1.3 Propositional Equivalences

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Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$.

We begin our discussion with a classification of compound propositions according to their possible truth values.

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

Logical Equivalences

Demo

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**. We can also define this notion as follows.

DEFINITION 2 The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions p and q are equivalent if and only if the columns

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE 2 De Morgan's Laws.

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Extra Examples

giving their truth values agree. Example 2 illustrates this method to establish an extremely important and useful logical equivalence, namely, that of $\neg(p \vee q)$ with $\neg p \wedge \neg q$. This logical equivalence is one of the two **De Morgan laws**, shown in Table 2, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.

EXAMPLE 2 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table 3. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent. ◀

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

EXAMPLE 3 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4. Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, they are logically equivalent. ◀

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We will now establish a logical equivalence of two compound propositions involving three different propositional variables p, q, and r. To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables. We symbolically represent these combinations by listing the truth values of p, q, and r, respectively. These eight combinations of truth values are TTT, TTF, TFT, TFF, FTT, FTF, FFT, and FFF; we use this order when we display the rows of the truth table. Note that we need to double the number of rows in the truth tables we use to show that compound propositions are equivalent for each additional propositional variable, so that 16 rows are needed to establish the logical equivalence of two compound propositions involving four propositional variables, and so on. In general, 2^n rows are required if a compound proposition involves n propositional variables.

TABLE 5 **A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.**

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

EXAMPLE 4 Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution: We construct the truth table for these compound propositions in Table 5. Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent. ◀

The identities in Table 6 are a special case of Boolean algebra identities found in Table 5 of Section 12.1. See Table 1 in Section 2.2 for analogous set identities.

Table 6 contains some important equivalences. In these equivalences, **T** denotes the compound proposition that is always true and **F** denotes the compound proposition that is always

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

false. We also display some useful equivalences for compound propositions involving conditional statements and biconditional statements in Tables 7 and 8, respectively. The reader is asked to verify the equivalences in Tables 6–8 in the exercises.

The associative law for disjunction shows that the expression $p \vee q \vee r$ is well defined, in the sense that it does not matter whether we first take the disjunction of p with q and then the disjunction of $p \vee q$ with r, or if we first take the disjunction of q and r and then take the disjunction of p with $q \vee r$. Similarly, the expression $p \wedge q \wedge r$ is well defined. By extending this reasoning, it follows that $p_1 \vee p_2 \vee \cdots \vee p_n$ and $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ are well defined whenever $p_1, p_2, \ldots, p_n$ are propositions.

Furthermore, note that De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n).$$

We will sometimes use the notation $\bigvee_{j=1}^{n} p_j$ for $p_1 \vee p_2 \vee \cdots \vee p_n$ and $\bigwedge_{j=1}^{n} p_j$ for $p_1 \wedge p_2 \wedge \cdots \wedge p_n$. Using this notation, the extended version of De Morgan's laws can be written concisely as $\neg\left(\bigvee_{j=1}^{n} p_j\right) \equiv \bigwedge_{j=1}^{n} \neg p_j$ and $\neg\left(\bigwedge_{j=1}^{n} p_j\right) \equiv \bigvee_{j=1}^{n} \neg p_j$. (Methods for proving these identities will be given in Section 5.1.)

Using De Morgan's Laws

When using De Morgan's laws, remember to change the logical connective after you negate.

The two logical equivalences known as De Morgan's laws are particularly important. They tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ tells us that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions. Similarly, the equivalence $\neg(p \wedge q) \equiv \neg p \vee \neg q$ tells us that the negation of a conjunction is formed by taking the disjunction of the negations of the component propositions. Example 5 illustrates the use of De Morgan's laws.

EXAMPLE 5 Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Assessment

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone or he does not have a laptop computer."

Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$. Consequently, we can express the negation of our original statement as "Heather will not go to the concert and Steve will not go to the concert." ◀

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Constructing New Logical Equivalences

The logical equivalences in Table 6, as well as any others that have been established (such as those shown in Tables 7 and 8), can be used to construct additional logical equivalences. The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition. This technique is illustrated in Examples 6–8, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent (see Exercise 56).

