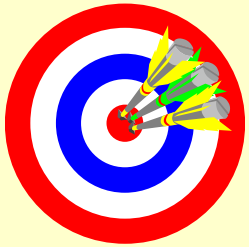
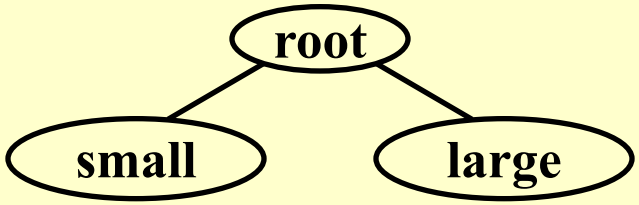
# AVL Trees, Splay Trees, and Amortized Analysis

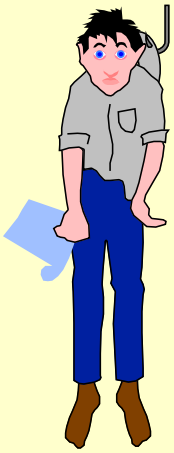
# AVL Trees



**Target :** Speed up searching (with insertion and deletion)

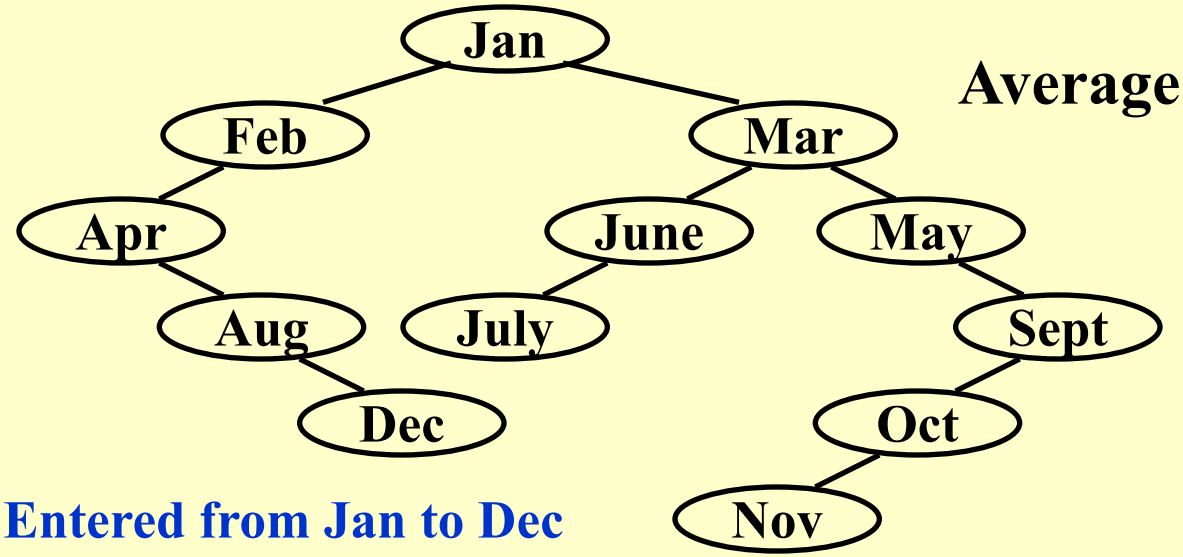
**Tool :** Binary search trees



**Problem :** Although $T_p$ = O( height ), but the height can be as bad as O( $N$ ).

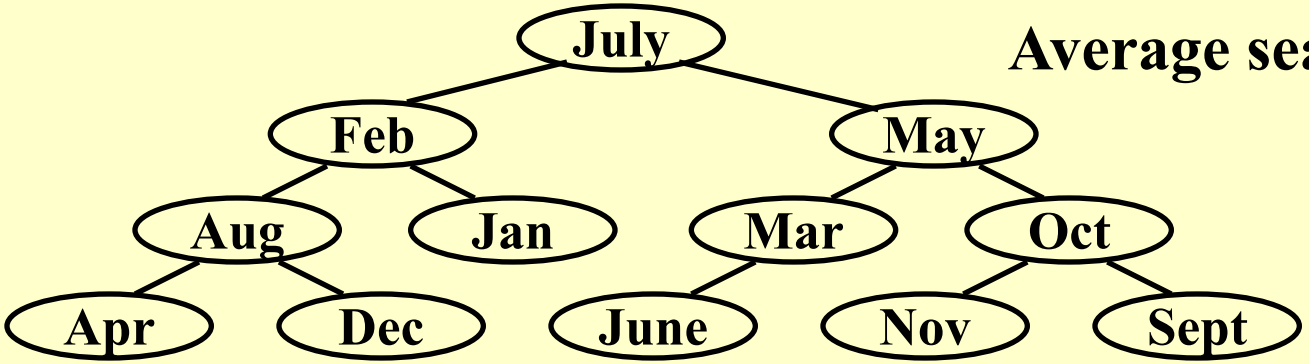
〖Example〗 2 binary search trees obtained for the months of the year




Entered from Jan to Dec

Average search time = 3.5

Average search time of the skew tree = 6.5




Average search time = 3.1

A balanced tree

# Adelson-Velskii-Landis (AVL) Trees (1962)

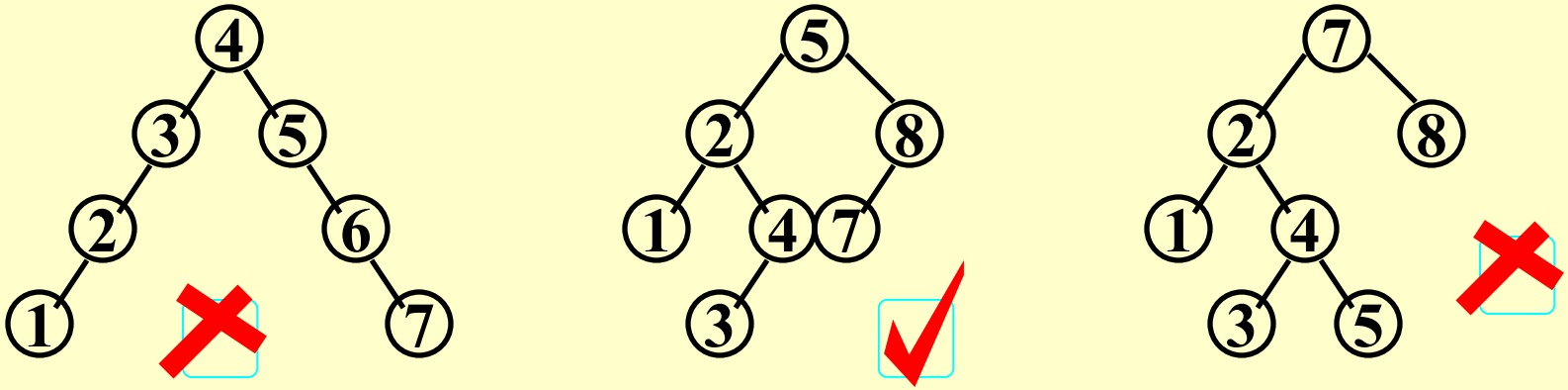
【Definition】 An empty binary tree is height balanced. If $T$ is a nonempty binary tree with $T_L$ and $T_R$ as its left and right subtrees, then $T$ is height bala

