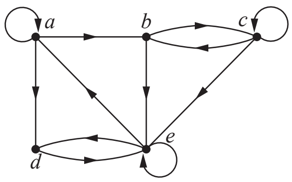
13. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^t$.

14. Show that the closure of a relation R with respect to a property **P**, if it exists, is the intersection of all the relations with property **P** that contain R.

15. When is it possible to define the "irreflexive closure" of a relation R, that is, a relation that contains R, is irreflexive, and is contained in every irreflexive relation that contains R?

16. Determine whether these sequences of vertices are paths in this directed graph.



a) a, b, c, e
b) b, e, c, b, e
c) a, a, b, e, d, e
d) b, c, e, d, a, a, b
e) b, c, c, b, e, d, e, d
f) $a, a, b, b, c, c, b, e, d$

17. Find all circuits of length three in the directed graph in Exercise 16.

18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.

a) a, b **b)** b, a **c)** b, b
d) a, e **e)** b, d **f)** c, d
g) d, d **h)** e, a **i)** e, c

19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), (5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4.
d) R^5. **e)** R^6. **f)** R^*.

20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

22. Suppose that the relation R is reflexive. Show that R^* is reflexive.

23. Suppose that the relation R is symmetric. Show that R^* is symmetric.

24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.

a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$
b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$
c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$
d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$

27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.

28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.

29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is

a) reflexive and transitive.
b) symmetric and transitive.
c) reflexive, symmetric, and transitive.

30. Finish the proof of the case when $a \neq b$ in Lemma 1.

31. Algorithms have been devised that use $O(n^{2.8})$ bit operations to compute the Boolean product of two $n \times n$ zero–one matrices. Assuming that these algorithms can be used, give big-O estimates for the number of bit operations using Algorithm 1 and using Warshall's algorithm to find the transitive closure of a relation on a set with n elements.

***32.** Devise an algorithm using the concept of interior vertices in a path to find the length of the shortest path between two vertices in a directed graph, if such a path exists.

33. Adapt Algorithm 1 to find the reflexive closure of the transitive closure of a relation on a set with n elements.

34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.

35. Show that the closure with respect to the property **P** of the relation $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ on the set $\{0, 1, 2\}$ does not exist if **P** is the property

a) "is not reflexive."
b) "has an odd number of elements."

9.5 Equivalence Relations

2/2 00:31
| read
01:20
| notes
02:04

Introduction

In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. (These characters are uppercase or lowercase letters, digits, or underscores.) Consequently, the compiler considers strings longer than eight characters that agree in their first eight characters the same. Let R be the relation on the set of strings of characters such that sRt, where s and t are two strings, if s and t are at least eight characters long and the first eight characters of s and t agree, or $s = t$. It is easy to see that R is reflexive, symmetric, and transitive. Moreover, R divides the set of all strings into classes, where all strings in a particular class are considered the same by a compiler for traditional C.

The integers a and b are related by the "congruence modulo 4" relation when 4 divides $a - b$. We will show later that this relation is reflexive, symmetric, and transitive. It is not hard to see that a is related to b if and only if a and b have the same remainder when divided by 4. It follows that this relation splits the set of integers into four different classes. When we care only what remainder an integer leaves when it is divided by 4, we need only know which class it is in, not its particular value.

These two relations, R and congruence modulo 4, are examples of equivalence relations, namely, relations that are reflexive, symmetric, and transitive. In this section we will show that such relations split sets into disjoint classes of equivalent elements. Equivalence relations arise whenever we care only whether an element of a set is in a certain class of elements, instead of caring about its particular identity.

Equivalence Relations

Links

In this section we will study relations with a particular combination of properties that allows them to be used to relate objects that are similar in some way.

DEFINITION 1 A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence relations are important in every branch of mathematics!

Equivalence relations are important throughout mathematics and computer science. One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

DEFINITION 2 Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

For the notion of equivalent elements to make sense, every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation. It makes sense to say that a and b are related (not just that a is related to b) by an equivalence relation, because when a is related to b, by the symmetric property, b is related to a. Furthermore, because an equivalence relation is transitive, if a and b are equivalent and b and c are equivalent, it follows that a and c are equivalent.

