1. Using Descartes' rule of signs, determine the number of real roots (both positive and negative) of the polynomials $p_6(x) = x^6 - x - 1$ and $p_4(x) = x^4 - x^3 - x^2 + x - 1$.

Solution:

- For $p_6(x) = x^6 - x - 1$: ignoring zeros, the sign pattern of coefficients is $+, -, -$, so there is 1 sign change. Hence p_6 has exactly 1 positive real root. For negative roots, consider $p_6(-x) = x^6 + x - 1$ whose nonzero signs are $+, +, -$ giving 1 sign change, so p_6 has exactly 1 negative real root (both counts are with multiplicity).
- For $p_4(x) = x^4 - x^3 - x^2 + x - 1$: signs $+, -, -, +, -$ yield 3 sign changes, so the number of positive real roots is 3 or 1. Actually, it is 1. For negative roots, $p_4(-x) = x^4 + x^3 - x^2 - x - 1$ has signs $+, +, -, -, -$ with 1 sign change, so there is exactly 1 negative real root (counting multiplicity).

2. Define an iteration formula by

$$x_{n+1} = z_{n+1} - \frac{f\left(z_{n+1}\right)}{f'\left(x_n\right)}, \quad z_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}.$$

Show that the order of convergence of $\{x_n\}$ to α is at least 3 .
Hint: Introduce the following two functions:

$$g(x) = h(x) - \frac{f(h(x))}{f'(x)}, \quad h(x) = x - \frac{f(x)}{f'(x)}$$

and take the required number of derivatives of g and use an appropriate theorem proved in class.

Proof. Let $f \in C^3$ near a simple root α, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Define

$$h(x) = x - \frac{f(x)}{f'(x)}, \qquad g(x) = h(x) - \frac{f(h(x))}{f'(x)}.$$

Then the iteration is $x_{n+1} = g(x_n)$. We will show $g(\alpha) = \alpha$, $g'(\alpha) = g''(\alpha) = 0$ (and generically $g^{(3)}(\alpha) \neq 0$), so by the stated theorem the order is at least 3.

Set

$$A = f'(\alpha), \quad B = f''(\alpha), \quad C = f^{(3)}(\alpha), \quad e = x - \alpha.$$

Using Taylor expansions about α,

$$f(x) = Ae + \frac{B}{2}e^2 + \frac{C}{6}e^3 + O(e^4), \qquad f'(x) = A + Be + \frac{C}{2}e^2 + O(e^3).$$

A straightforward series division gives

$$\frac{f(x)}{f'(x)} = e - \frac{B}{2A}e^2 + \left(-\frac{C}{3A} + \frac{B^2}{2A^2}\right)e^3 + O(e^4).$$

Hence the Newton map satisfies

$$\begin{aligned} h(x) &= x - \frac{f(x)}{f'(x)} \\ &= \alpha + \frac{B}{2A}e^2 + \Big(\frac{C}{3A} - \frac{B^2}{2A^2}\Big)e^3 + O(e^4), \end{aligned}$$

so

$$h(x) - \alpha = c_2 e^2 + c_3 e^3 + O(e^4), \qquad c_2 = \frac{B}{2A}, \quad c_3 = \frac{C}{3A} - \frac{B^2}{2A^2}.$$

Next, expand $f(h(x))$ and divide by $f'(x)$:

$$f(h(x)) = A\,(h(x) - \alpha) + \frac{B}{2}(h(x) - \alpha)^2 + O(e^5) = A(c_2 e^2 + c_3 e^3) + O(e^4),$$

and

$$\frac{f(h(x))}{f'(x)} = \frac{A(c_2 e^2 + c_3 e^3)}{A + Be + \frac{C}{2}e^2 + O(e^3)} = c_2 e^2 + \Big(c_3 - \frac{B}{A}c_2\Big)e^3 + O(e^4).$$

Therefore

$$\begin{aligned} g(x) - \alpha &= (h(x) - \alpha) - \frac{f(h(x))}{f'(x)} \\ &= \frac{B}{A}c_2\, e^3 + O(e^4) = \frac{B^2}{2A^2}\, e^3 + O(e^4). \end{aligned}$$