EXAMPLE 6 Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Extra Examples

Solution: We could use a truth table to show that these compound propositions are equivalent (similar to what we did in Example 4). Indeed, it would not be hard to do so. However, we want to illustrate how to use logical identities that we already know to establish new logical identities, something that is of practical importance for establishing equivalences of compound propositions with a large number of variables. So, we will establish this equivalence by developing a series of

Links

AUGUSTUS DE MORGAN (1806–1871) Augustus De Morgan was born in India, where his father was a colonel in the Indian army. De Morgan's family moved to England when he was 7 months old. He attended private schools, where in his early teens he developed a strong interest in mathematics. De Morgan studied at Trinity College, Cambridge, graduating in 1827. Although he considered medicine or law, he decided on mathematics for his career. He won a position at University College, London, in 1828, but resigned after the college dismissed a fellow professor without giving reasons. However, he resumed this position in 1836 when his successor died, remaining until 1866.

De Morgan was a noted teacher who stressed principles over techniques. His students included many famous mathematicians, including Augusta Ada, Countess of Lovelace, who was Charles Babbage's collaborator in his work on computing machines (see page 31 for biographical notes on Augusta Ada). (De Morgan cautioned the countess against studying too much mathematics, because it might interfere with her childbearing abilities!)

De Morgan was an extremely prolific writer, publishing more than 1000 articles in more than 15 periodicals. De Morgan also wrote textbooks on many subjects, including logic, probability, calculus, and algebra. In 1838 he presented what was perhaps the first clear explanation of an important proof technique known as *mathematical induction* (discussed in Section 5.1 of this text), a term he coined. In the 1840s De Morgan made fundamental contributions to the development of symbolic logic. He invented notations that helped him prove propositional equivalences, such as the laws that are named after him. In 1842 De Morgan presented what is considered to be the first precise definition of a limit and developed new tests for convergence of infinite series. De Morgan was also interested in the history of mathematics and wrote biographies of Newton and Halley.

In 1837 De Morgan married Sophia Frend, who wrote his biography in 1882. De Morgan's research, writing, and teaching left little time for his family or social life. Nevertheless, he was noted for his kindness, humor, and wide range of knowledge.

Solving Satisfiability Problems

A truth table can be used to determine whether a compound proposition is satisfiable, or equivalently, whether its negation is a tautology (see Exercise 60). This can be done by hand for a compound proposition with a small number of variables, but when the number of variables grows, this becomes impractical. For instance, there are $2^{20} = 1,048,576$ rows in the truth table for a compound proposition with 20 variables. Clearly, you need a computer to help you determine, in this way, whether a compound proposition in 20 variables is satisfiable.

When many applications are modeled, questions concerning the satisfiability of compound propositions with hundreds, thousands, or millions of variables arise. Note, for example, that when there are 1000 variables, checking every one of the 2^{1000} (a number with more than 300 decimal digits) possible combinations of truth values of the variables in a compound proposition cannot be done by a computer in even trillions of years. No procedure is known that a computer can follow to determine in a reasonable amount of time whether an arbitrary compound proposition in such a large number of variables is satisfiable. However, progress has been made developing methods for solving the satisfiability problem for the particular types of compound propositions that arise in practical applications, such as for the solution of Sudoku puzzles. Many computer programs have been developed for solving satisfiability problems which have practical use. In our discussion of the subject of algorithms in Chapter 3, we will discuss this question further. In particular, we will explain the important role the propositional satisfiability problem plays in the study of the complexity of algorithms.

Links

10/29 5:36 – 7:15 exercise 8:30 – 9:20

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Exercises

1.

(a)

p	T	p∧T
T	T	T
F	T	F

same ✓

(b)

p	F	p∨F
T	F	T
F	F	F

same ✓

(c)

p	F	p∧F
T	F	F
F	F	F

same ✓

(d)

p	T	p∨T
T	T	T
F	T	T

same ✓

(e)

p	p∨p
T	T
F	F

same ✓

(f)

p	p∧p
T	T
F	F

same ✓

1. Use truth tables to verify these equivalences.
   a) $p \wedge \mathbf{T} \equiv p$ b) $p \vee \mathbf{F} \equiv p$
   c) $p \wedge \mathbf{F} \equiv \mathbf{F}$ d) $p \vee \mathbf{T} \equiv \mathbf{T}$
   e) $p \vee p \equiv p$ f) $p \wedge p \equiv p$
2. Show that $\neg(\neg p)$ and p are logically equivalent.
3. Use truth tables to verify the commutative laws
   a) $p \vee q \equiv q \vee p$. b) $p \wedge q \equiv q \wedge p$.
4. Use truth tables to verify the associative laws
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
   b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
5. Use a truth table to verify the distributive law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

5.

