1. (a) Let $\lambda$ be an eigenvalue of $A$. Show that $|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|$ for some index $i$. (This is Gershgorin's theorem).

*Proof.* Let $Ax = \lambda x$, $x \neq 0$, pick index $k$ s.t. $|x_k| = \max_i |x_i|$.
Consider the $k$-th row of the $Ax = \lambda x$:

$$\sum_{j=1}^{n} a_{kj} x_j = \lambda x_k \implies (\lambda - a_{kk}) x_k = \sum_{j \neq k} a_{kj} x_j$$

Taking absolute values and using the triangle inequality, we have:

$$|\lambda - a_{kk}||x_k| = \left| \sum_{j \neq k} a_{kj} x_j \right| \leq \sum_{j \neq k} |a_{kj}||x_j| \leq \sum_{j \neq k} |a_{kj}||x_k|$$

$$|\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$$

□

(b) Show that if $A$ is strictly diagonally dominant (that is, $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all $i$ ), then $A$ is invertible.

*Proof.* If $Ax = 0, x \neq 0$, pick index $k$ s.t. $|x_k| = \max_i |x_i|$. Then,

$$|a_{kk}||x_k| \leq \sum_{j \neq k} |a_{kj}||x_j| \leq \Big( \sum_{j \neq k} |a_{kj}| \Big) |x_k|$$

$$\implies |a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$$

contradicting the strictly diagonally dominant condition. Thus, $Ax = 0 \Rightarrow x = 0$, so $A$ is invertible. □

2. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(a) What does Gershgorin's theorem say about the e-values of $A$ ?
For the first row, we have $|\lambda - 2| \leq 1$, i.e. $\lambda \in [1, 3]$.
For the second row, we have $|\lambda - 2| \leq 2$, i.e. $\lambda \in [0, 4]$.
So, the union of all Gershgorin discs is $[0, 4]$. Thus, all eigenvalues of $A$ lie in $[0, 4]$.

(b) Let $\boldsymbol{e}_i$ be the standard unit vector in the $i$ th direction. Calculate the Rayleigh quotients of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1+\boldsymbol{e}_3, \boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4, \boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4$. Use this information to estimate $\lambda_1$ and $\lambda_4$, the smallest and the largest e-values of $A$.

$$
\begin{aligned}
R_A(\boldsymbol{e}_1) &= \frac{\boldsymbol{e}_1^T A \boldsymbol{e}_1}{\boldsymbol{e}_1^T \boldsymbol{e}_1} = 2 \\
R_A(\boldsymbol{e}_2) &= \frac{\boldsymbol{e}_2^T A \boldsymbol{e}_2}{\boldsymbol{e}_2^T \boldsymbol{e}_2} = 2 \\
R_A(\boldsymbol{e}_1+\boldsymbol{e}_3) &= \frac{(\boldsymbol{e}_1+\boldsymbol{e}_3)^T A (\boldsymbol{e}_1+\boldsymbol{e}_3)}{(\boldsymbol{e}_1+\boldsymbol{e}_3)^T (\boldsymbol{e}_1+\boldsymbol{e}_3)} = \frac{4}{2} = 2 \\
R_A(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4) &= \frac{(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)^T A (\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)}{(\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)^T (\boldsymbol{e}_1-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4)} = \frac{14}{4} = 3.5 \\
R_A(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4) &= \frac{(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)^T A (\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)}{(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)^T (\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)} = \frac{2}{4} = 0.5
\end{aligned}
$$

Thus, we estimate

$$0 < \lambda_1 \leq 0.5, \quad 3.5 \leq \lambda_4 \leq 4$$

.

(c) Find $\|A\|_1, \|A\|_2, \|A\|_\infty, \|A\|_E, \rho(A), \operatorname{tr}(A), \kappa_2(A)$.

