Ch5.1 Mathematical Induction (Week 11)

Introduction

- Can be used only to prove results obtained in some other way
- *not* a tool for discovering formulae or theorems

> *Principle of Mathematical Induction* To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we compelte two steps:
> *Basis Step:* We verify that $P(1)$ is true.
> *Inductive Step:* We show that the conditional statement $P(k) \to P(k+1)$ is true for all positvie integers k.

The Good and the Bad of Mathematical Induction

- Good: can be used to prove a conjecture once it has been made (and is true)
- Bad: cannot be used to find new theorems

Examples of Proofs by Mathematical Induction

▼ Example 6

Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$, and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.
Basis Step: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. $P(4)$ is true because $2^4 = 16 < 24 = 4!$
Inductive Step: Assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, $2^k < k!$ for the positive integer k with $k \geq 4$. Prove that under this hypothesis, $P(k+1)$ is also true, namely, $2^{k+1} < (k+1)!$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\ &< 2 \cdot k! && \text{by the induction hypothesis} \\ &< (k+1)k! && \text{because } 2 < \text{k} + 1 \\ &= (k+1)! && \text{by definition of factorial function} \end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.
We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$.

Guidelines for Proofs by Mathematical Induction

> ***Template for Proofs by Mathematical Induction***
> 1. Express the statement that is to be proved in the form "for all $n \geq b,\ P(n)$" for a fixed integer b.
> 2. Write out the words "**Basis Step**." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
> 3. Write out the words "**Inductive Step**."
> 4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
> 5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
> 6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
> 7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
> 8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.