

# Proof of Irrationality of $\pi$

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## Abstract

$\pi$  is known as a familiar irrational number. However, in the high school curriculum and undergraduate courses, we accept the fact that  $\pi$  is irrational without any proof. In this article, we review the three proofs of Ivan Niven, Lambert and Cartwright.

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## 1 Introduction

In 1761, Johann Heinrich Lambert<sup>1</sup> reported the first proof that the number  $\pi$  is irrational, in other words  $\pi$  cannot be expressed as a fraction  $\frac{a}{b}$ , where  $a$  and  $b$  are both non-zero integers. In 1873, Charles Hermite<sup>23</sup> found another proof which only uses basic calculus. Three simplifications of Hermite's proof are due to Mary Cartwright, Ivan Niven<sup>4</sup>, and Nicolas Bourbaki<sup>5</sup>. Another proof, which is a simplification of Lambert's proof, is due to Miklós Laczkovich<sup>6</sup>. Many of these are *reductio ad absurdum*.

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<sup>1</sup>Lambert, Johann Heinrich (1768), "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques", in Berggren.

<sup>2</sup>Hermite, Charles (1873). "Extrait d'une lettre de Monsieur Ch. Hermite à Monsieur Paul Gordan". Journal für die reine und angewandte Mathematik. 76: 303–311.

<sup>3</sup>Hermite, Charles (1873). "Extrait d'une lettre de Mr. Ch. Hermite à Mr. Carl Borchardt". Journal für die reine und angewandte Mathematik. 76: 342–344.

<sup>4</sup>Niven, Ivan (1947), "A simple proof that  $\pi$  is irrational", Bulletin of the American Mathematical Society, vol. 53, no. 6, p. 509.

<sup>5</sup>Bourbaki, Nicolas (1949), Fonctions d'une variable réelle, chap. I–II–III, Actualités Scientifiques et Industrielles, vol. 1074, Hermann, pp. 137–138.

<sup>6</sup>Laczkovich, Miklós (1997), "On Lambert's proof of the irrationality of  $\pi$ ", American Mathematical Monthly, vol. 104, no. 5, pp. 439–443.

## 2 Ivan Niven's proof

**Lemma 2.1.**  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$  for all  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$

*Proof.* Note that Taylor series of  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . By the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x+1}{n+1} \right| = 0.$$

Therefore for any  $x \in \mathbb{R}$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges, which means  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$  for any  $x \in \mathbb{R}$ .  $\square$

Suppose that  $\pi$  is rational, such that  $\pi = \frac{a}{b}$ . ( $a, b \in \mathbb{Z}^+$ ,  $a, b$  are rel. prime.)

**Definition 2.1.** Let  $f \in C^{2n+2}$  be

$$f(x) = \frac{x^n(a-bx)^n}{n!} \quad (1)$$

where  $n \in \mathbb{N}$ .

**Definition 2.2.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be

$$F(x) = f(x) - f''(x) - \dots + (-1)^n f^{(2n)}(x) \quad (2)$$

where  $n \in \mathbb{N}$ .

In this section,  $f$  and  $F$  are used as defined above.

**Lemma 2.2.**  $f(x) = f(\pi - x)$  for all  $x \in \mathbb{R}$ .

*Proof.*

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{x^n b^n (\frac{a}{b} - x)^n}{n!} = \frac{(\pi - x)^n (bx)^n}{n!}$$

$$bx = a - a + bx = a - b(\frac{a}{b} - x) = a - b(\pi - x)$$

$$f(x) = \frac{(\pi - x)^n (bx)^n}{n!} = \frac{(\pi - x)^n (a - b(\pi - x))^n}{n!} = f(\pi - x).$$

Therefore,  $f(x) = f(\pi - x)$  for all  $x \in \mathbb{R}$ .  $\square$

Also, since  $f(x)$  is a polynomial of degree  $2n$ ,  $f(x)$  can be expressed as

$$f(x) = \frac{x^n(a-bx)^n}{n!} = \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \dots + \frac{c_n}{n!}x^{2n},$$

where  $c_n \in \mathbb{Z}$ .

