ON THE STRUCTURE OF THE BLOCH–KATO SELMER GROUPS OF MODULAR FORMS OVER ANTICYCLOTOMIC $\mathbf{Z}_p$-TOWERS

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ABSTRACT. Let p be an odd prime number and let K be an imaginary quadratic field in which p is split. Let f be a modular form with good reduction at p. We study the variation of the Bloch–Kato Selmer groups and the Bloch–Kato–Shafarevich–Tate groups of f over the anticyclotomic $\mathbf{Z}_p$-extension K_∞ of K. In particular, we show that under the generalized Heegner hypothesis, if the localization of the generalized Heegner cycle attached to f at one of the primes above p is primitive and certain local conditions hold, then the Pontryagin dual of the Selmer group of f over K_∞ is free over the Iwasawa algebra. Consequently, the Bloch–Kato–Shafarevich–Tate groups of f vanish. This generalizes earlier works of Matar and Matar–Nekovář on elliptic curves. Furthermore, our proof applies uniformly to the ordinary and non-ordinary settings.

1. INTRODUCTION

1.1. **Setting and notation.** Fix once and for all a prime number $p > 2$, an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, and an embedding $\iota_p \colon \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $N \geq 1$ be an integer and $f = \sum_{n>0} a_n(f) q^n$ be a normalized cuspidal eigen-newform of even weight $k \geq 2$ and level $\Gamma_0(N)$, and let $\mathfrak{F} = \mathbf{Q}_p\left(a_n(f) : n > 0\right)$ be the finite extension of $\mathbf{Q}_p$ generated by the Fourier coefficients of f under the embedding ι_p; denote by $\mathfrak{o}$ the ring of integers of $\mathfrak{F}$ and fix a uniformizer ϖ of $\mathfrak{o}$. Deligne [Del71] attached to f a p-adic representation

$$\rho_f \colon G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathfrak{F})$$

that is unramified outside of pN. Denote by V_f the underlying vector space, and let $V = V_f(k/2)$ be its central critical twist. When $p \nmid 2(k-2)! N \varphi(N)$, where $\varphi(N)$ is Euler's totient function, Nekovář [Nek92, §3] constructed a lattice $T \subset V$ that is endowed with a $G_{\mathbf{Q}}$-equivariant skew-symmetric perfect pairing

$$T \times T \to \mathfrak{o}(1),$$

making T, and hence V, self-dual. That is, $T \simeq \mathrm{Hom}_{\mathfrak{o}}(T, \mathfrak{o}(1))$ and $V \simeq \mathrm{Hom}_{\mathfrak{F}}(V, \mathfrak{F}(1))$; *cf.*, [Kat04, (14.10.1)]. We put $A = V/T$.

Let K be an imaginary quadratic field of discriminant $d_K \neq -3, -4$ coprime to Np. Write N as the product $N = N^+ N^-$ with N^+ (resp. N^-) divisible only by primes that split (resp. remain inert) in K. We assume that (K, N) satisfies the *generalized Heegner hypothesis*:

(Heeg.) N^- is a squarefree product of an even number of primes.

Next, let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension of K, and K_n be the intermediate extension with $\mathrm{Gal}(K_n/K) \simeq \mathbf{Z}/p^n\mathbf{Z}$. We will assume $p \nmid h_K$, so that both places above p in K are totally ramified in K_∞/K. By an abuse of notation, we will denote by v and $\bar{v}$ the two places above p in K_n for $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. Put $\Gamma = \mathrm{Gal}(K_\infty/K)$ and $\Lambda = \mathfrak{o}[[\Gamma]]$.

Finally, if M is a $\mathbf{Z}_p$-module, we denote by $M^\vee$ its Pontryagin dual $\mathrm{Hom}(M, \mathbf{Q}_p/\mathbf{Z}_p)$, and by M_{div} its maximal divisible subgroup.

1.2. **Background.** In this paper, we are interested in the anticyclotomic Iwasawa theory of Selmer groups attached to the form f, assuming local primitivity at v of the associated generalized Heegner class. We shall begin with a brief historical account which will provide some context to our results to be stated below. We invite the intrigued reader to consult [MN19, §0] for a much more detailed overview.

Our study of the Selmer groups is deeply rooted in Kolyvagin's original breakthrough [Kol90], where, under some assumptions, a bound of the Shafarevich–Tate group over K is given when f corresponds to a rational non-CM elliptic curve E for which the Heegner point $y_K \in E(K)$ is nontorsion. Gross [Gro91] subsequently gave a self-contained proof in the simplified setting when $y_K \notin pE(K)$, in which case the p-part of the Shafarevich–Tate group of E over K is trivial, and hence $\mathrm{Sel}_{p^\infty}(E/K)$ coincides with $E(K) \otimes \mathbf{Q}_p/\mathbf{Z}_p$, generated by y_K.

Assuming furthermore that E has good ordinary reduction at p, the work of Matar–Nekovář [MN19], among other things, put the result of Kolyvagin–Gross in the Iwasawa theoretic context, and proved the vanishing of the p-primary Shafarevich–Tate groups of E in the anticyclotomic $\mathbf{Z}_p$-tower. Furthermore, they studied the structure of $\mathrm{Sel}_{p^\infty}(E/K_n)$ and showed that the Pontryagin dual of $\mathrm{Sel}_{p^\infty}(E/K_\infty)$ is a free Λ-module of rank one. When E has good supersingular reduction at p, building on the work of Longo–Vigni [LV19b] and the plus/minus theory of Kobayashi [Kob03], Matar [Mat21a, Mat21b] extended the results of [MN19] to show that the p-primary Shafarevich–Tate groups of E in the anticyclotomic $\mathbf{Z}_p$-tower are once again trivial, whereas the Pontryagin dual of $\mathrm{Sel}_{p^\infty}(E/K_\infty)$ is a free Λ-module of rank two.

Recently, the second-named author [Mas25] established the analogous result of Kolyvagin–Gross in the higher weight case, with the Heegner point replaced by the generalized Heegner class $z_{f,K} \in H^1(K, T)$ (the exact definition is recalled in §3). In this vein, our goal here is to study the variation of Selmer groups of higher weight modular forms in the anticyclotomic tower.

1.3. **Main result.** We now state our main theorem, which corresponds to Proposition 5.1, Corollary 5.4 and Theorem 6.2 in the main body of the article. Below, for a number field L, $\mathrm{Sel}(L, A)$ denotes the usual Bloch–Kato Selmer group.

Theorem A. *Let f be a normalized cuspidal eigen-newform of even weight k and level $\Gamma_0(N)$. Let K be an imaginary quadratic field with coprime-to-Np discriminant $d_K \neq -3, -4$ and satisfying (Heeg.). Assume moreover $N \geq 5$ if $N^- = 1$, and $N^+ > 3$ and $k \geq 4$ if $N^- \neq 1$. Let p be a prime such that:*

(i) *p splits in K, and does not divide the class number h_K;*
(ii) *$p \nmid 2(k-2)!N\varphi(N)$;*
(iii) *if f has weight 2 and is p-ordinary, then $a_p \not\equiv 1 \pmod{\varpi}$;*
(iv) *for any $w \mid N$ in K and any $w' \mid w$ in K_∞, the group $A^{G_{K_\infty,w'}}$ is divisible;*
(v) *the class $z_{f,K}$ is locally primitive at v, i.e., its restriction $\mathrm{loc}_v(z_{f,K}) \in H^1_{\mathbf{f}}(K_v, T)$ is not in $\varpi H^1_{\mathbf{f}}(K_v, T)$;*
(vi) *$\mathrm{Sel}(K, A) = \mathfrak{F}/\mathfrak{o} \cdot z_{f,K}$.*

Then we have:

(1) *for $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, the relaxed-strict (or BDP) Selmer group $\mathrm{Sel}^{\emptyset,0}(K_n, A)$ is trivial;*
(2) *for $n \in \mathbf{Z}_{\geq 0}$, $\mathrm{Sel}(K_n, A) \simeq (\mathfrak{F}/\mathfrak{o})^{\oplus p^n}$;*
(3) *the Pontryagin dual $\mathrm{Sel}(K_\infty, A)^\vee$ is a free Λ-module of rank one (resp. two) if f is ordinary (resp. non-ordinary) at p.*

Remark 1.1. We now explain the *raisons d'être* of the hypotheses of Theorem A. The splitting assumption (i) is inherent in our approach, namely the use of BDP Selmer groups in the study

of usual Selmer groups; see §1.4 for more details. As mentioned earlier, condition (ii) originates from Nekovář's construction of the lattice T. Entry (iii) gives the triviality of $A[\varpi]^{G_{K_{\infty,w}}}$ for $w \mid p$, which ensures certain desirable properties of the local and global cohomology groups under consideration (see Lemma 4.1). Condition (iv) is a higher-weight variant of the p-freeness of Tamagawa numbers (*cf.* [Gre99, remark following Lemma 3.3]), and is needed for the exact control theorem Proposition 4.2. The premises (v)-(vi) are key inputs of our proof of the vanishing of the BDP Selmer groups. The local primitivity statement (v) is analogous to the assumption that the Heegner point y_K of an elliptic curve E satisfies $y_K \notin pE(K_v)$ in [Mat21a, Theorem 1.4]. Note that (v) implies $z_{f,K} \notin \varpi H^1(K, T)$, and thus if f does not have complex multiplication in the sense of [Rib77, p. 34] and p is large enough, (vi) is true by a result of the second-named author (see Theorem 3.1 and Remark 3.2).

Remark 1.2. As the reader will notice, our proof of Theorem A in §5 and §6 does not employ the generalized Heegner class in an essential way. Rather, we rely on the existence of a cohomology class $z \in H^1_{\mathbf{f}}(K, T)$ for which the conditions given by hypotheses (v) and (vi) above are met; note that any two such classes z, z' differ only by an element of $\mathfrak{o}^\times$. As such, when $N^- = 1$, Theorem A holds when $z_{f,K}$ is replaced by the classical Heegner class $P(1) \in H^1_{\mathbf{f}}(K, T)$ constructed by Nekovář [Nek92], where the corresponding statement of (vi) is a result of Besser [Bes97, proof of Theorem 1.2] under some assumptions on p (when $N^- \neq 1$, there is also a partial result in this direction given in [EdVP18], where the bound of Shafarevich–Tate groups is not made explicit). In this article, we settle for using the class $z_{f,K}$, because

- condition (vi) is known to be valid by the work of the second-named author under the generalized Heegner hypothesis together with certain additional assumptions; see §3 for an extensive discussion of this result;
- from the perspective of the BDP Iwasawa main conjecture, the vanishing of $\mathrm{Sel}^{\emptyset,0}(K_\infty, A)$ is tied with $z_{f,K}$ being globally primitive in $H^1(K, T)$, as explained in Remark 5.3. Therefore, it is natural to choose $z_{f,K}$ as our "prima donna".

1.4. **Strategy of proof.** Our proof can be regarded as a synthesis of two important philosophies: that of Matar–Nekovář [MN19], whose insight is that precise information of Selmer modules varying in the tower can be obtained from that at the bottom via purely Iwasawa-theoretical methods; and that of Kobayashi–Ota [KO20], which emphasizes the utility of the BDP Selmer groups that can be handled uniformly for both ordinary and non-ordinary primes, making them remarkable surrogates to the usual Selmer groups whose behaviors are far more intricate.

More precisely, our proof of Theorem A proceeds as follows. First, in §5, we establish the vanishing of BDP, i.e., relaxed-strict, Selmer groups, under the assumption of local primitivity of $z_{f,K}$ at the prime v; along the way, we review the control theorem for BDP Selmer groups in §4. The rest of §5 then uses the vanishing to prove that the localizations $\mathrm{Sel}(K_n, A) \to H^1_{\mathbf{f}}(K_{n,w}, A)$ are isomorphisms for $n \in \mathbf{Z}_{\geq 0}$ and $w \mid p$ (the subscript $\mathbf{f}$ denotes the Bloch–Kato local condition, and is recalled in §2). On the one hand, this gives the growth formula for Selmer groups. On the other, it converts the study of Selmer groups to that of local cohomology groups, which we investigate in §6 using the theory of universal norms from p-adic Hodge theory.

1.5. **Obiter dictum.** As an aside, in Appendix A, we present an alternative approach to results similar to Theorem A in the special case where $a_p(f) = 0$. This approach is modeled on that of Matar [Mat21a], which is based on the study of the (generalized) Heegner modules and crucially uses the plus/minus theory originally developed by Kobayashi in [Kob03] (and later extended to modular forms in [Lei11, Lei10]). In this case, under some extra condition on the first and second generalized Heegner classes, the corank growth formula can be proved by assuming the global primitivity of $z_{f,K}$ (see Theorem B.(II)), which complements Theorem A. As part of the proof, we

show that the Pontryagin duals of the plus and minus Selmer groups are free of rank one over Λ, generalizing [Mat21a, Theorem 3.1]. When $a_p(f) \neq 0$, even though Sprung's $\sharp/\flat$ theory [Spr12] (and the higher weight generalization) is available in the anticyclotomic setting (see [BL21]), it is not clear to the authors how to extend the proof of Theorem B to this generality. The main obstacle is the lack of an explicit description of the $\sharp/\flat$ Selmer groups over finite extensions K_n. Such description plays a crucial role in the proof of Theorem B.

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2. Selmer groups

From now on, we will always assume that p splits in K and $p \nmid 2(k-2)!N\varphi(N)h_K$.

In this section, we introduce the various notions of Selmer groups that will be utilized throughout the paper. Let $S = \{\ell : \ell \mid N\}$ and for any finite extension E/K, let S_E denote the set of primes of E above those of S. For $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ we also use the shorthand S_n in place of S_{K_n}. Write $\Sigma = S \cup \{p, \infty\}$, $\Sigma_E = S_E \cup \{w \mid p\infty\}$ and let E_Σ be the maximal unramified extension of E unramified outside Σ_E.

Recall that for any place w of E the Bloch–Kato finite local condition $\mathbf{f}$ on V is defined as

$$H^1_{\mathbf{f}}(E_w, V) = \begin{cases} \ker\left(H^1(E_w, V) \to H^1(E_w, V \otimes B_{\mathrm{cris}})\right) & \text{if } w \mid p; \\ \ker\left(H^1(E_w, V) \to H^1(I_w, V)\right) & \text{if } w \nmid p, \end{cases}$$

where $I_w \subset G_{E_w}$ is the inertia subgroup at w and B_{cris} is Fontaine's ring of cristalline periods. This defines also local subgroups $H^1_{\mathbf{f}}(E_w, A) \subseteq H^1(E_w, A)$ and $H^1_{\mathbf{f}}(E_w, T) \subseteq H^1(E_w, T)$ taking respectively its image and preimage via the morphisms induced by the natural projection $V \twoheadrightarrow A$ and inclusion $T \subset V$.

For $M = T, V, A$ the Bloch–Kato Selmer group is defined as

$$\mathrm{Sel}(E, M) = \ker\left(H^1(E_\Sigma, M) \to \prod_{\substack{w \in \Sigma_E \\ w \nmid \infty}} \frac{H^1(E_w, M)}{H^1_{\mathbf{f}}(E_w, M)}\right).$$

Note that the infinite places are ignored since $p > 2$. For $E = K_n$ we will consider additional Selmer groups by varying the local conditions at places dividing p. Thus for $w \in \{v, \bar{v}\}$ a place of K_n we introduce:

- the relaxed condition $\emptyset$ defined by

$$H^1_\emptyset(K_{n,w}, A) = H^1(K_{n,w}, A);$$

- the strict condition 0 defined by

$$H^1_0(K_{n,w}, A) = 0 \subseteq H^1(K_{n,w}, A).$$

We define for $\star, \bullet \in \{\mathbf{f}, 0, \emptyset\}$, the $(\star, \bullet)$-Selmer group of A as

$$\mathrm{Sel}^{\star, \bullet}(K_n, A) = \ker\left(H^1(K_{n,\Sigma}, A) \to \prod_{w \in S_n} \frac{H^1(K_{n,w}, A)}{H^1_{\mathbf{f}}(K_{n,w}, A)} \times \frac{H^1(K_{n,v}, A)}{H^1_{\star}(K_{n,v}, A)} \times \frac{H^1(K_{n,\bar{v}}, A)}{H^1_{\bullet}(K_{n,\bar{v}}, A)} \right).$$

Note that $\mathrm{Sel}^{\mathbf{f}, \mathbf{f}}(K_n, A) = \mathrm{Sel}(K_n, A)$. In addition, we put $\mathrm{Sel}^{\star, \bullet}(K_\infty, A) = \varinjlim_n \mathrm{Sel}^{\star, \bullet}(K_n, A)$.

We will call $\mathrm{Sel}^{\emptyset, 0}(K_n, A)$ the BDP-Selmer group of A and will denote it by $\mathrm{Sel}^{\mathrm{BDP}}(K_n, A)$, its name derived from its link with the p-adic L-function of Bertolini–Darmon–Prasanna [BDP13] and Brakočević [Bra11] via the corresponding Iwasawa main conjecture; see [Cas17, Theorem 3.4], [KO20, Theorem 1.5] and [LZ24, Theorem A], as well as our discussion in Remark 5.3 below.