Examples 1–5 illustrate the notion of an equivalence relation.

EXAMPLE 1 Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 9.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation. ◀

EXAMPLE 2 Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Extra Examples

Solution: Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation. ◀

One of the most widely used equivalence relations is congruence modulo m, where m is an integer greater than 1.

EXAMPLE 3 **Congruence Modulo m** Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall from Section 4.1 that $a \equiv b \pmod{m}$ if and only if m divides $a - b$. Note that $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$. Adding these two equations shows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation. ◀

EXAMPLE 4 Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

???

Solution: Because $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb, so that $l(a) = l(b)$. Then bRa, because $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$, so aRc. Consequently, R is transitive. Because R is reflexive, symmetric, and transitive, it is an equivalence relation. ◀

EXAMPLE 5 Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let $n = 3$ and let S be the set of all bit strings. Then sR_3t either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_301$ and $00111R_300101$, but $01 \not R_3 010$ and $01011 \not R_3 01110$.

Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Solution: The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S. If sR_nt, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that tR_ns. We conclude that R_n is symmetric.

Now suppose that sR_nt and tR_nu. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters (because in this case we know that s, t, and u are all at least n characters long and both s and u begin with the same n characters as t does). Consequently, R_n is transitive. It follows that R_n is an equivalence relation. ◀

In Examples 6 and 7 we look at two relations that are not equivalence relations.

EXAMPLE 6 Show that the "divides" relation is the set of positive integers in not an equivalence relation.

Solution: By Examples 9 and 15 in Section 9.1, we know that the "divides" relation is reflexive and transitive. However, by Example 12 in Section 9.1, we know that this relation is not symmetric (for instance, $2 \mid 4$ but $4 \nmid 2$). We conclude that the "divides" relation on the set of positive integers is not an equivalence relation. ◀

EXAMPLE 7 Let R be the relation on the set of real numbers such that $x R y$ if and only if x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if $x R y$, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that $y R x$. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8 \, R \, 1.9$, $1.9 \, R \, 1.1$, but $2.8 \not R \, 1.1$. ◀

Equivalence Classes

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y), where x and y graduated from the same high school. Given a student x, we can form the set of all students equivalent to x with respect to R. This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

DEFINITION 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

EXAMPLE 8 What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$. ◀

EXAMPLE 9 What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}.$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod{4}$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}.$$ ◀

In Example 9 the equivalence classes of 0 and 1 with respect to congruence modulo 4 were found. Example 9 can easily be generalized, replacing 4 with any positive integer m. The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo** m. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$. For instance, from Example 9 it follows that $[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ and $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.

EXAMPLE 10 What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 from Example 5 on the set of all bit strings? (Recall that $s R_3 t$ if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.)

Solution: The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on. Consequently,

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\}.$$ ◀

EXAMPLE 11 **Identifiers in the C Programming Language** In the C programming language, an **identifier** is the name of a variable, a function, or another type of entity. Each identifier is a nonempty string of characters where each character is a lowercase or an uppercase English letter, a digit, or an underscore, and the first character is a lowercase or an uppercase English letter. Identifiers can be any length. This allows developers to use as many characters as they want to name an entity, such as a variable. However, for compilers for some versions of C, there is a limit on the number of characters checked when two names are compared to see whether they refer to the same thing. For example, Standard C compilers consider two identifiers the same when they agree in their first 31 characters. Consequently, developers must be careful not to use identifiers with the same initial 31 characters for different things. We see that two identifiers are considered the same when they are related by the relation R_{31} in Example 5. Using Example 5, we know that R_{31}, on the set of all identifiers in Standard C, is an equivalence relation.

What are the equivalence classes of each of the identifiers Number_of_tropical_storms, Number_of_named_tropical_storms, and Number_of_named_tropical_storms_in_the_Atlantic_in_2005?

Solution: Note that when an identifier is less than 31 characters long, by the definition of R_{31}, its equivalence class contains only itself. Because the identifier Number_of_tropical_storms is 25 characters long, its equivalence class contains exactly one element, namely, itself.