**Lemma 2.3.**  $F(0) + F(\pi) \in \mathbb{Z}$

*Proof.*

**Claim 1)**  $F(0) \in \mathbb{Z}$ .

It can be easily seen that

$$F(0) = f(0) - f''(0) - \dots + (-1)^n f^{(2n)}(0).$$

by Eq. (2). Therefore, if  $f^{(k)}(0) \in \mathbb{Z}$  for all  $k \in \mathbb{N}$ ,  $F(0) \in \mathbb{Z}$ .

**Case 1-1)** For  $k < n$ ,

$$\begin{aligned} f(x) &= \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \dots + \frac{c_n}{n!}x^{2n} \\ f^{(k)}(x) &= C_0x^{n-k} + C_1x^{n+1-k} + \dots + C_nx^{2n-k}, \end{aligned}$$

where  $C_i = \frac{c_i}{n!} \prod_{j=1}^k (n+i+1-j)$  for  $i \in \{0, 1, 2, \dots, n\}$  and  $j \in \{1, 2, \dots, k\}$ .

$$\therefore f^{(k)}(x) = C_0 0^{n-k} + C_1 0^{n+1-k} + \dots + C_n 0^{2n-k} = 0.$$

Therefore, if  $k < n$ ,  $f^{(k)}(0) = 0$ .

**Case 1-2)** For  $n \leq k \leq 2n$ ,

$$\begin{aligned} f(x) &= \frac{c_0}{n!}x^n + \frac{c_1}{n!}x^{n+1} + \dots + \frac{c_n}{n!}x^{2n} \\ f^{(k)}(x) &= 0 + 0 + \dots + \frac{c_k}{n!}k! + \frac{c_k}{n!}(k+1)!x + \dots \\ f^{(k)}(0) &= \frac{c_k}{n!}k! \end{aligned}$$

Therefore, if  $n \leq k \leq 2n$ ,  $f^{(k)}(0) = \frac{c_k}{n!}k! \in \mathbb{Z}$ .

**Case 1-3)** For  $k > 2n$ ,

Since  $\deg f(x) = 2n$ , it is trivial that  $f^{(k)}(x) = 0$ . Therefore, if  $k > 2n$ ,  $f^{(k)}(x) = 0 \in \mathbb{Z}$ .

Summarizing the above discussion, it can be shown that  $f^{(k)}(0) \in \mathbb{Z}$  for all  $k \in \mathbb{N}$  and

$$f^{(k)}(x) = \begin{cases} 0, & \text{if } k < n, \\ \frac{c_k}{n!}k!, & \text{if } n \leq k \leq 2n, \\ 0, & \text{if } k > 2n. \end{cases}$$

**Claim 2)**  $F(\pi) \in \mathbb{Z}$ .

$$F(\pi) = f(\pi) - f''(\pi) - \dots + (-1)^n f^{(2n)}(\pi).$$

Therefore, If  $f^{(k)}(\pi) \in \mathbb{Z}$  for all  $k \in \mathbb{N}$ ,  $F(\pi) \in \mathbb{Z}$ .

By lemma 2.2 and the chain rule,

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x).$$

Substituting  $\pi$  to  $x$  in (2), we get

$$f^{(k)}(\pi) = (-1)^k f^{(k)}(0). \quad (3)$$

By Eq. (3) and the Claim 1, we get  $f^{(k)}(\pi) \in \mathbb{Z}$  for all  $k \in \mathbb{N}$ .

