24. **a)** Explain how graphs can be used to model e[redacted] mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

directed yes yes

b) Describe a graph that models the e[redacted] mail sent in a network in a particular week.

25. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?

same communication pattern & where it goes (where its from perhaps when a vertex has the same ins & outs but less (or more?) inputs and outputs

26. How can a graph that models e-mail messages sent in a network be used to find electronic mail mailing lists used to send the same message to many different e-mail addresses?

27. Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

simple directed graph No No

28. Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

29. For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are prerequisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?

30. Describe a graph model that represents the positive recommendations of movie critics, using vertices to represent both these critics and all movies that are currently being shown.

31. Describe a graph model that represents traditional marriages between men and women. Does this graph have any special properties?

32. Which statements must be executed before S_6 is executed in the program in Example 8? (Use the precedence graph in Figure 10.)

33. Construct a precedence graph for the following program:

$$S_1\colon x := 0$$
$$S_2\colon x := x + 1$$
$$S_3\colon y := 2$$
$$S_4\colon z := y$$
$$S_5\colon x := x + 2$$
$$S_6\colon y := x + z$$
$$S_7\colon z := 4$$

34. Describe a discrete structure based on a graph that can be used to model airline routes and their flight times. [*Hint:* Add structure to a directed graph.]

35. Describe a discrete structure based on a graph that can be used to model relationships between pairs of individuals in a group, where each individual may either like, dislike, or be neutral about another individual, and the reverse relationship may be different. [*Hint:* Add structure to a directed graph. Treat separately the edges in opposite directions between vertices representing two individuals.]

36. Describe a graph model that can be used to represent all forms of electronic communication between two people in a single graph. What kind of graph is needed?

10.2 Graph Terminology and Special Types of Graphs

Introduction

12/10 6:35 | read 8:54 10:04 | Qs 10:21 | note

Links

We introduce some of the basic vocabulary of graph theory in this section. We will use this vocabulary later in this chapter when we solve many different types of problems. One such problem involves determining whether a graph can be drawn in the plane so that no two of its edges cross. Another example is deciding whether there is a one-to-one correspondence between the vertices of two graphs that produces a one-to-one correspondence between the edges of the graphs. We will also introduce several important families of graphs often used as examples and in models. Several important applications will be described where these special types of graphs arise.

Basic Terminology

First, we give some terminology that describes the vertices and edges of undirected graphs.

DEFINITION 1

Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

We will also find useful terminology describing the set of vertices adjacent to a particular vertex of a graph.

what does it mean A is the subset of V ?
V is the set of all vertices ?

DEFINITION 2

(not number) neighbor

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

To keep track of how many edges are incident to a vertex, we make the following definition.

DEFINITION 3

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

EXAMPLE 1 What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$. In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$. ◀

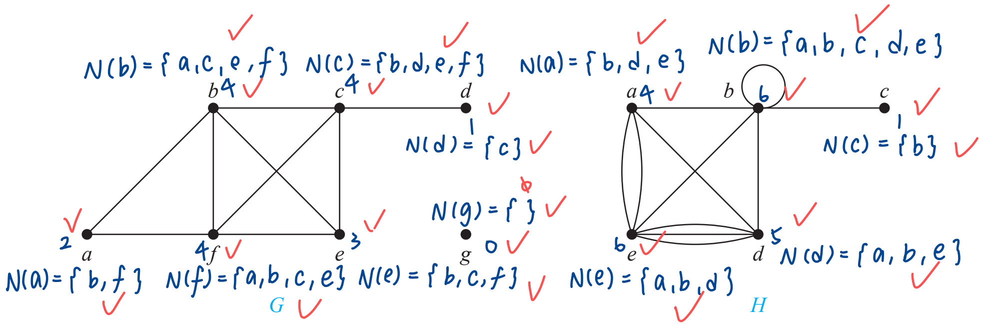


FIGURE 1 **The Undirected Graphs G and H.**

A vertex of degree zero is called **isolated**. It follows that an isolated vertex is not adjacent to any vertex. Vertex g in graph G in Example 1 is isolated. A vertex is **pendant** if and only if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other vertex. Vertex d in graph G in Example 1 is pendant.

Examining the degrees of vertices in a graph model can provide useful information about the model, as Example 2 shows.

