Approximation

What for?

— Dealing with HARD problems

Getting around NP-completeness

- **If *N* is small, even O(2^N) is acceptable**
- **Solve some important special cases in polynomial time**
- **Find *near-optimal* solutions in polynomial time**

  — *approximation* algorithm

Approximation Ratio

【Definition】 An algorithm has an *approximation ratio* of $\rho(n)$ if, for any input of size n, the cost C of the solution produced by the algorithm is within a factor of $\rho(n)$ of the cost C^* of an optimal solution:

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \le \rho(n)$$

If an algorithm achieves an approximation ratio of $\rho(n)$, we call it a $\rho(n)$-*approximation algorithm*.

【Definition】 An *approximation scheme* for an optimization problem is an approximation algorithm that takes as input not only an instance of the problem, but also a value $\varepsilon > 0$ such that for any fixed ε, the scheme is a $(1+\varepsilon)$-approximation algorithm.

We say that an approximation scheme is a *polynomial-time approximation scheme* (*PTAS*) if for any fixed $\varepsilon > 0$, the scheme runs in time polynomial in the size n of its input instance.

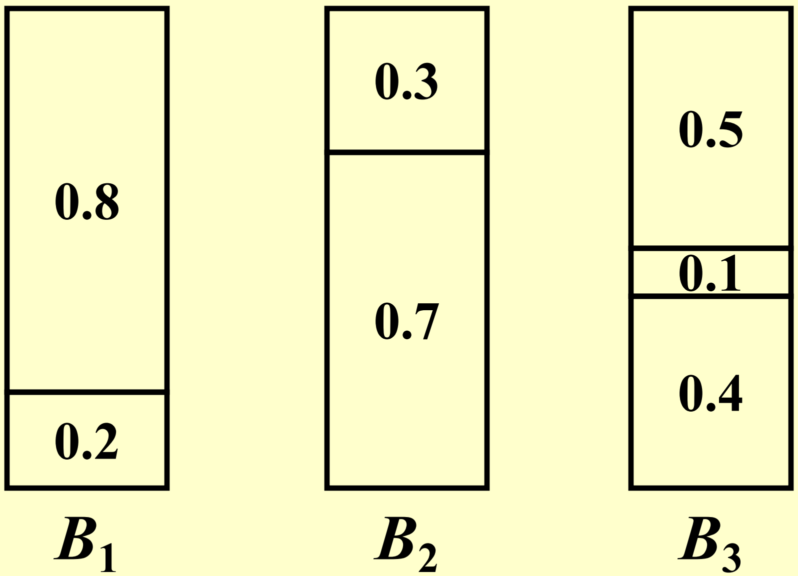
$$O(n^{2/\varepsilon}) \qquad O((1/\varepsilon)^2 n^3)$$

fully polynomial-time approximation scheme (*FPTAS*)

Approximate Bin Packing

Given N items of sizes $S_1, S_2, \ldots, S_N$, such that $0 < S_i \le 1$ for all $1 \le i \le N$. Pack these items in the **fewest** number of bins, each of which has **unit capacity**.

〖Example〗 $N = 7$; $S_i = 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8$



An Optimal Packing

❖ Next Fit

```
void NextFit ( )
{   read item1;
    while ( read item2 ) {
        if ( item2 can be packed in the same bin as item1 )
              place item2 in the bin;
        else
              create a new bin for item2;
        item1 = item2;
    } /* end-while */
}
```

【Theorem】 **Let M be the optimal number of bins required to pack a list I of items. Then *next fit* never uses more than $2M - 1$ bins. There exist sequences such that *next fit* uses $2M - 1$ bins.**

A simple proof for Next Fit:

If Next Fit generates $2M$ (or $2M+1$) bins, then the optimal solution must generate at least $M+1$ bins.

Let $S(B_i)$ be the size of the ith bin. Then we must have:

$S(B_1) + S(B_2) > 1$
$S(B_3) + S(B_4) > 1$
…………
$S(B_{2M-1}) + S(B_{2M}) > 1$

$$\Rightarrow \sum_{i=1}^{2M} S(B_i) > M$$

The optimal solution needs at least ⌈total size of all the items / 1⌉ bins

$$\Rightarrow \lceil \text{total size of all the items} / 1 \rceil = \left\lceil \sum_{i=1}^{2M} S(B_i) \right\rceil \geq M + 1$$

❖ First Fit

```
void FirstFit ( )
{   while ( read item ) {
        scan for the first bin that is large enough for item;
        if ( found )
            place item in that bin;
        else
            create a new bin for item;
    } /* end-while */
}
```

Can be implemented in O(N log N)

【Theorem】 Let M be the optimal number of bins required to pack a list I of items. Then *first fit* never uses more than $17M / 10$ bins. There exist sequences such that *first fit* uses $17(M – 1) / 10$ bins.

