1. Let us consider the quadrature formula

$$Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$$

for the approximation of

$$I(f) = \int_0^1 f(x)\,dx,$$

where $f \in C^1([0,1])$. Determine the coefficients α_1, α_2, and α_3 in such a way that Q has degree of exactness 2.

Solution. To determine the coefficients $\alpha_1, \alpha_2, \alpha_3$ such that the quadrature formula $Q(f)$ has a degree of exactness of 2, we require that the formula is exact for the monomials $f(x) = 1, x, x^2$.

- For $f(x) = 1$:

$$\int_0^1 1\,dx = [x]_0^1 = 1.$$

The quadrature gives:

$$Q(1) = \alpha_1 \cdot 1 + \alpha_2 \cdot 1 + \alpha_3 \cdot 0 = \alpha_1 + \alpha_2.$$

Thus, $\alpha_1 + \alpha_2 = 1$.

- For $f(x) = x$:

$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Note that $f'(x) = 1$, so $f'(0) = 1$. The quadrature gives:

$$Q(x) = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 + \alpha_3 \cdot 1 = \alpha_2 + \alpha_3.$$

Thus, $\alpha_2 + \alpha_3 = \frac{1}{2}$.

- For $f(x) = x^2$:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

Note that $f'(x) = 2x$, so $f'(0) = 0$. The quadrature gives:

$$Q(x^2) = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 + \alpha_3 \cdot 0 = \alpha_2.$$

Thus, $\alpha_2 = \frac{1}{3}$.

Now we solve the system of linear equations:

$$\begin{aligned}
\alpha_2 &= \frac{1}{3} \\
\alpha_2 + \alpha_3 &= \frac{1}{2} \implies \frac{1}{3} + \alpha_3 = \frac{1}{2} \implies \alpha_3 = \frac{1}{6} \\
\alpha_1 + \alpha_2 &= 1 \implies \alpha_1 + \frac{1}{3} = 1 \implies \alpha_1 = \frac{2}{3}
\end{aligned}$$

Therefore, the coefficients are:

$$\alpha_1 = \frac{2}{3}, \quad \alpha_2 = \frac{1}{3}, \quad \alpha_3 = \frac{1}{6}.$$

□

2. Apply the midpoint, trapezoidal, and Simpson's composite rules to approximate the integral

$$\int_{-1}^1 |x| e^x\,dx,$$

and discuss their convergence (both theoretically predicted and practically achieved) as a function of the size h of the subintervals.

Solution.

- **Midpoint Rule:** Let $h = \frac{2}{n}$ and $x_j = -1 + jh$, $j = 0, \dots, n$, and denote

$$I = \int_{-1}^{1} |x| e^{x}\, dx = 2\Big(1 - \frac{1}{e}\Big).$$

  The composite midpoint rule gives

$$M_h = h \sum_{j=1}^{n} f\Big(x_{j-\frac{1}{2}}\Big), \qquad x_{j-\frac{1}{2}} = -1 + \Big(j - \tfrac{1}{2}\Big)h, \quad f(x) = |x|e^{x}.$$

  For f smooth on each subinterval, the error satisfies

$$I - M_h = \mathcal{O}(h^2),$$

  which is confirmed numerically by observing that $|I - M_h| \approx C_M h^2$ on a log–log plot of the error vs. h.

- **Trapezoidal Rule:** The composite trapezoidal rule reads

$$T_h = \frac{h}{2}\Big(f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\Big).$$

  Since f is C^∞ on $[-1, 0]$ and $[0, 1]$ and 0 is a grid point for even n, the global error is again

$$I - T_h = \mathcal{O}(h^2).$$

  Numerically, $|I - T_h|$ decays approximately like $C_T h^2$ as $h \to 0$.

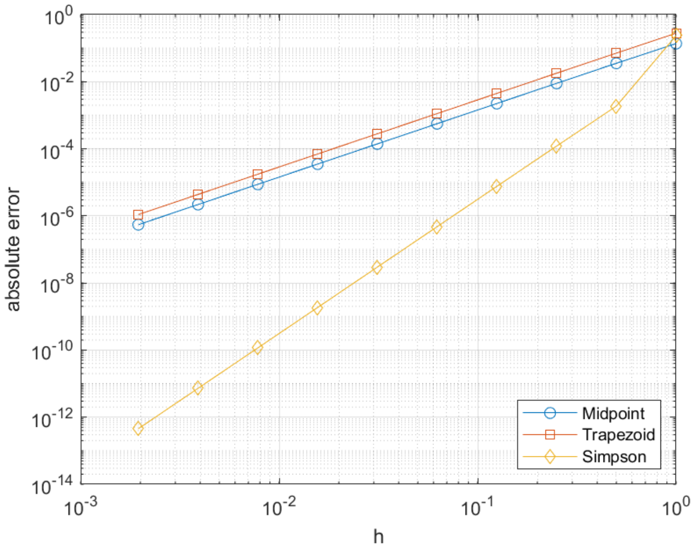
- **Simpson's Rule:** For even n, the composite Simpson rule is