Occasionally, we shall also consider compact Selmer groups over K_n:

$$\mathrm{Sel}^{\star, \bullet}(K_n, T) = \varprojlim_m \mathrm{Sel}^{\star, \bullet}(K_n, A)[p^m].$$

3. Review of generalized Heegner cycles

In what follows, we work with a distinguished set of cohomology classes $z_{f,c} \in \mathrm{Sel}(K[c], T)$, where $K[c]$ denotes the ring class field of K of conductor $c \in \mathbf{Z}_{>0}$. Throughout this section, we shall assume $k > 2$ when $N^- \neq 1$, as this is the setting of [Mag22], where such classes are constructed in the quaternionic case.

In the case $N^- = 1$, we define $z_{f,c} \in H^1(K[c], T)$ to be the class $z_{f,\chi,c}$ of [CH18, (4.6)] for χ the trivial character. These classes are the image of (a subclass of) the generalized Heegner cycles of [BDP13] via a suitable étale Abel–Jacobi map (see [CH18, §4.2, 4.4] and [Mas25, §3.3])

$$\mathrm{AJ}_{K[c]} \colon \mathrm{CH}^{k-1}(X_{k-2}/K[c])_0 \otimes \mathfrak{o} \to H^1(K[c], T).$$

Here, X_{k-2} is the so called *generalized Kuga–Sato variety*, namely the product over the modular curve $X_1(N)$ of the Kuga–Sato variety W_{k-2} and $(k-2)$-copies of a CM elliptic curve A defined over $K[1]$. As explained in [CH18, Remark 4.8], the image of $\mathrm{AJ}_{K[c]}$ is contained in $\mathrm{Sel}(K[c], T)$, so in particular $z_{f,c} \in \mathrm{Sel}(K[c], T)$.

When $N^- \neq 1$, we can define such classes in a similar way, after replacing W_{k-2} with the Kuga–Sato variety defined over a (quaternionic) Shimura curve and A with a false elliptic curve with CM by $\mathcal{O}_K$. This construction is carried out in [HB15] and [Mag22]. We write $z_{f,c}$ for the class $z_{\chi,c}$ of [Mag22, (5.2)] with χ taken to be the trivial character. We shall abuse the notation and use the same letters $\mathrm{AJ}_{K[c]}$ to denote the étale Abel–Jacobi map of [Mag22, §5.8-5.9]; again its image is contained in $\mathrm{Sel}(K[c], T)$ and in particular $z_{f,c} \in \mathrm{Sel}(K[c], T)$ as noted in [Mag22, proof of Proposition 7.9].

In both cases, we let

$$z_{f,K} = \mathrm{Cores}_{K[1]/K}(z_{f,1}) \in \mathrm{Sel}(K, T).$$

The classes $z_{f,c}$ satisfy certain norm relations as c varies, forming an (anticyclotomic) Euler system (see [CH18, Proposition 7.4] for the case $N^- = 1$, [Mag22, Proposition 7.9] for the case $N^- \neq 1$). Let $\mathscr{K}$ be the set of square-free integers n that are products of primes $\ell \nmid 2Np$ that are inert in K. For any $n = m\ell \in \mathscr{K}$, with ℓ a prime, we have

(E1) $\mathrm{Cores}_{K[n]/K[m]}(z_{f,n}) = a_\ell z_{f,m};$

(E2) for any compatible choice of primes λ_n of $K[n]$ and λ_m of $K[m]$ lying above ℓ, we have $\mathrm{loc}_{\lambda_n}(z_{f,n}) = \mathrm{Res}_{K[m]_{\lambda_m}/K[n]_{\lambda_n}}(\mathrm{Frob}_\ell \cdot \mathrm{loc}_{\lambda_m}(z_{f,m}))$. Here, $\mathrm{loc}_{\lambda_s}: H^1(K[s], T) \to H^1(K[s]_{\lambda_s}, T)$ denotes the localization at λ_s for $s = n, m$, and $\mathrm{Frob}_\ell \in \mathrm{Gal}(\mathbf{Q}_\ell^{\mathrm{ur}}/\mathbf{Q}_\ell)$ is the Frobenius element at ℓ;

(E3) $\tau \cdot z_{f,n} = w_f(\sigma \cdot z_{f,n})$, where τ denotes the complex conjugation, $w_f \in \{\pm 1\}$ is the eigenvalue of f with respect to the Atkin–Lehner involution and $\sigma \in \mathrm{Gal}(K[n]/K)$ is the image of $\bar{\mathfrak{N}}$ via the Artin map.

Using these properties, Castella–Hsieh and Magrone (see [CH18, Theorem 7.7] and [Mag22, Theorem 7.11]) showed that if $z_{f,K}$ is not $\mathfrak{o}$-torsion, then $\mathrm{Sel}(K, V) = \mathfrak{F} z_{f,K}$. More precisely, they prove that there exists a constant $C > 0$ such that p^C annihilates the quotient $\mathrm{Sel}(K, A)/(\mathfrak{F}/\mathfrak{o})z_{f,K}$ (see in particular [CH18, Theorem 7.19]). The second-named author of the present article improved this result in [Mas25, Theorem 0.2] by computing the constant C in the case $N^- = 1$, $k > 2$, $p \neq 3$ and f is p-ordinary together with certain assumptions on p. It was shown that if $z_{f,K}$ is not p-divisible, then $C = 0$, or equivalently $\mathrm{Sel}(K, A) = (\mathfrak{F}/\mathfrak{o})z_{f,K}$ (cf. *op. cit.* Theorem 0.1). The following theorem is a slight extension of the aforementioned result. Note that we drop the hypotheses of p-ordinarity and the condition that $k > 2$ since they are not used in the proof therein. Moreover, the exclusion of $p = 3$ ensured the validity of Lemma 4.5 *ibid.* (as explained in Remark 4.4 *ibid.*), for which we will provide a different proof below (see Lemma 3.4). Furthermore, we relax the hypothesis $N^- = 1$ using the quaternionic generalized Heegner cycles of [Mag22].

Theorem 3.1. *Let p be a prime such that*

- *$p \nmid 2N\varphi(N)(k-2)!$;*
- *p is unramified in $\mathfrak{F}$;*
- *ρ_f has big image, i.e., its image contains the set of matrices*

$$\left\{ g \in \mathrm{GL}_2(\mathfrak{o}) : \det g \in (\mathbf{Z}_p^\times)^{k-1} \right\}.$$

If $z_{f,K}$ is not $\mathfrak{o}$-torsion and is not divisible by p as an element of $\mathrm{Sel}(K, T)$, then

$$\mathrm{Sel}(K, A) = (\mathfrak{F}/\mathfrak{o})z_{f,K}.$$

Remark 3.2. Our assumption on the big image implies that f is non-CM for the following reason. As discussed in [Rib85, first paragraph of p. 186], if f were a CM form, $\mathrm{Lie}(\rho_f)$ could be identified with $L \otimes \mathbf{Q}_p$, where L is the field of "complex multiplication" of f [Rib77, p. 34, Remark 1]. However, the big image condition implies that this Lie algebra should contain $\mathfrak{sl}_2(\mathfrak{F})$, which is a contradiction. Consequently, [Rib85, discussion at the end of §3] tells us that Theorem 3.1 applies to all but finitely many p.

Proof of Theorem 3.1. The proof follows the same line of the argument of [Mas25, Theorem 0.2]. We give a brief summary here for the convenience of the reader and explain why the condition $N^- = 1$ and the p-ordinarity of f are not necessary. Let $M \geq 1$ be an integer and write $\mathscr{K}_M$ for the set of square-free products of prime numbers ℓ such that

(1) $\ell \nmid Npd_K$;
(2) ℓ inert in K;
(3) $p^M \mid a_\ell, \ell + 1$;
(4) $p^{M+1} \nmid \ell + 1 \pm a_\ell$.

Applying the Kolyvagin derivative operator to the classes $z_{f,n}$, we may construct as in [Mas25, §4.2] a family of classes $P(n) \in H^1(K, A[p^M])$, $n \in \mathscr{K}_M$ and show that the properties (E1)-(E3) and the inclusion $z_{f,n} \in \mathrm{Sel}(K[n], T)$ imply that [Bes97, Proposition 3.2] applies to our current setting. In other words, for any $n \in \mathscr{K}_M$, we have:

(1) $P(n)$ belongs to the ϵ_n-eigenspace of the complex conjugation acting on $H^1(K, A[p^M])$, where $\epsilon_n = (-1)^{\omega(n)} w_f \in \{\pm 1\}$ (here, $\omega(n)$ denotes the number of prime factors of n);
(2) for any $v \nmid Nn$, $\mathrm{loc}_v(P(n)) \in H^1_{\mathbf{f}}(K_v, A[p^M])$;
(3) if $n = m \cdot \ell$, there is an isomorphism $H^1_{\mathbf{f}}(K_\lambda, A[p^M]) \cong H^1_{\mathbf{s}}(K_\lambda, A[p^M])$, where $H^1_{\mathbf{s}} = H^1/H^1_{\mathbf{f}}$ is the singular quotient, such that $\mathrm{loc}_\lambda P(m)$ corresponds to $[\mathrm{loc}_\lambda P(n)]_s$ (the image of $\mathrm{loc}_\lambda P(n)$ in $H^1_{\mathbf{s}}(K_\lambda, A[p^M])$).

The fact that the classes $P(n)$ satisfy properties (1)-(3) and our assumptions on p are the input of [Bes97, §6] (where again f is not assumed to be ordinary at p). We can therefore apply verbatim the results therein. In particular, the quotient

$$\mathrm{Sel}(K, A)/(\mathfrak{F}/\mathfrak{o})z_{f,K}$$

is annihilated by $p^{2\mathcal{I}_p}$, where $\mathcal{I}_p$ denotes the smallest non-negative integer such that $z_{f,K}$ is non-zero in $H^1(K, A[p^{\mathcal{I}_p}])$. In particular, if $z_{f,K}$ is not divisible by p, we have

$$\mathrm{Sel}(K, A) = (\mathfrak{F}/\mathfrak{o})z_{f,K}. \qquad \square$$

We discuss consequences of Theorem 3.1 on Shafarevich–Tate groups.

Definition 3.3. Let E/K be a finite extension of fields. We define the Bloch–Kato–Shafarevich–Tate group of f over E as

$$\text{Ш}_{\mathrm{BK}}(E, A) = \frac{\mathrm{Sel}(E, A)}{\mathrm{Sel}(E, A)_{\mathrm{div}}}.$$

Note in particular that, for any finite extension E/K, the group $\text{Ш}_{\mathrm{BK}}(E, A)$ is finite since the Pontryagin dual $\mathrm{Sel}(E, A)^\vee$ is a finitely generated $\mathbf{Z}_p$-module (see [Rub00, Proposition B.2.7]). In what follows, we give the definition of another type of Shafarevich–Tate groups via the Abel–Jacobi map, similar to the ones considered in [Nek92]. We first state the following technical lemma.

Lemma 3.4. *The restriction maps*

$$\mathrm{Res}_{K[1]/K} \colon \mathrm{Sel}(K, T) \to \mathrm{Sel}(K[1], T)^{\mathrm{Gal}(K[1]/K)},$$

$$\mathrm{Res}_{K[p^{n+1}]/K_n} \colon \mathrm{Sel}(K_n, T) \to \mathrm{Sel}(K[p^{n+1}], T)^{\mathrm{Gal}(K[p^{n+1}]/K_n)}, \ n \geq 1$$

are isomorphisms.

Proof. We follow a similar argument given in [Pra25, Lemma 5.5.9]. Let (L, F) be either $(K[1], K)$ or $(K[p^{n+1}], K_n)$ and write $G = \mathrm{Gal}(L/F)$. Under the assumption $p \nmid h_K$, we have $p \nmid \#G$. consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Sel}(F, A) & \longrightarrow & H^1(F_\Sigma, A) & \longrightarrow & \prod_v H^1_{\mathbf{s}}(F_v, A) \\
& & \downarrow & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \prod_v g_v} \\
0 & \longrightarrow & \mathrm{Sel}(L, A)^G & \longrightarrow & H^1(L_\Sigma, A)^G & \longrightarrow & \prod_w H^1_{\mathbf{s}}(L_w, A)^G
\end{array}
$$

By the inflation-restriction sequence, the kernel and cokernel of Res are given by $H^i(G, H^0(L, A))$, for $i = 1, 2$, respectively. As $H^0(L, A)$ is a pro-p group while $p \nmid \#G$, [AW67, Corollary 6.1] tells us that Res is an isomorphism. Hence, by the snake lemma, it suffices to show that g_w is injective.

Let v be a place of F and let w be a place of L above v. Let $D = \mathrm{Gal}(L_w/F_v)$. We have the following commutative diagram where the rows are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathbf{f}}(F_v, A) & \longrightarrow & H^1(F_v, A) & \longrightarrow & H^1_{\mathbf{s}}(F_v, A) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h_v} & & \downarrow{\scriptstyle r_{A,v}} & & \downarrow{\scriptstyle g_v} & & \\
0 & \longrightarrow & H^1_{\mathbf{f}}(L_w, A)^D & \longrightarrow & H^1(L_w, A)^D & \longrightarrow & H^1_{\mathbf{s}}(L_w, A)^D. & &
\end{array}
$$

As the order of D is coprime to p, the map $r_{A,v}$ is an isomorphism, as before. Thus, by the snake lemma again, it suffices to show that h_v is surjective.

Suppose that $v|p$ and consider the following commutative diagrams

$$
(3.1) \qquad
\begin{array}{ccc}
H^1(F_v, V) & \xrightarrow{\ \ r_V\ \ } & H^1(L_w, V)^D \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b'} \\
H^1(F_v, V \otimes \mathbb{B}_{\mathrm{cris}}) & \xrightarrow{\ r_B\ } & H^1(L_w, V \otimes \mathbb{B}_{\mathrm{cris}})^D,
\end{array}
$$

$$
(3.2) \qquad
\begin{array}{ccc}
H^1(F_v, V) & \xrightarrow{\ r_V\ } & H^1(L_w, V)^D \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
H^1(F_v, A) & \xrightarrow{\ r_A\ } & H^1(L_w, A)^D,
\end{array}
$$

where we have written r_A for $r_{A,v}$ to simplify the notation. Note that r_V and r_B are all isomorphisms as $p \nmid \#D$.

Recall that $H^1_{\mathbf{f}}(L_w, A)$ is defined as the image of $H^1_{\mathbf{f}}(L_w, V)$ under the map induced by the natural map $V \to A$. This gives a surjection $H^1_{\mathbf{f}}(L_w, V) \to H^1_{\mathbf{f}}(L_w, A)$. As the kernel of this map is pro-p, the long exact sequence with respect to the group cohomology of D gives a surjection $\pi'_{\mathbf{f}} : H^1_{\mathbf{f}}(L_w, V)^D \to H^1_{\mathbf{f}}(L_w, A)^D$.

Let $x_0 \in H^1_{\mathbf{f}}(L_w, A)^D$. Since h_v is the restriction of r_A to $H^1_{\mathbf{f}}(F_v, A)$, the bijectivity of r_A tells us that there exists $x \in H^1(F_v, A)$ such that $r_A(x) = x_0$. In order to show the surjectivity of h_v, we show that x is, in fact, an element of $H^1_{\mathbf{f}}(F_v, A)$. This is equivalent to finding an element $z \in H^1_{\mathbf{f}}(F_v, V)$ such that $\pi(z) = x$.

By the surjectivity of $\pi'_{\mathbf{f}}$, there exists $y \in H^1_{\mathbf{f}}(L_w, V)^D$ such that $\pi'(y) = x_0 = r_A(x)$. In particular, $b'(y) = 0$ by the definition of $H^1_{\mathbf{f}}(L_w, V)$. Since r_V is a bijection, we can find $z \in H^1(F_v, V)$ such that $r_V(z) = y$. Then $r_B \circ b(z) = 0$ from the diagram (3.1). The injectivity of r_B implies that $b(z) = 0$. So, $z \in H^1_{\mathbf{f}}(K_v, V)$. From (3.2), $r_A(x) = r_A \circ \pi(z)$. As r_A is bijective, we have $x = \pi(z)$, as desired.