From both cases, we prove that

$$F(0) + F(\pi) \in \mathbb{Z}.$$

□

**Lemma 2.4.**  $F(0) + F(\pi) = \int_0^\pi f(x) \sin x \, dx$ .

*Proof.* By Eq. (2),

$$\begin{aligned} F''(x) &= f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{(n)} f^{(2n)}(x) \\ F(x) &= f(x) - \{f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots + (-1)^{(2n)}(x)\} \\ &= f(x) - F''(x) \\ f(x) &= F(x) + F''(x). \end{aligned} \tag{4}$$

Multiply  $\sin x$  on both sides of Eq. (4), we get

$$f(x) \sin x = F(x) \sin x + F''(x) \sin x. \tag{5}$$

When we integrate both sides of Eq. (5) over  $(0, \pi)$ , we get the desired identity:

$$\begin{aligned} &\int_0^\pi f(x) \sin x \, dx \\ &= \int_0^\pi F(x) \sin x \, dx + \int_0^\pi F''(x) \sin x \, dx \\ &= [-F(x) \cos x]_0^\pi + \int_0^\pi F'(x) \cos x \, dx \\ &+ [F'(x) \cos x]_0^\pi - \int_0^\pi F'(x) \cos x \, dx \\ &= F(0) + F(\pi). \end{aligned}$$

□

**Theorem 2.5** (Niven's Proof).  $\pi$  is irrational.

*Proof.* With some algebraic manipulations we get the following inequality.

$$\begin{aligned} (\pi - x)bx &= -b(x^2 - \pi x) \\ &= -b \left(x - \frac{\pi}{2}\right)^2 + \frac{b\pi^2}{4} \\ &= -b \left(x - \frac{\pi}{2}\right)^2 + \frac{a\pi}{4} \leq \frac{a\pi}{4}. \end{aligned}$$

By the above inequality, we can show that

$$\begin{aligned} 0 &< (\pi - x)(bx) \leq \frac{a\pi}{4}, \\ 0 &< \frac{(\pi - x)^n (bx)^n}{n!} \leq \frac{\pi}{n!} \left(\frac{a\pi}{4}\right)^n, \\ 0 &< f(x) \leq \frac{\pi}{n!} \left(\frac{a\pi}{4}\right)^n. \end{aligned} \tag{6}$$

for  $x \in (0, \pi)$ .

For  $x \in (0, \pi)$ ,  $0 < \sin x < 1$ . Using this fact with Eq. (6), this leads

$$\begin{aligned} 0 &< f(x) \sin x < \frac{1}{n!} \left( \frac{a\pi}{4} \right)^n, \\ 0 &< \int_0^\pi f(x) \sin x \, dx < \int_0^\pi \frac{1}{n!} \left( \frac{a\pi}{4} \right)^n \, dx = \frac{\pi}{n!} \left( \frac{a\pi}{4} \right)^n. \end{aligned}$$

Let  $a_n = \frac{\pi}{n!} \left( \frac{a\pi}{4} \right)^n$ . We can see that

$$\lim_{n \rightarrow \infty} a_n = 0$$

by lemma 2.1 or by the monotone convergence theorem.

Hence, for sufficiently large  $n$ ,

$$\frac{\pi}{n!} \left( \frac{a\pi}{4} \right)^n < 1.$$

By two inequalities we stated above, we show that

$$0 < \int_0^\pi f(x) \sin x \, dx < \frac{\pi}{n!} \left( \frac{a\pi}{4} \right)^n < 1. \quad (7)$$

The Eq. (7) contradicts with lemma 2.3 and 2.4 because there is no integer between 0 and 1. Therefore,  $\pi$  is irrational.  $\square$

### 3 Lambert's proof

**Definition 3.1.** A finite continued fraction is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}},$$

where  $a_0 \in \mathbb{R}$ ,  $a_k \in \mathbb{R}^+$  for  $k \in \{1, 2, \dots, n\}$ . The continued fraction is called simple if  $a_0, a_1, \dots, a_n \in \mathbb{Z}$ .

