Mathematica Code

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Title	SNNs in the Alexiewicz Topology: A new Perspective to Analysis and Error Bounds
for	neurocomputing; arXiv
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1. Spike Functions and LIF Model

A spike s with amplitude a at time t is represented as pair $s = (a, t) = a * \delta_t$.
A spike train is a sequence of spikes: $\eta = \sum_i a_i \delta_{t_i} \equiv \{(a_i, t_i)_i\}$

```
(* general functions *)

(* quantization and ceiling by integer truncation, a = {a1, a2, ...}
Remark: Rationalize is used for numerical reasons                    *)
q[a_]:=             IntegerPart[Rationalize /@ a];
ceil[a_]:=          Ceiling[Rationalize /@ a];
Bernoulli[p_]:=     If[RandomReal[]<p, 1, 0];

(* spike functions *)

(* set amplitude to 0 of spike s                                     *)
keepTime[s_]:=              {0,s[[2]]};
weightSpike[s_,w_]:=        {w*s[[1]],s[[2]]};


(* membrane's potential                                              *)
causSignal[t_,t0_,a_,alpha_]:=    If[t≥ t0, Exp[-alpha*(t-t0)]*a, 0];
(* resulting potential                                               *)
feta[t_,eta_,alpha_]:=            Sum[causSignal[t, eta[[k,2]], eta[[k,1]],alpha],
                                  {k,1,Length[eta]}];
(* add to spike trains                                               *)
AddWeightedSpikeTrains[w1_,eta1_,w2_,eta2_]:= Module[{etaNew, eta},
  eta = Join[weightSpike[#,w1]& /@ eta1, weightSpike[#,w2]& /@ eta2];
  eta = Sort[eta, #1[[2]]≤  #2[[2]]&];
  etaNew = {};
  While[Length[eta]≥2,
   If[eta[[1,2]]==eta[[2,2]],
     etaNew = Append[etaNew, {eta[[1,1]] + eta[[2,1]], eta[[1, 2]]}];
     eta  = Drop[eta,2],
     etaNew = Append[etaNew, eta[[1]]];
     eta  = Drop[eta,1];
     ];
   ];
  If[Length[eta]==1, etaNew = Append[etaNew, eta[[1]]]];
  etaNew
  ]

(* Alex norm                                                         *)
AalphaNorm[eta_,alpha_]:=  Max[Table[Abs[Sum[Exp[-alpha*(eta[[k,2]]-
                           eta[[i,2]])]*eta[[i,1]],{i,1,k}]],{k,1,Length[eta]}]];
L2alphaNorm[eta_,alpha_]:= (Sum[(Sum[Exp[-alpha*(eta[[k,2]]-
                           eta[[i,2]])]*eta[[i,1]],{i,1,k}])^2,
                           {k,1,Length[eta]}]])^(1/2);


(* membrane's potential at end of eta                                *)
Conv[eta_, alpha_]:=
  Sum[Exp[-alpha*(eta[[Length[eta],2]]- eta[[i,2]])]*eta[[i,1]],{i,1,Length[eta]}];


(* generate DN-1 spike train                                         *)
GetNu[Nr_, ExpPar_]:= Module[{a, nu, time},
  time = RandomVariate[ExponentialDistribution[ExpPar],Nr];
  time = Accumulate[time]; (* increasing sequence *)
  a = Table[RandomInteger[{0,1}],{k,1,Nr+1}];
  nu = Table[{a[[k+1]]-a[[k]], time[[k]]},{k,1,Nr}]
  ]
(* generate spike train                                              *)
```

```
GetEta[Nr_, ExpPar_,  pBernoulli_,amplitude_, discreteSteps_]:=
       Module[{a, nu, time},
       time = Table[RandomReal[{0.2,1}], {k, 1, Nr}];
       time = Accumulate[time]; (* increasing sequence *)
       a = Table[RandomInteger[{0,1}],{k,1,Nr+1}];
       nu = Table[{RandomInteger[{-amplitude,amplitude}]/discreteSteps
              *Bernoulli[pBernoulli], time[[k]]},{k,1,Nr}]]
```

