1. Compute the condition number $K(d)$ of the following equations:

   (a) $x - a^d = 0, \quad a > 0$

$$x = a^d \quad G(d) = a^d \quad G'(d) = a^d \ln a$$
$$K(d) = \|G'(d)\| \frac{\|d\|}{\|G(d)\|} = |d \ln a|$$
$$K_{\text{abs}}(d) = \|G'(d)\| = |a^d \ln a|$$

   (b) $d - x + 1 = 0$

$$x = d + 1 \quad G(d) = d + 1 \quad G'(d) = 1$$
$$K(d) = \frac{|d|}{|d+1|}$$
$$K_{\text{abs}}(d) = 1$$

2. Study the well posedness and the conditioning in the infinity norm of the following problem: for the datum d, find x and y such that:

$$\begin{cases} x + dy = 1, \\ dx + y = 0. \end{cases}$$

Solution: First, when $d = \pm 1$, then problem is ill-posed. Otherwise, when $d \neq \pm 1$, the problem is well-posed.Then

$$x = -\frac{1}{d^2 - 1}, \quad y = \frac{d}{d^2 - 1}$$

So, $G(d) = \begin{pmatrix} -\frac{1}{d^2-1} \\ \frac{d}{d^2-1} \end{pmatrix}, \quad \|G(d)\|_\infty = \frac{\max(1,|d|)}{|d^2-1|}$

$G'(d) = \begin{pmatrix} \frac{2d}{(d^2-1)^2} \\ -\frac{d^2+1}{(d^2-1)^2} \end{pmatrix}, \quad \|G'(d)\|_\infty = \frac{d^2+1}{(d^2-1)^2}$

$$K(d) = \|G'(d)\|_\infty \frac{\|d\|}{\|G(d)\|_\infty} = \begin{cases} \frac{d^2+1}{|d^2-1|}, & |d| > 1, \\ \frac{(d^2+1)|d|}{|d^2-1|}, & |d| < 1. \end{cases}$$

$$K_{\text{abs}}(d) = \|G'(d)\|_\infty = \frac{d^2+1}{(d^2-1)^2}$$

3. Study the conditioning of the solving formula $x_\pm = -p \pm \sqrt{p^2 + q}$ for the quadratic equation $x^2 + 2px - q = 0$ w.r.t. changes in the date p and q separately.

Solution:

$$G(p) = -p \pm \sqrt{p^2 + q} \quad G'(p) = -1 \pm \frac{p}{\sqrt{p^2+q}} = \frac{-\sqrt{p^2+q} \pm p}{\sqrt{p^2+q}}$$
$$K_{x_\pm}(p) = \|G'(p)\| \frac{\|p\|}{\|G(p)\|} = \frac{|-\sqrt{p^2+q} \pm p|}{\sqrt{p^2+q}} \cdot \frac{|p|}{|-p \pm \sqrt{p^2+q}|} = \frac{|p|}{\sqrt{p^2+q}}$$
$$K_{\text{abs}x_+}(p) = \|G'(p)\| = \frac{|-\sqrt{p^2+q} + p|}{\sqrt{p^2+q}} \quad K_{\text{abs}x_-}(p) = \frac{|-\sqrt{p^2+q} - p|}{\sqrt{p^2+q}}$$

$$G(q) = -p \pm \sqrt{p^2+q} \quad G'(q) = \pm\frac{1}{2\sqrt{p^2+q}}$$
$$K_{x_\pm}(q) = \|G'(q)\|\frac{\|q\|}{\|G(q)\|} = \left|\frac{1}{2\sqrt{p^2+q}}\right| \cdot \left|\frac{q}{-p\pm\sqrt{p^2+q}}\right| = \left|\frac{q}{2\sqrt{p^2+q}(-p\pm\sqrt{p^2+q})}\right|$$
$$K_{\mathrm{abs}x_\pm}(q) = \|G'(q)\| = \left|\frac{1}{2\sqrt{p^2+q}}\right|$$

4. Consider the IVP:
$$\begin{cases} x' = x_0 e^{at}(a\cos t - \sin t), & t > 0 \\ x(0) = x_0 \end{cases}$$
whose solution is $x(t) = x_0 e^{at}\cos t$ (a is a given real number). Study the conditioning of this IVP with respect to the choice of the initial datum and verify that on unbounded intervals it is well conditioned if $a < 0$, while it is ill conditioned if $a > 0$.

Solution:
$$G(x_0) = x(t) = x_0 e^{at}\cos t \quad G'(x_0) = e^{at}\cos t$$
$$\begin{aligned} K_{\mathrm{abs}}(x_0) &= |G'(x_0)| = |e^{at}\cos t| \\ K(x_0) &= 1 \end{aligned}$$
So, if $a < 0$, then $K_{\mathrm{abs}}(x_0)$ is bounded as $t \to \infty$, so it is well conditioned. But if $a > 0$, then $K_{\mathrm{abs}}(x_0)$ is unbounded as $t \to \infty$, so it is ill conditioned.

