Emerging Directions in Deep Learning

Manifolds - key concepts and examples

Mihai-Sorin Stupariu, 2024-2025

Introduction

Motivation - why do we need Geometric Machine Learning?

Face detection. 3D shapes and lack of flatness.

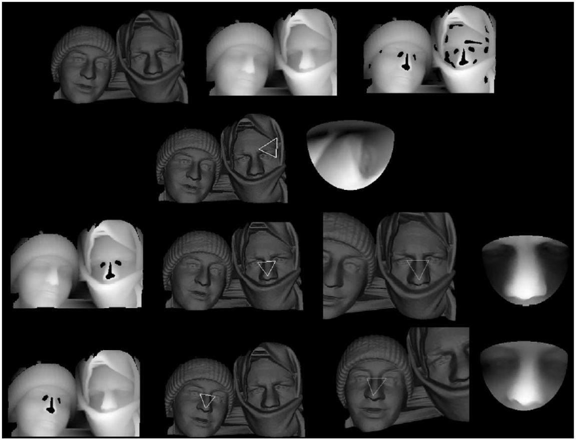

Fig. 10. An example of the system in action. Top row: input polygonal mesh, projected range image and eyes and noses candidates. Second row: a candidate face triangle and the associated range image that was classified as non-face. Third and fourth rows: the two face triangles correctly classified as faces.

[Colombo et al., 2006]

Motivation - why do we need Geometric Machine Learning?

Facial expressions. Again curvatures, this time in a ML pipeline!

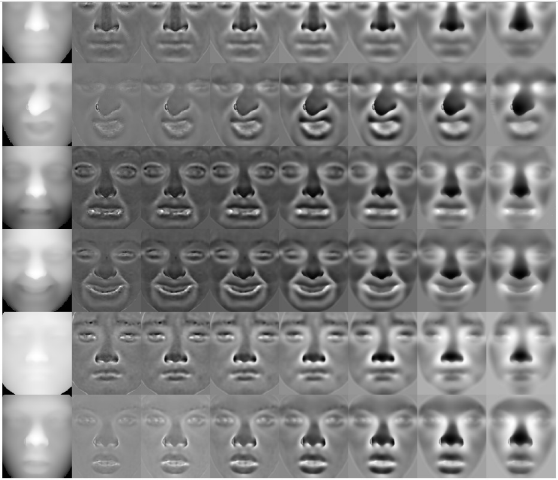

Fig. 2. Various examples of DMCMs after normalization, applied to 3D expressionnal faces. From top to bottom, expressions are Anger (AN), Disgust (DI), Fear (FE), Happinness (HA), Sadness (SA) and Surprise (SU). From left to the right, images are the original range image, then the DMCMs following sets of radiuses S_1, S_2, S_3, S_4, S_5, S_6 and S_7 according to section III-B.

[Lemaire et al., 2013]

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Facial expressions. Use convolutions or 3D irregular structures.

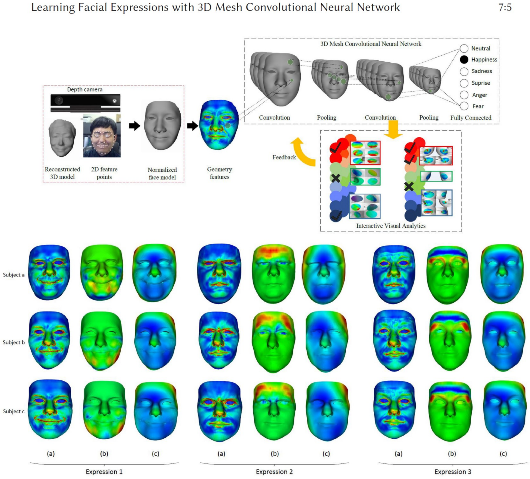

Fig. 11. The geometry signatures on the 3D face. Each row shows different expressions of the same subject.

[Jin et al., 2018]

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Facial expressions. Use a manifold embedded in a high dimensional feature space.

