$

$$\theta = A,\ I(\theta) = \frac{1}{\sigma^2},\ g(x[0]) = x[0]$$

Hence, $\hat{A} = g(x[0]) = x[0]$ is the MVU estimator.

Example – DC Level in White Gaussian Noise

Consider the multiple observations

$$x[n] = A + w[n], \quad n = 0,1,\ldots,N-1$$

where $w[n] \sim N(0,\sigma^2)$, and it is desired to estimate A

$$p(x;A) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x[n]-A)^2\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

$$\frac{\partial \ln p(x;A)}{\partial A} = \frac{\partial}{\partial A}\left[-\ln(2\pi\sigma^2)^{\frac{N}{2}} - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A) = \frac{N}{\sigma^2}(\bar{x}-A)$$

$$-\frac{\partial^2 \ln p(x;A)}{\partial A^2} = \frac{N}{\sigma^2}$$

$$E\left[\frac{\partial \ln p(x[0];A)}{\partial A}\right] = E\left[\frac{N}{\sigma^2}(\bar{x}-A)\right]$$

$$= \frac{N}{\sigma^2}E(\bar{x}) - \frac{N}{\sigma^2}A$$

$$= \frac{N}{\sigma^2}A - \frac{N}{\sigma^2}A = 0$$

$p(x;A)$ *satisfies the "regularity" condition*

$$\mathrm{var}(\hat{A}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;A)}{\partial A^2}\right]} = \frac{\sigma^2}{N} \quad \text{for all } A$$

Thus, no unbiased estimator can exist whose variance is lower than σ^2 / N for even a single value of A. If $\hat{A} = \bar{x}$, then $\mathrm{var}(\hat{A}) = \sigma^2 / N$. Since $\bar{x}$ is unbiased and attains the CRLB, it must therefore be the MVU estimator.

$$\frac{\partial \ln p(x;A)}{\partial A} = \frac{N}{\sigma^2}(\bar{x} - A) \qquad\qquad \frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)(g(x) - \theta)$$

$$\theta = A,\ I(\theta) = \frac{N}{\sigma^2},\ g(x) = \bar{x}$$

Hence, $\hat{A} = g(x) = \bar{x}$ is the MVU estimator.

We now prove that when the CRLB is attained

$$\mathrm{var}(\hat{\theta}) = \frac{1}{I(\theta)}$$

where

$$I(\theta) = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]$$

Proof:

The CRLB is attained

$$\mathrm{var}(\hat{\theta}) = \frac{1}{-E\left[\dfrac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]}$$

Since

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = I(\theta)(g(x)-\theta) = I(\theta)(\hat{\theta}-\theta)$$

$$\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2} = \frac{\partial I(\theta)}{\partial \theta}(\hat{\theta}-\theta) - I(\theta)$$

$$-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right] = -\frac{\partial I(\theta)}{\partial \theta}\left[E(\hat{\theta})-\theta\right] + I(\theta)$$

$$= -\frac{\partial I(\theta)}{\partial \theta}\left[\theta-\theta\right] + I(\theta) = I(\theta)$$

Example – Phase Estimation (the CRLB is not always satisfied)

Assume that we wish to estimate the phase ϕ of a sinusoid embedded in WGN or

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n], \quad n = 0,1,\ldots,N-1$$

The amplitude A and frequency f_0 are assumed known. The PDF is

$$p(x;\phi) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} \left[x[n] - A\cos(2\pi f_0 n + \phi) \right]^2 \right]$$

$$\frac{\partial \ln p(x;\phi)}{\partial \phi} = -\frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left[x[n] - A\cos(2\pi f_0 n + \phi) \right] A\sin(2\pi f_0 n + \phi)$$

$$= -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} \left[x[n]\sin(2\pi f_0 n + \phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi) \right]$$

$$\frac{\partial^2 \ln p(x;\phi)}{\partial \phi^2} = -\frac{A}{\sigma^2} \sum_{n=0}^{N-1} \left[x[n]\cos(2\pi f_0 n + \phi) - A\cos(4\pi f_0 n + 2\phi) \right]$$