❖ Best Fit

Place a new item in the tightest spot among all bins.

T = O(N log N) and bin no. $\leq 1.7M$

〖Example〗 S_i = 0.2, 0.5, 0.4, 0.7, 0.1, 0.3, 0.8

Next Fit | First Fit | Best Fit

Discussion 14: Please show the results.

〖Example〗 S_i = 1/7+ε, 1/7+ε, 1/7+ε, 1/7+ε, 1/7+ε, 1/7+ε,
1/3+ε, 1/3+ε, 1/3+ε, 1/3+ε, 1/3+ε, 1/3+ε,
1/2+ε, 1/2+ε, 1/2+ε, 1/2+ε, 1/2+ε, 1/2+ε

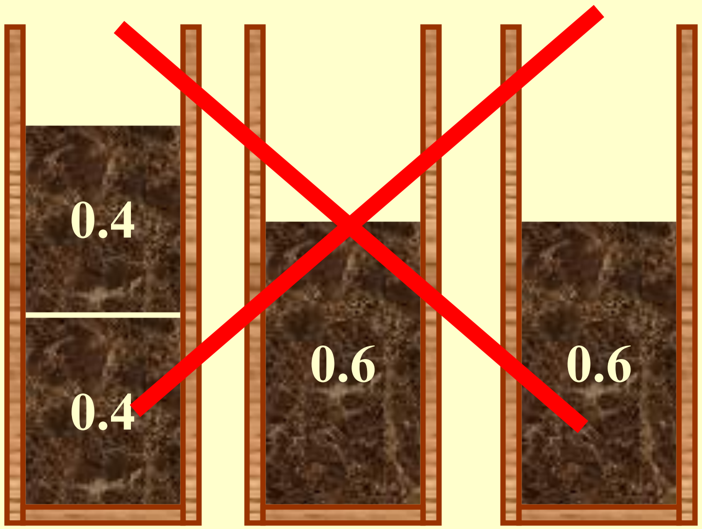
where ε = 0.001.

☞ The optimal solution requires 6 bins.
However, all the three on-line algorithms require 10 bins.

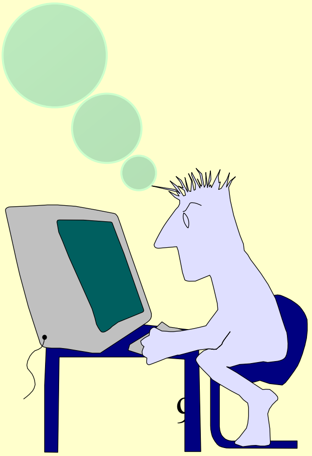
☞ On-line Algorithms

Place an item before processing the next one, and can **NOT change** decision.

〖Example〗 $S_i = 0.4, 0.4, 0.6, 0.6$



You never know
when the input might end.
No on-line algorithm
can always give
an optimal solution.



【Theorem】There are inputs that force any on-line bin-packing algorithm to use at least 5/3 the optimal number of bins.

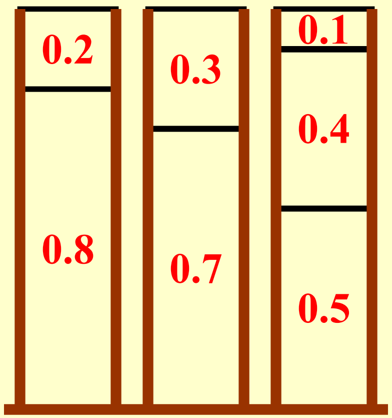
☞ Off-line Algorithms

View the entire item list before producing an answer.

Trouble-maker: The large items

Solution: Sort the items into non-increasing sequence of sizes. Then apply first (or best) fit – *first* (or *best*) *fit decreasing*.

〖Example〗 S_i = 0.8, 0.7, 0.5, 0.4, 0.3, 0.2, 0.1



【Theorem】Let M be the optimal number of bins required to pack a list I of items. Then *first fit decreasing* never uses more than $11M / 9 + 6/9$ bins. There exist sequences such that *first fit decreasing* uses $11M / 9 + 6/9$ bins.

Simple greedy heuristics can give good results.

The Knapsack Problem — fractional version

A knapsack with a capacity M is to be packed. Given N items. Each item i has a weight w_i and a profit p_i. If x_i is the percentage of the item i being packed, then the packed profit will be $p_i x_i$.