5. Derive geometrically the sequence of the first iterates computed by bisection, secant, false position and Newton's methods in the approximation of the zero of $f(x) = x^2 - 2$ in the interval $[1, 3]$. **Solution:** See Figure 1.

6. Let $f(x) = \cos^2(2x) - x^2$ be the function in the interval $0 \le x \le 1.5$. Having fixed a tolerance $\varepsilon = 10^{-10}$ on the absolute error, determine experimentally the subintervals for which Newton's method is convergent to the zero $\alpha \approx 0.5149$.

Solution: We search 1e6 point in the interval [0,1.5], and max iteration = 200. Within the 200 iteration, if $|x_n - \alpha| < \epsilon$ for one initial guess, we mark this initial guess that leads to convergence. The we have the subintervals experimentally, see Figure 2

7. Check the following properties of the fixed-point method:

   (a) $0 < g'(\alpha) < 1$: monotone convergence, that is, the error $x_n - \alpha$ maintains a constant sign as n varies;

   (b) $-1 < g'(\alpha) < 0$: oscillatory convergence, that is, $x_n - \alpha$ changes sign as n varies.

Solution: Assume $g(\alpha) = \alpha$ and g is differentiable at α with $|g'(\alpha)| < 1$. Consider the fixed-point iteration $x_{n+1} = g(x_n)$ and set $e_n := x_n - \alpha$.

Mean Value Theorem. For each n, there exists ξ_n between x_n and α such that
$$e_{n+1} = g(x_n) - g(\alpha) = g'(\xi_n)\,(x_n - \alpha) = g'(\xi_n)\,e_n.$$

