Faculty of Science

Breaks For Additive Season and Trend (BFAST)

Theoretical background and results

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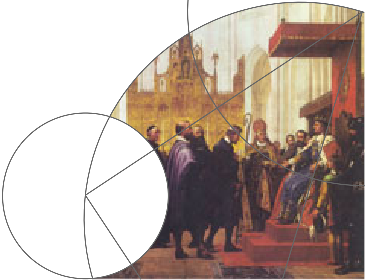


Introduction

- 7.5 ECTS project from Block 5
- In this presentation, I will:
  - Introduce Breaks For Additive Season and Trend (BFAST) sturcutral change detection algorithm (not to be confused with BFAST Monitor)
  - Show examples
  - Describe the algorithm steps of BFAST
  - Elaborate on most important steps of the algorithm and the underlying theory

Structural Changes in Time Series Models

There are three types of changes over time that are of interest to the earth science research:

1. **Seasonal change**: changes that happen within a season (e.g. year)
2. **Gradual change**: changes that are caused by interannual climate variability.
3. **Abrupt change**: rapid changes that are triggered by deforestation, floods, fires and similar.

BFAST

- In the paper from 2010, Verbesselt et. al outline a generic change detection approach that combines iterative decomposition into trend, seasonal and remainder components and detection and characterizing of breakpoints within the trend and seasonal components.
- For BFAST, we consider a following data model:

$$Y_t = T_t + S_t + R_t, \quad t = 1, ..., n$$

  where:
  - Y_t is the observation at time t
  - T_t is the trend component at time t
  - S_t is the seasonal component at time t
  - R_t is the remainder component at time t
  - n is the number of observations in time series

BFAST Example - `harvest`

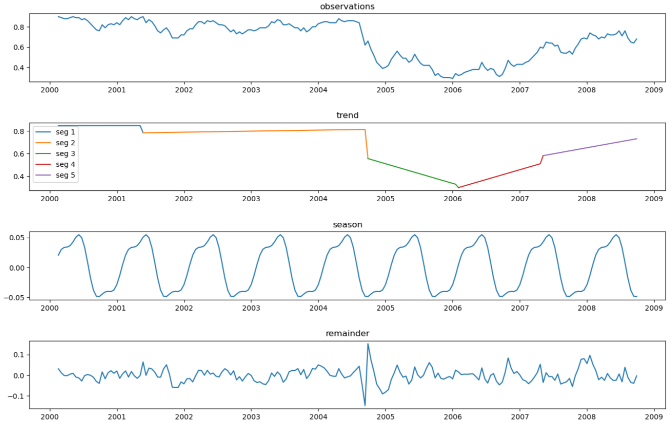


Figure: Normalized Difference Vegetation Index (NDVI) time series for a pine plantation, with 23 observation per year. NDVI is an estimate of the density of green on an area of land. There are 4 breakpoints in the trend component. Harmonic seasonal model

BFAST Example - `simts`

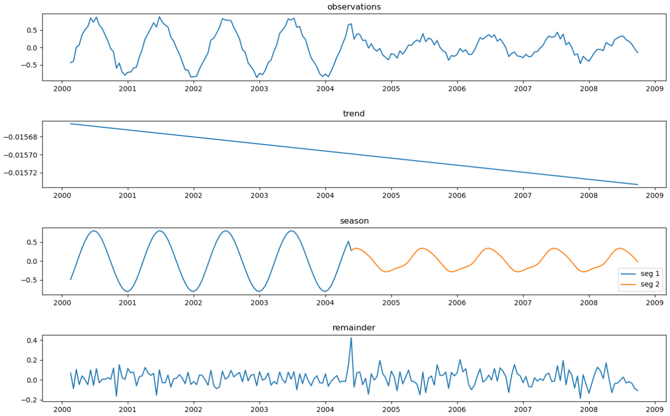


Figure: Simulated seasonal 16-day NDVI time series. The seasonal component has a single breakpoint. Harmonic seasonal model

