Ch9.3 Representing Relations (Week 18)

Representing Relations Using Matrices

Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

The relation R can be represented by the matrix $M_R = [m_{ij}]$, where $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$.

▼ Example 1

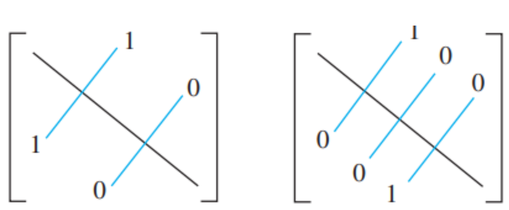
Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1, a_2 = 2, a_3 = 3, b_1 = 1, b_2 = 2$?

Solution:

Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix for R is $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Properties when Using Matrices

- *Reflexive:* R is reflexive *iff* $m_{ii} = 1$ for $i = 1, 2, \ldots, n$. i.e. R is reflexive if all the elements on the main diagonal of M_R are equal to 1.
- *Symmetric:* R is symmetric *iff* $m_{ji} = 1$ whenever $m_{ij} = 1$. i.e. $m_{ji} = 0$ whenever $m_{ij} = 0$. ⇒ R is symmetric *iff* $m_{ij} = m_{ji}$.
- *Antisymmetric:* R is antisymmetric *iff* $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. i.e. if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$ (or either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$).



(a) Symmetric (b) Antisymmetric

▼ Example 3

Suppose that the relations R on a set if represented by the matrix $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

R is reflexive → all diagonal elements $= 1$.

R is symmetric → M_R is symmetric.

R is not anti-symmetric.

- *Union*: $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$
- *Intersection*: $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$
- *Composite*: $M_{S \circ R} = M_R \odot M_S$

▼ Example 5

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

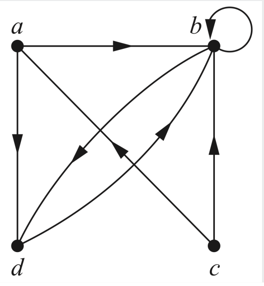
Solution: The matrix for $S \circ R$ is $M_{S\circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

→ The matrix representing the composite of two relations can be used to find the matrix for M_{R^n}, $M_{R^n} = M_R^{[n]}$. (works the same way as multiplication for matrices)

Representing Relations Using Digraphs

▼ Example 7

The directed graph with vertices a, b, c and d and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is displayed below.

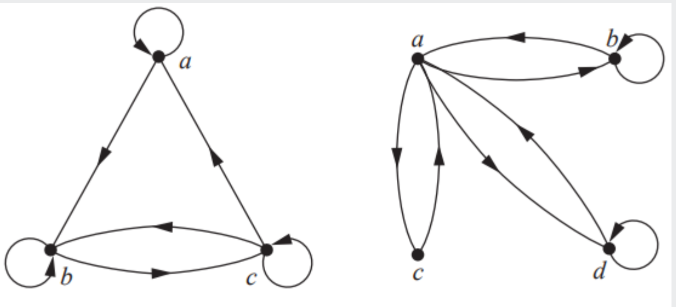


Properties when Using Digraphs

- *Reflexive*: *iff* there is a loop at every vertex of the directed graph.
- *Symmetric: iff* for every edge between distinct vertices in its digraph there is an edge in the opposite direction.
- *Antisymmetric: iff* there are never two edges in opposite directions between distinct vertices.
- *Transitive: iff* whenever there is an edge $x \rightarrow y, y \rightarrow z$, there is an edge $x \rightarrow z$. (completing a triangle w/ the correct direction)

▼ Example 10

Determine whether the relations for the directed graphs shown in Figure 6 are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of R (b) Directed graph of S

R is reflexive → every vertex there are a loop.

R is neither symmetric nor antisymmetric → there is an edge $a \rightarrow b$, but not $b \rightarrow a$, and there are edges in both directions $b \leftrightarrow c$.

R is not transitive → there is $a \rightarrow b,\ b \rightarrow c$, but no $a \rightarrow c$.

S is not reflexive → no loops in all vertices.

S is symmetric and not antisymmetric → every edge between distinct vertices is accompanied by an edge in the opposite direction.

S is not transitive.