1. for $y'(t) = f(t, y(t))$

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))ds \quad \text{by 梯形}$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, \underbrace{y_{n+1}}_{\to\, y_n + hf(t_n, y_n)})\right)$$

by hint notice that

$$h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h)$$

$$= y(t_{n+1}) - y(t_n) - \frac{h}{2}\left(f(t_n, y(t_n)) + f(t_{n+1}, y_n(t_n) + hf(t_n, y(t_n)))\right)$$

$$= \underbrace{\int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}\left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right)}_{=E_1}$$

$g(t)$

$$+ \underbrace{\frac{h}{2}\left[f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, y(t_n) + hf(t_n, y(t_n)))\right]}_{=E_2}$$

$$E_1 = \int_{t_n}^{t_{n+1}} g(s)ds - \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right) = -\frac{h^3}{12} g''(\xi_n) \quad \text{(梯形誤差)}$$

$$= O(h^3)$$

$E_2 \Rightarrow$ by forward Euler method

$$y(t_{n+1}) = y(t_n) + h\underline{y'(t_n)} + \underline{\frac{h^2}{2} y''(\eta_n)}$$

$$y(t_{n+1}) - y(t_n) - h\underline{f(t_n, y(t_n))} = \underline{O(h^2)}$$

$$|E_2| = \frac{h}{2} O(h^2) = O(h^3)$$

$$\therefore h\tau_{n+1} = E_1 + E_2 = O(h^3) \Rightarrow \tau_{n+1} = O(h^2) \#$$

2. for $y'(t) = f(t, y(t))$

by Crank-Nicolsen

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

by 梯形誤差 for $[t_n, t_{n+1}]$

$$\int_{t_n}^{t_{n+1}} g(s)\,ds = \frac{h}{2}\left[g(t_n) + g(t_{n+1})\right] - \frac{h^3}{12} g''(\xi_n), \quad \xi_n \in (t_n, t_{n+1})$$

$$\Rightarrow y(t_{n+1}) - y(t_n) = \frac{h}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right] - \frac{h^3}{12} g''(\xi_n)$$

$$\Rightarrow \frac{y(t_{n+1}) - y(t_n)}{h} = \frac{1}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right] - \frac{h^2}{12} g''(\xi_n)$$

$$\tau_{n+1} = \frac{y(t_{n+1}) - y(t_n)}{h} - \frac{1}{2}\left[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right] = -\frac{h^2}{12} g''(\xi_n) = O(h^2) \#$$