THE PREDICTABILITY OF HYDRODYNAMIC FLOW*†

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Among the innumerable processes which take place in our universe, there are many whose future behavior we often wish to predict. Some of these are hydrodynamical processes; that is, they depend upon the motion of a fluid. Let us begin by looking at a few of these processes and some of the methods which have been used to predict them.

Following the occurrence of a heavy rain, or the melting of a deep snow cover, the threat of a flood is often imminent. It is then a matter of great concern to predict the intensity of the flood crest as it moves downstream. Considerable success has been obtained by using laboratory models of river-drainage basins, in which actual floods may be simulated.

On a larger scale, the ever-present problem of weather prediction is now being tackled by dynamical methods. Mathematical equations, supposedly governing the behavior of certain features of the atmosphere, are solved by high-speed computing machines. Forecasts of upper-level wind patterns made in this manner compare favorably with those produced by other methods.

On a still larger scale, we sometimes wish to predict the behavior of sun's atmosphere, as manifested by sunspots and other occurrences. The approximate eleven-year period in sunspot activity became evident many years ago. Even without any physical theory, we can make reasonably good forecasts of sunspot numbers during the next decade or two simply by extrapolating the eleven-year cycle. We can gain further improvements by using more sophisticated statistical techniques.

Other problems of a less serious nature also present themselves. A morning commuter who has stopped for a quick cup of coffee may wonder how soon the coffee will become cool enough for him to drink it, before he rejoins the rush to the office. He might

The Division of Biochemistry held a meeting on January 22, 1963, at which Bernard L. Horecker of New York University School of Medicine, New York, N.Y., presented a paper entitled "Mechanism of Action of Aldolases." This paper will not be published by the Academy.

seem to have a problem in simple thermal conduction, rather than fluid motion. But actually there may be strong convective over-turning, which brings hot coffee from the bottom of the cup to the surface, and enables it to cool faster than it otherwise would. This convection often becomes visible if the cream fails to mix thoroughly.

In the earlier examples the predictions are never perfect, and they become successively poorer as the range of prediction increases. Because the period between successive sunspot maxima varies considerably about its eleven-year average value, we cannot say with any confidence what the phase of the sunspot cycle will be 100 years from now. The mathematically predicted weather diverges farther and farther from the real weather, as the cumulative effect of the approximations becomes more and more dominant. A laboratory model cannot predict next month's floods without any knowledge of next month's rainstorms.

The lack of perfection of current methods of predicting such phenomena as the weather has often been contrasted with the deadly accurancy of predictions of solar eclipses. Sometimes, it is concluded that meteorologists are not in a class with astonomers. Another conclusion, which is certainly as well justified by present evidence, is that no method of predicting the weather can ever compare in accuracy to the prediction of eclipses. The motion of a fluid and the motion of discrete heavenly bodies are simply not the same thing.

In this talk I want to explore the question, "How predictable is hydrodynamic flow?" Specifically, are there definite limits upon the accuracy with which a particular hydrodynamical process can be predicted? If there are, how is the accuracy of prediction related to the range of prediction?

The answer is obviously going to depend upon the particular flow. I should like first to distinguish between permanently established or statistically stationary flow on the one hand, and transient flow on the other.

In a permanently established flow, states closely resembling the present state have a positive probability of occurring again. In a transient flow, states which have recently occurred will eventually be avoided altogether. If the motion ultimately dies out completely, the flow is purely transient.

The cooling cup of coffee is an example of a transient system. The convective overturning will cease permanently as the coffee approaches room temperature, assuming that nobody drinks the coffee first. We might have trouble forecasting the temperature of

the coffee one minute in advance, but we should have little difficulty in forecasting it an hour ahead.

The other systems which we have considered are essentially permanent. Differences in the typical behavior of the weather from one year to another, or of sunspots from one century to another, presumably represent very-long-period fluctuations, and do not imply permanent changes in the nature of the atmosphere or the sun. The flood problem might seem to resemble the coffee problem. But even though each individual flood is temporary, the level of the river, which subsides as the flood recedes, will rise again with the next flood. Two-month forecasts of the river level may be more accurate than two-day forecasts when a flood is imminent, but two-day forecasts are certainly superior when a flood is not imminent.

We might make the coffee problem look more like the flood problem by considering the life history of an individual coffee-cup, as it is continually refilled with coffee. We could probably present plausible arguments for statistical stationarity, but in considering the relation between successive cupfuls of coffee, we should have to pass far beyond the realm of hydrodynamics.

During the rest of this discussion I shall not be concerned with transient phenomena, and shall limit its scope to the predictability of permanently established hydrodynamic flow with no transient component.

I should next like to distinguish between periodic or quasi-periodic flow on the one hand, and nonperiodic flow on the other. A quasi-periodic flow with no transient component eventually comes arbitrarily close to assuming a state which it has assumed before, and the history following the latter occurrence remains arbitrarily close to the history following the former. If the repetition is exact, the flow is strictly periodic. Mathematically a quasi-periodic function may be expressed as the sum of a finite or denumerable number of strictly periodic functions.

A nonperiodic flow may also come close to repeating a previous state, but the histories following the two occurrences do not remain close, except temporarily. Sometimes the two histories eventually lose all resemblance to each other. The flow is then purely nonperiodic. A general nonperiodic flow may be expressed as the linear sum of a quasi-periodic flow and a purely nonperiodic flow.

Quasi-periodicity is illustrated by the oceanic tides. Whether the tides are strictly quasi-periodic may be a matter of semantics. If we agree to talk about the vertical motion of the ocean surface, with the effect of individual waves averaged out, the flow we are considering has a nonperiodic component, which is due largely to the influence of winds. The major component is nevertheless the

quasi-periodic component, which in turn is a sum of strictly periodic components.

As a result, we can predict the tides many years in advance just about as accurately as we can predict them a few days in advance, simply by extrapolating the known periodic components, and recombining them. Indeed, the times of high and low tide, as they appear in the daily newspapers in coastal cities, are probably regarded by most readers as statements of fact rather than as predictions.

The nonperiodic component of the tides is to some extent predictable, and the statement that "tides will be from two to three feet above normal" is a familiar one to coastal dwellers. But more than a few days ahead, abnormal tides cannot be more predictable than the wind systems which cause them.

