Chapter 8

Graphs

Euler calculated without effort, as men breathe,
or as eagles sustain themselves in the wind.

— FRANÇOIS ARAGO

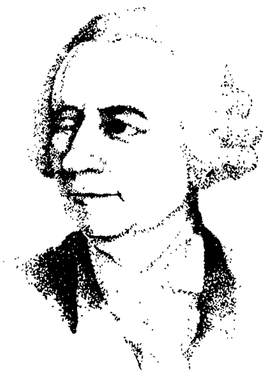
Graph theory, a fascinating branch of mathematics, has numerous applications to such diverse areas as computer science, engineering, linguistics, and management science, as well as the natural and social sciences.

Like many important discoveries, graph theory grew out of an interesting physical problem, the celebrated Königsberg Bridge Puzzle (see Section 8.1). The outstanding Swiss mathematician Leonhard Euler solved the puzzle in 1736, thus laying the foundation for graph theory and earning his title as the father of graph theory.

This chapter presents the fundamentals of the field he created, with its assortment of new terms. Since graph terminology is not yet standard, definitions of basic terms can vary from book to book, an important fact to remember.

We will study the following interesting problems as well as others:

- The City of Königsberg (see Figure 8.1) comprises the river banks A and C and the islands B and D. These four land areas are connected by seven bridges. Could a Königsbergian take a walk through his beloved city, passing over each bridge exactly once? Could he take a walk through the city passing over each bridge exactly once and return home?
- At a sesquicentennial ball, there are n guests and each person shakes hands with everybody else exactly once. How would you represent the handshakes pictorially? How many handshakes are made?
- Three married couples want to cross a river in a rowboat which can carry only two people at a time. No husband will allow his wife to be in the boat or on the shore in the presence of another man unless he is also present. The women can, of course, row well. How can they all cross? (S. Gudder, 1976)



Leonhard Euler *(1707–1783) was born in Basel, Switzerland. His father, a mathematician and a Calvinist pastor, wanted him also to become a pastor. Although Euler had different ideas, he followed his father's wishes, entering the University of Basel to study Hebrew and theology. His hard work at the University and remarkable ability brought him to the attention of the well-known mathematician Johann Bernoulli (1667–1748). Recognizing young Euler's talents, Bernoulli persuaded the boy's father to change his mind, and Euler was allowed to pursue his studies in mathematics.*

At age 19 Euler brought out his first paper. Although it failed to win the Paris Academy Prize in 1727, he eventually won the prize 12 times.

Euler was the most prolific mathematician, significantly contributing to every branch of mathematics. With his phenomenal memory, he had perfect recall for every formula. A genius, he could work anywhere and under any conditions. Euler belongs in a class by himself.

Week 13 (I) ↓

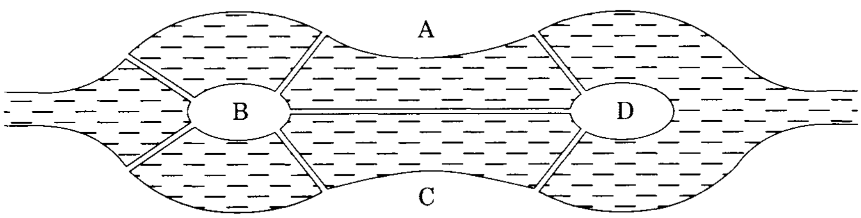
- A developer is building three new houses on one side of a street. If she would like to connect three utilities to each of them from the other side, can she lay the utility lines without any crossings?

8.1 Graphs

The Prussian city of Königsberg (called Kaliningrad during the era of the Soviet Union) lies on the Pregel river (see Figure 8.1). It consists of the two river banks A and C, and the two islands B and D. Seven bridges connect the four land areas of the city.

Figure 8.1

The City of Königsberg.



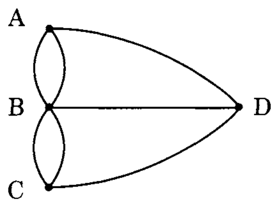
Residents of the city used to take evening walks from one part of the city to another. This, naturally, suggested the following question: *Is it possible to walk through the city, traversing each bridge exactly once*?

The problem sounds simple, and you might want to try a few possible paths before going any further. After all, by the multiplication principle, the maximum number of possible paths is $7! = 5040$.