(1) $T_L$ and $T_R$ are height b

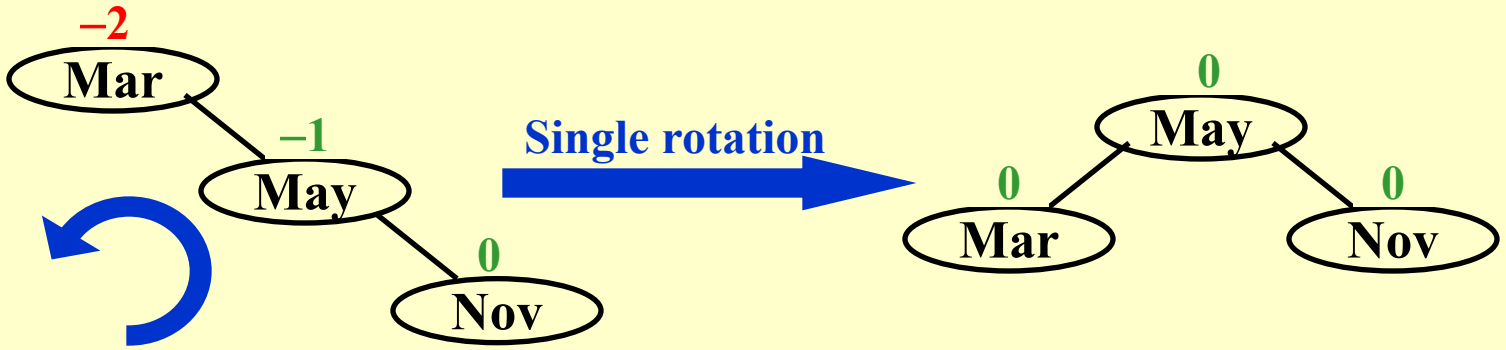
(2) $|h_L - h_R| \leq 1$ where $h$ and $T_R$, respectively.

The height of an empty tree is defined to be –1.

【Definition】 The balance factor $BF(\text{node}) = h_L - h_R$. In an AVL tree, $BF(\text{node}) = -1, 0, \text{or } 1$.

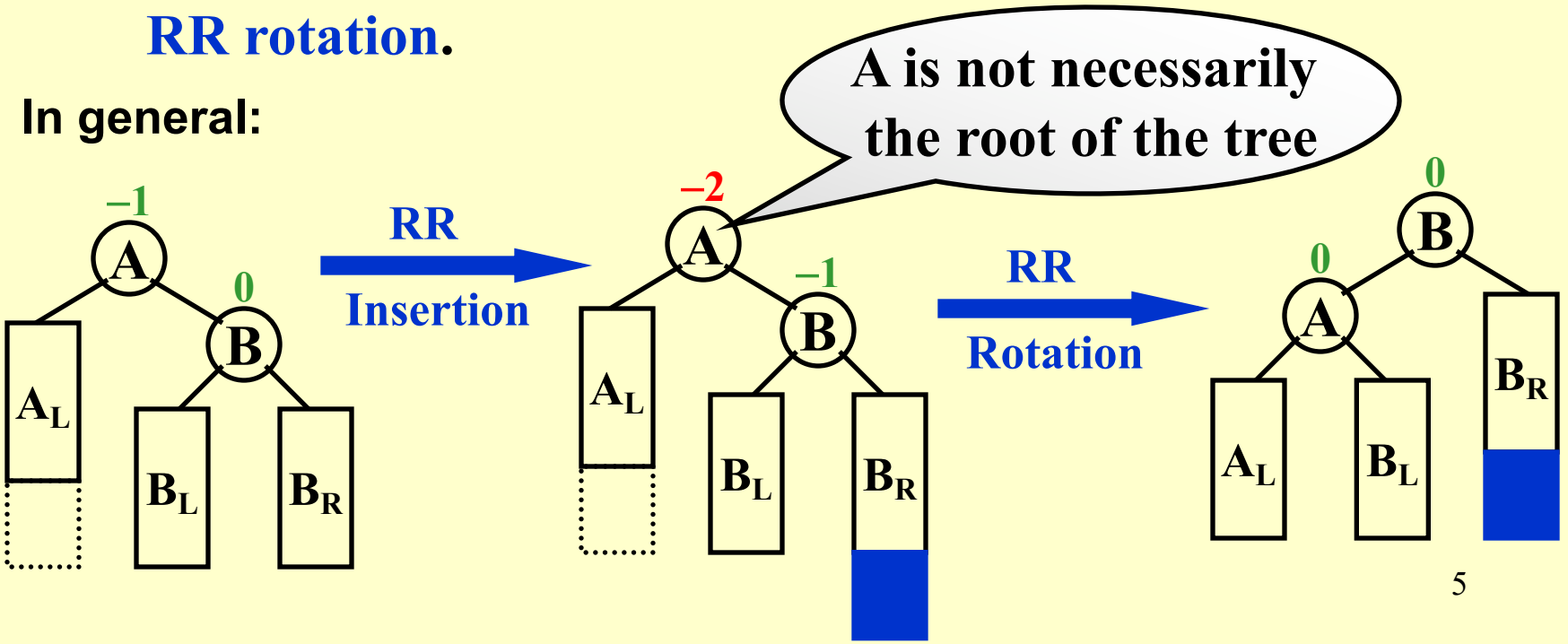


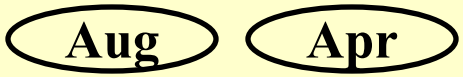
〖Example〗 Input the months Mar May Nov

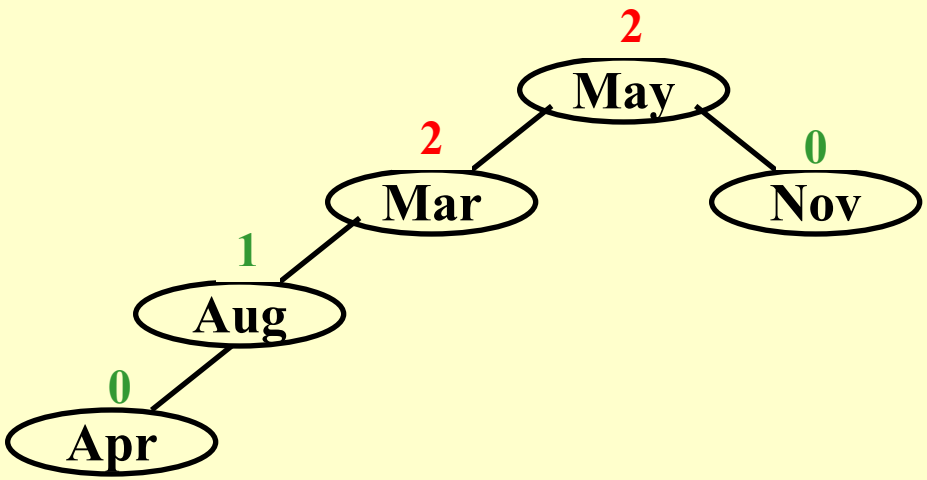



The trouble maker Nov is in the right subtree's right subtree of the trouble finder Mar. Hence it is called an RR rotation.

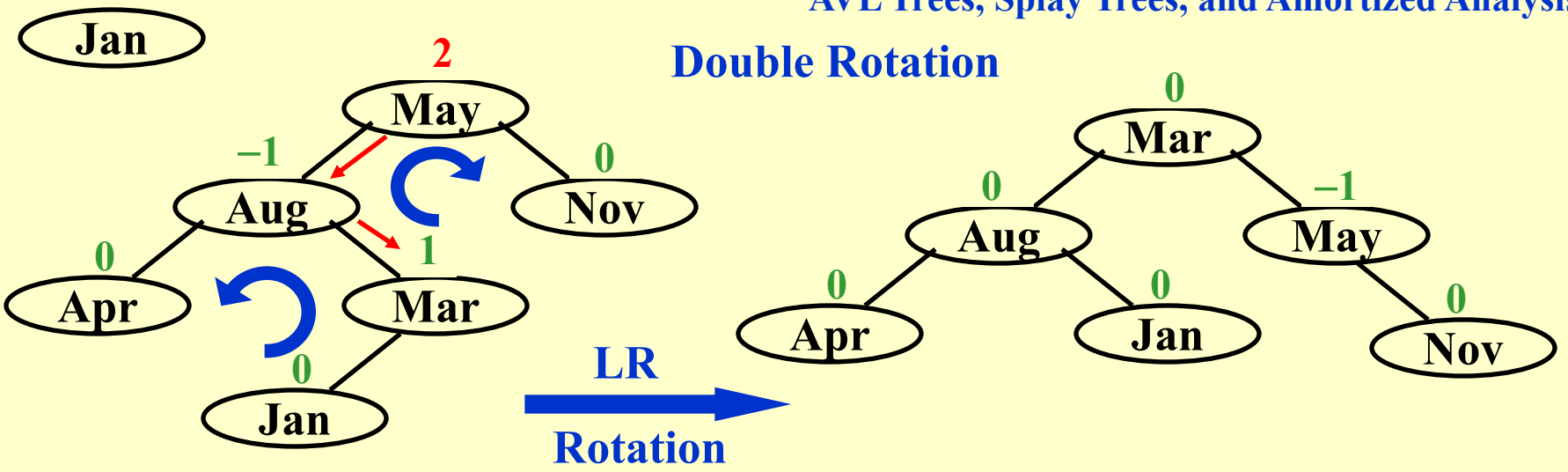
In general:



