Ch2.5 Induction (Oscar)(Week 11)

Induction Proof Structure

Start by saying what the statement is that you want to prove: "Let $P(n)$ be the statement..." To prove that $P(n)$ is true for all $n \geq 0$, you must prove two facts:

1. Base case: Prove that $P(0)$ is true. You do this directly. This is often easy.
2. Inductive case: Prove that $P(k) \to P(k+1)$ for all $k \geq 0$. That is, prove that for any $k \geq 0$ if $P(k)$ is true, then $P(k+1)$ is true as well. this is the proof of an *if... then...* statement, so you can assume $P(k)$ is true ($P(k)$ is called the *inductive hypothesis*). You must then explain why $P(k+1)$ is also true, given that assumption.

Assuming you are successful on both parts above, you can conclude, "Therefore by the principle of mathematical induction, the statement $P(n)$ is true for all $n \geq 0$."

Example 2.5.3

Prove that $n^2 < 2^n$ for all integers $n \geq 5$.

Understand:

What if increase n by 1 ??
LHS → increase the base number, go to the next square number
RHS → increase the power of 2, double the number.
How does doubling a number relate to increasing to the next square ??
The difference of two consecutive squares → $(n+1)^2 - n^2 = (n+1-n)(n+1+n) = 2n+1$.
But doubling RHS increases it by 2^n, since $2^{n+1} = 2^n + 2^n$. When n is large enough, $2^n > 2n+1$.
Each time n increases, LHS grows by less than RHS (never catch up)

Proof:

Let $P(n)$ be the statement $n^2 < 2n$. We will prove $P(n)$ is true for all integers $n \geq 5$.

Base case: $P(5)$ is the statement $5^2 < 2^5$. Since $5^2 = 25$ and $2^5 = 32$, we see that $P(5)$ is indeed true.

Inductive case: Let $k \geq 5$ be an arbitrary integer. Assume, for induction that $P(k)$ is true. That is, assume $k^2 < 2^k$. We will prove that $P(k+1)$ is true, i.e., $(k+1)^2 < 2^{k+1}$. To prove such an inequality, start with LHS and work towards RHS:

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< 2^k + 2k + 1 && \text{...by the inductive hypothesis} \\ &< 2^k + 2^k && \text{...since } 2\mathrm{k} + 1 < 2^{\mathrm{k}} \text{ for } \mathrm{k} \geq 5 \\ &= 2^{k+1}.\end{aligned}$$

Following the equalities and inequalities through, we get $(k+1)^2 < 2^{k+1}$, in other words, $P(k+1)$. Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 5$. QED

Strong Induction Proof Structure

Again, start by saying what you want to prove: "Let $P(n)$ be the statement..." Then establish two facts:

1. Base case: Prove that $P(0)$ is true.
2. Inductive case: Assume $P(k)$ is true for all $k < n$. Prove that $P(n)$ is true.

Conclude, "therefore, by strong induction, $P(n)$ is true for all $n > 0$."

- Technically, strong induction doesn't require a separate base case, but when proving the inductive case, we must show that $P(0)$ is true, assuming $P(k)$ is true for all $k < 0$. We end up proving $P(0)$ anyway, include the base case to be safe.