$$
\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq 4} \sum_{i=1}^{4} |a_{ij}| = 4 \\
\|A\|_\infty &= \max_{1\leq i\leq 4} \sum_{j=1}^{4} |a_{ij}| = 4 \\
\|A\|_E &= \sqrt{\sum_{i,j=1}^{4} |a_{ij}|^2} = \sqrt{22} \\
\operatorname{tr}(A) &= \sum_{i=1}^{4} a_{ii} = 8 \\
\rho(A) &= \max_{1\leq i\leq 4} |\lambda_i| = \frac{5+\sqrt{5}}{2} \\
\|A\|_2 &= \sqrt{\rho\left(A^T A\right)} = \frac{5+\sqrt{5}}{2} \\
\kappa_2(A) &= \|A\|_2 \left\|A^{-1}\right\|_2 = \frac{\frac{5+\sqrt{5}}{2}}{\frac{3-\sqrt{5}}{2}} = 5+2\sqrt{5}
\end{aligned}
$$

3. (a) Suppose $A\boldsymbol{x} = \boldsymbol{b}$. Let $\widetilde{\boldsymbol{x}}$ be an approximation of the exact solution $\boldsymbol{x}$. The error is defined by $\boldsymbol{e} = \boldsymbol{x} - \widetilde{\boldsymbol{x}}$, and the residual is $\boldsymbol{r} = \boldsymbol{b} - A\widetilde{\boldsymbol{x}}$. Show that $A\boldsymbol{e} = \boldsymbol{r}$ and $\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{x}\|} \leq \kappa(A)\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}$.

*Proof.*

$$A\tilde{\boldsymbol{x}} = \boldsymbol{b} - \boldsymbol{r}, \qquad A\boldsymbol{x} = \boldsymbol{b} \;\Rightarrow\; A(\boldsymbol{x} - \tilde{\boldsymbol{x}}) = \boldsymbol{r} \;\Rightarrow\; A\boldsymbol{e} = \boldsymbol{r}.$$

$$\|\boldsymbol{e}\| = \|A^{-1}\boldsymbol{r}\| \le \|A^{-1}\|\,\|\boldsymbol{r}\|, \quad \|\boldsymbol{b}\| = \|A\boldsymbol{x}\| \le \|A\|\,\|\boldsymbol{x}\| \Rightarrow \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{x}\|} \le \|A\|\,\|A^{-1}\|\,\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} = \kappa(A)\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}.$$

□

(b) It follows that if $A$ is invertible, then $\boldsymbol{e} = \boldsymbol{0}$ if and only if $\boldsymbol{r} = \boldsymbol{0}$. However, if $A$ is ill-conditioned, the error $\boldsymbol{e}$ may be large even though the residual $\boldsymbol{r}$ is small. This occurs in the example below. Show that $A\boldsymbol{x} = \boldsymbol{b}$. Find $\|\boldsymbol{e}\|_\infty$ and $\|\boldsymbol{r}\|_\infty$ for the vectors $\widetilde{\boldsymbol{x}}_1, \widetilde{\boldsymbol{x}}_2$. Find $\kappa_\infty(A)$. (Use exact arithmetics). $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \boldsymbol{b} = \binom{0.8642}{0.1440}, \boldsymbol{x} = \binom{2}{-2}, \widetilde{\boldsymbol{x}}_1 = \binom{0}{1}, \widetilde{\boldsymbol{x}}_2 = \binom{0.9911}{-0.4870}$

**Solution:**

$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

$$A\boldsymbol{x} = \begin{pmatrix} 1.2969 \cdot 2 + 0.8648(-2) \\ 0.2161 \cdot 2 + 0.1441(-2) \end{pmatrix} = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix} = \boldsymbol{b}.$$

**For** $\tilde{\boldsymbol{x}}_1 = \binom{0}{1}$:

$$\boldsymbol{e}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \;\; \|\boldsymbol{e}_1\|_\infty = 3, \qquad A\tilde{\boldsymbol{x}}_1 = \begin{pmatrix} 0.8648 \\ 0.1441 \end{pmatrix}, \; \boldsymbol{r}_1 = \begin{pmatrix} -0.0006 \\ -0.0001 \end{pmatrix}, \;\; \|\boldsymbol{r}_1\|_\infty = 6 \times 10^{-4}.$$

**For** $\tilde{\boldsymbol{x}}_2 = \binom{0.9911}{-0.4870}$:

$$\boldsymbol{e}_2 = \begin{pmatrix} 1.0089 \\ -1.5130 \end{pmatrix}, \;\; \|\boldsymbol{e}_2\|_\infty = 1.5130,$$

$$A\tilde{\boldsymbol{x}}_2 = \begin{pmatrix} 0.86419999 \\ 0.14400001 \end{pmatrix}, \; \boldsymbol{r}_2 = \begin{pmatrix} 10^{-8} \\ -10^{-8} \end{pmatrix}, \;\; \|\boldsymbol{r}_2\|_\infty = 10^{-8}.$$