Other periodic systems, such as the annual and diurnal components of the weather, are equally predictable. It requires little skill to conclude that, during the twenty-fifth century A.D., New York will be warmer in July than in January. But if we are to make more definitive statements even about the predictability of quasi-periodic flow in general, and certainly about nonperiodic flow, we must become more specific in our concept of predictability. Predictability at a given range is essentially a measure of the errors in predicting at that range. In order to avoid drawing incorrect conclusions because of confusion in terminology, we must qualify predictability according to the sources of the errors in prediction.

Let us call a flow *deterministic* if it is governed by a deterministic dynamics, or if for any other reason there exist formulas which express without error the future of the flow in terms of the present and past. The word formula should be interpreted liberally. It may denote a set of mathematical equations, a set of tables or graphs, a set of verbal rules, or a combination of these. Strictly periodic flow is certainly deterministic in this sense. If a flow is not deterministic, there will be prediction errors which are intrinsic properties of the flow itself.

It is unrealistic to assume that measurements of any flow will ever be exact. Even if a flow is deterministic, predictions of the future may contain errors, as a result of substituting incorrectly measured present and past states into the formulas. If the flow is not deterministic, the errors due to measurement will be combined with the intrinsic errors. Moreover the formula which would be most appropriate if measurements were exact may not be the best formula with measurements as they actually are.

Here we must interpret "errors in measurement" as including

not only instrumental errors, but errors due to incorrect interpolation over regions where there are no measurements at all. In the atmosphere the latter errors are presumably the more serious.

The mere existence of a perfect or nearly perfect formula does not guarantee a method by which we can deduce the formula. Ideally the dynamics of the flow should reveal the formula, but frequently, as, for example, in the case of sunspots, the dynamics are imperfectly understood. Statistical procedures, based upon the recorded past history of the flow, are then available, but frequently not enough past history has been recorded. The very real possibility that somebody might guess the proper formula is of little comfort, unless there is some basis for selecting this formula in preference to an inferior formula which somebody else has guessed. If we do not wish to wait while more history accumulates, we can only hope that with an apparently exponential growth in the world's population, there may also be an exponential growth in the number of hydrodynamicists, until one of them finds the proper dynamics.

In summary, we must distinguish between intrinsic predictability, which depends only upon the flow itself; attainable predictability, which is limited also by the inevitable inaccuracies in measurement; and practical predictability, which is further limited by our present inability to identify the most suitable formulas.

We may now state that there are no intrinsic errors in predicting periodic flow. Errors in measurement may lead to comparable errors in predicting periodic flow, but if the errors in measurement are random, their effect may be largely eliminated by averaging the past history over a number of cycles. However, a periodic flow need not be practically predictable at all, if neither theory nor observation has yet revealed that it is periodic.

In the case of the oceanic tides, experience indicates that the quasi-periodic component is practically predictable, with little error. This predictability stems partly from the theoretical knowledge that the frequencies involved are those which characterize the relative motions of the moon, earth, and sun. The determination of the amplitudes and phases in a particular tidal basin from a sample of observations would be far less precise, if this same sample had to be used to determine the frequencies as well.

It is not only less simple to predict nonperiodic systems, but it is also less simple to make general statements about their predictability. Later on I shall discuss the extent to which a general theory of the predictability of hydrodynamic flow may be formulated. For the time being let me simply say that such a theory is incomplete. In the absence of a complete theory, it is wise to base our

estimate of predictability at least in part upon the success in prediction which we have had so far.

I should therefore like to spend some time discussing some procedures which have been used to predict what is probably the most predicted of all hydrodynamic phenomena – the weather. Let us first look at dynamical prediction.

As early as 1904, V. Bjerknes identified the problem of weather forecasting with the problem of solving the governing dynamic equations. In 1922 Richardson presented a detailed scheme for solving the equations by numerical procedure. The method was exceedingly lengthy, and Richardson visualized a weather center in which 64,000 persons working together could produce weather forecasts more rapidly than the weather itself took place.

Richardson himself spent many months in a single-handed attempt to produce a forecast over a limited portion of Europe. His prediction failed completely, and he attributed the failure to inaccuracies in the observations of the initial wind field.

During the 1920's and 1930's, there was widespread doubt that the dynamic equations could be used to predict the weather. It was known that the variations of both the wind field and the pressure field depended upon quantities which could not in the foreseeable future be measured with sufficient accuracy. Optimism gradually returned following the suggestion by Rossby (1939) that the large-scale upper level wind systems might be governed approximately by the two-dimensional vorticity equation, which expresses the conservation of the absolute vorticity of individual parcels of fluid as they move horizontally. Computation of the advection of vorticity was feasible with existing measurements.

With the arrival of high-speed digital computing machines, the vorticity equation was solved numerically (see Charney, Fjörtoft, and von Neumann, 1950). After some technical refinements, encouragingly good twenty-four hour forecasts were produced. Thus the first moderately successful dynamical weather forecasts were made not with the most precise form of the dynamic equations possible, but with a truly crude approximation.

Much effort has been devoted since then to developing ostensibly more realistic approximations to the dynamic equations, which nevertheless retain realistic solutions. The numerical problems involved in approximating nonlinear partial differential equations by algebraic difference equations are numerous, and, in the absence of a complete theory, new computation schemes must generally be tested before their worth is known. These processes require much time. Thus the improvements in prediction which we have gained

during ten years fall far short of what many dynamic meteorologists had anticipated.

On a routine basis, we have yet to reintroduce such obvious physical processes as the mechanical and thermal interaction between the atmosphere and its environment, and the evaporation and condensation of water. It may be that within a few years, when even bigger and faster computers are available, and we have included the physical processes which we know about now, we shall obtain highly accurate short-range forecasts. Perhaps, on the other hand, we have nearly reached the limit which current errors in measurement place on predictability by dynamical methods.

Let us now turn our attention to statistical prediction — prediction by means of formulas derived from the observed past behavior of the atmosphere. A statistical formula may happen to coincide with a governing dynamical equation, although in general it will not do so. I shall confine my remarks to formulas which specify a single value for the predictand, although much effort has been devoted to probability forecasting, where a formula may give a complete probability distribution for the predictand.