The identifier Number_of_named_tropical_storms is exactly 31 characters long. An identifier is equivalent to it when it starts with these same 31 characters. Consequently, every identifier at least 31 characters long that starts with Number_of_named_tropical_storms is equivalent to this identifier. It follows that the equivalence class of Number_of_named_tropical_storms is the set of all identifiers that begin with the 31 characters Number_of_named_tropical_storms.

An identifier is equivalent to the Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it begins with its first 31 characters. Because these characters are Number_of_named_tropical_storms, we see that an identifier is equivalent to Number_of_named_tropical_storms_in_the_Atlantic_in_2005 if and only if it is equivalent to Number_of_named_tropical_storms. It follows that these last two identifiers have the same equivalence class. ◀

Equivalence Classes and Partitions

Let A be the set of students at your school who are majoring in exactly one subject, and let R be the relation on A consisting of pairs (x, y), where x and y are students with the same major. Then R is an equivalence relation, as the reader should verify. We can see that R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major. For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history. Furthermore, these subsets are equivalence classes of R. This example illustrates how the equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets. We will make these notions more precise in the following discussion.

Let R be a relation on the set A. Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

THEOREM 1 Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb $\quad$ (ii) $[a] = [b]$ $\quad$ (iii) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (i) implies (ii). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (ii) implies (iii). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. ◁

We are now in a position to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

Recall that an *index set* is a set whose members label, or index, the elements of a set.

These two observations show that the equivalence classes form a partition of A, because they split A into disjoint subsets. More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset \text{ when } i \neq j,$$

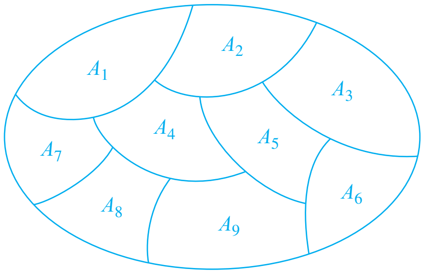


FIGURE 1 **A Partition of a Set.**

and

$$\bigcup_{i \in I} A_i = S.$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 12 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S. ◀

We have seen that the equivalence classes of an equivalence relation on a set form a partition of the set. The subsets in this partition are the equivalence classes. Conversely, every partition of a set can be used to form an equivalence relation. Two elements are equivalent with respect to this relation if and only if they are in the same subset of the partition.

To see this, assume that $\{A_i \mid i \in I\}$ is a partition on S. Let R be the relation on S consisting of the pairs (x, y), where x and y belong to the same subset A_i in the partition. To show that R is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

We see that $(a, a) \in R$ for every $a \in S$, because a is in the same subset as itself. Hence, R is reflexive. If $(a, b) \in R$, then b and a are in the same subset of the partition, so that $(b, a) \in R$ as well. Hence, R is symmetric. If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset X in the partition, and b and c are in the same subset Y of the partition. Because the subsets of the partition are disjoint and b belongs to X and Y, it follows that $X = Y$. Consequently, a and c belong to the same subset of the partition, so $(a, c) \in R$. Thus, R is transitive.

It follows that R is an equivalence relation. The equivalence classes of R consist of subsets of S containing related elements, and by the definition of R, these are the subsets of the partition. Theorem 2 summarizes the connections we have established between equivalence relations and partitions.

THEOREM 2 Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example 13 shows how to construct an equivalence relation from a partition.

EXAMPLE 13 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 12.

Solution: The subsets in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the partition. The pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 2)$, and $(3, 3)$ belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs $(4, 4)$, $(4, 5)$, $(5, 4)$, and $(5, 5)$ belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair $(6, 6)$ belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R. ◀

The congruence classes modulo m provide a useful illustration of Theorem 2. There are m different congruence classes modulo m, corresponding to the m different remainders possible when an integer is divided by m. These m congruence classes are denoted by $[0]_m, [1]_m, \ldots, [m-1]_m$. They form a partition of the set of integers.

EXAMPLE 14 What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: There are four congruence classes, corresponding to $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$. They are the sets

$$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\},$$
$$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\},$$
$$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\},$$
$$[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}.$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition. ◀

We now provide an example of a partition of the set of all strings arising from an equivalence relation on this set.