BFAST Example - `nile`

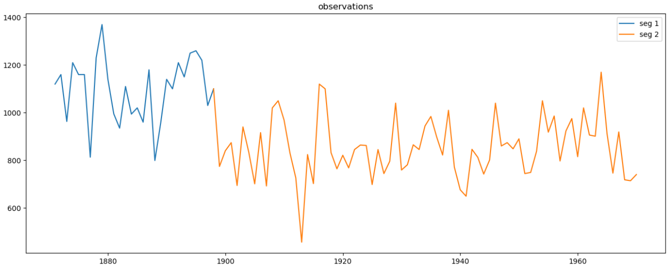


Figure: Measurements of the annual flow of the river Nile with apparent breakpoint near 1898 when the dam was built. There is no seasonal component, since there is one observation each year

BFAST Example - `ndvi`

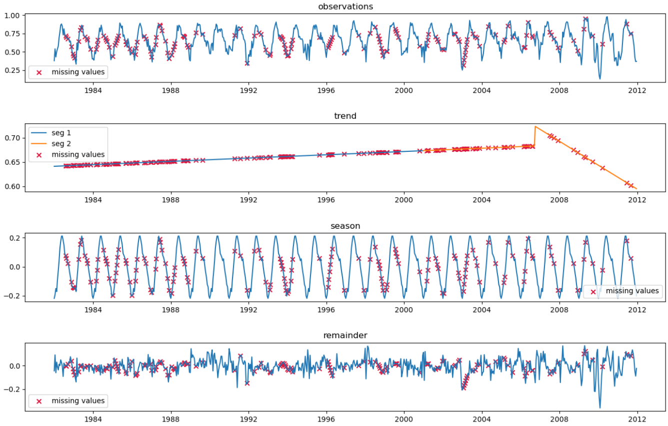


Figure: A random NDVI time series with missing values. Frequency is set to 24. There is a single breakpoint in the trend component. “dummy” seasonal model.

Trend Component

- We assume that T_t is piecewise linear and has breakpoints $t_1^*, \ldots, t_m^*$, where m is the number of breakpoints in the trend component and set $t_0^* = 0$. Then, the trend component can be described as

$$T_t = \alpha_j + \beta_j t \quad \text{for} \quad t_{j-1}^* < t \leq t_j^*$$

  where:
  - j is the number of the next breakpoint, i.e. $j = 1, \ldots, m$.
  - α_j and β_j are the corresponding linear coefficients

Seasonal Component

- There are two seasonal models that were introduced by Verbesselt et al.: Harmonic and “dummy”. This presentation only covers the former.
- The breakpoints in the seasonal component can occur at different times than the breaks in the trend component. Let $t_1^{\#}, \ldots, t_p^{\#}$ be the breakpoints in the seasonal component, where p is the number of breakpoints and $t_0^{\#} = 0$.
- For $t_{j-1}^{\#} < t \leq t_j^{\#}$, the seasonal term can be expressed as:

$$S_t = \sum_{k=1}^{K} \left[\gamma_{j,k} \sin\left(\frac{2\pi kt}{s}\right) + \theta_{j,k} \cos\left(\frac{2\pi kt}{s}\right) \right]$$

  where
  - K is the harmonic term, i.e. the number of pairs of harmonic terms: ($K = 3$ is used in BFAST)
  - $\gamma_{j,k}$ and $\theta_{j,k}$ are the seasonal coefficients
  - t is the observation time.
  - s is the period of seasonality (e.g. number of observations per year)