In 1736, Euler, while at St. Petersburg Academy, published a solution to the problem: No such walk is possible. In fact, he proved a far more general result, of which the Königsberg bridge puzzle is a special case. Euler constructed a mathematical model like Figure 8.2 for the problem in which points A and C represent the two river banks; B and D the two islands. The arcs or line segments joining them represent the seven bridges.

Figure 8.2

A mathematical model for the Königsberg bridge puzzle.



The Königsberg bridge problem can now be stated in layman's language as follows: *Beginning at one of the points A, B, C, or D, is it possible to trace the figure without lifting your pencil or traversing the same edge twice?* Section 8.5 will explore this further.

The Königsberg bridge model in Figure 8.2 consists of four points — A, B, C, and D — and the arcs or line segments joining them. Such a figure is called a graph.

Graph

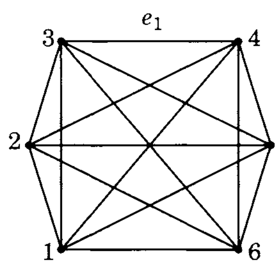
A **graph** (or **undirected graph**) G consists of a nonempty finite set V of points (called **vertices** or **nodes**) and a set E of unordered pairs of elements in V (called **edges**). The graph G is the ordered pair $(V,E): G = (V,E)$. An edge connecting the vertices u and v is denoted by $\{u,v\}$, u-v, or some label. Geometrically, edges are denoted by arcs or line segments.

The next example uses these terms to apply graphs to the theory of communications.

EXAMPLE 8.1

A taxpayer files his federal tax return at an Internal Revenue Service (IRS) center located in the region in which he lives. Six of the IRS centers in the continental United States are (1) Atlanta, (2) Holtsville, (3) Philadelphia, (4) Cincinnati, (5) Austin, and (6) Fresno. A computer at each center can communicate with a computer at any other center over a telephone line. This network of computers can be modeled by a graph, as in Figure 8.3.

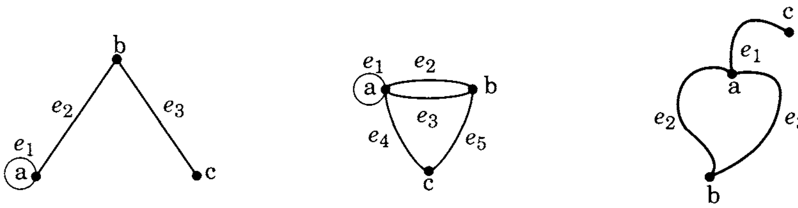
Figure 8.3



The graph has six vertices: 1, 2, 3, 4, 5, and 6. Each vertex represents a computer and each edge a telephone link. Since each computer can communicate with every other computer, an edge runs between any two vertices; e_1 denotes edge $\{3, 4\}$. ■

Figure 8.4 displays more graphs. The one in Figure 8.4a contains three vertices—a, b, and c: $V = \{a, b, c\}$. Its three edges are $e_1 = \{a, a\}$, $e_2 = \{a, b\}$, and $e_3 = \{b, c\}$: $E = \{e_1, e_2, e_3\} = \{\{a, a\}, \{a, b\}, \{b, c\}\}$.

Figure 8.4



(a) (b) (c)

範例

Airline route maps provide a fine paradigm of a graph, with each vertex

a typical example or pattern of something; a model

representing a city and every edge a direct flight from one city to another.

We now introduce several special classes of graphs: simple, complete, bipartite, complete bipartite, and weighted graphs.

Simple Graph

An edge $\{a, a\}$ emanating from and terminating at the same vertex a is a **loop**.* **Parallel edges** have the same vertices. A **simple graph** contains no loops or parallel edges.

For example, the graphs in Figures 8.4a and b display a loop at a, while the ones in Figures 8.4b [redacted] and 8.4c [redacted] have two parallel edges, e_2 and e_3, connecting vertices a and b. Figure 8.3 is a simple graph, unlike the graphs in Figure 8.4 (why?).

By means of graphs, modular arithmetic can construct aesthetically

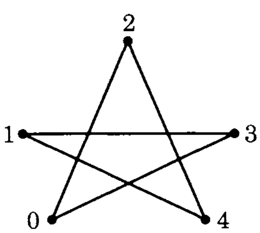
the use of numbers in counting and calculation

pleasing designs, as the next example demonstrates.