The main stumbling block in statistical forecasting is the necessity for any sample of observations of the past behavior of a system to be finite in size. As a result, we can always find an infinity of formulas which the sample of data fits exactly, provided that we allow the formulas to become complicated enough. The sample itself provides no basis for selection among these formulas, which ordinarily give widely varying results when applied to new data, and which probably do not include the most appropriate formula, anyway.

There is therefore no such thing as a "best" formula, from the point of view of a single sample, and we must seek instead the best formula of a specified restricted type. In all probability the arbitrary restrictions will exclude the formula which is really most appropriate.

The simplest and most widely used procedure is to restrict the formula to be linear. Specifically, we consider formulas of the form

$$y = a_1 x_1 + \cdots + a_M x_M + e , \qquad (1)$$

where y is the predictand, x_1, ---, x_M are the predictors, and e is the error in predicting y. The coefficients a_1, ---, a_M are to be chosen to minimize the mean square of e. We can include a constant term in formula (1) by appending one more "predictor," whose value is always unity.

To establish the formula, we choose a sample of data consisting of N values of y and corresponding values of x_1, ---, x_M, and arrange the data in tabular form, with the values of y in one column, and the values of x_1, ---, x_M in adjacent columns. The data for the predictors then constitute a rectangular array of numbers, and may be treated as a matrix X, with N rows and M columns. Similarly, the data for the predictand may be treated as a matrix Y, with N rows and one column. Formula (1) may then be rewritten

$$Y = X A + E, \tag{2}$$

where E is a matrix of N rows and one column, whose elements are the prediction errors, and A is a matrix of M rows and one column, whose elements are the prediction coefficients. The matrix A which minimizes the mean-square error in predicting y is simply

$$A = (X^T X)^{-1} X^T Y, \tag{3}$$

where the superscript T denotes the transpose of a matrix, and the exponent -1 denotes the inverse.

Formula (3) shows that the necessary computations to establish a linear prediction formula, once the data have been tabulated, consist only of three matrix multiplications and a matrix inversion. If M and N are large, the number of individual scalar operations included in these few matrix operations might well tax the capabilities of Richardson's team of 64,000. However, if a digital computer is available, and if the computer can interpret individual instructions as commands to perform specific matrix operations, only a handful of instructions is needed to write a complete program for the linear prediction problem.

It is not surprising, then, that numerous formulas have been produced by this method. Some of them, however, have failed completely when applied to further prediction. When this has happened, the failure can often be traced to the excessive freedom still allowed in the formulas.

If M is nearly as large as N, there is a high probability that the formula selected by the sample will differ from the desired formula by an accidental linear combination of the predictors, whose value will happen to be small within the sample, but will be large in general. If M is larger than N, the matrix $X^T X$ will become singular, and equation (3) will yield no prediction formula at all.

This difficulty may be largely avoided by making M small compared to N. Sometimes it is not feasible to collect large samples,

and at the same time, there may be many predictors which one is reluctant to discard. In this case procedures are available for decreasing M as much as possible, while eliminating as little significant information as possible. One of the simplest and most successful of these is the screening procedure, which Miller (1960) has adapted for meteorological purposes. In this procedure, data for a large number of predictors are first collected. Predictors are then selected from this set, one at a time, in order of their ability to reduce the error further. When the additional reduction in the error is no greater than that expected by chance, the procedure is terminated, and the chosen predictors are used in an ordinary linear prediction scheme.

Linear procedures have by now been applied on so many occasions to the prediction of various weather elements at various ranges, that there is some justification for claiming that the results of these predictions reveal the linear predictability of the atmosphere. Experience in predicting the sea-level pressure over the United States, twenty-four hours in advance, may be rather revealing. Some of the earliest attempts gave mean-square errors which were somewhat less than one half the variances of the predictands. These results were highly gratifying. Subsequent attempts, however, failed to show any marked improvement. The formulas certainly reveal positive skill, but I am sure that any practicing forecaster, after seeing prognostic pressure charts prepared by statistical formulas, would be convinced that he could do better by the usual subjective methods.

Thus there seems to be a close parallelism between the histories of dynamical prediction and statistical prediction, as they apply to short-range forecasting. Both methods contain inherent pitfalls; in statistical prediction, too many predictors can lead to disaster, while in dynamical prediction, too precise a formulation of the equations can be fatal. In either case, the pitfalls may be avoided by suppressing some information which should have been of real use.

Both methods achieved comparable early successes, as soon as the original pitfalls were circumvented. Much effort has since been devoted to reincorporating some of the information which was suppressed. In spite of these refinements, improvements in prediction have not been great. We are left in the position of not knowing whether short-range prediction of the quality we really desire is possible or not.

Because I have so far said little about subjective prediction, I do not wish to belittle its importance. Certainly it has been

practiced far more than dynamical or statistical forecasting. We can assess subjective forecasting on the basis of its results. We know that some forecasters achieve significantly better results than others, so that at least the better forecasters possess real skill. We know that good forecasters achieve results at least comparable to those which have so far been obtained by dynamical and statistical means. Yet there is nothing in the history of subjective forecasting to hint that great improvements are close at hand.

Let us now return to the question of building a theory of hydrodynamic predictability. This may be accomplished by first compiling some known results, and then adding some new results. Many of these results are not restricted to hydrodynamical systems.

Concerning subjective prediction, it is a bit discouraging to try to formulate a theory of predictability by a particular method, when we cannot even formulate the method. I shall therefore confine my remarks to dynamical and statistical methods.

First let us consider statistical predictability. To make full use of any physical quantity as a predictor, we must allow its past values, measured at two or more different times relative to the time of prediction, to enter the formula as separate predictors. We can do this by using time-series methods.

