1.1 Probability Space

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1.1.1

(i) If $\mathcal{F}_i$, $i \in I$ are σ-fields, then $\bigcap_{i \in I} \mathcal{F}_i$ is. Here $I = \emptyset$ is an arbitrary index set (i.e., possibly uncountable). (ii) Use the result in (i) to show that if we are given a set Ω and a collection A of subsets of $\mathcal{A}$, then there is a smallest σ-field containing $\mathcal{A}$. We will call this the σ-field generated by $\mathcal{A}$ and denote it by $\sigma(\mathcal{A})$.

(ii) Denote $\mathcal{F} := \bigcap_{i \in I} \mathcal{F}_i$.

Take $A \in \mathcal{F}$, then $A \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$. Then $A^C \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$, because $\mathcal{F}_i$'s are σ-fields. Therefore, $A^C \in \bigcap_{i \in I} \mathcal{F}_i = \mathcal{F}$.

Take a countable sequence of sets $A_j \in \mathcal{F}$ for $j = 1, 2, \ldots$, then $A_j \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$. Then $\bigcup_j A_j \in \mathcal{F}_i$ for $i = 1, 2, \ldots$. Therefore, $\bigcup_j A_j \in \bigcap_i \mathcal{F}_i = \mathcal{F}$.

Then we can conclude that $\mathcal{F}$ is a σ-field.

(ii) Define $\mathcal{F}$ to be the intersection of all σ-field that contains $\mathcal{A}$. Then by (i), $\mathcal{F}$ is a σ-field. And by its definition, every σ-field that contains $\mathcal{F}$ contains $\mathcal{F}$. That means: $\mathcal{F}$ is the smallest σ-field containing $\mathcal{A}$.

1.1.2

Let $\Omega = \mathbb{R}$,$\mathcal{F}$ = all subsets so that A or A^C is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Proof: First we show that $\mathcal{F}$ is a σ-field:

Take $A \in \mathcal{F}$, then $A = (A^C)^C$ or A is countable. This means that $A^C \in \mathcal{F}$.

Take a countable sequence of sets $A_i \in \mathcal{F}$ for all $i = 1, 2, \ldots$. Then either A_i or A_i^C is countable. If all A_i's are countable, then $\bigcup_i A_i$ is countable, because countable union of countable sets is still countable. Otherwise, there exists uncountable A_j for some j, then we have A_j^C is countable. Then $(\bigcup_i A_i)^C = \bigcap_i A_i^C \subset A_j^C$ is countable. Then $\bigcup_i A_i \in \mathcal{F}$.

Therefore, $\mathcal{F}$ is a σ-field.

Second, we show that P is a measure:

For any $A \in \mathcal{F}$, $P(A) \geq 0 = P(\emptyset)$ by the definition of P. Take a countable sequence of disjoint sets $A_i \in \mathcal{F}$. If all A_i's are countable, then $\bigcup_i A_i$ is also countable. Then $P(\bigcup_i A_i) = 0$, and $P(A_i) = 0$ for all i. Therefore, we have $P(\bigcup_i A_i) = 0 = \sum_i P(A_i)$. Otherwise, there exists an uncountable A_j for some j. Because A_i's are disjoint, $A_i \subset A_j^C$ for all $i \neq j$. $A_j \in \mathcal{F}$ and A_j is uncountable, then A_j^C is countable. Then $A_i \subset A_j^C$ is countable for all $i \neq j$, which means that $P(A_j) = 1$ and $P(A_i) = 0$ for all $i \neq j$. Moreover, we have $A_j \subset \bigcup_i A_i$, and therefore, $\bigcup_i A_i$ is uncountable, and then $P(\bigcup_i A_i) = 1$. Then we have $P(\bigcup_i A_i) = 1 = 1 + 0 = P(A_j) + \sum_{i \neq j} P(A_i) = \sum_i P(A_i)$.

Therefore, P is a measure with $P(\Omega) = P(\mathbb{R}) = 1$. Then P is a probability.