This implies

$$g(\alpha) = \alpha, \qquad g'(\alpha) = g''(\alpha) = 0, \qquad g^{(3)}(\alpha) = 3! \cdot \frac{B^2}{2A^2} = \frac{3B^2}{A^2} \quad (B \neq 0).$$

By the supplied fixed-point theorem, the iteration $x_{n+1} = g(x_n)$ has order at least 3 (and exactly 3 if $f''(\alpha) \neq 0$; if $f''(\alpha) = 0$ the order is higher). □

3. There is another modification of Newton's method, similar to the secant method, but using a different approximation of the derivative $f'(\xi_n)$. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)}, \quad D(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}.$$

This one-point method is called Steffenson's method. Assuming $f'(\alpha) \neq 0$, show that this is a second-order method.

Hint: Write the iteration as $x_{n+1} = g(x_n)$. Use $f(x) = (x - \alpha)h(x)$ with $h(\alpha) \neq 0$, and then compute the formula for $g(x)$ in terms of $h(x)$. Having done so, take the required number of derivatives of g and use an appropriate theorem proved in class.

Proof. Let α be a simple root of f, so $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Write

$$f(x) = (x - \alpha)h(x), \qquad h \in C^2 \text{ near } \alpha, \quad h(\alpha) =: h_0 \neq 0.$$

Define the Steffensen iteration $x_{n+1} = g(x_n)$ with

$$g(x) = x - \frac{f(x)}{D(x)}, \qquad D(x) = \frac{f(x + f(x)) - f(x)}{f(x)}.$$

First express g in terms of h. Using $f(x) = (x - \alpha)h(x)$,

$$f(x + f(x)) = \big((x - \alpha) + f(x)\big)\, h(x + f(x)) = (x - \alpha)\big(1 + h(x)\big)h\big(x + f(x)\big).$$

Hence

$$D(x) = \frac{(x - \alpha)(1 + h(x))h(x + f(x)) - (x - \alpha)h(x)}{(x - \alpha)h(x)} = \frac{(1 + h(x))h(x + f(x)) - h(x)}{h(x)}.$$

Therefore

$$\frac{f(x)}{D(x)} = \frac{(x - \alpha)h(x)}{\dfrac{(1 + h(x))h(x + f(x)) - h(x)}{h(x)}} = (x - \alpha)\,\frac{h(x)^2}{(1 + h(x))h(x + f(x)) - h(x)}.$$

It follows that

$$g(x) - \alpha = (x - \alpha)\left[1 - \frac{h(x)^2}{(1 + h(x))h(x + f(x)) - h(x)}\right].$$

Set

$$Q(x) := 1 - \frac{N(x)}{M(x)}, \qquad N(x) := h(x)^2, \qquad M(x) := (1 + h(x))h(x + f(x)) - h(x),$$

so that $g(x) - \alpha = (x - \alpha)Q(x)$.

Step 1: $g'(\alpha) = 0$. Since $f(\alpha) = 0$, we have $x + f(x) \to \alpha$ as $x \to \alpha$, hence

$$N(\alpha) = h_0^2, \qquad M(\alpha) = (1 + h_0)h_0 - h_0 = h_0^2,$$

and thus $Q(\alpha) = 1 - \dfrac{N(\alpha)}{M(\alpha)} = 0$. Consequently,

$$g'(\alpha) = \lim_{x\to\alpha} \frac{g(x) - \alpha}{x - \alpha} = \lim_{x\to\alpha} Q(x) = Q(\alpha) = 0.$$

Step 2: $g''(\alpha) = 2\,\dfrac{h'(\alpha)}{h(\alpha)}\big(1+h(\alpha)\big)$. Since $g(x)-\alpha = (x-\alpha)Q(x)$, we have $g'(\alpha) = Q(\alpha) = 0$ and $g''(\alpha) = 2Q'(\alpha)$. Differentiate

$$Q(x) = 1 - \frac{N(x)}{M(x)} \quad \Rightarrow \quad Q'(x) = -\frac{N'(x)M(x) - N(x)M'(x)}{M(x)^2}.$$