The theory of linear prediction of stationary time series, initiated by Kolmogorov (1941) and Wiener (1949), has reached a rather advanced state of development. The simplest and most thoroughly studied case is that of simple time series, where the future of a single quantity is to be expressed as a linear function of its own present and past. We let $---, y_{-1}, y_0, y_1, ---$ be values of the predictand at equally spaced time intervals, and seek formulas of the form

$$y_{n+k} = c_{ko}\, y_n + c_{kl}\, y_{n-1} + --- + e_{k,n+k}\,. \qquad (4)$$

Here the constants c_{ko}, c_{kl}, --- are to be chosen to minimize the the mean square of $e_{k,n+k}$, the error in predicting y_{n+k}, k steps in advance. For completeness, we include as a formula a sequence of successively better formulas which do not approach a particular formula as a limit, provided that the sequence of errors approaches a limit.

In the general linear prediction scheme, the errors are completely determined by the variances and covariances of the predictors and the predictand. Likewise, in the present problem, the errors are completely determined by the covariance function

$$R_k = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=o}^{N} y_n \, y_{n+k} \quad , \tag{5}$$

which depends upon the lag k. The autocorrelation is obtained by dividing the covariance R_k by the variance R_o. I shall enumerate some of the principal results relating the covariance function to the intrinsic linear predictability. A detailed presentation is given by Doob (1953, ch. 12).

First of all, any stationary time series may be expressed as sum of two series, one of which is quasi-periodic, and one of which is purely nonperiodic. These two series are completely uncorrelated with each other at all lags. Hence the separate covariance functions of the two series may be added to give the covariance function for the total series. (If the mean of the series is not zero, it is treated as a periodic component having an infinite period. The quasi-periodic component or the nonperiodic component sometimes vanishes identically).

The autocorrelation function of the quasi-periodic series returns arbitrarily close to one, arbitrarily often, as the lag approaches infinity. The autocorrelation function of the nonperiodic series approaches zero as the lag approaches infinity. For the total series, then, the fraction of the variance accounted for by the quasi-periodic component equals the upper bound of the autocorrelation function, as the lag approaches infinity.

The intrinsic linear predictability is most readily expressed in terms of the spectrum, which is a Fourier transform of the covariance function. The covariance function of the quasi-periodic component does not possess a Fourier integral, but is expressible as the sum of a denumerable number of cosine functions. Its spectrum is therefore a denumerable set of lines. The quasi-periodic component is intrinsically linearly predictable without error, at all lags.

The covariance function of the nonperiodic component possesses a Fourier integral, and its Fourier integral transform is its spectrum. The spectrum is absolutely continuous. The intrinsic linear predictability is completely determined by the spectrum. In particular, aside from the factor 2π, the mean square error in predicting one step ahead is equal to the geometric mean of the spectrum, while the variance itself is equal to the arithmetic mean. If the spectrum does not approach zero anywhere, or, more generally, if the logarithm of the spectrum is integrable, there is a positive

error in predicting one step ahead, and the predictability falls to zero as the range of prediction becomes infinite. In the extreme case of white noise, where the spectrum is constant, the arithmetic and geometric means are equal, and the series is completely unpredictable. If, on the other hand, the spectrum vanishes along a continuum, or, more generally, if the logarithm of the spectrum is not integrable, the series is intrinsically predictable without error at all lags, that is, it is linearly deterministic.

Finally, in the general case, a single formula will simultaneously predict the quasi-periodic component without error, and yield the optimum prediction of the nonperiodic component.

Random errors in measurement add their own spectrum to the spectrum of a perfectly measured series. They therefore do not interfere with the predictability of the quasi-periodic component, but they increase the errors in predicting the nonperiodic component. In particular, they make a linearly deterministic nonperiodic series indistinguishable from a nondeterministic series, so that the attainable predictability at infinite range, instead of being perfect, falls to zero. Only quasi-periodic flow is attainably predictable from its own past, by linear means, at extremely long range.

The procedure for computing the series of prediction coefficients is rather lengthy. I am not aware of any instance where it has been carried to completion with meteorological data. A straightforward procedure for estimating the spectrum has been presented by Blackman and Tukey (1958), and a truly vast number of meteorological spectra have been evaluated in the past few years.

The computed spectra of atmospheric quatities frequently show a number of rather narrow bands. Whether or not these bands indicate real phenomena or sampling errors, there is no suggestion that the spectra lie close to zero in the continua which separate the bands. The atmosphere is thus revealed as having a strong nonperiodic component, which is nondeterministic from the linear point of view.

Often the predictability revealed by simple time series methods does not compare favorably with that obtained by the far simpler screening procedure. Linear predictions of tomorrow's pressure at New York, for example, are better when the predictors are today's and yesterday's pressures at New York and Cleveland, than when they are today's and an unlimited number of past pressures at New York alone.

We must therefore consider the theory of predictability of multiple time series. This theory is less complete. The prediction errors in this case are determined by the autocovariances and cross-covariances. The predictability may be expressed in terms

of the spectra and cross-spectra, which are Fourier transforms of the autocovariance and cross-covariance functions.

As in the case of simple time series, periodic components are intrinsically predictable without error at all lags. The predictability of the nonperiodic components is determined by the matrix whose diagonal elements are the spectra and whose off-diagonal elements are the cross-spectra. If this matrix is not singular anywhere, or, more generally, if the logarithm of its determinant is integrable, there are positive errors in predicting one step ahead, and the predictability falls to zero as the range becomes infinite. The converse of this theorem is not true, however, as it would be in the simple-time-series case.

The formal problem of determining the optimum prediction coefficients has been solved, for the case where the matrix is non-singular, but the precedure is extremely cumbersome. A formula of Wiener and Masani (1958), for example, expresses each coefficient as the sum of an infinite series, plus the sum of a doubly infinite series, plus the sum of a triply infinite series, and so on, to infinity. This formula has yet to be applied to meteorological data.

Random errors will always render the spectral matrix non-singular, so that the attainable predictability of nonperiodic flow always falls to zero as the range becomes infinite.

Thus there exists a fairly well developed theory of predictability by linear means. This theory has not been applied to the atmosphere enough to indicate whether we have approached the limit of linear predictability. Let us therefore look at dynamical predictability.

A few years ago the Statistical Forecasting Project at M.I.T. conducted an investigation which was aimed at determining whether a hydrodynamical flow which is intrinsically predictable by means of its own governing equations is also intrinsically predictable by linear statistical means. It was planned to study a simplified system of differential equations, which nevertheless would retain the nonlinear process of advection, and would possess nonperiodic solutions. Numerical solutions were to be generated on a digital computing machine. These solutions were then to be regarded as data, which would be used to determine linear prediction formulas. The ability of these formulas to reproduce the solutions of the differential equations could then be ascertained.