p	q	r	p∧q	p∧r	q∨r	p∧(q∨r)	(p∧q)∨(p∧r)
F	F	F	F	F	F	F	F
F	F	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	T	T	F	F	T	F	F
T	F	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	T	F	T	F	T	T	T
T	T	T	T	T	T	T	T

same ✓

6. Use a truth table to verify the first De Morgan law $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
7. Use De Morgan's laws to find the negation of each of the following statements.
   a) Jan is rich and happy. Jan is not rich or happy ✓ Jan is not
   b) Carlos will bicycle or run tomorrow.
   Carlos will not bicycle and run tomorrow. ✗ neither nor

Links

HENRY MAURICE SHEFFER (1883–1964) Henry Maurice Sheffer, born to Jewish parents in the western Ukraine, emigrated to the United States in 1892 with his parents and six siblings. He studied at the Boston Latin School before entering Harvard, where he completed his undergraduate degree in 1905, his master's in 1907, and his Ph.D. in philosophy in 1908. After holding a postdoctoral position at Harvard, Henry traveled to Europe on a fellowship. Upon returning to the United States, he became an academic nomad, spending one year each at the University of Washington, Cornell, the University of Minnesota, the University of Missouri, and City College in New York. In 1916 he returned to Harvard as a faculty member in the philosophy department. He remained at Harvard until his retirement in 1952.

Sheffer introduced what is now known as the Sheffer stroke in 1913; it became well known only after its use in the 1925 edition of Whitehead and Russell's *Principia Mathematica*. In this same edition Russell wrote that Sheffer had invented a powerful method that could be used to simplify the *Principia*. Because of this comment, Sheffer was something of a mystery man to logicians, especially because Sheffer, who published little in his career, never published the details of this method, only describing it in mimeographed notes and in a brief published abstract.

Sheffer was a dedicated teacher of mathematical logic. He liked his classes to be small and did not like auditors. When strangers appeared in his classroom, Sheffer would order them to leave, even his colleagues or distinguished guests visiting Harvard. Sheffer was barely five feet tall; he was noted for his wit and vigor, as well as for his nervousness and irritability. Although widely liked, he was quite lonely. He is noted for a quip he spoke at his retirement: "Old professors never die, they just become emeriti." Sheffer is also credited with coining the term "Boolean algebra" (the subject of Chapter 12 of this text). Sheffer was briefly married and lived most of his later life in small rooms at a hotel packed with his logic books and vast files of slips of paper he used to jot down his ideas. Unfortunately, Sheffer suffered from severe depression during the last two decades of his life.

1.6 Rules of Inference

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Introduction

Later in this chapter we will study proofs. Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an **argument**, we mean a sequence of statements that end with a conclusion. By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument. That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments. Rules of inference are our basic tools for establishing the truth of statements.

繞口

演義,推論

Before we study mathematical proofs, we will look at arguments that involve only compound propositions. We will define what it means for an argument involving compound propositions to be valid. Then we will introduce a collection of rules of inference in propositional logic. These rules of inference are among the most important ingredients in producing valid arguments. After we illustrate how rules of inference are used to produce valid arguments, we will describe some common forms of incorrect reasoning, called **fallacies**, which lead to invalid arguments.

謬論

After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements. We will describe how these rules of inference can be used to produce valid arguments. These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.

Finally, we will show how rules of inference for propositions and for quantified statements can be combined. These combinations of rule of inference are often used together in complicated arguments.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,

"You can log onto the network."

We would like to determine whether this is a valid argument. That is, we would like to determine whether the conclusion "You can log onto the network" must be true when the premises "If you have a current password, then you can log onto the network" and "You have a current password" are both true.

Before we discuss the validity of this particular argument, we will look at its form. Use p to represent "You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

where $\therefore$ is the symbol that denotes "therefore."

We know that when p and q are propositional variables, the statement $((p \to q) \wedge p) \to q$ is a tautology (see Exercise 10(c) in Section 1.3). In particular, when both $p \to q$ and p are true, we know that q must also be true. We say this form of argument is **valid** because whenever all its premises (all statements in the argument other than the final one, the conclusion) are true, the conclusion must also be true. Now suppose that both "If you have a current password, then you can log onto the network" and "You have a current password" are true statements. When we replace p by "You have a current password" and q by "You can log onto the network," it necessarily follows that the conclusion "You can log onto the network" is true. This argument is **valid** because its form is valid. Note that whenever we replace p and q by propositions where $p \to q$ and p are both true, then q must also be true.

What happens when we replace p and q in this argument form by propositions where not both p and $p \to q$ are true? For example, suppose that p represents "You have access to the network" and q represents "You can change your grade" and that p is true, but $p \to q$ is false. The argument we obtain by substituting these values of p and q into the argument form is

"If you have access to the network, then you can change your grade."
"You have access to the network."