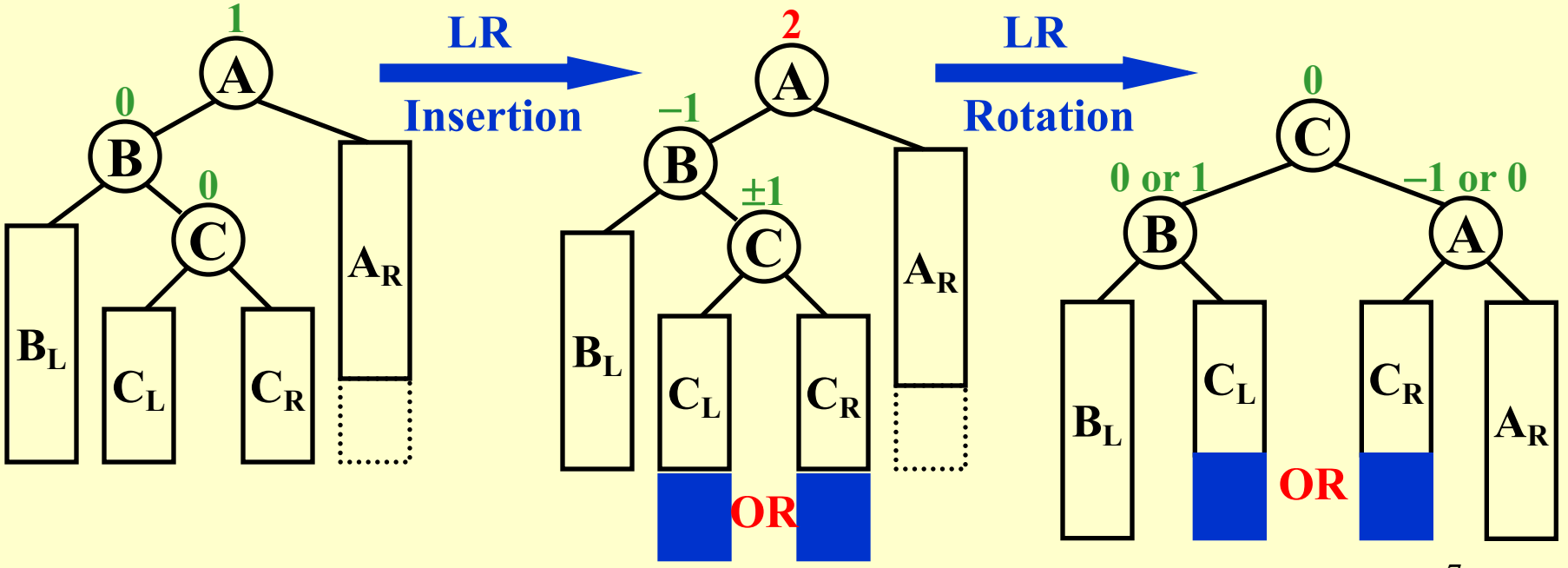


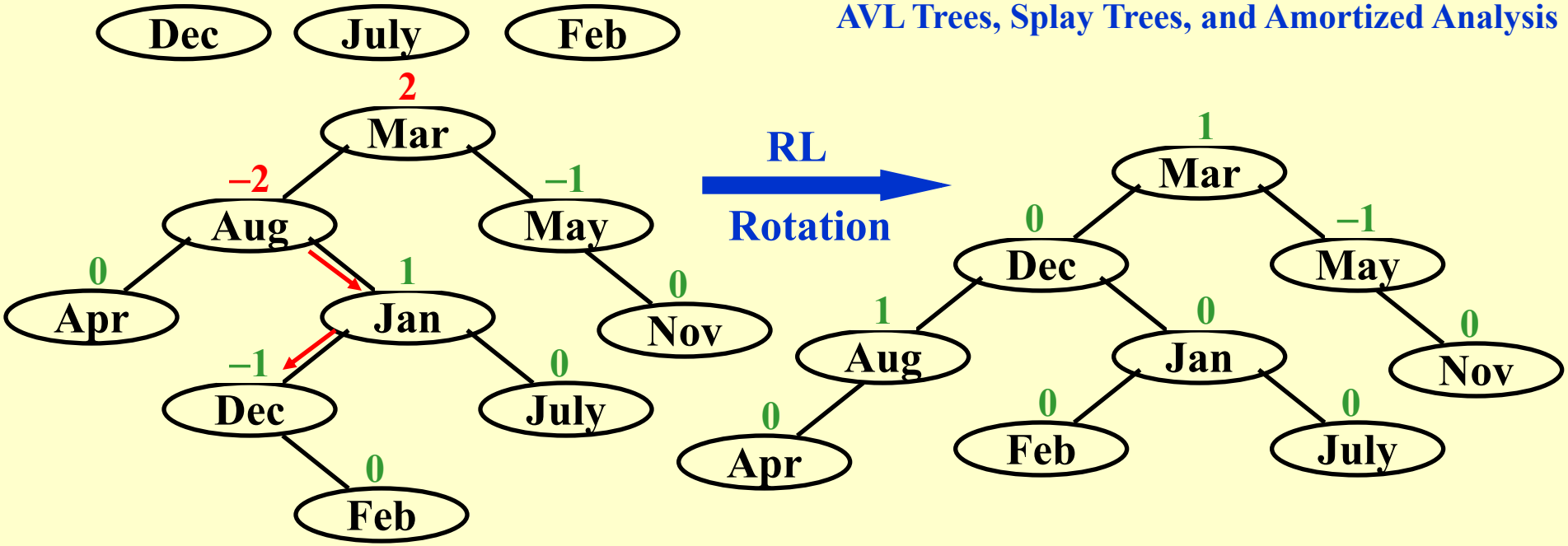




**Discussion 1:** What can we do now?

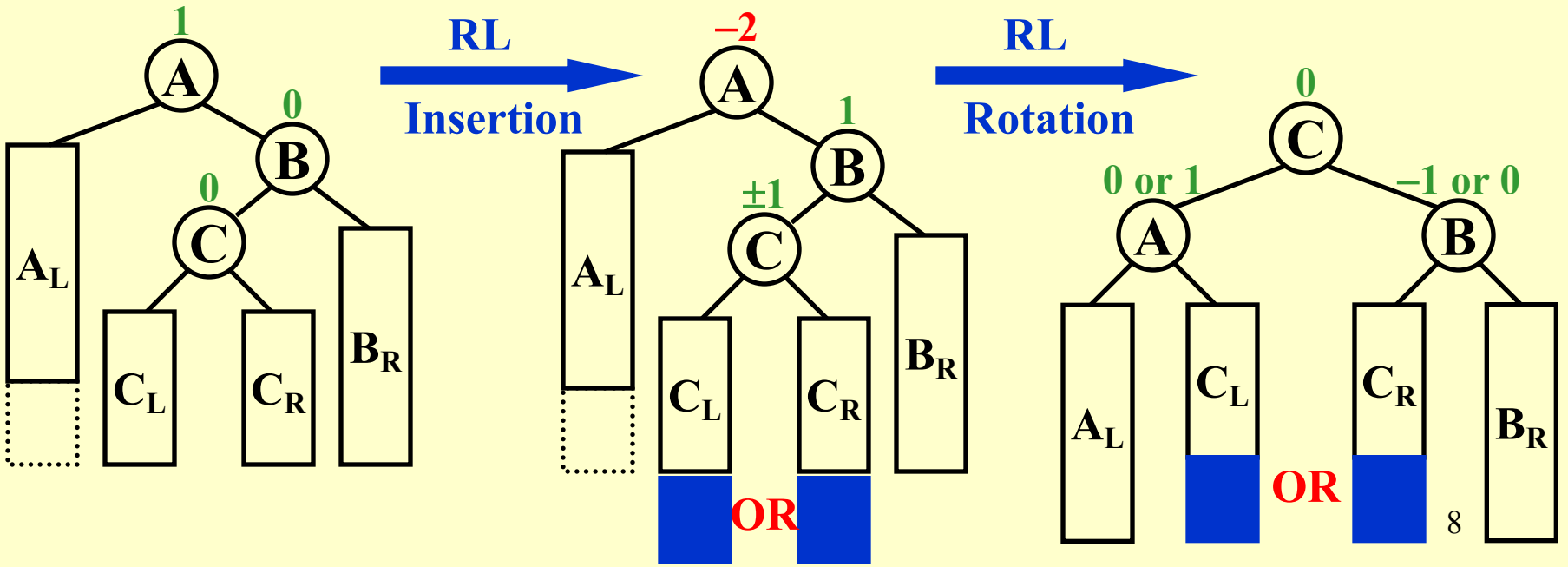

Jan
2
May
–1
Aug
0
Nov
0
Apr
1
Mar
0
Jan
Double Rotation
LR
Rotation
0
Mar
0
Aug
–1
May
0
Apr
0
Jan
0
Nov

**In general:**

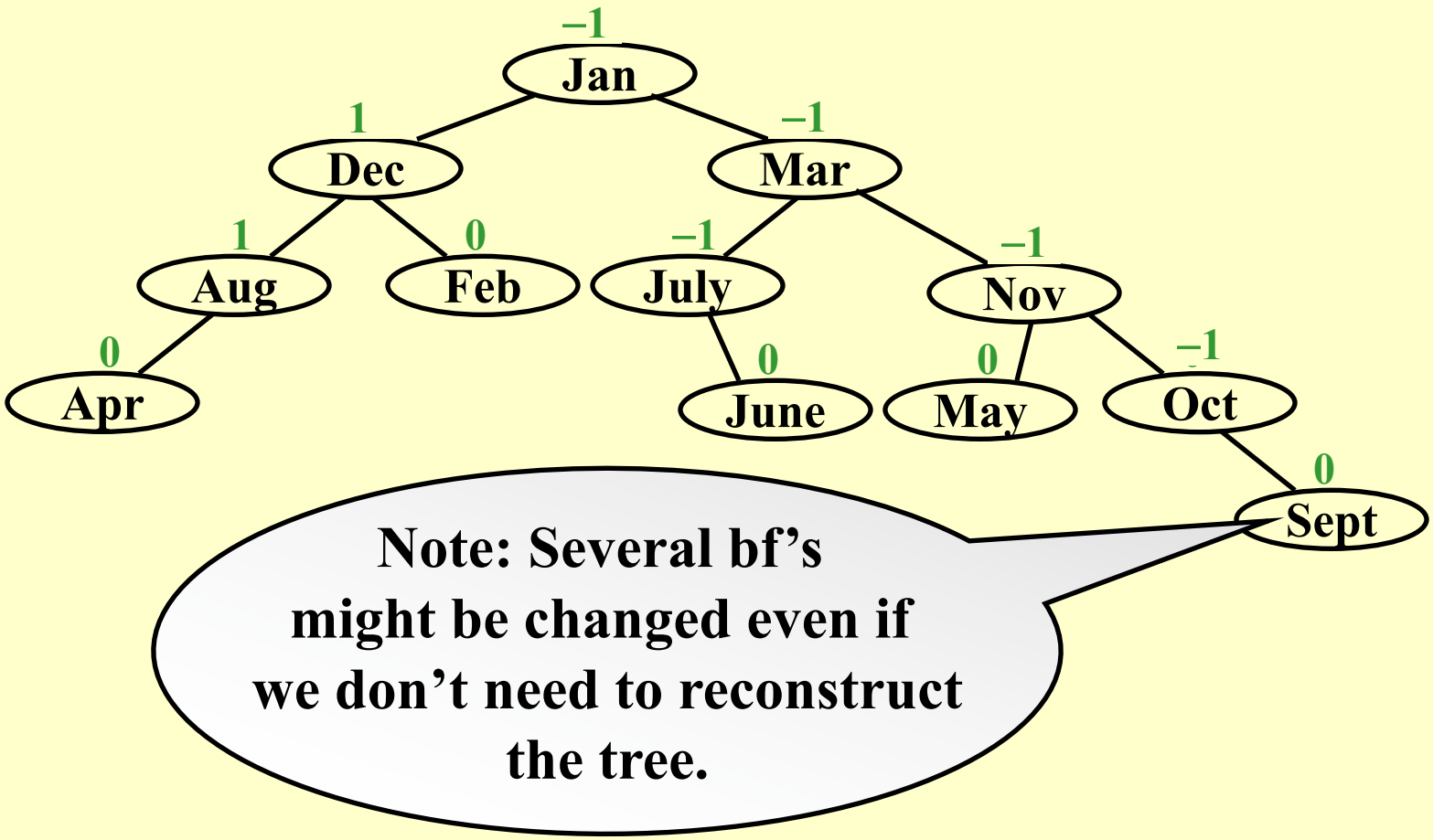

1
A
0
B
0
C
$A_R$
$B_L$
$C_L$
$C_R$
LR
Insertion
2
A
–1
B
±1
C
$A_R$
$B_L$
$C_L$
$C_R$
OR
LR
Rotation
0
C
0 or 1
–1 or 0
B
A
$B_L$
$C_L$
$C_R$
$A_R$
OR


Dec
July
Feb
2
Mar
–2
Aug
–1
May
0
Apr
1
Jan
0
Nov