$$S_h = \frac{h}{3}\Big(f(x_0) + f(x_n) + 4\sum_{\substack{j=1 \\ j \text{ odd}}}^{n-1} f(x_j) + 2\sum_{\substack{j=2 \\ j \text{ even}}}^{n-2} f(x_j)\Big).$$

  Applied separately on $[-1, 0]$ and $[0, 1]$, where f is smooth, the theoretical error satisfies

$$I - S_h = \mathcal{O}(h^4).$$

This theoretical convergence rate is also observed in practice, with the following picture log-log plot of the error vs. h:



And the fitted slopes are approximately:

- Midpoint Rule: slope ≈ 1.995

- Trapezoidal Rule: slope ≈ 1.997
- Simpson's Rule: slope ≈ 4.156

So the numerical experiments confirm the theoretical convergence rates.

□

3. Consider the integral

$$I(f) = \int_0^1 e^x \, dx$$

and estimate the minimum number m of subintervals that is needed for computing $I(f)$ up to an absolute error $\leq 5 \cdot 10^{-4}$ using the composite trapezoidal and Simpson's rules. Evaluate in both cases the absolute error that is actually made.

Solution. The exact value is:

$$I = \int_0^1 e^x dx = e - 1$$

- For composite trapezoidal rule,

$$E_T = I - T_m = -\frac{b-a}{12} h^2 f''(\xi), \quad h = \frac{b-a}{m}$$

So,

$$|E_T| \leq \frac{b-a}{12} h^2 \max_{x \in [a,b]} |f''(x)| = \frac{1}{12} \left(\frac{1}{m}\right)^2 e = \frac{e}{12m^2}$$

To ensure $|E_T| \leq 5 \cdot 10^{-4}$, we need:

$$\frac{e}{12m^2} \leq 5 \cdot 10^{-4} \implies m^2 \geq \frac{e}{12 \cdot 5 \cdot 10^{-4}} \implies m \geq \sqrt{\frac{e}{0.006}} \approx 21.28$$

Thus, we take $m = 22$. After computing with MATLAB, the actual error is:

$$|I - T_{22}| \approx 2.958 \times 10^{-4}$$

- For composite Simpson's rule,

$$E_S = I - S_m = -\frac{b-a}{180} h^4 f^{(4)}(\xi), \quad h = \frac{b-a}{m}$$

So,

$$|E_S| \leq \frac{b-a}{180} h^4 \max_{x \in [a,b]} |f^{(4)}(x)| = \frac{1}{180} \left(\frac{1}{m}\right)^4 e = \frac{e}{180m^4}$$

To ensure $|E_S| \leq 5 \cdot 10^{-4}$, we need:

$$\frac{e}{180m^4} \leq 5 \cdot 10^{-4} \implies m^4 \geq \frac{e}{180 \cdot 5 \cdot 10^{-4}} \implies m \geq \left(\frac{e}{0.09}\right)^{1/4} \approx 2.34.$$

But since Simpson's rule requires an even number of intervals, we take $m = 4$. After computing with MATLAB, the actual error is:

$$|I - S_4| \approx 3.701 \times 10^{-5}$$

□

4. (a) Assume that $f(x)$ is continuous and that $f'(x)$ is integrable on $[0, 1]$. Show that the error in the trapezoidal rule for calculating

$$\int_0^1 f(x) \, dx$$

has the form

$$E_n(f) = \int_0^1 K(x) f'(x) \, dx,$$

where

$$K(x) = \frac{x_{j-1} + x_j}{2} - x, \qquad x_{j-1} \leq x \leq x_j, \quad j = 1, \ldots, n.$$

Proof. Consider the integral over a single subinterval $[x_{j-1}, x_j]$, the error in the trapezoidal rule is given by:

$$\begin{aligned}
E_j(f) &= \int_{x_{j-1}}^{x_j} f(x)\,dx - \frac{x_j - x_{j-1}}{2}\left(f(x_{j-1}) + f(x_j)\right) \\
&= \int_{x_{j-1}}^{x_j} f(x)\,dx + \left(\frac{x_{j-1}+x_j}{2} - x_j\right) f(x_j) - \left(\frac{-x_{j-1}+x_j}{2}\right) f(x_{j-1}) \\
&= \left(\frac{x_{j-1}+x_j}{2} - x\right) f(x)\Big|_{x_{j-1}}^{x_j} + \int_{x_{j-1}}^{x_j} f(x)\,dx \\
&= \int_{x_{j-1}}^{x_j} \left(\frac{x_{j-1}+x_j}{2} - x\right) f'(x)\,dx
\end{aligned}$$