BFAST Algorithm Steps

- **Estimate S_t using STL decomposition, resulting in $\hat{S}_t$**
- Iterate, until the number and position of the breakpoints do not change during the iteration or the maximum allowed number of iterations is reached:
  1. **Calculate the deasonalized time series:** $V_t = Y_t - \hat{S}_t$
  2. **Apply the OLS-MOSUM test** to V_t If the returned p-value is lower than the significance level α, **estimate the number and position of the trend components** using the breakpoint estimation algorithm by Bai and Perron.
  3. **Compute the trend coefficients** α_j and β_j for $j = 1, \ldots, m$ using linear regression. Set the trend estimate $\hat{T}_t = \hat{\alpha}_j + \hat{\beta}_j t$ for $t = t^*_{j-1} + 1, \ldots, t^*_j$.
  4. **Calculate the detrended time series**: $W_t = Y_t - \hat{T}_t$
  5. **Apply the OLS-MOSUM test** to W_t If the test results show that the breakpoints are present in the seasonal data, **estimate the number and position of the breakpoints in the seasonal component** using the breakpoint estimation algorithm
  6. **Compute the coefficients for the seasonal component and reconstruct** $\hat{S}_t$ from the chosen seasonal model and seasonal coefficient

Ordinary Least Squares (OLS) Regression - Quick Recap

- A crucial component of BFAST
- Linear Model

$$\mathrm{Y} = \mathrm{X}\beta + \varepsilon$$

- Wish to find a solution to a quadratic minimization problem

$$\hat{\beta} = \mathrm{argmin}_{\beta} \left\| Y - \mathrm{X}\beta \right\|^2$$

- $\hat{\beta}$ is the OLS estimator for β and can be found using the explicit formula:

$$\hat{\beta} = \left(\mathrm{X}^\top \mathrm{X}\right)^{-1} \mathrm{X}^\top \mathrm{Y}$$

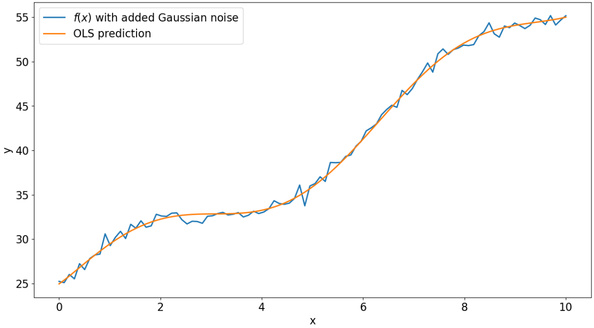
- A numerically stable solution can be obtained using QR-decomposition or Moore-Penrose pseudoinverse of X (an expansive topic in itself).

OLS-Regression for Non-linear Functions

- Linear regression can be used to estimate linear parameters for non-linear functions.
- E.g. $f(x) = 25 + 2x^{1.2} + 3\sin x$, then:

$$\mathrm{X} = \begin{bmatrix} 1 & x_1^{1.2} & \sin x_1 \\ 1 & x_2^{1.2} & \sin x_2 \\ \vdots & \vdots & \vdots \\ 1 & x_n^{1.2} & \sin x_n \end{bmatrix} \quad \beta = \begin{bmatrix} 25 \\ 2 \\ 3 \end{bmatrix}$$

OLS Example



f(x) with added Gaussian noise
OLS prediction
55
50
45
40
35
30
25
y
0
2
4
6
8
10
x

STL Intro

- Seasonal and Trend decomposition using Loess (STL), as first described by Cleveland et al.
- Decomposition of a time series (Y_v) into a trend (T_v), seasonal (S_v) and remainder (R_v) s.t:

$$Y_v = T_v + S_v + R_v \text{ for } v \in 1 \dots N$$

STL Example

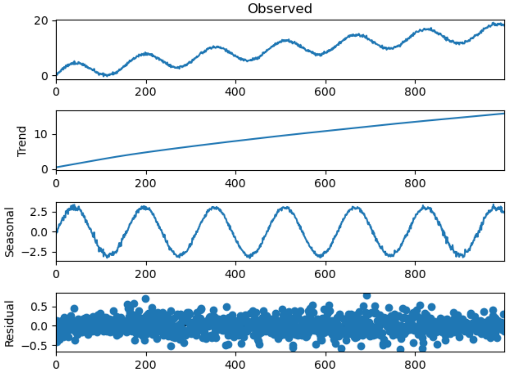


Figure: $f(x) = x^{0.75} + 2\sin(x)$ with added noise $\sim \mathcal{N}(0, 0.25)$