EXAMPLE 2 What does the degree of a vertex in a niche overlap graph (introduced in Example 11 in Section 10.1) represent? Which vertices in this graph are pendant and which are isolated? Use the niche overlap graph shown in Figure 11 of Section 10.1 to interpret your answers.

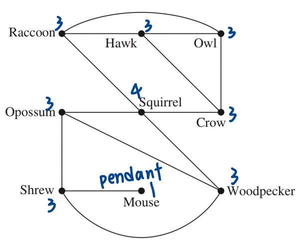


FIGURE 11 **A Niche Overlap Graph.**

Solution: There is an edge between two vertices in a niche overlap graph if and only if the two species represented by these vertices compete. Hence, the degree of a vertex in a niche overlap graph is the number of species in the ecosystem that compete with the species represented by this vertex. A vertex is pendant if the species competes with exactly one other species in the ecosystem. Finally, the vertex representing a species is isolated if this species does not compete with any other species in the ecosystem.

For instance, the degree of the vertex representing the squirrel in the niche overlap graph in Figure 11 in Section 10.1 is four, because the squirrel competes with four other species: the crow, the opossum, the raccoon, and the woodpecker. In this niche overlap graph, the mouse is the only species represented by a pendant vertex, because the mouse competes only with the shrew and all other species compete with at least two other species. There are no isolated vertices in the graph in this niche overlap graph because every species in this ecosystem competes with at least one other species. ◀

What do we get when we add the degrees of all the vertices of a graph $G = (V, E)$? Each edge contributes two to the sum of the degrees of the vertices because an edge is incident with exactly two (possibly equal) vertices. This means that the sum of the degrees of the vertices is twice the number of edges. We have the result in Theorem 1, which is sometimes called the handshaking theorem (and is also often known as the handshaking lemma), because of the analogy between an edge having two endpoints and a handshake involving two hands.

THEOREM 1

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

EXAMPLE 3 How many edges are there in a graph with 10 vertices each of degree six?

$2m = 6 \cdot 10$
$m = 30$

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$. ◀

Theorem 1 shows that the sum of the degrees of the vertices of an undirected graph is even. This simple fact has many consequences, one of which is given as Theorem 2.

THEOREM 2

An undirected graph has an <u>even number of vertices of odd degree</u>. 有偶数个奇数の点？

Proof: Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

not fully understand yet

Because $\deg(v)$ is even for $v \in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is $2m$. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree. ◁

Terminology for graphs with directed edges reflects the fact that edges in directed graphs have directions.

DEFINITION 4 When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

(u, v)
u adjacent to v
v adjacent from u
initial vertex : u
terminal / end vertex : v
loop : initial = end vertex

Because the edges in graphs with directed edges are ordered pairs, the definition of the degree of a vertex can be refined to reflect the number of edges with this vertex as the initial vertex and as the terminal vertex.

DEFINITION 5 In a graph with directed edges the *in-degree of a vertex v*, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of v*, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

EXAMPLE 4 Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.

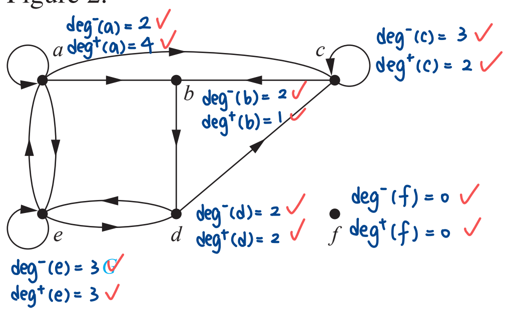


FIGURE 2 **The Directed Graph *G*.**

Solution: The in-degrees in G are $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, and $\deg^-(f) = 0$. The out-degrees are $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, and $\deg^+(f) = 0$. ◀

Because each edge has an initial vertex and a terminal vertex, the sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same. Both of these sums are the number of edges in the graph. This result is stated as Theorem 3.

THEOREM 3 Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

in-degree = out-degree

There are many properties of a graph with directed edges that do not depend on the direction of its edges. Consequently, it is often useful to ignore these directions. The undirected graph that results from ignoring directions of edges is called the **underlying undirected graph**. A graph with directed edges and its underlying undirected graph have the same number of edges.

Some Special Simple Graphs

We will now introduce several classes of simple graphs. These graphs are often used as examples and arise in many applications.

