1.1 Probability Space

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1.1.1

(i) If $\mathcal{F}_i$, $i \in I$ are σ-fields, then $\bigcap_{i\in I} \mathcal{F}_i$ is. Here $I = \emptyset$ is an arbitrary index set (i.e., possibly uncountable). (ii) Use the result in (i) to show that if we are given a set Ω and a collection A of subsets of $\mathcal{A}$, then there is a smallest σ-field containing $\mathcal{A}$. We will call this the σ-field generated by $\mathcal{A}$ and denote it by $\sigma(\mathcal{A})$.

(ii) Denote $\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i$.

Take $A \in \mathcal{F}$, then $A \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$. Then $A^C \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$, because $\mathcal{F}_i$'s are σ-fields. Therefore, $A^C \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}$.

Take a countable sequence of sets $A_j \in \mathcal{F}$ for $j = 1, 2, \ldots$, then $A_j \in \mathcal{F}_i$ for all $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$. Then $\bigcup_j A_j \in \mathcal{F}_i$ for $i = 1, 2, \ldots$. Therefore, $\bigcup_j A_j \in \bigcap_i \mathcal{F}_i = \mathcal{F}$.

Then we can conclude that $\mathcal{F}$ is a σ-field.

(ii) Define $\mathcal{F}$ to be the intersection of all σ-field that contains $\mathcal{A}$. Then by (i), $\mathcal{F}$ is a σ-field. And by its definition, every σ-field that contains $\mathcal{F}$ contains $\mathcal{F}$. That means: $\mathcal{F}$ is the smallest σ-field containing $\mathcal{A}$.

1.1.2

Let $\Omega = \mathbb{R}$,$\mathcal{F}$ = all subsets so that A or A^C is countable, $P(A) = 0$ in the first case and $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Proof: First we show that $\mathcal{F}$ is a σ-field:

Take $A \in \mathcal{F}$, then $A = (A^C)^C$ or A is countable. This means that $A^C \in \mathcal{F}$.

Take a countable sequence of sets $A_i \in \mathcal{F}$ for all $i = 1, 2, \ldots$. Then either A_i or A_i^C is countable. If all A_i's are countable, then $\bigcup_i A_i$ is countable, because countable union of countable sets is still countable. Otherwise, there exists uncountable A_j for some j, then we have A_j^C is countable. Then $(\bigcup_i A_i)^C = \bigcap_i A_i^C \subset A_j^C$ is countable. Then $\bigcup_i A_i \in \mathcal{F}$.

Therefore, $\mathcal{F}$ is a σ-field.

Second, we show that P is a measure:

For any $A \in \mathcal{F}$, $P(A) \geq 0 = P(\emptyset)$ by the definition of P. Take a countable sequence of disjoint sets $A_i \in \mathcal{F}$. If all A_i's are countable, then $\bigcup_i A_i$ is also countable. Then $P(\bigcup_i A_i) = 0$, and $P(A_i) = 0$ for all i. Therefore, we have $P(\bigcup_i A_i) = 0 = \sum_i P(A_i)$. Otherwise, there exists an uncountable A_j for some j. Because A_i's are disjoint, $A_i \subset A_j^C$ for all $i \neq j$. $A_j \in \mathcal{F}$ and A_j is uncountable, then A_j^C is countable. Then $A_i \subset A_j^C$ is countable for all $i \neq j$, which means that $P(A_j) = 1$ and $P(A_i) = 0$ for all $i \neq j$. Moreover, we have $A_j \subset \bigcup_i A_i$, and therefore, $\bigcup_i A_i$ is uncountable, and then $P(\bigcup_i A_i) = 1$. Then we have $P(\bigcup_i A_i) = 1 = 1 + 0 = P(A_j) + \sum_{i\neq j} P(A_i) = \sum_i P(A_i)$.

Therefore, P is a measure with $P(\Omega) = P(\mathbb{R}) = 1$. Then P is a probability.

1.1.3

Let $\mathcal{S}_d$ be the empty set plus all sets of the form $(a_i, b_1] \times \ldots \times (a_d, b_d]$ where $-\infty \le a_i < b_i \le \infty$. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}_d$.

Proof: Before I prove this problem, i would like to prove the following lemma: $(a_1, b_1) \times \ldots \times (a_d, b_d)$ where $-\infty \le a_i < b_i \le \infty$ generates $\mathcal{R}^d$.

proof of the lemma:

Denote $\mathcal{F}$ as the σ-algebra generated by $(a_1, b_1) \times \ldots \times (a_d, b_d)$ where $-\infty \le a_i < b_i \le \infty$. Then easy to show that $\mathcal{F} \subset \mathcal{R}^d$. For an open set $A \in \mathbb{R}^d$, and any point $x \in A$, we know there exists rational numbers $a_1^x, \ldots, a_d^x, b_1^x, \ldots, b_d^x$ such that $x \in (a_1^x, b_1^x) \times \ldots \times (a_d^x, b_d^x) \subset A$. Then $A = \bigcup_{x \in A}(a_1^x, b_1^x) \times \ldots \times (a_d^x, b_d^x)$ and there are only countably many sets of the form $(a_1^x, b_1^x) \times \ldots \times (a_d^x, b_d^x)$, which means A is a countable union of sets of form $(a_1, b_1) \times \ldots \times (a_d, b_d)$. Therefore, $A \in \mathcal{F}$. Together with definition of $\mathcal{R}^d$(the σ-algebra generated by all open sets in $\mathbb{R}^d$), we have $\mathcal{R}^d \subset \mathcal{F}$. In conclusion, $\mathcal{F} = \mathcal{R}^d$.

