Ch 10.5 Euler and Hamilton Paths (Week 13)

Euler Paths and Circuits

1 An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

- **Path/Walk:** edges travel from vertex to vertex
  Simple Path/Trial: walk with no repeated edge (path → trail with no repeated vertices) (when trail is used)
- **Circuit/Cycle:** a path of length > 0 + start = end
  Simple Circuit: no repeated edge
- **Euler circuit:**
  simple circuit with every edge | start $=$ end + no repeated edge
- **Euler path:**
  simple path with every edge | start $\neq$ end + no repeated vertex

1 **Theorem 1** A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

2 **Theorem 2** A connected multigraph has an Euler path but not an Euler circuit *iff* it has exactly two vertices of odd degree.

Hamilton Paths and Circuits

2 A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, ..., x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, ..., x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, ..., x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, ..., x_{n-1}, x_n$ is a Hamilton path.

- a graph with a vertex of degree one cannot have a Hamilton circuit
- the more edges a graph has, the more likely it is to have a Hamilton circuit

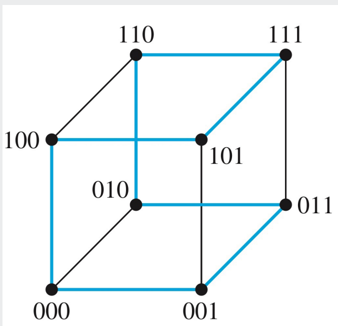
3 **Dirac's Theorem** If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

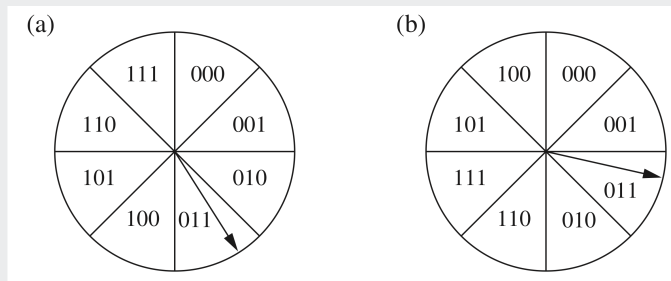
4 **Ore's Theorem** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq \mathrm{n}$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Applications of Hamilton Circuits

▼ Example 8

Gray Codes: a labeling of the arcs of the circle such that adjacent arcs are labeled with bit strings that differ in eactly one bit.





(b) is a Gray code

We can model this problem using the n-cube Q_n.