(* LIF Models *)

```
LeakyIntFireMod[eta_,alpha_,th_]:= Module[{out,pot,potFunction, checkInterval,a},
  pot = 0;
  out = keepTime[#]& /@ eta;
  potFunction = eta;
  a = 1;
  For[i=1, i ≤ Length[potFunction], i++,
   checkInterval = Take[potFunction, {a, i}];
   pot = Conv[checkInterval, alpha];
   If[Abs[pot]≥ th,  out[[i,1]]= q[pot/th]*th; potFunction[[i,1]] = pot-
q[pot/th]*th; a=i;];
   ];
  out
  ]

LeakyIntFireSub[eta_,alpha_,th_]:= Module[{out,pot,potFunction, checkInterval,a},
  pot = 0;
  out = keepTime[#]& /@ eta;
  potFunction = eta;
  a = 1;
  For[i=1, i ≤ Length[potFunction], i++,
   checkInterval = Take[potFunction, {a, i}];
   pot = Conv[checkInterval, alpha];
   If[Abs[pot]≥ th,  out[[i,1]]= If[pot>0, th, -th]; potFunction[[i,1]] = pot-
If[pot>0, th, -th]; a=i;];
   ];
  out
  ]

LeakyIntFireZero[eta_,alpha_,th_]:= Module[{out,pot,potFunction, checkInterval,a},
  pot = 0;
  out = keepTime[#]& /@ eta;
  potFunction = eta;
  a = 1;
  For[i=1, i ≤ Length[potFunction], i++,
   checkInterval = Take[potFunction, {a, i}];
   pot = Conv[checkInterval, alpha];
   If[Abs[pot]≥ th,  out[[i,1]]= If[pot>0, th, -th]; potFunction[[i,1]] =0; a=i;];
   ];
  out
  ]
```

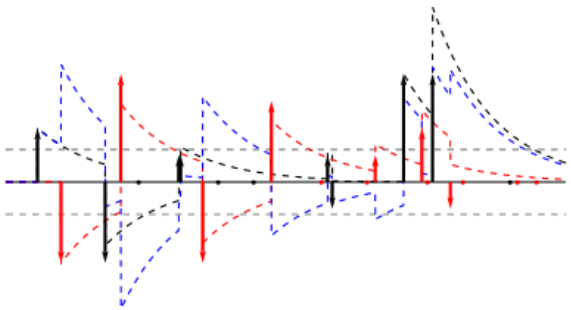
(* Threshold Error *)

```
ThrMod[eta_,Dth_,alpha_,th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1,LeakyIntFireMod[eta,alpha,th+Dth],
       -1,LeakyIntFireMod[eta10,alpha,th]],alpha];

ThrSub[eta_,Dth_,alpha_,th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1,LeakyIntFireSub[eta, alpha, th+Dth],
       -1,LeakyIntFireSub[eta10,alpha,th]],alpha];
```

```
ThrZero[eta_,Dth_,alpha_,th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1,LeakyIntFireZero[eta,alpha,th+Dth],
       -1, LeakyIntFireZero[eta10, alpha, th]], alpha];


(* Graphics *)
(* define Arrow from (0, t) to (a, t)                                    *)
setY0[s_]:=          {s[[1]],0};
showSpike[s_]:=      {setY0[Reverse[s]],Reverse[s]};
```

1.1 Membrane ‘s Potential

```
(* spikes:   eta1= GetEta[15, 2, 1]; eta2= GetEta[15, 2, 1]              *)
(* for single spike train                                                *)
plotSingleSpikeTrain=Plot[{1,0,-1,1.5,-1.5},{t,-0.1,10}, PlotRange→All,
PlotStyle→{{Dashed,Gray},{Black},{Dashed,Gray},{White},{White}}, Axes→False,
Epilog→Join[Join[{Arrowheads[.02],Black},Table[Arrow[showSpike[eta10[[k]]]],
{k,1,Length[eta10]}]]]];
(* for two spike trains                                                  *)
alpha = 3;
plotRange = 13;
plotTwoSpikeTrains = Plot[{0.3,0,-0.3, 1.5, -1.5},{t, -0.1,plotRange},
       PlotRange→All, PlotStyle→{{Dashed,Gray},{Black},{Dashed,Gray},{White},
       {White}}, Axes→False,Epilog→Join[Join[{Thick, Arrowheads[.02],
       Black},Table[Arrow[showSpike[eta1[[k]]]],{k,1,Length[eta1]}]],
       Join[{Thick,Arrowheads[.02], Red},Table[Arrow[showSpike[eta2[[k]]]],
       {k,1,Length[eta2]}]]]];
(* membrane                                                              *)
plotMembrane=Plot[{feta[t, eta1, alpha], feta[t, eta2,
       alpha],feta[t,AddWeightedSpikeTrains[1,eta1,-1,eta2],alpha]},
       {t,-0.1,plotRange},PlotStyle→{{Dashed,Black},{Dashed,Red},
       {Dashed, Blue}},PlotRange→All];
Show[{pl1, pl2}]
```