When $v \nmid p$, we have analogous commutative diagrams as (3.1) and (3.2), with the map r_B replaced by $r_I : H^1(I_v, V) \to H^1(I_w, V)^{I_w/I_v}$. Thus, the same conclusion holds, as desired. $\qquad\square$

Definition 3.5. We define the 'modified' image of the Abel–Jacobi map

$$
\Psi(K) = \mathrm{Res}^{-1}_{K[1]/K}\left((\mathrm{im}\,\mathrm{AJ}_{K[1]})^{\mathrm{Gal}(K[1]/K)} \right) \subseteq \mathrm{Sel}(K, T)
$$

and for any $n \geq 1$,

$$
\Psi(K_n) = \mathrm{Res}^{-1}_{K[p^{n+1}]/K_n}\left((\mathrm{im}\,\mathrm{AJ}_{K[p^{n+1}]})^{\mathrm{Gal}(K[p^{n+1}]/K_n)} \right) \subseteq \mathrm{Sel}(K_n, T).
$$

The Abel–Jacobi–Shafarevich–Tate group of f over K_n, for $n \geq 0$, is defined as

$$
\mathrm{III}_{\mathrm{AJ}}(K_n, A) = \frac{\mathrm{Sel}(K_n, A)}{\Psi(K_n) \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o}}.
$$

The reader is referred to [Mas25, Definitions 3.7 and 5.19], where Ψ is denoted by $\widetilde{\Lambda}_{\mathfrak{p}}$ and III_{AJ} by $\widetilde{\text{III}}_{\mathfrak{p}^\infty}$, for further details. As remarked above, the definition of $\text{III}_{\text{AJ}}(K_n, A)$ is inspired by [Nek92]. Here, the modified image $\Psi(K_n)$ is introduced to ensure that $z_{f,K} \in \Psi(K)$ and $\alpha_n := \text{Cores}_{K[p^{n+1}]/K_n}(z_{f,p^{n+1}}) \in \Psi(K_n)$, for any $n \geq 0$. Since classical Heegner cycles are known to lie inside the image Abel–Jacobi map, no modification of the image was necessary in *loc. cit.* This inclusion facilitates the use of generalized Heegner cycles in the study of Shafarevich–Tate groups, which we will do in the appendix of the article.

Note that $\Psi(K_n) \otimes \mathfrak{F}/\mathfrak{o} \subseteq \text{Sel}(K_n, A)_{\text{div}}$, by the commutativity of the following diagram for $n \geq 0$

$$
\begin{array}{ccc}
\Psi(K_n) \otimes \mathfrak{F} & \longrightarrow & \text{Sel}(K_n, V) \\
\downarrow & & \downarrow \\
\Psi(K_n) \otimes (\mathfrak{F}/\mathfrak{o}) & \longrightarrow & \text{Sel}(K_n, A).
\end{array}
$$

Hence, there is a projection

$$\text{III}_{\text{AJ}}(K_n, A) \twoheadrightarrow \text{III}_{\text{BK}}(K_n, A).$$

As far as the authors are aware, it is not known whether this map is an isomorphism in general. However, under an additional assumption on $z_{f,K}$, we may prove the following result, which is analogous to [LV23, Proposition 4.20] (see also Remark 3.8).

Proposition 3.6. *Suppose that $z_{f,K}$ is not $\mathfrak{o}$-torsion. Then, $\text{III}_{\text{AJ}}(K, A) = \text{III}_{\text{BK}}(K, A)$.*

Proof. When $N^- = 1$, [CH18, Theorem 7.19] tells us that $\text{Sel}(K, A)/(\mathfrak{F}/\mathfrak{o})z_{f,K}$ is annihilated by a power of p and thus finite. In particular, it follows that $\text{Sel}(K, A)$ has corank 1 over $\mathfrak{o}$. Since $z_{f,K} \in \Psi(K)$, the group $\text{III}_{\text{AJ}}(K, A)$ is a quotient of $\text{Sel}(K, A)/(\mathfrak{F}/\mathfrak{o})z_{f,K}$ and therefore again finite. Consequently, $\Psi(K) \otimes_\mathfrak{o} \mathfrak{F}/\mathfrak{o}$ and $\text{Sel}(K, A)$ both have corank 1 over $\mathfrak{o}$. The same can be said when $N^- > 1$ since the results of [CH18, §7.5] can be readily generalized after replacing the anticyclotomic Euler system in *loc. cit.* by the one constructed in [Mag22]. Therefore, in both cases, we deduce that the $\mathfrak{o}$-modules $\Psi(K) \otimes_\mathfrak{o} \mathfrak{F}/\mathfrak{o}$ and $\text{Sel}(K, A)_{\text{div}}$ are cofree of corank one. In particular, they must be equal to each other. As $\text{III}_{\text{AJ}}(K, A) = \text{Sel}(K, A)/(\Psi(K) \otimes_\mathfrak{o} \mathfrak{F}/\mathfrak{o})$ and $\text{III}_{\text{BK}}(K, A) = \text{Sel}(K, A)/\text{Sel}(K, A)_{\text{div}}$, the proposition follows. $\square$

From Theorem 3.1 we deduce that

Corollary 3.7. *Let p be as in the statement of Theorem 3.1. Suppose that $z_{f,K}$ is not $\mathfrak{o}$-torsion and that it is not divisible by p as an element of $\text{Sel}(K, T)$. Then,*

$$\text{III}_{\text{AJ}}(K, A) = \text{III}_{\text{BK}}(K, A) = 0.$$

Remark 3.8. If f is p-ordinary, Theorem 3.1 can be used to prove (see [Mas25, Theorem 5.23]) that

$$\text{III}_{\text{AJ}}(K_\infty, A) := \frac{\text{Sel}(K_\infty, A)}{\left(\varinjlim_n \Psi(K_n)\right) \otimes \mathfrak{F}/\mathfrak{o}} = 0.$$

In the current article, we prove that the same vanishing holds for the Bloch–Kato–Shafarevich–Tate group over K_n for all $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ under suitable assumptions for both ordinary and non-ordinary primes. Moreover, in the appendix, we extend the III_{AJ} vanishing results of [Mas25] to non-ordinary primes assuming $a_p = 0$ (see Theorem B.(V)). It should become apparent to the reader that the proofs we present here utilize techniques of a very different nature.

In general, it is not known whether $\text{III}_{\text{AJ}}(K_n, A)$ and $\text{III}_{\text{BK}}(K_n, A)$ coincide. In the ordinary setting, Theorem A together with [Mas25, Theorem 5.23] imply that $\text{III}_{\text{AJ}}(K_\infty, A) = \text{III}_{\text{BK}}(K_\infty, A) = 0$ under appropriate hypotheses. In the non-ordinary setting, when $a_p = 0$, Theorem B.(III) shows that $\text{III}_{\text{AJ}}(K_n, A)$ and $\text{III}_{\text{BK}}(K_n, A)$ coincide under the hypotheses therein

(and vanish simultaneously under an additional condition). Such agreements obtained in this article seem to be a remarkable serendipity.

4. CONTROL THEOREM

We first introduce some notation. Recall that K_n is the n-th layer of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K, and we denote $\Gamma_n = \mathrm{Gal}(K_\infty/K_n) = \Gamma^{p^n}$. Given $w \nmid p$ a place of K, let $w' \mid w$ be a place of K_∞, and write $G_{K_\infty,w'}$ for the decomposition group of w' in G_{K_∞}. Define

$$B_w = A^{G_{K_\infty,w'}}/\left(A^{G_{K_\infty,w'}}\right)_{\mathrm{div}};$$

note that $b_w = |B_w|$ depends on w and not on w'. Also, since B_w is a p-group, $p \nmid b_w$ if and only if $B_w = 0$, namely $A^{G_{K_\infty,w'}}$ is divisible. In what follows, for a Selmer condition $\star = \mathbf{f}, 0, \emptyset$, we will write $H^1_{/\star}$ for $H^1/H^1_\star$ for local and global cohomology groups.

We will work under the following hypotheses:

(Tors.) if f has weight 2 and is p-ordinary, then $a_p \not\equiv 1 \pmod{\varpi}$;

(Tama.) for any place $w \mid N$ of K, $p \nmid b_w$.

The assumption (Tors.) should be thought of as a restraint on the torsion group A, as the following lemma shows. We note that in what follows, when referring to (Tors.), we are often having Lemma 4.1.(a) in mind.

Lemma 4.1. *Suppose* (Tors.) *holds. Let w be a place of K lying above p. Then*

(a) $H^0(K_{\infty,w}, A) = 0$;

(b) *suppose that f is ordinary at p and let T^+ denote the unique rank-one sub-representation of $T|_{G_{K_w}}$ whose Hodge–Tate weight is $k/2$. Let $A^- = \mathrm{Hom}(T^+, \mathfrak{F}/\mathfrak{o})(1)$. Then, we have $H^0(K_{\infty,w}, A^-) = 0$.*

Proof. If f is non-ordinary at p, the assertion in (a) is a consequence of a result of Fontaine (see, e.g., Lemma A.3). In the remainder of the proof, we assume that f is ordinary at p. The representation $T|_{G_{K_w}}$ is of the form

$$\begin{bmatrix} \delta\chi_{\mathrm{cyc}}^{k/2} & * \\ 0 & \delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2} \end{bmatrix},$$

where χ_{cyc} is the cyclotomic character, and δ is an unramified character that sends the geometric Frobenius to the unit root of $X^2 - a_p(f)X + p^{k-1}$ (see [LV21, p. 445]). In particular,

$$A^- \simeq (\mathfrak{F}/\mathfrak{o})\left(\delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2}\right).$$

Let I_w be the inertia group of G_{K_w}. Then the image of I_w under χ_{cyc} is $\mathbf{Z}_p^\times$. If $k \geq 4$, we have $-(p-1) < 1 - k/2 < 0$ (thanks to our running hypothesis that $p \nmid 2(k-2)!N\varphi(N)$). Thus, $\delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2}(I_w) \not\subseteq 1 + p\mathbf{Z}_p$. This implies that $H^0(I_w, A^-) = 0$. If $k = 2$, the hypothesis that $a_p(f) \not\equiv 1 \mod \varpi$ ensures that $\delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2} = \delta^{-1}$ is not the trivial character modulo ϖ, so $H^0(K_w, A^-[\varpi]) = 0$ still. In both cases, we have $H^0(K_w, A^-) = 0$. Consequently, we can apply an orbit-stabilizer argument, similar to the one given in [DR25, proof of Lemma 5.2] to deduce that $H^0(K_{n,w}, A^-) = 0$ for all $n \in \mathbf{Z}_{\geq 0}$: Suppose for the sake of contradiction that there is some nonzero $x \in H^0(K_{n,w}, A^-)$. Let $M_x = \mathfrak{o}[\mathrm{Gal}(K_{n,w}/K_w)] \cdot x \subseteq H^0(K_{n,w}, A^-)$. Since x is

torsion over $\mathbf{Z}_p$, the module M_x is finite. Consider the partition of M_x into the orbits under the $\mathrm{Gal}(K_{n,w}/K_w)$-action. We have

$$(4.1) \qquad |M_x| = |\mathrm{Gal}(K_{n,w}/K_w) \cdot 0| + \sum_{x' \neq 0} |\mathrm{Gal}(K_{n,w}/K_w) \cdot x'|,$$

where the sum runs over the orbits in M_x generated by nonzero elements. Note that if $x' \neq 0$, then the orbit containing x' contains more than one element, as otherwise $x' \in H^0(K_w, A^-) = 0$, a contradiction. In particular, as $\mathrm{Gal}(K_{n,w}/K_w)$ is a p-group, $|\mathrm{Gal}(K_{n,w}/K_w) \cdot x'|$ is divisible by p. It follows that

$$|M_x| \equiv |\mathrm{Gal}(K_{n,w}/K_w) \cdot 0| = |\{0\}| = 1 \pmod{p},$$

which is impossible since M_x is a nontrivial p-group. This shows that $H^0(K_{n,w}, A^-)$ is identically zero. Now, recall that the Galois action on V is continuous [Nek92, Proposition 3.1.(1)], so we have a continuous map $\mathcal{J} : G_{K_w} \times V \to V$ given by the group action. Hence, if $x \in V$, then $\mathcal{J}^{-1}(x + T)$ must contain $U \times (x + T)$ for some open subset U of G_{K_w}. Thus any element of $A = V/T$ has a stabilizer that contains an open subset of G_{K_w}, whereby it must be open itself. As such, we find that any element in $H^0(K_{\infty,w}, A^-)$ is stabilized by an open subgroup containing $G_{K_{\infty,w}}$, meaning it is invariant under $G_{K_{n,w}}$ for some n. In sum, $H^0(K_{\infty,w}, A^-) = \cup_{n \geq 0} H^0(K_{n,w}, A^-) = 0$, which proves part (b) of the lemma.

We now turn our attention to part (a). As above, it suffices to show that $H^0(K_w, A) = 0$. Let $\{e_1, e_2\}$ be a basis of $\mathfrak{o}^{\oplus 2}$ on which G_{K_w} acts through the matrix $\begin{bmatrix} \delta\chi_{\mathrm{cyc}}^{k/2} & \Xi \\ 0 & \delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2} \end{bmatrix}$. Suppose $H^0(K_w, A) \neq 0$. Then there exists an element $xe_1 + ye_2 \in \mathfrak{o}^{\oplus 2}$ such that, for all $\sigma \in G_w$,

$$\sigma \cdot (xe_1 + ye_2) \equiv xe_1 + ye_2 \pmod{\varpi}.$$

In other words, for all $\sigma \in G_w$,

$$\left(x\delta\chi_{\mathrm{cyc}}^{k/2}(\sigma) + y\Xi(\sigma) \right) e_1 + y\delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2}(\sigma)e_2 \equiv xe_1 + ye_2 \pmod{\varpi}.$$

In particular, for all $\sigma \in G_w$,

$$y\delta^{-1}\chi_{\mathrm{cyc}}^{1-k/2}(\sigma)e_2 \equiv ye_2 \pmod{\varpi}.$$

Our proof for part (b) tells us that we must have $y \equiv 0 \mod \varpi$. This in turn implies that

$$x\delta\chi_{\mathrm{cyc}}^{k/2}(\sigma) \equiv xe_1 \pmod{\varpi} \text{ for all } \sigma \in G_{K_w}.$$

As $0 < k/2 < (p-1)$, we have once again $\delta\chi_{\mathrm{cyc}}^{k/2}(I_w) \not\subset 1 + p\mathbf{Z}_p$. Therefore, $H^0(I_w, \mathfrak{F}/\mathfrak{o}(\delta\chi_{\mathrm{cyc}}^{k/2})) = 0$, which implies that $x \equiv 0 \mod \varpi$. Hence, we conclude that $H^0(K_w, A) = 0$, which implies the assertion of part (a). $\qquad\square$

Proposition 4.2. *Suppose A satisfies assumptions* (Tors.) *and* (Tama.)*. Let $\star, \bullet \in \{0, \emptyset\}$. The restriction map on cohomology groups induces an isomorphism $\Phi_n : \mathrm{Sel}^{\star,\bullet}(K_n, A) \to \mathrm{Sel}^{\star,\bullet}(K_\infty, A)^{\Gamma_n}$ for all $n \in \mathbf{Z}_{\geq 0}$.*

While control theorems of this flavor are standard (see, e.g., [Mat21a, Lemma 4.3], [LLM23, Theorem 4.1] and [Pon20, Lemma 2.3]), we give a proof here for the reader's convenience. We begin with two lemmas.

Lemma 4.3. *Let $w \mid N$ be a place of K and F be an unramified finite extension of K_w. Then A^{G_F} is finite.*

Proof. For the sake of contradiction, suppose instead that there exist infinitely many $n \in \mathbf{Z}_{\geq 0}$ such that $x_n \in A^{G_F}$ and the annihilator ideal of x_n is $(\varpi^n) \subseteq \mathfrak{o}$; we extend x_n to all of $n \in \mathbf{Z}_{\geq 0}$ by setting $x_{n-1} = \varpi x_n$ if x_n is defined and x_{n-1} is not. Let e be the ramification index of $\mathfrak{F}/\mathbf{Q}_p$ and put $y_n = x_{en}$. Denote by $(-, -)$ the pairing from [Nek92, Proposition 3.1.(2)] on $T \times T$ and also $A[p^n] \times A[p^n]$ for $n \in \mathbf{Z}_{\geq 0}$ (note that since T is self-dual, $A[p^n] \simeq (T/p^n)^{\vee}(1)$). Then we have $(p^{m-n} y_m, y_n) \in \mu_{p^n}$ for all $n, m \in \mathbf{Z}_{\geq 0}, m > n$.

For m, n as above, denote by $D_{m,n}$ the $\mathfrak{o}/p^n$-submodule generated by $p^{m-n} y_m$ and y_n in $A[p^n]$. If $D_{m,n} = A[p^n]$, then the same argument as [Sil09, Corollary III.8.1.1] shows that $\mu_{p^n} \subset F$, which cannot happen for $n \gg 0$. Thus there exists $n_0 \in \mathbf{Z}_{\geq 0}$ such that for all $n \geq n_0$, $y_{n_0} = a_n p^{n-n_0} y_n$ for some $a_n \in \mathfrak{o}^{\times}$. Put

$$\tilde{y}_n = \begin{cases} p^{n_0-n} y_{n_0}, & \text{if } n \leq n_0; \\ a_n y_n, & \text{if } n > n_0. \end{cases}$$

It follows from our construction that $(\tilde{y}_n)_{n \geq 0} \in \varprojlim_n A[p^n]^{G_F}$; thus we have a nonzero element in T^{G_F}, and hence in V^{G_F}.