An optimal packing is a feasible one with maximum profit. That is, we are supposed to find the values of x_i such that $\sum_{i=1}^{n} p_i x_i$ obtains its maximum under the constrains

$$\sum_{i=1}^{n} w_i x_i \le M \quad \text{and} \quad x_i \in [0, 1] \text{ for } 1 \le i \le n$$

Q: What must we do in each stage?

A: Pack one item into the knapsack.

Q: On which criterion shall we be greedy?

① maximum profit ② minimum weight

③ maximum profit density p_i / w_i

Example :

$n = 3$, $M = 20$,

$(p1, p2, p3) = (25, 24, 15)$

$(w1, w2, w3) = (18, 15, 10)$

Solution is...?

(0, 1, 1/2)

P = 31.5

The Knapsack Problem — 0-1 version

NP-hard r 0

Example :

$n = 5$, $M = 11$,

$(p_1, p_2, p_3, p_4, p_5) = (1, 6, 18, 22, 28)$

$(w_1, w_2, w_3, w_4, w_5) = (1, 2, 5, 6, 7)$

Solution is...? (0, 0, 1, 1, 0) P = 40

The greedy solution is (1, 1, 0, 0, 1) P = 35

What if we are greedy on taking the maximum profit *or* profit density?

The approximation ratio is 2.

Proof: $p_{max} \leq P_{opt} \leq P_{frac}$

$p_{max} \leq P_{greedy}$ $\Rightarrow$ $P_{opt} / P_{greedy} \leq 1 + p_{max} / P_{greedy} \leq 2$

$P_{opt} \leq P_{greedy} + p_{max}$

☞ A Dynamic Programming Solution

$W_{i,\,p}$ = the minimum weight of a collection from {1, …, i } with total profit being exactly p

① take i : $W_{i,p} = w_i + W_{i-1,p-p_i}$

② skip i : $W_{i,p} = W_{i-1,p}$

③ impossible to get p : $W_{i,p} = \infty$

$$W_{i,p} = \begin{cases} \infty & i = 0 \\ W_{i-1,p} & p_i > p \\ \min\{W_{i-1,p}, w_i + W_{i-1,p-p_i}\} & \textit{otherwise} \end{cases}$$

$i = 1, \ldots, n$; $p = 1, \ldots,$ $n\, p_{max}$ ➡ $O(\, n^2 p_{max}\,)$

☞ **What if p_{max} is LARGE?**

Item	Profit	Weight
1	134,221	1
2	656,342	2
3	1,810,013	5
4	22,217,800	6
5	28,343,199	7

