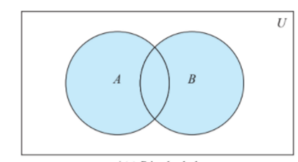
Review Notes

Ch 2.2 Set Operations

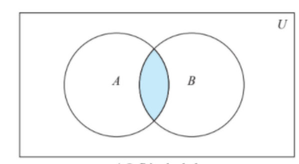
Introduction

1 Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.



$A \cup B$ is shaded.

2 Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.



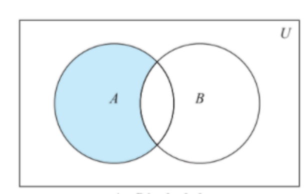
$A \cap B$ is shaded.

3 Two sets are called *disjoint* if their intersection is the empty set.

The principle of inclusion-exclusion

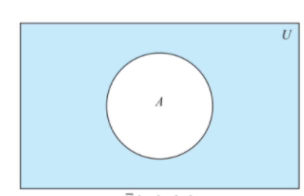
The generalization of the result to unions of an arbitrary number of sets. An important technique used in enumeration. (Discussed in Ch6 & 8)

4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$ (A/B), is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.



$A - B$ is shaded.

5 Let U be the universal set. The complement of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.



$\overline{A}$ is shaded.

> $A - B = A \cap \overline{B}$

Set Identities

▼ [EXAMPLE 10] Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Prove that A ∩ B = A ∪ B.

Solution: We will prove that the two sets $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A) or \neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. We do this by showing that if x is in $\overline{A} \cup \overline{B}$, then it must also be in $\overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, we know that $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, the proposition $\neg(x \in A) \vee \neg(x \in B)$ is true.

By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved.

▼ [EXAMPLE 11] Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We can prove this ientity with the following steps.

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$

A ∩ B = {x | x ∈ A ∩ B} by definition of complement
= {x | ¬(x ∈ (A ∩ B))} by definition of does not belong symbol
= {x | ¬(x ∈ A ∧ x ∈ B)} by definition of intersection
= {x | ¬(x ∈ A) ∨ ¬(x ∈ B)} by the first De Morgan law for logical equivalences
= {x | x /∈ A ∨ x /∈ B} by definition of does not belong symbol
= {x | x ∈ A ∨ x ∈ B} by definition of complement
= {x | x ∈ A ∪ B} by definition of union
= A ∪ B by meaning of set builder notation

▼ [EXAMPLE 12]

Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.
Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then x ∈ A and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and$x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that$((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$.We conclude that either x ∈ A and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$.

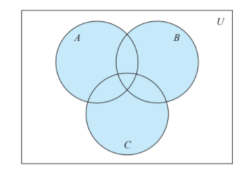
TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity.

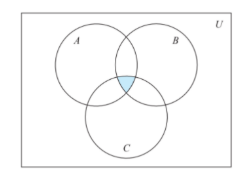
Generalized Unions and Intersections

6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.



(a) $A \cup B \cup C$ is shaded.

7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.



(b) $A \cap B \cap C$ is shaded.

Computer Representation of Sets (Bit Strings)

▼ [EXAMPLE18]

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

(1) $\{1, 3, 5, 7, 9\}$ → 1010101010

(2) $\{2, 4, 6, 8, 10\}$ → 0101010101

(3) $\{1, 2, 3, 4, 5\}$ → 1111100000

▼ [EXAMPLE20]

The bit strings for the sets$\{1, 2, 3, 4, 5\}$ and$\{1, 3, 5, 7, 9\}$ are 1111100000 and 1010101010, respectively. Use bit strings to find the union and intersection of these sets.

- Union → 1111101010 → $\{1, 2, 3, 4, 5, 7, 9\}$
- Intersection → 1010100000 → $\{1, 3, 5\}$