Let $h_1 := h'(\alpha)$. Then $N'(x) = 2h(x)h'(x)$, so $N'(\alpha) = 2h_0h_1$. For M,

$$M'(x) = h'(x)\, h(x + f(x)) + (1 + h(x))\, h'(x + f(x))\, (1 + f'(x)) - h'(x),$$

and since $f'(x) = h(x) + (x-\alpha)h'(x)$, we have $f'(\alpha) = h_0$. Passing to the limit $x \to \alpha$ gives

$$M'(\alpha) = h_1 h_0 + (1+h_0)h_1(1+h_0) - h_1 = h_1(3h_0 + h_0^2).$$

Using $N(\alpha) = M(\alpha) = h_0^2$, we obtain

$$Q'(\alpha) = -\frac{(2h_0h_1)h_0^2 - h_0^2\, h_1(3h_0 + h_0^2)}{h_0^4} = \frac{h_1(1+h_0)}{h_0}.$$

Therefore

$$g''(\alpha) = 2Q'(\alpha) = 2\,\frac{h'(\alpha)}{h(\alpha)}\big(1 + h(\alpha)\big).$$

Conclusion. We have $g \in C^2$ near α, $g(\alpha) = \alpha$, $g'(\alpha) = 0$, and generically $g''(\alpha) \neq 0$. By the stated theorem (if $g^{(k)}(\alpha) = 0$ for $1 \le k \le p$ and $g^{(p+1)}(\alpha) \neq 0$, then the fixed-point iteration $x_{n+1} = g(x_n)$ has order $p+1$), the iteration $x_{n+1} = g(x_n)$ is of order 2. Hence Steffensen's method is (at least) second order. □

4. (a) Show that an eigenvalue λ of an orthogonal matrix satisfies $|\lambda| = 1$

   (b) Show that $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal. Find the eigenvalues of R_θ.

   (c) A permutation matrix is a matrix whose elements are either 0 or 1 , such that every row and column has precisely one nonzero element. Show that a permutation matrix is orthogonal. Give an example of a 4×4 permutation matrix whose eigenvalues are $\pm 1, \pm i$.

Solution:

(a) Let Q be orthogonal, so $Q^TQ = I$. If λ is an eigenvalue with eigenvector $v \neq 0$, then $Qv = \lambda v$. Taking conjugate transpose, $v^*Q^T = \bar{\lambda}\, v^*$. Hence

$$\bar{\lambda}\lambda\, v^*v = v^*Q^TQv = v^*v.$$

Since $v^*v = \|v\|_2^2 > 0$, we get $|\lambda|^2 = 1$, i.e., $|\lambda| = 1$.

(b) Compute $R_\theta^T R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, so R_θ is orthogonal. And

$$\det(R_\theta - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\cos\theta\,\lambda + 1.$$

Solving, $\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

(c) Let P be a permutation matrix. Each row has exactly one 1 (say in column $\sigma(i)$), so row i is $e_{\sigma(i)}^T$, where e_j is the j-th standard basis vector. Then $(P^TP)_{jk} = \sum_i p_{ij}p_{ik}$. Since each row has only one 1, this sum equals 1 if $j = k$ (that column's single 1) and 0 otherwise. Hence $P^TP = I$, so P is orthogonal.

For an example with eigenvalues $\pm 1, \pm i$, take the 4×4 permutation matrix corresponding to the cyclic permutation $(1\,2\,3\,4)$:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\det(\lambda I - P) = \lambda^4 - 1$, with roots $\lambda = \pm 1, \pm i$.

5. For each of the following matrices, determine whether it is normal, diagonalizable. For those which are diagonalizable, find the e-values and e-vectors.

   (a) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$: normal (real symmetric); diagonalizable over $\mathbb{R}$. Eigenvalues 3, 1 with eigenvectors e.g. $v_3 = (1,-1)^T$, $v_1 = (1,1)^T$.

   (b) $\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$: not normal; not diagonalizable. Single eigenvalue 2 (alg. mult. 2, geom. mult. 1); eigenspace $\ker(A-2I) = \{(0,y)^T\}$, e.g. $v = (0,1)^T$.