M = 11

Item	Profit	Weight
1	2	1
2	7	2
3	19	5
4	223	6
5	284	7

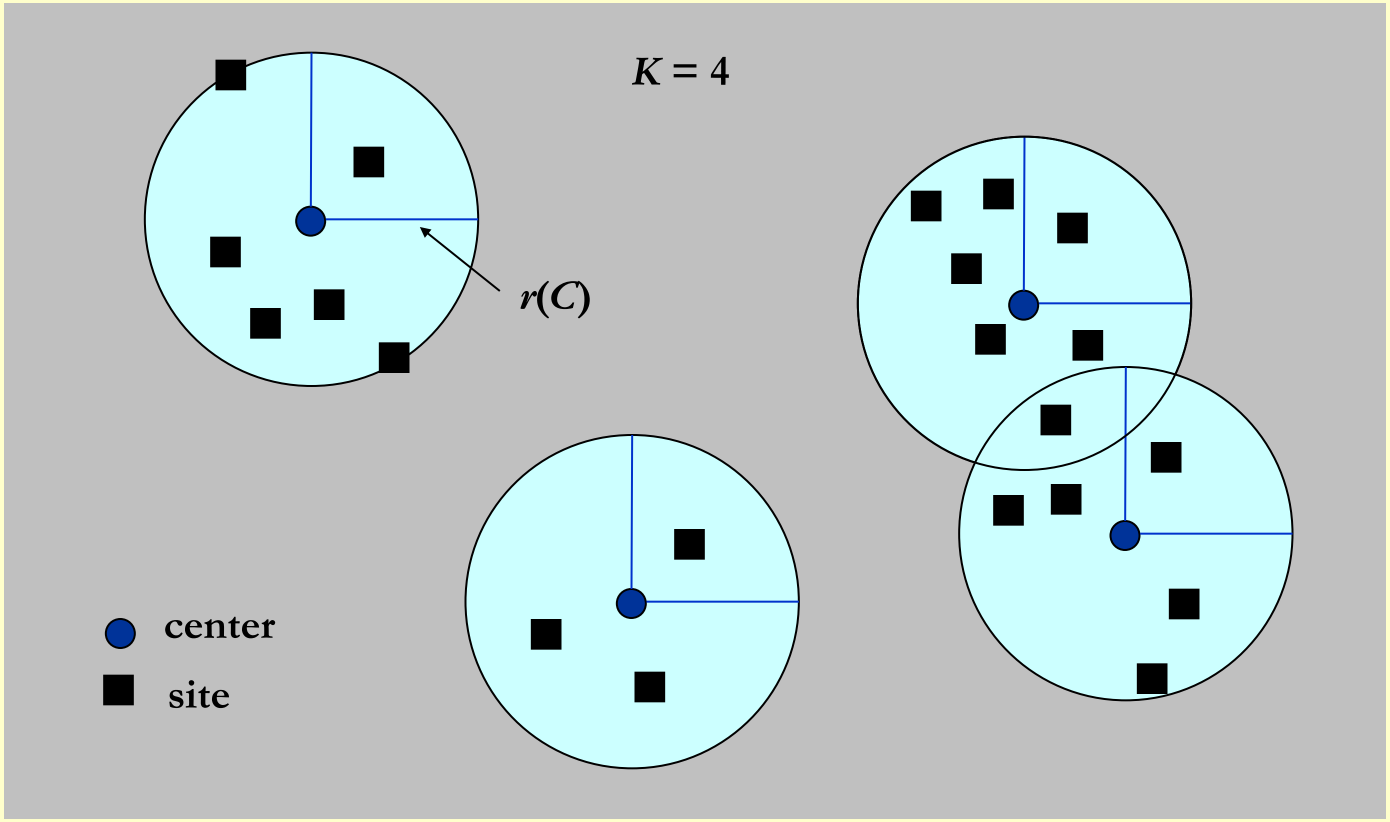
M = 11

☝ **Round all profit values up to lie in smaller range!**

$$(1+\varepsilon)\, P_{alg} \le P$$ **for any feasible solution P**

precision parameter

The K-center Problem



Input: Set of n sites $s_1, \ldots, s_n$

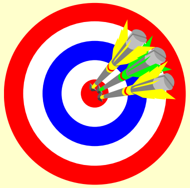
Center selection problem: Select K centers C so that the maximum distance from a site to the nearest center is minimized.

☞ **What is a *distance*?**

- ✓ **dist(x, x) = 0** **(identity)**
- ✓ **dist(x, y) = dist(y, x)** **(symmetry)**
- ✓ **dist(x, y) ≤ dist(x, z) + dist(z, y)** **(triangle inequality)**

$$\mathrm{dist}(s_i, C) = \min_{c\in C} \mathrm{dist}(s_i, c)$$
$$= \text{distance from } s_i \text{ to the closest center}$$

$$r(C) = \max_i \mathrm{dist}(s_i, C) = \text{smallest covering radius}$$

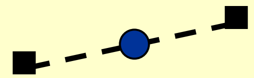


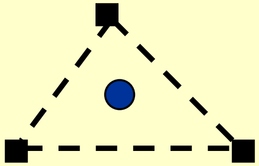
Find a set of centers C that minimizes $r(C)$, subject to $|C| = K$.

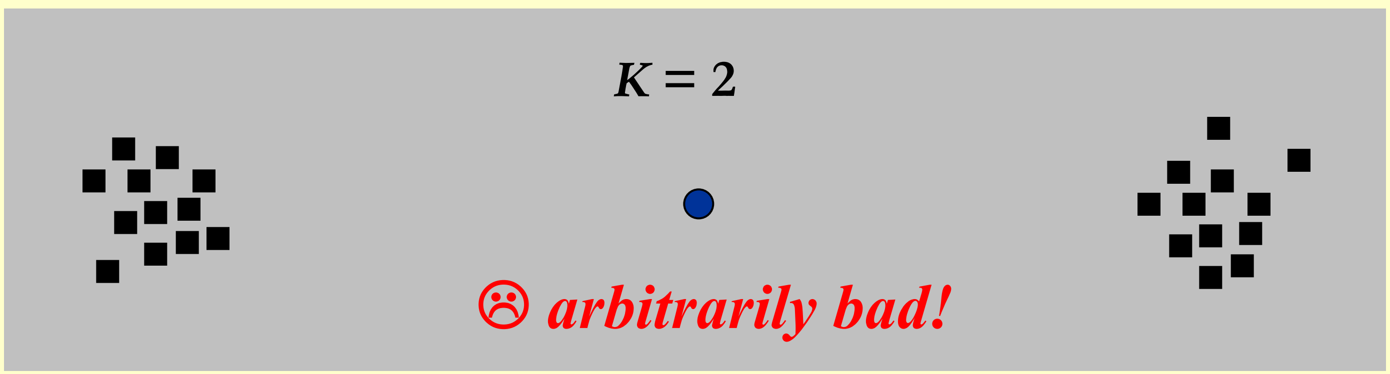
Number of candidate centers = ∞

☞ A Greedy Solution

Put the first center at the ***best possible*** location for a single center, and then keep adding centers so as to ***reduce the covering radius*** each time by as much as possible.