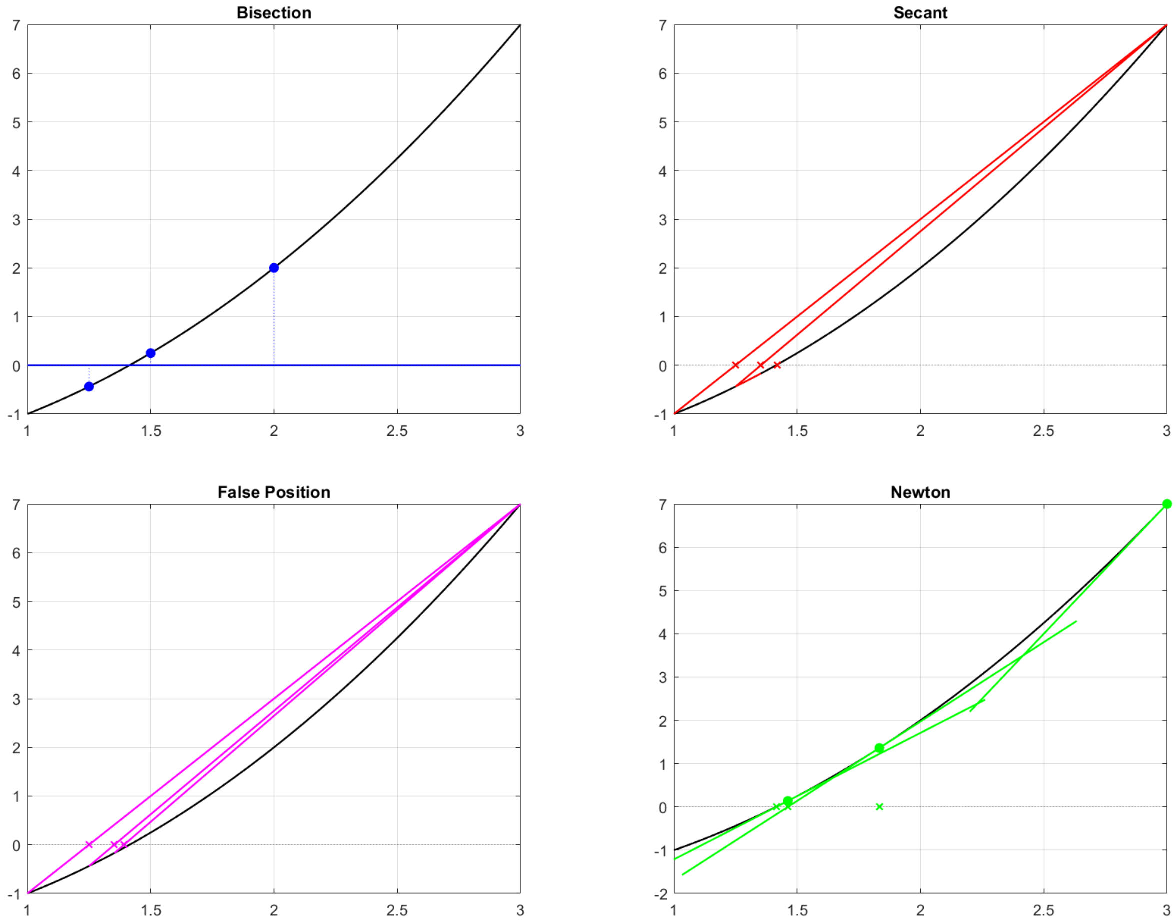

Figure 1: The figures of question5

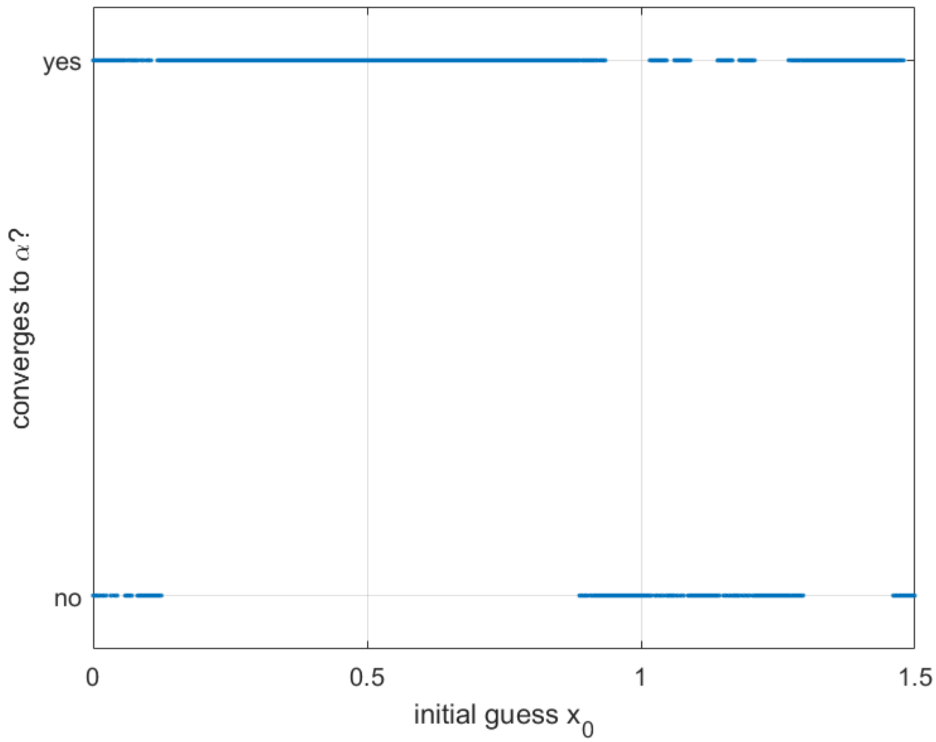

Figure 2: The figure of question6

Since g' is continuous at α, choose q and $r > 0$ so that

$$|g'(\alpha)| < q < 1 \quad \text{and} \quad |x - \alpha| < r \implies |g'(x)| \le q.$$

Take x_0 with $|x_0 - \alpha| < r$. If the iterates remain in this neighborhood, then

$$|e_{n+1}| = |g'(\xi_n)|\,|e_n| \le q\,|e_n| \quad \Rightarrow \quad |e_n| \le q^n |e_0|,$$

so $x_n \to \alpha$. Moreover,

$$\operatorname{sign}(e_{n+1}) = \operatorname{sign}(g'(\xi_n))\ \operatorname{sign}(e_n).$$

Because $\xi_n \to \alpha$ and g' is continuous, we have $g'(\xi_n) \to g'(\alpha)$, hence for all sufficiently large n:

$$\begin{cases} 0 < g'(\alpha) < 1 \ \Rightarrow\ \operatorname{sign}(e_{n+1}) = \operatorname{sign}(e_n) & \text{(one-sided/monotone convergence);} \\ -1 < g'(\alpha) < 0 \ \Rightarrow\ \operatorname{sign}(e_{n+1}) = -\operatorname{sign}(e_n) & \text{(oscillatory convergence).} \end{cases}$$

8. Analyze the convergence of the fixed-point method for computing the zeros $\alpha_1 = -1$ and $\alpha_2 = 2$ of the function $f(x) = x^2 - x - 2$, when the following iteration functions are used:

   (a) $g(x) = x^2 - 2$

   - *Fixed points:* Solve $x = g(x) \Leftrightarrow x^2 - x - 2 = 0$; hence $\alpha_1 = -1$, $\alpha_2 = 2$.
   - *Derivative:* $g'(x) = 2x$.

