Spectral Deferred Correction Methods for Ordinary Differential Equations
Final Report

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1 Introduction and Problem Statement

Numerical solution of ordinary differential equations requires methods that are accurate, stable, and computationally efficient. Given an initial-value problem:

$$\varphi'(t) = F(t, \varphi(t)), \qquad \varphi(a) = \varphi_a,$$

the goal is to approximate the solution on an interval while controlling numerical error.

Classical high-order solvers may face stability limitations or high computational cost, especially when very high accuracy is needed. Deferred correction methods aim to improve a low-order approximation by repeatedly correcting an error equation. However, classical deferred correction suffers from the instability of interpolation at equispaced points and from numerical differentiation errors.

Spectral Deferred Correction (SDC) addresses these issues by working with the Picard integral equation and using stable Gauss–Legendre nodes for spectral integration, providing a route to arbitrarily high-order accuracy.

2 Method Explanation

SDC is a high-order time integration framework that improves an approximate numerical solution through a sequence of correction sweeps. The method is fundamentally built on the Picard formulation of an ODE, combined with high-accuracy spectral quadrature using Gauss–Legendre nodes. By repeatedly correcting the numerical solution through a low-order solver while relying on high-order quadrature for the integral, SDC achieves flexible and arbitrarily high-order accuracy.

2.1 Picard Integral Equation

Consider the initial-value problem

$$\varphi'(t) = F(t, \varphi(t)), \qquad \varphi(a) = \varphi_a.$$

Integrating from a to t gives the Picard representation

$$\varphi(t) = \varphi_a + \int_a^t F(s, \varphi(s))\, ds.$$

SDC is formulated on this integral form rather than on the differential equation itself because integration is naturally a smoothing operation and does not amplify local perturbations in the same way differentiation does. The Picard representation also turns the solution of the ODE

into a fixed-point problem, making it well suited for iterative refinement procedures. In this perspective, SDC may be interpreted as applying a high-order fixed-point scheme to the Picard equation, where each correction step increases the overall accuracy of the numerical solution.

2.2 Residual and Error Equation

Let $\varphi_0(t)$ denote an approximate solution. Substituting φ_0 into the Picard equation produces the residual

$$\varepsilon(t) = \varphi_a + \int_a^t F(s, \varphi_0(s))\, ds - \varphi_0(t),$$

which describes how far φ_0 is from satisfying the integral equation. Define the true error by $\delta(t) = \varphi(t) - \varphi_0(t)$. Substituting $\varphi = \varphi_0 + \delta$ into the Picard equation yields the nonlinear error equation

$$\delta(t) = \int_a^t [F(s, \varphi_0(s) + \delta(s)) - F(s, \varphi_0(s))]\, ds + \varepsilon(t).$$

Introducing

$$G(t, \delta) = F(t, \varphi_0(t) + \delta) - F(t, \varphi_0(t)),$$

the above becomes

$$\delta(t) = \int_a^t G(s, \delta(s))\, ds + \varepsilon(t).$$

SDC proceeds by constructing discrete approximations of this equation and computing successive updates of the form $\varphi \leftarrow \varphi + \delta$. Because the Picard operator is contractive for many classes of problems, this iterative correction process gradually reduces the approximation error.

2.3 Spectral Quadrature with Gauss–Legendre Nodes

To approximate the Picard integral, SDC employs Gauss–Legendre nodes $t_1, \ldots, t_m$ on the interval $[a, b]$, together with the corresponding spectral integration matrix Q_m. The matrix approximates the integral

$$\int_a^{t_i} f(s)\, ds \approx \sum_{j=1}^{m} (Q_m)_{ij}\, f(t_j),$$

where $(Q_m)_{ij}$ is the integral of the j-th Lagrange basis function over $[a, t_i]$. This spectral quadrature achieves very high accuracy because Gauss–Legendre rules integrate polynomials up to degree $2m - 1$ exactly and provide excellent stability properties. The use of a spectral integration matrix is essential in SDC because it ensures that the quadrature error is sufficiently small, allowing the correction sweeps to focus on reducing the discretization error of the differential operator rather than being dominated by inaccuracies in the numerical integral.

2.4 Explicit and Implicit Euler Correction

Although the quadrature used in SDC is high-order, the correction equation is solved using a deliberately simple low-order method. The philosophy is to apply a basic solver repeatedly so that each sweep moves the solution closer to the true fixed point of the Picard operator. For non-stiff problems, explicit Euler is commonly used. The discrete correction step has the form

$$\delta_{i+1} \approx \delta_i + h_i\, G(t_i, \delta_i) + \text{residual correction},$$

where $h_i = t_{i+1} - t_i$. For stiff ODEs, implicit Euler is preferred, leading to a correction of the form

$$\delta_{i+1} = \delta_i + h_i\, G(t_{i+1}, \delta_{i+1}) + \text{residual correction}.$$

In each sweep, the updated error δ is added to the current approximation, producing a new, more accurate iterate.

2.5 Overall Structure of SDC

An SDC method proceeds by first selecting Gauss–Legendre nodes and constructing the spectral integration matrix. An initial approximation is obtained using a low-order solver. At each correction sweep, the residual corresponding to the Picard equation is computed using the spectral quadrature. The discrete error equation is then solved with an explicit or implicit Euler method, and the resulting correction is added to the approximate solution. Repeating this process for K sweeps yields a method whose order is approximately

$$\min(m,\, K+1),$$

where m is the number of quadrature nodes. The accuracy can therefore be increased either by refining the quadrature or by performing more correction sweeps, giving SDC a flexible structure that can achieve very high-order time integration while maintaining numerical stability.