–1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
–1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

**In general:**


1
A
0
B
0
C
$A_L$
$C_L$
$C_R$
$B_R$
RL
Insertion
–2
A
1
B
±1
C
$A_L$
$C_L$
$C_R$
$B_R$
OR
RL
Rotation
0
C
0 or 1
A
–1 or 0
B
$A_L$
$C_L$
$C_R$
$B_R$
OR

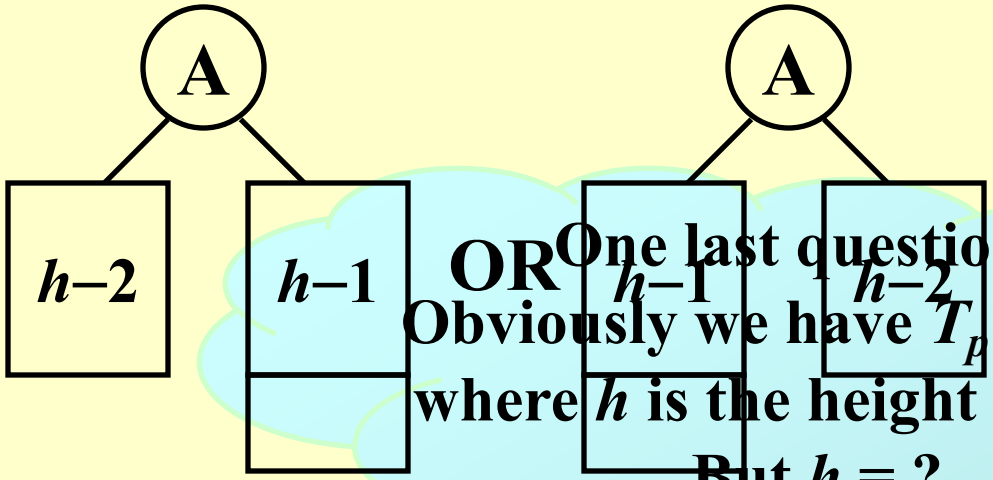
June Oct Sept




**Another option is to keep a *height* field for each node.**

**Read the declaration and functions in [1] Figures 4.42 – 4.48**

Let $n_h$ be the minimum number of nodes in a height balanced tree of height $h$. Then the tree must look like




