HW6

$A \in H_n$, then $A \geq 0 \overset{?}{\Leftrightarrow}$ all its leading principle minors are nonnegative.

Sol. $A \geq 0 \Rightarrow$ leading principle minors nonnegative

Suppose $A = [a_{ij}]$, $A_k = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$; $k = 1, \cdots, n$

For any $y \in \mathbb{C}^k$, construct $x \in \mathbb{C}^n$ as $x = \begin{bmatrix} y \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ by adding n-k zeros.

$A \geq 0 \Rightarrow$ ~~$\forall x$~~ $\forall y$, $x^* A x \geq 0$

$x^* A x = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{x_i} a_{ij} x_j = \sum_{i=1}^{k} \sum_{j=1}^{k} \overline{y_i} a_{ij} y_j = y^* A_k y$

We get $\forall y$, $y^* A_k y \geq 0$, so A_k ~~$\geq$~~ ≥ 0, and $k = 1, \cdots, n$

Hence all of A's leading principle minors are nonnegative.

$A \geq 0 \not\Leftarrow$ leading principle minors nonnegative

Suppose $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. $D_1 = |a_{11}| = 0$, $D_2 = \det A = 0$, A's leading principle minors are nonnegative.

But when $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x^* A x = -1 < 0$, A is not ~~semi~~ positive semi-definite.