Problem1 (1.1.4) Proof: Denote $\mathbb{Q}$ = set of all rational numbers
$\mathbb{R}$ = set of all real numbers.

Let $\mathcal{C} = \{(a_1,b_1) \times \cdots \times (a_d,b_d) : a_i, b_i \in \mathbb{Q}, -\infty \le a_i < b_i \le +\infty \text{ for all } i\}$

Step 1: Show that $\sigma(\mathcal{C}) \subseteq \mathcal{R}^d$

Take $a_i, b_i \in \mathbb{Q}$ with $-\infty \le a_i < b_i \le +\infty$ for $1 \le i \le d$;

then the set $(a_1,b_1) \times \cdots \times (a_d,b_d)$ is an open set in $\mathbb{R}^d$.

$\Rightarrow (a_1,b_1) \times \cdots \times (a_d,b_d) \in \mathcal{R}^d$, because $\mathcal{R}^d$ contains all open sets in $\mathbb{R}^d$.

$\Rightarrow \mathcal{C} \subseteq \mathcal{R}^d$

$\sigma(\mathcal{C})$ is the smallest σ-field that contains $\mathcal{C}$,

which means: if $\mathcal{F}$ is a σ-field containing $\mathcal{C}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

This implies that $\sigma(\mathcal{C}) \subseteq \mathcal{R}^d$

Step 2: Show that $\mathcal{R}^d \subseteq \sigma(\mathcal{C})$:

Take an open set $U \subseteq \mathbb{R}^d$.

$\mathcal{U}$ is open

$\Rightarrow$ For any point $x = (x_1, \cdots, x_d) \in \mathcal{U}$, we can find $a_1^x, \cdots, a_d^x, b_1^x, \cdots, b_d^x \in \mathbb{Q}$ $\to$ {rational numbers} such that

① $-\infty \le a_i^x < x_i < b_i^x \le +\infty$ for all $i \in \{1, 2, \cdots, d\}$

② $x \in (a_1^x, b_1^x) \times \cdots \times (a_d^x, b_d^x) \subseteq \mathcal{U}$.

$\Rightarrow \mathcal{U} = \bigcup_{x \in \mathcal{U}} \{x\} \subseteq \bigcup_{x \in \mathcal{U}} (a_1^x, b_1^x) \times \cdots \times (a_d^x, b_d^x) \subseteq \mathcal{U}$

i.e. $\mathcal{U} = \bigcup_{x \in \mathcal{U}} (a_1^x, b_1^x) \times \cdots \times (a_d^x, b_d^x)$.

But $a_1^x, \cdots, a_d^x, b_1^x, \cdots, b_d^x \in \mathbb{Q}$ which implies that there are only countably many possible choices for $a_1^x, \cdots, a_d^x, b_1^x, \cdots, b_d^x$.

$\Rightarrow$ There are countably many sets of the form

$$(a_1^x, b_1^x) \times \cdots \times (a_d^x, b_d^x)$$

$\Rightarrow \mathcal{U} = \bigcup_{x \in \mathcal{U}} (a_1^x, b_1^x) \times \cdots \times (a_d^x, b_d^x)$ is actually a countable union of sets from $\mathcal{C}$, where

$\mathcal{C} = \{(a_1, b_1) \times \cdots \times (a_d, \cdots, b_d) : a_i, b_i \in \mathbb{Q}, -\infty \le a_i < b_i \le \infty \text{ for } 1 \le i \le d\}$

$\Rightarrow U \in \sigma(\mathcal{C})$ because σ-field is closed under countable union.

U is an arbitrarily chosen open set of $\mathbb{R}^d$

$\Rightarrow \sigma(\mathcal{C})$ contains all open sets in $\mathbb{R}^d$.

$\Rightarrow \mathcal{R}^d \subseteq \sigma(\mathcal{C})$ because $\mathcal{R}^d$ is the smallest σ-field containing all open sets of $\mathbb{R}^d$.

Therefore, by Step 1 and step 2, we can conclude that

$$\mathcal{R}^d = \sigma(\mathcal{C})$$

Also $a_i, b_i \in \mathbb{Q} \cup \{+\infty, -\infty\}$ for all $1 \le i \le d$

$\Rightarrow$ there are countably many possible choices for $a_1, \cdots, a_d$ and $b_1, \cdots, b_d$.

$\Rightarrow$ there are countably many sets in the form

$$(a_1, b_1) \times \cdots \times (a_d, b_d)$$

with $-\infty \le a_i < b_i \le +\infty$ and $a_i, b_i \in \mathbb{Q}$

$\Rightarrow \mathcal{C}$ is a countable collection.