$$E\left[\frac{\partial \ln p(x;\phi)}{\partial \phi}\right] = -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[\sin(2\pi f_0 n+\phi)E\left[x[n]\right] - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\right]$$

$$= -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[\sin(2\pi f_0 n+\phi)A\cos(2\pi f_0 n+\phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\right]$$

$$= 0$$

$p(x;\phi)$ *satisfies the "regularity" condition*

$$\mathrm{var}(\hat{\phi}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\phi)}{\partial \phi^2}\right]} \quad \text{for all } \phi$$

$$-E\left[\frac{\partial^2 \ln p(x;\phi)}{\partial \phi^2}\right] = \frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[E[x[n]]\cos(2\pi f_0 n+\phi) - A\cos(4\pi f_0 n + 2\phi)\right]$$

$$\cos(2x) = 2\cos^2 x - 1$$

$$= \frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[A\cos^2(2\pi f_0 n+\phi) - A\cos(4\pi f_0 n + 2\phi)\right]$$

$$= \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1}\left[\frac{1}{2}+\frac{1}{2}\cos(4\pi f_0 n+2\phi) - \cos(4\pi f_0 n + 2\phi)\right] \approx \frac{NA^2}{2\sigma^2}$$

Since

$$\frac{1}{N}\sum_{n=0}^{N}\cos(4\pi f_0 n + 2\phi) \approx 0$$

for f_0 not near 0 or ½ (see Problem 3.7). Therefore

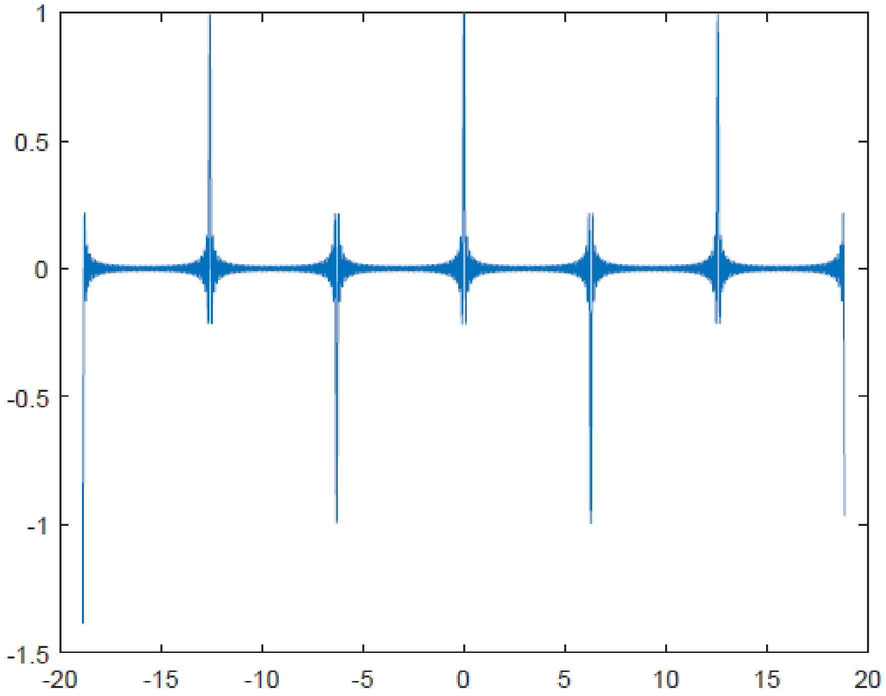
$$\mathrm{var}(\hat{\phi}) \geq \frac{2\sigma^2}{NA^2}$$

$$\frac{\partial \ln p(x;\phi)}{\partial \phi} = -\frac{A}{\sigma^2}\sum_{n=0}^{N-1}\left[x[n]\sin(2\pi f_0 n + \phi) - \frac{A}{2}\sin(4\pi f_0 n + 2\phi)\right]$$
$$\neq I(\phi)(g(x)-\phi)$$

The condition for the bound to hold is not satisfied. Hence, a phase estimator does not exist which is unbiased and attains the CRLB. It is still possible, however, that an MVU estimator may exist.

