Ch10.4 Connectivity (Week 14)

Paths

- **path:** a sequence of edges
- **circuit:** a path that begin = end & length > 0

⇒ *pass through* vertices or *traverse* edges

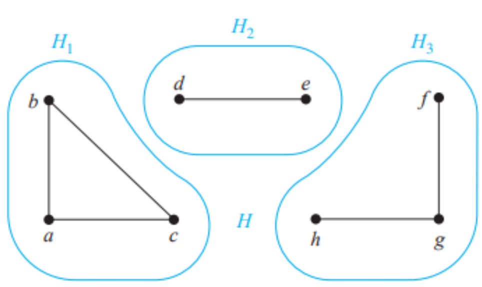
Connectedness in Undirected Graphs

- *connected:* there is a path between every pair of distinct vertices

Theorem 1 There is a simple path between every pair of distinct vertices of connected undirected graph.

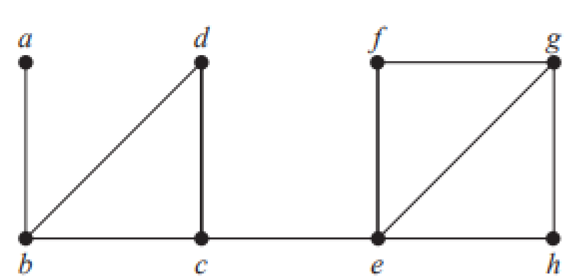
Connected Components

- a graph is a connected subgraph that is not a proper subgraph of another connected subgraph



How Connected is a Graph ??

- **cut vertices** (or **articulation points**): the vertex that if removed itself and all incident edges, produces a subgraph with more connected components
- **cut edge** (or **bridge**): the edge that if removed, produces a graph with more connected components



Example 7 Find the cut vertices & cut edges in the graph.

cut vertices → b, c, e (either one of them can disconnect the graph)

cut edges → $\{a, b\}$ and $\{c, e\}$ (same, either one can)

Vertex Connectivity $\kappa(G)$

- **nonseparable graphs:** connected graphs without cut vertices (complete graph: K_n $n \geq 3$)
- **vertex cut** (or **separating set**): if $G - V'$ is disconnected (V': subset of the vertex)
- **vertex connectivity:** the minimum number of vertices in a vertex cut

$\kappa(K_n) = n - 1,$

the number of vertices needed to be removed to produce a graph with a single vertex.

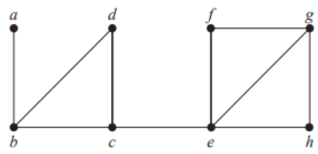
$\kappa(G) \uparrow$ $\rightarrow$ more connected G is

k-**connected** (or k-**vertex-connected**): $\kappa(G) \geq k$

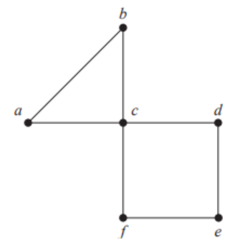
Every connected graph, except a complete graph, has a vertex cut.

Edge Connectivity $\lambda(G)$

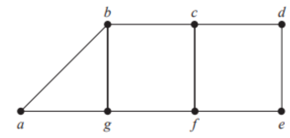
- **edge cut:** if $G - E'$ is disconnected, then the set of edges E' is an edge cut
- **edge connectivity:** minimum number of edges in an edge cut



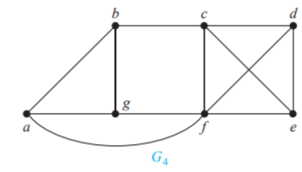
G_1

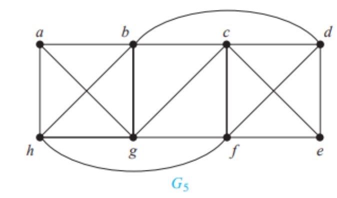


G_2



G_3





Example 8 Find the **vertex connectivity** for each graphs.

G_1: one cut vertex, $\kappa(G_1) = 1$

G_2: one cut vertex c, $\kappa(G_2) = 1$

G_3: no cut vertices, vertex cut: $\{b, g\}$, $\kappa(G_3) = 2$

G_4: one vertex cut of size two, $\{c, f\}$, no cut vertices, $\kappa(G_4) = 2$

G_5: no vertex cut of size two,vertex cut: $\{b, c, f\}$, $\kappa(G_5) = 3$

Example 9 Find the **edge connectivity** for each graphs.

G_1: one cut vertex (??), $\lambda(G_1) = 1$

G_2: no cut vertex c, $\lambda(G_2) = 2$

G_3: no cut edges, $\lambda(G_3) = 2$

G_4: the removal of no two edges disconnects, does, $\lambda(G_4) = 3$

G_5: same as G_4, $\{a, b\}, \{a, g\}, \{a, h\}$ does, $\lambda(G_5) = 3$

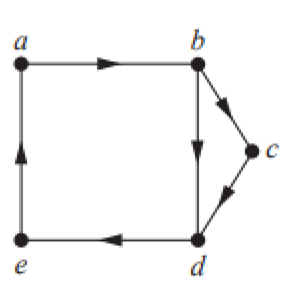
An Inequality for Vertex Connectivity and Edge Connectivity

$\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$.

Connectedness in Directed Graphs

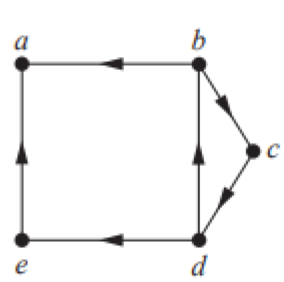
4 A directed graph is *strongly connected* if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

5 A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.



G

G: strongly connected

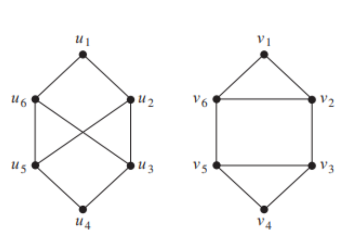


H

H: weakly connected

- **strongly connected components/strong components:** maximal strongly connected subgraphs

Paths and Isomorphism

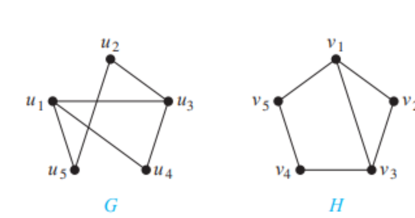


not isomorphic

$\rightarrow$ # of vertices, edges, and degree of vertices $\Rightarrow$ same

H has a simple circuit of length three, G has no.

5 vertices, 6 edges, 2 vertices of degree three, 3 vertices of degree 2, a simple circuit of length three, four, and five $\leftarrow$

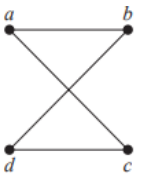


isomorphic

Counting Paths Between Vertices

2 **Theorem 2** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undireted edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of $\mathbf{A}^r$.

Example 15 How many paths of length four are there from a to d in the graph ??



The adjacency matrix is $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Hence, the number of paths of length four from a to d is the $(1, 4)$th entry of A^4. Becuase $A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$, there are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d.