Then $\sigma(\mathcal{C}) = \mathcal{R}^d$ implies that $\mathcal{R}^d$ is countably generated. $\blacksquare$

Problem 2. (1.2.3)

Proof: Let F be a distribution function.

Let $A = \{x \in \mathbb{R} : F \text{ is discontinuous at } x\}$

$B = [0,1] \cap \mathbb{Q}$ ($\mathbb{Q}$: set of all rational numbers)

I want to construct an injective (one-to-one) function φ from A to B.

First I need the following Lemmas

Lemma 2.1 If F is discontinuous at x, then $F(x-) < F(x)$

proof: F discontinuous at x, then $F(x-) \neq F(x+)$

By right-continuity of F, we have $F(x+) = F(x)$

$\Rightarrow$ $F(x-) \neq F(x)$ ①

Also F is non-decreasing

$\Rightarrow$ If $y < x$, then $F(y) \leq F(x)$

$\Rightarrow \lim_{y \uparrow x} F(y) \leq F(x)$ i.e. $F(x-) \leq F(x)$ ②

① ② $\Rightarrow$ $F(x-) < F(x)$. ▨

Lemma 2.2 If $x_1, x_2 \in A$ w/ $x_1 < x_2$

then $F(x_1-) < F(x_1) \leq F(x_2-) < F(x_2)$.

Proof: Given $x_1 < x_2$, then for any real number y such that $x_1 < y < x_2$, we have:

$F(x_1) \le F(y) \le F(x_2)$ because F is non-decreasing.

Let $y \uparrow x_2$, we have $F(x_1) \le \lim_{y \uparrow x_2} F(y) \le F(x_2)$.

i.e. $F(x_1) \le F(x_2-) \le F(x_2)$.

Also by lemma 2.1, $x_1, x_2 \in A$ implies that

$F(x_1-) < F(x_1)$ and $F(x_2-) < F(x_2)$.

Together we have $F(x_1-) < F(x_1) \le F(x_2-) < F(x_2)$ ∎

Now I want to construct function $\varphi : A \mapsto B$:

$\forall\ x \in A$, then F is discontinuous at x.

Then by lemma 2.1, we have $F(x-) < F(x)$.

F is a distribution function $\Rightarrow 0 \le F(y) \le 1$ for all $y \in \mathbb{R}$.

$\Rightarrow$ then we can choose a rational number $q_x \in [0,1]$

such that $F(x^-) < q_x < F(x)$.

Define $\varphi(x) = q_x$.

Then the φ that we want is constructed.

Now I want to show that this φ is strictly increasing, and therefore, it is injective:

Take any $x_1, x_2 \in A$ with $x_1 \neq x_2$

Without loss of generality, we can assume that

$$x_1 < x_2.$$

Then by our construction of φ, we have

$$F(x_1-) < \varphi(x_1) < F(x_1)$$

and $F(x_2-) < \varphi(x_2) < F(x_2)$

By lemma 2.2 $x_1 < x_2$ implies that

$$F(x_1-) < F(x_1) \leq F(x_2-) < F(x_2)$$

Therefore, we have

$$F(x_1-) < \varphi(x_1) < F(x_1) \leq F(x_2-) < \varphi(x_2) < F(x_2)$$

$\Rightarrow \varphi(x_1) < \varphi(x_2)$

$\Rightarrow \varphi$ is strictly increasing, and this means that φ must be injective.

Now we find an injective function $\varphi: A \longrightarrow B$.

Then we have $|A| \leq |B|$ where $|A|$ means the cardinality of A.

i.e. number of discontinuities of $F = |A|$

$$\leq |B|$$

$$= |[0,1] \cap \mathbb{Q}|$$

$\mathbb{Q}$ is countable $\Rightarrow |[0,1] \cap \mathbb{Q}|$ is countably infinite.

$\Rightarrow F$ has at most countably many discontinuities.

F is an arbitrarily chosen distribution function; therefore, we can conclude that every distribution function has at most countably many discontinuities. $\blacksquare$

Problem 3. (1.2.5)

g is strictly increasing on $(\alpha, \beta) \Rightarrow g$ is invertible on (α, β).