Locally Estimated Scatterplot Smoothing (LOESS)

- For all x fit a curve $\hat{g}(x)$ by giving the other points x_i a weight v_i.
- Select the value of the smoothing factor $q \in \mathbb{Z}^+$ and let $\lambda_q(x)$ be the distance from x to q'th closest x_i. For $q > n$:

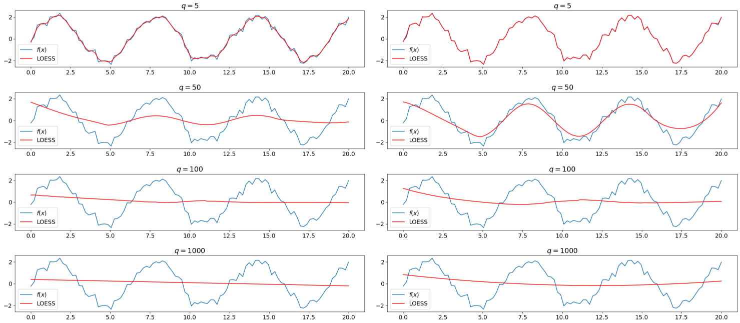
$$\lambda_q(x) = \frac{q \cdot \lambda_n(x)}{n}$$

- We calculate the weights using the tricube weight function:

$$v_i = \left(1 - \left(\frac{|x_i - x|}{\lambda_q(x)}\right)^3\right)^3$$

  for $|x_i - x| \geq \lambda_q(x)$, set $v_i = 0$
- Use locally-linear or locally quadratic fitting with weights $\rho_i v_i$, where ρ_i are the robustness weights that make it possible to ignore the outliers in the dataset.
- The value of x after the application of LOESS is $\hat{g}(x)$,

LOESS Examples



(a) $d = 1$ (b) $d = 2$

Figure: LOESS applied to $f(x) = 2\sin(x)$ with added Gaussian noise and $n = 100$. d is the degree of the fitted polynomial, q is the smoothing factor, when $q \to \infty$, LOESS would be equivalent to an ordinary OLS fit of degree d

STL - Simplified Algorithm Steps

- We begin by setting:

$$T_v^0 = 0, \quad R_v^0 = 0, \quad \rho_v = 1$$

- Then for k in $(0, \ldots, n_{\text{iter}} - 1)$:
  1. **Detrending**: $V_v = Y_v - T_v^k$, where k is the iteration number of the inner loop
  2. **Cycle-Subseries Smoothing**: V_v is split into cycle-subseries, calculate the mean average subseries resulting in C^{k+1}.
  3. **Low-pass Filter of Smoothed Cycle-Subseries**: Apply the low-pass filter to C_{k+1}. This is accomplished by application of two moving averages of lag equal to 3 followed by LOESS smoothing with $q = n_l$ and $d = 1$. The result is saved as L^{k+1}
  4. **Detrending of the Smoothed Cycle-Subseries**: $S^{k+1} = C^{k+1} - L^{k+1}$
  5. **Deseasoning**: $W_v = Y_v - S_v^{k+1}$.
  6. **Trend Smoothing**: Apply LOESS to W_v with $q = n_t$, resulting in T^{k+1}
- Return $T_v^{n_{\text{iter}}}$, $S_v^{n_{\text{iter}}}$ and $R_v^{n_{\text{iter}}} = Y_v - T_v^{n_{\text{iter}}} - S_v^{n_{\text{iter}}}$