A suitable set of twelve ordinary differential equations was derived from the partial differential equations of a two-layer baroclinic model used in dynamical weather prediction, by expanding the original variables in double Fourier series, and then truncating

the series. The twelve terms which were retained represented a
rectilinear zonal flow, with finite waves of a single wave length
superposed. The nonlinear advection process appeared as nonlinear
interactions between the variables.

Solutions of the equations, covering a total span of more than
twenty years, were generated on a Royal-McBee LGP-30 electronic
computer. A six-hour time increment was used. The solutions were
judged to be nonperiodic simply because no periodicity was ev-
ident.

FIGURE 1 shows the variation of one of the variables during
an interval of seven months. It reveals the occurrence of certain
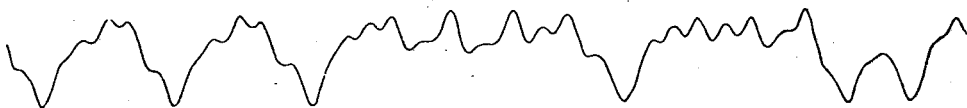typical features, but shows no tendency for these features to recur
after fixed periods.

FIGURE 1. Time series of one of twelve simultaneous numerically-
generated meteorological variables, for a particular seven-month interval.

FIGURE 2 shows a spectrum of the same variable. The spec-
trum was based upon covariances at lags up to 200 days, which
were estimated from about eight years of daily values. There is no
suggestion of any spectral lines. However the spectrum lies near
zero over a considerable range of frequencies so that the series of
daily values is predictable at short range from its own past, with
relatively little error.

The spectrum of the series of values at two-day intervals is
equal to the left-hand half of the spectrum shown in FIGURE 2,
with the right-hand half, reversed in direction, added to it. This
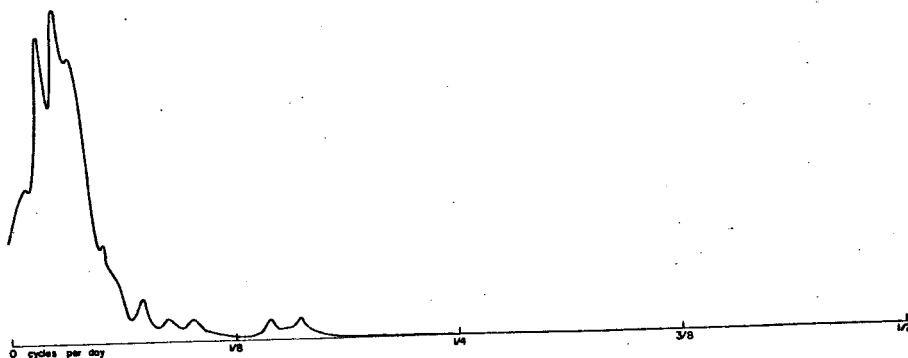

FIGURE 2. Numerically estimated spectrum of variable shown in
FIGURE 1, based upon eight years of "data."

spectrum does not approximate zero over a continuum, so that the series of values at two-day intervals is only moderately predictable from its own past, even at short range.

Attempts to predict one variable from the present and past of all twelve variables could not be exhaustive. There was nearly perfect predictability one day in advance, using observations separated by one day as predictors. When the values of the predictors were separated by at least two days, and the range of prediction was at least two days, the predictability was far short of perfection.

It thus appears that the solutions of deterministic dynamic equations cannot in general be reproduced by linear formulas which are easily deduced from moderate-sized samples of data. If, however, the spectra of the continuous solutions are almost devoid of variance at high frequencies, a linear prediction scheme based upon closely spaced observations may give nearly perfect results at short range. In the atmosphere, no such absence of variance at high frequencies has been detected. Turbulence of virtually all measurable scales seems to be present.

During the course of this investigation we encountered a rather striking phenomenon. At various times we found it desirable to repeat portions of previous computations. For this purpose we took the values which the machine had printed at one particular time step, and entered these values into the machine as new initial conditions. Sometimes the machine did not repeat its previous performance. Fairly soon, small differences between the solutions would appear, and these would grow until eventually there was no resemblance between the two solutions.

The cause of the initial discrepancies was soon evident. The numbers had been carried in the machine to six significant figures, but only three figures had been printed in the output. The new initial values had thus been rounded off to three figures, and we were unwittingly superposing a small disturbance upon the earlier conditions. In comparing the two solutions, we were observing the growth of a small disturbance. In some sense, the original solution was unstable.

Following this observation, we made further runs in which the initial conditions represented small departures from conditions previously encountered. In every case the new solution eventually diverged from the old one, and finally lost all resemblance to it.

This result had obvious implications for the atmosphere, in view of the inevitable inaccuracies of observed initial conditions. It suggested that two indistinguishable states could eventually

evolve into entirely different states, and that a long-range predic-
tion would fail completely in at least one instance.

Although the study was based entirely upon the numerical
solution of a mathematical model, the results seem to be empirical
rather than theoretical. The underlying theory is the theory of
dynamical systems, as presented by Birkhoff (1927).

Consider a system of M ordinary differential equations in M
dependent variables, with time as the single independent variable.
A particular set of values of the dependent variables represents a
possible state of the system.

A convenient geometrical representation of such a system is a
Euclidean space of M dimensions. Each state of the system is
represented by a point, whose coordinates are the values of the M
variables. A state which is varying in accordance with the govern-
ing equations is then represented by a moving point or particle,
and the trajectory on which this particle travels represents a par-
ticular time-dependent solution. This space has been called *phase
space*. Even when we are not visualizing phase space, we often
borrow the geometrical terminology. For instance, we may talk
about the distance between two states, meaning the square root of
the sum of the squares of the differences of the variables.

In a hydrodynamical system, the nonlinear process of advection
does not alter the total energy, but simply increases the confusion.
If the remaining processes together dissipate energy whenever the
total energy already exceeds a certain value, the total energy, and
hence the value of each variable, will remain forever within fixed
limits. In phase space, each trajectory will be confined within a
fixed volume.

