7. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$

$\therefore \Gamma(z+1) = \int_0^\infty e^{-t} t^z\, dt$ let $u = t^z$, $du = z t^{z-1} dt$, $dv = e^{-t} dt$, $v = -e^{-t}$

$= \left(-e^{-t} t^z\right)_0^\infty + \int_0^\infty z\, \underline{e^{-t} t^{z-1} dt} = \Gamma(z)$

$= \left(-e^{-t} t^z\right)_0^\infty + z\Gamma(z)$

$= \left(\lim_{t\to\infty} -e^{-t} t^z\right) - \left(\lim_{t\to 0} -e^{-t} t^z\right) + z\Gamma(z)$

跑比較快 $\quad \underset{\to 0}{e^{-t}} \quad \underset{1}{e^{-t}}\ \underset{\to 0}{t^z}$

$= 0 - 0 + z\Gamma(z)$

$= z\Gamma(z)$ #

9. For $u_{n+1} = u_n + h\left[(1-\frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})\right]$

let $\theta = \frac{\alpha}{2}$

$u_{n+1} = u_n + h\left[(1-\theta) f_n + \theta f_{n+1}\right]$

by Taylor

$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + O(h^3)$

$f(x_{n+1}, y(x_{n+1})) = y'(x_{n+1})$

$= y'(x_n) + h y''(x_n) + O(h^2)$

$\Rightarrow$ Local Truncation Error

$h\tau_{n+1} = y(x_{n+1}) - y(x_n) - h\left[(1-\theta) y'(x_n) + \theta y'(x_{n+1})\right]$

$= \left(h\cancel{y'} + \frac{h^2}{2} y'' + O(h^3)\right) - h\left[\cancel{y'} - \cancel{\theta y'} + \cancel{\theta y'} + \theta h y'' + \theta O(h^2)\right]$

$= h^2\left(\frac{1}{2} - \theta\right) y''(x_n) + O(h^3)$

$\therefore \tau_{n+1} = \left(\frac{1}{2} - \theta\right) h y''(x_n) + O(h^2)$

① for $\alpha = 1$ $\theta = \frac{1}{2}$ $\tau_{n+1} = O(h^2) \Rightarrow$ 二階

② for any α $(\theta = \frac{\alpha}{2})$ $\tau_{n+1} = O(h) \Rightarrow$ 一階

when $\alpha = 1$ $\quad u_{n+1} = u_n + \frac{h}{2}\left(f(x_n, u_n) + f(x_{n+1}, u_{n+1})\right)$

$\because y' = -10y \Rightarrow f(x, y) = -10y$

$\Rightarrow u_{n+1} = u_n + \frac{h}{2}\left[-10u_n - 10u_{n+1}\right]$

$\Rightarrow u_{n+1} + 5h u_{n+1} = u_n - 5h u_n$

$\Rightarrow u_{n+1} = \underbrace{\frac{1-5h}{1+5h}}_{=R} u_n$

$|R(z)| = \frac{|1-5h|}{1+5h} < 1 \quad \forall h > 0$ is absolutely stable

for $y' = \lambda y$

$\Rightarrow u_{n+1} = u_n + \frac{h}{2}\left[\lambda u_n + \lambda u_{n+1}\right]$

$\Rightarrow \left(1 - \frac{h}{2}\lambda\right) u_{n+1} = \left(1 + \frac{h}{2}\lambda\right) u_n$

$\Rightarrow u_{n+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} u_n$

$R(z) = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$