OLS-MOSUM Test - The Model

- One step of the BFAST algorithm is to detect structural change in the trend and seasonal components before we commit to the heavy-weight estimation of the number and location of the breakpoints.
- For each observation $i \in (1, \ldots, n)$ we consider a following linear model:

$$y_i = x_i^\top \beta_i + u_i$$

  where:
  - $x_i = (1, x_{i2}, x_{i3}, ..., x_{ik})^T \in \mathbb{R}^k$
  - $u_i \in \mathbb{R}$ is the residual term that is independently and identically distributed with mean $\mu = 0$ and variance σ^2.
- We can then test for structural change by testing the null hypothesis:

$$H_0 : \quad \beta_i = \beta_0 \quad (i = 1, \ldots, n)$$

OLS-MOSUM Test - Continued

- $\hat{\beta}^{(n)}$ is the ordinary least squares (OLS) estimate of the regression coefficients based on all the observations up to n,
- Let the OLS residuals (estimates of u_i) be defined as:

$$\hat{u}_i = y_i - x_i^T \hat{\beta}^{(n)} \tag{1}$$

the variance estimate would then be:

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{i=1}^{n} \hat{u}_i^2 \tag{2}$$

Empirical Fluctuation Process (OLS-MOSUM)

- It is possible to detect structural change by analyzing moving sum of residuals ($\hat{u}$)
- The resulting empirical fluctuation process consists of a sum of a fixed number of residuals in a data interval, which size is determined by the value of the parameter $h \in (0,1)$ (bandwidth).
- The OLS-based MOSUM process at time t is given by:

$$M(t) = \frac{1}{\hat{\sigma}\sqrt{n}} \left(\sum_{i=\lfloor N_n t \rfloor + 1}^{\lfloor N_n t \rfloor + \lfloor nh \rfloor} \hat{u}_i \right) \quad (0 \leq t \leq 1 - h) \tag{3}$$

where $N_n = (n - \lfloor nh \rfloor)/(1-h)$ and $\lfloor nh \rfloor$ is the size of the window.

Significance Testing

- A key observation is that if a structural change takes place at t_0, the OLS-MOSUM path would also have a strong shift at t_0. We reject the null hypothesis of there being no structural change, when the fluctuation of the OLS-MOSUM process becomes too large.
- In practice, we determine whether the null hypothesis can be rejected using a significance test (also called statistical hypothesis testing).
- First, we calculate the test statistic. For the residual-based OLS-MOSUM process, it is defined as:

$$S_{\text{MOSUM}} = \max(|M(t)|) \quad \text{for } 0 \leq t \leq (1-h) \tag{4}$$

Significance Testing - Continued

- This formulation is not usable in a context of an implementation, since we are working with infinite set of real numbers from 0 to $1-h$.
- Another key observation is that $M(t)$ returns $n-\lfloor nh \rfloor+1$ unique values for $0 \leq t \leq (1-h)$:
- Let:

$$\bar{M}(t') = \frac{1}{\hat{\sigma}\sqrt{n}} \left(\sum_{i=t}^{t'+\lfloor nh \rfloor} \hat{u}_i \right) \quad (t' = 1, 2, \ldots, n - \lfloor nh \rfloor + 1)) \tag{5}$$

- Then: $\max(|M(t)|) = \max(|\bar{M}(t')|)$ and we have:

$$S_{\text{MOSUM}} = \max(|\bar{M}(t')|) \quad \text{for } t' \text{ in } 1, 2, .., (n - \lfloor nh \rfloor + 1) \tag{6}$$

Significance Testing - Continued 2

- We use the value of S_{MOSUM} to calculate the probability of getting such sample, given that the null hypothesis holds, using the critical values approach. The p-value is calculated from the table of simulated asymptotic critical values of the Moving Estimate (ME) tests with the maximum norm, given by Chu et al. (1995).
- We then compare the resulting probability with a chosen confidence interval, which is a value $0 < \alpha < 1$
- If the resulting probability is below the value of α, we reject the null hypothesis, hence a structural change is detected in the time series.