A solution is *stable* if any other solution which approaches
sufficiently close to it must remain arbitrarily close. Otherwise it
is *unstable*. The solutions which we obtained in our numerical
study were evidently unstable. FIGURE 3 shows schematically
some trajectories in phase space which represent stable and un-
stable solutions.

The main result is based upon the necessity for any trajectory
which remains within fixed bounds to pass arbitrarily close to the
same point on more than one occasion, and, indeed on arbitrarily
many occasions. If it possesses no transient component, it must
pass arbitrarily close, arbitrarily often, to any point through which
it itself has previously passed. In the language of the meteorologist,
an arbitrarily good analogue of any previous situation must even-
tually occur.

Now suppose that a solution is stable. Then when an approx-
imate repetition of a previous state occurs, the subsequent history

must forever remain arbitrarily close to the history following the previous occurrence. It follows that the solution is quasi-periodic.

Equivalently, if the solution is nonperiodic, it is necessarily unstable. This situation is also shown schematically in FIGURE 3.
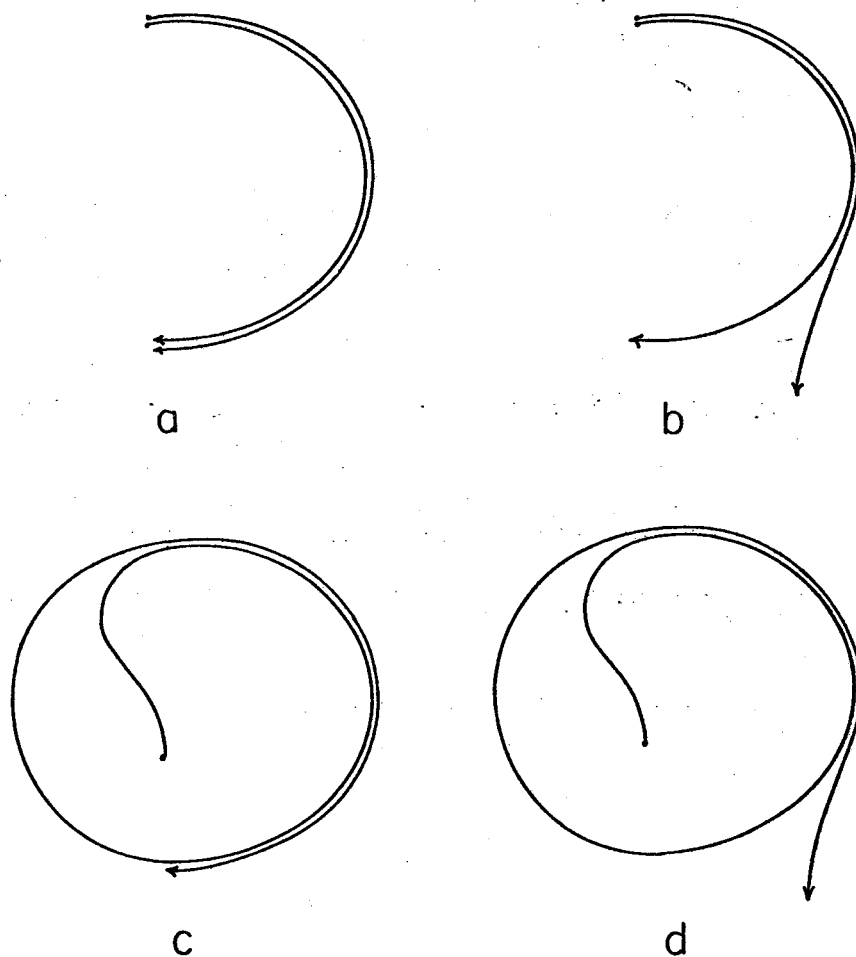

FIGURE 3. Schematic trajectories in phase space; (a) neighboring stable trajectories; (b) neighboring unstable trajectories; (c) stability implying periodicity (after the transient flow has died out); (d) nonperiodicity implying instability.

These results, together with rigorous proofs, are discussed by Nemytskii and Stepanov (1960, ch. 5). They have been applied to hydrodynamical flow by the writer (1963). They imply that attainable predictability of a finite nonperiodic hydrodynamical system, by dynamical means, fall considerably short of perfection.

Our arguments so far have been based upon point-set concepts. We have regarded a solution as unstable if two arbitrarily close states do not remain arbitrarily close. We have left open the possibility that the subsequent histories of two sufficiently close states may continue to bear some resemblance to each other, even though they do not remain arbitrarily close. In our numerical study, pairs of neighboring solutions appeared to lose all resemblance after a few months. Let us see whether this result is true in general.

In the complete future history of any solution, there will be an average distance D between two arbitrarily chosen states. In fact, D^2 will be twice the sum of the variances of the M variables. Suppose that the average square of the distance between two solutions, as they evolve together, is less than D^2. In this case, the value of at least one of the variables in one solution is positively correlated with the value of the same variable in the other solution. If the two solutions are actually two portions of the same solution separated by a time lag T, the variable has a positive autocorrelation at lag T.

As mentioned earlier, any solution must pass arbitrarily close arbitrarily often to any point through which it has previously passed, say after times T_1, T_2, T_3, ---- . If sufficiently close states must evolve into states retaining some resemblance, in the sense of remaining closer together than randomly chosen states, there will be a sequence of lags T_1, T_2, T_3, --- , approaching infinity, for which the corresponding autocorrelations do not approach zero. But then the solution has a quasi-periodic component.

Equivalently, if a solution is purely nonperiodic, that is, if it has no periodic components, it is not only unstable, but two neighboring solutions must eventually become as far separated as two randomly chosen solutions.

This result has immediate implications concerning not only dynamical prediction, but also all other method of prediction, provided that the system merely has a dynamics and is nonperiodic. Errors in observation will always prevent us from distinguishing between two sufficiently similar states. If these two states eventually evolve in different ways, no method of predicting the distant future from the present can give good predictions in both instances.

Likewise, errors in observation will always prevent us from distinguishing between the present and past of two states which were so nearly alike in the past that their differences have not yet grown to observable size. Hence no method of predicting the distant future from the present and past can give good predictions in both instances.