**Lemma 3.1.** Every irrational number can be written as a infinite continued fraction.

*Proof.*

**Claim 1)** Every finite simple continued fraction can be written as a rational number.

We will prove it using mathematical induction. For  $n = 1$ , we have

$$[a_0; a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1},$$

which is rational. Now, we assume that for the positive integer  $k$  the simple continued fraction  $[a_0; a_1, a_2, \dots, a_k]$  is rational whenever  $a_0 \in \mathbb{Z}, a_i \in \mathbb{Z}^+$  for  $i \in \{1, 2, \dots, k\}$ . Let  $a_0 \in \mathbb{Z}, a_i \in \mathbb{Z}^+$  for  $i \in \{1, 2, \dots, k+1\}$ . Note that

$$[a_0; a_1, \dots, a_{k+1}] = a_0 + \frac{1}{[a_1; a_2, \dots, a_{k+1}]}.$$

By the induction hypothesis,  $[a_1; a_2, \dots, a_{k+1}]$  is rational; hence, there are integers  $r$  and  $s$ , with  $s \neq 0$ , such that this continued fraction equals  $r/s$ . Then

$$[a_1; a_2, \dots, a_{k+1}] = a_0 + \frac{1}{r/s} = \frac{a_0 r + s}{r},$$

which is again a rational number.

**Claim 2)** Every irrational number can be written as a *infinite continued fraction*.

The contrapositive of Claim 1 is Claim 2. Therefore, every irrational number can be written as a infinite continued fraction.  $\square$

**Lemma 3.2.** 
$$\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \ddots}}}}$$

for  $x \in \mathbb{R}$ .

*Proof.* Using Maclaurin series,  $\tan x$  can be expressed as

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots} \\
&= \frac{x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots} = \frac{x}{\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} \\
&= \frac{x}{\frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) + \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} \\
&= \frac{x}{\frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) + \left(-\frac{2}{3!}x^2 + \frac{4}{5!}x^4 - \frac{6}{7!}x^6 + \dots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} \\
&= \frac{x}{1 - \frac{\left(\frac{2}{3!}x^2 - \frac{4}{5!}x^4 + \frac{6}{7!}x^6 - \dots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} = \frac{x}{1 - \frac{x^2 \left(\frac{2}{3!} - \frac{4}{5!}x^2 + \frac{6}{7!}x^4 - \dots\right)}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}} \\
&= \frac{x}{1 - \frac{\frac{x^2}{\left(\frac{2}{3!} - \frac{4}{5!}x^2 + \frac{6}{7!}x^4 - \dots\right)}}{1 - \frac{x^2}{\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}}}} = \frac{x}{1 - \frac{x^2}{\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{\frac{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) - 3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right) + 3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{\frac{3\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right) + \left(-\frac{2}{5} \frac{x^2}{3!} + \frac{4}{7} \frac{x^4}{5!} - \dots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{3 - \frac{\left(\frac{2}{5} \frac{x^2}{3!} - \frac{4}{7} \frac{x^4}{5!} + \dots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}}} = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2 \left(\frac{2}{5} \frac{1}{3!} - \frac{4}{7} \frac{x^2}{5!} + \dots\right)}{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}}} \\
&= \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{\frac{\left(\frac{1}{3} - \frac{1}{53!}x^2 + \frac{1}{75!}x^4 - \dots\right)}{\left(\frac{2}{5} \frac{1}{3!} - \frac{4}{7} \frac{x^2}{5!} + \dots\right)}}}} \\
&\dots
\end{aligned}$$

Repeating this procedure, we can prove that

$$\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \ddots}}}},$$

for  $x \in \mathbb{R}$ . □

**Lemma 3.3.** *If  $x$  is rational,  $\tan x$  is irrational.*

*Proof.* Let  $x = \frac{p}{q}$ , such that  $p, q \in \mathbb{Z}$ ,  $p, q$  are rel. prime.

$$\tan x = \frac{\left(\frac{p}{q}\right)}{1 - \frac{\left(\frac{p}{q}\right)^2}{3 - \frac{\left(\frac{p}{q}\right)^2}{5 - \frac{\left(\frac{p}{q}\right)^2}{7 - \ddots}}}}.$$