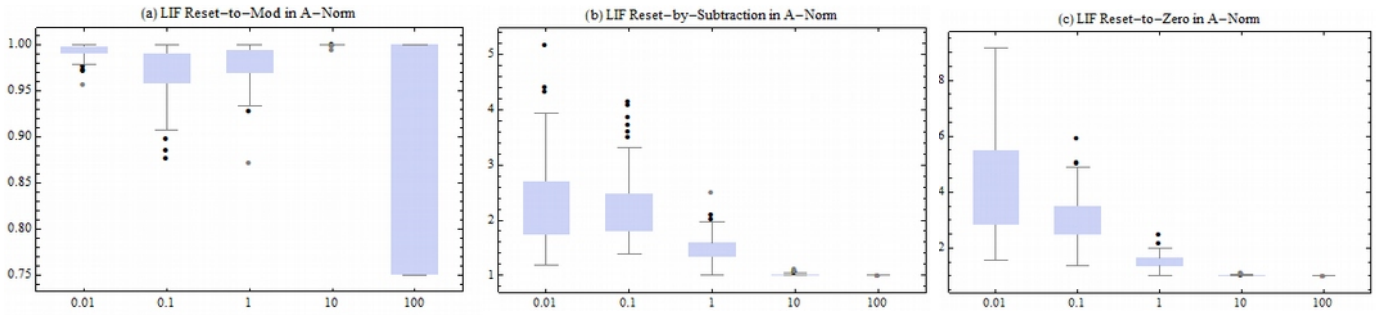


2. Experiments

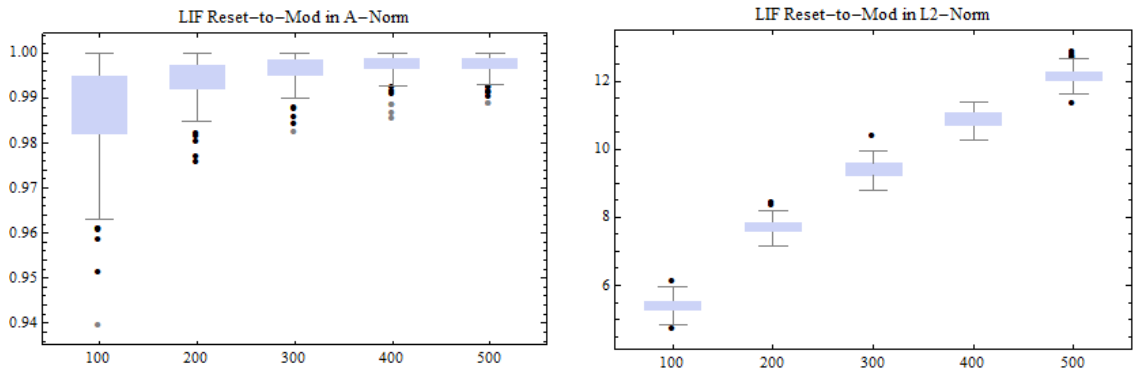
2.1 Quantization Error

```
(* Add. Error, LIF, A norm, depending on alpha                                *)
K = 30; (* runs for different nu *)
M = 4;  (* alphas *)
Nr= 30; (* spikes *)
amplitude= 8;
discreteSteps = 4;
outLeakyModList = Table[Table[0, {k,1, K}], {i,1,M}];
outLeakySubList = Table[Table[0, {k,1, K}], {i,1,M}];
outLeakyZeroList = Table[Table[0, {k,1, K}], {i,1,M}];
alpha = 0.01;
th = 1;
(* alpha *)
For[n=1, n≤M, n++,
 If[n ==1,alpha =0];
 If[n ==2,alpha =0.01];
 alpha = alpha*10;
 (* spike trains eta *)
 For[k = 1,k<=K, k++,
  eta =  GetEta[Nr, 1, 1,amplitude, discreteSteps];
  outLeakyMod = LeakyIntFireMod[eta, alpha, th];
  outLeakySub = LeakyIntFireSub[eta, alpha, th];
  outLeakyZero = LeakyIntFireZero[eta, alpha,th];

  (* added spike trains nu                                                    *)
  For[m = 1,m<=K, m++,
   nu = GetNu[Nr, 1];
   etaPlusNu = AddWeightedSpikeTrains[1, eta, 1, nu];
   outLeakyModNu = LeakyIntFireMod[etaPlusNu, alpha, th];
   outLeakySubNu = LeakyIntFireSub[etaPlusNu, alpha, th];
   outLeakyZeroNu = LeakyIntFireZero[etaPlusNu, alpha, th];

   outLeakyModList[[n,k]] =
       AalphaNorm[AddWeightedSpikeTrains[1,outLeakyModNu,-1,outLeakyMod],alpha];
   outLeakySubList[[n,k]]=
       AalphaNorm[AddWeightedSpikeTrains[1,outLeakySubNu,-1,outLeakySub],alpha];
   outLeakyZeroList[[n,k]]=
       AalphaNorm[AddWeightedSpikeTrains[1,outLeakyZeroNu,-1,outLeakyZero],alpha];
   ]];
]

(* Graphics: Quantization Error w.r.t alpha                                  *)
data=outLeakyModList;
LIFModPlot=  BoxWhiskerChart[data,"Outliers",
             ChartLabels→{"0","0.1","1", "10", "100"},
             PlotLabel→"(a) LIF Reset-to-Mod in A-Norm"]

data=outLeakySubList;
LIFModPlot=  BoxWhiskerChart[data,"Outliers",
             ChartLabels→{"0","0.1","1", "10", "100"},
             PlotLabel→"(b) LIF Reset-by-Sub in A-Norm"]

data=outLeakyZeroList;
LIFModPlot=  BoxWhiskerChart[data,"Outliers",
```