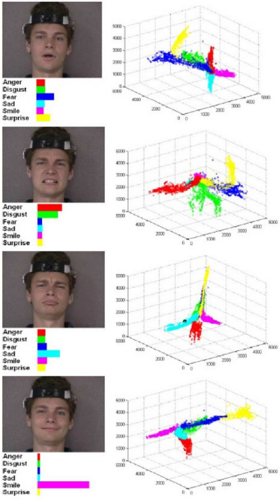
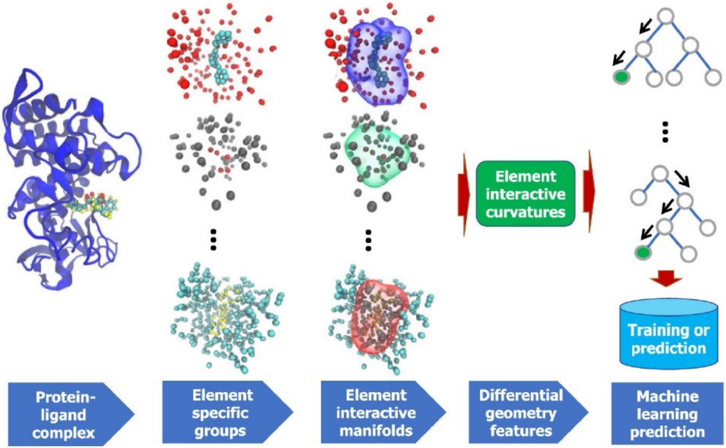


Fig. 6. Facial expression recognition result with manifold visualization.

[Chang et al., 2006]

Motivation - why do we need Geometric Machine Learning?

Molecular Biology. DG-GL. Again curvatures and other feature embedding.



[Nguyen & Wei, 2019]

Motivation - why do we need Geometric Machine Learning?

Pedestrian detection. The manifold of interest is known - space of matrices.

[Tuzel et al., 2008]

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- A **(smooth) manifold of dimension** n (Romanian=*varietate diferențiabilă n-dimensională*): topological space M that 'locally' (around each point) is homeomorphic to $\mathbb{R}^n$ (via maps called **charts**) and such that the overlapping charts induce smooth functions between open sets of $\mathbb{R}^n$.

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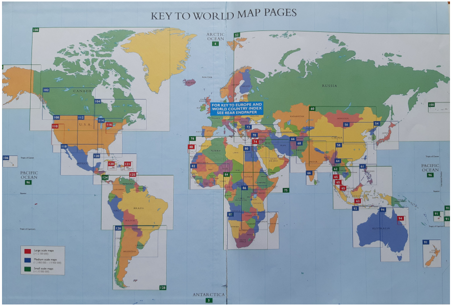
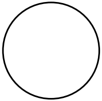
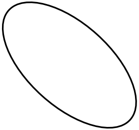