**Condition number** $\kappa_\infty(A)$.

$$\|A\|_\infty = \max\{1.2969 + 0.8648,\; 0.2161 + 0.1441\} = 2.1617,$$

$$\det(A) = 1.2969 \cdot 0.1441 - 0.8648 \cdot 0.2161 = 10^{-8},$$

$$A^{-1} = \frac{1}{10^{-8}} \begin{pmatrix} 0.1441 & -0.8648 \\ -0.2161 & 1.2969 \end{pmatrix}, \quad \|A^{-1}\|_\infty = \max\{1.0089, 1.5130\} \times 10^8 = 1.5130 \times 10^8,$$

$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 2.1617 \cdot 1.5130 \times 10^8 = 3.27065210 \times 10^8.$$

4. Compute the solution of the $2 \times 2$ system:

$$\begin{aligned} 10^{-3}x + y &= 5 \\ x - y &= 6 \end{aligned}$$

using standard Gaussian elimination and Gaussian elimination with pivoting. Conduct all computations with two significant digits (decimal). Compare and explain the results.

**(A) Standard Gaussian elimination (no pivoting).**

Start with (to two sig. figs.)

$$\left[\,1.0\times10^{-3}\;\;1.0\;\;\middle|\;\;5.0\,\right],\qquad\left[\,1.0\;\;-1.0\;\;\middle|\;\;6.0\,\right].$$

Eliminate $a_{21}$ using multiplier $m=\dfrac{1.0}{1.0\times10^{-3}}=1.0\times10^{3}$.

$$(1.0\times10^{3})\cdot\mathrm{row}_1=\left[\,1.0\;\;1.0\times10^{3}\;\;\middle|\;\;5.0\times10^{3}\,\right].$$

Row operation (to two sig. figs.):

$$\mathrm{row}_2\leftarrow\mathrm{row}_2-(1.0\times10^{3})\cdot\mathrm{row}_1\Rightarrow\left[\,0.0\;\;-1.0\times10^{3}\;\;\middle|\;\;-5.0\times10^{3}\,\right].$$

Thus $-1.0\times10^{3}\,y=-5.0\times10^{3}\Rightarrow y=5.0$. Back–substitute into the first equation:

$$1.0\times10^{-3}x+5.0=5.0\;\Rightarrow\;1.0\times10^{-3}x=0.0\;\Rightarrow\;x=0.0.$$

$$\boxed{(x,y)_{\text{no pivot}}=(0.0,\,5.0)}.$$

**(B) Gaussian elimination with partial pivoting.**

Swap the rows so the first pivot is 1.0:

$$\left[\,1.0\;\;-1.0\;\;\middle|\;\;6.0\,\right],\qquad\left[\,1.0\times10^{-3}\;\;1.0\;\;\middle|\;\;5.0\,\right].$$

Eliminate $a_{21}$ with $m=\dfrac{1.0\times10^{-3}}{1.0}=1.0\times10^{-3}$:

$$(1.0\times10^{-3})\cdot\mathrm{row}_1=\left[\,1.0\times10^{-3}\;\;-1.0\times10^{-3}\;\;\middle|\;\;6.0\times10^{-3}\,\right].$$

Row operation (rounded to two sig. figs.):

$$\mathrm{row}_2\leftarrow\mathrm{row}_2-(1.0\times10^{-3})\cdot\mathrm{row}_1\Rightarrow\left[\,0.0\;\;1.0\;\;\middle|\;\;5.0\,\right],$$

so $y=5.0$. Back–substitute in the first (pivoted) row:

$$x-y=6.0\;\Rightarrow\;x=11.0.$$

$$\boxed{(x,y)_{\text{pivot}}=(11.0,\,5.0)}.$$

**Comparison and explanation.** The exact solution is $(x,y)=(10.989\ldots,\,4.989\ldots)$. Without pivoting the tiny pivot $10^{-3}$ produces a huge multiplier $10^{3}$, forcing subtraction of nearly equal large numbers and catastrophic cancellation, yielding $x=0.0$. Pivoting selects the well–scaled pivot 1.0, avoids amplification of rounding, and returns the correct two–digit result (11.0, 5.0).