To derive a contradiction, we will now recall some property of the local Galois representation at the bad prime w; we thank Ming-Lun Hsieh for explaining this to us. Denote by $I \subset G_F$ the inertia subgroup, and recall $V_f = V(-k/2)$ is Deligne's cohomological Galois representation. Since $w \nmid p$, taking the I-invariant commutes with the p-cyclotomic twists. Note also that, with ℓ being the rational prime under w, $I \subset G_F \subset G_{\mathbf{Q}_\ell}$ is the full inertia group of ℓ as K is unramified at places that divide N. Then, as is standard, we have $V^I = V_f^I(-k/2)$ is either trivial or one-dimensional. In the latter case, we have Frob_ℓ acts on V^I as $-\ell \varepsilon_{f,\ell}$, following Atkin–Lehner [AL70] and Carayol [Car86] (see also [Nek92, Proposition 3.1.(4)]); here $\varepsilon_{f,\ell}$ is the Atkin–Lehner sign. Therefore, no power of Frob_ℓ is 1 and we have $V^{G_F} = 0$. $\square$

Lemma 4.4. *Let $w \mid N$ be a place of K_n that is finitely decomposed in K_∞, and let $w' \mid w$ be a place in K_∞. Denote by $\Gamma_{n,w'} \subset \Gamma_n$ the decomposition group of w'. We have*

$$\# \ker \left(H^1_{/\mathbf{f}}(K_{n,w}, A) \to H^1_{/\mathbf{f}}(K_{\infty,w'}, A)^{\Gamma_{n,w'}} \right) \leq b_w.$$

In particular, when $p \nmid b_w$, the above kernel is trivial.

Proof. The proof is a variant of [Gre99, Lemma 3.3]. Recall that by [PR92, §2.1,7, §2.2.4], we have $H^1_{\mathbf{f}}(K_{\infty,w'}, A) = 0$. Next, consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1_{\mathbf{f}}(K_{n,w}, A) & \longrightarrow & H^1(K_{n,w}, A) & \longrightarrow & H^1_{/\mathbf{f}}(K_{n,w}, A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g_{w'}} & & \\
0 & \longrightarrow & H^1(K_{\infty,w'}, A)^{\Gamma_{n,w'}} & = & H^1(K_{\infty,w'}, A)^{\Gamma_{n,w'}} & \longrightarrow & 0
\end{array}$$

By the snake lemma, we find

$$\# \ker(g_{w'}) \leq \# \ker \left(H^1(K_{n,w}, A) \to H^1(K_{\infty,w'}, A)^{\Gamma_{n,w'}} \right) = \# H^1(\Gamma_{n,w'}, A^{G_{K_{\infty,w'}}}).$$

As $\Gamma_{n,w'} \neq 0$ is pro-cyclic, if we fix a topological generator $\gamma_{w'} \in \Gamma_{n,w'}$, then

$$H^1(\Gamma_{n,w'}, A^{G_{K_{\infty,w'}}}) \simeq A^{G_{K_{\infty,w'}}} / (\gamma_{w'} - 1) A^{G_{K_{\infty,w'}}}$$

(see [Gre01, Exercise 2.2] for example). Lemma 4.3 tells us that $(A^{G_{K_{\infty,w'}}})_{\mathrm{div}} \subseteq (\gamma_{w'} - 1) A^{G_{K_{\infty,w'}}}$. Recall that $B_w = A^{G_{K_{\infty,w'}}} / \left(A^{G_{K_{\infty,w'}}} \right)_{\mathrm{div}}$. It follows that $A^{G_{K_{\infty,w'}}} / (\gamma_{w'} - 1) A^{G_{K_{\infty,w'}}}$ is a quotient of B_w. Thus, $\# \ker(g_{w'}) \leq b_w$. $\square$

Proof of Proposition 4.2. We first note that the injectivity of Φ_n follows from the inflation-restriction exact sequence

$$0 \to H^1(\Gamma_n, A^{G_{K_\infty}}) \to H^1(K_n, A) \to H^1(K_\infty, A),$$

and our assumption (Tors.).

For the surjectivity, we consider the commutative diagram with exact rows

$$
\begin{array}{ccccc}
\mathrm{Sel}^{\star,\bullet}(K_n, A) & \hookrightarrow & H^1(K_\Sigma/K_n, A) & \longrightarrow & H^1_{/\star}(K_{n,v}, A) \times H^1_{/\bullet}(K_{n,\bar{v}}, A) \times \prod_{w \in S_n} H^1_{/\mathbf{f}}(K_{n,w}, A) \\
\downarrow{\scriptstyle \Phi_n} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\
\mathrm{Sel}^{\star,\bullet}(K_\infty, A)^{\Gamma_n} & \hookrightarrow & H^1(K_\Sigma/K_\infty, A)^{\Gamma_n} & \longrightarrow & \left(H^1_{/\star}(K_{\infty,v}, A) \times H^1_{/\bullet}(K_{\infty,\bar{v}}, A) \times \prod_{w' \in S_\infty} H^1_{/\mathbf{f}}(K_{\infty,w'}, A) \right)^{\Gamma_n}
\end{array}.
$$

By the inflation-restriction exact sequence and (Tors.) again, we have $\mathrm{coker}(h) = 0$. Thus, by the snake lemma, the surjectivity of Φ_n would follow from $\ker(g) = 0$. Note that g is given by the local maps

$$g_w : H^1_{/\heartsuit}(K_{n,w}, A) \to \left(\prod_{w'|w} H^1_{/\heartsuit}(K_{\infty,w'}, A) \right)^{\Gamma_n},$$

where

$$\heartsuit = \begin{cases} \mathbf{f} & w \mid N; \\ \star & w = v; \\ \bullet & w = \bar{v}. \end{cases}$$

The inflation-restriction exact sequence and (Tors.) imply that $\ker(g_w) = 0$ with $w \mid p$. It remains to show that $\ker(g_w) = 0$ for $w \mid N$. For this, we enlarge g_w to

$$g'_w : H^1_{/\mathbf{f}}(K_{n,w}, A) \to \left(\prod_{w'|w} H^1_{/\mathbf{f}}(K_{\infty,w'}, A) \right)^{\Gamma_n} \hookrightarrow \prod_{w'|w} H^1_{/\mathbf{f}}(K_{\infty,w'}, A)^{\Gamma_{n,w'}}$$

(notation as in Lemma 4.4). We then have a dichotomy: either w is infinitely decomposed in K_∞, in which case $\ker(g'_w) = 0$; or w is finitely decomposed, in which scenario the vanishing of $\ker(g'_w)$ remains valid thanks to Lemma 4.4 and (Tama.). $\qquad\square$

5. Vanishing of $\mathrm{Sel}^{\mathrm{BDP}}$ and its consequences

Recall that $z_{f,K} \in H^1_{\mathbf{f}}(K, T)$ is the generalized Heegner class over K; we use the same letter for its image in $H^1_{\mathbf{f}}(K, A)$. The following hypotheses are in effect:

(Loc.prim.) $$\mathrm{loc}_v(z_{f,K}) \notin \varpi H^1_{\mathbf{f}}(K_v, T);$$

(Sel.) $$\mathrm{Sel}(K, A) = \mathfrak{F}/\mathfrak{o} \cdot z_{f,K}.$$

Note also that, by local Tate duality, the assumption (Tors.) implies

(Tors'.) $$H^2(K_{\bar{v}}, T) = 0.$$

The proof of the following proposition is inspired by [Mat21a, proof of Theorem 3.2].

Proposition 5.1. *Assume* (Tors.), (Tama.), (Loc.prim.) *and* (Sel.). *For all* $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$,

$$\mathrm{Sel}^{\mathrm{BDP}}(K_n, A) = 0.$$

Proof. By Proposition 4.2, we have $\mathrm{Sel}^{\mathrm{BDP}}(K_n, A) = \mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A)^{\Gamma_n}$. As $\mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A)$ is cofinitely generated [Gre89, Proposition 3], by Nakayama's lemma it suffices to prove $\mathrm{Sel}^{\mathrm{BDP}}(K, A) = 0$. Equivalently, we show that

$$(5.1) \qquad \mathrm{Sel}^{\emptyset, \emptyset}(K, A) \to H^1(K_{\bar{v}}, A)$$

is injective.

We start with a bisection of $\mathrm{Sel}^{\emptyset, \emptyset}(K, A)$. We have the following Cassels–Poitou–Tate exact sequence (see [Was97, p. 119] for a detailed discussion over $\mathbf{Q}$; the argument can be applied to general number fields with minor adjustments):

$$0 \to \mathrm{Sel}(K, A) \to \mathrm{Sel}^{\emptyset, \emptyset}(K, A) \xrightarrow{\psi} H^1_{/\mathbf{f}}(K_v, A) \times H^1_{/\mathbf{f}}(K_{\bar{v}}, A) \xrightarrow{\theta} \mathrm{Sel}(K, T)^\vee,$$

where $\mathrm{Sel}(K, T) = \varprojlim_n \mathrm{Sel}(K, A)[p^n]$. This gives rise to the following short exact sequence:

$$(5.2) \qquad 0 \to \mathrm{Sel}(K, A) \to \mathrm{Sel}^{\emptyset, \emptyset}(K, A) \xrightarrow{\psi} \ker(\theta) \to 0.$$

Consequently, the injectivity of (5.1) follows from the following two claims:

(a) if $z \in \mathrm{Sel}^{\emptyset, \emptyset}(K, A)$ is such that $\mathrm{loc}_{\bar{v}}(z) = 0$, then $z \in \mathrm{Sel}(K, A)$;
(b) the localization map $\mathrm{Sel}(K, A) \to H^1(K_{\bar{v}}, A)$ is injective.

We begin with the proof of (b). First, note that the localization map can be decomposed as

$$(5.3) \qquad \mathrm{loc}_{\bar{v}} : \mathrm{Sel}(K, A) \simeq \mathfrak{F}/\mathfrak{o} \cdot z_{f,K} \xrightarrow{\psi_{0,\bar{v}}} H^1_{\mathbf{f}}(K_{\bar{v}}, A) \hookrightarrow H^1(K_{\bar{v}}, A),$$

where $\psi_{0,\bar{v}}$ is the built-in morphism from the definition of $\mathrm{Sel}(K, A)$, and the first isomorphism is guaranteed by (Sel.). Suppose $z \in \ker(\psi_{0,\bar{v}})$ is of the form $z_{f,K} \otimes a/\varpi^n$ for some $a \in \mathfrak{o}^\times$ and $n \in \mathbf{Z}_{\geq 0}$. By (Tors.), we have an exact sequence

$$0 \to H^1_{\mathbf{f}}(K_{\bar{v}}, T) \to H^1_{\mathbf{f}}(K_{\bar{v}}, V) \to H^1_{\mathbf{f}}(K_{\bar{v}}, A).$$

Hence, we may and will regard $H^1_{\mathbf{f}}(K_{\bar{v}}, T)$ as a subgroup of $H^1_{\mathbf{f}}(K_{\bar{v}}, V)$. Since $\psi_{0,\bar{v}}(z) = \mathrm{loc}_{\bar{v}}(z_{f,K}) \otimes a/\varpi^n = 0 \in H^1_{\mathbf{f}}(K_{\bar{v}}, A)$, we have $\mathrm{loc}_{\bar{v}}(z_{f,K}) \in \varpi^n H^1_{\mathbf{f}}(K_{\bar{v}}, T)$. Now, by applying the complex conjugation to (Loc.prim.), we find $\mathrm{loc}_{\bar{v}}(z_{f,K}) \notin \varpi H^1_{\mathbf{f}}(K_{\bar{v}}, T)$, which forces $n = 0$ and hence $z = 0$.

As for (a), we note that by (Loc.prim.) and (Sel.), we have $\mathrm{Sel}(K, T) = \varprojlim_n \mathrm{Sel}(K, A)[p^n] = \mathfrak{o} \cdot z_{f,K}$, and that, by definition, $\ker(\theta) \subseteq H^1_{/\mathbf{f}}(K_v, A) \times H^1_{/\mathbf{f}}(K_{\bar{v}}, A)$ is orthogonal to $\mathrm{im}(\theta^\vee) = \mathrm{im}(\mathrm{Sel}(K, T)) \subseteq H^1_{\mathbf{f}}(K_v, T) \times H^1_{\mathbf{f}}(K_{\bar{v}}, T)$. Now, if $z \in \mathrm{Sel}^{\emptyset, \emptyset}(K, A)$ is such that $\mathrm{loc}_{\bar{v}}(z) = 0$, then $\psi(z) \in \ker(\theta)$ is also orthogonal to $H^1_{\mathbf{f}}(K_{\bar{v}}, T)$, and thus to

$$0 \times H^1_{\mathbf{f}}(K_{\bar{v}}, T) + \mathrm{im}(\mathrm{Sel}(K, T)) \subseteq H^1_{\mathbf{f}}(K_v, T) \times H^1_{\mathbf{f}}(K_{\bar{v}}, T).$$

By (Tors.), $H^1_{\mathbf{f}}(K_v, T) \times H^1_{\mathbf{f}}(K_{\bar{v}}, T)$ is torsion-free and thus isomorphic to $\mathfrak{o}^2$ (see [BK90, Corollary 3.8.4]). Now, (Loc.prim.) tells us that the projection of $z_{f,K}$ in $H^1_{\mathbf{f}}(K_v, T)$ is a generator, whereby $0 \times H^1_{\mathbf{f}}(K_{\bar{v}}, T) + \mathrm{im}(\mathrm{Sel}(K, T)) = H^1_{\mathbf{f}}(K_v, T) \times H^1_{\mathbf{f}}(K_{\bar{v}}, T)$. This shows $\psi(z) = 0$ and hence $z \in \mathrm{Sel}(K, A)$. $\qquad\square$

Remark 5.2. Note that we proved that

(i) $\mathrm{Sel}(K, A) \to H^1_{\mathbf{f}}(K_{\bar{v}}, A)$ is an isomorphism;
(ii) $\ker(\theta) \simeq \mathfrak{F}/\mathfrak{o}$, as its Pontryagin dual is $\left(H^1_{\mathbf{f}}(K_v, T) \times H^1_{\mathbf{f}}(K_{\bar{v}}, T)\right)/\mathrm{im}(\mathrm{Sel}(K, T))$, which is isomorphic to $\mathfrak{o}$ by (Loc.prim.). In particular, $\mathrm{Sel}^{\emptyset, \emptyset}(K, A) \simeq (\mathfrak{F}/\mathfrak{o})^2$ by (5.2).

Additionally, as one can check, our proof works with $v, \bar{v}$ switched under the same set of assumptions.

Remark 5.3. We discuss a possible alternative proof of Proposition 5.1 utilizing the Iwasawa main conjecture that relates the BDP p-adic L-function of [BDP13, Bra11, HB15] (which we denote by $L_p^{\mathrm{BDP}} \in \widehat{\mathfrak{o}^{\mathrm{ur}}}[[\Gamma]]$, where $\widehat{\mathfrak{o}^{\mathrm{ur}}}$ denotes the ring of integers of the completion of the maximal unramified extension of $\mathfrak{F}$) to the BDP Selmer group. Throughout this remark, we assume that the following inclusion holds

$$(5.4) \qquad \left(L_p^{\mathrm{BDP}} \right)^2 \in \mathrm{char}_\Lambda \, \mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A)^\vee \otimes \widehat{\mathfrak{o}^{\mathrm{ur}}}.$$

In the ordinary case, this is a consequence of [LV19a, Theorem 3.5] under mild hypotheses; see the discussion in [HL23, Remark 13], see also [BCK21, Theorem B] and [Swe24] for more recent developments. When f is non-ordinary at p, $k = 2$ and $N^- = 1$, (5.4) has been studied in [LZ24], which built on the prior work of Kobyashi–Ota [KO20].

One can check, using the interpolation formula for L_p^{BDP} at the trivial character (see [JLZ21, Theorem 7.2.4], [BDP13, Theorem 5.13] and [HB15, Theorem 1.1]), that L_p^{BDP} is a unit under an appropriate hypothesis on the image of the $z_{f,K}$ under the Bloch–Kato logarithm map. Consequently, (5.4) implies that $\mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A)^\vee$ is a pseudo-null Λ-module. However, $\mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A)^\vee$ does not contain any non-trivial pseudo-null submodule (see [LMX24, Lemma 3.4]). Therefore, we deduce $\mathrm{Sel}^{\mathrm{BDP}}(K_\infty, A) = 0$. If (Tama.) also holds, we can combine the last equation with the exact control theorem Proposition 4.2 to conclude that $\mathrm{Sel}^{\mathrm{BDP}}(K_n, A) = 0$ for all n.

