MA113 线性代数真题手册

期中考试

24 Spring/23 Fall/23 Spring

前言

妮可 MA113 线性代数 **24 Spring/23 Fall/23 Spring** 期中试题
强烈建议扫描下方二维码反馈意见与建议。此问卷将长期有效。

目录（试题/答案）

- 24 Spring Midterm Question 01/65
- 24 Spring Midterm Answer 20/65
- 23 Fall Midterm Question 27/65
- 23 Fall Midterm Answer 43/65
- 23 Spring Midterm Question 49/65
- 23 Spring Midterm Answer 63/65

24 Spring Midterm Question

24 Spring Midterm Question 中英试题分离 分页留空版本
线性代数2023-2024学年春季学期期中考试

---

1. (共15分，每小题3分)选择题，只有一个选项是正确的.

(1) 假定

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

若$\alpha_1, \alpha_2, \alpha_3$线性相关，则$c$的取值为

$(A)5.$
$(B)6.$
$(C)7.$
$(D)8.$

(2) 设 A 为 一 个 $m \times n$ 实 矩 阵 , b 为 一 个 m 维 实 列 向 量 , 以 下 说 法 一 定 是 **正 确** 的 是 ?

(A)若$A\mathbf{x} = \mathbf{b}$无解，则$A\mathbf{x} = \mathbf{0}$只有零解.
(B)若$A\mathbf{x} = \mathbf{0}$有无穷多解，则$A\mathbf{x} = \mathbf{b}$有无穷多解。
(C)若$m < n$,则$A\mathbf{x} = \mathbf{b}$和$A\mathbf{x} = \mathbf{0}$都有无穷多解。
(D)若A的秩为n,则$A\mathbf{x} = 0$只有零解.

(3) 如果以下线性方程组有两个自由变量

$$\begin{cases} x_1+2x_2-4x_3+3x_4=0, \\ x_1+3x_2-2x_3-2x_4=0, \\ x_1+5x_2+(5-k)x_3-12x_4=0, \end{cases}$$

k的取值为

$(A)5.$
$(B)4.$
$(C)3.$
$(D)2.$

(4) 设 $u, v \in \mathbb{R}^3, \lambda \in \mathbb{R}$. 以下说法**错误**的是?

(A)如果u和v为满足$u^T v=0$的非零向量，则u和v线性无关.
(B)如果$u+v$和$u-v$正交，则$||u||=||v||$.
$(C)u^T v=0$当且仅当 $u=0$ or $v=0$.
$(D)\lambda v=0$当且仅当 $v=0$ or $\lambda=0$.

(5) 设A和B都为n阶矩阵.以下说法**错误**的是?

(A)如果A, B为对称矩阵，则AB也为一个对称矩阵。
(B)如果A, B 为可逆矩阵，则 AB 也为一个可逆矩阵.
(C)如果A, B为置换矩阵，则AB也为一个置换矩阵.
(D)如果A, B为上三角矩阵，则AB也为上三角矩阵。

2. (20 points, 5 points each) 填空，共4题。

(1)$A=\begin{bmatrix}1&0&0\\a&1&0\\b&3&2\end{bmatrix},a,b\in\mathbb{R},$则$A^{-1}=$______.

(2)设A为一个4×3的实矩阵，B为$\begin{bmatrix}1&0&0\\0&2&0\\-1&0&3\end{bmatrix}$.

如果矩阵A的秩为 2,则AB的秩为______.

(3)设 $A=\begin{bmatrix}1&-1&1\\-1&1&-1\\2&-2&2\end{bmatrix}$,则 $A^{2024}=$______.

(4)考虑以下线性方程组：

$$A\mathbf{x}=\mathbf{b}:\begin{cases}x&=&2\\y&=&3\\x+y&=&6\end{cases}$$

该线性方程组的最小二乘解为______.

3. (10points)设

$$A=\begin{bmatrix}1 & -2 & 3\\ 2 & -5 & 1\\ 1 & -4 & -7\end{bmatrix}.$$

求矩阵A的一个LU分解

4. 考虑以下 4× 5 矩阵 A 以及 4 维列向量 $\mathbf{b}$:

$$A=\begin{bmatrix}0&2&4&1&6\\0&1&1&1&3\\0&4&10&1&2\\0&-1&-5&1&7\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}3\\2\\-5\\10\end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量。

(b)求$Ax=b$的所有解.

5. (20 points)设 Let $A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$,$T$为按照以下方式定义的从$\mathbb{R}^{2\times 2}$到 $\mathbb{R}^{2\times 2}$线性变换:

$$T\left(X\right)=XA+AX, X\in\mathbb{R}^{2\times 2}.$$

其中$\mathbb{R}^{2\times 2}$表示所有2×2实矩阵构成的向量空间.

(a)求T在以下有序基

$$v_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, v_3=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

下的矩阵表示.

(b)求一个矩阵B使得

$$T(B)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

(c)求一个矩阵C使得

$$T\left(C\right)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}.$$

6. (5 points) 设A, B为满足$A^2 = A$和$B^2 = B$的n阶实矩阵.证明：如果$(A+B)^2 = A + B$,则$AB = O$.其中O 表示n阶零矩阵。

7. (6 points)设A为 3$\times$2矩阵，B 为 2$\times$3矩阵，并且

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a)计算$(AB)^2$.

(b)求BA.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共15分，每小题3分)选择题，只有一个选项是正确的.

