Review Notes

Chapter 2.1 Sets

> A set is countable when it is finite OR has the same size as the set of positive integers.

The set of rational numbers is countable, the set of real numbers is not.

Introduction

1 A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Roster Method

- Listing the elements inside braces

N= {0, 1, 2, 3,...}, the set of **natural numbers**
Z = {..., −2, −1, 0, 1, 2,...}, the set of **integers**
Z^+ = {1, 2, 3,...}, the set of **positive integers**
Q = {p/q | p ∈ Z, q ∈ Z, and q "= 0}, the set of **rational numbers**
R, the set of **real numbers**
R^+, the set of **positive real numbers**
C, the set of **complex numbers**

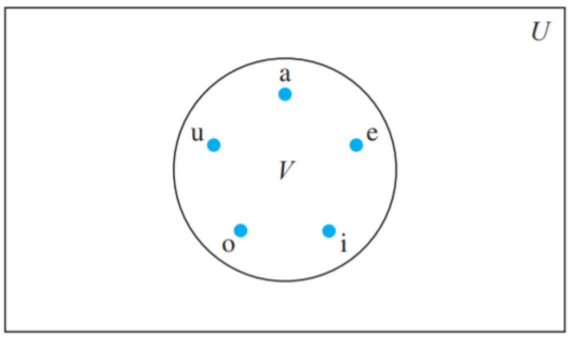
$[a, b]$ is called the **closed interval**; (a, b) is called the **open interval**

2 Two sets are *equal iff* they have the same elements. Therefore, if A and B are sets, then A and B are equal *iff* $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

The Empty Set (or **null set**) : $\emptyset$ or $\{\ \}$

A Singleton Set: A set with one element Ex. $\{\emptyset\}$ is a singleton set, not an empty set

Venn Diagrams



Venn Diagram for the Set of Vowels

Subsets

3 The set A is a subset of B *iff* every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of set B.

T1 For every set S, $(i)\ \emptyset \subseteq S$ and $(ii)\ S \subseteq S$

▼ Proof

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \to x \in S)$ is true. Because the empty set contains no elements, it follows that x ∈ ∅ is always false. It follows that the conditional statement $x \in \emptyset \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \to x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof.

Try it yourself:

Proper Subset: A is a subset of B, but $A \neq B \to A \subset B$

A is a proper subset of B *iff* $\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$

To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

The Size of a Set

4 Let S be a set. If there are exactly n distinct elements in S where n is a non-negative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

5 A set is said to be *infinite* if it is not finite.

Power Sets

6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Cartesian Products

7 The *ordered n-tuple* $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ..., and a_n as its nth element.

Ordered Pairs (Ordered 2-tuples):

- $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ *iff* $a_i = b_i$, for $i = 1, 2, ..., n$.
- (a, b) and (c, d) are equal *iff* $a = c$ and $b = d$

8 Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$. Hence,
$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

- Cartesian products $A \times B$ and $B \times A$ are not equal unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$

9 The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for i=1,2,...,n. In other words,
$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \ for \ i = 1, 2, \ldots, n\}$

- When A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$
- $A^2 = A \times A$, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, etc.

a **Relation** from the set A to the set B

- A subset ***R*** of the Cartesian product $A \times B$
- **R** belongs to $A \times B$, no need to equal.
- A relation from a set A to itself is called a relation on A.

Using Set Notation with Quantifiers

Shorthand Notations

- $\forall x \in S(P(x))$ = $\forall x(x \in S \rightarrow P(x))$
- $\exists x \in S(P(x))$ = $\exists x(x \in S \land P(x))$

Truth Sets and Quantifiers

- The truth set of P(x) is denoted by $\{x \in D \mid P(x)\}$
- Truth set is the range of x according to the property of P.

  Ex. Domain is integers:
  $P(x)$ is "$|x| = 1$" → {-1,1}
  $Q(x)$ is "$x^2 = 2$" → $\emptyset$
  $R(x)$ is "$|x| = x$" → $\mathbb{N}$