Perturbation

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DSGE Model Framework

$$E\left[f_\theta\left(y_{t-1}, y_t, y_{t+1}, u_t \,|\, \Omega_t\right)\right] = 0$$
$$u_s \sim WN(0,\Sigma_u)$$

$t, s \in \mathbb{T}$: discrete time set, typically $\mathbb{N}$ or $\mathbb{Z}$

y_t: n endogenous variables (declared in *var* block)

u_t: n_u exogenous variables (declared in *varexo* block)

Σ_u: covariance matrix of invariant distribution of exogenous variables (declared in *shocks* block)

θ: n_θ model parameters (declared in *parameters* block)

f: n model equations (declared in *model* block)

f_θ is a continuous non-linear function indexed by a vector of parameters θ

$$E\left[f_\theta\left(y_{t-1}, y_t, y_{t+1}, u_t \,|\, \Omega_t\right)\right] = 0$$
$$u_s \sim WN(0, \Sigma_u)$$

Rational Expectations

- information set includes model equations f, value of parameters θ, value of current state y_{t-1}, value of current exogenous variables u_t, invariant distribution (but not values!) of future exogenous variables u_{t+s}
- Ω_t: information set (*filtration*, i.e. $\Omega_t \subseteq \Omega_{t+s} \forall s \geq 0$)
- $\Omega_t = \{f, \theta, y_{t-1}, u_t, u_{t+s} \sim N(0, \Sigma)\}$ for all $t \in \mathbb{T}$, $s > 0$
- $E[\,\cdot\,|\,\Omega_t]$: conditional expectation operator, typically we use shorthand E_t

$$E\left[f_{\theta}\left(y_{t-1}, y_t, y_{t+1}, u_t \,|\, \Omega_t\right)\right] = 0$$

$$E_t\left[f\left(y_{t-1}, y_t, y_{t+1}, u_t\right)\right] = 0$$

Perturbation approach

General idea

Step 1: Introduce perturbation parameter

- scale u_t by a parameter $\sigma \geq 0$: $u_t = \sigma \ \varepsilon_t$ with $\varepsilon_t \sim WN(0, \Sigma_\varepsilon)$
- note that this implies $\Sigma_u = \sigma^2 \Sigma_\varepsilon$
- σ is called the *perturbation parameter*
  - non-stochastic, i.e. static model: $\sigma = 0$
  - stochastic, i.e. dynamic model: $\sigma > 0$

General idea

Step 2: *define* dynamic solution

- invariant mapping between y_t and (y_{t-1}, u_t):

$$y_t = g(y_{t-1}, u_t, \sigma)$$

- $g(\cdot)$ is called the *policy-function* or *decision rule*

- $g(\cdot)$ is unknown, i.e. we need to solve a *functional equation*

General idea

Idea: Maybe we can get g from $E_t\left[f\left(y_{t-1}, y_t, y_{t+1}, u_t\right)\right] = 0$?

From the policy function we can define

- $y_t = g(y_{t-1}, u_t, \sigma)$
- $y_{t+1} = g(y_t, u_{t+1}, \sigma) = g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma)$

General idea

Rewrite dynamic model: $$f\left(y_{t-1} \,,\, y_t \,,\, y_{t+1} \,,\, u_t \right)$$

$$= f\left(y_{t-1},\ g(y_{t-1}, u_t, \sigma),\ g(g(y_{t-1}, u_t \sigma), u_{t+1}, \sigma),\ u_t \right)$$

$$\equiv F(y_{t-1}, u_t, u_{t+1}, \sigma)$$

General idea

Perturbation is based on the *implicit function theorem:*

$$E_t F(y_{t-1}, u_t, u_{t+1}, \sigma) = 0 \quad \text{[known]}$$

implicitly defines

$$g(y_{t-1}, u_t, \sigma) \text{ [unknown]}$$

General idea

We know how to solve for the non-stochastic ($\sigma = 0$) steady-state $\bar{y}$ by solving the *static* model:

$$\bar{f}(\bar{y}) \equiv f(\bar{y}, \bar{y}, \bar{y}, 0) = F(\bar{y}, 0, 0, 0) = 0$$

which provides us with the non-stochastic steady-state for $\bar{y}$

Even though we do not know $g(\cdot)$ explicitly, we do know its value at $\bar{y}$:

$$\bar{y} = g(\bar{y}, 0, 0)$$

Taylor approximation of g

$$y_t = g(y_{t-1}, u_t, \sigma)$$

Let's approximate $g(\cdot)$ around $\bar{y}$ with a 1st order Taylor expansion:

$$y_t \approx \bar{y} + \left[\frac{\partial g(\bar{y},0,0)}{\partial y'_{t-1}}\right](y_{t-1} - \bar{y}) + \left[\frac{\partial g(\bar{y},0,0)}{\partial u'_t}\right](u_t - 0) + \left[\frac{\partial g(\bar{y},0,0)}{\partial \sigma}\right](\sigma - 0)$$

Some progress: instead of an infinite unknown number of parameters for g, we have now only <u>*three*</u> unknown matrices

Taylor approximation of g

But: how do we obtain these?

