Ch6.4 Binomial Coefficients and Identities

The Binomial Theorem

- **binomial**: the sum of two terms, i.e. $x + y$.

THEOREM 1 THE BINOMIAL THEOREM
Let x and y be variables, and let n be a nonnegative integer.
Then $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$.

▼ Example 4

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

From example 3, we know that the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$ is $\binom{25}{13} = \frac{25!}{13!12!} =$ $5,200,300$.

Also, $(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$. Consequently, the coeffient is $\binom{25}{13} 2^{12}(-3)^{13} = -5,200,300 \cdot 2^{12} \cdot 3^{13}$.

COROLLARY 1 Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

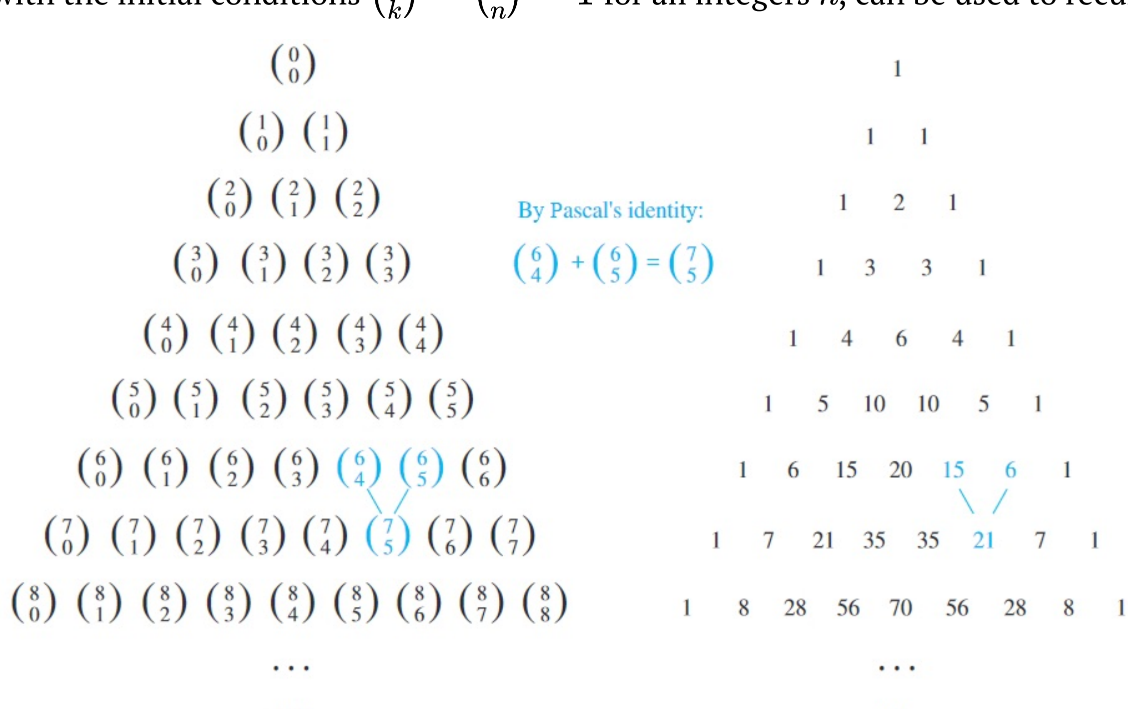
COROLLARY 2 Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Remark: Corollary 2 implies that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$.

COROLLARY 3 Let n be a nonnegative integer. Then $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

THEOREM 2 PASCA'S IDENTITY
Let n and k be positive integers with $n \geq k$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Remark: Pascal's identity, together with the initial conditions $\binom{n}{k} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients.



Other Identities Involving Binomial Coefficients

3 **THEOREM 3** VANDERMONDE'S IDENTITY

Let m, n and r be nonnegative integers with r not exceeding either m or n. Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} + \binom{n}{k}$.

4 **COROLLARY 4** Let n be a nonnegative integer, then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

4 **THEOREM 4**

Let n and r be nonnegative integers with $r \leq n$. Then $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$.