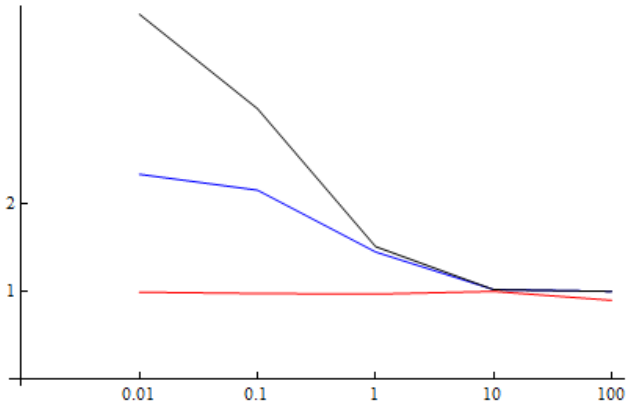
```
ChartLabels→{"0","0.1","1", "10", "100"},
PlotLabel→"(c) LIF Reset-to-Zero in A-Norm"]
```




```
(* Quantization Error w.r.t nr of spikes                                    *)
K = 100;
M = 5;
th = 1;
amplitude = 8;
discreteSteps = 4;
outLeakyListL2 = Table[Table[0, {k,1, K}], {i,1,M}];
outLeakyListA = Table[Table[0, {k,1, K}], {i,1,M}];
alpha = 0.001;
For[n=1, n≤M, n++,
 alpha =1;
 For[k = 1,k<=K, k++,
  Nr = n*100;
  eta =GetEta[Nr, 1, 1,amplitude, discreteSteps];
  outLeaky = LeakyIntFireMod[eta, alpha,th];
  outLeakyListL2[[n,k]] = L2alphaNorm[AddWeightedSpikeTrains[1,eta,-
1,outLeaky],alpha];
  outLeakyListA[[n,k]] = AalphaNorm[AddWeightedSpikeTrains[1,eta,-
1,outLeaky],alpha];
  ];
 ]

data=outLeakyListA;
LIFModPlot= BoxWhiskerChart[data,"Outliers",ChartLabels→{"100","200","300", "400",
"500"}, PlotLabel→"LIF Reset-to-Mod in A-Norm"]

data=outLeakyListL2;
LIFModPlot=  BoxWhiskerChart[data,"Outliers",ChartLabels→{"100","200","300",  "400",
             "500"}, PlotLabel→"LIF Reset-to-Mod in L2-Norm"]
```

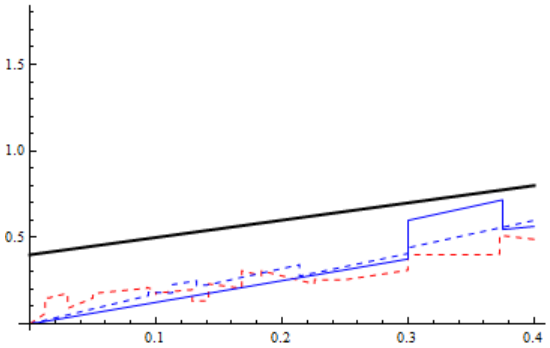


```
(* Quantization Error: mean w.r.t alpha                                    *)
ListLinePlot[{Mean[#] & /@ outLeakyModList,
              Mean[#]  & /@ outLeakySubList,
              Mean[#]  & /@ outLeakyZeroList},
              PlotStyle→{{Red, Blue, Black}},
              Ticks→{{{1, 0.01},{2,0.1},{3,1},{4,10}, {5,100}},{1,2, 10}}]
```



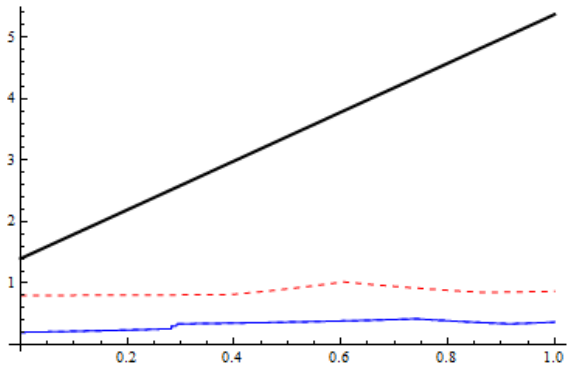
2.2 Threshold Errors

```
eta1 =GetEta[15, 2, 1];
ThrMod[Dth_,alpha_,th_,eta_]:=AalphaNorm[
            AddWeightedSpikeTrains[1,LeakyIntFireMod[eta,alpha,th+Dth],
            -1, LeakyIntFireMod[eta, alpha, th]], alpha];
ThrSub[Dth_, alpha_, th_,eta_]:=AalphaNorm[
             AddWeightedSpikeTrains[1, LeakyIntFireSub[eta, alpha, th+ Dth] ,
             -1, LeakyIntFireSub[eta, alpha, th]], alpha];
ThrZero[Dth_, alpha_, th_,eta_]:=AalphaNorm[
             AddWeightedSpikeTrains[1, LeakyIntFireZero[eta, alpha, th+ Dth] ,
             -1, LeakyIntFireZero[eta, alpha, th]], alpha];

th=0.2; alpha = 0.8;
plotThresholdError =Plot[{0,1.8, 2*th + Dth,
   ThrMod[Dth, alpha,th, eta1],
   ThrSub[Dth, alpha,th, eta1],
   ThrZero[Dth, alpha,th, eta1]},{Dth,0,0.4},
       PlotStyle→{{White},{White},{Thick,Black},{Dashed,Red},{Dashed,Blue},{Blue}}]
```