Write the inverse of g on (α, β) as g^{-1} and its domain should be $g((\alpha, \beta)) = (g(\alpha), g(\beta))$.

g is differentiable on (α, β), then by Inverse Function Theorem, we have g^{-1} is differentiable on $(g(\alpha), g(\beta))$ and

$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}.$$

Note: $g(X) \leq y \iff X \leq g^{-1}(y)$ for all $y \in (g(\alpha), g(\beta))$;

this is because g is strictly increasing and invertible on (α, β).

The domain of X is $(\alpha, \beta) \Rightarrow$ the domain of $g(X)$ is $(g(\alpha), g(\beta))$.

Now we compute the density function of $g(X)$ by computing its distribution function:

$$F_{g(X)}(y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$
$$= \int_{\alpha}^{g^{-1}(y)} f(t)\,dt \quad , \quad y \in (g(\alpha), g(\beta))$$

then the density function of $g(X)$ on $(g(\alpha), g(\beta))$ is:

$$\frac{d}{dy} F_{g(X)}(y) = \frac{d}{dy} \int_{\alpha}^{g^{-1}(y)} f(t)\,dt.$$

(By Fundamental Theorem of Calculus)

$$= f(g^{-1}(y)) \cdot (g^{-1})'(y)$$
$$= \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}.$$

$g(X)$ has density function $\dfrac{f(g^{-1}(y))}{g'(g^{-1}(y))}$ with $y \in (g(\alpha), g(\beta))$.

When $g(x) = ax+b$, with $a > 0$, then g is strictly increasing and differentiable on $\mathbb{R}$ with $g^{-1}(y) = \frac{y-b}{a}$ and $g'(x) = a$.

$\Rightarrow$ The density function of $g(x)$ is $\frac{1}{a} f(\frac{y-b}{a})$.

$\blacksquare$

Problem 4. (1.3.1).

Proof: Let $\mathcal{F}$ be the σ-field generated by $X^{-1}(\mathcal{A})$.

Then we need to show that $\mathcal{F} = \sigma(X)$.

① Show that $\mathcal{F} \subseteq \sigma(X)$:

$\mathcal{A}$ generates $\mathcal{S}$ $\Rightarrow$ $\mathcal{A} \subseteq \mathcal{S}$.

$\Rightarrow \forall A \in \mathcal{A}$, we must have $A \in \mathcal{S}$.

$$\Rightarrow X^{-1}(\mathcal{A}) = \{\{X \in A\} : A \in \mathcal{A}\}$$
$$\subseteq \{\{X \in A\} : A \in \mathcal{S}\}$$
$$= \sigma(X).$$

$\mathcal{F}$ is generated by $X^{-1}(\mathcal{A})$ and $\sigma(X)$ is a σ-field containing $X^{-1}(\mathcal{A})$.

$\Rightarrow$ $\mathcal{F} \subseteq \sigma(X)$, because $\mathcal{F}$ is the smallest σ-field containg $X^{-1}(\mathcal{A})$.

② Show that $\sigma(X) \subseteq \mathcal{F}$:

Now we have $\{X \in A\} \in \mathcal{F}$ for all $A \in \mathcal{A}$,

because $X^{-1}(\mathcal{A}) = \{\{X \in A\} : A \in \mathcal{A}\}$ generates $\mathcal{F}$.

Moreover, we are given that $\mathcal{A}$ generates $\mathcal{S}$.

Then by Theorem 1.3.1 from textbook,

$X : (\Omega, \mathcal{F}) \mapsto (S, \mathcal{S})$ is measurable.

i.e. if $B \in \mathcal{S}$, then $\{X \in B\} \in \mathcal{F}$.

$\Rightarrow \sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\} \subseteq \mathcal{F}$.

Therefore, we have $\sigma(X) = \mathcal{F}$

i.e. $X^{-1}(\mathcal{A})$ generateds $\sigma(X)$.

∎

Problem 5 (1.3.5)

Proof:

① Suppose $\liminf_{y\to x} f(y) \geq f(x)$ for all x, I want to show that $\{x : f(x) \leq a\}$ is closed for each $a \in \mathbb{R}$:

Take $a \in \mathbb{R}$.

Take a convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\{x : f(x) \leq a\}$.

(where $\mathbb{N}$ = the set of all natural numbers)

i.e. $f(x_n) \leq a$ for all $n \in \mathbb{N}$ and $x_n \to x$.

Then $f(x) \leq \liminf_{x_n\to x} f(x_n)$ by lower semicontinuity of f.

$\leq a$ because $f(x_n) \leq a$ for all $n \in \mathbb{N}$.

$f(x) \leq a \Rightarrow x \in \{x : f(x) \leq a\}$.