EXAMPLE 5 **Complete Graphs** A **complete graph on n vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete**. ◀

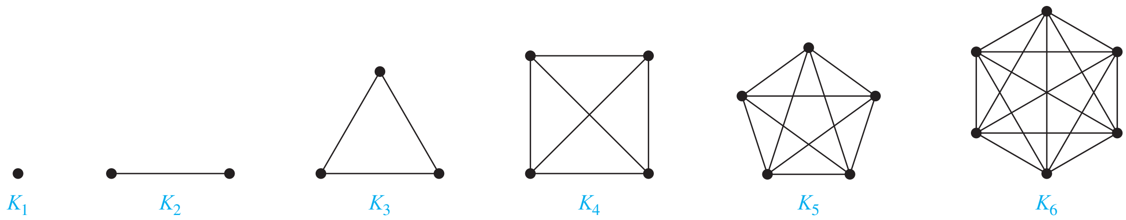


FIGURE 3 **The Graphs K_n for $1 \le n \le 6$.**

EXAMPLE 6 **Cycles** A **cycle C_n**, $n \ge 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4. ◀

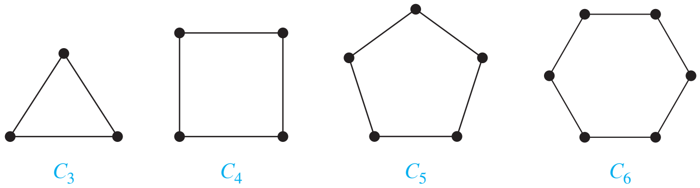


FIGURE 4 **The Cycles C_3, C_4, C_5, and C_6.**

EXAMPLE 7 **Wheels** We obtain a **wheel W_n** when we add an additional vertex to a cycle C_n, for $n \ge 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5, and W_6 are displayed in Figure 5. ◀

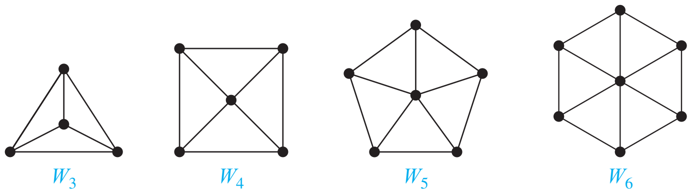


FIGURE 5 **The Wheels W_3, W_4, W_5, and W_6.**

EXAMPLE 8 ***n*-Cubes** An ***n*-dimensional hypercube**, or ***n*-cube**, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. We display Q_1, Q_2, and Q_3 in Figure 6.

Note that you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. In Figure 6, Q_3 is constructed from Q_2 by drawing two copies of Q_2 as the top and bottom faces of Q_3, adding 0 at the beginning of the label of each vertex in the bottom face and 1 at the beginning of the label of each vertex in the top face. (Here, by *face* we mean a face of a cube in three-dimensional space. Think of drawing the graph Q_3 in three-dimensional space with copies of Q_2 as the top and bottom faces of a cube and then drawing the projection of the resulting depiction in the plane.) ◀

W13 ↑

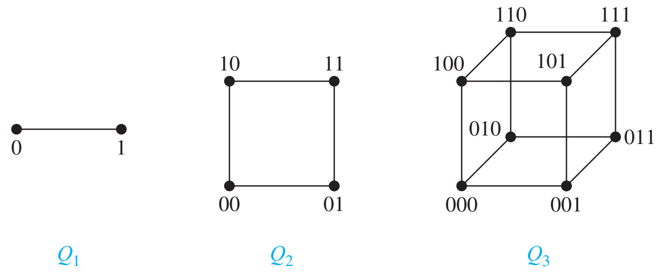


FIGURE 6 The n-cube Q_n, $n = 1, 2, 3$.

W14 ↓

Bipartite Graphs

Links

Sometimes a graph has the property that its vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset. For example, consider the graph representing marriages between men and women in a village, where each person is represented by a vertex and a marriage is represented by an edge. In this graph, each edge connects a vertex in the subset of vertices representing males and a vertex in the subset of vertices representing females. This leads us to Definition 5.

DEFINITION 6 A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

In Example 9 we will show that C_6 is bipartite, and in Example 10 we will show that K_3 is not bipartite.