Corollary 5.4. *Suppose that the assumptions of Proposition 5.1 hold. Then, for all $n \in \mathbf{Z}_{\geq 0}$, the map $\mathrm{Sel}(K_n, A) \to H_{\mathbf{f}}^1(K_{n,w}, A)$ is an isomorphism for $w \mid p$. Consequently,*

(1) $\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}(K_n, A)) = p^n$;

(2) *the Bloch–Kato–Shafarevich–Tate group* $\mathrm{Ш}_{\mathrm{BK}}(K_n, A) = \mathrm{Sel}(K_n, A)/\mathrm{Sel}(K_n, A)_{\mathrm{div}} = 0$.

Proof. Without loss of generality, assume $w = \bar{v}$. We show that there is an exact sequence

$$(5.5) \qquad \mathrm{Sel}^{\mathbf{f},0}(K_n, A) \to \mathrm{Sel}(K_n, A) \xrightarrow{\alpha} H_{\mathbf{f}}^1(K_{n,\bar{v}}, A) \xrightarrow{\beta} \mathrm{Sel}^{0,\emptyset}(K_n, T)^\vee.$$

Consider the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Sel}^{\mathbf{f},0}(K_n, A) & \longrightarrow & \mathrm{Sel}(K_n, A) & \xrightarrow{\alpha'} & H_{\mathbf{f}}^1(K_{n,v}, A) \times H_{\mathbf{f}}^1(K_{n,\bar{v}}, A) & \xrightarrow{\beta'} & \mathrm{Sel}^{\emptyset,\emptyset}(K_n, T)^\vee \\
& & & \searrow{\alpha} & \downarrow{\pi} & & \downarrow{\iota^\vee} \\
& & & & H_{\mathbf{f}}^1(K_{n,\bar{v}}, A) & \xrightarrow{\beta} & \mathrm{Sel}^{0,\emptyset}(K_n, T)^\vee,
\end{array}$$

where π is the natural projection and $\iota : \mathrm{Sel}^{0,\emptyset}(K_n, T) \to \mathrm{Sel}^{\emptyset,\emptyset}(K_n, T)$ is the natural injection. By the Cassels–Poitou–Tate exact sequence, the first row is exact at $H_{\mathbf{f}}^1(K_{n,v}, A) \times H_{\mathbf{f}}^1(K_{n,\bar{v}}, A)$. Suppose $z \in H_{\mathbf{f}}^1(K_{n,\bar{v}}, A)$ belongs to $\ker(\beta)$. Put $z' = (0, z) \in H_{\mathbf{f}}^1(K_{n,v}, A) \times H_{\mathbf{f}}^1(K_{n,\bar{v}}, A)$, then $\iota^\vee \circ \beta'(z') = 0$, so $\beta'(z')$, as a function on $\mathrm{Sel}^{\emptyset,\emptyset}(K_n, T)$, is orthogonal to $\mathrm{Sel}^{0,\emptyset}(K_n, T)$. It follows that $\beta'(z')$ is actually a linear function on $\mathrm{coker}(\iota) \subseteq H^1(K_{n,v}, T)$. Since z' has zero v-component, we also have $\beta'(z')$ vanishes on $H^1(K_{n,v}, T)$. Hence, $\beta'(z') = 0$. This shows that $z' = \alpha'(w)$ for some $w \in \mathrm{Sel}(K_n, A)$, and thus $z = \pi(z') = \pi\alpha'(w) = \alpha(w)$. This finishes the proof of the exactness of (5.5)

We now show that the map α is an isomorphism. By (5.5) and Proposition 5.1, α is injective. As for the surjectivity, invoking Proposition 5.1 again with $v, \bar{v}$ switched, we have

$$\mathrm{Sel}^{0,\emptyset}(K_n, A) = \bigcup_{m > 0} \mathrm{Sel}^{0,\emptyset}(K_n, A)[p^m] = 0.$$

Thus, $\mathrm{Sel}^{0,\emptyset}(K_n, A)[p^m] = 0$ for all m, whereby $\mathrm{Sel}^{0,\emptyset}(K_n, T) = \varprojlim_m \mathrm{Sel}^{0,\emptyset}(K_n, A)[p^m] = 0$.

Finally, we turn our attention to the "therefore" part. For (1), we have $\mathrm{corank}_{\mathfrak{o}}(H_{\mathbf{f}}^1(K_{n,w}, A)) = p^n$ by [BK90, Corollary 3.8.4] and the base change property $(B_{\mathrm{dR}} \otimes V)^{G_{K_{n,w}}} = (B_{\mathrm{dR}} \otimes V)^{G_{K_w}} \otimes_{K_w}$

$K_{n,w}$. The assertion (2) follows from the fact that $H^1(K_{n,w}, A)$ is divisible (see [Gre01, proof of Theorem 2.9]).   □

6. Results on universal norms

Throughout §6, we let $w \in \{v, \bar{v}\}$ be a place of K. The objective of this section is to study the structure of $\mathrm{Sel}(K_\infty, A)$. It follows from Corollary 5.4 that under the assumptions of Proposition 5.1,

(6.1) $$\mathrm{Sel}(K_\infty, A) = \varinjlim_n \mathrm{Sel}(K_n, A) = \varinjlim_n H^1_{\mathbf{f}}(K_{n,w}, A).$$

We shall prove part (3) of Theorem A, which describes the structure of this Λ-module. We do so via the theory of local universal norms à la Perrin-Riou [PR00] who studied modules over cyclotomic towers analogous to the second direct limit in (6.1).

Lemma 6.1. *There is a Λ-isomorphism*

$$\left(\varinjlim_n H^1_{\mathbf{f}}(K_{n,w}, A)\right)^{\vee} \simeq \varprojlim_n H^1(K_{n,w}, T)/\varprojlim_n H^1_{\mathbf{f}}(K_{n,w}, T).$$

Proof. By local Tate duality, there is a perfect pairing

$$\langle -, - \rangle : \varinjlim_n H^1(K_{n,w}, A) \times \varprojlim_n H^1(K_{n,w}, T) \to \mathfrak{F}/\mathfrak{o}.$$

As explained in [PR92, §2.1.8], $\langle -, - \rangle$ induces a perfect pairing

$$\left(\varinjlim_n H^1(K_{n,w}, A)/\varinjlim_n H^1_{\mathbf{f}}(K_{n,w}, A)\right) \times \varprojlim_n H^1_{\mathbf{f}}(K_{n,w}, T) \to \mathfrak{F}/\mathfrak{o}.$$

In other words, $\varinjlim_n H^1_{\mathbf{f}}(K_{n,w}, A)$ and $\varprojlim_n H^1_{\mathbf{f}}(K_{n,w}, T)$ are the annihilator of each other under $\langle -, - \rangle$, from which the lemma follows.   □

We introduce the notation that will be used in our proof of Theorem A(3). Let G be a topological group that is isomorphic to $\mathbf{Z}_p$. Let γ be a topological generator of G. We write $\mathscr{H}_\infty(G)$ for the algebra of tempered distributions on G. More explicitly, it is the set of power series $f(\gamma - 1)$ where $f(X) \in \mathbf{Q}_p[[X]]$ that converges on the open unit disk. Given a real number $r > 0$, we write $\mathscr{H}_r(G)$ for the subset of $\mathscr{H}_\infty(G)$ consisting of the elements that are $O(\log_p^r)$, i.e., those arise from power series $f(X) = \sum_{n \geq 0} c_n X^n$ satisfying $\sup\{|c_n|_p/n^r\} < \infty$. Let $\tilde{\mathfrak{o}}$ denote the ring of integers of the completion of the maximal unramified extension of $\mathfrak{F}$. We write $\tilde{\mathscr{H}}_\infty(G) = \mathscr{H}_\infty(G) \otimes_{\mathbf{Z}_p} \tilde{\mathfrak{o}}$ and $\tilde{\mathscr{H}}_r(G)$ is defined similarly.

Given a G_{K_w}-representation M and a p-adic Lie extension $\mathcal{K}$ of K_w, we write

$$H^1_{\mathrm{Iw}}(\mathcal{K}, M) = \varprojlim H^1(K', M),$$

where M runs through finite extension of K_w contained inside $\mathcal{K}$ and the connecting maps are corestrictions.

Let $K_{\mathrm{cyc},w}$ be the cyclotomic $\mathbf{Z}_p$-extension of K_w and let $\mathcal{K}$ denote the compositum of $K_{\mathrm{cyc}.w}$ and $K_{\infty,w}$. We write $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(K_{\mathrm{cyc},w}/K_w)$ and $\Gamma' = \mathrm{Gal}(\mathcal{K}/K_{\mathrm{cyc},w})$, the latter of which can be identified with the Galois group of the unramified $\mathbf{Z}_p$-extension of K_w. Let α and β be the two roots of $X^2 - a_p(f)X + p^{k-1}$. Let $r = \max(\mathrm{ord}_p(\alpha), \mathrm{ord}_p(\beta))$. Since the Hodge–Tate weights of $T(k/2 - 1)$ are 0 and $k - 1$, there is a two-variable Perrin-Riou map

$$\mathcal{L}_{\mathcal{K}, T(k/2-1)} : H^1_{\mathrm{Iw}}(\mathcal{K}, T(k/2 - 1)) \to \tilde{\mathfrak{o}}[[\Gamma']] \widehat{\otimes} \mathscr{H}_r(\Gamma_{\mathrm{cyc}}) \otimes \mathbb{D}_{\mathrm{cris}}(V(k/2 - 1))$$

defined in [LZ14, Theorem 4.7] (the fact that the image consists of elements that are $O(\log_p^r)$ is proven in Proposition 4.8 of *op. cit.*). There is a natural isomorphism $H^1_{\mathrm{Iw}}(\mathcal{K}, T) \otimes \mathbf{Z}_p(k/2-1) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(\mathcal{K}, T(k/2-1))$ (see [PR95, Corollary A.4.4]). This induces a corresponding Perrin-Riou map

$$\mathcal{L}_{\mathcal{K},T} : H^1_{\mathrm{Iw}}(\mathcal{K}, T) \to \tilde{\mathfrak{o}}[[\Gamma']]\widehat{\otimes}\mathscr{H}_r(\Gamma_{\mathrm{cyc}}) \otimes \mathbb{D}_{\mathrm{cris}}(V).$$

As in [LZ24, Theorem A.2], we may quotient out by the kernel of the projection map

$$\tilde{\mathfrak{o}}[[\mathrm{Gal}(\mathcal{K}/K_w)]] \to \tilde{\mathfrak{o}}[[\Gamma]],$$

resulting in

$$\mathcal{L}_{K_{\infty,w},T} : H^1_{\mathrm{Iw}}(K_{\infty,w}, T) \to \tilde{\mathscr{H}}_r(\Gamma) \otimes \mathbb{D}_{\mathrm{cris}}(V).$$

Theorem 6.2. *Assume that* (Tors.) *holds. Then:*

i) *the inverse limit $H^1_{\mathrm{Iw}}(K_{\infty,w}, T)$ is free of rank two over Λ;*

ii) *if f is non-ordinary at p, then $\varprojlim_n H^1_{\mathbf{f}}(K_{n,w}, T) = 0$;*

iii) *if f ordinary at p, then $H^1_{\mathrm{Iw}}(K_{\infty,w}, T)/\varprojlim_n H^1_{\mathbf{f}}(K_{n,w}, T)$ is free of rank one over Λ.*

Proof. We first prove part i). By [PR92, Proposition 2.1.3 and 2.1.6] and Lemma 4.1, we see that $H^1_{\mathrm{Iw}}(K_{\infty,w}, T)$ is of Λ-rank 2 and isomorphic to $\mathrm{Hom}_\Lambda(\mathcal{Z}, \Lambda)$ for some finitely generated Λ-module $\mathcal{Z}$. We contend that $\mathrm{Hom}_\Lambda(\mathcal{Z}, \Lambda)$ is a free Λ-module, which is enough to establish i). Indeed, let $\mathcal{N}$ be the maximal torsion Λ-submodule of $\mathcal{Z}$, then $\mathrm{Hom}_\Lambda(\mathcal{Z}/\mathcal{N}, \Lambda) = \mathrm{Hom}_\Lambda(\mathcal{Z}, \Lambda)$, so we may assume $\mathcal{Z}$ is torsion-free over Λ. As $\mathcal{Z}$ is finitely generated, there exists a pseudo-null Λ-module $\mathcal{M}$ such that

$$0 \to \mathcal{Z} \to \Lambda^{\oplus 2} \to \mathcal{M} \to 0$$

is exact. Taking $\mathrm{Hom}_\Lambda(-, \Lambda)$, we have the exact sequence $0 \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus 2}, \Lambda) \to \mathrm{Hom}(\mathcal{Z}, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathcal{M}, \Lambda)$. Thus, we are left to show $\mathrm{Ext}^1_\Lambda(\mathcal{M}, \Lambda) = 0$; namely, any Λ-module extension $0 \to \Lambda \to J \to \mathcal{M} \to 0$ splits.

Since $\mathcal{M}$ is pseudo-null, there exists a non-zero element $g \in \Lambda$ that is an annihilator of $\mathcal{M}$. After multiplying by g on the short exact sequence of $0 \to \Lambda \to J \to \mathcal{M} \to 0$ and applying snake lemma, we have another exact sequence $0 \to J^{g=0} \to \mathcal{M} \to \Lambda/(g)$. Since $\Lambda/(g)$ has no pseudo-null submodule, we conclude that there is a canonical lifting $\mathcal{M} \simeq J^{g=0} \hookrightarrow J$; hence, the extension is trivial, which concludes the proof of part i).

We turn our attention to ii). In particular, we assume that f is non-ordinary at p. If $\mathbf{z} = (z_n) \in \varprojlim H^1_{\mathbf{f}}(K_{n,w}, T)$, then the image of z_n under the Bloch–Kato dual exponential map is zero. Thus, the interpolation formula of the Perrin-Riou map (see [LZ14, Theorem 4.15]) implies that $\mathcal{L}_{K_{\infty,w},T}(\mathbf{z})$ is an element of the following set:

$$\mathcal{A} = \big\{ g \in \tilde{\mathscr{H}}_r \otimes \mathbb{D}_{\mathrm{cris}}(V) : g(\chi^{-j}\theta) \in K_n \otimes \varphi^n \mathrm{Fil}^j \mathbb{D}_{\mathrm{cris}}(V), g(\theta) = 0,$$
$$\text{for all finite characters } \theta \text{ of } \Gamma \text{ and } j \le k/2 \big\}.$$

It follows from the calculations in [PR00, §3] that when $V|_{G_{K_w}}$ is irreducible, the set $\mathcal{A} = 0$, which in turn implies that $\mathbf{z} = 0$ by Proposition 2.5.4 of *op. cit.*

We now prove iii). Since f is assumed to be ordinary at p, $T|_{G_{K_w}}$ admits a one-dimensional sub-representation T^+ so that

$$H^1_{\mathbf{f}}(L', T) = H^1(L', T^+)$$

for any finite extension L' of K_w (see [PR92, Remark 2.3.4], which says that $H^1_{\mathbf{f}}(L', T)$ and $H^1(L', T^+)$ agree up to torsion; Lemma 4.1 tells us that the torsion subgroups are trivial). Next, we claim that the natural map $H^1_{\mathrm{Iw}}(K_{\infty,w}, T)/H^1_{\mathrm{Iw}}(K_{\infty,w}, T^+) \to H^1_{\mathrm{Iw}}(K_{\infty,w}, T^-)$ is an isomorphism. By dualizing, this boils down to the exactness of the following sequence

$$0 \to H^1(K_{\infty,w}, A^+) \to H^1(K_{\infty,w}, A) \to H^1(K_{\infty,w}, A^-) \to 0,$$

which in turn is guaranteed by Lemma 4.1(b) (note that by local duality, $H^2(K_{\infty,w}, A^+) = H^0(K_{\infty,w}, T^-)^\vee$). Now, we may proceed as in the proof of i) to conclude that $H^1_{\mathrm{Iw}}(K_{\infty,w}, T^-)$ is free of rank one over Λ, again using Lemma 4.1(b). $\qquad\square$

Remark 6.3. The calculations in this section are carried out purely locally at p and do not depend on the hypotheses (Tama.) and (Loc.prim.) assumed in Proposition 5.1. Their sole role is in enabling the passage between the Selmer group and the local cohomology groups via (6.1).

APPENDIX A. GROWTH OF SELMER GROUPS: THE CASE OF $a_p = 0$

A.1. **Summary of results.** In this appendix, following the line of argument of Matar presented in [Mat21a], we give an alternative proof of Theorem A for modular forms f with $a_p(f) = 0$ utilizing the plus and minus Selmer groups. As in §3, when $N^- > 1$, we will need to assume that the weight k is greater than 2, as this is the setting of [Mag22].