EXAMPLE 15 Let R_3 be the relation from Example 5. What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings? (Recall that $s\,R_3\,t$, where s and t are bit strings, if $s = t$ or s and t are bit strings with at least three bits that agree in their first three bits.)

Solution: Note that every bit string of length less than three is equivalent only to itself. Hence $[\lambda]_{R_3} = \{\lambda\}$, $[0]_{R_3} = \{0\}$, $[1]_{R_3} = \{1\}$, $[00]_{R_3} = \{00\}$, $[01]_{R_3} = \{01\}$, $[10]_{R_3} = \{10\}$, and $[11]_{R_3} = \{11\}$. Note that every bit string of length three or more is equivalent to one of the eight bit strings 000, 001, 010, 011, 100, 101, 110, and 111. We have

$$[000]_{R_3} = \{000, 0000, 0001, 00000, 00001, 00010, 00011, \ldots\},$$

$$[001]_{R_3} = \{001, 0010, 0011, 00100, 00101, 00110, 00111, \ldots\},$$

$$[010]_{R_3} = \{010, 0100, 0101, 01000, 01001, 01010, 01011, \ldots\},$$

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\},$$

$$[100]_{R_3} = \{100, 1000, 1001, 10000, 10001, 10010, 10011, \ldots\},$$

$$[101]_{R_3} = \{101, 1010, 1011, 10100, 10101, 10110, 10111, \ldots\},$$

$$[110]_{R_3} = \{110, 1100, 1101, 11000, 11001, 11010, 11011, \ldots\},$$

$$[111]_{R_3} = \{111, 1110, 1111, 11100, 11101, 11110, 11111, \ldots\}.$$

These 15 equivalence classes are disjoint and every bit string is in exactly one of them. As Theorem 2 tells us, these equivalence classes partition the set of all bit strings. ◀

Exercises

2/2 3:10
| do
4:14
| correct
4:41

reflexive (a,a),(b,b)
symmetric (a,b),(b,a)
transitive (a,b),(b,c),(a,c)

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$
b) $\{(a, b) \mid a$ and b have the same parents$\}$
c) $\{(a, b) \mid a$ and b share a common parent$\}$
d) $\{(a, b) \mid a$ and b have met$\}$
e) $\{(a, b) \mid a$ and b speak a common language$\}$

3. Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1$ for all $x \in \mathbf{Z}\}$
d) $\{(f, g) \mid$ for some $C \in \mathbf{Z}$, for all $x \in \mathbf{Z}$, $f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of **F** and of **T**?

8. Let R be the relation on the set of all sets of real numbers such that $S\,R\,T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\mathbf{Z}$?

9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n, the nth characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

17. (*Requires calculus*)

a) Show that the relation R on the set of all differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.

b) Which functions are in the same equivalence class as the function $f(x) = x^2$?

18. (*Requires calculus*)

a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the nth derivative of $f(x)$.]

b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

19. Let R be the relation on the set of all URLs (or Web addresses) such that $x\,R\,y$ if and only if the Web page at x is the same as the Web page at y. Show that R is an equivalence relation.

20. Let R be the relation on the set of all people who have visited a particular Web page such that $x\,R\,y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

1. (a) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(b) ~~reflexive~~ (1,1) X, symmetric, ~~transitive~~ (0,2). (2,3) X (0,3) ⇒ ~~equivalence relation~~.
(c) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(d) reflexive, symmetric, not transitive → (2,3) X (2,1) X (1,2) X ✓
(e) reflexive, not symmetric → (2,1) X, ~~transitive~~.

2. (a) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(b) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(c) reflexive, symmetric, not transitive. ✓
(d) reflexive, symmetric, not transitive. ✓
(e) reflexive, symmetric, transitive ⇒ ~~equivalence relation~~.