EXAMPLE 9 C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2. ◀

EXAMPLE 10 K_3 is not bipartite. To verify this, note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge. ◀

EXAMPLE 11 Are the graphs G and H displayed in Figure 8 bipartite?

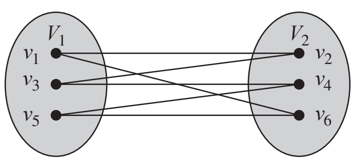


FIGURE 7 Showing That C_6 Is Bipartite.

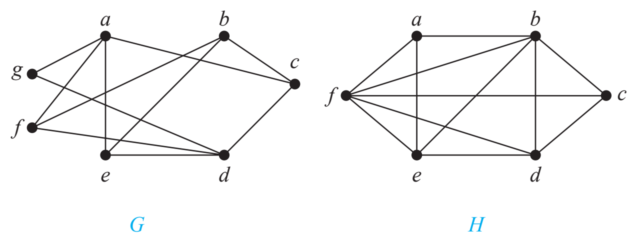


FIGURE 8 The Undirected Graphs G and H.

Hence, by the inductive hypothesis, the graph K has a complete matching. Combining this complete matching with the complete matching from W_1' to W_2', we obtain a complete matching from W_1 to W_2.

Wk14

We have shown that in both cases there is a complete matching from W_1 to W_2. This completes the inductive step and completes the proof. ◁

We have used strong induction to prove Hall's marriage theorem. Although our proof is elegant, it does have some drawbacks. In particular, we cannot construct an algorithm based on this proof that finds a complete matching in a bipartite graph. For a constructive proof that can be used as the basis of an algorithm, see [Gi85].

Wk13

Some Applications of Special Types of Graphs

We conclude this section by introducing some additional graph models that involve the special types of graph we have discussed in this section.

EXAMPLE 16 **Local Area Networks** The various computers in a building, such as minicomputers and personal computers, as well as peripheral devices such as printers and plotters, can be connected using a *local area network*. Some of these networks are based on a *star* 拓撲 *topology*, where all the way in which constituent parts are interrelated or arranged devices are connected to a central control device. A local area network can be represented using 組成の,構成の a complete bipartite consisting of two parts graph $K_{1,n}$, as shown in Figure 11(a). Messages are sent from device to device through the central control device.

Links

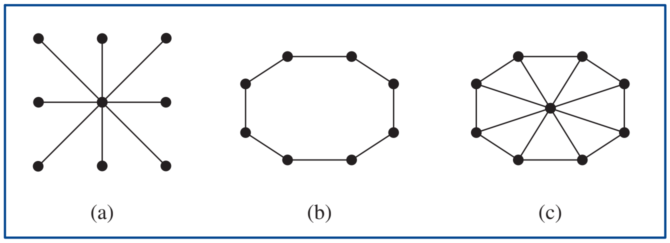


FIGURE 11 **Star, Ring, and Hybrid Topologies for Local Area Networks.**

Other local area networks are based on a *ring topology*, where each device is connected to exactly two others. Local area networks with a ring topology are modeled using n-cycles, C_n, as shown in Figure 11(b). Messages are sent from device to device around the cycle until the intended recipient of a message is reached.

Finally, some local area networks use a hybrid of these two topologies. Messages may be sent around the ring, or through a central device. This redundancy makes the network more reliable. Local area networks with this redundancy can be modeled using wheels W_n, as shown in Figure 11(c). ◀

EXAMPLE 17 **Interconnection Networks for Parallel Computation** For many years, computers executed programs one operation at a time. Consequently, the algorithms written to solve problems were designed to perform one step at a time; such algorithms are called **serial**. (Almost all algorithms described in this book are serial.) However, many computationally intense problems, such as weather simulations, medical imaging, and cryptanalysis, cannot be solved in a reasonable amount of time using serial operations, even on a supercomputer. Furthermore, there is a physical limit to how fast a computer can carry out basic operations, so there will always be problems that cannot be solved in a reasonable length of time using serial operations.

Parallel processing, which uses computers made up of many separate processors, each with its own memory, helps overcome the limitations of computers with a single processor. **Parallel algorithms**, which break a problem into a number of subproblems that can be solved

FIGURE 12 A Linear Array for Six Processors.