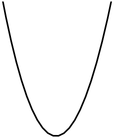
Examples (I) Smooth curves



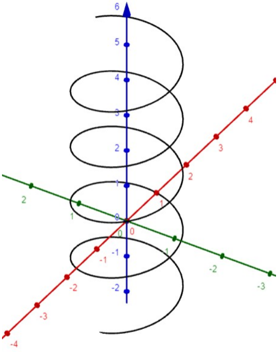
Circle



Ellipse

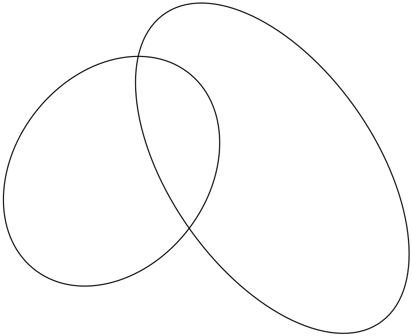


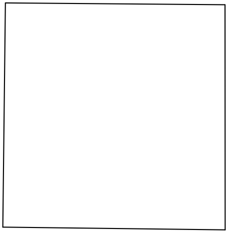
Parabola



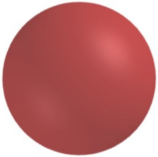
Helix

Counterexamples



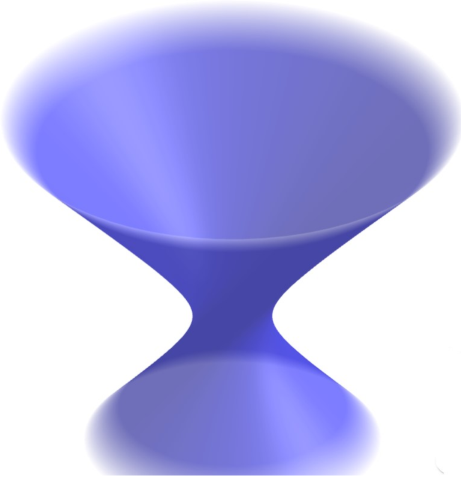


Examples (II) Smooth surfaces



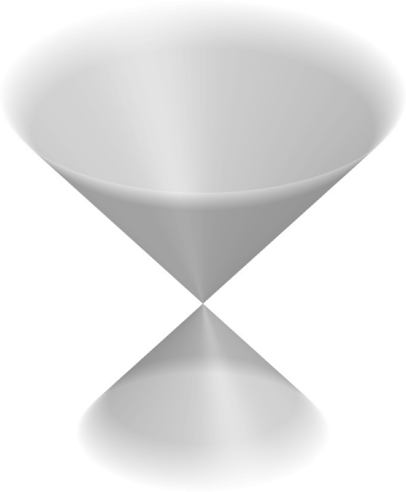
Sphere

Cylinder



Hyperboloid

Is this a smooth surface?



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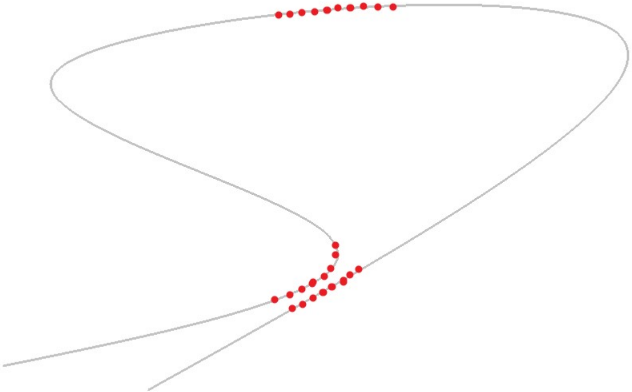
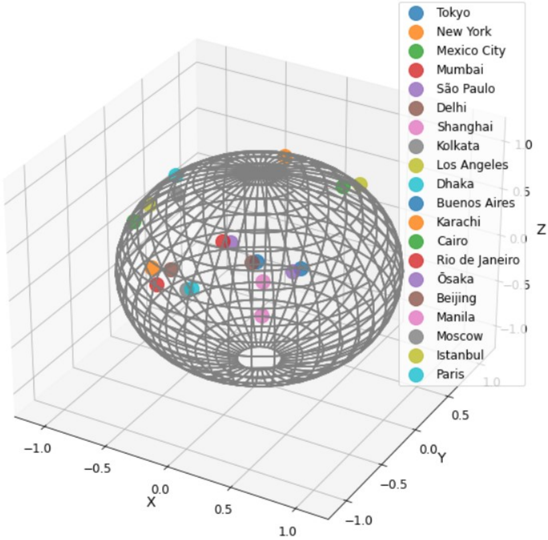


... or other practical problems



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- If a manifold is embedded in $\mathbb{R}^d$, although one may consider vectors with origin at a given point, the end point of the vector might not be situated on the manifold.

Tangent vectors, tangent space

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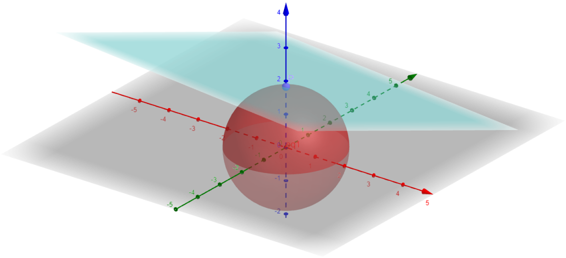
- Examples: (i) $\mathbb{R}^d$ (in this case for each point the tangent space is $\mathbb{R}^d$ itself), (ii) the circle, (iii) the sphere



A point P on the sphere and the tangent plane to the sphere at P.