令 $\alpha = 4\pi f_0, \beta = 2\phi$

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\cos(\alpha n + \beta) &= \frac{1}{N}Re(\sum_n e^{j(\alpha n+\beta)}) \\ &= \frac{1}{N}Re(e^{j\beta}\frac{1-e^{j\alpha N}}{1-e^{j\alpha}}) \\ &= \frac{1}{N}Re(e^{j\beta}\frac{e^{j\alpha N/2}}{e^{j\alpha/2}}\frac{e^{-j\alpha N/2}-e^{j\alpha N/2}}{e^{-j\alpha/2}-e^{j\alpha/2}}) \\ &= \frac{1}{N}Re(e^{j\beta}e^{j\alpha(\frac{N-1}{2})}\frac{\sin N\alpha/2}{\sin\alpha/2}) \\ &= \frac{\sin N\alpha/2}{N\sin\alpha/2}\cos[\alpha(\frac{N-1}{2})+\beta]\end{aligned}$$



除非 α 在 2π 的整数倍附近，否则 $\frac{\sin N\alpha/2}{N\sin\alpha/2} \approx 0$。$\alpha$ 取 2π 时，f_0 取 $1/2$。

图为 N=100 时，$\frac{\sin Nx/2}{N\sin x/2}$ 的取值。

An estimator which is unbiased and attains the CRLB is said to be *efficient* in that it efficiently uses the data. An MVU estimator may or may not be efficient.

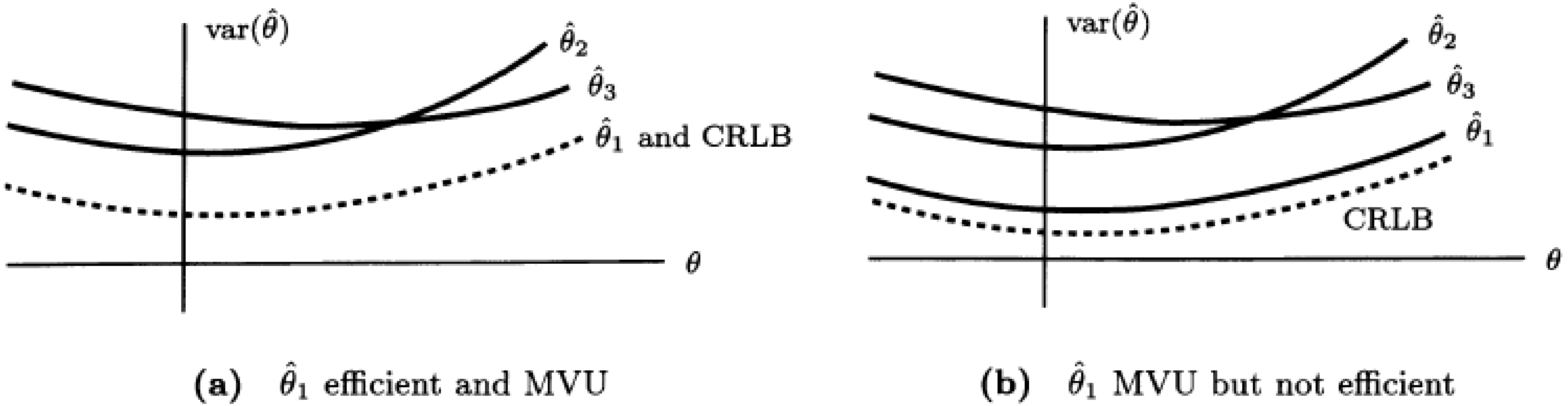


Figure 3.2 Efficiency vs. minimum variance

Efficient Estimator: an unbiased estimator that attains CRLB, meaning its variance is the smallest theoretically achievable, hence it is called "efficient".

MVU estimator: the estimator with the smallest variance among all unbiased estimators, but it may not necessarily attain the CRLB, and thus may not be efficient.