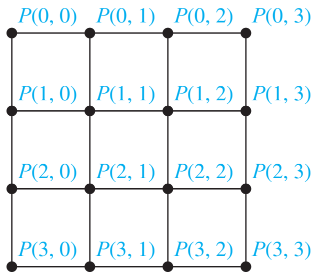


FIGURE 13 A Mesh Network for 16 Processors.

concurrently, can then be devised to rapidly solve problems using a computer with multiple processors. In a parallel algorithm, a single instruction stream controls the execution of the algorithm, sending subproblems to different processors, and directs the input and output of these subproblems to the appropriate processors.

When parallel processing is used, one processor may need output generated by another processor. Consequently, these processors need to be interconnected. We can use the appropriate type of graph to represent the interconnection network of the processors in a computer with multiple processors. In the following discussion, we will describe the most commonly used types of interconnection networks for parallel processors. The type of interconnection network used to implement a particular parallel algorithm depends on the requirements for exchange of data between processors, the desired speed, and, of course, the available hardware.

The simplest, but most expensive, network-interconnecting processors include a two-way link between each pair of processors. This network can be represented by K_n, the complete graph on n vertices, when there are n processors. However, there are serious problems with this type of interconnection network because the required number of connections is so large. In reality, the number of direct connections to a processor is limited, so when there are a large number of processors, a processor cannot be linked directly to all others. For example, when there are 64 processors, $C(64, 2) = 2016$ connections would be required, and each processor would have to be directly connected to 63 others.

On the other hand, perhaps the simplest way to interconnect n processors is to use an arrangement known as a **linear array**. Each processor P_i, other than P_1 and P_n, is connected to its neighbors P_{i-1} and P_{i+1} via a two-way link. P_1 is connected only to P_2, and P_n is connected only to P_{n-1}. The linear array for six processors is shown in Figure 12. The advantage of a linear array is that each processor has at most two direct connections to other processors. The disadvantage is that it is sometimes necessary to use a large number of intermediate links, called **hops**, for processors to share information.

The **mesh network** (or **two-dimensional array**) is a commonly used interconnection network. In such a network, the number of processors is a perfect square, say $n = m^2$. The n processors are labeled $P(i, j)$, $0 \le i \le m - 1$, $0 \le j \le m - 1$. Two-way links connect processor $P(i, j)$ with its four neighbors, processors $P(i \pm 1, j)$ and $P(i, j \pm 1)$, as long as these are processors in the mesh. (Note that four processors, on the corners of the mesh, have only two adjacent processors, and other processors on the boundaries have only three neighbors. Sometimes a variant of a mesh network in which every processor has exactly four connections is used; see Exercise 72.) The mesh network limits the number of links for each processor. Communication between some pairs of processors requires $O(\sqrt{n}) = O(m)$ intermediate links. (See Exercise 73.) The graph representing the mesh network for 16 processors is shown in Figure 13.

One important type of interconnection network is the hypercube. For such a network, the number of processors is a power of 2, $n = 2^m$. The n processors are labeled $P_0, P_1, \ldots, P_{n-1}$. Each processor has two-way connections to m other processors. Processor P_i is linked to the processors with indices whose binary representations differ from the binary representation of i

in exactly one bit. The hypercube network balances the number of direct connections for each processor and the number of intermediate connections required so that processors can communicate. Many computers have been built using a hypercube network, and many parallel algorithms have been devised that use a hypercube network. The graph Q_m, the m-cube, represents the hypercube network with $n = 2^m$ processors. Figure 14 displays the hypercube network for eight processors. (Figure 14 displays a different way to draw Q_3 than was shown in Figure 6.) ◀

New Graphs from Old

Sometimes we need only part of a graph to solve a problem. For instance, we may care only about the part of a large computer network that involves the computer centers in New York, Denver, Detroit, and Atlanta. Then we can ignore the other computer centers and all telephone lines not linking two of these specific four computer centers. In the graph model for the large network, we can remove the vertices corresponding to the computer centers other than the four of interest, and we can remove all edges incident with a vertex that was removed. When edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such a graph is called a **subgraph** of the original graph.

DEFINITION 7 A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

Given a set of vertices of a graph, we can form a subgraph of this graph with these vertices and the edges of the graph that connect them.