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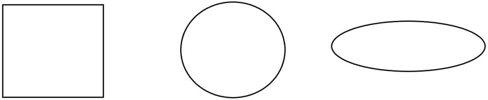

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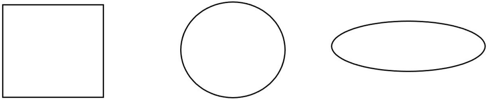
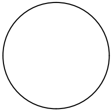
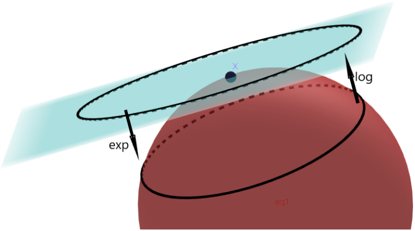


- Metrics on a circle (sphere): the chordal distance and the arc (geodesic) distance



Distance measurement on manifolds: geodesics, exponential and logarithm

- Given a Riemannian manifold (M, g), take $x \in M$ arbitrary and $v \in T_x M$. There exists a unique **geodesic** γ_v such that $\gamma_v(0) = x$ and $\gamma_v'(0) = v$. There exists a map, called **exponential map** (it is defined locally!), $\exp_x : T_x M \to M$, such that $\exp_x(v) = \gamma_v(1)$. Its inverse is called **logarithm**. A geodesic is, locally, a path of shortest length.



The exponential and the logarithm map are locally defined and they are inverse to each other.

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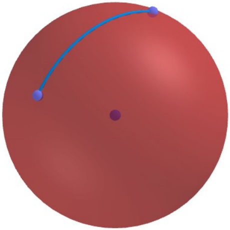

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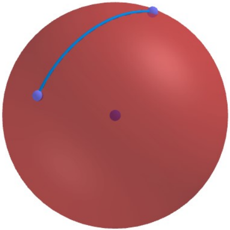
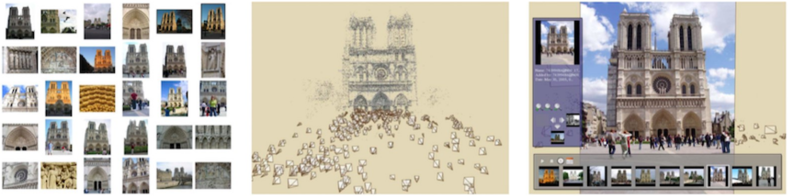


- Given two points on a cylinder: how can one construct a geodesic?

The groups $\mathrm{SO}(3)$, $\mathrm{E}(3)$ and movements

Structure from motion. (Relative) rotations can be modeled by using the group $\mathrm{SO}(3)$ (rotations in the 3-dimensional space), which also has a (natural) structure of 3-dimensional manifold. Could you describe the group $\mathrm{SO}(2)$ (plane rotations)?



(a) (b) (c)

Source: [Özyeşil et al., 2017, cf. Snavely et al., 2006], see also Hartley

SPD-matrices

SPD (Symmetric positive definite) matrices

$$SPD(n) = \{S \in \mathcal{M}_n(\mathbb{R}) | S^T = S, \ \forall x \neq 0, \ x^T S x > 0\}.$$

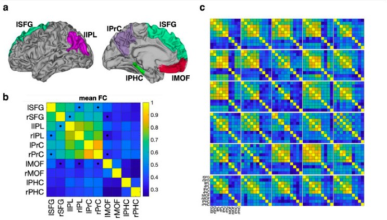


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Fig. 13. The brain regions used in the evaluation of the default mode network. a) The brain network includes the superior frontal gyrus (SFG), medial orbitofrontal cortex (MOF), parahippocampus (PHC), precuneus (PrC) and inferior parietal lobe (IPL) according to the Desikan-Killiany atlas (Desikan et al., 2006). The brain regions in the left hemisphere are presented. The first letters "l" and "r" in each region's name indicate left and right hemisphere. b) The geometric average of a group of 734 functional connectivity matrices (FC) is presented. All square pixels in the connectivity matrix indicate edges among the brain regions. Dots in (b) indicate edges that show 5% differences between Euclidean average and geometric average of group connectivity matrices. c) Functional connectivity matrices of 30 exemplary subjects are displayed.

Source: [You and Park, 2021]

Reference

[Miolane et al., 2020] and the references therein. This work is the reference paper for the `geomstats` package.