EXAMPLE 8.2 Choose $V = \{0, 1, 2, 3, 4\}$, the set of integers modulo 5.

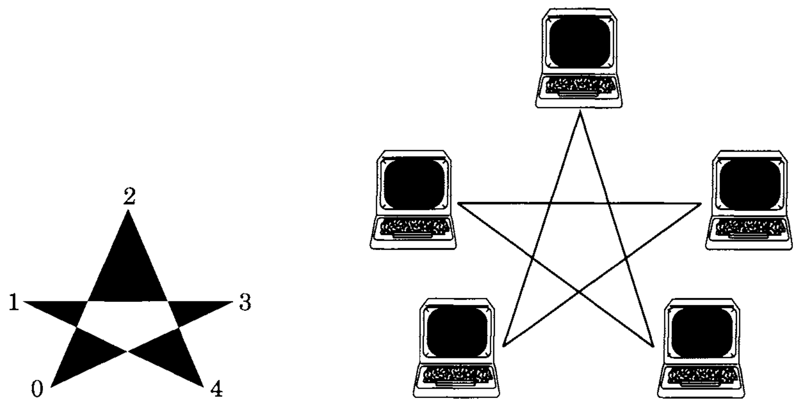
Figure 8.5

A pentagram.



*Although $\{a, a\} = \{a\}$ as sets, the loop at a is denoted by $\{a, a\}$ or a-a.

Figure 8.6



(a) (b)

Mark the vertices at equal intervals on a circle. An edge exists between the vertices x and y if $y \equiv (x+2) \pmod 5$. For instance, $1 \equiv (4+2) \pmod 5$; so an edge runs between the vertices 4 and 1. A simple graph, called a **pentagram,** materializes in Figure 8.5. Coloring the various wedge-shaped regions creates the pleasing design in Figure 8.6a. The pentagram reminds us of the point-to-point communication system in Figure 8.6b. ■

Graphs can also facilitate the study of hydrocarbons.

EXAMPLE 8.3

Arthur Cayley used graphs in studying isomers of hydrocarbons. A hydrocarbon molecule consists of carbon and hydrogen atoms. Each hydrogen atom (H) is bonded to a single carbon atom (C), whereas a carbon atom bonds with two, three, or four atoms which can be carbon or hydrogen.

Figure 8.7

Ethane, C_2H_6.

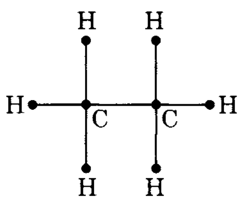
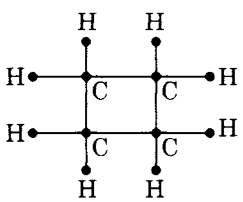


Figure 8.8

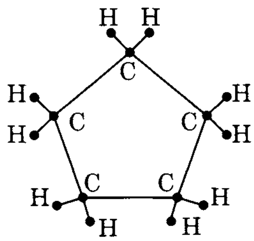
Cyclobutane, C_4H_8.



An ethane molecule, for instance, consists of two carbon atoms and six hydrogen atoms. Its **structural formula** appears as the graph in Figure 8.7, representing the **molecular formula** C_2H_6. Figures 8.8 and 8.9

Figure 8.9

Cyclopentane, C_5H_{10}.



show the structural formulas of cyclobutane and cyclopentane molecules. ■

It is not necessary to have edges between any two vertices in a graph, so we make the following definition.

Adjacency and Incidence

Two vertices v and w in a graph are **adjacent** if an edge runs between them; if a loop occurs at v, v is adjacent to itself. An **isolated vertex** is not adjacent to any vertex. **Adjacent edges** have a common vertex. An edge is **incident** with a vertex v if v is an endpoint of the edge.

For example, in the graph in Figure 8.4a, vertices a and b are adjacent, but a and c are not. Edges $\{a, b\}$ and $\{b, c\}$ are adjacent. Edge e_2 is incident with vertices a and b. The graph contains no isolated vertices.

The concept of the degree of a vertex is important in the study of graphs, as will be seen later.