DEFINITION 8 Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

don't understand

EXAMPLE 18 The graph G shown in Figure 15 is a subgraph of K_5. If we add the edge connecting c and e to G, we obtain the subgraph induced by $W = \{a, b, c, e\}$. ◀

REMOVING OR ADDING EDGES OF A GRAPH Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e$, has the same vertex set V as G. Its edge set is $E - e$. Hence,

$$G - e = (V, E - \{e\}).$$

Similarly, if E' is a subset of E, we can produce a subgraph of G by removing the edges in E' from the graph. The resulting subgraph has the same vertex set V as G. Its edge set is $E - E'$.

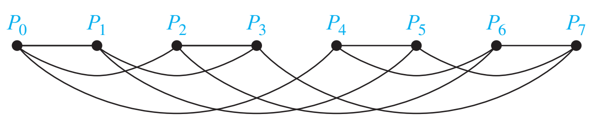


FIGURE 14 **A Hypercube Network for Eight Processors.**

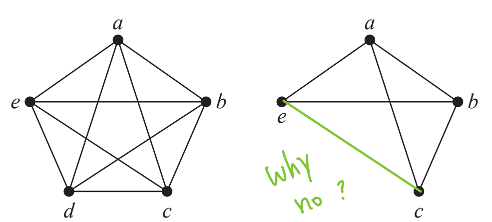


FIGURE 15 **A Subgraph of K_5.**

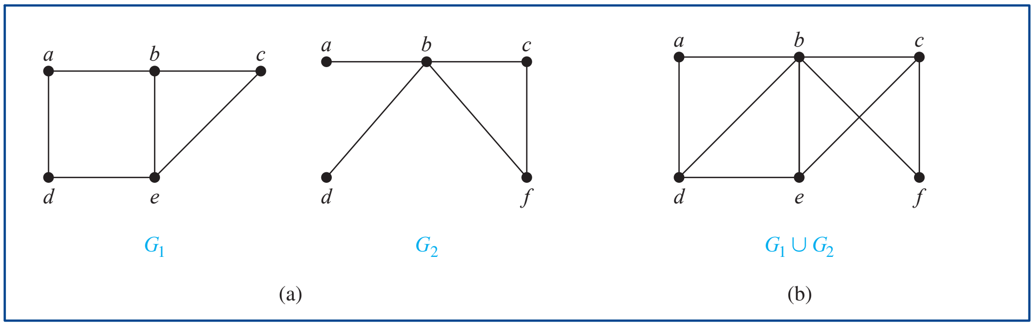


FIGURE 16 **(a) The Simple Graphs G_1 and G_2; (b) Their Union $G_1 \cup G_2$.**

We can also add an edge e to a graph to produce a new larger graph when this edge connects two vertices already in G. We denote by $G + e$ the new graph produced by adding a new edge e, connecting two previously nonincident vertices, to the graph G Hence,

$$G + e = (V, E \cup \{e\}).$$

The vertex set of $G + e$ is the same as the vertex set of G and the edge set is the union of the edge set of G and the set $\{e\}$.

EDGE CONTRACTIONS Sometimes when we remove an edge from a graph, we do not want to retain the endpoints of this edge as separate vertices in the resulting subgraph. In such a case we perform an **edge contraction** which removes an edge e with endpoints u and v and merges u and w into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u or v and with the same second endpoint. Hence, the contraction of the edge e with endpoints u and v in the graph $G = (V, E)$ produces a new graph $G' = (V', E')$ (which is not a subgraph of G), where $V' = V - \{u, v\} \cup \{w\}$ and E' contains the edges in E which do not have either u or v as endpoints and an edge connecting w to every neighbor of either u or v in V. For example, the contraction of the edge connecting the vertices e and c in the graph G_1 in Figure 16 produces a new graph G_1' with vertices a, b, d, and w. As in G_1, there is an edge in G_1' connecting a and b and an edge connecting a and d. There also is an edge in G_1' that connects b and w that replaces the edges connecting b and c and connecting b and e in G_1 and an edge in G_1' that connects d and w replacing the edge connecting d and e in G_1.

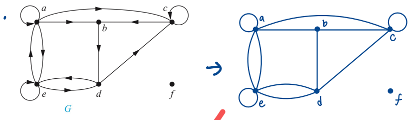
REMOVING VERTICES FROM A GRAPH When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that $G - v = (V - v, E')$, where E' is the set of edges of G not incident to v. Similarly, if V' is a subset of V, then the graph $G - V'$ is the subgraph $(V - V', E')$, where E' is the set of edges of G not incident to a vertex in V'.

