Ch9.5 Equivalence Relations

Equivalence Relations

1 A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

2 Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

▼ Example 3 **Congruence Modulo** m

Let m be an integer with $m > 1$. Show that the relation $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers.

Solution:

[From Ch4.1 $a \equiv b \pmod{m}$ *iff* m divides $a - b$]

Note that $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation.

▼ Example 7 (not equivalence relations)

Let R be the relation on the set of real numbers such that xRy *iff* x and y are real numbers that differ by less than 1, that is $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbb{R}$. R is symmetric, for if xRy, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that yRx. However, R is not equivalence relation because it is not transitive. Take $x = 2.8,\ y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1,\ |y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8R1.9,\ 1.9R1.1$, but $2.8\not{R}1.1$.

Equivalence Classes

3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

If R is an equivalence relation on a set A, the equivalence class of the element a is $[a]_R = \{s \mid (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

▼ Example 9

What are the equivalence classes of 0 and 1 for congruence modulo 4 ?

Solution:

The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod{4}$.

$\Rightarrow [0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.

The equivalence class of 1 contains all integers a such that $a \equiv 1 \pmod{4}$.

$\Rightarrow [1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$.

The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a - 2m, a - m, a, a + m, a + 2m, \ldots\}$.

Equivalence Classes and Partitions

1 **Theorem 1** Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:
(i) aRb (ii) $[a]=[b]$ (iii) $[a]\cap[b]\neq\emptyset$

2 **Theorem 2** Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i\in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i,\ i\in I$, as its equivalence classes.

▼ Example 13

List the ordered pairs in the equivalence relation R produced by the partition $A_1=\{1,2,3\},\ A_2=\{4,5\}$, and $A_3=\{6\}$ of $S=\{1,2,3,4,5,6\}$.

Solution:

$A_1 \Rightarrow (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$.

$A_2 \Rightarrow (4,4),(4,5),(5,4),(5,5)$.

$A_3 \Rightarrow (6,6)$.

No pair other than those listed belongs to R.