➡ Let's approximate $F(\,\cdot\,)$ around $\bar{y}$ with a 1st order Taylor expansion!

More Notation

$$u := u_t \, , \, u_+ := u_{t+1}$$

$$y_- := y_{t-1} \, , \, y_0 := y_t \, , \, y_+ := y_{t+1}$$

$$r := \begin{pmatrix} y_- \\ u \\ u_+ \\ \sigma \end{pmatrix} \qquad z := \begin{pmatrix} y_- \\ y \\ y_+ \\ u \end{pmatrix} = \begin{pmatrix} y_- \\ g(y_-, u, \sigma) \\ g(g(y_-, u, \sigma), u_+, \sigma) \\ u \end{pmatrix}$$

Notation Jacobian Matrices

$$g_y := \left[\frac{\partial g(\bar{y},0,0)}{\partial y'_{t-1}}\right] \quad g_u := \left[\frac{\partial g(\bar{y},0,0)}{\partial u'_t}\right] \quad g_\sigma := \left[\frac{\partial g(\bar{y},0,0)}{\partial \sigma}\right] \quad \text{[unknown]}$$

$$f_{y_-} := \left[\frac{\partial f(\bar{z})}{\partial y'_{t-1}}\right] \quad f_{y_0} := \left[\frac{\partial f(\bar{z})}{\partial y'_t}\right] \quad f_{y_+} := \left[\frac{\partial f(\bar{z})}{\partial y'_{t+1}}\right] \quad f_u := \left[\frac{\partial f(\bar{z})}{\partial u'_t}\right] \quad \text{[known]}$$

$$F_y := \left[\frac{\partial F(\bar{r})}{\partial y'_{t-1}}\right] \quad F_u := \left[\frac{\partial F(\bar{r})}{\partial u'_t}\right] \quad F_{u_+} := \left[\frac{\partial F(\bar{r})}{\partial u'_{t+1}}\right] \quad F_\sigma := \left[\frac{\partial F(\bar{r})}{\partial \sigma}\right] \quad \text{[implicit]}$$

All derivatives are evaluated at the non-stochastic steady-state

Taylor approximation of F

Let's approximate $F(r) = F(y_{t-1}, u_t, u_{t+1}, \sigma)$ around $\bar{r}$ at 1st order:

$$F(r) \approx F(\bar{r}) + F_y\, \hat{y}_- + F_u\, \hat{u} + F_{u_+}\, \hat{u}_+ + F_\sigma\, \hat{\sigma}$$

with $\hat{y} = (y_- - \bar{y})$, $\hat{u} = (u - 0) = u$, $\hat{u}_+ = (u_+ - 0) = \sigma\varepsilon_+$, $\hat{\sigma} = (\sigma - 0) = \sigma$

Taylor approximation of F

Our model implies that $E_t F(r) = 0$, so let's use this on the first-order approximation:

$$0 = E_t F(r) \approx 0 \; + \; F_y \, \hat{y}_- \; + \; F_u \, u \; + \; F_{u_+} \, E_t \, \sigma \, \varepsilon_+ \; + \; F_\sigma \, \sigma$$

$$0 \approx F_y \, \hat{y}_- \; + \; F_u \, u \; + \; \left(F_\sigma \; + \; F_{u_+} \, E_t \, \varepsilon_+ \right) \sigma$$

Insight: this equation needs to be satisfied for *any* value of $\hat{y}_-$, u and σ; hence:

$$F_y = 0 \quad \text{and} \quad F_u = 0 \quad \text{and} \quad F_\sigma + F_{u_+} \, E_t \, \varepsilon_+ = 0$$

Taylor approximation of F

We have 3 (multivariate) equations:

$$F_y = 0 \quad \text{and} \quad F_u = 0 \quad \text{and} \quad F_\sigma + F_{u_+} E_t \varepsilon_+ = 0$$

to recover three unknown matrices

- g_y from $F_y = 0$
- g_u from $F_u = 0$
- g_σ from $F_\sigma + F_{u_+} E_t \varepsilon_+ = 0$