∴ "You can change your grade."

The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is false, we cannot conclude that the conclusion is true. (Most likely, this conclusion is false.)

In our discussion, to analyze an argument, we replaced propositions by propositional variables. This changed an argument to an **argument form**. We saw that the validity of an argument follows from the validity of the form of the argument. We summarize the terminology used to discuss the validity of arguments with our definition of the key notions.

DEFINITION 1

An *argument* in propositional logic is a sequence of propositions. All but the final proposition in the argument are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology.

The key to showing that an argument in propositional logic is valid is to show that its argument form is valid. Consequently, we would like techniques to show that argument forms are valid. We will now develop methods for accomplishing this task.

Rules of Inference for Propositional Logic

We can always use a truth table to show that an argument form is valid. We do this by showing that whenever the premises are true, the conclusion must also be true. However, this can be a tedious approach. For example, when an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows. Fortunately, we do not have to resort to truth tables. Instead, we can first establish the validity of some relatively simple argument forms, called **rules of inference**. These rules of inference can be used as building blocks to construct more complicated valid argument forms. We will now introduce the most important rules of inference in propositional logic.

The tautology $(p \wedge (p \to q)) \to q$ is the basis of the rule of inference called **modus ponens**, or the **law of detachment**. (Modus ponens is Latin for *mode that affirms*.) This tautology leads to the following valid argument form, which we have already seen in our initial discussion about arguments (where, as before, the symbol $\therefore$ denotes "therefore"):

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion. In particular, modus ponens tells us that if a conditional statement and the hypothesis of this conditional statement are both true, then the conclusion must also be true. Example 1 illustrates the use of modus ponens.

EXAMPLE 1 Suppose that the conditional statement "If it snows today, then we will go skiing" and its hypothesis, "It is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the conditional statement, "We will go skiing," is true. ◀

As we mentioned earlier, a valid argument can lead to an incorrect conclusion if one or more of its premises is false. We illustrate this again in Example 2.

EXAMPLE 2 Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

"If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $\left(\sqrt{2}\right)^2 = 2 > \left(\frac{3}{2}\right)^2 = \frac{9}{4}$."

Solution: Let p be the proposition "$\sqrt{2} > \frac{3}{2}$" and q the proposition "$2 > (\frac{3}{2})^2$." The premises of the argument are $p \to q$ and p, and q is its conclusion. This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$, is false. Consequently, we cannot conclude that the conclusion is true. Furthermore, note that the conclusion of this argument is false, because $2 < \frac{9}{4}$. ◀

There are many useful rules of inference for propositional logic. Perhaps the most widely used of these are listed in Table 1. Exercises 9, 10, 15, and 30 in Section 1.3 ask for the verifications that these rules of inference are valid argument forms. We now give examples of arguments that use these rules of inference. In each argument, we first use propositional variables to express the propositions in the argument. We then show that the resulting argument form is a rule of inference from Table 1.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

EXAMPLE 3 State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\begin{array}{rl} & p \\ \hline \therefore & p \lor q \end{array}$$

This is an argument that uses the addition rule. ◀

EXAMPLE 4 State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{rl} & p \land q \\ \hline \therefore & p \end{array}$$

This argument uses the simplification rule. ◀

EXAMPLE 5 State which rule of inference is used in the argument:

> If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated by Examples 6 and 7, where the steps of arguments are displayed on separate lines, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Extra Examples

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$ and conclusion t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Note that we could have used a truth table to show that whenever each of the four hypotheses is true, the conclusion is also true. However, because we are working with five propositional variables, p, q, r, s, and t, such a truth table would have 32 rows. ◀

EXAMPLE 7 Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	Reason
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

◀

Resolution

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution**. This rule of inference is based on the tautology

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$$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r).$$

(Exercise 30 in Section 1.3 asks for the verification that this is a tautology.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent**. When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \to q$. Furthermore, when we let $r = \mathbf{F}$, we obtain $(p \vee q) \wedge (\neg p) \to q$ (because $q \vee \mathbf{F} \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8 Use resolution to show that the hypotheses "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey."

Extra Examples

Solution: Let p be the proposition "It is snowing," q the proposition "Jasmine is skiing," and r the proposition "Bart is playing hockey." We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, "Jasmine is skiing or Bart is playing hockey," follows. ◀

Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses**, where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan's law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace a conditional statement $p \to q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the premises $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the premises $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$. ◀

Fallacies

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.