$\Rightarrow n_h = n_{h-1} + n_{h-2} + 1$

One last question:
Obviously we have $T_p = O(h)$
where $h$ is the height of the tree.
But $h = ?$

**Fibonacci numbers:**

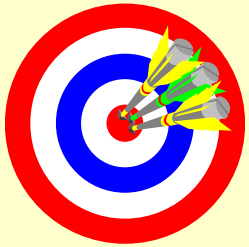
$$F_0 = 0,\ F_1 = 1,\ F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1$$

$\Rightarrow \quad n_h = F_{h+2} - 1, \quad \text{for } h \geq 0$

Fibonacci number theory gives that $F_i \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i$

$$\Rightarrow \quad n_h \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+2} - 1 \quad \Rightarrow \quad h = O(\ln n)$$

# Splay Trees

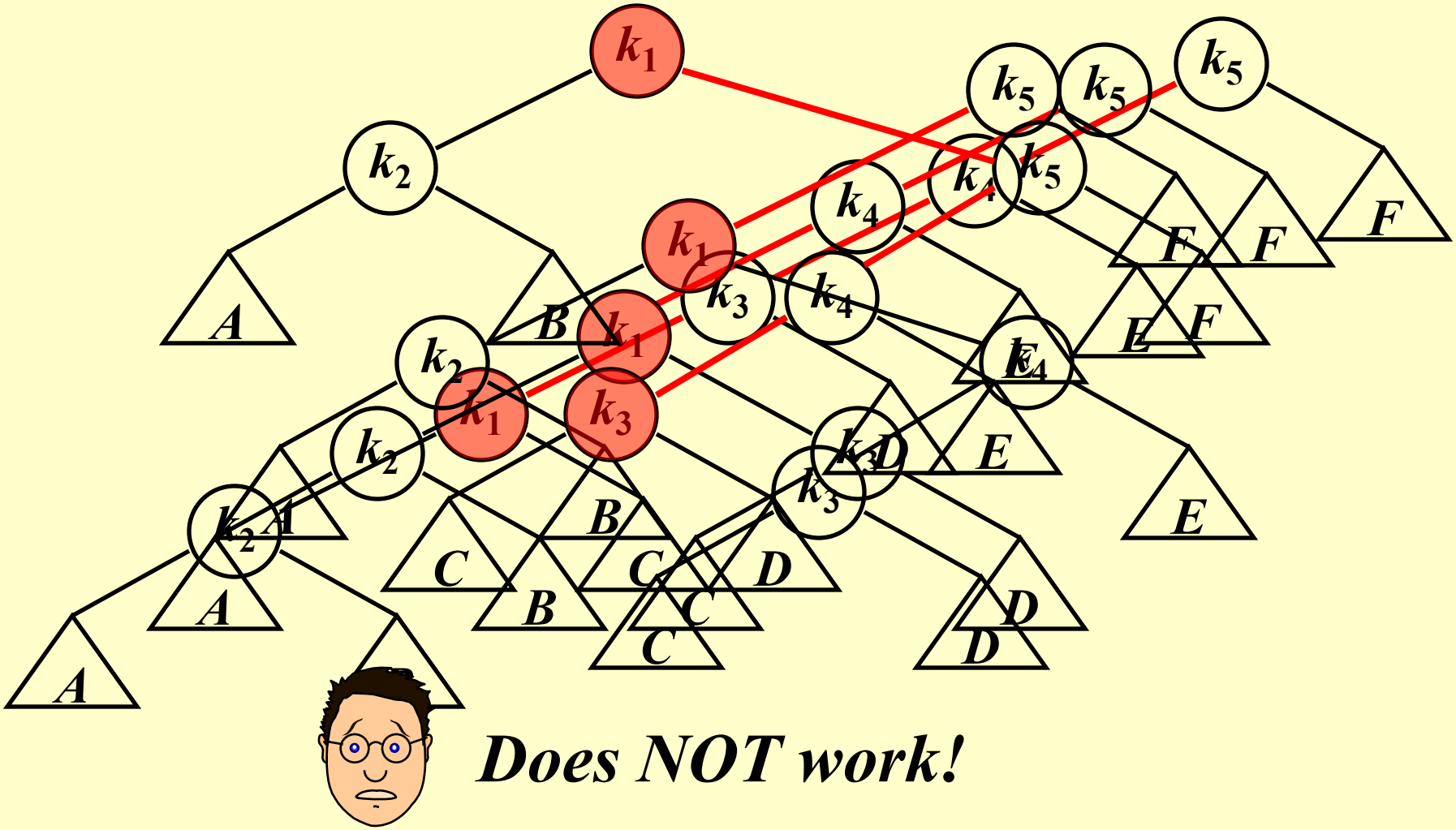


**Target :** Any $M$ consecutive tree operations starting from an empty tree take at most $O(M \log N)$ time.

Sure we can – that only means that whenever a node is accessed, it must be moved.

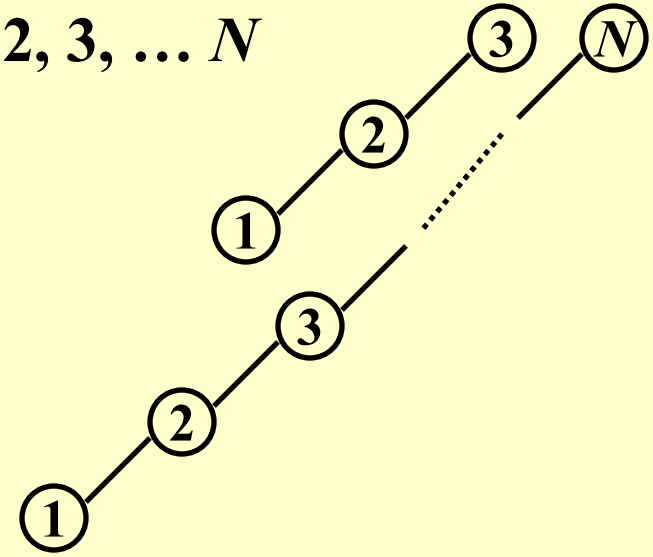
ne node takes O(*N*) time
we can keep accessing it
imes, can't we?

**Idea :** After a node is accessed, it is pushed to the root by a series of AVL tree rotations.

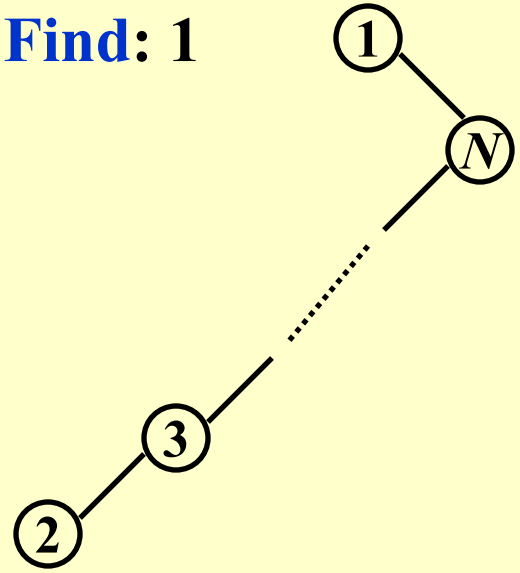

$k_1$
$k_2$
$k_5$
$k_5$
$k_5$
$k_5$
$k_4$
$k_4$
$k_1$
$k_3$
$k_4$
$k_1$
$k_2$
$k_1$
$k_3$
$k_2$
$k_3$
$k_2$
A
B
C
D
E
F
Does NOT work!