5. (a) Show that for any matrix $A$, there exists a permutation matrix $P$ such that $PA=LU$, where $L$ is a unit lower triangular matrix and $U$ is an upper triangular matrix.

   Hint: $P$ is a composition, that is, a product of all permutation matrices used at every stage of the Gaussian elimination algorithm.

*Proof.* Let Gaussian elimination with (partial) pivoting act on $A$. At step $k$ a permutation $P_k$ swaps rows to place a nonzero pivot and a unit lower–triangular elimination matrix $E_k$ zeroes entries below it. After $m$ steps,

$$P_m \cdots P_1 \, A = LU, \qquad E_m \cdots E_1 \, P_m \cdots P_1 \, A = U.$$

Set $P = P_m \cdots P_1$ and $L = (E_m \cdots E_1)^{-1}$ (unit lower triangular). Then $PA = LU$. □

(b) For the following matrix $A$, find $P, L$ and $U$. Use the $LU$ factorization to solve $A\boldsymbol{x} = \boldsymbol{b}$.

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

For $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, partial pivoting on column 1 then column 2 gives

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{so } PA = LU.$$

Solve $A\boldsymbol{x} = \boldsymbol{b}$ via $LU\boldsymbol{x} = P\boldsymbol{b}$:

$$P\boldsymbol{b} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad L\boldsymbol{y} = P\boldsymbol{b} \Rightarrow \boldsymbol{y} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}, \quad U\boldsymbol{x} = \boldsymbol{y} \Rightarrow \boldsymbol{x} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

6. Let $M$ be an invertible matrix. Show that $\|\boldsymbol{x}\|_M := \|M\boldsymbol{x}\|_\infty$ defines a vector norm for which the subordinate matrix norm is $\|A\|_M := \|MAM^{-1}\|_\infty$.

*Proof.* **1) $\|\cdot\|_M$ is a norm.**

- (Positivity) $\|x\|_M = \|Mx\|_\infty \geq 0$, and $\|x\|_M = 0 \iff Mx = 0 \iff x = 0$ (since $M$ is invertible).
- (Absolute homogeneity) $\|\alpha x\|_M = \|M(\alpha x)\|_\infty = |\alpha| \, \|Mx\|_\infty = |\alpha| \, \|x\|_M$.
- (Triangle inequality) $\|x + y\|_M = \|M(x + y)\|_\infty \leq \|Mx\|_\infty + \|My\|_\infty = \|x\|_M + \|y\|_M$, using the triangle inequality of $\|\cdot\|_\infty$.

**2) Subordinate matrix norm.** Define the operator norm subordinate to $\|\cdot\|_M$ by

$$\|A\|_M := \sup_{x \neq 0} \frac{\|Ax\|_M}{\|x\|_M} = \sup_{x \neq 0} \frac{\|MAx\|_\infty}{\|Mx\|_\infty}.$$

Let $y = Mx$ (bijection since $M$ is invertible). Then

$$\|A\|_M = \sup_{y \neq 0} \frac{\|MAM^{-1}y\|_\infty}{\|y\|_\infty} = \|MAM^{-1}\|_\infty.$$

Hence the subordinate matrix norm to $\|\cdot\|_M$ is exactly $\|A\|_M := \|MAM^{-1}\|_\infty$. □

7. Consider the 2 -point boundary value problem

$$-\phi''(x) + x^2\phi(x) = \left(1 + 4x + 2x^2 - x^4\right) e^x, \quad \phi(0) = 1, \phi(1) = 0.$$

Check that the solution is $\phi(x) = (1 - x^2)\, e^x$. Write a computer program to solve this problem using the second-order finite-difference scheme discussed in class. Solve the tridiagonal system using the LU factorization. Run the code with mesh size $h = 1/2^p$ for $p = 1, \ldots, 14$. Output the results in the following format,

$$\begin{aligned} &\text{column1: } h \\ &\text{column2: } \|u_h - \phi_h\|_\infty \\ &\text{column3: } \|u_h - \phi_h\|_\infty / h^2 \end{aligned}$$

Describe and explain what happens to the error for small values of $h$.