Degree of a Vertex

The **degree** of a vertex v in a graph is the number of edges meeting at v; it is denoted by $\deg(v)$.

Clearly, a vertex v is isolated if $\deg(v) = 0$. In addition, a loop at v contributes two to its degree.

For example, in Figure 8.4b, $\deg(a) = 5$, $\deg(b) = 3$, and $\deg(c) = 2$. In Figure 8.4c, $\deg(a) = 3$, $\deg(b) = 2$, and $\deg(c) = 1$.

We have seen that digraphs can arise from matrices; graphs also can arise from them.

Adjacency Matrix

The **adjacency matrix** of a graph with n vertices $v_1, v_2, \ldots, v_n$ is an $n \times n$ matrix $A = (a_{ij})$, where a_{ij} = number of edges from v_i to v_j.

Because every edge in a graph is undirected, $a_{ij} = a_{ji}$ for every i and j, so the adjacency matrix of every graph is symmetric. If the graph is simple, A is a boolean matrix.

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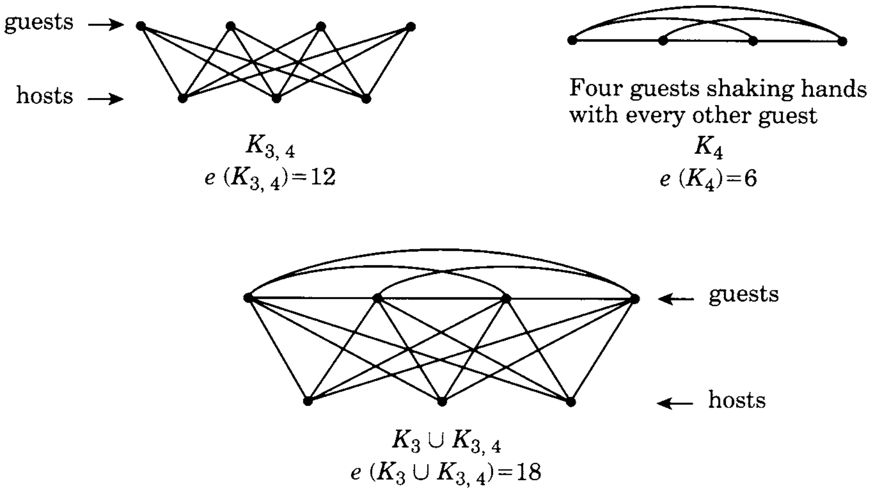
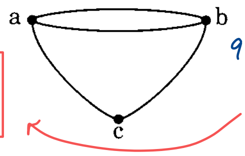
Figure 8.31



Exercise

1.

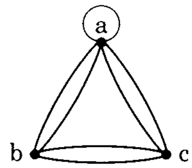


$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 2 & 1 \\ b & 2 & 0 & 1 \\ c & 1 & 1 & 0 \end{array}$$

5. $n = 3$
$m = 4$

9. $N(a) = \{b, c\}$
$N(b) = \{a, c\}$
$N(c) = \{a, b\}$

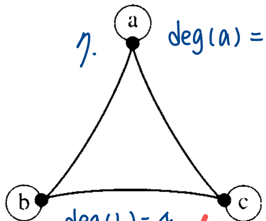
2.



6. $n = 3$
$m = 6$

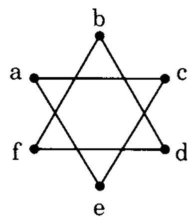
10. $N(a) = \{a, b, c\}$
$N(b) = \{a, c\}$
$N(c) = \{a, b\}$

3.



7. $\deg(a) = 4$
$\deg(b) = 4$
$\deg(c) = 4$

4.



8. $\deg(a) = 2$
$\deg(b) = 2$
$\deg(c) = 2$
$\deg(d) = 2$
$\deg(e) = 2$
$\deg(f) = 2$

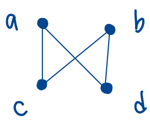
5–6. Find the number of vertices (n) and edges (m) of the graphs in Exercises 1–2.

7–8. Find the degrees ($\deg(\ )$) of the vertices of the graphs in Exercises 3–4.

9–10. Find the adjacency matrix of the graphs in Exercises 1 and 2.

Draw the graph with the given adjacency matrix.

11.
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



12.
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$