$(1) Let$

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent, then c equals
$(A) 5.$
$(B) 6.$
$(C) 7.$
$(D) 8.$

(2) let A be an $m \times n$ real matrix and b be an $m \times 1$ real column vector. Which of the following statements is correct?
(A) If $A\mathbf{x} = \mathbf{b}$ does not have any solution, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.
(B) If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
(C) If $m < n$, both $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have infinitely many solutions.
(D) If the rank of A is n, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(3) For which value of k does the system

$$\begin{cases} x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\ x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\ x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0, \end{cases}$$

have exactly two free variables?

$(A)5.$

$(B)4.$

$(C)3.$

$(D)2.$

(4) Let $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Which of the following statements is false?

(A) If u and v are nonzero vectors satisfying $u^T v = 0$, then u and v are linearly independent.

(B)If $u+v$ is orthogonal to $u-v$, then $||u|| = ||v||$.

$(C)u^T v = 0$ if and only if $u = 0$ or $v = 0$.

$(D)\lambda v = 0$ if and only if $v = 0$ or $\lambda = 0$.

(5) Let A and B be two $n \times n$ matrices. Which of the following assertions is **false**?

(A) If A, B are symmetric matrices, then AB is a symmetric matrix.

(B) If A, B are invertible matrices, then AB is an invertible matrix.

(C) If A, B are permutation matrices, then AB is a permutation matrix.

(D) If A, B are upper triangular matrices, then AB is an upper triangular matrix.

2. (20 points, 5 points each) Fill in the blanks.

(1)Let $A=\begin{bmatrix}1&0&0\\a&1&0\\b&3&2\end{bmatrix}, a,b\in\mathbb{R}$. Then $A^{-1}=$ ______.

(2) Let A be a 4×3 real matrix with rank 2 and $B=\begin{bmatrix}1&0&2\\0&2&0\\-1&0&3\end{bmatrix}$.

Then the rank AB is ______.

(3)Let $A=\begin{bmatrix}1&-1&1\\-1&1&-1\\2&-2&2\end{bmatrix}$. Then $A^{2024}=$ ______.

(4) Consider the system of linear equations:

$$Ax=b:\begin{cases}x&=&2\\y&=&3\\x+y&=&6\end{cases}$$

The least-squares solution for the system is ______.

3. (10points)Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}.$$

Find an LU factorization of A.

4. (24 points) Consider the following 4×5 matrix A and 4-dimensional column vector b:

$$A = \begin{bmatrix} 0 & 2 & 4 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 4 & 10 & 1 & 2 \\ 0 & -1 & -5 & 1 & 7 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 10 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

5. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and T be the linear transformation from $R^{2\times 2}$ to $R^{2\times 2}$ defined by

$$T(X) = XA + AX,\ X \in \mathbb{R}^{2\times 2}.$$

Where $\mathbb{R}^{2\times 2}$ denotes the vector space consisting of all 2×2 real matrices.

(a) Find the matrix representation of T with respect to the following ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$(\mathbf{b})$ Find a matrix B such that

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Find a matrix C such that

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) Let A, B be two $n \times n$ real matrices satisfying $A^2 = A$ and $B^2 = B$. Show that if $(A+B)^2 = A+B$, then $AB = O$. Where O denotes the $n \times n$ zero matrix.

7. (6 points) Let A be a 3×2 matrix, B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a)Compute$(AB)^2$.
(b) Find BA.

24 Spring Midterm Answer

线性代数2023-2024学年春季学期期中考试

快速对答案（详解在之后）

一、D D C C A

二、

$$(1)A^{-1}=\begin{bmatrix}1 & & \\ -a & 1 & \\ (3a-b)/2 & -3/2 & 1/2\end{bmatrix} \qquad (2)2$$

$$(3)4^{2023}A=4^{2023}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 2 & -2 & 2\end{bmatrix} \qquad (4)\begin{bmatrix}\frac{7}{3}\\ \frac{10}{3}\end{bmatrix}$$

三、$A=LU=\begin{bmatrix}1 & & \\ 2 & 1 & \\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ & -1 & -5\\ & & 0\end{bmatrix}$

四、(a)

$$(1)C(A)=span\left\{\begin{bmatrix}2\\1\\4\\-1\end{bmatrix},\begin{bmatrix}4\\1\\10\\-5\end{bmatrix},\begin{bmatrix}6\\3\\2\\7\end{bmatrix}\right\}$$

$$(2)\, C\left(A^T\right) = \text{Span}\left[\begin{bmatrix}0\\2\\4\\1\\6\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\4\\10\\1\\2\end{bmatrix}\right]$$

$$(3)\, X = K_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + K_4\begin{bmatrix}0\\-\frac{3}{2}\\\frac{1}{2}\\1\\0\end{bmatrix}(k_1, k_2 \in \mathbb{R})$$

$$(4) y = k_0\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}, k_0 \in R$$

(b)

$$x = x_p + x_n = \begin{bmatrix}1\\-5\\1\\3\\1\end{bmatrix} + k_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + k_4\begin{bmatrix}0\\-3/2\\1/2\\1\\0\end{bmatrix} \quad k_1, k_4 \in \mathbb{R}$$

