Real Analysis Final Exam 2025 Fall

Question 1

Show that for measurable $f(x)$, $f(x)$ is integrable on [0,1] iff $g(x, y) = |f(x) - f(y)|$ is integrable on $[0, 1] \times [0, 1]$.

Question 2

Chapter 3 Exercise 4

Question 3

Recall Fatou's lemma, for measurable functions $f_n \geq 0$ with limit $f = \lim_{n\to\infty} f_n$,

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

raise examples:

1. The inequality can be strict for some f_n.
2. If we remove the non-negativity assumption, the inequality may fail.

Question 4

Show $\sqrt{x}$ is absolutely continuous on $[0, 1]$.

Question 5

1. Show the integral version of Cauchy-Schwarz inequality on $[a, b]$.
2. If f is absolutely continuous on $[a, b]$ and $(f')^2$ is integrable, show that there exists a constant C such that
$$|f(x) - f(y)| \leq C|x - y|^{1/2}, \quad \forall x, y \in [a, b].$$