Summing over all subintervals, we have:

$$E_n(f) = \sum_{j=1}^{n} E_j(f) = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \left(\frac{x_{j-1}+x_j}{2} - x\right) f'(x)\,dx = \int_0^1 K(x) f'(x)\,dx$$

□

(b) Apply the result from part (a) to $f(x) = x^\alpha$ and to $f(x) = x^\alpha \ln x$, $0 < \alpha < 1$. This gives an order of convergence, although it is less than the true order.

Solution.

$$|E_n(f)| \le \int_0^1 |K(x)||f'(x)|\,dx \le \max_{x\in[0,1]} |K(x)| \int_0^1 |f'(x)|\,dx = \frac{h}{2}\int_0^1 |f'(x)|\,dx$$

For $f(x) = x^\alpha$, we have $f'(x) = \alpha x^{\alpha-1}$. Thus,

$$\int_0^1 |f'(x)|\,dx = \int_0^1 \alpha x^{\alpha-1}\,dx = \alpha\left[\frac{x^\alpha}{\alpha}\right]_0^1 = 1$$

Therefore,

$$|E_n(f)| \le \frac{h}{2} = \frac{1}{2n}$$

For $f(x) = x^\alpha \ln x$, we have $f'(x) = x^{\alpha-1}(\alpha \ln x + 1)$. Thus,

$$\int_0^1 |f'(x)|\,dx = \int_0^1 x^{\alpha-1}|\alpha \ln x + 1|\,dx \le \alpha \int_0^1 x^{\alpha-1}|\ln x|\,dx + \int_0^1 x^{\alpha-1}\,dx$$

Calculate the integrals, we have:

$$\int_0^1 |f'(x)|\,dx \le \alpha\cdot\frac{1}{\alpha^2} + \frac{1}{\alpha} = \frac{2}{\alpha}$$

Therefore,

$$|E_n(f)| \le \frac{h}{2}\cdot\frac{2}{\alpha} = \frac{1}{\alpha n}$$

So both of them have order of convergence $\mathcal{O}(1/n)$. □

5. Using the trapezoidal rule with $n = 2, 4, 8, 16, 32, 64, 128, 256$, and 512 subdivisions, determine empirically its rate of convergence for the evaluation of the integral

$$\int_0^1 x^\alpha \ln x\,dx$$

for $\alpha = 0.25, 0.5, 0.75$ and 1.

Solution. Assume

$$E(h) \approx Ch^p$$

, where $h = \frac{1}{n}$, then:

$$E(\frac{h}{2}) \approx C(\frac{h}{2})^p = \frac{E(h)}{2^p} \implies p \approx \log_2(\frac{E(h)}{E(h/2)})$$

The empirical p's are shown in the following tables:

□

Table 1: Trapezoidal rule errors and empirical rates for $\alpha = 0.25$.

n	h	$\lvert E_n\rvert$	p
2	0.5	3.485675e-01	–
4	0.25	1.822896e-01	0.9352
8	0.125	9.226608e-02	0.9824
16	0.0625	4.551250e-02	1.0195
32	0.03125	2.199860e-02	1.0489
64	0.015625	1.046245e-02	1.0722
128	0.0078125	4.911363e-03	1.0910
256	0.00390625	2.281021e-03	1.1064
512	0.00195312	1.050036e-03	1.1192

Table 2: Trapezoidal rule errors and empirical rates for $\alpha = 0.50$.

n	h	$\lvert E_n\rvert$	p
2	0.5	1.993799e-01	–
4	0.25	8.634039e-02	1.2074
8	0.125	3.635440e-02	1.2479
16	0.0625	1.496986e-02	1.2801
32	0.03125	6.054958e-03	1.3059
64	0.015625	2.413761e-03	1.3268
128	0.0078125	9.507895e-04	1.3441
256	0.00390625	3.708078e-04	1.3585
512	0.00195312	1.434066e-04	1.3706

Table 3: Trapezoidal rule errors and empirical rates for $\alpha = 0.75$.

n	h	$\lvert E_n\rvert$	p
2	0.5	1.204567e-01	–
4	0.25	4.299868e-02	1.4862
8	0.125	1.497044e-02	1.5222
16	0.0625	5.112788e-03	1.5499
32	0.03125	1.719798e-03	1.5719
64	0.015625	5.714162e-04	1.5896
128	0.0078125	1.879382e-04	1.6043
256	0.00390625	6.128770e-05	1.6166
512	0.00195312	1.984172e-05	1.6271