Recovering g_σ

Recovering g_σ

$$F = f\left(y_- \ , \ \underbrace{g(y_-, u, \sigma)}_{y_0} \ , \ \underbrace{g(\ \overbrace{g(y_-, u, \sigma)}^{y_0} \ , u_+ \ , \sigma \ )}_{y_+} \ , u \right)$$

First order derivative with respect to σ yields:

$$F_\sigma = f_{y_0}\, g_\sigma \ + \ f_{y_+}\, (g_y\, g_\sigma \ + \ g_\sigma)$$

First order derivative with respect to u_+ yields:

$$F_{u_+} = f_{y_+}\, g_u$$

Recovering g_σ

$$f_{y_0} g_\sigma + f_{y_+}(g_x g_\sigma + g_\sigma) + f_{y_+} g_u E_t \varepsilon_+ = 0$$

$$\Leftrightarrow g_\sigma = -\left(f_{y_0} + f_{y_+} g_x + f_{y_+}\right)^{-1} f_{y_+} g_u E_t \varepsilon_+$$

Of course, we know that $E_t \varepsilon_{t+1} = 0$, which implies:

$$g_\sigma = 0$$

Certainty Equivalence $g_\sigma = 0$

When we derived the optimality conditions (aka model equations) agents do take into account the effect of future uncertainty when optimizing

BUT: the policy function is independent of the size of the stochastic innovations:

$$\hat{y}_t = g_y \hat{y}_{t-1} \ + \ g_u u_t \ + \ 0 \cdot \sigma$$

Future uncertainty does not matter for the decision rules of the agents!

Certainty equivalence is a result of the first-order perturbation approximation, we can break it with e.g. higher-order perturbation approximation

Recovering g_u

Recovering g_u

$$F = f\left(y_- \, , \, \underbrace{g(y_-, u, \sigma)}_{y_0} \, , \, \underbrace{g(\, \overbrace{g(y_-, u, \sigma)}^{y_0} \, , u_+ \, , \sigma \,)}_{y_+} \, , u \right)$$

First order derivative with respect to u yields:

$$F_u = f_{y_0} g_u + f_{y_+} g_y g_u + f_u$$

$F_u = 0$ implies:

$$g_u = -\left(f_{y_0} + f_{y_+} g_y\right)^{-1} f_u$$

Recovering g_u

$$g_u = -\left(f_{y_0} + f_{y_+} g_y\right)^{-1} f_u$$

This is a linear equation which requires computing an inverse involving g_y

Therefore: once we know g_y, we can easily compute g_u.

Recovering g_y

Recovering g_y

$$F = f\left(y_- \;,\; \underbrace{g(y_-, u, \sigma)}_{y_0} \;,\; \underbrace{g(\overbrace{g(y_-, u, \sigma)}^{y_0} \;,\; u_+ \;,\; \sigma\,)}_{y_+} \;,\; u \right)$$

First order derivative with respect to $y\_$ and setting it to zero yields:

$$F_y = f_{y_-} + f_{y_0} g_y + f_{y_+} g_y g_y \stackrel{!}{=} 0$$

This is a *quadratic equation*, but the unknown g_y is a matrix!

It is generally impossible to solve this equation analytically, but there are several ways to deal with this as this boils down to solving so-called *Linear Rational Expectations Models*

Linear Rational Expectations Model

Re-consider original dynamic model:

$$E_t f(y_{t-1}, y_t, y_{t+1}, u_t) = 0$$

Take first-order Taylor expansion:

$$f_{y_-}\hat{y}_{t-1} + f_{y_0}\hat{y}_t + f_{y_+}E_t\hat{y}_{t+1} + f_u u_t = 0$$

In the literature this is known as a *Linear Rational Expectations Model*

Linear Rational Expectations Model

$$f_{y_-}\hat{y}_{t-1} \; + \; f_{y_0}\hat{y}_t \; + \; f_{y_+}E_t\hat{y}_{t+1} \; + \; f_u u_t = 0$$

Using the first-order policy function:

$$\hat{y}_t = g_y\hat{y}_{t-1} + g_u u_t$$

$$E_t\hat{y}_{t+1} = g_y\hat{y}_t + g_u E_t u_{t+1} = g_y(g_y\hat{y}_{t-1} + g_u u_t) = g_y g_y \hat{y}_{t-1} + g_y g_u u_t$$

Rewriting the above equation we see the connection to perturbation:

$$\underbrace{(f_{y_-} + f_{y_0}g_y + f_{y_+}g_y g_y)}_{F_y=0}\;\hat{y}_{t-1} = -\,\underbrace{(f_{y_0}g_u + f_{y_+}g_y g_u + f_u)}_{F_u=0}\;u_t = 0$$