- *At the roots:* $g'(-1) = -2$ and $g'(2) = 4$.
- *Verdict:* $|g'(\alpha_1)| = 2 > 1$ and $|g'(\alpha_2)| = 4 > 1 \Rightarrow$ both fixed points are **repelling** $\Rightarrow$ the iteration **diverges**.

(b) $g(x) = \sqrt{x+2}$

- *Fixed points:* $x = \sqrt{x+2} \Rightarrow x^2 - x - 2 = 0$, but the branch requires $x \geq 0$, so only $\alpha_2 = 2$ is a fixed point; $\alpha_1 = -1$ is not.
- *Derivative:* $g'(x) = \dfrac{1}{2\sqrt{x+2}}$.
- *At the root:* $g'(2) = \dfrac{1}{4}$.
- *Verdict:* $|g'(2)| = \frac{1}{4} < 1 \Rightarrow$ **locally convergent** to α_2 , and since $g'(2) > 0$, the approach is **one-sided**. It cannot converge to α_1 under this branch because -1 is not a fixed point of g.

(c) $g(x) = -\sqrt{x+2}$

- *Fixed points:* $x = -\sqrt{x+2} \Rightarrow x^2 - x - 2 = 0$ with $x \leq 0$, so only $\alpha_1 = -1$ is a fixed point; $\alpha_2 = 2$ is not.
- *Derivative:* $g'(x) = -\dfrac{1}{2\sqrt{x+2}}$.
- *At the root:* $g'(-1) = -\dfrac{1}{2}$.
- *Verdict:* $|g'(-1)| = \frac{1}{2} < 1 \Rightarrow$ **locally convergent** to α_1, and since $g'(-1) < 0$, the approach is **oscillatory**. It cannot converge to α_2 because 2 is not a fixed point of g.

(d) $g(x) = 1 + \dfrac{2}{x}$

- *Fixed points:* $x = 1 + \dfrac{2}{x} \iff x^2 - x - 2 = 0$; hence $\alpha_1 = -1$, $\alpha_2 = 2$.
- *Derivative:* $g'(x) = -\dfrac{2}{x^2}$.
- *At the roots:* $g'(2) = -\dfrac{1}{2}$ and $g'(-1) = -2$.
- *Verdict:* At α_2: $|g'(2)| = \frac{1}{2} < 1 \Rightarrow$ **locally convergent** to 2 with **linear**, **oscillatory** behavior. At α_1: $|g'(-1)| = 2 > 1 \Rightarrow$ **diverges** (oscillatory).

9. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = \sqrt{1+x^2}$. Show that the iterates of Newton's method for the equation $g'(x) = 0$ satisfy the following properties:

(a) $|x_0| < 1 \Longrightarrow g(x_{n+1}) < g(x_n), n \geq 0, \quad \lim_{n\to\infty} x_n = 0,$

(b) $|x_0| > 1 \Longrightarrow g(x_{n+1}) > g(x_n), n \geq 0, \quad \lim_{n\to\infty} |x_n| = \infty.$

Solution: First compute

$$f(x) := g'(x) = \frac{x}{\sqrt{1+x^2}}, \qquad f'(x) = \left(1+x^2\right)^{-3/2}.$$

Hence the Newton iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\dfrac{x_n}{\sqrt{1+x_n^2}}}{(1+x_n^2)^{-3/2}} = x_n - x_n(1+x_n^2) = -x_n^3.$$

Therefore

$$|x_{n+1}| = |x_n|^3 \qquad (n \geq 0).$$

(a) If $|x_0| < 1$

Then $|x_1| = |x_0|^3 < |x_0| < 1$. By induction, $|x_n| < 1$ and the sequence $(|x_n|)$ is strictly decreasing to 0 because repeatedly cubing a number in $(0, 1)$ drives it to 0. Thus $x_n \to 0$.

Moreover,

$$g(x_{n+1}) = \sqrt{1+x_{n+1}^2} = \sqrt{1+x_n^6} < \sqrt{1+x_n^2} = g(x_n) \qquad (\text{since } 0 < x_n^6 < x_n^2),$$

so $g(x_{n+1}) < g(x_n)$ for all n.

(b) If $|x_0| > 1$

Then $|x_1| = |x_0|^3 > |x_0| > 1$. By induction, $|x_{n+1}| = |x_n|^3 > |x_n|$, hence $|x_n| \to \infty$. Since $g(x) = \sqrt{1+x^2}$ is strictly increasing in $|x|$, we get

$$g(x_{n+1}) = \sqrt{1+x_{n+1}^2} > \sqrt{1+x_n^2} = g(x_n),$$

so $g(x_{n+1}) > g(x_n)$ for all n.