Table 4: Trapezoidal rule errors and empirical rates for $\alpha = 1.00$.

n	h	$\lvert E_n\rvert$	p
2	0.5	7.671320e-02	–
4	0.25	2.277282e-02	1.7522
8	0.125	6.594733e-03	1.7879
16	0.0625	1.874254e-03	1.8150
32	0.03125	5.249679e-04	1.8360
64	0.015625	1.453438e-04	1.8528
128	0.0078125	3.986148e-05	1.8664
256	0.00390625	1.084675e-05	1.8777
512	0.00195312	2.932033e-06	1.8873

6. Assume that the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots .$$

Generalize the Richardson extrapolation process to obtain formulae for C_1 and C_2. Assume that three values I_n, I_{2n} and I_{4n} have been computed, and use them to compute C_1, C_2 and to estimate I with an error of order $1/(n^2\sqrt{n})$.

Solution.

$$I = I_n + \frac{C_1}{n^{3/2}} + \frac{C_2}{n^2} + \mathcal{O}(n^{-5/2}) \tag{1}$$

$$I = I_{2n} + \frac{C_1}{(2n)^{3/2}} + \frac{C_2}{(2n)^2} + \mathcal{O}(n^{-5/2}) \tag{2}$$

$$I = I_{4n} + \frac{C_1}{(4n)^{3/2}} + \frac{C_2}{(4n)^2} + \mathcal{O}(n^{-5/2}) \tag{3}$$

Seek a linear combination

$$I \approx aI_n + bI_{2n} + cI_{4n}$$

such that the coefficients of C_1 and C_2 vanish:

$$\begin{aligned} a + b + c &= 1, \\ a + \frac{b}{2^{3/2}} + \frac{c}{4^{3/2}} &= 0, \\ a + \frac{b}{4} + \frac{c}{16} &= 0. \end{aligned}$$

Solve the system, we have

$$a = \frac{1+2\sqrt{2}}{21}, \quad b = -\frac{12+10\sqrt{2}}{21}, \quad c = \frac{32+8\sqrt{2}}{21}.$$

Based on (1), (2), and (3), we can estimate C_1 and C_2 as follows:

$$\begin{aligned} C_1 &= \frac{12+10\sqrt{2}}{7} n^{3/2}(I_n - 5I_{2n} + 4I_{4n}) + \mathcal{O}(n^{-\frac{1}{2}}), \\ C_2 &= \frac{4}{3} n^2 \left((6+5\sqrt{2})I_{2n} - (2+\sqrt{2})I_n - 4(1+\sqrt{2})I_{4n} \right) + \mathcal{O}(n^{-\frac{1}{2}}) \end{aligned}$$

After obtaining C_1 and C_2, we can estimate I with an error of order $1/(n^2\sqrt{n})$ using:

$$I \approx I_n + \frac{C_1}{n^{3/2}} + \frac{C_2}{n^2}.$$

□

7. Compute weights and nodes of the following quadrature formulae

$$\int_a^b w(x)f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i),$$

in such a way that the order is maximum for:

(a) $w(x) = \sqrt{x}$, $a = 0$, $b = 1$, $n = 1$;

(b) $w(x) = 2x^2 + 1$, $a = -1$, $b = 1$, $n = 0$;

(c) $w(x) = \begin{cases} 2, & 0 < x \le 1, \\ 1, & -1 \le x \le 0, \end{cases} \quad a = -1,\ b = 1,\ n = 1.$

Solution. (a)

$$\int_0^1 \sqrt{x} f(x)\,dx \approx w_0 f(x_0) + w_1 f(x_1)$$

The maximum degree of exactness for $n = 1$ is $2n + 1 = 3$. So we need it to be exact for $f(x) = 1, x, x^2, x^3$.

$$\begin{aligned}
\int_0^1 \sqrt{x}\,dx &= \frac{2}{3} = w_0 + w_1,\\
\int_0^1 x^{3/2}\,dx &= \frac{2}{5} = w_0 x_0 + w_1 x_1,\\
\int_0^1 x^{5/2}\,dx &= \frac{2}{7} = w_0 x_0^2 + w_1 x_1^2,\\
\int_0^1 x^{7/2}\,dx &= \frac{2}{9} = w_0 x_0^3 + w_1 x_1^3.
\end{aligned}$$