Finally, it is not even necessary that the dynamics of the system be deterministic, provided that any component of the future of the system which is not determined by the present and past of the system is not determined by anything else observable.

Thus the attainable predictability of a purely nonperiodic flow by any means approaches zero as the range approaches infinity. If periodic components are present, these are of course predictable, even by linear means.

Let us try to see to what extent this theory may be applied to real hydrodynamical systems in general, and to the atmosphere in particular. Our really crucial assumption is the assumption that the system must eventually come arbitrarily close to repeating a previous state. This assumption is necessarily correct for finite-parameter systems with bounded solutions, but not necessarily for infinite-parameter solutions. It may not be an easy task to establish that analogues in the atmosphere are required.

Since we have assumed that any method of prediction will give the same forecast in two indistinguishable situations, any information of predictive value must enter the definition of an analogue. This means in the case of the atmosphere that the environment must be included as part of the system, and two situations in which the environment differs cannot be regarded as analogues.

Finally, if some part of the atmosphere, or particularly some part of the environment, is unobservable, it need not be considered in judging analogues, but if the state of the unobservable part is implied by the observable past history, this past history must be considered in judging analogues.

If it is safe to say that measurements can never become exact, it should be safe to say that we can never measure an infinite number of features of the atmosphere and its environment. We are therefore justified in regarding the observable atmosphere as a finite system, and, unless the atmosphere is to blow up, we can say that analogues will eventually occur in the observable atmosphere.

Following these occurrences of apparently similar situations, the subsequent developments will either be the same, or they will not be the same. If they are the same, the atmosphere will continue to repeat its past history, and from then on will be quasi-periodic. The more reasonable alternative is that the atmosphere will not repeat its past history, in which case no method of forecasting can succeed in both instances.

What about predicting some special function of the system at long range, as opposed to predicting the complete system? Can we, for example, predict annual average temperatures many years in

advance? Any such function may be appended to the system as an additional variable, and if this new variable is not periodic, it too cannot be predicted at very long range. The range at which it can be predicted, however, may far exceed the range at which instantaneous states are predictable.

. In summary, we have found certain analogies between attainable predictability by linear methods, and attainable predictability by unrestricted methods. In either case the quasi-periodic component is predictable with relatively little error at all ranges. The nonperiodic component is not predictable, without some error, even at short range, and its attainable predictability approaches zero as the range approaches infinity.

There is one important difference. Except in special cases, there are intrinsic errors in predicting nonperiodic flow by linear methods. The intrinsic errors in dynamic prediction, if any, result only from possible physical indeterminism, and are not comparable to the intrinsic errors in linear prediction. With sufficient improvement in measurement, the attainable predictability by dynamic methods at short and intermediate ranges should exceed the attainable predictability by linear methods. The ultimate hope for weather prediction may therefore lie in dynamical or quasi-dynamical procedures.

One word of caution should be added. The preference for dynamic over statistical methods is a preference for theory over observation. Theory itself has dictated this preference. Perhaps the referee is also one of the contestants.

I should like to conclude by describing another numerical experiment, which was aimed at finding the rate of decay of predictability in the atmosphere. This work was performed at the Meteorological Institute in Oslo, Norway.

As in the study mentioned earlier, the equations were derived from a two-layer baroclinic model used in numerical weather prediction. Long waves of three different wave lengths, together with a rectilinear zonal flow, were included. The interactions among these waves, and the interaction of each wave with the zonal current, were represented in the equations. Altogether there were 28 dependent variables, and the 28 equations contained a total of 336 quadratic terms, 56 linear terms, and one constant term. The computations were performed on a FACIT-EDB electronic computer. A time increment of three hours proved satisfactory.

Most of the study was confined to one particular solution, and other solutions which initially differed from it by small amounts. FIGURE 4 shows the graph of one of the variables, over a period of four months. The straight-line segments connecting the plotted
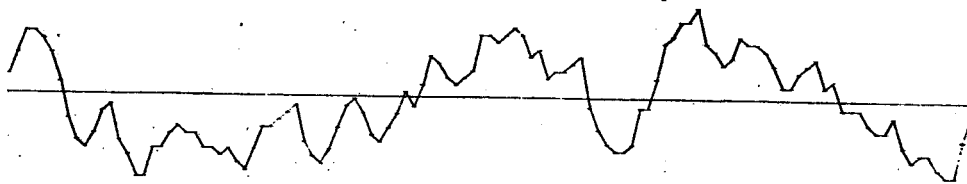
FIGURE 4. Time series of one of 28 simultaneous numerically-generated variables, for a particular four-month interval.

daily values are merely intended to make the chronological order more apparent, and do not imply any straight-line variations. The variables appear to oscillate irregularly. Extension of these graphs to a three-year span failed to suggest any periodicity.

Resemblance between this system and the atmosphere is further suggested by FIGURE 5, which shows two maps of the 500-millibar contour height, separated by one day. The eastward displacement of the principal features is obvious, and there is a suggestion that in high latitudes the predominant wave number is decreasing.
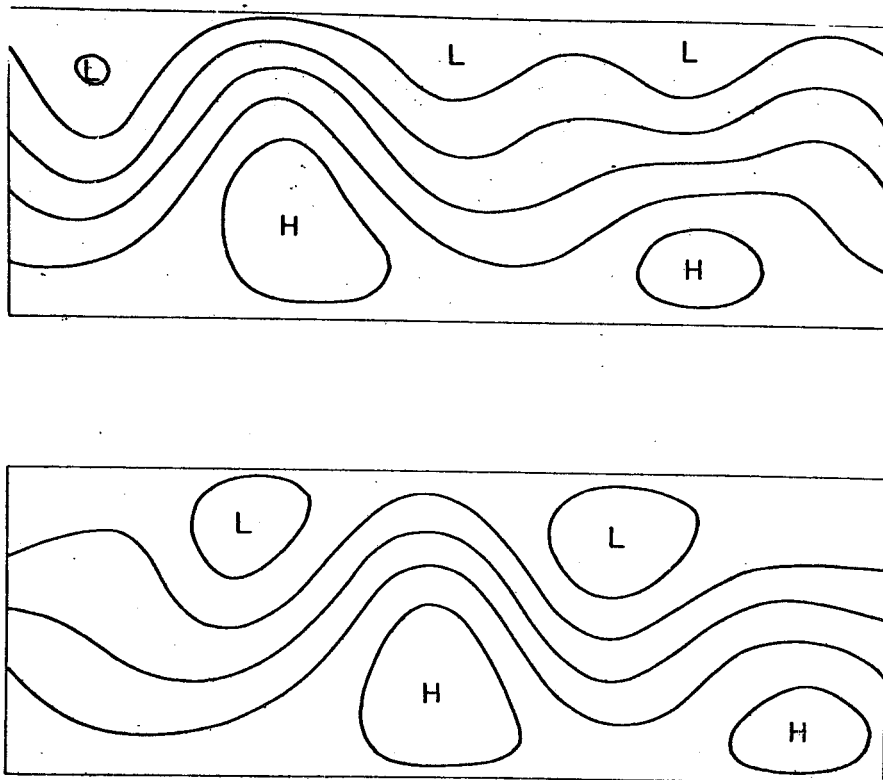