2.3 Time Delay

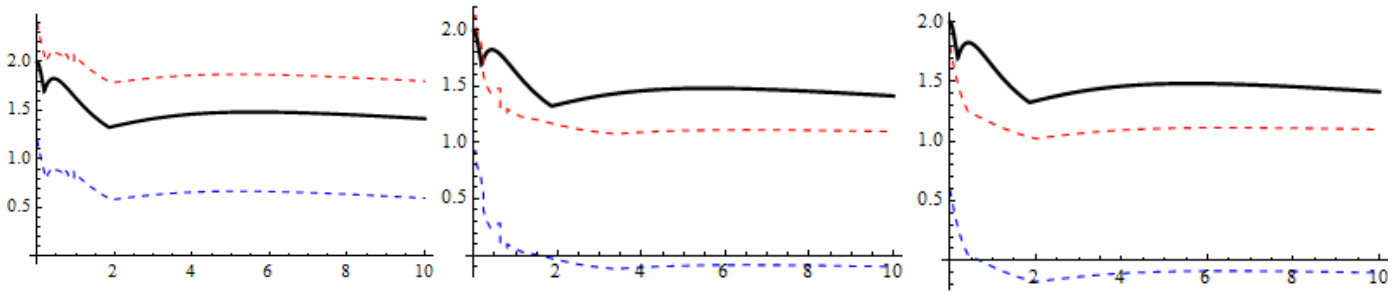
```
DiffEtaDt[alpha_,eta1_,Dt_]:=     AalphaNorm[AddWeightedSpikeTrains[1, eta1, -1,
                                  etaLag[eta1, Dt]], alpha];

DiffLIFEtaDtMod[alpha_,eta1_,Dt_,th_]:=       AalphaNorm[AddWeightedSpikeTrains[1,
                          LeakyIntFireMod[eta1, alpha, th], -1,
                          LeakyIntFireMod[etaLag[eta1, Dt], alpha,th]], alpha];

DiffLIFEtaDtSub[alpha_,eta1_, Dt_, th_]:=     AalphaNorm[AddWeightedSpikeTrains[1,
                          LeakyIntFireSub[eta1, alpha, th], -1,
                          LeakyIntFireSub[etaLag[eta1, Dt], alpha,th]], alpha];

DiffLIFEtaDtZero[alpha_,eta1_, Dt_, th_]:=    AalphaNorm[AddWeightedSpikeTrains[1,
                          LeakyIntFireZero[eta1, alpha, th], -1,
                          LeakyIntFireZero[etaLag[eta1, Dt], alpha,th]], alpha];

th=0.2; alpha = 2;
plotDelayError =Plot[{0,2*th + 1 + alpha*Dt* (AalphaNorm[eta1,alpha]+ 1),
                    DiffLIFEtaDtMod[alpha,eta1,  Dt, th],
                    DiffLIFEtaDtSub[alpha,eta1,  Dt, th],
                    DiffLIFEtaDtZero[alpha,eta1,  Dt, th]}, {Dt,0,1},
              PlotStyle→{{White},{Thick,Black},{Dashed,Red},{Dashed,Blue},{Blue}}]
```