Solving this system, we find:

$$x_0 = \frac{5}{9} - \frac{2\sqrt{70}}{63}, \quad x_1 = \frac{5}{9} + \frac{2\sqrt{70}}{63}, \quad \omega_0 = \frac{1}{3} - \frac{\sqrt{70}}{150}, \quad \omega_1 = \frac{1}{3} + \frac{\sqrt{70}}{150}.$$

(b)

$$\int_{-1}^1 (2x^2 + 1) f(x)\,dx \approx w_0 f(x_0)$$

The maximum degree of exactness for $n = 0$ is 1. So we need it to be exact for $f(x) = 1, x$.

$$\begin{aligned}
\int_{-1}^1 (2x^2 + 1)\,dx &= \frac{10}{3} = w_0,\\
\int_{-1}^1 (2x^2 + 1) x\,dx &= 0 = w_0 x_0.
\end{aligned}$$

Thus, we have:

$$x_0 = 0, \quad w_0 = \frac{10}{3}.$$

(c) The maximum degree of exactness for $n = 1$ is $2n + 1 = 3$. So we need it to be exact for $f(x) = 1, x, x^2, x^3$.

$$\begin{aligned}
\int_{-1}^0 1\,dx + \int_0^1 2\,dx &= 3 = w_0 + w_1,\\
\int_{-1}^0 x\,dx + \int_0^1 2x\,dx &= \frac{1}{2} = w_0 x_0 + w_1 x_1,\\
\int_{-1}^0 x^2\,dx + \int_0^1 2x^2\,dx &= 1 = w_0 x_0^2 + w_1 x_1^2,\\
\int_{-1}^0 x^3\,dx + \int_0^1 2x^3\,dx &= \frac{1}{4} = w_0 x_0^3 + w_1 x_1^3.
\end{aligned}$$

Solving this system, we find:

$$x_0 = \frac{1 - \sqrt{155}}{22}, \quad x_1 = \frac{1 + \sqrt{155}}{22}, \quad \omega_0 = \frac{3}{2} - \frac{4\sqrt{155}}{155}, \quad \omega_1 = \frac{3}{2} + \frac{4\sqrt{155}}{155}.$$

□

8. Compute, with an error less than 10^{-4}, the following integrals:

   (a) $\displaystyle\int_0^\infty \frac{\sin x}{1 + x^4}\,dx$;

(b) $\displaystyle\int_0^{\infty} e^{-x}(1+x)^{-5}\,dx$;

(c) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx$.

Solution. (a) Let $x = \exp(\sinh t)$, then

$$\frac{dx}{dt} = \exp(\sinh t)\cosh t = x\cosh t,$$

So the integral becomes

$$\int_0^{\infty} \frac{\sin x}{1+x^4}\,dx = \int_{-\infty}^{\infty} \frac{\sin(\exp(\sinh t))}{1+\exp(4\sinh t)} \exp(\sinh t)\cosh t\,dt.$$

The we use trapezoidal rule and truncate to $[-7, 7]$ to compute the integral. The result is $I_1 \approx 0.5698$.

(b) Use the same trick as in (a), the result is $I_2 \approx 0.1915$.

(c) Use

$$x = \sinh t, \quad dx = \cosh t dt$$

to transform the integral to

$$\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx = \int_{-\infty}^{\infty} e^{-\sinh^2 t}\cos(\sinh t)\cosh t\,dt.$$

The we use trapezoidal rule and truncate to $[-8, 8]$ to compute the integral. The result is $I_3 \approx 1.3804$.

□

9. Decrease the singular behavior of the integrand in

$$\int_0^1 f(x)\ln x\,dx$$

by using the change of variables $x = t^r$, $r > 0$. Analyze the smoothness of the resulting integrand. Also explore the empirical behavior of the trapezoidal and Simpson's rules for various r.

Solution. With the change of variables $x = t^r$, we have $dx = rt^{r-1}dt$. The integral becomes:

$$\int_0^1 f(x)\ln x\,dx = \int_0^1 f(t^r)\ln(t^r)rt^{r-1}\,dt = \int_0^1 r^2 f(t^r)t^{r-1}\ln t\,dt$$

The new integrand is $g(t) = r^2 f(t^r)t^{r-1}\ln t$. To analyze the smoothness of $g(t)$ at $t = 0$, we consider:

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+} r^2 f(t^r)t^{r-1}\ln t$$

If f is continuous at 0, then $f(t^r) \to f(0)$ as $t \to 0^+$. The term $t^{r-1}\ln t$ behaves like $t^{r-1}\cdot(-\infty)$ as $t \to 0^+$. For the limit to exist and be finite, we need $r > 1$. Thus, for $r > 1$, the integrand $g(t)$ is smooth on $[0, 1]$.

Compared with the original integrand $f(x)\ln x$, which has a logarithmic singularity at $x = 0$, the transformed integrand $g(t)$ is smoother for $r > 1$.