Structural State-Space System

$$f_{y_-}\hat{y}_{t-1} + f_{y_0}\hat{y}_t + f_{y_+}E_t\hat{y}_{t+1} + f_u u_t = 0$$

$$\underbrace{\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix}}_{:=D} \underbrace{\begin{pmatrix} \hat{y}_t \\ E_t\hat{y}_{t+1} \end{pmatrix}}_{:=Y_t} = \underbrace{\begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix}}_{:=E} \underbrace{\begin{pmatrix} \hat{y}_{t-1} \\ \hat{y}_t \end{pmatrix}}_{:=Y_{t-1}} + \underbrace{\begin{pmatrix} -f_u \\ 0 \end{pmatrix} u_t}_{U_t}$$

$$D \cdot Y_t = E \cdot Y_{t-1} + U_t$$

D and E are by construction square matrices

Stability

$$D \cdot Y_t = E \cdot Y_{t-1} + U_t$$

IF D is invertible, then:

$$Y_t = (D^{-1}E)Y_{t-1} + D^{-1}U_t$$

$$= (D^{-1}E)^0 D^{-1}U_t + (D^{-1}E)^1 D^{-1}U_{t-1} + (D^{-1}E)^2 D^{-1}U_{t-2} + (D^{-1}E)^3 D^{-1}U_{t-3} + \ldots$$

Stable solution if and only if all Eigenvalues λ_i of $(D^{-1}E)$ are inside unit circle

Stability

REMINDER: Eigenvalue λ_i and corresponding eigenvector v_i of $(D^{-1}E)$ satisfy:

$$\lambda_i v_i = (D^{-1}E)v_i$$

BUT: D is typically singular and non-invertible!

THEREFORE: use Generalized Eigenvalues λ_i that satisfy:

$$\lambda_i D v_i = E v_i$$

SAME IDEA: stability only for $|\lambda_i| < 1$ (inside unit circle)

MATLAB: *Lambda = eig(E,D)*

Generalized Schur Decomposition

Eigenvalue is defined via a zero determinant of matrix pencil: $det(D + \lambda E) = 0$

So instead of inverse we'll use a Schur decomposition on matrix pencil:

$$D = Q'TZ' \quad \text{and} \quad E = Q'SZ'$$

Q is orthogonal: $Q' = Q^{-1}$ and $Q'Q = QQ' = I$

Z is orthogonal: $Z' = Z^{-1}$ and $Z'Z = ZZ' = I$

T is upper triangular and S is quasi-upper triangular

MATLAB: *[S,T,Q,Z] = qz(E,D)*

Generalized Eigenvalues

Stability: look at *Generalized Eigenvalues* of D and E:

$$\lambda_i D v_i = E v_i$$

which can be found on the diagonal of S and T: $\lambda_i = \dfrac{S_{ii}}{T_{ii}}$

If $T_{ii} = 0$, then: $S_{ii} > 0 \rightarrow \lambda_i = \infty$ and $S_{ii} < 0 \rightarrow \lambda_i = -\infty$

Structural State-Space System

$$\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix} \begin{pmatrix} \hat{y}_t \\ E_t \hat{y}_{t+1} \end{pmatrix} = \begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{y}_{t-1} \\ \hat{y}_t \end{pmatrix} + \begin{pmatrix} -f_u \\ 0 \end{pmatrix} u_t$$

Insert the policy functions:

$$\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix} \begin{pmatrix} g_y \hat{y}_{t-1} + g_u u_t \\ g_y (g_y \hat{y}_{t-1} + g_u u_t) + g_u \underbrace{E_t u_{t+1}}_{=0} \end{pmatrix} = \begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{y}_{t-1} \\ g_y \hat{y}_{t-1} + g_u u_t \end{pmatrix} + \begin{pmatrix} -f_u u_t \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1} + \begin{pmatrix} -f_{y_+} g_y g_u \\ -g_u \end{pmatrix} u_t + \begin{pmatrix} -f_{y_0} g_u \\ g_u \end{pmatrix} u_t + \begin{pmatrix} -f_u \\ 0 \end{pmatrix} u_t$$

$$\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1} + \begin{pmatrix} -F_u \\ -g_u + g_u \end{pmatrix} u_t$$

$$\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

Schur Decomposition on Structural State-Space System

$$\underbrace{\begin{pmatrix} 0 & f_{y_+} \\ I & 0 \end{pmatrix}}_{D} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \underbrace{\begin{pmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{pmatrix}}_{E} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

$$Q'TZ' \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = Q'SZ' \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