**An even worse case:**

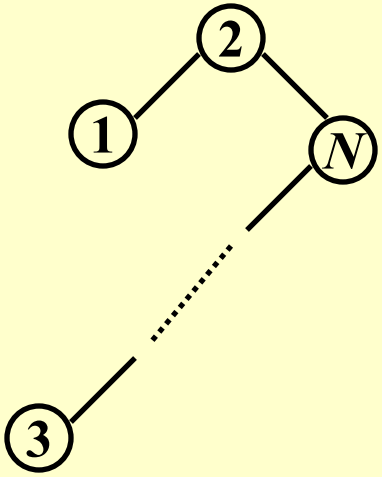
**Insert**: 1, 2, 3, … *N*



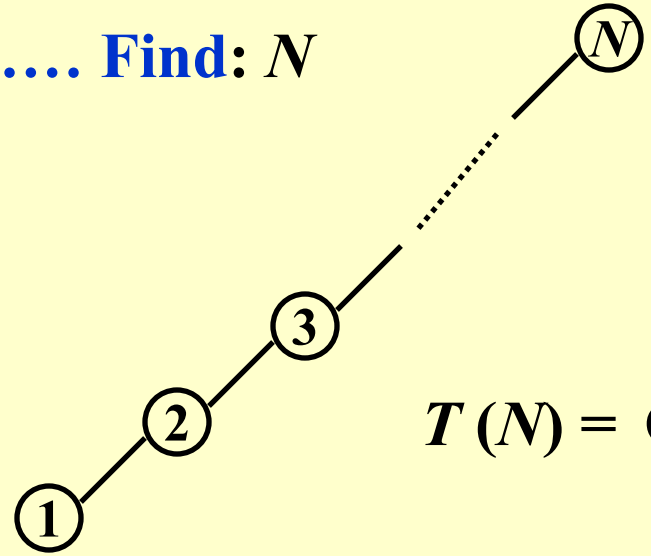

**Find**: 1




**Find**: 2




…… **Find**: *N*



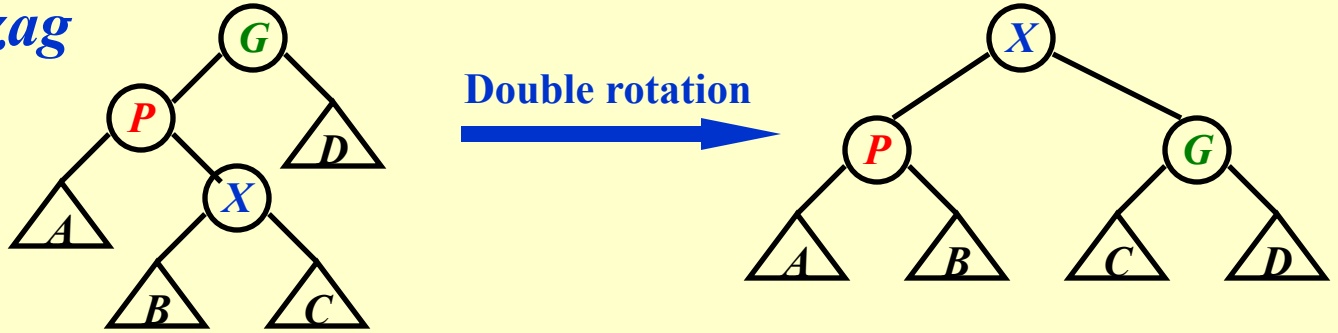

$T(N) = O(N^2)$

**Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :**

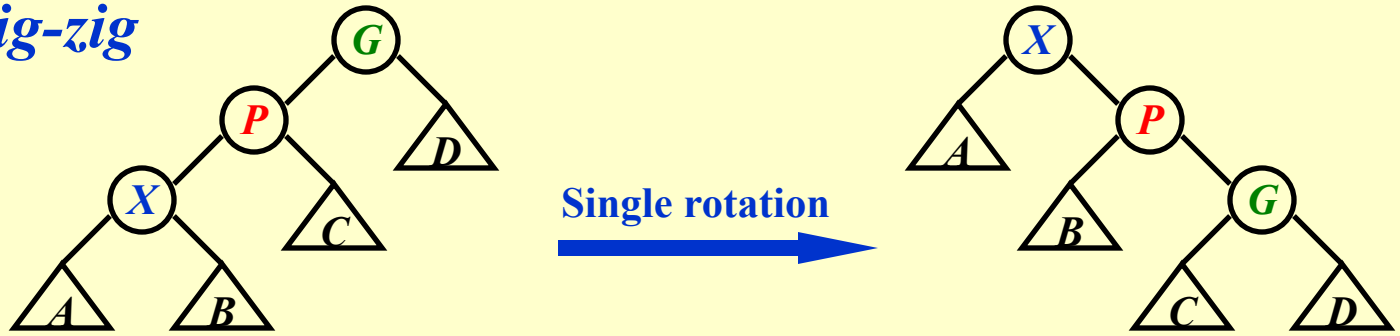
**Case 1: *P* is the root → Rotate *X* and *P***