We will keep the same notation from the main body of the article, with the following additions

- If R is an $\mathfrak{o}$-algebra, we write $\Lambda_R = \Lambda \otimes_{\mathfrak{o}} R$.
- From the generalized Heegner classes z_{f,p^n} [CH18, Mag22], we form
  $$\alpha_n = \mathrm{Cores}_{K[p^{n+1}]/K_n}(z_{f,p^{n+1}}) \in H^1(K_n, T).$$

- For $n \geq 1$, we denote $\Phi_n(X) = \frac{X^{p^n}-1}{X^{p^{n-1}}-1}$, $\omega_n(X) = (X+1)^{p^n} - 1$,
  $$\tilde{\omega}_n^+(X) = \prod_{2 \leq i \leq n, i \equiv 0 \bmod 2} \Phi_i(X+1), \quad \tilde{\omega}_n^-(X) = \prod_{1 \leq i \leq n, i \equiv 1 \bmod 2} \Phi_i(X+1),$$
  and $\omega_n^\pm(X) = X\tilde{\omega}_n^\pm(X)$.

Our main result is the following

Theorem B. *Let f be a normalized cuspidal eigen-newform of even weight k and level $\Gamma_0(N)$. Let K be an imaginary quadratic field with coprime-to-Np discriminant $d_K \neq -3, -4$ and satisfying (Heeg.). Assume moreover $N \geq 5$ if $N^- = 1$, and $N^+ > 3$ and $k \geq 4$ if $N^- \neq 1$. Let p be a prime such that:*

(i) *p splits in K, and does not divide the class number h_K;*

(ii') *$p \nmid 2(k-1)!N\varphi(N)$;*

(iii') *$a_p(f) = 0$;*

(iv) *For any $w \mid N$ in K and any $w' \mid w$ in K_∞, the group $A^{G_{K_{\infty,w'}}}$ is divisible;*

(v') *The class $z_{f,K}$ is globally primitive, i.e., $z_{f,K} \notin \varpi H^1(K, T)$. Moreover, $p\alpha_1 \neq p^{(k-2)/2}(p-1)/2 \cdot \alpha_0$ and $p\alpha_2 \neq -p^{k-2}\alpha_0$;*

(vi) *$\mathrm{Sel}(K, A) \simeq \mathfrak{F}/\mathfrak{o} \cdot z_{f,K}$;*

(vii) *p is unramified in $\mathfrak{F}$.*

Then:

(I) *the Selmer modules $\mathrm{Sel}^+(K_\infty, A)$ and $\mathrm{Sel}^-(K_\infty, A)$, whose definitions are recalled in §A.2.4, are cofree of corank one over Λ;*

(II) *for all $n \geq 0$, $\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}(K_n/A)) = p^n$;*

(III) *the Shafarevich–Tate groups $\Sha_{\mathrm{BK}}(K_n, A)$ and $\Sha_{\mathrm{AJ}}(K_n, A)$ introduced in §3 coincide. In particular, $\Sha_{\mathrm{AJ}}(K_n, A)$ is finite for all $n \geq 0$.*

Suppose in addition that $\mathrm{loc}_v(z_{f,K}) \notin \varpi H^1_{\mathbf{f}}(K_v, T)$, then we have

(IV) *$\mathrm{Sel}(K_\infty, A)^\vee$ is free of rank 2 over Λ;*

(V) *$\Sha_{\mathrm{AJ}}(K_n, A) = 0$ for all $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$.*

Remark A.1. Some comments for the assumptions:

(1) we need the Fontaine–Laffaille condition $p > k - 1$ in (ii') so as to construct a strongly divisible lattice (in the sense of [FL82, Définition 7.7]; see also [BK90, Theorem 4.3]) inside the crystalline Dieudonné module of V; this allows us to define integral Coleman maps in §A.2;

(2) in the supersingular setting, the assumption (iii) in Theorem A, as well as its consequence Lemma 4.1.(a), is automatically valid (see Lemma A.3);

(3) we continue to assume the p-freeness of Tamagawa numbers, condition (iv), to ensure exact control theorems of plus/minus Selmer groups as well as that of their intersection (see Proposition A.9, Lemma A.10 below);

(4) condition (v') allows us to study Selmer groups explicitly by exploiting the Heegner modules sitting inside, an approach also employed by [PR87, LV19b, Mat21a]. Note that if $k = 2$, then the conditions on α_1, α_2 are superfluous as $\alpha_0 = -2z_{f,K} \notin pH^1(K,T)$;

(5) under the assumption $z_{f,K} \notin \varpi H^1(K,T)$ from (v'), we see that (vi) actually asserts that $z_{f,K}$ is a generator of $\mathrm{Sel}(K,A)$, namely, the map $\mathfrak{F}/\mathfrak{o} \to \mathrm{Sel}(K,A)$ sending t to $z_{f,K} \otimes t$ is an isomorphism;

(6) the additional assumption (vii) is required for our argument in §A.3, which guarantees the irreducibility of cyclotomic polynomials $\Phi_n(X)$.

A.1.1. *Organization.* This appendix is arranged as follows: In §A.2 we recall the definition of plus/minus Selmer groups and establish some basic properties of them in parallel with those studied by Kobayashi [Kob03]. In §A.3, we adapt an argument of Matar to prove the nontorsionness (Corollary A.14) and the lower bound of the corank of $\mathrm{Sel}(K_n, A)$ (Proposition A.15). In §A.4, we replicate Matar's argument in our setting and establish the Λ-structures of $\mathrm{Sel}^{\pm}(K_\infty, A)^\vee$ and the upper bound of $\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}(K_n, A))$. Combined with the study of generalized Heegner modules, these are enough to deduce the finiteness of Shafarevich–Tate groups. Finally, assuming $\mathrm{loc}_v(z_{f,K}) \notin \varpi H^1_{\mathbf{f}}(K_v, T)$, we borrow some results from §5 to deduce the rest of the theorem.

A.2. **Review of plus/minus theory.**

A.2.1. *Setup.* Throughout the appendix, we assume $p \nmid 2N\varphi(N)(k-1)!h_K$, p splits in K and $a_p(f) = 0$. We recall here Lei's theory of plus/minus Coleman maps for modular forms [Lei10, Lei11]. As the Galois representation V is crystalline, we can form the corresponding Dieudonné module $D(V) = \mathbb{D}_{\mathrm{cris}}(V)$, which is equipped with the decreasing de Rham filtration and the semi-linear operator φ. For $i \in \mathbf{Z}$, we write $D^i(V)$ for the i-th filtration of $D(V)$. In addition, since we assume the Fontaine–Laffaille condition $p > k - 1$ holds, there is a strongly divisible lattice $D(T) \subset D(V)$ (see [FL82, Définition 7.7] or [BK90, Theorem 4.3]), which inherits a filtration from $D(V)$. We denote this filtration by $D^\bullet(T)$.

Throughout §A.2 we will fix $w \in \{v, \bar{v}\}$. As $K_{\infty,w}/K_w$ is a totally ramified $\mathbf{Z}_p$-extension of $K_w = \mathbf{Q}_p$, we have the Perrin-Riou regulator as constructed in [Lei10, §2]:

$$\mathcal{L}_T : H^1(K_{\infty,w}, T) = \varprojlim_n H^1(K_{n,w}, T) \longrightarrow D(V) \otimes \mathscr{H}_\infty,$$

where $\mathscr{H}_\infty$ is the algebra of tempered distributions on $\Gamma \simeq \mathrm{Gal}(K_{\infty,w}/K_w)$, which we may identify with the set of power series in $\mathfrak{F}[[X]]$ that converge on the p-adic open unit disc $\{z \in \mathbf{C}_p : |z| < 1\}$.

A.2.2. *Coleman maps.* As V has Hodge–Tate weights $\{r, 1 - r\}$, where $r = k/2$, we see that

$$D^i(T) = \begin{cases} D(T), & i \leq -r; \\ \mathfrak{o} \cdot \omega, & -r < i \leq r - 1; \\ 0, & i \geq r. \end{cases}$$

Here, ω is a chosen generator of $D^0(T)$.

Lemma A.2. *We have $\varphi(\omega)/p^{r-1} \in D(T)$, and $D(T)$ is freely spanned by ω and $\varphi(\omega)/p^{r-1}$.*

Proof. We argue in the same way as [LLZ17, Lemma 3.1]. The Fontaine–Laffaille condition guaranties the following:

- $\varphi(\omega) \in p^{r-1}D(T)$ since $\omega \in D^{r-1}(T)$.
- There exists $\xi \in D(T)$ such that $D(T) = \mathfrak{o}\xi \oplus \mathfrak{o}\omega$.
- $D(T) = \mathfrak{o} \cdot p^{1-r}\varphi(\omega) + p^r\varphi(D(T))$.

Let D be the $\mathfrak{o}$-submodule of $D(T)$ spanned by ω and $\varphi(\omega)/p^{r-1}$. The second and third properties then implies $\xi \in D + \mathfrak{o} \cdot p^r\varphi(\xi)$. Iterating, we find $\xi \in D + p^r\varphi(D) + \mathfrak{o} \cdot p^k\varphi^2(\xi)$. As $V = V_f(r)$, we have $\varphi^2 = -p^{-1}$. So $p^r\varphi(D) \subset \mathfrak{o} \cdot p^r\varphi(\omega) + \mathfrak{o}\omega \subset D$, and thus $\xi \in D + \mathfrak{o}p^{k-1}\xi$. This forces $\xi \in D$. $\qquad\square$

Now, put $v_1 = \omega, v_2 = \varphi(\omega)/p^{r-1}$. Then under this basis, $\mathcal{L}_T$ admits the following decomposition for all $z \in H^1(K_{\infty,v}, T)$ [Lei10, Definition 2.8]:

$$\mathcal{L}_T(z) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \log_{p,k}^+ & 0 \\ 0 & \log_{p,k}^- \end{bmatrix} \begin{bmatrix} \mathrm{Col}^+(z) \\ \mathrm{Col}^-(z) \end{bmatrix}.$$

Here,

- for $\varepsilon \in \{\pm 1\}$,

$$\log_{p,k}^\varepsilon = \prod_{1-r \le j \le r-1} \frac{1}{p} \prod_{\substack{n \ge 1 \\ (-1)^n = \varepsilon}} \frac{\Phi_n(\gamma^{-j}(1+X))}{p}.$$

  As shown by Pollack [Pol03, Lemma 4.1], both products converge in $\mathbf{Q}_p[[X]]$;
- the Coleman maps, $\mathrm{Col}^\pm$, are Λ-morphisms from $H^1(K_{\infty,w}, T^*)$ to $\Lambda_{\mathfrak{F}}$. By [BL17, Proposition 2.20], it is known that the image of $\mathrm{Col}^\pm$ is a finite-index submodule of some free Λ-submodule of rank 1 inside $\Lambda_{\mathfrak{F}}$.

Next, we recall the following lemma that will be crucially used in our proof of the main theorem.

Lemma A.3. *We have $A[\varpi]^{G_{K_{\infty,w}}} = 0$, and hence $A^{G_{K_{\infty,w}}} = 0$.*

Proof. This follows from the same proof of [LZ24, Lemma 4.4] and [Lei11, Lemma 4.4]. By [Edi92, Theorem 2.6], the inertia group at p acts on $A[\varpi]$ as $\begin{bmatrix} \psi^{k-1} & 0 \\ 0 & \psi'^{k-1} \end{bmatrix}$, where ψ and ψ' are fundamental characters of level 2. As ψ^{k-1} and ψ'^{k-1} are non-trivial characters, it follows that $A[\varpi]^{G_{K_{\infty,w}}} = 0$. $\qquad\square$

Thanks to Lemma A.3, we have an identification

$$H^1(K_{n,w}, A) = H^1(K_{\infty,w}, A)^{\Gamma_n}$$

for all $n \in \mathbf{Z}_{\ge 0}$. By local Tate duality, we derive

$$H^1(K_{n,w}, T) = H^1(K_{\infty,w}, T)_{\Gamma_n}.$$

As such, we can define $\mathfrak{Col}_n^\pm$ to be $\mathrm{Col}^\pm$ modulo Γ_n, namely the induced map

$$\mathfrak{Col}_n^\pm : H^1(K_{n,w}, T) \to \Lambda_{\mathfrak{F}}/\omega_n\Lambda_{\mathfrak{F}}.$$

A.2.3. *Plus/minus local subgroups.* We put

$$H_\pm^1(K_{\infty,w}, T) = \ker(\mathrm{Col}^\pm) \subset H^1(K_{\infty,w}, T),$$

and write $H_\pm^1(K_{\infty,w}, A) \subset H^1(K_{\infty,w}, A)$ for its orthogonal complement under the local Tate pairing; put also

$$H_\pm^1(K_{n,w}, A) = H^1(K_{\infty,w}, A)^{\Gamma_n}.$$

By local duality, we have

$$H_\pm^1(K_{n,w}, A) = \ker(\mathfrak{Col}_n^\pm)^\perp.$$

We recall the following description given in [Lei10, Proposition 3.3]; note that for $m \in \mathbf{Z}_{\geq 0}$, the restriction map $H^1(K_{m,w}, A) \to H^1(K_{m+1,w}, A)$ is injective, by Lemma A.3.

Lemma A.4. *Let $\varepsilon \in \{\pm 1\}$. The local subgroup $H_\varepsilon^1(K_{n,w}, A)$ coincides with*

$$\left\{ z \in H_{\mathbf{f}}^1(K_{n,w}, A) : \mathrm{Cores}_{K_n/K_{m+1}}(z) \in H^1(K_{m,v}, A) \text{ for all } m \in \mathbf{Z}_{\geq 0}, m < n, (-1)^m = \varepsilon \right\}.$$

Corollary A.5. *The image of $\mathfrak{Col}_n^\pm$ is annihilated by $\omega_n^\pm$.*

Proof. Let $\varepsilon \in \{\pm 1\}$. Since $\mathfrak{Col}_n^\varepsilon$ is valued in $\Lambda_{\mathfrak{F}}/\omega_n \Lambda_{\mathfrak{F}}$, it suffices to show that there exists $t \in \mathbf{Z}_{\geq 0}$ such that $p^t \omega_n^\varepsilon$ annihilates $\mathrm{im}(\mathfrak{Col}_n^\varepsilon)$. Or, by duality, any $z \in H_\varepsilon^1(K_{m,v}, A)$ is killed by $p^t \omega_n^\varepsilon$. By Lemma A.4, we have

$$p^{t(n,\varepsilon)} \omega_n^\varepsilon \cdot z = X \cdot \mathrm{Tr}_{K_n/K}(z) = 0,$$

where

$$t(n,\varepsilon) = \begin{cases} n/2, & \text{if } 2 \mid n; \\ (n+1)/2, & \text{if } 2 \nmid n \text{ and } \varepsilon = +1; \\ (n-1)/2, & \text{otherwise.} \end{cases}$$

$\square$

We will thus denote by $\mathrm{Col}_n^\pm$ the composition

$$\mathrm{Col}_n^\pm : H^1(K_{n,w}, T) \xrightarrow{\mathfrak{Col}_n^\pm} \Lambda_{\mathfrak{F}}/\omega_n \Lambda_{\mathfrak{F}} \twoheadrightarrow \Lambda_{\mathfrak{F}}/\omega_n^\pm \Lambda_{\mathfrak{F}},$$

where we identify $\Lambda_{\mathfrak{F}}/\omega_n \simeq \Lambda_{\mathfrak{F}}/\omega_n^\pm \oplus \Lambda_{\mathfrak{F}}/\tilde{\omega}_n^\mp$ by the Chinese remainder theorem and the second map is the natural projection.

Proposition A.6. *We have $\ker(\mathrm{Col}_n^\pm) = \ker(\mathfrak{Col}_n^\pm)$.*

Proof. We will prove the plus part; the minus part can be handled similarly. By definition, it boils down to establishing the reverse inclusion $\ker(\mathrm{Col}_n^+) \subseteq \ker(\mathfrak{Col}_n^+)$. Applying the Chinese remainder theorem, we have $F, G \in \mathbf{Z}_p[[X]]$ and $m \in \mathbf{Z}_{\geq 0}$ such that $F\omega_n^+ + G\tilde{\omega}_n^- = p^m$; note that this forces G to be prime to ω_n^+ in $\Lambda_{\mathfrak{F}}$. Now, suppose $z \in \ker(\mathrm{Col}_n^+)$. It follows from Corollary A.5 that

$$p^m \cdot \mathfrak{Col}_n^+(z) = G\tilde{\omega}_n^- \cdot \mathfrak{Col}_n^+(z).$$

So, $G(X)\tilde{\omega}_n^-(X)\mathfrak{Col}_n^+(z) = 0 \in \Lambda_{\mathfrak{F}}/\omega_n^+ \Lambda_{\mathfrak{F}}$. As $\Lambda_{\mathfrak{F}}$ is a unique factorization domain, this shows $\mathfrak{Col}_n^+(z) \in \omega_n^+ \Lambda_{\mathfrak{F}}$, whereby $p^m \mathfrak{Col}_n^+(z) = G\tilde{\omega}_n^- \cdot \mathfrak{Col}_n^+(z) = 0 \in \Lambda_{\mathfrak{F}}/\omega_n \Lambda_{\mathfrak{F}}$. That is, $z \in \ker(\mathfrak{Col}_n^+)$.