Proof of the problem:

First note that $(a_1, b_1] \times \ldots \times (a_d, b_d] = (a_1, b_1 + 1) \times \ldots \times (a_d, b_d + 1) \cap (b_1, b_1 + 1) \times \ldots \times (b_d, b_d + 1)$. The latter two sets are open set, so they are in $\mathcal{R}^d$; therefore, we can conclude that their intersection $(a_i, b_1] \times \ldots \times (a_d, b_d] \in \mathcal{R}^d$. So $\sigma(\mathcal{S}^d) \subset \mathcal{R}^d$. Second, for an arbitrary given open interval (a, b), we can write $(a, b) = \bigcup_{n=1}^{\infty}(a, b - \frac{1}{n}] \in \sigma(\mathcal{S}^d)$. All open intervals generates $\mathcal{R}^d$. So $\mathcal{R}^d \subset \sigma(\mathcal{S}^d)$. Then we can conclude that $\sigma(\mathcal{S}^d) = \mathcal{R}^d$.

1.1.4

A σ-field $\mathcal{F}$ is said to be countably generated if there is a countable collection $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is countably generated.

Proof: Idea: show that $\mathcal{R}^d$ is generated by $(a_i, b_1] \times \ldots \times (a_d, b_d]$ where $-\infty \le a_i < b_i \le \infty$ and $a_i, b_i \in \mathbb{Q}$. Detail proof please refer to file Homework1.pdf.

1.1.5

(i) Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ are σ-algebras, then $\cup_i \mathcal{F}_i$ is an algebra.

(ii) Give an example to show that $\cup_i \mathcal{F}_i$ need not be a σ-algebra.

Proof:

(i) Take $A \in \cup_i \mathcal{F}_i$, then $A \in \mathcal{F}_i$ for some i. $\mathcal{F}_i$ is a σ-algebra, therefore, $A^c \in \mathcal{F}_i \subset \cup_i \mathcal{F}_i$. Take a finite sequence of sets $A_1, A_2, \ldots, A_n \subset \cup_i \mathcal{F}_i$. Then for each i in $\{1, 2, \ldots, n\}$, $A_i \in \mathcal{F}_{k_i}$ for some k_i. Take $k = max\{k_1, \ldots, k_n\}$, then $A_i \in \mathcal{F}_k$ for all i. Then $\bigcup_i A_i \in \mathcal{F}_k \subset \cup_i \mathcal{F}_i$. Therefore, $\cup_i \mathcal{F}_i$ is an algebra.

(ii) We define the σ-algebra $\mathcal{F}_i$ to be the σ-algebra generated by all the open sets in $[\frac{1}{i}, 1]$. Then clearly we have $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$. Take $A_i = (\frac{1}{i}, 1) \in \mathcal{F}_i$. Then $A_i \in \bigcup_n \mathcal{F}_n$ for all i. But $\bigcup_i A_i = (0, 1) \notin \mathcal{F}_n$ for any n. Therefore, $\bigcup_i A_i = (0, 1) \notin \bigcup_n \mathcal{F}_n$, which means that $\cup_i \mathcal{F}_i$ is not a σ-algebra.

1.1.6

1.1.6. A set $\mathcal{A} \subset \{1, 2, \ldots\}$ is said to have asymptotic density θ if $\lim_{n\to\infty} \mid A \cap \{1, 2, \ldots, n\} \mid /n = \theta$. Let $\mathcal{A}$ be the collection of sets for which the asymptotic density exists. Is $\mathcal{A}$ a σ-algebra? an algebra?

Proof: It is not even an algebra. Let's assume that $C_k = [2^k, 2^{k+1}]$ for $k = 0, 1, \ldots$. Define A be the set of all odd numbers in C_k for odd k and all even numbers in C_k for even k. Then $\lim_{n\to\infty} \mid A \cap \{1, 2, \ldots, n\} \mid /n = \frac{1}{2}$. Let $B = \{2, 4, 6, \ldots\}$, then $\lim_{n\to\infty} \mid B \cap \{1, 2, \ldots, n\} \mid /n = \frac{1}{2}$. Now consider $A \cap B$: This is the set of all even numbers in $[2^{2k}, 2^{2k+1}]$. In each interval like this, there are $2^k + 1$ even numbers. If n approaches ∞ in form of 2^{2k+1}, then this limit turns to be $\frac{1}{3}$; and if n approaches ∞ in form of 2^{2k}, this limit turns to be $\frac{1}{6}$.

This means this limit must diverge. i.e. $A \cap B$ does not have asymptotic density. Therefore, our set is not closed under finite intersection, it cannot be an algebra.

Finished.