   (c) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$: normal; diagonalizable over $\mathbb{C}$ (not over $\mathbb{R}$). Eigenvalues $2 \pm i$ with eigenvectors $v_{2+i} = (1,i)^T$, $v_{2-i} = (1,-i)^T$.

   (d) $\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$: not normal; diagonalizable over $\mathbb{R}$. Eigenvalues 1, -1 with eigenvectors $v_1 = (1,0)^T$, $v_{-1} = (1,2)^T$.

6. Prove the following assertions.

   (a) $\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\|_2 \le \|\boldsymbol{x}\|_1$ for any vector $\boldsymbol{x}$.

   Proof. Let $x = (x_1, \dots, x_n)^T$. Since $\|x\|_2^2 = \sum_i |x_i|^2 \ge \max_i |x_i|^2 = \|x\|_\infty^2$, we get $\|x\|_\infty \le \|x\|_2$. Moreover,

$$\|x\|_2^2 = \sum_i |x_i|^2 \le \Big(\sum_i |x_i|\Big)^2 = \|x\|_1^2,$$

   hence $\|x\|_2 \le \|x\|_1$. □

   (b) Given a vector norm $\|\boldsymbol{x}\|$, the formula $\|A\| = \sup_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$ defines a matrix norm.

   Proof. (i) Nonnegativity and definiteness: the ratio is ≥ 0, so $\|A\| \ge 0$. If $\|A\| = 0$, then for every $x \neq 0$, $\|Ax\|/\|x\| \le 0$, hence $Ax = 0$ for all x, so $A = 0$. (ii) Homogeneity: $\|\alpha A\| = \sup_{x\neq 0} \dfrac{\|\alpha Ax\|}{\|x\|} = |\alpha| \sup_{x\neq 0} \dfrac{\|Ax\|}{\|x\|} = |\alpha|\,\|A\|$. (iii) Triangle inequality: $\|A+B\| = \sup_{x\neq 0} \dfrac{\|(A+B)x\|}{\|x\|} \le \sup_{x\neq 0} \dfrac{\|Ax\| + \|Bx\|}{\|x\|} \le \|A\| + \|B\|$. (iv) Submultiplicativity: for any $y \neq 0$, by the definition of $\|A\|$ we have $\|Ay\| \le \|A\|\,\|y\|$. Hence for any $x \neq 0$, $\dfrac{\|ABx\|}{\|x\|} \le \|A\| \dfrac{\|Bx\|}{\|x\|} \le \|A\|\,\|B\|$. Taking supremum over $x \neq 0$ gives $\|AB\| \le \|A\|\,\|B\|$. Therefore this formula defines a matrix norm. (v) Consistency: for any $x \neq 0$, $\|Ax\|/\|x\| \le \|A\|$ by definition, so $\|Ax\| \le \|A\|\,\|x\|$. □

   (c) $\|A\|_\infty = \max_i \sum_j |a_{ij}|$.

   Proof. Let $\|A\|_\infty := \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$ with $\|x\|_\infty = \max_j |x_j|$. For any x,

$$\|Ax\|_\infty = \max_i \Big|\sum_j a_{ij}x_j\Big| \le \max_i \sum_j |a_{ij}|\,|x_j| \le \Big(\max_i \sum_j |a_{ij}|\Big)\|x\|_\infty,$$

hence $\|A\|_\infty \leq \max_i \sum_j |a_{ij}|$.
Then we only need to show

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \max_i \sum_j |a_{ij}|, \quad \text{for some } x \neq 0.$$

Choose $\hat{x}$ s.t.

$$x_j = \begin{cases} 1 & a_{i^*j} \geq 0 \\ -1, & a_{i^*j} < 0 \end{cases}$$

Then $\|\hat{x}\|_\infty = 1$, we only need to show:

$$\|A\hat{x}\|_\infty \geq \max_i \sum_j |a_{ij}|.$$

Fix a row i^*, s.t. $i^* \in \arg\max_i \sum_j |a_{ij}|$. Then,

$$\|Ax\|_\infty \geq |(Ax)_{i^*}| = \left| \sum_j a_{i^*j} x_j \right| = \sum_j |a_{i^*j}| = \max_i \sum_j |a_{ij}|.$$

$\square$