$\square$

Remark A.7. If we consider $\mathrm{Col}^\pm$ to be valued in $\mathfrak{F}[[\Gamma]]$, then from our construction we have $\mathrm{Col}^\pm = \varprojlim_n \mathrm{Col}_n^+$. In particular, this equality holds in $\Lambda_{\mathfrak{F}}$; *cf.*, [Kob03, Definition 8.22].

A.2.4. *Selmer groups and their control theorems.* For $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, we define

$$\mathrm{Sel}^{\pm}(K_n, A) = \ker\left(H^1(K_n, A) \to \prod_{w|p} \frac{H^1(K_{n,w}, A)}{H^1_{\pm}(K_{n,w}, A)} \times \prod_{w \nmid p} \frac{H^1(K_{n,w}, A)}{H^1_{\mathbf{f}}(K_{n,w}, A)}\right).$$

As both $\mathrm{Sel}^{\pm}(K_\infty, A)$ are contained in $\mathrm{Sel}(K_\infty, A)$, they are cofinitely generated Λ-modules [Gre89, Proposition 3].

Remark A.8. From Lemma A.4, we see that

$$\mathrm{Sel}^{\pm}(K, A) = \mathrm{Sel}(K, A).$$

Next, we consider a special case of the control theorem of Ponsinet [Pon20, Lemma 2.3]:

Proposition A.9. *Assume* (Tama.) *holds. For $n \geq 0$, the map*

$$\mathrm{Sel}^{\pm}(K_n, A) \to \mathrm{Sel}^{\pm}(K_\infty, A)^{\omega_n = 0}$$

is an isomorphism.

Proof. By Lemma A.3 the map is injective. For the surjectivity, the proof is more or less in the same lines as those of Proposition 4.2, except that at $w \mid p$, one observes instead

$$\frac{H^1(K_{n,w}, A)}{H^1_{\pm}(K_{n,w}, A)} \to \left(\frac{H^1(K_{\infty,w}, A)}{H^1_{\pm}(K_{\infty,w}, A)}\right)^{\Gamma_n}$$

is injective (*cf., loc. cit.*). $\square$

Next, after Iovita–Pollack [IP06, Definition 7.3], for $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ we define

$$\mathrm{Sel}^1(K_n, A) = \ker\left(\mathrm{Sel}(K_n, A) \to \prod_{w|p} \frac{H^1(K_{n,w}, A)}{H^1_{\mathbf{f}}(K_w, A)}\right).$$

We first note the following

Lemma A.10. *The intersection* $\mathrm{Sel}^+(K_n, A) \cap \mathrm{Sel}^-(K_n, A)$ *equals* $\mathrm{Sel}^1(K_n, A)$.

Proof. We have $\mathrm{Sel}^1(K_n, A) \subseteq \mathrm{Sel}^+(K_n, A) \cap \mathrm{Sel}^-(K_n, A)$ by definition. Now suppose $x \in \mathrm{Sel}^+(K_n, A) \cap \mathrm{Sel}^-(K_n, A)$. Then for $w \mid p$, we have $\mathrm{loc}_w(x) \in H^1_+(K_{n,w}, A) \cap H^1_-(K_{n,w}, A)$. We claim that $\mathrm{loc}_w(x) \in H^1_{\mathbf{f}}(K_w, A)$. Otherwise, let $t \in \mathbf{Z}_{\geq 1}$ be smallest such that $\mathrm{loc}_v(x) \in H^1_{\mathbf{f}}(K_{t,w}, A) \setminus H^1_{\mathbf{f}}(K_{t-1,w}, A)$. As $\mathrm{loc}_w(x) \in H^1_{(-1)^{t-1}}(K_{n,w}, A)$, by Lemma A.4, we have

$$\mathrm{Cores}_{K_n/K_t}(\mathrm{loc}_w(x)) \in H^1_{\mathbf{f}}(K_{t-1,w}, A).$$

Since

$$\mathrm{loc}_v(x) = \frac{1}{p^{n-t}}\mathrm{Cores}_{K_n/K_t}(\mathrm{loc}_w(x)),$$

we find $p^{n-t}\mathrm{loc}_w(x) \in H^1_{\mathbf{f}}(K_{t-1,w}, A)$. However, as $H^1_{\mathbf{f}}(K_{t-1}, A)$ is divisible (see [Gre01, proof of Theorem 2.9]), we deduce that $\mathrm{loc}_w(x) \in H^1_{\mathbf{f}}(K_{t-1}, A)$, which is a contradiction. $\square$

Lemma A.11. *Assume* (Tama.) *holds. For $n \in \mathbf{Z}_{\geq 0}$, the injection*

$$\mathrm{Sel}^1(K_n, A) \hookrightarrow \mathrm{Sel}^1(K_\infty, A)^{\Gamma_n}$$

is an isomorphism.

Proof. The proof is similar to that of Proposition 4.2. In the snake diagram, at a place $w \mid p$ we use the identification

$$\frac{H^1(K_{n,w}, A)}{H^1_{\mathbf{f}}(K_w, A)} = \frac{H^1(K_{\infty,w}, A)^{\Gamma_n}}{H^1_{\mathbf{f}}(K_w, A)},$$

valid by Lemma A.3, to bound the kernel of the map on the product of local cohomology groups. □

Below, for a Λ-module M and $n \in \mathbf{Z}_{\geq 0}$, we denote $M^{\omega_n^+ = \omega_n^- = 0}$ simply by $M^{\omega_n^{\pm} = 0}$.

Corollary A.12. *Assume* (Tama.) *and* (Sel.) *hold. We have* $\mathrm{corank}(\mathrm{Sel}^1(K_n, A)^{\omega_n^{\pm} = 0}) \leq 1$.

Proof. The proof is due to Matar [Mat21a, pp. 140-141], which we record here for completeness. As $\gcd(\omega_n^+, \omega_n^-) = X$ in $\mathbf{Q}_p[X]$, we have $A(X), B(X) \in \mathbf{Z}_p[X]$ and $m \in \mathbf{Z}_{\geq 0}$ such that $A\omega_n^+ + B\omega_n^- = p^m X$. It follows that

$$\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^1(K_n, A)^{\omega_n^{\pm} = 0}) \leq \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^1(K_n, A)^{p^m(\gamma-1)=0}) = \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^1(K_n, A)^{\Gamma}).$$

By Lemma A.11, we conclude the last quantity is $\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^1(K, A))$, and thus is no greater than $\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}(K, A))$, which is ≤ 1 by (Sel.). □

A.3. Nontorsionness of $\mathcal{X}_{\infty}^{\pm}$ via generalized Heegner modules. From now on we will always assume that p is unramified in the local Hecke field $\mathfrak{F}$; this ensures that all the cyclotomic polynomials $\Phi_n(X)$ are irreducible. For $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, denote $\mathcal{X}_n^{\pm} = \mathrm{Sel}_{\pm}(K_n, A)^{\vee}$. In this section we modify an argument of Matar to prove both Λ-modules $\mathcal{X}_{\infty}^{\pm}$ are nontorsion.

Recall we have assumed that $a_p = a_p(f) = 0$, $p \nmid 2N\varphi(N)(k-1)!h_K$ and the weight of the modular form to be $k = 2r \in 2 \cdot \mathbf{Z}_{>0}$. For $n \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, let $\Psi(K_n) \subseteq H^1_{\mathbf{f}}(K_n, T)$ be the image of the p-adic Abel–Jacobi map recalled in Definition 3.5. Since $\Psi(K_n)$ is an $\mathfrak{o}$-submodule of $H^1(K_n, T)$, we see that $\Psi(K_n)$ is finite free over $\mathfrak{o}$ since $H^1(K_n, T)$ is (using Lemma A.3 again and the inclusion $A^{G_{K_n}} \subseteq A^{G_{K_{n,w}}}$ for $w \mid p$).

Next, let $z_{f,p^n} \in H^1(K[p^n], T)$ be the generalized Heegner classes recalled in §3, where $K[p^n]$ is the ring class field of K of conductor p^n. Recall we write $\mathrm{Cores}_{K[1]/K}(z_{f,1})$ as $z_{f,K}$. For $n \in \mathbf{Z}_{\geq 0}$, denote $\alpha_n = \mathrm{Cores}_{K[p^{n+1}]/K_n}(z_{f,p^{n+1}}) \in \Psi(K_n)$, which in fact belongs to $\mathrm{Sel}(K_n, T)$ as explained in §3. We have the following norm relations (see [LV19a, Lemma 4.1] when $N^- = 1$ and [Pat25, Lemma 4.1] when $N^- > 1$):

(H1) for $n \geq 2$, $\mathrm{Cores}_{K_n/K_{n-1}}(\alpha_n) = -p^{k-2}\alpha_{n-2}$;
(H2) $\mathrm{Cores}_{K_1/K_0} \alpha_1 = p^{r-1}(p-1)/2 \cdot \alpha_0$;
(H3) $\alpha_0 = -2p^{r-1}z_{f,K}$.

Throughout this section, we will assume the following linear non-incidence condition:

$$\text{(Inc.)} \qquad \alpha_0 \neq 0, \ p\alpha_1 \neq \frac{p^{r-1}(p-1)}{2}\alpha_0, \ p\alpha_2 \neq -p^{k-2}\alpha_0.$$

In particular, since in our setting $H^1(K, T)[p^{\infty}] = 0$, $\alpha_0 \neq 0$ if and only if $z_{f,K} \neq 0$, which is further equivalent to $z_{f,K}$ being nontorsion.

Define $\mathcal{H}_n \subseteq \Psi(K_n)$ to be the $\mathfrak{o}$-span of α_n^{σ} for σ ranging in $\mathrm{Gal}(K_n/K)$. If M is a $\mathfrak{o}$-module write $M_{\mathfrak{F}} = M \otimes_{\mathfrak{o}} \mathfrak{F}$. For $n \in \mathbf{Z}$, put $\varepsilon(n) = (-1)^n$. We prove

Proposition A.13. *Suppose* (Inc.) *holds. Then the* $\mathfrak{F}[\mathrm{Gal}(K_n/K)]$-*module* $\mathcal{H}_{n,\mathfrak{F}}$ *is isomorphic to* $\Lambda_{\mathfrak{F}}/(\omega_n^{\varepsilon(n)}(X))$.

Proof. By the norm relations (H1) and (H2), we have

$$\omega_n^{\varepsilon(n)}(X) \cdot \alpha_n \in \mathbf{Z}_p(\gamma - 1) \cdot \alpha_0.$$

As $(\gamma-1)\alpha_0 = 0$, we deduce that $\mathrm{Ann}(\mathcal{H}_{n,\mathfrak{F}}) \supseteq (\omega_n^{\varepsilon(n)}(X))$ since the $\Lambda_{\mathfrak{F}}$-module $\mathcal{H}_{n,\mathfrak{F}}$ is generated by α_n. We now prove the reverse inclusion. For this, we choose a generator F of $\mathrm{Ann}(\mathcal{H}_{n,\mathfrak{F}})$, which exists as $\Lambda_{\mathfrak{F}}$ is a principal ideal domain.

We shall prove the reverse inclusion by induction. The case $n = 0$ follows from the assumption $\alpha_0 \neq 0$. For $n = 1$, we have $\omega_n^{\varepsilon(n)} = \omega_1(X) = X^p - 1$. Suppose $F \in \Lambda_{\mathfrak{F}}$ is a proper divisor of ω_1, then $F = X$ or $F = \Phi_1(X)$. In the former scenario, we find $\gamma\alpha_1 = \alpha_1$, so

$$p^{r-1}(p-1)/2 \cdot \alpha_0 = \mathrm{Cores}_{K_1/K}(\alpha_1) = p\alpha_1,$$

contravening our hypothesis. Next, suppose $\Phi_1(X)\alpha_1 = 0$, which, by the norm relation, is equivalent to $p^{r-1}(p-1)/2 \cdot \alpha_0 = 0$. Again this is ruled out as $\Psi(K)$ is free over $\mathfrak{o}$ and we assumed $\alpha_0 \neq 0$. Thus for $n = 1$, we must have $F = \omega_1(X)$.

Now suppose we are done with $n < t$ for some $t \in \mathbf{Z}_{\geq 2}$. For $n = t$, first suppose for the sake of contradiction that $\Phi_t \nmid F$. Then we have $\omega_{t-1}(X) \cdot \alpha_t = 0$, namely $\alpha_t \in H^1(K_t, T)^{\Gamma_{t-1}} = H^1(K_{t-1}, T)$ by Lemma A.3. Thus,

$$-p^{k-2}\alpha_{t-2} = \mathrm{Cores}_{K_t/K_{t-1}}(\alpha_t) = p\alpha_t.$$

Taking corestrictions on both sides, we find

$$\begin{cases} p\alpha_2 = -p^{k-2}\alpha_0, & \text{if } t \equiv 0 \bmod 2, \\ p\alpha_1 = p^{r-1}(p-1)/2 \cdot \alpha_0, & \text{if } t \equiv 1 \bmod 2. \end{cases}$$

As both possibilities are excluded from our assumption, we conclude that Φ_t must divide F.

Now, for any $\sigma \in \mathrm{Gal}(K_n/K)$, we have

$$0 = (p^{k-2}F/\Phi_t)(\Phi_t/p^{k-2})(\alpha_n^\sigma) = -(p^{k-2}F/\Phi_t)\alpha_{n-2}^\sigma.$$

Thus $F/\Phi_t \in \mathrm{Ann}(\mathcal{H}_{t-2,\mathfrak{F}})$, which by inductive hypothesis implies that $\omega_{t-2}^{\varepsilon(t)} \mid F/\Phi_t$. It follows that $\omega_t^{\varepsilon(t)} \mid F$, as desired. $\qquad\square$

Corollary A.14. *Assume* (Inc.) *holds. The Λ-modules $\mathcal{X}_\infty^\pm$ are not torsion.*

Proof. For $n \in \mathbf{Z}_{\geq 0}$, the norm relations imply that $\mathcal{H}_n \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o} \subseteq \mathrm{Sel}^{(-1)^n}(K_n, A)$. By Proposition A.13 we have an identification $\mathcal{H}_n \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o} \simeq (\mathfrak{F}/\mathfrak{o})[X]/(\omega_n^{\varepsilon(n)})$. Thus $\mathcal{X}_\infty^+$ admits a surjective map to $\mathfrak{o}[X]/(\omega_{2n}^+)$. Now, if $\mathcal{X}_\infty^+$ is Λ-torsion, then there exists $F \in \mathfrak{o}[[X]]$ such that F annihilates $(\mathcal{H}_{2n} \otimes \mathfrak{F}/\mathfrak{o})^\vee$ for all $n \in \mathbf{Z}_{\geq 0}$. This forces $\omega_{2n}^+ \mid F$ in $\Lambda_{\mathfrak{F}}$ for all such n, which is impossible. For the same reason, $\mathcal{X}_\infty^-$ is also nontorsion. $\qquad\square$

Now, for $n \in \mathbf{Z}_{\geq 0}$ we put $H(K_n) = \sum_{0 \leq m \leq n} \mathcal{H}_m$, $H(K_n) = \mathcal{H}(K_n) \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o}$, $H(K_n)^{\varepsilon(n)} = \mathcal{H}_n \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o}$ and $H(K_n)^{-\varepsilon(n)} = \mathcal{H}_{n-1} \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o}$. By the norm relations, we have

$$(\mathrm{A.1}) \qquad\qquad H(K_n) = H(K_n)^+ + H(K_n)^-.$$

Proposition A.15. *Assume* (Tama.), (Sel.) *and* (Inc.) *hold. For $n \in \mathbf{Z}_{\geq 0}$, we have* $\mathrm{rank}_{\mathfrak{o}} \mathcal{H}(K_n) \geq p^n$ *and* $\mathrm{rank}_{\mathfrak{o}} \Psi(K_n) \geq p^n$.