$\{x_n\}_{n\in\mathbb{N}}$ is an arbitrarily chosen convergent sequence in $\{x : f(x) \leq a\}$

$\Rightarrow$ The limit of any convergent sequence in $\{x : f(x) \leq a\}$ lies inside $\{x : f(x) \leq a\}$.

$\Rightarrow \{x : f(x) \leq a\}$ is closed.

② Suppose $\{x : f(x) \le a\}$ is closed for all $a \in \mathbb{R}$, I want to show that $\lim_{y \to x} f(y) \ge f(x)$ for all x.

Take x in the domain of f.

Fix $\epsilon > 0$, then the set $S_1 = \{y : f(y) \le f(x) - \epsilon\}$ is closed.

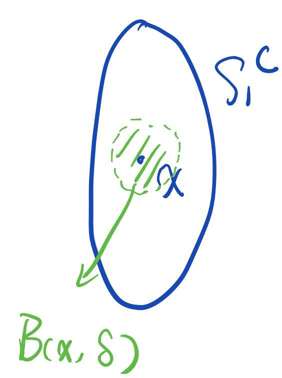
$\Rightarrow S_1^c = \{y : f(y) > f(x) - \epsilon\}$ is open.

In particular, $x \in S_1^c$ because $f(x) > f(x) - \epsilon$.

By definition of open sets, there exists $\delta > 0$ such that

that $$x \in B(x, \delta) \subseteq S_1^c,$$

where $B(x, \delta)$ denotes the open ball centered at x with radius δ.



$\Rightarrow \forall y \in B(x, \delta), \quad f(y) > f(x) - \epsilon.$

$$\Rightarrow \liminf_{y \to x} f(y) \ge f(x) - \epsilon$$

Let $\epsilon \downarrow 0$, we get $\liminf\limits_{y \to x} f(y) \geq f(x)$.

x is arbitrarily chosen from domain of f

$\Rightarrow \liminf\limits_{y \to x} f(y) \geq f(x)$ for all x

$\Rightarrow f$ is lower semicontinuous.

By ① and ② f is lower semicontinuous if and only if $\{x : f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$.

Suppose $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{R})$ is semicontinuous, we have the following senarios:

Senario ① f is lower semicontinuous,

then $\{x : f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$

$\Rightarrow \{x : f(x) > a\}$ is open for all $a \in \mathbb{R}$

i.e. $f^{-1}((a, +\infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.

Also by Real Analysis, we have:

$\{(a,+\infty), a\in\mathbb{R}\}$ generates $\mathcal{R}$.

Then we can apply Theorem 1.3.1 and conclude that f is measurable

Senario ② If f is upper semicontinuous, then $-f$ is lower semicontinuous.

By arguments in Senario ①, we have

$\{x: -f(x) > a\}$ is open for all $a\in\mathbb{R}$

$\Rightarrow \{x: f(x) < -a\}$ is open for all $a\in\mathbb{R}$

i.e. $f^{-1}((-\infty,-a))\in\mathcal{F}$ for all $a\in\mathbb{R}$

$\{(-\infty,-a): a\in\mathbb{R}\} = \{(-\infty,a): a\in\mathbb{R}\}$ generates $\mathcal{R}$.

$\Rightarrow f$ is measurable by Theorem 1.3.1 from our textbook.

In conclusion, if f is semicontinuous, then f is measurable.

Problem 6 (1.3.7)

Proof:

Let C_1 = the smallest class of functions on $(\Omega, \mathcal{F})$ that containing the simple functions and closed under pointwise limits.

C_2 = the class of $\mathcal{F}$ measurable functions.

Want to show: $C_1 = C_2$.

Step 1: Show that every simple function is measurable:

Take a simple function $\varphi: (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{R})$.

Then by definition, we can write φ as:

$\varphi(\omega) = \sum_{m=1}^{n} c_m I_{A_m}(\omega)$, for all $\omega \in \Omega$,

where $c_m \in \mathbb{R} - \{0\}$, $A_m \in \mathcal{F}$ for $1 \le m \le n$.

$c_m I_{A_m}$ is measurable because for any $B \in \mathcal{R}$,

$$I_{A_m}^{-1}(B) = \begin{cases} A_m & \text{if } c_m \in B, 0 \notin B \\ A_m^c & \text{if } c_m \notin B, 0 \in B. \\ \Omega & \text{if } c_m \in B, 0 \in B. \end{cases}$$

We know $A_m \in \mathcal{F}$ by our setup,

then $A_m^c \in \mathcal{F}$.