$\therefore \tan\left(\frac{p}{q}\right)$  is an infinite continued fraction. Therefore, by Theorem 3.2., if  $x$  is a rational number,  $\tan x$  is irrational. □

**Theorem 3.4** (Lambert's Proof).  *$\pi$  is irrational.*

Suppose that  $\pi$  is a rational number. Then,  $\frac{\pi}{4}$  is also a rational number. Put  $x = \frac{\pi}{4}$  into  $\tan x$ ,

$$\tan \frac{\pi}{4} = 1.$$

However, it contradicts with lemma 3.3 Therefore,  $\pi$  is irrational. □

## 4 Cartwright's proof

Define  $I_n : \mathbb{R} \rightarrow \mathbb{R}$  as  $I_n(x) = \int_{-1}^1 (1 - z^2)^n \cos(xz) dz$ , where  $n$  is a non-negative integer.

**Lemma 4.1.** *For  $n \leq 2$ ,  $x^2 I_n(x) = 2n(2n - 1)I_{n-1}(x) - 4n(n - 1)I_{n-2}(x)$ .*



*Proof.*

$$\begin{aligned}
I_n(x) &= \int_{-1}^1 (1-z^2)^n \cos(xz) dz = \left[ \frac{(1-z^2)^n \sin(xz)}{x} \right]_{-1}^1 + \int_{-1}^1 \frac{2nz(1-z^2)^{n-1} \sin(xz)}{x} dz \\
&= 2n \int_{-1}^1 \frac{2nz(1-z^2)^{n-1} \sin(xz)}{x} dz \\
&= -2n \left[ \frac{(1-z^2)^{n-1} \cos(xz)}{x} \right]_{-1}^1 + 2n \int_{-1}^1 \frac{\{(1-z^2)^{n-1} - 2(n-1)z^2(1-z^2)^{n-2}\} \cos(xz)}{x^2} dz \\
&= \frac{2n}{x^2} \int_{-1}^1 \{(1-z^2)^{n-1} - 2(n-1)z^2(1-z^2)^{n-2}\} \cos(xz) dz \\
&= \frac{2n}{x^2} \int_{-1}^1 (1-z^2)^{n-1} \cos(xz) dz - \frac{2n(n-1)}{x^2} \int_{-1}^1 2z^2(1-z^2)^{n-2} \cos(xz) dz. \\
\therefore I_n(x) &= \frac{2n}{x} I_{n-1}(x) - \frac{4n(n-1)}{x^2} \int_{-1}^1 z^2(1-z^2)^{n-2} \cos(xz) dz.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
I_{n-1}(x) - I_{n-2}(x) &= \int_{-1}^1 (1-z^2)^{n-1} \cos(xz) dz - \int_{-1}^1 (1-z^2)^{n-2} \cos(xz) dz \\
&= \int_{-1}^1 (1-z^2)^{n-2} (1-z^2 - 1) \cos(xz) dz \\
&= - \int_{-1}^1 z^2 (1-z^2)^{n-2} \cos(xz) dz.
\end{aligned}$$

$$\begin{aligned}
\therefore I_n(x) &= \frac{2n}{x} I_{n-1}(x) + \frac{4n(n-1)}{x^2} \{I_{n-1}(x) - I_{n-2}(x)\} \\
&= \frac{2n(2n-1)}{x^2} I_{n-1}(x) - \frac{4n(n-1)}{x^2} I_{n-2}(x).
\end{aligned}$$

Therefore, For  $n \leq 2$ ,  $x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$ .  $\square$

If  $J_n(x) = x^{2n+1} I_n(x)$ , then this becomes

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).$$

for  $n \geq 2$ . Furthermore,

$$J_0(x) = 2 \sin x, J_1(x) = -4x \cos x + 4 \sin x.$$

**Lemma 4.2.** For  $n \in \mathbb{Z}^+$ ,  $J_n(x) = n!(P_n(x) \cos x + Q_n(x) \sin x)$ , where  $P_n(x), Q_n(x)$  are polynomials of degree  $\leq n$  with integer coefficients (depending on  $n$ ).