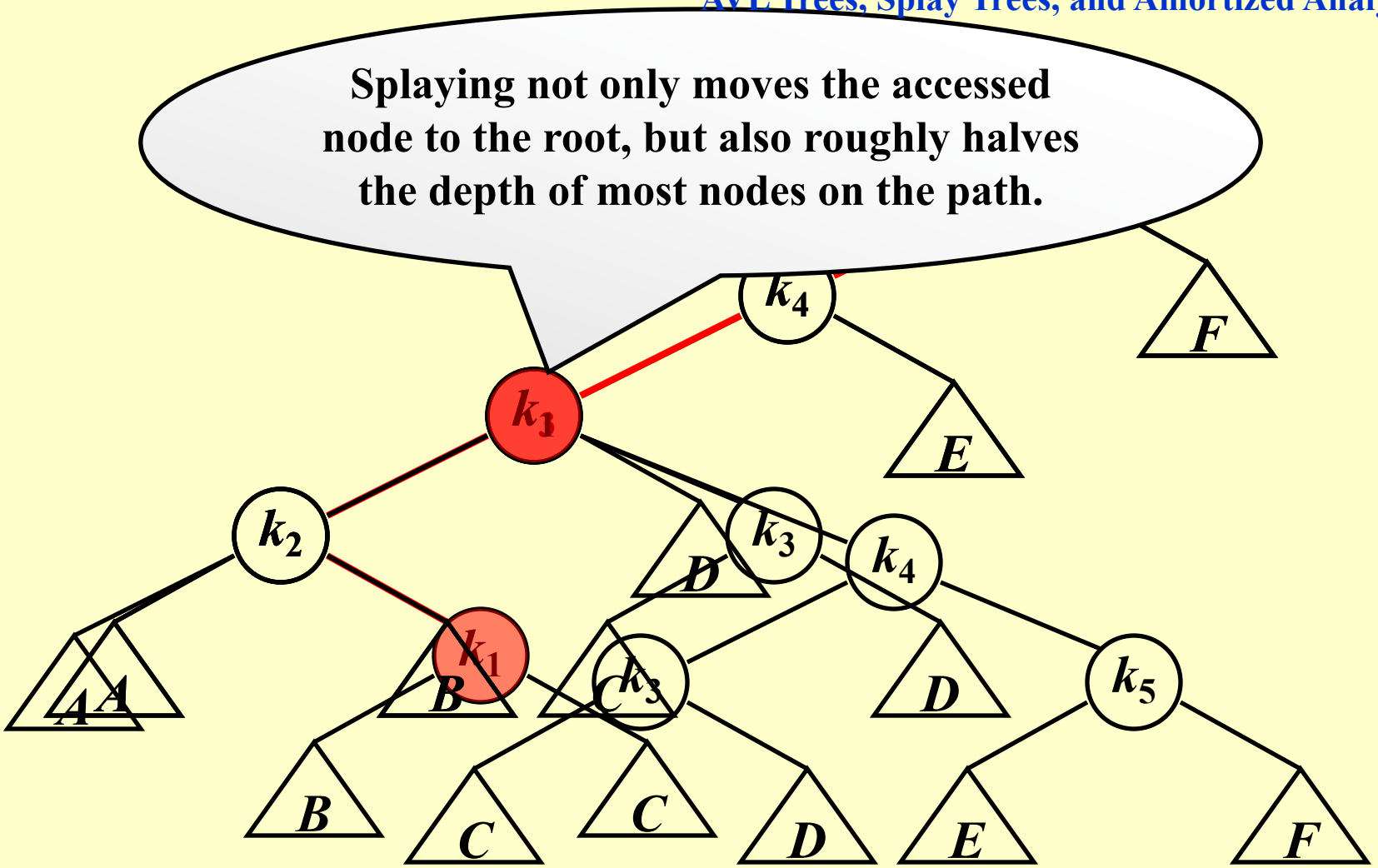
**Case 2: *P* is not the root**

***Zig-zag***

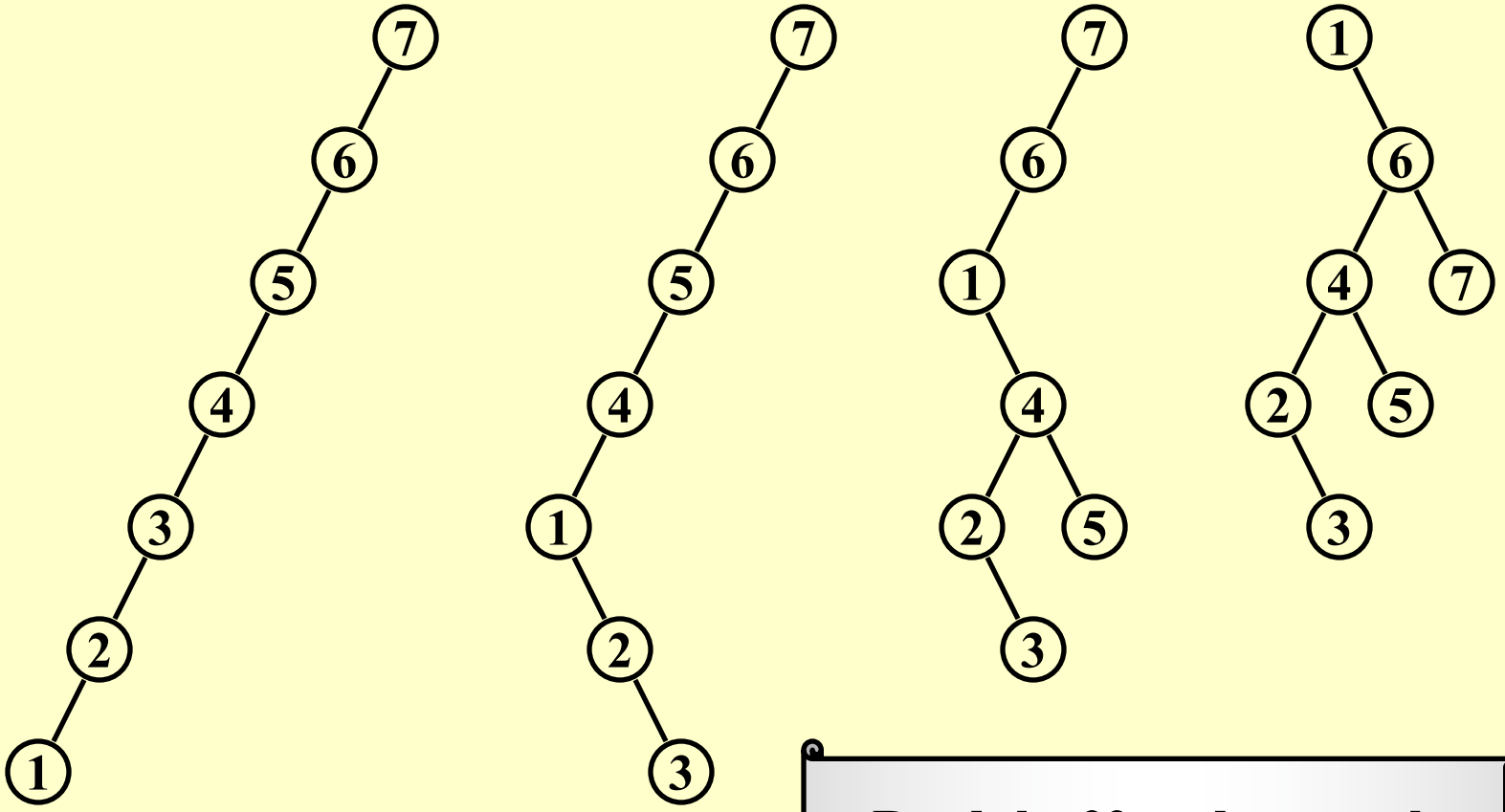



***Zig-zig***




Splaying not only moves the accessed node to the root, but also roughly halves the depth of most nodes on the path.
$k_4$
F
$k_3$
E
$k_2$
$k_3$
$k_4$
D
$k_1$
$k_3$
$k_5$
A
A
B
C
D
B
C
C
D
E
F

**Insert**: 1, 2, 3, 4, 5, 6, 7 **Find**: 1




**Read the 32-node example given in Figures 4.52 – 4.60**

## Deletions:

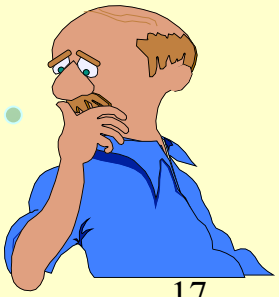
- **Step 1:** Find $X$ ;
- **Step 2:** Remove $X$ ;
- **Step 3:** FindMax ( $T_L$ ) ;
- **Step 4:** Make $T_R$ the right child of the root of $T_L$ .

$X$ will be at the root.

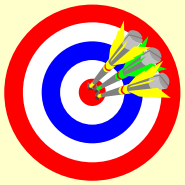
There will be two subtrees $T_L$ and $T_R$ .

The largest element will be the root of $T_L$ , and *has no right child.*

Are splay trees really better than AVL trees?



# Amortized Analysis



**Target:** Any $M$ consecutive operations take at most $O(M \log N)$ time.

-- *Amortized* time bound

worst-case bound $\geq$ amortized bound $\geq$ average-case bound

Probability is *not* involved

- Aggregate analysis
- Accounting method
- Potential method

## ☞ Aggregate analysis

**Idea :** Show that for all $n$, a sequence of $n$ operations takes *worst-case* time $T(n)$ in ...se, the average cost, ...ration is

We can pop each object from the stack *at most once* for each time we have pushed it onto the stack

*Total* = $O(n^2)$ ?