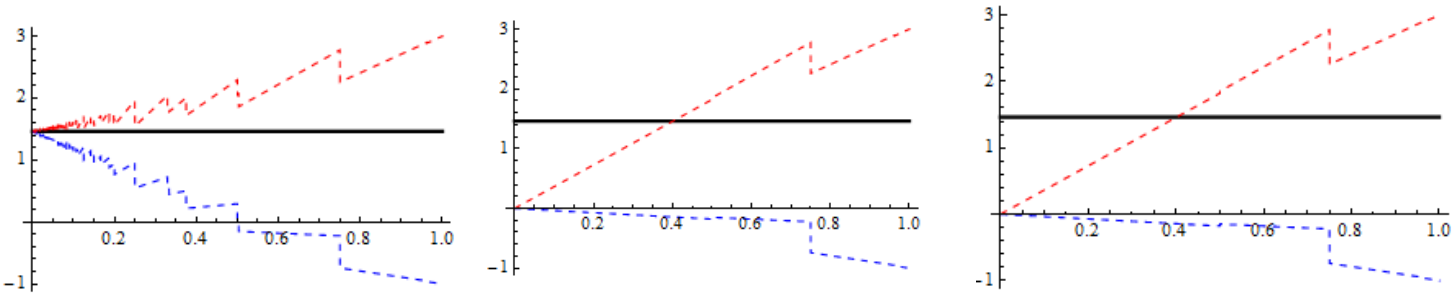
2.4 Quasi Isometry

```
DiffEta[alpha_, eta1_, eta2_]:=
     AalphaNorm[AddWeightedSpikeTrains[1, eta1, -1, eta2], alpha];

DiffLIFEtaMod[alpha_,eta1_, eta2_, th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1, LeakyIntFireMod[eta1, alpha, th],-1,
       LeakyIntFireMod[eta2, alpha,th]], alpha];

DiffLIFEtaSub[alpha_,eta1_, eta2_, th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1, LeakyIntFireSub[eta1, alpha, th], -1,
       LeakyIntFireSub[eta2, alpha,th]], alpha];

DiffLIFEtaZero[alpha_,eta1_, eta2_, th_]:=
       AalphaNorm[AddWeightedSpikeTrains[1, LeakyIntFireZero[eta1, alpha, th], -1,
       LeakyIntFireZero[eta2, alpha,th]], alpha];
```

```
(* th fix, alpha running                                                    *)
th=0.3;
Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaMod[alpha,eta1, eta2, th]-2*th,
       DiffLIFEtaMod[alpha,eta1, eta2, th]+2*th}, {alpha, 0,10}, PlotRange→All,
       PlotStyle→{{White},{Thick,Black}, {Dashed,Blue}, {Dashed,Red}}]

Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaSub[alpha,eta1, eta2, th]-2*th,
       DiffLIFEtaSub[alpha,eta1, eta2, th]+2*th}, {alpha, 0,10}, PlotRange→All,
       PlotStyle→{{White},{Thick,Black}, {Dashed,Blue}, {Dashed,Red}}]

Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaZero[alpha,eta1, eta2, th]-2*th,
       DiffLIFEtaZero[alpha,eta1, eta2, th]+2*th}, {alpha, 0,10}, PlotRange→All,
       PlotStyle→{{White},{Thick, Black}, {Dashed,Blue}, {Dashed,Red}}]
```

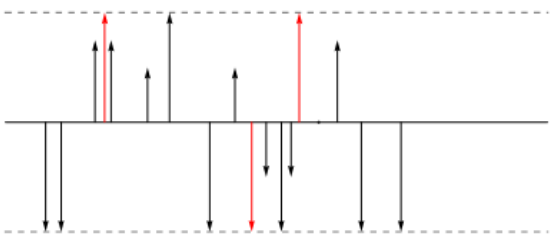


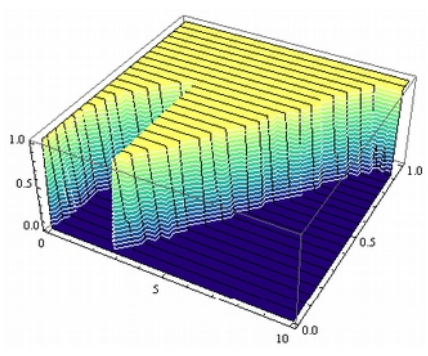

```
(* alpha fix, th running                                                    *)
alpha = 4;
Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaMod[alpha,eta1, eta2, th]-2*(th),
       DiffLIFEtaMod[alpha,eta1, eta2, th]+2*(th)}, {th, 0,1}, PlotRange→All,
       PlotStyle→{{White},{Thick,Black}, {Dashed,Blue}, {Dashed,Red}}]

Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaSub[alpha,eta1, eta2, th]-2*(th),
       DiffLIFEtaSub[alpha,eta1, eta2, th]+2*(th)}, {th, 0,1}, PlotRange→All,
       PlotStyle→{{White},{Thick,Black}, {Dashed,Blue}, {Dashed,Red}}]


Plot[{0, DiffEta[alpha, eta1, eta2],DiffLIFEtaZero[alpha,eta1, eta2, th]-2*(th),
     DiffLIFEtaZero[alpha,eta1, eta2, th]+2*(th)}, {th, 0,1}, PlotRange→All,
     PlotStyle→{{White},{Thick, Black}, {Dashed,Blue}, {Dashed,Red}}]
```