五、略，解析部分有方法

六、证明略

七、

$$(AB)^2 = \begin{bmatrix}72 & 0 & -36\\-\frac{27}{2} & 81 & -54\\-18 & 0 & 9\end{bmatrix}$$

$$BA = 9I$$

填空及大题详解

二、(1)

$$[A\quad I]=\begin{bmatrix}1 & & & \vdots & 1 & & \\ a & 1 & & \vdots & & 1 & \\ b & 3 & 2 & \vdots & & & 1\end{bmatrix}\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 3 & 2 & \vdots & -b & 0 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ 0 & 1 & & \vdots & -a & 1 & \\ 0 & 0 & 2 & \vdots & 3a-b & -3 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}1 & & & \vdots & 1 & & \\ & 1 & & \vdots & -a & 1 & \\ & & 1 & \vdots & (3a-b)/2 & -3/2 & 1/2\end{bmatrix}=[I\quad A^{-1}]$$

(2)法1令

$$A_{4\times 3}=\begin{bmatrix}1 & & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & & \\ & & 1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots\end{bmatrix}\begin{bmatrix}1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$
$$\to R(AB)=2$$

法2

$$R(A)=2, R(B)=3$$
$$R(AB) \geq R(A)+R(B)-n=2+3-3=2$$
$$R(AB) \leq min\{R(A), R(B)\}=2$$

$$\Longrightarrow R(AB)=2$$

(3)剥蒜（爆算）法 直接计算A^2, A^3的得出规律

(4)$A=\begin{bmatrix}0 & 1\\ 0 & 1\\ 1 & 1\end{bmatrix}, b=\begin{bmatrix}2\\ 3\\ 6\end{bmatrix}$

三、

$$A=\begin{bmatrix}1 & -2 & 3\\ 2 & -5 & 1\\ 4 & -4 & -7\end{bmatrix}$$
$$E_{21}(-2)A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & -5\\ 1 & -4 & -7\end{bmatrix}$$
$$E_{31}(-1)E_{21}(-2)A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & -5\\ 0 & -2 & -10\end{bmatrix}$$

$$E_{32}(-2)E_{31}(-1)E_{31}(-2)A=\begin{bmatrix}1 & -2 & -3\\ 0 & -1 & -5\\ 0 & 0 & 0\end{bmatrix}$$
$$A=E_{21}^{-1}(-2)E_{31}(-1)^{-1}E_{32}^{-1}(-2)\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & -5\\ 0 & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}1 & & \\ 2 & 1 & \\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & -5\\ 0 & 0 & 0\end{bmatrix}=LU$$

四、(a) (1)C(A),对A行变换

$$A \rightarrow \begin{bmatrix} 0 & \boxed{2} & 0 & 3 & 0 \\ 0 & 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow C(A) = span\left\{\begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 10 \\ -5 \end{bmatrix},\right.$$

$(2)C(A^T)$ 对A^T行变换

$$A^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 4 & -1 \\ 4 & 1 & 10 & -5 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & \cancel{3} & -\cancel{6} & \cancel{9} \\ 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ & \boxed{1} & -2 & 3 \\ & & \boxed{10} & 10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \Rightarrow C(A^T) = Span\left\{\begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 10 \\ 1 \\ 2 \end{bmatrix}\right\}$$

$(3)N(A)$

$$Ax = 0 \Rightarrow \begin{bmatrix} 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1, x_4 \in \mathbb{R} \\ x_2 = -\frac{3}{2}x_4 \\ x_3 = \frac{1}{2}x_4 \\ x_5 = 0 \end{cases}$$

$$\Rightarrow x = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} (k_1, k_2 \in \mathbb{R})$$

$(4) N(A^T)$

$$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & -2 & 3 \\ & & 10 & -10 \\ & & & 0 \\ & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} y_1 = -y_4 \\ y_2 = -y_4 \\ y_3 = y_4 \\ y_4 \in R \end{cases} \Rightarrow y = k_0 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, k_0 \in R$$

$(b) Ax = b$ 时 利用高斯消元化简增广矩阵(略)

令出一特解 $x_p = [1\ \ -5\ \ 1\ 3\ 1]^T$

$$x = x_p + x_n = \begin{bmatrix} 1 \\ -5 \\ 1 \\ 3 \\ 1 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 \\ -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \quad k_1, k_4 \in \mathbb{R}$$

五、令 $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\text{则 XA} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & a+2b \\ c & c+2a \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 2c & 2d \end{bmatrix}$$

$$T(x) = XA + AX = \begin{bmatrix} 2a+c & a+3b+d \\ 3c & c+4d \end{bmatrix}$$

$$= (2a+c)V_1 + (a+3b+d)V_2 + (3c)V_3 + (c+4d)V_4$$

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (V_1\ V_2\ V_3\ V_4)\begin{pmatrix} 2a+c \\ a+3b+d \\ 3c \\ c+4d \end{pmatrix}$$

六、$(A+B)^2$

$= (A+B)(A+B) = A^2 + AB + BA + B^2 = A + B$

$\because A^2 = A, B^2 = B$

$\therefore AB + BA = 0 \quad ....①$

$B(AB+BA) = BAB + B^2A = (BA)B + B^2A = -AB^2 + B^2A$

又 $\because B^2 = B$

$\therefore -AB^2 + B^2A = -AB + BA = 0 \quad ....②$

联立①② $\begin{cases} AB + BA = 0 \\ -AB + BA = 0 \end{cases} \Rightarrow AB = 0$, 得证

七、

$$(a)(AB)^2 = (AB)(AB) = \begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & 0 & -36 \\ -\frac{27}{2} & 81 & -54 \\ -18 & 0 & 9 \end{bmatrix}$$

(b)

$(AB)^2 = 9(AB)$

$R(A) \geq R(AB) = 2 \Rightarrow R(A) = 2$, 同理 $R(B) = 2$

A 行满秩, 令 $XA = I_2$, X 为A左逆

B 列满秩, 令 $BY = I_2$, Y 为B右逆

$\therefore BA = (XA)(BA)(BY) = X(AB)^2Y = 9XABY = 9I$

23 Fall Midterm Question

线代23秋期中试题 发布版 中英分离

1.(共 15 分，每小题 3 分) 选择题，只有一个选项是正确的.