〖Example〗 Stack with ...op( int k, Stack S )

```
Algorithm {
   while ( !IsEmpty(S) && k>0 ) {
      Pop(S);
      k - -;
   } /* end while-loop */
}
```

$T = \min(\text{sizeof}(S), k)$

...Consider a seque...ce of $n$ Push, Pop, and MultiPop operations on an initially empty stack.

$$\text{sizeof}(S) \le n$$

$$T_{amortized} = O(n)/n = O(1)$$

## Accounting method


**When an operation's *amortized cost* $\hat{c}_i$ exceeds its *actual cost* $c_i$, we assign the difference to specific objects in the data structure as *credit*. Credit can help *pay* for later operations whose amortized cost is less than their actual cost.**

**Note: For all sequences of $n$ operations, we must have**

$$T_{amortized} = \frac{\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i}{n}$$

〖Example〗 Stack with MultiPop( int k, Stack S )

$c_i$ for Push: 1 ; Pop: 1 ; and MultiPop: min ( sizeof($S$), $k$ )

$\hat{c}_i$ for Push: 2 ; Pop: 0 ; and MultiPop: 0

Starting [illegible] empty [illegible] *credits* for

Push: +1 ; Po[illegible] [illegible]r each +1

sizeof($S$) ≥ 0 [illegible]

The amortized costs of the operations may *differ* from each other

$$\Rightarrow O(n) = \sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i$$

$$\Rightarrow T_{amortized} = O(n)/n = O(1)$$

## Potential method

**Idea:** Take a closer look at the *credit* --

$$\hat{c}_i - c_i = Credit_i = \Phi(D_i) - \Phi(D_{i-1})$$

*Potential function*

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left( c_i + \Phi(D_i) - \Phi(D_{i-1}) \right)$$

$$= \left( \sum_{i=1}^{n} c_i \right) + \underbrace{\Phi(D_n) - \Phi(D_0)}_{\geq 0}$$

**In general, a good potential function should always assume its minimum at the start of the sequence.**

〖Example〗 Stack with MultiPop( int k, Stack S )

$D_i$ = the stack that results after the $i$-th operation

$\Phi(D_i)$ = the number of objects in the stack $D_i$

$$\Phi(D_i) \geq 0 = \Phi(D_0)$$

**Push:** $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)+1) - sizeof(S) = 1$

$$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$$

**Pop:** $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)-1) - sizeof(S) = -1$

$$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$$

**MultiPop:** $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)-k') - sizeof(S) = -k'$

$$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k' - k' = 0$$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} O(1) = O(n) \geq \sum_{i=1}^{n} c_i \Rightarrow T_{amortized} = O(n)/n = O(1)$$

〖Example〗 Splay Trees: $T_{amortized} = O(\log N)$

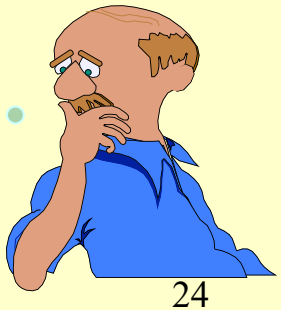
$D_i$ = the root of the resulting tree

$\Phi(D_i)$ = must increase by at most $O(\log N)$ over $n$ steps, AND will also cancel out the number of rotations (*zig*:1; *zig-zag*:2; *zig-zig*:2).

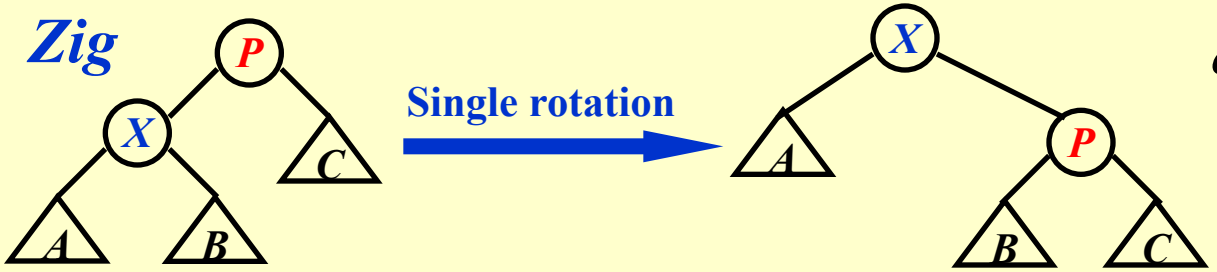
$\Phi(T) = \sum_{i \in T} \log S(i)$ where $S(i)$ is the number of descendants of $i$ ($i$ included).

*Rank of the subtree*
*≈ Height of the tree*

Why not simply use the heights of the trees?

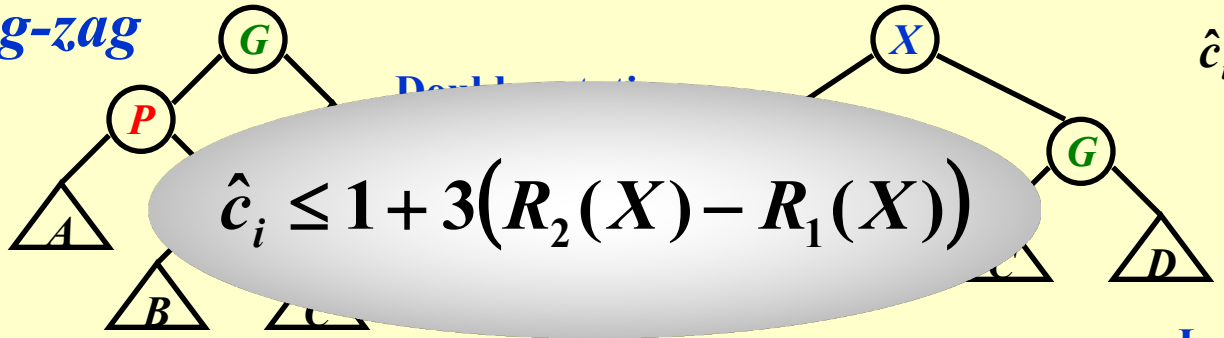


**Zig**




$$\hat{c}_i = 1 + R_2(X) - R_1(X) + R_2(P) - R_1(P) \le 1 + R_2(X) - R_1(X)$$

**Zig-zag**

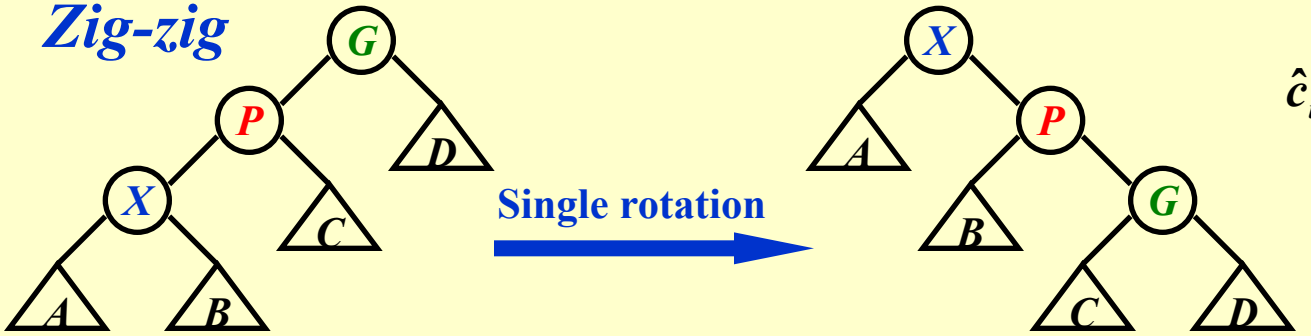



$$\hat{c}_i \le 1 + 3\left(R_2(X) - R_1(X)\right)$$

$$\hat{c}_i = 2 + R_2(X) - R_1(X) + R_2(P) - R_1(P) + R_2(G) - R_1(G) \le 2\left(R_2(X) - R_1(X)\right)$$

Lemma 11.4 on [1] p.448

**Zig-zig**




$$\hat{c}_i = 2 + R_2(X) - R_1(X) + R_2(P) - R_1(P) + R_2(G) - R_1(G) \le 3\left(R_2(X) - R_1(X)\right)$$

**【Theorem】** The amortized time to splay a tree with root $T$ at node $X$ is at most $3(R(T) - R(X)) + 1 = O(\log N)$.

# Reference:

**Data Structure and Algorithm Analysis in C (2nd Edition): Ch.4，p.106-128； Ch.11，p.447-451；*M.A.Weiss著、陈越改编，人民邮件出版社，2005***

**Introduction to Algorithms, 3rd Edition: Ch.17，p. 451-478； *Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***