#5 $W_{n+1}(x)=\prod_{j=0}^{n}(x-x_j)=(x-x_0)\cdots(x-x_j)$ for $N=\frac{n}{2}$ 對稱原點

$\Rightarrow x_0=-Nh \qquad x_{N-1}=-h \qquad x_{n-1}=(N-1)h$

$x_1=-(N-1)h \qquad x_N=0 \qquad x_n=Nh$

$x_{N+1}=h$

$\Rightarrow W_{n+1}(x)=(x+Nh)(x+(N-1)h)\cdots(x+h)x(x-h)\cdots(x-(N-1)h)(x-Nh)$

Now let $x=rh$

$W_{n+1}(rh)=\prod_{j=-N}^{N}(rh-jh)=\underline{h^{2N+1}\prod_{j=-N}^{N}(r-j)}$

$\because N-1<r<N$ ① when $r\to N$, $\underline{\prod_{j=-N}^{N}(r-j)}\approx\underline{\prod_{j=-N}^{N-2}(N-j)}\cdot(r-(N-1))\cdot(r-N)$

$\approx\underline{\prod_{k=2}^{2N}k=(2N)!}\cdot(r-(N-1))\cdot(r-N)$

② when $r\to N-1$, $\underline{\prod_{j=-N}^{N}(r-j)}\approx\underline{\prod_{j=-N}^{N-2}((N-1)-j)}\cdot(r-(N-1))\cdot(r-N)$

$\approx\underline{\prod_{k=2}^{2N-1}k=(2N-1)!}\cdot(r-(N-1))\cdot(r-N)$

$\Rightarrow (2N-1)!\,h^{2N+1}\underbrace{(r-(N-1))}_{h^2}|r-N|\le|W_{n+1}(x)|\le(2N)!\,h^{2N+1}(r-(N-1))|r-N|$

$=(2N-1)!\,h^{2N-1}|(x-x_{n-1})(x-x_n)|\le|W_{n+1}(x)|\le(2N)!\,h^{2N-1}|(x-x_{n-1})(x-x_n)|\quad\because N=\frac{n}{2}$

$\therefore\Rightarrow(n-1)!\,h^{n-1}|(x-x_{n-1})(x-x_n)|\le|W_{n+1}(x)|\le n!\,h^{n-1}|(x-x_{n-1})(x-x_n)|$

#6 for $\left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right|=\left|\prod_{j=-N}^{N}\frac{x+h-jh}{x-jh}\right|=\left|\prod_{j=-N}^{N}\frac{x-(j-1)h}{x-jh}\right|=\left|\frac{x-(-N-1)h}{x-(-N)h}\cdots\frac{x-(N-1)h}{x-Nh}\right|=\left|\frac{x+(N+1)h}{x-Nh}\right|$

$\because x_N=Nh=1\Rightarrow\left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right|=\left|\frac{x+(N+1)h}{x-1}\right|\underset{x<1}{=}\frac{x+(N+1)h}{1-x}$

$\because x+(N+1)h-(1-x)\underset{x>0}{=}2x+(N+1)h-1=2x-1+\underbrace{(\tfrac{n}{2}+1)\cdot\tfrac{2}{n}}_{>1}>0\quad\therefore|W_{n+1}(x+h)|>|W_{n+1}(x)|$ is increasing

$\therefore|W_{n+1}|$ is maximum

#8 Consider $Hf(x)=\sum_{j=0}^{n}a_j(x-x_0)^j$

$(Hf(x))^{(k)}(x)=\sum_{j=k}^{n}a_j\cdot j\cdot(j-1)\cdots(j-k+1)(x-x_0)^{j-k}$

when $x=x_0$, $(x-x_0)^{j-k}=0$ when $j>k$

$\therefore(Hf)^{(k)}(x_0)=a_k\cdot k\cdot(k-1)\cdots1=a_k\cdot k!$

$\therefore(Hf)^{(k)}(x_0)=f^{(k)}(x_0)\Rightarrow a_k\cdot k!=f^{(k)}(x_0)\Rightarrow a_k=\frac{f^{(k)}(x_0)}{k!}$

$\therefore Hf(x)=\sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$

#2 for $w_i = \frac{1}{\omega'(x_i)}$ and $\omega_{n+1}(x) = \frac{1}{2^{n-1}}(x^2-1)U_{n-1}(x)$

on $x_i = \cos\left(\frac{i}{n}\pi\right)$ $\omega'(x_i) = \frac{(-1)^i n}{2^{n-1} d_i}$, $d_i = \begin{cases} \frac{1}{2} & i=0 \text{ or } i=n \\ 1 & \text{other} \end{cases}$ rescaling

$\therefore$ $W_i = \frac{2^{n-1} d_i}{(-1)^i n} \propto \frac{d_i}{(-1)^i}$ $\begin{cases} \text{(i)} \quad i=0 & W_0 \propto \frac{\frac{1}{2}}{(-1)^0} \Rightarrow \frac{1}{2} \\ \text{(ii)} \quad i=1\cdots n-1 & W_i \propto \frac{1}{(-1)^i} \Rightarrow (-1)^i \\ \text{(iii)} \quad i=n & W_n \propto \frac{\frac{1}{2}}{(-1)^n} \Rightarrow \frac{(-1)^n}{2} \end{cases}$ #