3. (a) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(b) reflexive, symmetric, ~~transitive~~ ⇒ ~~equivalence relation~~.
$f(x)=0$, $g(x)=x$, $h(x)=1$
$f(0)=g(0)$, $g(1)=h(1) \to f(0) \neq h(1)$
(c) not reflexive, not symmetric, not transitive. ✓
(d) reflexive, symmetric, transitive ⇒ equivalence relation. ✓
(e) ~~reflexive~~, symmetric, ~~transitive~~ ⇒ ~~equivalence relation~~.
a lot of function doesn't have the property $f(0)=f(1)$ ex $f(x)=x$
ex. $f(0)=g(1)=h(0)=7$, $f(1)=g(0)=h(1)=3$, $f \to g$, $g \to h$, $f \not\to h$

4. {(a,b) | a and b have passed Midterm}
{(a,b) |
{

(a) $A_1=\{0\}$, $A_2=\{1\}$, $A_3=\{2\}$, $A_4=\{3\}$.
(b) $B_1=\{0,2\}$, $B_2=\{2,3\}$
(c) $C_1=\{0\}$, $C_2=\{1,2\}$, $C_3=\{3\}$.

26. in 1.
a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

27. in 2. ??
a) {(a, b) | a and b are the same age}
b) {(a, b) | a and b have the same parents}
c) {(a, b) | a and b share a common parent}
d) {(a, b) | a and b have met}
e) {(a, b) | a and b speak a common language}

(a) An equivalence class is the set of all people who are the same age. (By the same age here mean identical official birthday)

(b) For each pair (m,f) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children.

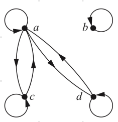
30. in 11. ??

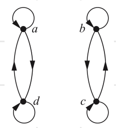
41. (b)(c) ✓

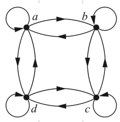
5. {Two buildings are equivalent if they were opened during the same year
{Two buildings are equivalent if they have the same number of stories
{Partition the set of all buildings into two classes - those in which you have a class this semester and those you don't. Every building that you don't have a class is equivalent to every building that you don't have a class.

7. ?? Two propositions are equivalent if their truth tables are identical. This relation is reflexive, since the truth table of a proposition is identical to itself. It is symmetric, since if p and q have the same truth table, then q and p have the same truth table. There is one technical point about transitivity that should be noted. We need to assume that the truth tables, as we consider them for three propositions p, q, and r, have the same atomic variables in them. If we make this assumption (and it cannot hurt to do so, since adding info about a extra variables that do not appear in a pair of propositions does not change the truth value of the propositions), then we argue in the usual way: if p and q have identical truth tables, and if q and r have identical truth tables, then p and r have the same common truth table. The proposition T is always true,. therefore the equivalence class of T is the set of all tautologies. Similarly, the equivalence class of F is the set of all contradictions.

11. ??This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to its first 3 bits.

21. 
→ reflexive (loop)
→ symmetric (both dir) ✓
→ transitive

22. 
→ reflexive (loop)
→ symmetric (both dir)
→ transitive

23. 
→ reflexive (loop)
→ symmetric (both dir) ✓
→ not transitive (a,c) X (c,a) X (b,d) X (d,b) X

24.
a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
→ reflexive (1,1)
→ not symmetric (0 1)
→ not transitive (2,1) X (2,3). (3,1)

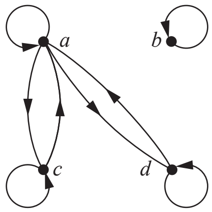
b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
→ reflexive (1,1)
→ symmetric
→ transitive
} equivalence relation

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
→ reflexive
→ symmetric
→ transitive
} equivalence relation

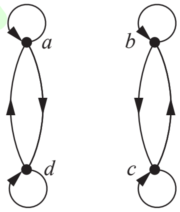
25. ?? This follows from Exercise 9, with f being the function from bit stings to nonnegative integers given by f(s)=the number of 1's in s.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

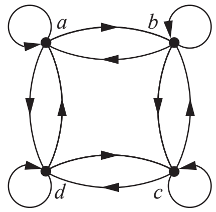
21.



22.



23.



24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s\,R\,t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 **b)** 1011 **c)** 11111 **d)** 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? **b)** 3? **c)** 6? **d)** -3?

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) *No* **b)** *Yes* **c)** *Help*

39. **a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to (a, b).]

40. **a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [*Hint:* Look at the ratio a/b corresponding to (a, b).]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ **b)** $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$

b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$

c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$

d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain $3k$ ones for some nonnegative integer k; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers

b) the set of positive integers and the set of negative integers