Ch9.1 Relations and Their Properties (Week 17)

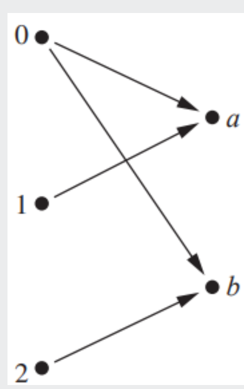
Introduction

1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

- a **related to** b by R: a R b → $(a, b) \in R$; on the other hand: a $\not R$ b → $(a, b) \notin R$

▼ Example 3

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that 0 R a, but that 1 $\not R$ b. Relations can be represented graphically, as shown in Figure 1.



R	a	b
0	×	×
1	×	
2		×

Figure 1 → Left: arrows; Right: table

Functions as Relations

→ A relation can be used to express a one-to-many relationship between the elements of the sets A and B, where A may be related to more than one element of B. A function represents a relation where exactly one element of B is related to each element of A.

→ Relations are a generalization of graphs of functions.

Relations on a Set

2 A *relation on a set A* is a relation from A to A.

(A relation on a set A is a subset of $A \times A$)

- There are 2^{n^2} relations on a set with n elements

Properties of Relations

3 A relation R on a set is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

▼ Example 7

Consider the following relations on $\{1, 2, 3, 4\}$:

$$
\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},\\
R_2 &= \{(1,1),(1,2),(2,1)\},\\
R_3 &= \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},\\
R_4 &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\},\\
R_5 &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\},\\
R_6 &= \{(3,4)\}.
\end{aligned}
$$

Which of these relations are reflexive?

Solution:
The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1), (2, 2), (3, 3)$, and $(4, 4)$. Other relations does not contain $(3, 3)$, hence not reflexive.

4 A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- The two terms are not opposites, a relation can have both or lack both.
- A relation cannot be both if it contains some pair of the form (a, b), where $a \neq b$.

▼ Example 10

Which of the relation from Example 7 are symmetric and which are antisymmetric?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
$R_6 = \{(3, 4)\}$.

Solution:
R_2 and R_3 are symmetric, each (b, a) belongs to the relation whenever (a, b) does.
For R_2 → check both $(2, 1)$ and $(1, 2)$ are in.
For R_3 → check both $(1, 2)$ and $(2, 1)$, $(1, 4)$ and $(4, 1)$ are in.
R_4, R_5, and R_6 are antisymmetric, there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) are in the relation.

5 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

▼ Example 13

Which of the relations in Example 7 are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
$R_6 = \{(3, 4)\}$.

Solutoin:
R_4, R_5, and R_6 are transitive.
For R_4 → $(3, 2)\&(2, 1)$, $(4, 2)\&(2, 1)$, $(4, 3)\&(3, 1)$, and $(4, 3)\&(3, 2)$ are the only sets of pairs. $(3, 1)$, $(4, 1)$, and $(4, 2)$ belong to R_4.
For R_1 → $(3, 4)\&(4, 1)$ belong to R_1, but $(3, 1)$ does not.
For R_2 → $(2, 1)\&(1, 2)$ belong to R_2, but $(2, 2)$ does not.
For R_3 → $(4, 1)\&(1, 2)$ belong to R_3, but $(4, 2)$ does not.

- There are $2^{n(n-1)}$ reflexive relations (# of ways to choose whether each element (a, b), with $a \neq b$ belongs to R)

Combining Relations

→ Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

▼ Example 17

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain:
$R_1 \cup R_2 = \{1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
$R_1 \cap R_2 = \{(1, 1)\}$,
$R_1 - R_2 = \{(2, 2), (3, 3)\}$,
$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$.

6 Let R be a relation from a set A to a set B and S (is) a relation from B to C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

▼ Example 20

What is the composite of the relations R and S, where R is a relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution:
$S \circ R$ is constructed using all ordered pairs in R and S, for example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$.
$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

7 Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by
$R^1 = R \quad$ and $\quad R^{n+1} = R^n \circ R$.

→ $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

▼ Example 22

Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$

Solution:
Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R^4 is the same as R^3, and so on.

1 **Theorem 1** The relation R on a set A is transitive *iff* $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$