*Proof.* We will prove the statement by mathematical induction. For  $n = 0$  and  $n = 1$ , we already proved them previously. Now, we assume that for the positive integers  $1, 2, \dots, k$ ,

$$J_k(x) = k!(P_k(x) \cos x + Q_k(x) \sin x).$$

Note that

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2J_{n-2}(x).$$

$$\begin{aligned} J_{k+1}(x) &= 2(k+1)(2k+1)J_k(x) - 4k(k+1)x^2J_{k-1}(x) \\ &= 2(k+1)(2k+1)k!(P_k(x)\cos x + Q_k(x)\sin x) \\ &\quad - 4k(k+1)x^2(k-1)!(P_{k-1}(x)\cos x + Q_{k-1}(x)\sin x) \\ &= (k+1)!\{(2(2k+1)P_k(x) - 4x^2P_{k-1}(x))\cos x \\ &\quad + (2(2k+1)Q_k(x) - 4x^2Q_{k-1}(x))\sin x\}. \end{aligned}$$

Let  $P_{k+1} = 2(2k+1)P_k(x) - 4x^2P_{k-1}(x)$ ,  $Q_{k+1} = 2(2k+1)Q_k(x) - 4x^2Q_{k-1}(x)$ . By the induction hypothesis,  $P_{k+1}(x), Q_{k+1}$  are polynomials of degree  $\leq n$ , and with integer coefficients. Therefore, by mathematical induction, for all  $n \in \mathbb{Z}^+$ ,

$$J_n(x) = n!(P_n(x)\cos x + Q_n(x)\sin x),$$

where  $P_n(x), Q_n(x)$  are polynomials of degree  $\leq n$ , and with integer coefficients.  $\square$

**Theorem 4.3** (Cartwright's Proof).  $\pi$  is irrational.

Suppose that  $\pi$  is rational. Then,  $\frac{\pi}{2}$  is also rational and can be expressed as  $\frac{a}{b}$  such that  $a, b \in \mathbb{Z}^+$ ,  $a, b$  are rel. prime. Note that

$$x^{2n+1}I_n(x) = n!(P_n(x)\cos x + Q_n(x)\sin x). \quad (8)$$

In (1), take  $x = \frac{\pi}{2}$ .

$$\begin{aligned} \left(\frac{a}{b}\right)^{2n+1} I_n\left(\frac{\pi}{2}\right) &= n! \cdot P_n\left(\frac{\pi}{2}\right). \\ \Rightarrow \frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) &= P_n\left(\frac{\pi}{2}\right) \cdot b^{2n+1}. \end{aligned}$$

The right side is an integer, so is the left side. However,  $0 < I_n\left(\frac{\pi}{2}\right) \leq 2$  since for  $z \in (-1, 1)$ ,

$$0 < (1 - z^2)^n \leq 1, 0 < \cos\left(\frac{\pi}{2}y\right) \leq 1.$$

$$\begin{aligned} 0 &< (1 - z^2)^n \cos\left(\frac{\pi}{2}y\right) \leq 1 \\ \Rightarrow 0 &< \int_{-1}^1 (1 - z^2)^n \cos\left(\frac{\pi}{2}y\right) \leq 1 \\ \Rightarrow 0 &< I_n\left(\frac{\pi}{2}\right) \leq 2. \end{aligned}$$

On the other hand,

$$\lim_{n \rightarrow \infty} \frac{a^{2n+1}}{n!} = 0.$$

Hence, for sufficiently large  $n$ ,

$$0 < \frac{a^{2n+1} I_n \left( \frac{\pi}{2} \right)}{n!} < 1.$$

That is, we could find an integer between 0 and 1. It contradicts with the assumption that  $\pi$  is rational. Therefore,  $\pi$  is irrational.