(1)设$\alpha_1,\alpha_2,\alpha_3$为矩阵$A$的零空间$N(A)$的一组基. 下列哪一组向量也是矩阵A的零空间的一组基?

(A)$\alpha_1+\alpha_2-\alpha_3,\alpha_1+\alpha_2+5\alpha_3,4\alpha_1+\alpha_2-2\alpha_3.$

(B)$\alpha_1+2\alpha_2+\alpha_3,2\alpha_1+\alpha_2+2\alpha_3,\alpha_3+\alpha_1+\alpha_2,$

(C)$\alpha_1+\alpha_2,\alpha_1+\alpha_2+\alpha_3.$

(D)$\alpha_1-\alpha_2,\alpha_2-\alpha_3,\alpha_3-\alpha_1.$

(2)以下说法一定是正确的是?

(A)如果矩阵A的列向量线性无关，那么对任意的$\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$有唯一的解.

(B)任意5×7矩阵的列向量一定是线性相关的.

(C)如果矩阵A 的列向量线性相关，该矩阵的行向量也线性相关.

(D)一个10×12矩阵的行空间和列空间可能具有不同的维数

(3)设 $\alpha_1=\begin{bmatrix}1\\4\\1\end{bmatrix},\alpha_2=\begin{bmatrix}2\\1\\-5\end{bmatrix},\alpha_3=\begin{bmatrix}6\\2\\-16\end{bmatrix},\beta=\begin{bmatrix}2\\t\\3\end{bmatrix}$

当$t=()$时,β可用$\alpha_1,\alpha_2,\alpha_3$线性表示

(A)1.

(B)3.

(C)6.

(D)9.

(4) 以下说法一定是正确的事？

(A) 设E为一个可逆矩阵.如果A, B矩阵满足$EA = B$,则A和B的列空间相同

(B) 设A为秩为1的n阶的方阵, 则$A^n = cA$, 其 中 n 为 正 整 数 , c 为 实 数 .

(C) 如果A, B 为对称矩阵, 则AB为对称矩阵. 如果矩阵 A 为一个行满秩矩阵, 那么 $Ax = 0$ 只 有 零 解 .

(D) 如果矩阵A为一个列满秩矩阵，那么$Ax = 0$只有零解。

设A与B都为n阶矩阵，A为非零矩阵，且AB=0，则

(1)$BA = 0$

(2)$B = 0$

(3)$(A + B)(A - B) = A^2 - B^2$

(4)$\mathrm{rank}\ B < n$.

2.(共 25 分，每小题 5 分)填空题.

(1)记所有7×7实矩阵构成的向量空间为$M_{7\times 7}(\mathbb{R}), W$为$M_{7\times 7}(\mathbb{R})$中所有斜对称矩阵构成的子空间，则 $\dim W =$ ______.

如果$A^T = -A, A$就称之为斜对称的。

(2)设A, B为两个 可逆矩阵，假设的逆矩阵为，期中O为的零矩阵，则$D =$ ______.

(3)设 $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$且 $rank(A) < 4$，则 $a =$ ______.

(4)考虑一下线性方程组：

$$A\mathbf{x}=\mathbf{b}:\begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

该线性方程组的最小二乘解为______.

(5)设H为如下定义的一个R^3中的子空间

$$H=\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1+2x_2+x_3=0\right\}.$$

一个和子空间H正交的**单位向量**为______.

3.(24 points)考虑以下这个4×5矩阵A以及他的简化阶梯形矩阵R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \ R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)分别求矩阵 A 的四个基本子空间的一组基向量.

(b)求出矩阵 A 的第三个列向量.

4.(15 points) 设

$$A=\begin{bmatrix}1 & -1 & -1\\ 2 & a & 1\\ -1 & 1 & a\end{bmatrix},\ B=\begin{bmatrix}2 & 2\\ 1 & a\\ -a-1 & -2\end{bmatrix}.$$

当a为何值时，矩阵方程$AX=B$无解、有唯一解、有无穷多解？
在有解时，求解此方程，这里的X为一个3×2矩阵

5.（15 points）设$M_{2\times 2}(\mathbb{R})$为所有$2\times 2$实矩阵构成的向量空间，设

$$A=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ B=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

考虑以下映射

$$T: M_{2\times 2}(\mathbb{R})\to\mathbb{R}^3,\ T(X)=\begin{bmatrix}tr(A^TX)\\ tr(B^TX)\\ tr(C^TX)\end{bmatrix},$$

对任意的 2×2实矩阵, 其中 $tr(D)$ 表示n阶矩阵D的迹.