Also $\Omega = A_m \cup A_m^c \in \mathcal{F}$.

$\Rightarrow c_m I_{A_m}$ is measurable.

(Moreover, $c_m I_{A_m}$ is a random variable)

Then by Theorem 1.3.4 from textbook,

$\varphi = \sum_{m=1}^{n} c_m I_{A_m}$ is a random variable, it is measurable.

$\Rightarrow \varphi \in C_2$ = class of $\mathcal{F}$ measurable functions.

φ is arbitrary $\Rightarrow$ Every simple function lies in C_2.

Step 2: Show that C_2 is closed under pointwise limit:

Suppose $\{f_n\}$ is a sequence of functions in C_2 with $f_n \to f$ pointwise.

i.e. $\forall \omega \in \Omega$, we have $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

We want to show that $f \in C_2$.

Fix $\omega \in \Omega$, $\{f_n(\omega)\}_{n\in\mathbb{N}}$ is a convergent sequence of real numbers with limit $f(\omega)$.

(Note: $f_n \to f$ pointwise also implies that $|f(\omega)| < \infty$)

$f_n(\omega) \longrightarrow f(\omega) \quad \Rightarrow \quad \liminf_{n\to\infty} f_n(\omega) = \lim_{n\to\infty} f_n(\omega) = \limsup_{n\to\infty} f_n(\omega)$

by Real Analysis. (Here we view $\{f_n(\omega)\}$ as sequence of real numbers.)

ω is arbitrary, we have $\liminf_{n\to\infty} f_n(\omega) = f(\omega)$ with $|f(\omega)| < \infty$ for all $\omega \in \Omega$.

$\Rightarrow f = \liminf_{n\to\infty} f_n$ with $\liminf_{n\to\infty} f_n(\omega)$ finite for all ω.

and each f_n is $\mathcal{F}$-measurable with image on $(\mathbb{R}, \mathcal{R})$ (therefore is a random variable).

$\Rightarrow$ $f = \liminf_{n\to\infty} f_n$ is a random variable by

Theorem 1.3.5 from textbook.

In particular, f is $\mathcal{F}$ measurable.

$\Rightarrow f \in C_2$

$\{f_n\}_{n\in\mathbb{N}}$ is an arbitrarily chosen pointwise convergent sequences of functions in C_2.

$\Rightarrow$ C_2 is closed under pointwise limits.

By step 1 and step 2, we have: the class of $\mathcal{F}$-measurable functions (C_2) contains all simple function and is closed under pointwise limit

Therefore, $C_1 \subset C_2$ where

C_1 = the smallest class containing the simple functions and closed under pointwise limits.

Step 3: We show that every $\mathcal{F}$ measurable function is the pointwise limit of a sequence of simple functions (i.e. $C_2 \subseteq C_1$)

Take $f \in C_2$, i.e. f is $\mathcal{F}$ measurable.

We will prove this by first proving this for non-negative f, and then generalizing it to all f.

(Here $f(\omega) \in \mathbb{R}$, $\forall \omega \in \Omega$ implies that $|f(\omega)| < \infty$ for all $\omega \in \Omega$)

<1> Suppose $f(\omega) \geq 0$ for all $\omega \in \Omega$.

Fix $n \in \mathbb{N}$

We define

$$A_k^n = \begin{cases} \{\omega : k2^{-n} \leq f(\omega) < (k+1)2^{-n}\}, & k = 0, 1, \cdots, n2^n - 1 \\ \{\omega : f(\omega) \geq n\}, & k = n2^n \end{cases}$$

Since $f(\omega) \geq 0$ for all $\omega \in \Omega$ by our definition

$\Rightarrow \bigcup_{k=0}^{n2^n} A_k^n = \{\omega : f(\omega) \geq 0\} = \Omega$

And define $f_n(\omega) := \sum_{k=0}^{n2^n} k2^{-n} I_{A_k^n}(\omega)$.

Note for fix n, A_k^n with $k = 0, 1, \cdots, n2^n$ are disjoint.

So if $\omega \in A_k^n$ for some k, then

$$f_n(\omega) = k2^{-n}.$$

Now I want to show that f_n converges to f pointwise.

Fix $x \in \Omega$, and then fix $\epsilon > 0$,

we want to show that there exists a natural number N such that

$$|f_n(x) - f(x)| \leq \epsilon \text{ for all } n \geq N.$$

$f(x) \in \mathbb{R}$, then we can find a natural number N_1

such that $f(x) < N_1$.