Empirically, we can test the trapezoidal and Simpson's rules for various values of r (e.g., $r = 0.5, 1, 2, 4, 6$) and observe the convergence rates.

Table 5: Empirical convergence rates p estimated from the last three refinements (error $\sim N^{-p}$).

r	Trapezoidal rule p	Simpson's rule p
0.5	0.41	0.41
1	0.90	0.91
2	1.92	2.00
4	2.00	3.92
6	2.00	4.00

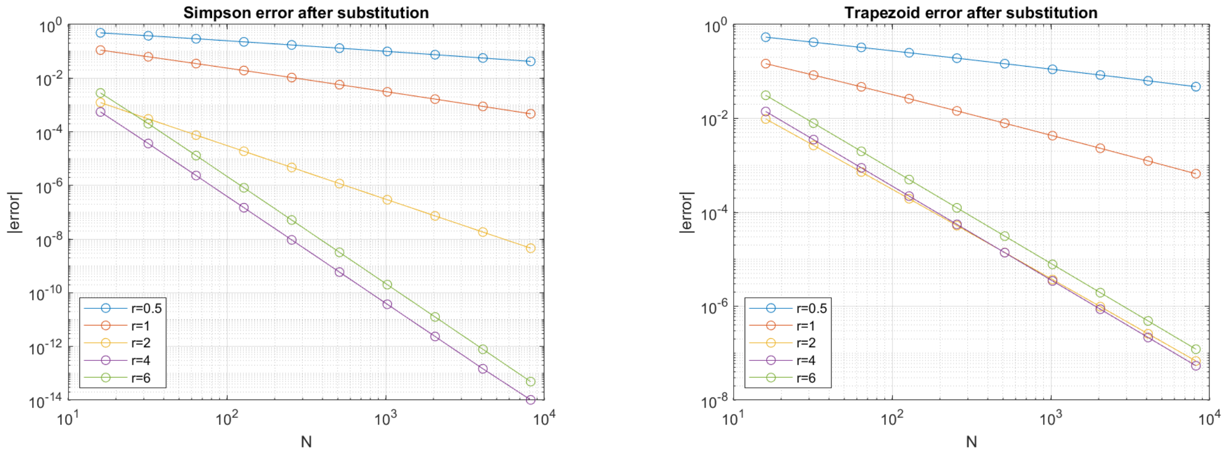
The empirical orders p are estimated from the last three refinements assuming $|E_N| \approx CN^{-p}$, so larger p means faster convergence.

For $r = 0.5$ the factor t^{r-1} is singular at $t = 0$, leading to very slow convergence ($p \approx 0.41$).

For $r = 1$ the logarithmic endpoint singularity persists, giving roughly first-order behavior ($p \approx 0.9$).

For $r = 2$ the transformed integrand is bounded and smoother near 0, so the trapezoidal rule recovers its expected second-order rate ($p \approx 2$), while Simpson's rule is still limited by insufficient endpoint smoothness.

For $r = 4$ and $r = 6$ the endpoint is sufficiently regularized so Simpson's rule approaches its ideal fourth-order rate ($p \approx 4$), while the trapezoidal rule saturates at second order ($p \approx 2$).



□

10. Prove that the Chebyshev polynomials $T_k(x) = \cos(k \arccos x)$ satisfy the three-term relation

$$\begin{cases} T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), & k \geq 1, \\ T_0(x) = 1, \quad T_1(x) = x. \end{cases}$$

Proof.

$$\begin{aligned} T_0(x) &= \cos(0) = 1 \\ T_1(x) &= \cos(\arccos x) = x \\ T_{k+1}(x) &= \cos(k \arccos x + \arccos x) \\ &= \cos(k \arccos x)\cos(\arccos x) - \sin(k \arccos x)\sin(\arccos x) \\ T_{k-1}(x) &= \cos(k \arccos x - \arccos x) \\ &= \cos(k \arccos x)\cos(\arccos x) + \sin(k \arccos x)\sin(\arccos x) \\ T_{k+1}(x) + T_{k-1}(x) &= 2\cos(k \arccos x)\cos(\arccos x) \\ &= 2xT_k(x) \end{aligned}$$

□

11. Verify that

$$\phi_n(x) = \frac{(-1)^n}{n!} e^x \frac{d^n}{dx^n}(x^n e^{-x}), \qquad n \geq 0,$$

are orthogonal on the interval $[0, \infty)$ with respect to the weight $w(x) = e^{-x}$.

$$\textit{Hint: Note that } \int_0^\infty e^{-x} x^m \, dx = m! \text{ for } m = 0, 1, 2, \ldots.$$