方阵D的迹是指D的对角元之和，也即

$$tr(D)=d_{11}+d_{22}+\cdots+d_{nn}$$

(a)证明T是一个线性变换
(b)求T在$M_{2\times 2}(R)$的一组基

$$v_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, v_3=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

以及R^3的标准基

$$e_1=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, e_2=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, e_3=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

下的矩阵表示.

(c)是否可以找到一个矩阵 X 使得$T(X)=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$?如果可以,请求出一个符合要求的矩阵$X$. 如果不存在,请说明理由.

6.（6 points）设A为$m \times n$矩阵，B为$m \times p$矩阵, C为$q \times p$矩阵.证明:

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

其中O为$q \times n$的零矩阵

1.(15 points, 3 points each) Multiple Choice. Only one choice is correct.

(1)Suppose that $\alpha_1, \alpha_2, \alpha_3$ are a basis for nullspace of a matrix $A, N(A)$. Which of the following lists of vectors is also a basis for $N(A)$?

(A)$\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + 5\alpha_3, 4\alpha_1 + \alpha_2 - 2\alpha_3$.

(B)$\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_3 + \alpha_1 + \alpha_2,$

(C)$\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$.

(D)$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1$.

(2)Which of the following statements is correct?

(A) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every b.

(B) Any 5×7 matrix has linearly dependent columns.

(C) If the columns of a matrix A are linearly dependent, so are the rows.

(D) The column space and row space of a 10×12 matrix may have different dimensions.

(3)Let

$$\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \ \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}.$$

β can be written as a linear combination of $\alpha_1, \alpha_2, \alpha_3$ if $t =$ ( )

(A)1.

(B)3.

(C)6.

(D)9.

(4) Which of the following statements is correct?

(A) Suppose that $EA = B$ and E is an invertible matrix, then the column space of $A$and the column space of B are the same.

(B) Let A be a $n \times n$ matrix with rank 1, then $A^n = cA$, where n is a positive integer and c is a real number.

(C) Let A, B be symmetric matrices, then AB is symmetric.

(D) If A is of full row rank, then $Ax = 0$ has only the zero solution.

(5)Let A and B be two $n \times n$ matrices. If A is a non-zero matrix and AB = 0, then

(1)$BA = 0$

(2)$B = 0$

(3)$(A+B)(A-B) = A^2 - B^2$

(4)$\text{rank } B < n$.

2.$(25 \text{ points, } 5 \text{ points each})$ Fill in the blanks.

(1)Denote the vector space of 7×7 real matrices by $M_{7\times 7}(\mathbb{R})$, and let W be the subspace of $M_{7\times 7}(\mathbb{R})$ consisting of skew-symmetric real matrices, then dim $W =$ ______.

A matrix A is called skew symmetric if $A^T = -A$.

(2)Let A, B be two $n \times n$ invertible matrices. Suppose the inverse of $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, where O is the $n \times n$ zero matrix. Then $D =$ ______.

(3)Let $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ with $rank(A) < 4$. Then $a =$ ______.

(4)Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ & & y & = & -1. \end{cases}$$

The least-squares solution for the system is______.

(5)

Let H be the subspace of R^3 be defined as follows:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 + 2x_2 + x_3 = 0 \right\}.$$

A **unit** vector orthogonal to H is ______.

3.(24 points) Consider the following 4×5 matrix A with its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \ R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.
(b) Find the third column of matrix A.

4.(15points)Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{bmatrix}, \ B = \begin{bmatrix} 2 & 2 \\ 1 & a \\ -a-1 & -2 \end{bmatrix}.$$

For which value(s) of a, the matrix equation $AX = B$ has no solution, a unique solution, or infinitely many solutions? Where X is a 3×2 matrix. Solve $AX = B$ if it has at least one solution.

5.(15 points) Let $M_2 \times 2(\mathbb{R})$ be the vector space of 2×2 real matrices. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the map

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3,\ T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

for any 2×2 real matrix X, where $tr(D)$ denotes the trace of a matrix D.

The trace of an $n \times n$ matrix D is defined to be the sum of all the diagonal entries of D, in other words,

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\text{for } M_{2\times 2}$ $(\mathbb{R})$and the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(c) Can we find a matrix X such that $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? If yes, please find one such matrix. Otherwise, give an explanation.

6.(6 points) Let A be an $m \times n$ matrix, B be an $m \times p$ matrix, and C be an $q \times p$ matrix. Show that

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

where O is the $q \times n$ zero matrix.

23 Fall Midterm Answer

线代23秋期中试题答案 发布版

Q1 (1)A (2)B（存疑, 一说D） (3)D (4)B (5)D

Q2(1)21

(2)$-A^{-1}CB^{-1}$

(3)1or-3 （存疑，一说 1 or -2）

(4)$\begin{bmatrix} -\frac{19}{11} \\ -\frac{5}{11} \end{bmatrix}$

(5)$\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ or } -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q3

A basis for $C(A)$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

A basis for $C(A^T)$

$$\left\{\begin{bmatrix}1\\0\\1\\0\\3\end{bmatrix},\begin{bmatrix}0\\1\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\right\}\text{or}\left\{\begin{bmatrix}1\\2\\5\\1\\6\end{bmatrix},\begin{bmatrix}0\\1\\2\\1\\2\end{bmatrix},\begin{bmatrix}-1\\1\\1\\3\\1\end{bmatrix}\right\}$$