Take $N = \max\{N_1, -\log_2 \epsilon\}$

then for each $n \geq N$,

$$n \geq N_1 \Rightarrow f(x) < N_1 \leq n$$

$\Rightarrow$ then $x \in A_k^n$ for some $0 \leq k \leq n2^n - 1$

$$\Rightarrow \quad k2^{-n} \leq f(x) < (k+1)2^{-n}.$$

and $f_n(x) = k2^{-n}$.

$$\begin{aligned} \Rightarrow |f(x) - f_n(x)| &= |f(x) - k2^{-n}| \\ &< (k+1)2^{-n} - k2^{-n} \\ &= 2^{-n} \leq \epsilon \end{aligned}$$

(This is because $n \geq N \geq -\log_2 \epsilon \Rightarrow 2^{-n} \leq 2^{\log_2 \epsilon} = \epsilon$)

Therefore, we have $f_n(x) \to f(x)$

Since x is arbitrarily chosen from Ω, $f_n \to f$ pointwise.

<2> For arbitrary f

Let $f_+(\omega) = \begin{cases} f(\omega) & \text{if } f(\omega) > 0 \\ 0 & \text{otherwise} \end{cases}$

and $f_-(\omega) = \begin{cases} -f(\omega) & \text{if } f(\omega) < 0 \\ 0 & \text{otherwise.} \end{cases}$

then $f_+(\omega) \geq 0$ and $f_-(\omega) \geq 0$ for all $\omega \in \Omega$,

and $f(\omega) = f_+(\omega) - f_-(\omega)$ for all $\omega \in \Omega$.

Also, fix $A \in \mathcal{R}$,

we can write $A = A_1 + A_2$ where

$$A_1 = A \cap [0, +\infty)$$

$$A_2 = A \cap (-\infty, 0)$$

Note $A_1, A_2 \in \mathcal{R}$ because σ-field $\mathcal{R}$ is closed under finite intersection, and $[0,+\infty), (-\infty,0) \in \mathcal{R}$.

then $\{\omega: f_+(\omega) \in A\}$

$= \{\omega: f_+(\omega) \in A_1\} + \{\omega: f_+(\omega) \in A_2\}$.

$= \{\omega: f_+(\omega) \in A_1\}$ because $f(\omega) \notin (-\infty, 0)$ for all $\omega \in \Omega$

$= \{\omega: f(\omega) \in A_1\}$ by definition of f_+

$\in \mathcal{F}$ because f is $\mathcal{F}$ measurable

Similarly, we can derive that

$\{\omega: f_-(\omega) \in A\} = \{\omega: f_-(\omega) \in A_1\}$

$= \{\omega: -f(\omega) \in A_1\}$

$\in \mathcal{F}$

because f is $\mathcal{F}$ measurable, which implies that $-f$ is $\mathcal{F}$ measurable as well.

A is arbitrarily chosen from $\mathbb{R}$

$\Rightarrow f_+, f_-$ are both $\mathcal{F}$ measurable.

and they are non-negative.

Then by previous argument, f_+, f_- are pointwise limits of simple functions.

$\Rightarrow$ For each $x \in \Omega$

$$f_+(x) = \lim_{n\to\infty} f_n^+(x) \quad \text{for some sequence of simple functions } \{f_n^+\}_{n\in\mathbb{N}}$$

$$f_-(x) = \lim_{n\to\infty} f_n^-(x) \quad \text{for some sequence of simple functions } \{f_n^-\}_{n\in\mathbb{N}}$$

$$\Rightarrow f(x) = f_+(x) - f_-(x) = \lim_{n\to\infty} f_n^+(x) - \lim_{n\to\infty} f_n^-(x)$$

$$= \lim_{n\to\infty} \left(f_n^+(x) - f_n^-(x)\right)$$

f is the pointwise limit of $(f_n^+ - f_n^-)$

where $f_n^+ - f_n^-$ is simple because both f_n^+ and f_n^- are simple.

$\Rightarrow f$ is the pointwise limit of a sequence of simple functions.

f is arbitrary

$\Rightarrow$ Every $\mathcal{F}$ measurable function is the pointwise limit of a sequence of simple functions

$\Rightarrow C_2 \subseteq C_1$

Therefore, we have proved

$$C_1 = C_2$$

i.e. the class of $\mathcal{F}$ measurable function is the smallest class containing the simple functions and closed under pointwise limit.