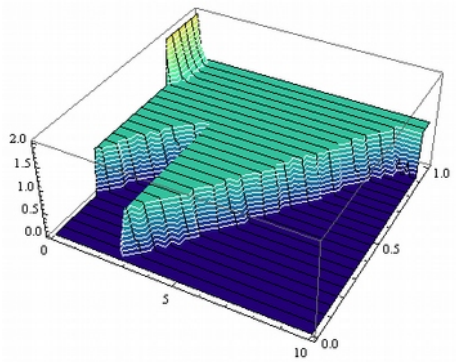


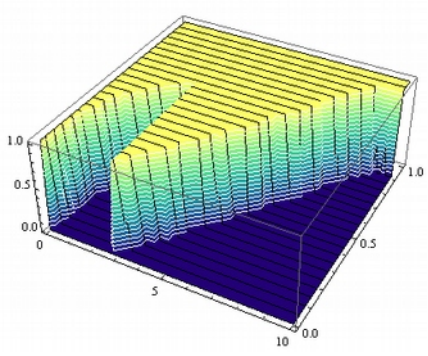
2.5 Resonance Phenomenon in the Context of Lipschitz-style Upper Bound

```
(* Resonance Phenomenon / Lipschitz Sytle Upper Bound                    *)
(* noise example                                                         *)
nu1p[p_]:=  {{1*p, 2.3},{-1*p, 5.85}, {1*p, 7}};
resFunZero[eta1_, alpha_,p_, th_]:=Module[{DeltaEta},
      DeltaEta = AddWeightedSpikeTrains[1,
      LeakyIntFireZero[AddWeightedSpikeTrains[1,eta1, 1,nu1p[p]], alpha, th], -1,
      LeakyIntFireZero[eta1, alpha,th]];
      AalphaNorm[DeltaEta, alpha]];

resFunMod[eta1_,alpha_,p_,th_]:=Module[{DeltaEta},
      DeltaEta = AddWeightedSpikeTrains[1,
      LeakyIntFireMod[AddWeightedSpikeTrains[1,eta1, 1,nu1p[p]], alpha, th], -1,
      LeakyIntFireMod[eta1, alpha,th]];
      AalphaNorm[DeltaEta, alpha]];

resFunSub[eta1_,alpha_,p_, th_]:=Module[{DeltaEta},
      DeltaEta = AddWeightedSpikeTrains[1,
      LeakyIntFireSub[AddWeightedSpikeTrains[1,eta1, 1,nu1p[p]], alpha, th], -1,
      LeakyIntFireSub[eta1, alpha, th]];
      AalphaNorm[DeltaEta, alpha]];

(* graphics                                                              *)
alpha = 0.1;
pl2Mod = Plot3D[resFunMod[alpha, p, th],{alpha,0,10},
{p,0,1},MeshStyle→{({Red,Tube@@#}&),Directive[White],Directive[Black]},
  MeshFunctions→{#1&, #3&,#2&},Mesh→{{0},20,20},ColorFunction→"BlueGreenYellow"]

pl2Zero = Plot3D[resFunZero[alpha, p, th],{alpha,0,10},
{p,0,1},MeshStyle→{({Red,Tube@@#}&),Directive[White],Directive[Black]},
  MeshFunctions→{#1&, #3&,#2&},Mesh→{{0},20,20},ColorFunction→"BlueGreenYellow"]

pl2Sub = Plot3D[resFunSub[alpha, p, th],{alpha,0,10},
{p,0,1},MeshStyle→{({Red,Tube@@#}&),Directive[White],Directive[Black]},
  MeshFunctions→{#1&, #3&,#2&},Mesh→{{0},20,20},ColorFunction→"BlueGreenYellow"]


plot = Plot[{1,0,-1, 1.5, -1.5},{t, -0.1,13}, PlotRange→All,
      PlotStyle→{{Dashed,Gray},{Black},{Dashed,Gray}, {White},{White}},Axes→False,
      Epilog→Join[Join[{Arrowheads[.02], Black},Table[Arrow[showSpike[eta1[[k]]]],
      {k,1,Length[eta1]}]], Join[{Arrowheads[.02],
      Red},Table[Arrow[showSpike[nu1p[1][[k]]]],{k,1,Length[nu1p[1]]}]]]]
```