OLS-MOSUM Test - Steps of the Algorithm

1. **Calculate $\hat{\beta}^{(n)}$ using OLS-based linear regression from matrix X and vector y**
2. **Calculate the vector of OLS residuals $\hat{\mu}$**
3. **Calculate the standard deviation $\hat{\sigma}$**
4. **Calculate the residual-based OLS-MOSUM process as a vector of size $n - \lfloor nh \rfloor + 1$**

```
nh = floor(n * h)
e = concat([0], residuals)
process = cumsum(e)
process = process[nh:n] - process[0:(n - nh + 1)]
process = process / (sigma * sqrt(n))
```

5. **Calculate the test statistic $S_{\mathbf{test}}$** = `max(abs(process))`
6. **Calculate the p-value using the table of critical values and linear interpolation**
7. **Return the result**:
   If $p \leq \alpha$, we reject the null-hypothesis. Structural change is detected

Breakpoint Estimation - Intro

- Described in the paper by Bai and Perron from 2003
- Estimate the number and position of breakpoints in a time series using a dynamic programming algorithm and Bayesian Information Criterion (BIC)

Breakpoint Estimation - The Model

- We assume a pure structural change model for m breaks ($m+1$ segments):
$$y_t = x_t^\top \beta_j + u_t \quad t = T_{j-1}+1, \ldots, T_j$$
for $j = 1, \ldots, m+1$, and where:
  - $x_t \in \mathbb{R}^q$ is the value of the independent variable at time $t = 1, \ldots, T$
  - $y_t \in \mathbb{R}$ is the observation at time t
  - $\beta_j : (j = 1, \ldots, m+1)$ is the vector of coefficients for the segment j
  - $u_t \in \mathbb{R}$ is the disturbance (error) at time t
  - $(T_1, \ldots, T_m)$ are the unknown indices of the m breakpoints. We additionally set $T_0 = 0$ and $T_{m+1} = T$
- In other words, we split the time series into $m+1$ segments (of potentially different sizes) and perform linear regression independently for each segment, where for segment i, we estimate a coefficient vector β_i. Hence, we find the estimate of the coefficient vector ($\hat{\beta}$) for each m-partition $(T_1, ..., T_m)$ by minimizing the sum of squared residuals:

$$\sum_{i=1}^{m+1} \sum_{i=T_{i-1}+1}^{T_i} \left[y_t - x_t^\top \beta_i \right]^2$$

- Let $\{T_j\}$ denote an m-partition $(T_1, ..., T_m)$ and $S_T(T_1, ..., T_m)$ denote the resulting sum of squared residuals. Since we can estimate the coefficient vector for each partition, we can minimize the sum of squared residuals by finding the optimal position for the breakpoints

$$(\hat{T}_1, ..., \hat{T}_m) = \text{argmin}_{T_1, ..., T_m} S_T(T_1, \ldots, T_m)$$

- There is a finite number of possible breakpoints, and only a subset of possible partitions is feasible. There are $T(T+1)/2$ (sum of integers from 1 to T) possible segments that can be chosen. This can be demonstrated by building a matrix of possible segments, with starting date on the y-axis and terminal date on the y-axis. A length of a segment is positive, hence we can eliminate one half of the potential segments. There are further reductions that can be made that reduce the number of feasible segments to $T(T+1)/2 - \left(T(h-1) - mh(h-1) - (h-1)^2 - h(h-1)/2\right)$, where h is the minimal segment length.

Upper-diagonal Matrix of Sums of Squared Residuals

stop

start	1	2	3	4	5	6	7	8	9
1	nf_1	nf_1	nf_1	f	f	f	nf_2	nf_2	nf_2
2		nf_1	nf_1	nf_1	nf_3	nf_3	nf_2	nf_2	nf_2
3			nf_1	nf_1	nf_1	nf_3	nf_2	nf_2	nf_2
4				nf_1	nf_1	nf_1	f	f	f
5					nf_1	nf_1	nf_1	f	f
6						nf_1	nf_1	nf_1	f
7							nf_1	nf_1	nf_1
8								nf_1	nf_1
9									nf_1

Table: An example of an upper-triangular matrix of sums of squared residuals for $T = 9$, $h = 3$, $m = 2$. We must compute sum of squared residuals for all feasible segments (f).

Links

- Full project report can be found here: `https://raw.githubusercontent.com/mortvest/bfast-py/master/report/main.pdf`
- The source code can be found in the GitHub repository: `http://github.com/mortvest/bfast-py`

Questions

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