Links

The proposition $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the **fallacy of affirming the conclusion**.

EXAMPLE 10 Is the following argument valid?

> If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.
>
> Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \rightarrow q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.) ◀

The proposition $((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis**.

EXAMPLE 11 Let p and q be as in Example 10. If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

Rules of Inference for Quantified Statements

We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how some of these rules of inference for quantified statements are used in Examples 12 and 13.

EXAMPLE 12 Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

Extra Examples

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3) ◂

EXAMPLE 13 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

◀

Combining Rules of Inference for Propositions and Quantified Statements

We have developed rules of inference both for propositions and for quantified statements. Note that in our arguments in Examples 12 and 13 we used both universal instantiation, a rule of inference for quantified statements, and modus ponens, a rule of inference for propositional logic. We will often need to use this combination of rules of inference. Because universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens**. This rule tells us that if $\forall x(P(x) \rightarrow Q(x))$ is true, and if $P(a)$ is true for a particular element a in the domain of the universal quantifier, then $Q(a)$ must also be true. To see this, note that by universal instantiation, $P(a) \rightarrow Q(a)$ is true. Then, by modus ponens, $Q(a)$ must also be true. We can describe universal modus ponens as follows:

$$\begin{array}{rl} & \forall x(P(x) \rightarrow Q(x)) \\ & P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & Q(a) \end{array}$$

Universal modus ponens is commonly used in mathematical arguments. This is illustrated in Example 14.

EXAMPLE 14 Assume that "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true. Use universal modus ponens to show that $100^2 < 2^{100}$.

Solution: Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$." The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(100)$ is true, namely that $100^2 < 2^{100}$. ◀

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens**. Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{l} \forall x(P(x) \to Q(x)) \\ \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore \neg P(a) \end{array}$$

The verification of universal modus tollens is left as Exercise 25. Exercises 26–29 develop additional combinations of rules of inference in propositional logic and quantified statements.

11/13 17:21 do 1 11/13 \\

Exercises

19:09 2h29m − 23 = 2h06m

19:33 correct 1 19:50

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal. $p \to q$
Socrates is human. p

$\therefore$ Socrates is mortal. $\therefore q$

modus ponens

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not a spider.
George is a spider.

$\therefore$ George has eight legs.

3. What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

$\dfrac{p}{\therefore p \lor q}$ ⇒ addition

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

$\dfrac{p \land q}{\therefore p}$ ⇒ simplification

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

$\dfrac{p \to q \quad p}{\therefore q}$ ⇒ modus ponens

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

$\dfrac{p \to q \quad \neg q}{\therefore \neg p}$ ⇒ modus tollens

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

$\dfrac{p \to q \quad q \to r}{p \to r}$ ⇒ hypothetical syllogism

4. What rule of inference is used in each of these arguments?

a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.

b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.

c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.

e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

5. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

1. p premise
2. $p \to q$ premise
3. $q \to r$ premise
4. $p \to r$ hypothetical syllogism using (2) and (3)
5. r modus ponens using (1) and (4)

6. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

1. $(\neg p \lor \neg q) \to r$ Premise
2. $\neg(p \land q) \to r$ De Morgan's law
3. $(p \land q) \lor r$ $p \to q \equiv \neg p \lor q$
4. $r \to s$ Premise
5. $\neg r \lor s$ $p \to q \equiv \neg p \lor q$
6. $(p \land q) \lor s$ Resolution using (3) and (5)
7. $p \lor s$ Simplification using (6)
8. $s \lor p$ $p \to q \equiv \neg p \lor q$

same

7. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal." Modus ponens

8. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."

9. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."

b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."

c) "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."

d) "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."

e) "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."

f) "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

13. a) P(x): x knows how to write programs in java
Q(x): x can get a high-paying job
C(x): x is in this class

1. $\forall x(P(x) \to Q(x))$ Premise
2. ~~$P(a) \to Q(a)$ Universal instantiation from (1)~~
3. $P(\text{Doug}) \land C(\text{Doug})$ Premise
4. ~~$\exists x(P(x) \land C(x))$ Existential generalization from (3)~~
5. ~~$P(a) \land C(a)$ Universal instantiation from (4)~~
6. ~~$\neg P(a) \lor Q(a)$ $p \to q \equiv \neg p \lor q$ from (2)~~
7. $C(a)$ ~~Simplification from (5)~~ Premise
8. $P(a)$ ~~Simplification from (5)~~ Premise
9. $Q(a)$ ~~Disjunctive syllogism from (6)(8)~~ Modus ponens (2)(3)
10. $C(a) \land Q(a)$ Conjunction from ~~(8)(9)~~ (4)(5)
11. $\exists x(C(x) \land Q(x))$ Existential generalization from ~~(10)~~ (6)

b) C(x): x is in this class
W(x): x enjoys whale watching
O(x): x cares about ocean pollution

premises
$\exists x(C(x) \land W(x))$
$\forall x(W(x) \to O(x))$
$\therefore \exists x(C(x) \land O(x))$

