#5

by hint $\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\Big[\underline{\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2 dt} - \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)^2 dt\Big]$

by $(\int ab)^2 \le \int a^2 \int b^2$

$\Rightarrow \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2 dt \le \left(\int_{x_j-h}^{x_j} |u''(t)|^2 dt\right)\left(\underline{\int_{x_j-h}^{x_j} (x_j-h-t)^4 dt}\right) \le \underline{\frac{h^5}{5}\int_{x_j-h}^{x_j} |u''(t)|^2 dt}$

$\Rightarrow$ let $x_j-h-t = y \quad x_j-h-y=t \quad -dy=dt$

$\Rightarrow \int_0^{-h} y^4 \cdot dy = \left[\frac{y^5}{5}\right]_0^h = \underline{\frac{h^5}{5}}$

by $(a+b+c)^2 \le 3(a^2+b^2+c^2)$

$\therefore \|\tau_h\|_h^2 = h\sum_{j=1}^{n-1}\tau_h(x_j)^2 \le 3h\sum_{j=1}^{n-1}[u''(x_j)]^2 + \frac{3h}{h^4}\sum_{j=1}^{n-1}\left(\frac{h^5}{5}\int_{x_j-h}^{x_j}|u''(t)|^2 dt + \frac{h^5}{5}\int_{x_j}^{x_j+h}|u''(t)|^2 dt\right)$

$\le \underbrace{3h\sum_{j=1}^{n-1}[u''(x_j)]^2}_{=\|f\|_h^2} + \frac{3}{5}h\sum_{j=1}^{n-1}\left(\underline{h\int_{x_j-h}^{x_j}|u''(t)|^2 dt} + \underline{h\int_{x_j}^{x_j+h}|u''(t)|^2 dt}\right)$

$\hookrightarrow \le \int_0^1 |u''(t)|^2 dt$

$\le 3\|f\|_h^2 + \frac{6}{5}\int_0^1 |u''(t)|^2 dt = 3\|f\|_h^2 + \frac{6}{5}\|f\|_{L^2(0,1)}^2 \le 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right)$

#7

$T_h g(x_j) = \sum_{k=1}^{n-1} \overset{=1}{\overline{g(x_k)}}\, G^k(x_j) \rightarrow h\,G(x_j, x_k)\begin{cases} s(1-x_j) & 0\le s\le x_j \\ x_j(1-s) & x_j\le s\le 1\end{cases} \begin{cases} x_k(1-x_j), & k\le j \\ x_j(1-x_k), & j<k\end{cases}$

$= h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right] \quad (x_k = kh,\ x_j = jh)$

$= h\left[(1-x_j)h\sum_{k=1}^{j} k + x_j\sum_{k=j+1}^{n-1}(1-x_k)\right] \quad h\sum_{k=j+1}^{n-1} k$

$= h\left[(1-x_j)h\frac{(1+j)j}{2} + x_j\left((n-1-j) - h\left(\frac{(n-1)n}{2} - \frac{j(j+1)}{2}\right)\right)\right] \quad h=\frac{1}{n},\ x_j=\frac{j}{n}$

$= \frac{1}{2n^3}\left[j(j+1)(n-j) + j(2n(n-1-j) - (n-1)n + j(j+1))\right]$

$= \frac{1}{2n^3}\left[j(j+1)n + j(2n(n-1-j) - (n-1)n)\right]$

$= \frac{1}{2n^3}\, jn(j+1) + jn(2(n-1)-j) - (n-1))$

$= \frac{1}{2n^3}\, jn(-j+n)$

$= \frac{j}{n}\cdot\frac{n-j}{2n} = \frac{1}{2}x_j(1-x_j)$

#8.

Young's Inequality: $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, $\forall a,b\in\mathbb{R}\ \forall \varepsilon>0$

$\because \left(\sqrt{\varepsilon}a - \frac{b}{2\sqrt{\varepsilon}}\right)^2 \ge 0$

$\Rightarrow \varepsilon a^2 - ab + \frac{b^2}{4\varepsilon} \ge 0 \Rightarrow \varepsilon a^2 + \frac{1}{4\varepsilon}b^2 \ge ab$

#9 Prove $\|V_h\|_h \le \|V_h\|_{h,\infty}$

$\forall\, j = 1 \cdots n-1 \quad |V_j| \le \max_{1\le k\le n-1} |V_k| = \|V_h\|_{h,\infty}$

$\therefore V_j^2 \le \|V_h\|_{h,\infty}^2$

$\Rightarrow \sum_{j=1}^{n-1} V_j^2 \le \sum_{j=1}^{n-1} \|V_h\|_{h,\infty}^2 = (n-1)\|V_h\|_{h,\infty}^2 \quad (\times h)$

$\|V_h\|_h^2 = h\sum_{j=1}^{n-1} V_j^2 \le \underbrace{h(n-1)}_{<1} \|V_h\|_{h,\infty}^2 \quad \left(h = \frac{1}{n} \therefore h(n-1) = \frac{n-1}{n} < 1 \right)$

$\Rightarrow \|V_h\|_h^2 \le \|V_h\|_{h,\infty}^2$

$\Rightarrow \|V_h\|_h \le \|V_h\|_{h,\infty}$

#11

by 12.9. $(L_h W_h)(x_j) = -\frac{W_{j+1} - 2W_j + W_{j-1}}{h^2}$ for $Lu = -u^{(ii)}$

let $z_j = (L_h W)_j = -\frac{W_{j+1} - 2W_j + W_{j-1}}{h^2}$

$(L_h z)_j = -\frac{z_{j+1} - 2z_j + z_{j-1}}{h^2}$

$= -\frac{1}{h^2}\left\{ -\frac{1}{h^2}\left[(W_{j+2} - 2W_{j+1} - W_j) - 2(W_{j+1} - 2W_j + W_{j-1}) + (W_j - 2W_{j-1} + W_{j-2}) \right] \right\}$

$= \frac{1}{h^4}\left[W_{j+2} - 4W_{j+1} - 6W_j - 4W_{j-1} + W_{j-2} \right]$

$\therefore (L_h^{(4)} w)_j := -(L_h^2 W)_j = -\frac{1}{h^4}\left[W_{j+2} - 4W_{j+1} - 6W_j - 4W_{j-1} + W_{j-2} \right] \quad j = 2 \cdots n-2$