

Ch6.4 Binomial Coefficients and Identities

The Binomial Theorem

- **binomial**: the sum of two terms, i.e. x + y.
- THEOREM 1 THE BINOMIAL THEOREM

Let x and y be variables, and let n be a nonnegative integer.

Then
$$(x+y)^n=\sum_{j=0}^n \binom{n}{j}x^{n-j}y^j=\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n.$$

▼ Example 4

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

From example 3, we know that the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$ is $\binom{25}{13}=\frac{25!}{13!12!}=5,200,300$.

Also, $(2x+(-3y))^{25}=\sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$. Consequently, the coefficient is $\binom{25}{13} 2^{12} (-3)^{13}=-5,200,300 + 2^{12} \cdot 3^{13}$.

- COROLLARY 1 Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^{n}$.
- **2** COROLLARY 2 Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Remark: Corollary 2 implies that $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots$.

- **3 COROLLARY 3** Let n be a nonnegative integer. Then $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.
- **THEOREM 2** PASCA'S IDENTITY Let n and k be positive integers with $n \ge k$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Remark: Pascal's identity, together with the initial conditions $\binom{n}{k} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \qquad 1$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \qquad \qquad 1 \qquad 1$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \text{By Pascal's identity:} \qquad 1 \qquad 2 \qquad 1$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \qquad 1 \qquad 3 \qquad 3 \qquad 1$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \qquad \qquad 1 \qquad 4 \qquad 6 \qquad 4 \qquad 1$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \qquad \qquad \qquad 1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \qquad \qquad 1 \qquad 6 \qquad 15 \qquad 20 \qquad 15 \qquad 6 \qquad 1$$

$$\begin{pmatrix} 7 \\ 0 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} \qquad \qquad 1 \qquad 7 \qquad 21 \quad 35 \quad 35 \quad 21 \qquad 7 \qquad 1$$

$$\begin{pmatrix} 8 \\ 0 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \qquad 1 \qquad 8 \qquad 28 \qquad 56 \qquad 70 \qquad 56 \qquad 28 \qquad 8 \qquad 1$$

Other Identities Involving Binomial Coefficients

- THEOREM 3 VANDERMONDE'S IDENTITY

 Let m, n and r be nonnegative integers with r not exceeding either m or n. Then $\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} + \binom{n}{k}$.
- **COROLLARY 4** Let n be a nonnegative integer, then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.
- THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then $\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}$.