A basis for $N(A)$

$$\left\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\-1\\0\\-1\end{bmatrix}\right\}.$$

A basis for $N(A^T)$

$$\left\{\begin{bmatrix}-5\\13\\-3\\1\end{bmatrix}\right\}.$$

$$(b)\ \begin{bmatrix}5\\2\\1\\2\end{bmatrix}$$

Q4

Gaussian Eliminations give:

$$\left[\begin{array}{ccc|cc}1 & -1 & -1 & 2 & 2\\ 2 & a & 1 & 1 & a\\ -1 & 1 & a & -a-1 & -2\end{array}\right]$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 & \vdots & 2 & 2 \\ 0 & a+2 & 3 & \vdots & -3 & a-4 \\ 0 & 0 & a-1 & \vdots & 0 & 0 \end{bmatrix}$$

If $a = -2$, then $rankA = 2 \neq 3 = rank(A \vdots B)$,$AX = B$ has no solution.

If $a \neq 1$ and $a \neq -1$, $AX = B$ has a unique solution.

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 2 \\ 0 & a+2 & 3 & 1 & -3 \\ 0 & 0 & a-1 & 1 & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 2 \\ 0 & a+2 & 3 & \vdots & -3 \\ 0 & 0 & a-1 & \vdots & 1-a \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & z \\ 0 & a+2 & 3 & \vdots & a-4 \\ 0 & 0 & a-1 & \vdots & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} \frac{3a}{a+2} \\ \frac{a-4}{a+2} \\ 0 \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & \frac{3a}{a+2} \\ 0 & \frac{a-4}{a+2} \\ -1 & 0 \end{bmatrix}$$

If $a = 1$, $Ax = B$ has infinitely many solutions

$$\begin{bmatrix} 1 & -1 & -1 & -2 & \\ 0 & 3 & 3 & -3 & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 2 & \\ 0 & 3 & 3 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 1 \\ -k_1 - 1 & -k_2 - 1 \\ k_1 & k_2 \end{bmatrix}, \quad k_1, k_2 \quad \text{anbitrary constants.}$$

Q5

(a)Let $X, Y \in M_{2\times 2}(R)$ and $C \in R$, than we have

$$\begin{aligned} T\left(CX+Y\right) &= \begin{bmatrix} tr & A^T\left(CX+Y\right) \\ tr & B^T\left(CX+Y\right) \\ tr & C^T\left(CX+Y\right) \end{bmatrix} \\ &= c \begin{bmatrix} tr(A^TX) \\ tr(B^TX) \\ tr(C^TX) \end{bmatrix} + \begin{bmatrix} tr(A^TX) \\ tr(B^TY) \\ tr(C^TY) \end{bmatrix} \\ &= cT(X) + T(Y) \end{aligned}$$

(b)

$$T\left(V_1\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=w_1} + 0 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{=w_2} + 0 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{=w_3}$$

$$T\left(V_2\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(V_3\right)=\begin{bmatrix}0\\1\\1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(V_4\right)=\begin{bmatrix}0\\1\\-1\end{bmatrix}=0\begin{bmatrix}1\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}+-1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Therefore,the matix representation of T with respect to V_1, V_2, V_3, V_4, V_4, and W_1, V_2, W_3 , is:

$$\begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & -1\end{bmatrix}.$$

(c)Since $T(A)=\begin{bmatrix}2\\0\\0\end{bmatrix}, T(B)=\begin{bmatrix}0\\1\\0\end{bmatrix}, T(C)=\begin{bmatrix}0\\0\\1\end{bmatrix}$,We can take X to be

$$\begin{aligned}&\frac{1}{2}A-2B+C\\&=\begin{bmatrix}y_2 & 0\\0 & -y_2\end{bmatrix}-\begin{bmatrix}0 & 2\\-0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\\&=\begin{bmatrix}y_2 & -2\\1 & -y_2\end{bmatrix}.\end{aligned}$$

Q6 Apply Elementary Row and Column Operations to A and C to obtain $D_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ for A and $D_2 = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$for C.

Where $r = rankA, s = rankC$.
Let $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$. Then M can be converted to $M_1 = \begin{bmatrix} D_1 & C_1 \\ 0 & D_2 \end{bmatrix}$ via elementary row and column operations.

Furthermore, the pivots in D_1 and D_2 can be used to eliminate the nonzero entries in C_1, to obtain

$$M_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In conclusion,

$$\begin{aligned} rankM &= rankM_1 = rankM_2 = r + s + rankC_2 \\ &\geq r + s = rankA + rankC \end{aligned}$$

23 Spring Midterm Question

线性代数 23春季 期中试题 发布版

Q1.(20 points, 4 points each)
暂无选择题。

Q2.(25 points, 5 points each) Fill in the blanks

(1)Let $u, v \in \mathbb{R}^n$ with $\|u\| = 2$, $\|v\| = 4$ and $u^T v = 6$. Then $\|3u - v\| =$ ______.

(2) Let A be an $n \times n$ matrix with $A^2 = -A$ and let I be the $n \times n$ identity matrix. Then $(A - I)^{-1} =$ ______.

(3)Let$A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$with $rank\,(A) = 1$. Then $a =$ ______.

(4)Let α be a nonzero 3-dimensional real column vector in $\mathbb{R}^3$with $\alpha^T \alpha \neq 1$, and I_3 be the 3×3 identity matrix. Then rank $\left(I_3 - \alpha\alpha^T\right) =$ ______.