GRAPH UNIONS Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs. We will give a more formal definition for the union of two simple graphs.

DEFINITION 9 The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

EXAMPLE 19 Find the union of the graphs G_1 and G_2 shown in Figure 16(a). ◀

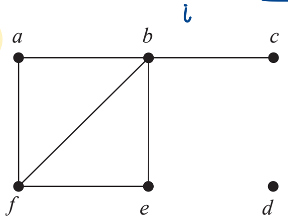
Solution: The vertex set of the union $G_1 \cup G_2$ is the union of the two vertex sets, namely, $\{a, b, c, d, e, f\}$. The edge set of the union is the union of the two edge sets. The union is displayed in Figure 16(b).

11.

same num of edges!

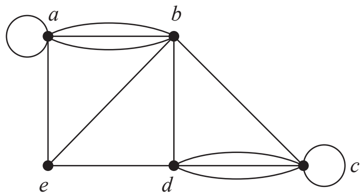
Exercises

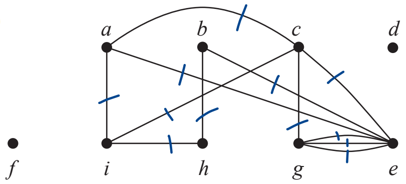
12/10 11:30 do 12:27 correct 12:44

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

1.


$n = 6$, $m = 6$
deg(a) = 2
deg(b) = 4
deg(c) = 1
deg(d) = 0
deg(e) = 2
deg(f) = 3
i = {d}
p = {c}

2.


3.


$n = 9$, $m = 12$
deg(a) = 3
deg(b) = 2
deg(c) = 4
deg(d) = 0
deg(e) = 6
deg(f) = 0
deg(g) = 4
deg(h) = 2
deg(i) = 3
i = {d, f}
p = ∅

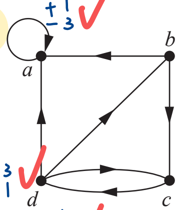
4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph. ① sum = 12 = 2m = 2·6 ③ sum = 24 = 2m = 2·12

5. Can a simple graph exist with 15 vertices each of degree five? No

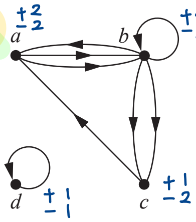
6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

3×2, 6×2, 10×2 (不行)
bc a directed graph must have even num of ins and outs

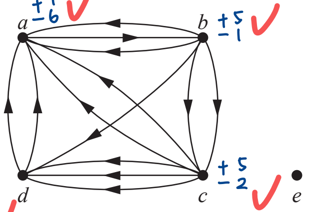
In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

7.


a: +1 −3; b: +2 −1; c: +1 −2; d: +3 −1
n = 4
m = 7

8.


a: +2 −2; b: +4 −3; c: +1 −2; d: +1 −1
n = 4
m = 8

9.


a: +1 −6; b: +5 −1; c: +5 −2; d: +2 −4; e: +0 −0
n = 5
m = 13

⑦ sum = 14 = 7·2 ⑧ sum = 16 = 8·2 ⑨ 26 = 13·2

10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.

11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.

12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood of a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

how many people each person knows
the two of them knows each other
a person knows no one
a person knows only one people on the world
The average amount one person knows around the world is 1000.

13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?

the number one had collaborate w/ people had collaborated
people never collab w/ others
people only collab w/ one person

~~14.~~ What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?

15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 4 of Section 10.1, represent? What does the degree of a vertex in the undirected version of this graph represent?

16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 5 of Section 10.1, represent?

17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?

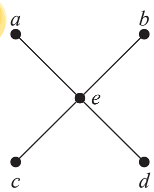
18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

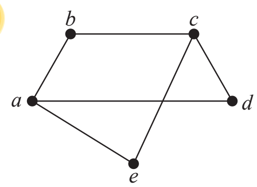
19. Use Exercise 18 to show that in a group of people, there must be two people who are friends with the same number of other people in the group.

20. Draw these graphs.

a) K_7 b) $K_{1,8}$ c) $K_{4,4}$
d) C_7 e) W_7 f) Q_4

In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

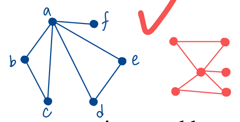
21.


