12.1

$$\begin{cases} -u''(x) = f(x) & 0<x<1 \\ u(0) = u(1) = 0 \end{cases}$$

by (12.3) $u(x) = \int_0^1 \underline{G(x,s)} f(s)\,ds = \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$

$$\underline{G(x,s)} = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$$

for $f(s) = \frac{1}{s}$ $\quad u(x) = \int_0^x s(1-x)\frac{1}{s}\,ds + \int_x^1 x(1-s)\frac{1}{s}\,ds$

$$= (1-x)\int_0^x 1\,ds + x\int_x^1 \left(\frac{1}{s}-1\right)ds$$

$$= (1-x)(x-0) + x(\ln s - s)\Big|_x^1$$

$$= \cancel{x} - \cancel{x^2} + x(\cancel{-1} - \ln x + \cancel{x}) = -x\ln x$$

when $0<x<1 \Rightarrow u = -x\ln x \in C^2(0,1)$

$u(x) = -x\ln x$

$u'(x) = -\ln x - 1$

$u''(x) = -\frac{1}{x}$

$\ln 0, \frac{1}{0}$ not exist $\Rightarrow u(0), u'(0), u''(0)$ not exist

2. $\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = \sum_{j=0}^{n-1} w_{j+1}v_j - \sum_{j=0}^{n-1} w_j v_j$

$$= \sum_{j=1}^{n} w_j v_{j-1} - \sum_{j=0}^{n-1} w_j v_j$$

$$= w_n v_{n-1} - w_0 v_0 + \sum_{j=1}^{n-1} w_j(v_{j-1}-v_j) + w_n v_n - w_n v_n$$

$$= w_n v_n - w_0 v_0 + \sum_{j=1}^{n-1} w_j(v_{j-1}-v_j) - (v_n - v_{n-1})w_n$$

$$= w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_j)w_{j+1} \quad \#$$

for $V_n^0 = \{v_n : v_0 = 0, v_n = 0\}$

$(L_n v)_j = -\frac{v_{j+1}-2v_j+v_{j-1}}{h^2}$

$(u,v)_h = h\sum_{j=1}^{n-1} u_j v_j$

$\Rightarrow (L_n v_n, v_n)_h = h\sum_{j=1}^{n-1}(L_n v)_j v_j = -\frac{h}{h^2}\sum_{j=1}^{n-1}(v_{j+1}-2v_j+v_{j-1})v_j$ $\quad$ let $w_j = v_j - v_{j-1}$

$$= -\frac{1}{h}\sum_{j=1}^{n-1}(w_{j+1}-w_j)v_j \quad \because v_0 = 0$$

$$= -\frac{1}{h}\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j \quad \text{by above}$$

$$= -\frac{1}{h}\left(w_n \cancel{v_n}^{0} - w_0\cancel{v_0}^{0} - \sum_{j=0}^{n-1}(v_{j+1}-v_j)w_{j+1}\right)$$

$$= -\frac{1}{h}\left[-\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2\right] = \frac{1}{h}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$$

#12.6

Know that $G(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$, $(L_h V)_j = -\dfrac{V_{j+1} - 2V_j + V_{j-1}}{h^2}$,

$G^k \in V_h^0$

$L_h G^k = e^k$, $G_0^k = G_{n+1}^k = 0$

let $\tilde{G}_j = hG(x_j, x_k)$ $j = 0 \cdots n+1$

$\because G(0, x_k) = G(1, x_k) = 0$

$\therefore \tilde{G}_0 = hG(0, x_k) = 0$. $\tilde{G}_{n+1} = hG(1, x_k) = 0$

$\because G(x, x_k) = \begin{cases} x(1-x_k) & 0 \le x \le x_k \\ x_k(1-x) & x_k \le x \le 1 \end{cases}$ when $x_j = jh$

$j \le k \Rightarrow G(x_j, x_k) = x_j(1-x_k) = jh(1-kh)$ linear

$j \ge k \Rightarrow G(x_j, x_k) = x_k(1-x_j) = kh(1-jh)$ linear

now for $j \ne k$ for $a+bj$ $(\cancel{a}+\cancel{b(j+1)}) - 2(\cancel{a}+\cancel{bj}) + (\cancel{a}+\cancel{b(j-1)}) = 0$

$\Rightarrow \tilde{G}_{j+1} - 2\tilde{G}_j + \tilde{G}_{j-1} = 0$

$\Rightarrow (L_h \tilde{G})_j = -\dfrac{\tilde{G}_{j+1} - 2\tilde{G}_j + \tilde{G}_{j-1}}{h^2} = 0$ $(j \ne k)$

for $j = k$ $\Rightarrow (L_h H)_k = -\dfrac{H_{j+1} - 2H_j + H_{j-1}}{h^2}$

$= -\dfrac{1}{h^2}\Big(x_k(1 - \overset{(x_k+h)}{x_{k+1}}) - 2x_k(1-x_k) + \overset{(x_k-h)}{x_{k-1}}(1-x_k) \Big)$

$= -\dfrac{1}{h^2}\left(\cancel{x_k} - \cancel{x_k^2} - \cancel{x_k h} - 2x_k + \cancel{2x_k^2} + \cancel{x_k} - \cancel{x_k^2} - h + \cancel{h x_k}\right)$

$= \dfrac{1}{h}$

$\Rightarrow (L_h \tilde{G})_k = h(L_h H)_k = h \cdot \dfrac{1}{h} = 1$

$\Rightarrow (L_h \tilde{G})_h = \begin{cases} 1 & j = k \\ 0 & j \ne k \end{cases}$, $\tilde{G}_0 = \tilde{G}_{n+1} = 0$

$\therefore G^k(x_j) = \tilde{G}_j = hG(x_j, x_k)$