Proof. We need to show that for $m \neq n$,

$$\int_0^\infty \phi_n(x)\phi_m(x)e^{-x} \, dx = 0.$$

Substituting the definition of $\phi_n(x)$ and $\phi_m(x)$, we have:

$$\begin{aligned}\int_0^\infty \phi_n(x)\phi_m(x)e^{-x}\,dx &= \int_0^\infty \frac{(-1)^n}{n!}e^x\frac{d^n}{dx^n}(x^ne^{-x})\cdot\frac{(-1)^m}{m!}e^x\frac{d^m}{dx^m}(x^me^{-x})e^{-x}\,dx\\ &= \frac{(-1)^{n+m}}{n!m!}\int_0^\infty e^x\frac{d^n}{dx^n}(x^ne^{-x})\cdot\frac{d^m}{dx^m}(x^me^{-x})\,dx\\ &= \frac{(-1)^{n+m}}{n!m!}\int_0^\infty e^x\left(m!e^{-x}\sum_{k=0}^m(-1)^k\binom{m}{k}\frac{x^k}{k!}\right)\cdot\left(n!e^{-x}\sum_{j=0}^n(-1)^j\binom{n}{j}\frac{x^j}{j!}\right)\end{aligned}$$

Cancel $n!, m!$ and combine exponentials to get

$$= (-1)^{n+m}\int_0^\infty e^{-x}\sum_{k=0}^m\sum_{j=0}^n(-1)^{k+j}\binom{m}{k}\binom{n}{j}\frac{x^{k+j}}{k!\,j!}\,dx.$$

Interchange sum and integral (finite sum) and use the hint $\int_0^\infty e^{-x}x^{k+j}\,dx = (k+j)!$:

$$= (-1)^{n+m}\sum_{k=0}^m\sum_{j=0}^n(-1)^{k+j}\binom{m}{k}\binom{n}{j}\frac{(k+j)!}{k!\,j!} = (-1)^{n+m}\sum_{k=0}^m\sum_{j=0}^n(-1)^{k+j}\binom{m}{k}\binom{n}{j}\binom{k+j}{k}.$$

Now evaluate the inner sum (standard identity)

$$\sum_{k=0}^m(-1)^k\binom{m}{k}\binom{k+j}{k} = \begin{cases}0, & j<m,\\ \binom{j}{m}, & j\geq m,\end{cases}$$

so

$$\int_0^\infty \phi_n(x)\phi_m(x)e^{-x}\,dx = (-1)^{n+m}\sum_{j=m}^n(-1)^j\binom{n}{j}\binom{j}{m}.$$

Use $\binom{n}{j}\binom{j}{m} = \binom{n}{m}\binom{n-m}{j-m}$ and set $r = j-m$:

$$= (-1)^{n+m}\binom{n}{m}\sum_{r=0}^{n-m}(-1)^{r+m}\binom{n-m}{r} = (-1)^n\binom{n}{m}(1-1)^{n-m}.$$

Hence for $n\neq m$, WLOG assume $n-m\geq 1$ then $(1-1)^{n-m}=0$, so

$$\int_0^\infty \phi_n(x)\phi_m(x)e^{-x}\,dx = 0,$$

which proves orthogonality. □

12. Let $f(x) = \arccos x$ for $-1\leq x\leq 1$. Find the polynomial of degree two,

$$p(x) = a_0 + a_1x + a_2x^2,$$

which minimizes

$$\int_{-1}^1 \frac{[f(x)-p(x)]^2}{\sqrt{1-x^2}}\,dx.$$

Solution. Since the weight function is $w(x) = \frac{1}{\sqrt{1-x^2}}$, we can use the Chebyshev polynomials of the first kind, which are orthogonal with respect to this weight function on the interval $[-1, 1]$.
The Chebyshev polynomials of the first kind are given by:

$$T_k(x) = \cos(k\arccos x).$$

The first three Chebyshev polynomials are:

$$\begin{aligned}T_0(x) &= 1,\\ T_1(x) &= x,\\ T_2(x) &= 2x^2-1.\end{aligned}$$

We need to find the minimum of $\|f(x) - p(x)\|_\omega$ by choosing $p(x)$ in the span of $\{T_0(x), T_1(x), T_2(x)\}$, the best approximation is given by orthogonal projection:

$$p(x) = \sum_{k=0}^{2} c_k T_k(x),$$

where

$$c_k = \frac{\langle f, T_k\rangle_\omega}{\langle T_k, T_k\rangle_\omega}.$$

We compute the inner products:

$$c_0 = \frac{\int_{-1}^{1} \frac{\arccos x \cdot 1}{\sqrt{1-x^2}}\, dx}{\int_{-1}^{1} \frac{1^2}{\sqrt{1-x^2}}\, dx} = \frac{\frac{\pi^2}{2}}{\pi} = \frac{\pi}{2},$$

$$c_1 = \frac{\int_{-1}^{1} \frac{\arccos x \cdot x}{\sqrt{1-x^2}}\, dx}{\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\, dx} = -\frac{4}{\pi},$$

$$c_2 = \frac{\int_{-1}^{1} \frac{\arccos x \cdot (2x^2-1)}{\sqrt{1-x^2}}\, dx}{\int_{-1}^{1} \frac{(2x^2-1)^2}{\sqrt{1-x^2}}\, dx} = 0.$$

So the polynomial that minimizes the integral is:

$$p(x) = \frac{\pi}{2} T_0(x) - \frac{4}{\pi} T_1(x) + 0 \cdot T_2(x) = \frac{\pi}{2} - \frac{4}{\pi} x.$$

□

13. Define

$$S_n(x) := \frac{1}{n+1} T'_{n+1}(x), \qquad n \geq 0,$$

with $T_{n+1}(x)$ the Chebyshev polynomial of degree $n+1$. The polynomials $S_n(x)$ are called Chebyshev polynomials of the second kind.

(a) Show that $\{S_n(x)\}$ is a family of orthogonal polynomials on $[-1, 1]$ with respect to the weight $w(x) = \sqrt{1-x^2}$.

Proof. We need to show that for $m \neq n$,

$$\int_{-1}^{1} S_n(x) S_m(x) w(x)\, dx = 0,$$

where $w(x) = \sqrt{1-x^2}$.
By definition,

$$\begin{aligned} S_n(x) &= \frac{1}{n+1} T'_{n+1}(x) \\ &= \frac{1}{n+1} \cdot (\sin((n+1)\arccos x)) \cdot \frac{n+1}{\sqrt{1-x^2}} \\ &= \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}. \end{aligned}$$

Therefore,

$$S_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Now, we compute the inner product:

$$\begin{aligned} \int_{-1}^{1} S_n(x) S_m(x) w(x)\, dx &= \int_0^{\pi} S_n(\cos\theta) S_m(\cos\theta) \sin^2\theta\, d\theta \\ &= \int_0^{\pi} \left(\frac{\sin((n+1)\theta)}{\sin\theta}\right)\left(\frac{\sin((m+1)\theta)}{\sin\theta}\right) \sin^2\theta\, d\theta \\ &= \int_0^{\pi} \sin((n+1)\theta)\sin((m+1)\theta)\, d\theta = 0, \quad \text{for } m \neq n. \end{aligned}$$

□

(b) Show that the family $\{S_n(x)\}$ satisfies the same triple recursion relation as the family $\{T_n(x)\}$.

Proof. Recall the three-term recurrence relation for Chebyshev polynomials of the first kind:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

For $S_n(cos\theta)$, we have:

$$\begin{aligned} S_{n+1}(\cos\theta) &= \frac{\sin((n+2)\theta)}{\sin\theta} \\ &= \frac{2\cos\theta\sin((n+1)\theta) - \sin(n\theta)}{\sin\theta} \\ &= 2\cos\theta S_n(\cos\theta) - S_{n-1}(\cos\theta). \end{aligned}$$

So,

$$S_{n+1}(x) = 2xS_n(x) - S_{n-1}(x)$$

□

(c) Given $f \in C[-1, 1]$, minimize

$$\int_{-1}^{1} \sqrt{1-x^2}\,[f(x) - P_n(x)]^2\,dx,$$

where $P_n(x)$ is allowed to range over all polynomials of degree $\leq n$.

Solution. The best approximation polynomial $P_n(x)$ can be expressed as a linear combination of the Chebyshev polynomials of the second kind since they are orthogonal with respect to the weight $w(x) = \sqrt{1-x^2}$:

$$P_n(x) = \sum_{k=0}^{n} c_k S_k(x),$$

where

$$c_k = \frac{\langle f, S_k\rangle_w}{\langle S_k, S_k\rangle_w}.$$

We compute the inner products:

$$c_k = \frac{\int_{-1}^{1} f(x)S_k(x)w(x)\,dx}{\int_{-1}^{1} S_k^2(x)w(x)\,dx}.$$

Thus, the polynomial $P_n(x)$ that minimizes the integral is given by:

$$P_n(x) = \sum_{k=0}^{n} \left(\frac{\int_{-1}^{1} f(x)S_k(x)w(x)\,dx}{\int_{-1}^{1} S_k^2(x)w(x)\,dx} \right) S_k(x).$$

□