22.


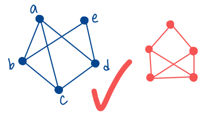
W13
W14

39. K_n sequence. (degree)
$K_1 = 0$ 1×0
$K_2 = 2$ 2×1
$K_3 = 6$ 3×2
$K_4 = 12$ 4×3
$K_5 = 20$ 5×4
$K_6 = 30$ 6×5
$n\times(n-1)$

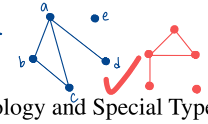
41. $n=6$, sum $= 14 \Rightarrow m=7$
5, 2, 2, 2, 2, 1



43. (a) 3, 3, 3, 3, 2
sum $= 14$, $m=7$, $n=5$



(e) 3, 2, 2, 1, 0
sum $= 8$, $m=4$, $n=5$



35. How many vertices and how many edges do these graphs have?

a) K_n b) C_n c) W_n
d) $K_{m,n}$ e) Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.

36. Find the degree sequences for each of the graphs in Exercises 21–25.

37. Find the degree sequence of each of the following graphs.

a) K_4 b) C_4 c) W_4
d) $K_{2,3}$ e) Q_3

38. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

39. What is the degree sequence of K_n, where n is a positive integer? Explain your answer. $K_n = n(n-1)$, bc each vertex needs to connect the all other $(n-1)$ vertex.

40. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

41. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.
$n=6$, sum $= 14 \Rightarrow m=7$

A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

42. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 b) 6, 5, 4, 3, 2, 1 c) 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 e) 3, 3, 2, 2, 2, 2 f) 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 h) 5, 5, 4, 3, 2, 1

43. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 yes b) 5, 4, 3, 2, 1 odd no c) 4, 4, 3, 2, 1 no can't have 2 that need to connect to all and have one w/ 1.
d) 4, 4, 3, 3, 3 no odd e) 3, 2, 2, 1, 0 yes f) 1, 1, 1, 1, 1 no odd

***44.** Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$.
?

***45.** Show that a sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

***46.** Use Exercise 45 to construct a recursive algorithm for determining whether a nonincreasing sequence of positive integers is graphic.

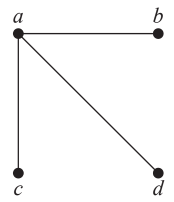
47. Show that every nonincreasing sequence of nonnegative integers with an even sum of its terms is the degree sequence of a pseudograph, that is, an undirected graph where loops are allowed. [*Hint:* Construct such a graph by first adding as many loops as possible at each vertex. Then add additional edges connecting vertices of odd degree. Explain why this construction works.]

48. How many subgraphs with at least one vertex does K_2 have?

49. How many subgraphs with at least one vertex does K_3 have?

50. How many subgraphs with at least one vertex does W_3 have?

51. Draw all subgraphs of this graph.



52. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. b) $2e/v \leq M$.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

53. For which values of n are these graphs regular?

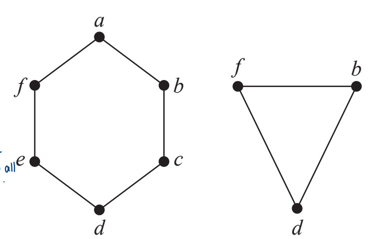
a) K_n b) C_n c) W_n d) Q_n

54. For which values of m and n is $K_{m,n}$ regular?

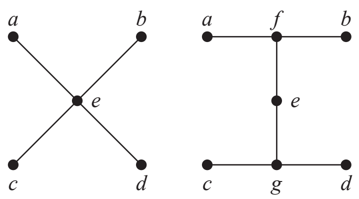
55. How many vertices does a regular graph of degree four with 10 edges have?

In Exercises 56–58 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

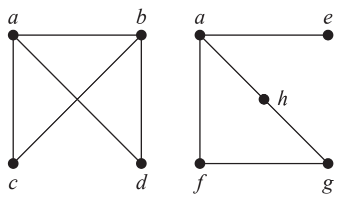
56.



57.



58.



59. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

a) $\overline{K_n}$ b) $\overline{K_{m,n}}$ c) $\overline{C_n}$ d) $\overline{Q_n}$

60. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?