Multiply by Q:

$$TZ' \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = SZ' \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

Re-ordering of Schur decomposition

$$TZ' \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = SZ' \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

Order stable Generalized Eigenvalues $|\lambda_i| < 1$ in the upper left corner of T and S:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

T_{11} and S_{11} are square matrices and contain stable Generalized Eigenvalues

T_{22} and S_{22} are square matrices and contain unstable Generalized Eigenvalues

Impose Stability

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} g_y \hat{y}_{t-1} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} \hat{y}_{t-1}$$

We ***DON'T WANT*** an explosive solution, so we rule this out by imposing:

$$\begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} = \begin{pmatrix} \mathrm{XXX} \\ 0 \end{pmatrix}$$

such that the lower (explosive) rows are always zero:

$$0 \cdot XXX + T_{22} \cdot 0 = 0 \cdot XXX + S_{22} \cdot 0 = 0$$

Impose Stability

$$Z' \begin{pmatrix} I \\ g_y \end{pmatrix} = \begin{pmatrix} XXX \\ 0 \end{pmatrix}$$

Pre-multiply by Z:

$$\underbrace{ZZ'}_{I} \begin{pmatrix} I \\ g_y \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} XXX \\ 0 \end{pmatrix}$$

Focusing on the upper rows we get:

$$Z_{11} \cdot XXX \;+\; Z_{12} \cdot 0 = I \Leftrightarrow XXX = (Z_{11})^{-1}$$

Recovering g_y

$$\begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} I \\ g_y \end{pmatrix} = \begin{pmatrix} (Z_{11})^{-1} \\ 0 \end{pmatrix}$$

From the lower rows we can recover g_y:

$$Z'_{12} \cdot I + Z'_{22} \cdot g_y = 0$$

$$g_y = -(Z'_{22})^{-1} Z'_{12}$$

Revisit Blanchard & Khan (1980) conditions

1. Order condition:
   Squareness of Z_{22}, i.e. requirement to have as many explosive Eigenvalues as **jumper variables** (forward or mixed endogenous variables)

2. Rank condition:
   Invertibility of Z_{22}, i.e. full rank of Z_{22}

Blanchard & Khan (1980) conditions

Provided that the rank condition is satisfied, three cases are possible:

UNIQUE STABLE SOLUTION

Number of forward or mixed variables == Number of explosive Eigenvalues

INDETERMINACY

Number of forward or mixed variables > Number of explosive Eigenvalues

EXPLOSIVENESS

Number of forward or mixed variables < Number of explosive Eigenvalues

Summary

Summary

Policy function / decision rule:

$$y_t = \bar{y} + g_y(y_{t-1} - \bar{y}) + g_u u_t$$

Algorithm:

1. create D and E matrices
2. do a QZ/Schur decomposition with re-ordering
3. $g_y = -(Z'_{22})^{-1} Z'_{12}$
4. $g_u = -(f_{y_0} + f_{y_+} g_y)^{-1} f_u$

Summary

g_y is a $n \times n$ matrix

- only columns wrt state (predetermined and mixed) variables are nonzero; Dynare's $oo\_.dr.ghx$ focuses only on states
- rows are in declaration order; rows in Dynare's $oo\_.dr.ghx$ are in DR order

g_u is a $n \times n_u$ matrix

- rows are in declaration order; rows in Dynare's $oo\_.dr.ghu$ are in DR order

Higher-order perturbation

Higher-order perturbation

$$y_t = \bar{y} + g_x(x_{t-1} - \bar{x}) + g_u u_t$$

$$+\frac{1}{2}\left[g_{xx}(x_{t-1} - \bar{x}) \otimes (x_{t-1} - \bar{x}) + 2g_{xu}(x_{t-1} \otimes u_t) + g_{uu}(u_t \otimes u_t) + g_{\sigma\sigma}\sigma^2\right]$$

$$+\frac{1}{6}[\ldots] + \ldots$$

Higher-order perturbation

- Computationally easier (no quadratic equations, only linear ones), but notation tedious (tensors, Einstein notation, Faá di Bruno formula)

- Available at arbitrary Taylor approximation order:

```
stoch_simul(order=4);
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- Break with certainty equivalence for order>1: constant (2), linear (3), quadratic (4) corrections for uncertainty (important for e.g. pre-cautionary savings)

- Subject to computational limits

- Problem of explosive simulation trajectories (use pruning or bounded shocks)