(5) Let$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
Then the least squares solution to $Ax = b$ is $\widehat{x} =$ ______.

Q3 (15 points) Let $\alpha \in R$, and

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a) By applying row operations, determine for which values of α is the matrix A_α invertible?
(b) Find the values of α such that the nullspace of $A_\alpha, N(A_\alpha)$, has dimension 1?
(c) Let $\alpha = 2$.Write down the matrix inverse of A_α.

Q4.(10points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Find an LU factorization of A.

Q5.(10 points)

Consider the following system ofline are quations:

$$(I):\begin{cases}x_1+x_2=0,\\ x_2-x_4=0.\end{cases}$$

Note that the above system (I) has four variables x_1,x_2,x_3,x_4.
Suppose another homogeneous
system of linear equations (II) has special solutions

$$u=\begin{bmatrix}0\\1\\1\\0\end{bmatrix},v=\begin{bmatrix}-1\\2\\2\\1\end{bmatrix}.$$

Find the common nonzero solutions of systems (I) and (II).

Q6.(8 points)

Let $R^{2\times 2}$ be the vector space consisting of all 2×2 real matrices.

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$E=\left\{E_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(a) Show that E is a basis for $\mathbb{R}^{2\times 2}$.

(b) Show that T:$\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}, X\mapsto XA$ is a linear transformation.

(c)Find the matrix representation of T with respect to the ordered basis $E_{11}, E_{12}, E_{21}, E_{22}$.

Q7.(6 points) Let A,B be two $m \times n$ matrices. Prove

(a)$rank\left(A+B\right) \leq rankA + rankB$

(b)$rank\left(A+B\right) \geq rankA - rankB$

Q8.(6 points)

Let A be an $m \times n$ matrix with rank r. Show that there exist an $m \times r$ matrix B and an $r \times n$ matrix C such that $A = BC$ and both B and C have rank r.

(共25分，每小题5分)填空题。

(1)设$u, v \in \mathbb{R}^n$且$\| u \| = 2, \| v \| = 4$以及$u^T v = 6$.则
$\|3u - v\| =$ ______.

(2)设A为一个n阶矩阵，且$A^2 = -A, I$表示n阶单位矩阵。则
$(A - I)^{-1} =$ ______.

(3)设$A = \begin{bmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{bmatrix}$且 $rank\,(A) = 1$. 则 $a =$ ______.

(4)设 $\alpha \in \mathbb{R}^3$为一个非零列向量且 $\alpha^T \alpha \neq 1$, I_3 为3×3 单位矩阵.则
$\operatorname{rank}\left(I_3 - \alpha\alpha^T\right) =$ ______

(5)
令$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
则 $Ax = b$ 的最小二乘解 $\widehat{x} =$ ______.

Q3 (15 points)设α为实数，A_α为

$$A_\alpha = \begin{bmatrix} 1 & -\alpha & 1+\alpha \\ \alpha & \alpha^2 & \alpha \\ -\alpha & 1 & -2 \end{bmatrix}.$$

(a)对矩阵A_α做初等行变换，α为何时时，A_α为可逆矩阵?

(b)α取何值时，矩阵A_α的零空间的维数等于1?

(c)设 $\alpha = 2$ ，求矩阵A_α的逆矩阵.

Q4.(10 points)设

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 9 & -3 & 1 \\ -1 & 2 & 2 \end{bmatrix}.$$

求A的一个LU分解

Q5.(10 points) 考虑以下线性方程组：

$$(I): \begin{cases} x_1 + x_2 = 0, \\ x_2 - x_4 = 0. \end{cases}$$

注意上述方程组(I)有四个变量x_1, x_2, x_3, x_4。假设另一个齐次线性方程组(II)有特殊解

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

找出方程组(I)和(II)的共同非零解。

Q6.(8 points)

设$R^{2\times 2}$为所有2×2实矩阵构成的向量空间. 设$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, 且

$$E=\left\{E_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21}=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(a)证明：E为$R^{2\times 2}$的一组基。

(b)证明：$T:\mathbb{R}^{2\times 2}\to\mathbb{R}^{2\times 2}, X\mapsto XA$为线性变换

(c)求 T 在有序基$E_{11},E_{12},E_{21},E_{22}$下的矩阵表示

Q7.(6 points)设A, B都为$m \times n$矩阵，证明：

(a)$rank\left(A+B\right) \leq rankA + rankB$

(b)$rank\left(A+B\right) \geq rankA - rankB$

Q8.(6 points)

设 A 为一个秩为r的$m \times n$矩阵. 证明：存在一个$m \times r$矩阵B和一个$r \times n$矩阵C,使得 $A = BC$,其中B, C的秩都为r.

23 Spring Midterm Answer

线性代数 23春季 期中试题答案 发布版

Q1 (1)A (2)D (3)C (4)B (5)B

Q2

(1)4

(2)$-\frac{1}{2}A-I$

(3)1

(4)3

(5)$\begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$

Q3

(a) $\alpha \neq 0, 1, -3$

(b) $\alpha = 0, 1, -3$

(c) $\alpha = 2$

$$A_{\alpha}^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{20} & -\frac{4}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{2} & \frac{3}{20} & \frac{2}{5} \end{bmatrix}$$

Q4

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ -1 & -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -12 & -8 \\ 0 & 0 & 1 \end{bmatrix}$$

Q5

$$k \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, k \neq 0.$$

Q6 (a)

- linear independent
- E spans $\mathbf{R}^{2\times 2}$.

