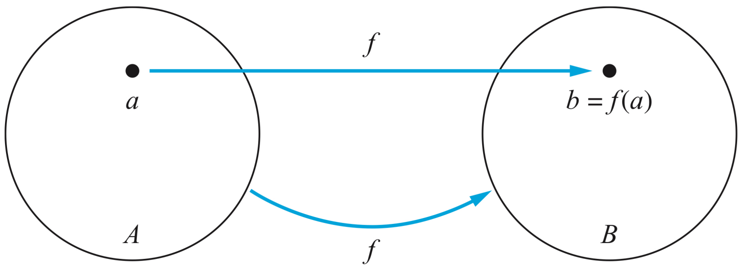
Ch 2.3 Functions

1 Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

- Functions are sometimes also called **mappings** or **tranformations**.

2 If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we way that b is the *image* of a and a is a *preimage* of b. The *range*, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

- Two functions are equal → Same domain, codomain, map each element of their common domain to the same element in their common codomain.



The Function f Maps A to B.

- **Real-valued:** If its codomain is the set of real numbers.
- **Integer-valued:** If its codomain is the set of integers.

3 Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

4 Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the elements of S. We denote the image of S by $f(S)$, so $f(S) = \{t \mid \exists s \in S(t = f(s))\}$.
We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

One-to-One and Onto Functions

- **One-to-one:** never assign the same value to two different domain elements.

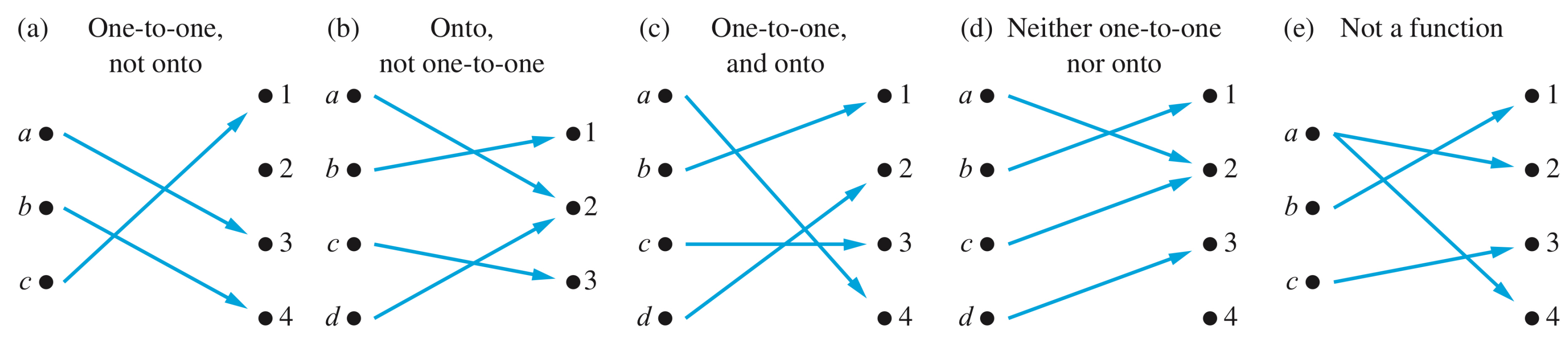
5 A function f is said to be *one-on-one*, or an *injunction*, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

6 A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

- A function that is increasing, but not strictly increasing OR decreasing, but not strictly decreasing, is not one-to-one.

7 A function f from A to B is called *onto*, or a *surjection*, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

- A function f is onto is $\forall y \exists x (f(x) = y)$



Examples of Different Types of Correspondences

8 The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

- All elements of the codomain are images of elements in the domain $\Rightarrow$ Bijection

Identity function

$\iota_A : A \to A$, where $\iota_A(x) = x$

A function assigns each element to itself. $\rightarrow$ one-to-one & onto $\Rightarrow$ Bijection

Summary

Suppose that $f : A \to B$.
To show that f is injective : Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.
To show that f is not injective : Find particular elemetns, $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
To show that f is surjective : Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
To show that f is not surjective : Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.