3 Simple Implementation / Experiment

To illustrate how spectral integration is incorporated into a deferred correction framework, we consider the simple test equation

$$y'(t) = -y(t), \qquad y(0) = 1,$$

whose exact solution is $y(t) = e^{-t}$.

The implementation begins by constructing m Gauss–Legendre nodes on the interval $[0, 1]$ together with the corresponding spectral integration matrix Q. The matrix is formed by integrating the Lagrange basis polynomials associated with the Gauss–Legendre points, allowing the approximation of integrals of the form

$$\int_0^{t_i} f(s)\, ds \approx \sum_{j=1}^{m} Q_{ij} f(t_j)$$

with very high accuracy. Once the matrix is available, the Picard equation

$$y(t) = 1 + \int_0^t F(s, y(s))\, ds, \qquad F(t, y) = -y,$$

is discretized at the nodes, leading to the system

$$y_i \approx 1 + \sum_{j=1}^{m} Q_{ij} F(t_j, y_j).$$

This formulation naturally yields a fixed-point iteration. Starting from an initial guess, which in this demonstration is simply the constant vector $y_i^{(0)} = 1$ on all nodes, the next iterate is obtained by evaluating the right-hand side using the current approximation. Specifically, the vector iteration

$$y^{(k+1)} = 1 + Q\, F(t, y^{(k)})$$

is applied repeatedly, and each sweep corresponds to one layer of correction in the deferred correction framework. Although this procedure does not explicitly solve the differential equation in the traditional sense, the successive iterations converge rapidly toward the exact nodal values of e^{-t} due to the accuracy of the spectral quadrature and the contractive nature of the Picard operator for this particular problem.

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Gauss-Legendre nodes:
[0.04691008 0.23076534 0.5        0.76923466 0.95308992]

Spectral integration matrix Q:
[[ 0.072384 -0.018979  0.010931 -0.005435  0.001543]
 [ 0.128191  0.10092  -0.024499  0.010296 -0.002764]
 [ 0.113627  0.260712  0.142222 -0.021398  0.004837]
 [ 0.121228  0.229018  0.308943  0.138394 -0.009727]
 [ 0.11692   0.24475   0.273514  0.258294  0.046079]]

SDC approximation: [1.       0.858199 0.640916 0.482595 0.415081]
Exact solution    : [0.954173 0.793926 0.606531 0.463368 0.385548]
Max error         : 0.06427370905928587
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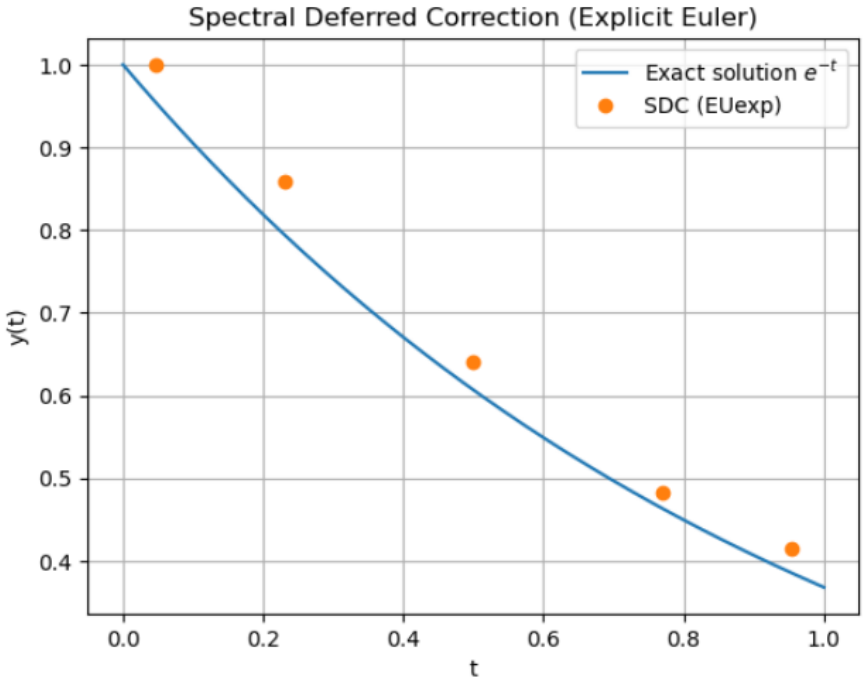

4 Summary and Conclusion

Spectral Deferred Correction provides a pathway to arbitrarily high-order accuracy by repeatedly correcting the solution of the Picard integral equation using stable spectral integration on Gauss–Legendre nodes.

Key strengths include:

- high-order accuracy controlled by nodes m and sweeps K,
- stable interpolation and quadrature,
- applicability to both stiff and non-stiff problems (depending on the Euler variant),
- modular structure suitable for adaptive implementations.

Limitations include:

- dense spectral matrices leading to $O(m^2)$ cost,
- need for nonlinear solves in implicit variants,
- stability properties that may vary with parameter choices.

Possible extensions involve alternative node sets, higher-order base solvers, or enhanced adaptive strategies.