(b)

$$\begin{aligned} &\cdot T(X+Y) = T(X) + T(Y), \\ &\cdot T(\lambda X) = \lambda T(X). \end{aligned}$$

(c)

$$M = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & 0 & d \end{bmatrix}.$$

Q7(a)

Pivot columns of A: $a_1, a_2, \cdots, a_r$;

Pivot columns of B : $b_1, b_2, \cdots, b_s$;

$rankA = r, rankB = s$.

$$
\begin{aligned}
V &= span\left(a_1, \cdots, a_s, b_1, \cdots, b_s\right). dimV \leq r + s \\
&= span\left(a_1, \cdots, a_s, b_1, \cdots, b_s\right) \supseteq C\left(A + B\right) \\
&\implies dimC(A + B) \leq dimV \\
&\implies rank(A + B) \leq rank(A) + rank(B)
\end{aligned}
$$

(b)

$A + B - B = A$

$rank(A + B - B) \leq rank(A + B) + rank(-B) \ldots \text{by (a)}$

$rank(A + B) + rank(-B) = rank(B)$

$\implies rankA - rankB \leq rank(A + B)$

Q8 P_1, Q_1 invertible.

$$
\begin{aligned}
A &= P_1 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_1 \\
&= \underbrace{P_1 \begin{bmatrix} I_r \\ 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} I_r & 0 \end{bmatrix} Q_1}_{C}
\end{aligned}
$$

项目情况说明 后记

感谢您使用这本小册子。您可扫描下方二维码进行反馈，您的意见对我们改进服务和拓展其他科目非常重要（此二维码与开头前言部分链接相同）：

Email:

- **huajiebridge34@gmail.com** （邮箱已更换，以此为准）

Github Repo:

- Open_Notes_SUSTech: 南方科技大学一位23级本科生的学习笔记，论文和项目
  https://github.com/LIUBINfighter/Open_Notes_SUSTech
  给个star吧我什么都会做的（bushi）

Blog:

- Huajie's Blog
  https://liubinfighter.github.io/Blog/
  更新风格：突发恶疾式

目前的工程文件以及草稿不定期上传到仓库线性代数栏目。你也可以下载往期结项的文件了解我的工作方式，欢迎来戳。

同时，向各位同学和高年级助教征求之后表格中留空的材料，包括照片，扫描件，手写件，markdown或TeX，演示文稿等文件，二版时会将您加入贡献者栏并赠与免费样书，如果你是愿意帮助的热心人，助教或互助课堂的主讲人，能够予以手稿提供，OCR，排版，校对，答案审核一类的协助就更好了。

关于贡献和署名

如果您认为您的材料被使用了却没有署名，请务必与我们联系；如果您认为您应当在贡献者栏却没有发现或您对展示自己的名称有异议，请联系我们。核实后，我们会在勘误表和新一版材料中更新，并联系赞助为您发放样书和奖励。

项目相关流程请务必通过邮件 huajiebridge34@gmail.com 联系。

项目进度

线性代数期中考试试题

	原卷	答案	完整度	发布时间
20Fall	可用	可用	✅	与印刷版同时
21Spring	可用	可用	✅	与印刷版同时
21Fall	可用	无	❌	搁置
22Spring	无	无	❌	搁置

	原卷	答案	完整度	发布时间
22Fall	相片质量	无	❌	搁置
23Spring	无选择题	手写答案	❌	与印刷版同时
23Fall	Released	Released	☑️	24.9.14
24Spring	Released	Released	☑️	24.9.10

致谢

感谢 对项目提供赞助。

感谢一直以来为同学们服务的教授们，欧阳大康团队，万里团队，洛酱数学仓库，维护网络服务的工友以及积极贡献资料的前辈们。

感谢Obsidian，几乎完美潇洒的笔记管理软件和写作工作流。

感谢SimpleTex提供准确高速LaTeX/OCR服务. 希望**SimpleTex**能被更多人认识和使用，发展越来越好！

感谢Pandoc提供便捷快速的markdown转pdf服务. 这在优化排版上起了非同小可的帮助。

感谢我的三位舍友，宽容我在寝室里进行战地厨房展开，每天睡过头和半夜敲键盘的行为，并和我在闲暇之余一起玩游戏解压。

以及我自己，做了一点微小的工作。

> **Quote**
>
> Per aspera ad astra.
> 颠簸路途通繁星.

Support Now, Buy me Tea.

用嗨了请点赞，更希望您留下合理化建议？hhh

打赏请随意。

推荐使用微信支付

格尔顿司马徽晟(**杰)

微信支付

支付宝

推荐使用支付宝

刘斌(**杰)

打开支付宝[扫一扫]

So you move on, until this page?

At the end of the end, thanks to @vida_bwe.

Your presence inspires me to keep moving forward.

Believe in the green light.

Gatsby believed in the green light,

The orgiastic future that year by year recedes before us.

It eluded us then, but that's no matter...

Tomorrow we will run faster,

Stretch out our arms farther...

And one fine morning...

So we beat on,

Boats against the current,

Borne back, ceaselessly

Into the past....

The Great Gatsby

不断前进。

刘华杰

Signature