**Model problem and exact solution.**

$$-\phi''(x) + x^2\phi(x) = (1 + 4x + 2x^2 - x^4)e^x, \quad \phi(0) = 1,\ \phi(1) = 0, \quad \phi(x) = (1 - x^2)e^x.$$

**Second-order FD scheme (interior nodes $x_i = ih$, $i = 1, \ldots, n$).**

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + x_i^2 u_i = f(x_i), \qquad u_0 = 1,\ u_{n+1} = 0,$$

$$A = \text{tridiag}\Big( -\tfrac{1}{h^2},\ \tfrac{2}{h^2} + x_i^2,\ -\tfrac{1}{h^2} \Big), \quad b_i = f(x_i) + \tfrac{1}{h^2}\mathbf{1}_{\{i=1\}}.$$

**Solve by LU.** Compute $[L, U, P] = \text{lu}(A)$, then $U\, u = L^{-1} P\, b$. Form $u_h = [1;\, u;\, 0]$ and the exact vector $\phi_h = \phi(x)$. Report:

| $h$ | $\|u_h - \phi_h\|_\infty$ | $\|u_h - \phi_h\|_\infty / h^2$ |
|---|---|---|
| 5.000000e−01 | 6.472527e−02 | 2.589011e−01 |
| 2.500000e−01 | 1.651983e−02 | 2.643173e−01 |
| 1.250000e−01 | 4.154789e−03 | 2.659065e−01 |
| 6.250000e−02 | 1.056844e−03 | 2.705520e−01 |
| 3.125000e−02 | 2.643184e−04 | 2.706620e−01 |
| 1.562500e−02 | 6.608632e−05 | 2.706896e−01 |
| 7.812500e−03 | 1.652200e−05 | 2.706965e−01 |
| 3.906250e−03 | 4.130527e−06 | 2.706982e−01 |
| 1.953125e−03 | 1.032633e−06 | 2.706986e−01 |
| 9.765625e−04 | 2.581585e−07 | 2.706988e−01 |
| 4.882812e−04 | 6.454092e−08 | 2.707042e−01 |
| 2.441406e−04 | 1.613556e−08 | 2.707097e−01 |
| 1.220703e−04 | 3.081321e−09 | 2.067840e−01 |
| 6.103516e−05 | 2.814358e−10 | 7.554734e−02 |

**Behavior for small $h$:** For $h$ down to $2.44 \times 10^{-4}$, the ratio $\|u_h - \phi_h\|_\infty / h^2$ stays essentially constant at $\approx 2.707 \times 10^{-1}$, confirming the expected second–order convergence:

$$\|u_h - \phi_h\|_\infty \approx \alpha\, h^2, \qquad \alpha \approx 2.707 \times 10^{-1}.$$

For the last two meshes, $h = 1.22 \times 10^{-4}$ and $6.10 \times 10^{-5}$, the ratio *drops* (to $2.07 \times 10^{-1}$ and $7.55 \times 10^{-2}$). This is a finite–precision effect: the total error is well modeled by

$$E(h) \approx \underbrace{\alpha h^2}_{\text{truncation}} + \underbrace{\beta\, \varepsilon_{\text{mach}}\, h^{-2}}_{\text{roundoff/backward error}},$$

with $\varepsilon_{\text{mach}} \approx 2.22 \times 10^{-16}$ in double precision. When $h$ is small, the two terms become comparable; if $\beta$ has opposite sign to $\alpha$, they partially cancel, producing the observed dip in $E(h)/h^2$. If you refine further, the roundoff term will eventually dominate and $E(h)$ will stagnate and then *increase* (so $E(h)/h^2$ rises), as the LU solve backward error scales like $O(\varepsilon_{\text{mach}}\|A\|) \sim O(\varepsilon_{\text{mach}}\, h^{-2})$.

**Summary.**

$$\begin{cases} \text{Second–order regime:} & h \gtrsim 2.4 \times 10^{-4}, \quad E(h)/h^2 \approx 0.2707, \\ \text{Roundoff interaction:} & h \lesssim 1.2 \times 10^{-4}, \quad \text{favorable cancellation lowers } E(h)/h^2, \\ \text{Prediction:} & \text{for even smaller } h, \text{ roundoff dominates } \Rightarrow E(h) \uparrow . \end{cases}$$