FIGURE 5. Numerically generated 500-millibar weather maps. Lower map follows upper map by 24 hr.

In order to study the behavior of random errors, a given initial
condition was subjected to small perturbations on 28 separate
occasions. In each of the 28 perturbed initial states, only one of
the 28 variables differed from its value in the unperturbed state.
In each case the equations were integrated over a period of two
days, and the 28 final states were compared with the undisturbed
final state. The mean growth rate of random perturbations during
the two-day period was given by the root-mean-square growth rate
of the 28 individual perturbations. This entire procedure was then
repeated at two-day intervals, for a total of 32 times. Thus the
time variations of the mean two-day growth rate could be studied.

Since an ensemble of perturbations which is initially random
will in general not be random after two days, the mean growth dur-
ing a four-day period is not in general the product of the mean
growths during successive two-day periods. Hence the average
growth rates during 4-, 8- and 16-day periods were also computed.

The growth rates during successive 2-, 4-, 8-, and 16-day pe-
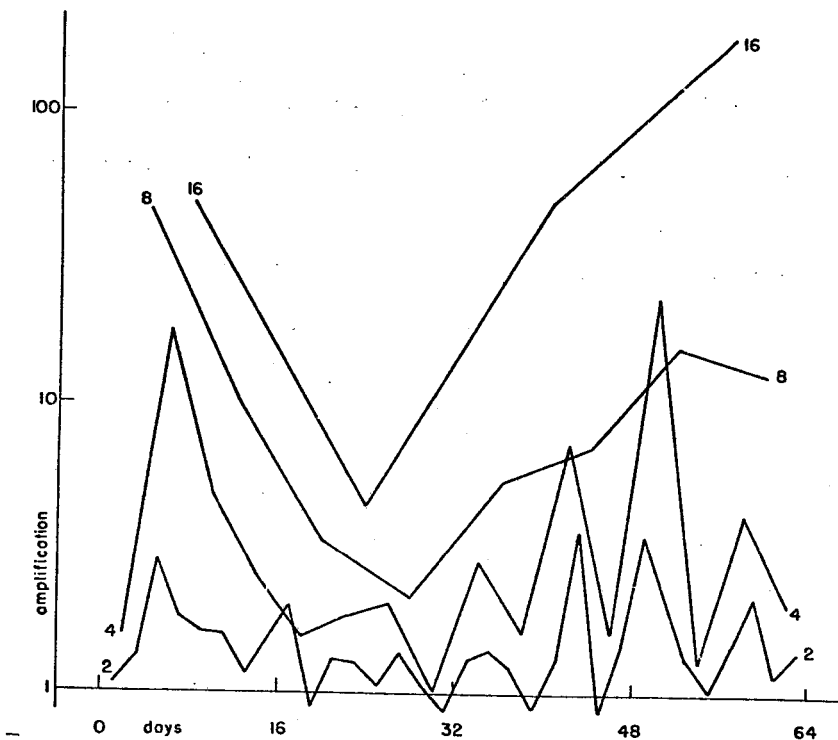riods are illustrated in FIGURE 6. Perhaps the outstanding feature

FIGURE 6. Mean amplification factors for small random errors dur-
ing a particular 64-day interval. Curves labeled 2,4,8, and 16 indicate
amplifications during successive 2-,4-,8-, and 16-day periods.

of the growth rate is its great variability. Most of the growth which might be termed "explosive" is confined to one rather brief interval early in the 64-day span, and a few brief intervals in the latter half. In a few cases random errors actually diminished in size during two-day periods. During the fourth eight-day period, random errors no more than doubled. By contrast, random errors increased forty-fold during the first eight-day period.

These growth rates apply only to small errors. Once they have become large enough to be serious in their own right, they grow less rapidly. They cease growing altogether when the disturbed and undisturbed solutions have lost all resemblance to each other.

In the real atmosphere, average errors in measurement can increase perhaps by a factor of five before a forecast becomes generally poor. Over regions like the United States and Europe, where observations are plentiful, the tolerable amplification is considerably larger; over the oceans it is presumably smaller. If the results of this numerical study are at all applicable to the atmosphere, they suggest a wide discrepancy between practical predictability and attainable predictability at ranges up to one week. Good forecasts several days in advance do not seem to be prevented simply by current errors in measurement. If, however, we are genuinely interested in forecasting a few weeks in advance, we should give serious consideration to enlarging our network of observing stations, particularly over the oceans.

Perhaps these conclusions are too optimistic. The real atmosphere possesses significant fluctuations of shorter period than any which occur in the numerical model. Maybe what we have called one week in the model is more like two or three days in the real atmosphere. If this is so, we have already reached the maximum range which present errors in measurement will allow.

When the instability of a uniform flow with respect to infinitesimal perturbations was first suggested as an explanation for the presence of cyclones and anticyclones in the atmosphere, the idea was not universally accepted. One meteorologist remarked that if the theory were correct, one flap of a sea gull's wings would be enough to alter the course of the weather forever. The controversy has not yet been settled, but the most recent evidence seems to favor the sea gulls.

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