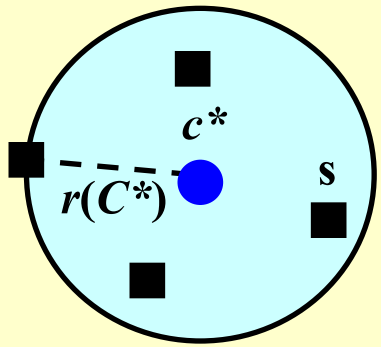






☞ **A Greedy Solution — try again …**

What if we know that $r(C^*) \le r$ where C^* is the optimal solution set?



Discussion 15:

Take s to be the center, how can we select r so that s can cover all the sites that are covered by c^*?

```
Centers  Greedy-2r ( Sites S[ ], int n, int K, double r )
{   Sites  S'[ ] = S[ ]; /* S' is the set of the remaining sites */
    Centers  C[ ] = ∅;
    while ( S'[ ] != ∅ ) {
        Select any s from S' and add it to C;
        Delete all s' from S' that are at dist(s', s) ≤ 2r;
    } /* end-while */
    if ( |C| ≤ K ) return C;
    else ERROR(No set of K centers with covering radius at most r);
}
```

【Theorem】 Suppose the algorithm selects more than K centers. Then for any set C^* of size at most K, the covering radius is $r(C^*) > r$.

Do we really know $r(C^*)$?

Binary search for r

$0 < r \le r_{max}$ **Guess:** $r = (0 + r_{max}) / 2$

$\Rightarrow \begin{cases} \text{Yes: } K \text{ centers found with } 2r \\ \text{or} \\ \text{No: } r \text{ is too small} \end{cases}$

$r_0 < r \le r_1$ $r = (r_0 + r_1) / 2$

$\Rightarrow$ **Solution radius = $2r_1$ —— 2-approximation**

A smarter solution — be far away

```
Centers  Greedy-Kcenter ( Sites S[ ], int n, int K )
{   Centers  C[ ] = ∅;
    Select any s from S and add it to C;
    while ( |C| < K ) {
        Select s from S with maximum dist(s, C);
        Add s it to C;
    } /* end-while */
    return C;
}
```

【Theorem】 The algorithm returns a set C of K centers such that $r(C) \leq 2r(C^*)$ where C^* is an optimal set of K centers.

—— 2-approximation

☞ **Is there a hope of a 3/2-approximation? Or 4/3?**

【Theorem】 Unless P = NP, there is no ρ-approximation for center-selection problem for any $\rho < 2$.

Sketch of the proof: By contradiction.

If we can obtain a (2-ε)-approximation in polynomial time, then we can solve DOMINATING-SET (which is NP-complete) in polynomial time.

Dominating set problem has a solution of size K iff there exists K centers C^* with $r(C^*) = 1$.

Then a (2-ε)-approximation must give the optimal solution since all the distances involved are integers.

Three aspects to be considered:

A: Optimality *-- quality of a solution*

B: Efficiency *-- cost of computations*

C: All instances

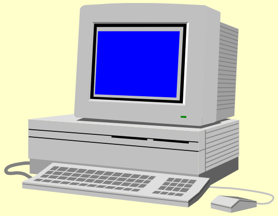
Researchers are working on

A+C: Exact algorithms for all instances

A+B: Exact and fast algorithms for special cases

B+C: Approximation algorithms

Even if P=NP, still we cannot guarantee **A+B+C .**



Research Project 5

Texture Packing (26)

Texture Packing is to pack multiple rectangle shaped textures into one large texture. The resulting texture must have a given width and a minimum height.

You are to design and analyze an approximation algorithm that runs in polynomial time.

Detailed requirements can be downloaded from https://pintia.cn/

Reference:

Data Structure and Algorithm Analysis in C (2nd Edition)：
Ch.10，p.359-366；*M.A.Weiss著、陈越改编，人民邮件出版社，2005*

Introduction to Algorithms, 3rd Edition: Ch.35, p.1106 - 1140； *Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009*