1. $\exists x(C(x) \land W(x))$ Premise
2. $C(a) \land W(a)$ Existential instantiation from (1)
3. $C(a)$ and $W(a)$ Simplification from (2)
4. $\forall x(W(x) \to O(x))$ Premise
5. $W(a) \to Q(a)$ Universal instantiation from (4)
6. ~~$\neg W(a) \lor Q(a)$ $p \to q \equiv \neg p \lor q$ from (5)~~
7. $Q(a)$ ~~Disjuntive syllogism from (3)(6)~~ Modus ponens (3)(5)
8. $C(a) \land Q(a)$ Conjunction from (3)(7)
9. $\exists x(C(x) \land Q(x))$ Existential generalization

c) C(x): x is one of the 93 students in this class
P(x): x owns a personal computer
W(x): x can use a word processing program

$\forall x(C(x) \to P(x))$ C(Zeke)
$\forall x(P(x) \to W(x))$
$\therefore$ ~~C(Zeke) ∧~~ W(Zeke)

1. $\forall x(C(x) \to P(x))$ Premise
2. ~~$C(a) \land P(a)$~~ $C(\text{Zeke}) \to P(\text{Zeke})$ Universal instantiation (1)
3. C(Zeke) ~~Simplification (2)~~ Premise
4. P(Zeke) ~~Simplification (2)~~ Modus ponens (2)(3)
5. $\forall x(P(x) \to W(x))$ Premise
6. $P(\text{Zeke}) \to W(\text{Zeke})$ Universal instantiation (5)
7. ~~$\neg P(a) \lor W(a)$ $p \to q \equiv \neg p \lor q$ (6)~~
8. ~~$W(a)$ Disjunctive syllogism (4)(7)~~
9. ~~$C(a) \land W(a)$ Conjunction (3)(8)~~
10. ~~$C(\text{Zeke}) \land W(\text{Zeke})$~~

W(Zeke) Modus ponens (4)(6)

$\forall x(J(x) \to Q(x))$
$\exists x(J(x) \land N(x))$
$\therefore \exists x(Q(x) \land N(x))$

d) J(x): x lives New Jersey
Q(x): x lives within 50 miles of the ocean
N(x): x has never seen the ocean

1. $\forall x(J(x) \to Q(x))$ Premise
2. $J(a) \to Q(a)$ Universal instantiation (1)
3. ~~$\neg J(a) \lor Q(a)$ $p \to q \equiv \neg p \lor q$ (2)~~
4. $\exists x(J(x) \land N(x))$ Premise
5. $J(a) \land N(a)$ Existential instantiation (4)
6. $J(a)$ Simplification (5)
7. $N(a)$ Simplification (5)
8. $Q(a)$ ~~Disjunctive syllogism (3)(6)~~ modus ponens (2)(6)
9. $Q(a) \land N(a)$ Conjunction (7)(8)
10. $\exists x(Q(x) \land N(x))$ Exisential generalization (9)

10. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

d) "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."

e) "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."

f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

11. Show that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q \to r$ is valid if the argument form with premises $p_1, p_2, \ldots, p_n, q$, and conclusion r is valid.

12. Show that the argument form with premises $(p \land t) \to (r \lor s)$, $q \to (u \land t)$, $u \to p$, and $\neg s$ and conclusion $q \to r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

13. For each of these arguments, explain which rules of inference are used for each step.

a) "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."

b) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."

c) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

d) "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."

14. For each of these arguments, explain which rules of inference are used for each step.

a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."

b) "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.

O a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic. Modus ponens & universal instantiation

X b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major. fallacy of affirming the conclusion

X c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit. fallacy of denying the hypothesis

O d) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day. Modus tollens & universal instantiation

16. For each of these arguments determine whether the argument is correct or incorrect and explain why.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.

b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

17. What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy. can't conclude Lola is one such x.

18. What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \le 9$. Then $n \le 3$.

c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \le 2$. Then $n^2 \le 4$.

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