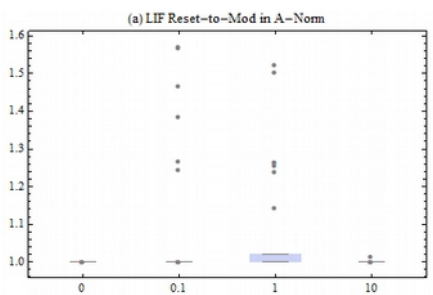


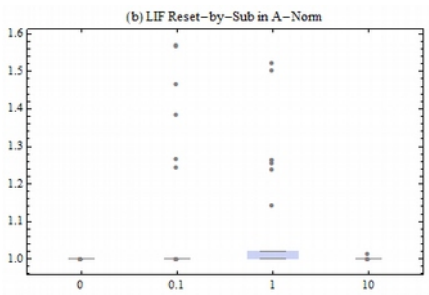


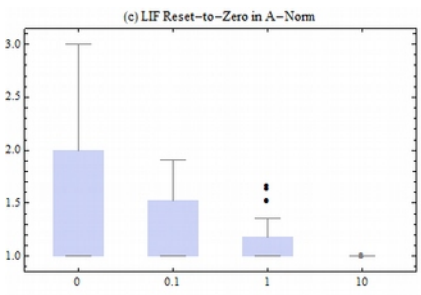




```
(* BoxWhisker Charts                                                    *)
K = 30; (* runs for different nu *)
M = 4;  (* alphas *)
Nr= 30; (* spike trains *)
outLeakyModList = Table[Table[0, {k,1, K}], {i,1,M}];
outLeakySubList = Table[Table[0, {k,1, K}], {i,1,M}];
outLeakyZeroList = Table[Table[0, {k,1, K}], {i,1,M}];
alpha = 0.01;
th = 1;
(* alpha *)
For[n=1, n≤M, n++,
 If[n ==1,alpha =0];
 If[n ==2,alpha =0.01];
alpha = alpha*10;
 (* spike train eta *)
 For[k = 1,k<=K, k++,
  eta =  GetEta[Nr, 1, 1];
  outLeakyMod = LeakyIntFireMod[eta, alpha,th];
  outLeakySub = LeakyIntFireSub[eta, alpha,th];
  outLeakyZero = LeakyIntFireZero[eta, alpha,th];
  (* added spike train nu *)
  For[m = 1,m<=K, m++,
   nu = GetNu[Nr, 1];
   etaPlusNu = AddWeightedSpikeTrains[1, eta, 1, nu];
   outLeakyModNu = LeakyIntFireMod[etaPlusNu, alpha, th];
   outLeakySubNu = LeakyIntFireSub[etaPlusNu, alpha, th];
   outLeakyZeroNu = LeakyIntFireZero[etaPlusNu, alpha,th];
   (* collect in list *)
   outLeakyModList[[n,k]] =
       AalphaNorm[AddWeightedSpikeTrains[1, outLeakyModNu,-1,outLeakyMod ], alpha];
   outLeakySubList[[n,k]]=
       AalphaNorm[AddWeightedSpikeTrains[1, outLeakySubNu,-1,outLeakySub ], alpha];
   outLeakyZeroList[[n,k]]=
       AalphaNorm[AddWeightedSpikeTrains[1, outLeakyZeroNu,-1,outLeakyZero ],
alpha];
   ]];
 ]

(* graphics *)

data=outLeakySubList;
LIFModPlot=   BoxWhiskerChart[data,"Outliers",
              ChartLabels→{"0.01","0.1","1", "10", "100"},
              PlotLabel→"(a) LIF Reset-to-Mod in A-Norm"]
data=outLeakySubList;
LIFModPlot=   BoxWhiskerChart[data,"Outliers",
              ChartLabels→{"0.01","0.1","1", "10", "100"},
              PlotLabel→"(b) LIF Reset-by-Sub in A-Norm"]
data=outLeakyZeroList;
LIFModPlot=BoxWhiskerChart[data,"Outliers",
              ChartLabels→{"0.01","0.1","1","10","100"},
              PlotLabel→"(c) LIF Reset-to-Zero in A-Norm"]
```

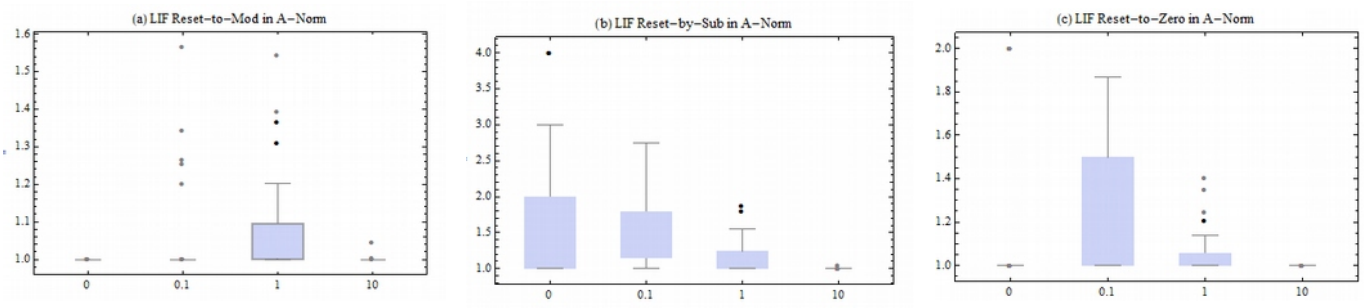






with max amplitude = 1

with max amplitude = 2




```
(* showing maxima illustrates resonance phenomenon *)

ListLinePlot[{Table[Max[outLeakyZeroList[[n,All]]],{n,1,4}],
              Table[Max[outLeakySubList[[n,All]]],{n,1,4}],
              Table[Max[outLeakyModList[[n, All]]],{n,1,4}]},PlotStyle→{Black, Blue,
              Red}, Ticks→{{{1,0}, {2,0.1}, {3, 1}, {4,10}}}]
```