3.5 General CRLB for Signals in White Gaussian Noise

Assume that a deterministic signal with an unknown parameter θ is observed in WGN as

$$x[n] = s[n;\theta] + w[n], \quad n = 0,1,\ldots,N-1$$

The likelihood function is

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-s[n;\theta]\right)^2\right\}$$

$$\frac{\partial \ln p(x;\theta)}{\partial\theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(x[n]-s[n;\theta]\right)\frac{\partial s[n;\theta]}{\partial\theta}$$

$$\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left\{\left(x[n]-s[n;\theta]\right)\frac{\partial^2 s[n;\theta]}{\partial\theta^2} - \left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2\right\}$$

$$E\left(\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right) = -\frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2$$

$$\mathrm{var}(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2}$$

This form of the bound demonstrates the importance of the signal dependence on θ. Signals that change rapidly as the unknown parameter changes result in accurate estimators.

The CRLB may also be expressed in a slightly different form. Since

$$E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}\right]$$ (see Appendix 3A)

we have

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial \theta}\right)^2\right]}$$ (see Problem 3.8)

Fisher information

$$I(\theta) = E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial\theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right]$$

$$\frac{\partial \ln p(x;\theta)}{\partial\theta} = \frac{1}{p(x;\theta)}\frac{\partial p(x;\theta)}{\partial\theta}$$

$$\int p(x;\theta)\,dx = 1 \qquad \int \frac{\partial p(x;\theta)}{\partial\theta}\,dx = 0 \qquad E\left[\frac{\partial \ln p(x;\theta)}{\partial\theta}\right] = \int \underline{\frac{\partial \ln p(x;\theta)}{\partial\theta}p(x;\theta)}\,dx = 0$$

"regularity" condition

$$I(\theta) = E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial\theta}\right)^2\right] = \int \left(\frac{\partial \ln p(x;\theta)}{\partial\theta}\right)^2 p(x;\theta)\,dx = \int \frac{\partial \ln p(x;\theta)}{\partial\theta}\,\underline{\frac{\partial \ln p(x;\theta)}{\partial\theta}p(x;\theta)}\,dx$$

$$I(\theta) = \int \frac{\partial \ln p(x;\theta)}{\partial\theta}\,\frac{\partial p(x;\theta)}{\partial\theta}\,dx$$

Fisher information

$$I(\theta) = \int \underline{\frac{\partial \ln p(x;\theta)}{\partial \theta} \frac{\partial p(x;\theta)}{\partial \theta}}\, dx$$

$$\frac{\partial}{\partial \theta}\left[p(x;\theta)\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] = \underline{\frac{\partial \ln p(x;\theta)}{\partial \theta} \frac{\partial p(x;\theta)}{\partial \theta}} + \frac{\partial^2 \ln p(x;\theta)}{\partial^2 \theta} p(x;\theta)$$

$$I(\theta) = \int \frac{\partial}{\partial \theta}\left[p(x;\theta)\frac{\partial \ln p(x;\theta)}{\partial \theta}\right] dx - \int \frac{\partial^2 \ln p(x;\theta)}{\partial^2 \theta} p(x;\theta)\, dx$$

$$\frac{\partial \ln p(x;\theta)}{\partial \theta} = \frac{1}{p(x;\theta)} \frac{\partial p(x;\theta)}{\partial \theta}$$

$$I(\theta) = \int \frac{\partial}{\partial \theta}\left(\frac{\partial p(x;\theta)}{\partial \theta}\right) dx - E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial^2 \theta}\right]$$

Fisher information

$$I(\theta) = \int \frac{\partial}{\partial\theta}\left(\frac{\partial p(x;\theta)}{\partial\theta}\right) dx - E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial^2\theta}\right]$$

$$I(\theta) = \frac{\partial}{\partial\theta}\int \frac{\partial p(x;\theta)}{\partial\theta}\, dx - E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial^2\theta}\right]$$

$$\int \frac{\partial p(x;\theta)}{\partial\theta}\, dx = 0 \qquad I(\theta) = E\left[\left(\frac{\partial \ln p(x;\theta)}{\partial\theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right]$$

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta^2}\right]} = \frac{1}{I(\theta)}$$