Proof. The argument is due to Matar [Mat21a, proof of Theorem 4.2]. By (A.1), we have

$$\mathrm{rank}_{\mathfrak{o}}(\mathcal{H}(K_n)) = \mathrm{corank}_{\mathfrak{o}}(H(K_n))$$

$$= \mathrm{corank}_{\mathfrak{o}}(H(K_n)^+) + \mathrm{corank}_{\mathfrak{o}}(H(K_n)^-) - \mathrm{corank}_{\mathfrak{o}}(H(K_n)^+ \cap H(K_n)^-)$$

$$= p^n + 1 - \mathrm{corank}_{\mathfrak{o}}(H(K_n)^+ \cap H(K_n)^-),$$

where the last equality follows from Proposition A.13 and the identity $\deg(\omega_n^+) + \deg(\omega_n^-) = p^n + 1$. As $H(K_n)^\varepsilon \subseteq \mathrm{Sel}^\varepsilon(K_n, A)^{\omega_n^\varepsilon=0}$ for $\varepsilon \in \{\pm 1\}$, we have the inclusion

$$H(K_n)^+ \cap H(K_n)^- \subseteq \mathrm{Sel}^1(K_n, A)^{\omega_n^\pm=0}.$$

As such, by Corollary A.12, we conclude that

$$\mathrm{rank}_{\mathfrak{o}}(\mathcal{H}(K_n)) \geq p^n + 1 - 1 = p^n.$$

$\square$

A.4. **Proof of Theorem B.**

Proposition A.16. *Assume* (Tama.), (Sel.) *and* (Inc.) *hold. The Λ-modules $\mathcal{X}_\infty^\pm$ are free of rank 1.*

Proof. By Proposition A.9, we find

$$\mathcal{X}_\infty^\pm/(\gamma - 1)\mathcal{X}_\infty^\pm \simeq \mathrm{Sel}^\pm(K, A)^\vee \simeq \mathfrak{o},$$

where the last identity is by assumption (Sel.). As recalled in §A.2.4, both $\mathcal{X}_\infty^\pm$ are finitely generated Λ-modules. Thus, $\mathcal{X}_\infty^\pm$ are cyclic Λ-modules by Nakayama's lemma. Corollary A.14 then implies that they are free. $\square$

Next, we record the following analogue to [Kob03, Proposition 10.1].

Lemma A.17. *Let $n \in \mathbf{Z}_{\geq 0}$. Let j be the map*

$$j : \mathrm{Sel}^+(K_n, A)^{\omega_n^+ = 0} \oplus \mathrm{Sel}^-(K_n, A)^{\omega_n^- = 0} \to \mathrm{Sel}(K_n, A)$$

induced from the inclusions $\mathrm{Sel}^\pm(K_n, A) \to \mathrm{Sel}(K_n, A)$, sending (P_1, P_2) to $P_1 + P_2$. Then $\mathrm{im}(j) \supseteq \mathrm{Sel}(K_n, A)_{\mathrm{div}}$.

Proof. The proof in *loc. cit.* can be carried verbatim to our situation: Let $F, G \in \mathbf{Z}_p[[X]]$ and $m \in \mathbf{Z}_{\geq 0}$ be such that $F\omega_n^+ + G\tilde{\omega}_n^- = p^m$. If $P \in \mathrm{Sel}(K_n, A)_{\mathrm{div}}$, choose $Q \in \mathrm{Sel}(K_n, A)$ such that $p^m Q = P$. Then $P^+ = G\tilde{\omega}_n^- \cdot Q \in \mathrm{Sel}^+(K, A_n)^{\omega_n^+ = 0}$ and $P^- = F\omega_n^+ \cdot Q \in \mathrm{Sel}^-(K, A_n)^{\omega_n^- = 0}$ are such that $P^+ + P^- = P$. $\square$

Proof of Theorem B, part 1. We have proved (I) in Proposition A.16. For (II), note that we have already the lower bound from Proposition A.15. To establish the upper bound, we recycle the following computation of Matar [Mat21a]:

$$\begin{aligned}
\mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}(K_n, A)) &= \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^+(K_n, A)^{\omega_n^+ = 0}) + \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^-(K_n, A)^{\omega_n^- = 0}) \\
&\quad - \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^+(K_n, A)^{\omega_n^+ = 0} \cap \mathrm{Sel}^-(K_n, A)^{\omega_n^- = 0}) \\
&= \mathrm{rank}_{\mathfrak{o}}(\mathcal{X}_\infty^+/\omega_n^+ \mathcal{X}_\infty^+) + \mathrm{rank}_{\mathfrak{o}}(\mathcal{X}_\infty^-/\omega_n^- \mathcal{X}_\infty^-) - \mathrm{corank}_{\mathfrak{o}}(\mathrm{Sel}^1(K_n, A)^{\omega_n^\pm = 0}) \\
&= \mathrm{rank}_{\mathfrak{o}}(\Lambda/\omega_n^+) + \mathrm{rank}_{\mathfrak{o}}(\Lambda/\omega_n^-) - 1 \\
&= p^n.
\end{aligned}$$

Here, the first equality uses Lemma A.17, the second uses Lemma A.10, the third uses Proposition A.16 and the inclusion $\mathfrak{F}/\mathfrak{o} \simeq \mathrm{Sel}(K, A) \hookrightarrow \mathrm{Sel}^1(K_n, A)^{\omega_n^\pm = 0}$.

Concerning (III): note that Proposition A.15 already shows that $\mathrm{corank}_{\mathfrak{o}}(\Psi(K_n) \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o}) \geq p^n$. As $\Psi(K_n) \otimes_{\mathfrak{o}} \mathfrak{F}/\mathfrak{o} \subseteq \mathrm{Sel}(K_n, A)$, their divisible parts must coincide, whereby $\mathrm{Ш}_{\mathrm{AJ}}(K_n, A) = \mathrm{Ш}_{\mathrm{BK}}(K_n, A)$. The finiteness of $\mathrm{Ш}_{\mathrm{AJ}}(K_n, A)$ then follows from that of $\mathrm{Ш}_{\mathrm{BK}}(K_n, A)$. $\square$

Proof of Theorem B, part 2. We now prove (IV) and (V) with the aid of results from §5. First note that (V) is immediate from Corollary 5.4.(2) and (III). Concerning (IV): from Remark 5.2.(ii) we have $\mathrm{Sel}(K_\infty, A)^\Gamma \subseteq \mathrm{Sel}^{\emptyset, \emptyset}(K, A) \simeq (\mathfrak{F}/\mathfrak{o})^{\oplus 2}$. Hence, by Nakayama's lemma we have $\mathcal{X}_\infty \simeq \Lambda^2/\mathfrak{a}$ for some Λ-submodule $\mathfrak{a} \subseteq \Lambda^2$. Now, dualizing the exact sequence

$$0 \to \varinjlim_n \mathrm{Sel}^1(K_n, A)^{\omega_n^\pm = 0} \to \varinjlim_n \mathrm{Sel}^+(K_n, A)^{\omega_n^+ = 0} \oplus \varinjlim_n \mathrm{Sel}^-(K_n, A)^{\omega_n^- = 0} \to \mathrm{Sel}(K_\infty, A),$$

we find a map $\mathcal{X}_\infty \to \Lambda^{\oplus 2}$ with cokernel isomorphic to a certain projective limit of $\mathfrak{o}$-modules of ranks bounded by 1 by Corollary A.12. This forces $\mathcal{X}_\infty$ to have Λ-rank at least 2 and thus $\mathfrak{a} = 0$ (otherwise $\mathcal{X}_\infty$ has a nontrivial annihilator), which gives (IV). $\qquad\square$

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References

[AL70]   A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann. **185** (1970), 134–160.

[AW67]   M. F. Atiyah and C. T. C. Wall, *Cohomology of groups*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Academic Press, London, 1967, pp. 94–115.

[BCK21]   Ashay Burungale, Francesc Castella, and Chan-Ho Kim, *A proof of Perrin-Riou's Heegner point main conjecture*, Algebra Number Theory **15** (2021), no. 7, 1627–1653.

[BDP13]   Massimo Bertolini, Henri Darmon, and Kartik Prasanna, *Generalized Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2013), no. 6, 1033–1148, With an appendix by Brian Conrad.

[Bes97]   Amnon Besser, *On the finiteness of* Ш *for motives associated to modular forms*, Doc. Math. **2** (1997), 31–46.

[BK90]   Spencer Bloch and Kazuya Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.

[BL17]   Kâzım Büyükboduk and Antonio Lei, *Integral Iwasawa theory of Galois representations for non-ordinary primes*, Math. Z. **286** (2017), no. 1-2, 361–398.

[BL21]   ———, *Iwasawa theory of elliptic modular forms over imaginary quadratic fields at non-ordinary primes*, Int. Math. Res. Not. IMRN (2021), no. 14, 10654–10730.

[Bra11]   Miljan Brakočević, *Anticyclotomic p-adic L-function of central critical Rankin-Selberg L-value*, Int. Math. Res. Not. IMRN (2011), no. 21, 4967–5018.

[Car86]   Henri Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468.

[Cas17]   Francesc Castella, *p-adic heights of Heegner points and Beilinson-Flach classes*, J. Lond. Math. Soc. (2) **96** (2017), no. 1, 156–180.

[CH18]   Francesc Castella and Ming-Lun Hsieh, *Heegner cycles and p-adic L-functions*, Math. Ann. **370** (2018), no. 1-2, 567–628.

[Del71]   Pierre Deligne, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Springer, Berlin, 1971, pp. Exp. No. 355, 139–172.

[DR25]   Cédric Dion and Jishnu Ray, *On the Mordell-Weil ranks of supersingular abelian varieties over* $\mathbb{Z}_p^2$-*extensions*, 2025, to appear in Israel J. Math., https://doi.org/10.1007/s11856-025-2794-2.

[Edi92]   Bas Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), no. 3, 563–594.

[EdVP18] Yara Elias and Carlos de Vera-Piquero, *CM cycles on Kuga-Sato varieties over Shimura curves and Selmer groups*, Forum Math. **30** (2018), no. 2, 321–346.

[FL82]   Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608.

[Gre89]   Ralph Greenberg, *Iwasawa theory for p-adic representations*, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 97–137.

[Gre99]   ———, *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., vol. 1716, Springer, Berlin, 1999, pp. 51–144.

[Gre01]   ———, *Introduction to Iwasawa theory for elliptic curves*, Arithmetic algebraic geometry (Park City, UT, 1999), IAS/Park City Math. Ser., vol. 9, Amer. Math. Soc., Providence, RI, 2001, pp. 407–464.

[Gro91]   Benedict H. Gross, *Kolyvagin's work on modular elliptic curves*, L-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 235–256.

[HB15]   Ernest Hunter Brooks, *Shimura curves and special values of p-adic L-functions*, Int. Math. Res. Not. IMRN (2015), no. 12, 4177–4241.

[HL23]   Jeffrey Hatley and Antonio Lei, *The vanishing of anticyclotomic μ-invariants for non-ordinary modular forms*, C. R. Math. Acad. Sci. Paris **361** (2023), 65–72.

[IP06]   Adrian Iovita and Robert Pollack, *Iwasawa theory of elliptic curves at supersingular primes over $\mathbb{Z}_p$-extensions of number fields*, J. Reine Angew. Math. **598** (2006), 71–103.

[JLZ21]   Dimitar Jetchev, David Loeffler, and Sarah Livia Zerbes, *Heegner points in Coleman families*, Proc. Lond. Math. Soc. (3) **122** (2021), no. 1, 124–152.

[Kat04]   Kazuya Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, no. 295, 2004, Cohomologies p-adiques et applications arithmétiques. III, pp. ix, 117–290. MR 2104361

[KO20]   Shinichi Kobayashi and Kazuto Ota, *Anticyclotomic main conjecture for modular forms and integral Perrin-Riou twists*, Development of Iwasawa theory—the centennial of K. Iwasawa's birth, Adv. Stud. Pure Math., vol. 86, Math. Soc. Japan, Tokyo, [2020] ©2020, pp. 537–594.

[Kob03]   Shin-ichi Kobayashi, *Iwasawa theory for elliptic curves at supersingular primes*, Invent. Math. **152** (2003), no. 1, 1–36.

[Kol90]   V. A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 435–483.

[Lei10]   Antonio Lei, *Coleman maps for modular forms at supersingular primes over Lubin-Tate extensions*, J. Number Theory **130** (2010), no. 10, 2293–2307.

[Lei11]   ———, *Iwasawa theory for modular forms at supersingular primes*, Compos. Math. **147** (2011), no. 3, 803–838.

[LLM23]   Antonio Lei, Meng Fai Lim, and Katharina Müller, *Asymptotic formula for Tate-Shafarevich groups of p-supersingular elliptic curves over anticyclotomic extensions*, Adv. Math. **434** (2023), Paper No. 109320, 28.

[LLZ17]   Antonio Lei, David Loeffler, and Sarah Livia Zerbes, *On the asymptotic growth of Bloch-Kato-Shafarevich-Tate groups of modular forms over cyclotomic extensions*, Canad. J. Math. **69** (2017), no. 4, 826–850.

[LMX24]   Antonio Lei, Katharina Müller, and Jiacheng Xia, *On the Iwasawa invariants of BDP Selmer groups and BDP p-adic L-functions*, Forum Math. **36** (2024), no. 1, 153–172.

[LV19a]   Matteo Longo and Stefano Vigni, *Kolyvagin systems and Iwasawa theory of generalized Heegner cycles*, Kyoto J. Math. **59** (2019), no. 3, 717–746.

[LV19b]   ———, *Plus/minus Heegner points and Iwasawa theory of elliptic curves at supersingular primes*, Boll. Unione Mat. Ital. **12** (2019), no. 3, 315–347.

[LV21]   ———, *On Bloch-Kato Selmer groups and Iwasawa theory of p-adic Galois representations*, New York J. Math. **27** (2021), 437–467.

[LV23]   ———, *The Tamagawa number conjecture and Kolyvagin's conjecture for motives of modular forms*, Preprint, arXiv:2211.04907v3, 2023.

[LZ14]   David Loeffler and Sarah Livia Zerbes, *Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions*, Int. J. Number Theory **10** (2014), no. 8, 2045–2095.

[LZ24]   Antonio Lei and Luochen Zhao, *On the BDP Iwasawa main conjecture for modular forms*, Manuscripta Math. **173** (2024), no. 3-4, 867–888.

[Mag22]   Paola Magrone, *Generalized Heegner cycles and p-adic L-functions in a quaternionic setting*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **23** (2022), no. 4, 1807–1870.

[Mas25]   Luca Mastella, *Vanishing of the $\mathfrak{p}$-part of the Shafarevich–Tate group of a modular form and its consequences for anticyclotomic Iwasawa theory*, ≥2025, to appear in Kyoto J. Math., arxiv:2307.13134v2.

[Mat21a]   Ahmed Matar, *Kolyvagin's work and anticyclotomic tower fields: the supersingular case*, Acta Arith. **201** (2021), no. 2, 131–147.

[Mat21b]   ———, *Plus/minus Selmer groups and anticyclotomic $\mathbb{Z}_p$-extensions*, Res. Number Theory **7** (2021), no. 3, Paper No. 43, 22.

[MN19]   Ahmed Matar and Jan Nekovář, *Kolyvagin's result on the vanishing of $Ш(E/K)[p^\infty]$ and its consequences for anticyclotomic Iwasawa theory*, J. Théor. Nombres Bordeaux **31** (2019), no. 2, 455–501.

[Nek92]   Jan Nekovář, *Kolyvagin's method for Chow groups of Kuga-Sato varieties*, Invent. Math. **107** (1992), no. 1, 99–125.

[Pat25]   Maria Rosaria Pati, *On the Iwasawa main conjecture for generalized Heegner classes in a quaternionic setting*, Forum Math. **37** (2025), no. 1, 225–245.

[Pol03]   Robert Pollack, *On the p-adic L-function of a modular form at a supersingular prime*, Duke Math. J. **118** (2003), no. 3, 523–558.

[Pon20]   Gautier Ponsinet, *On the structure of signed Selmer groups*, Math. Z. **294** (2020), no. 3-4, 1635–1658.

[PR87]   Bernadette Perrin-Riou, *Fonctions L p-adiques, théorie d'Iwasawa et points de Heegner*, Bull. Soc. Math. France **115** (1987), no. 4, 399–456.

[PR92]   ———, *Théorie d'Iwasawa et hauteurs p-adiques*, Invent. Math. **109** (1992), no. 1, 137–185.

[PR95]   ———, *Fonctions L p-adiques des représentations p-adiques*, Astérisque (1995), no. 229, 198.

[PR00]   _____, *Représentations p-adiques et normes universelles. I. Le cas cristallin*, J. Amer. Math. Soc. **13** (2000), no. 3, 533–551.

[Pra25]   Naman Pratap, *Iwasawa invariants of Mazur–Tate elements of elliptic curves and modular forms*, Ph.D. thesis, IISER Pune, 2025, http://dr.iiserpune.ac.in:8080/xmlui/handle/123456789/10021.

[Rib77]   Kenneth A. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Lecture Notes in Math., Vol. 601, Springer, Berlin-New York, 1977, pp. 17–51.

[Rib85]   _____, *On l-adic representations attached to modular forms. II*, Glasgow Math. J. **27** (1985), 185–194.

[Rub00]   Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study.

[Sil09]   Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009.

[Spr12]   Florian E. Ito Sprung, *Iwasawa theory for elliptic curves at supersingular primes: a pair of main conjectures*, J. Number Theory **132** (2012), no. 7, 1483–1506.

[Swe24]   Naomi Sweeting, *Kolyvagin's conjecture and patched Euler systems in anticyclotomic Iwasawa theory*, Preprint, arXiv:2012.11771, 2024.

[Was97]   Lawrence C. Washington, *Galois cohomology*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 101–120.

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