Bài tập Phương pháp lặp đơn

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

a. $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$

b. $g_2(x) = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2}$

c. $g_3(x) = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2}$

d. $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

2. **a.** Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

b. Which function do you think gives the best approximation to the solution?

3. The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = p_{n-1}\left(1 + \dfrac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$

b. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

d. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

5. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

7. Use Theorem 2.3 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

8. Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.5 to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.

9. Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 12 of Section 2.1.

10. Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 13 of Section 2.1.

11. For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a. $x = \dfrac{2 - e^x + x^2}{3}$ **b.** $x = \dfrac{5}{x^2} + 2$

c. $x = (e^x/3)^{1/2}$ **d.** $x = 5^{-x}$

e. $x = 6^{-x}$ **f.** $x = 0.5(\sin x + \cos x)$

12. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.

a. $2 + \sin x - x = 0$ use $[2, 3]$ **b.** $x^3 - 2x - 5 = 0$ use $[2, 3]$

c. $3x^2 - e^x = 0$ **d.** $x - \cos x = 0$

13. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

14. Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

15. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2\sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

16. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

17. Find a function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.3 but still has a unique fixed point on $[0, 1]$.

18. **a.** Show that Theorem 2.2 is true if the inequality $|g'(x)| \le k$ is replaced by $g'(x) \le k$, for all $x \in (a, b)$. [*Hint:* Only uniqueness is in question.]

b. Show that Theorem 2.3 may not hold if inequality $|g'(x)| \le k$ is replaced by $g'(x) \le k$. [*Hint:* Show that $g(x) = 1 - x^2$, for x in $[0, 1]$, provides a counterexample.]

19. **a.** Use Theorem 2.4 to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \ge 1,$$

converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

b. Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \ne \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

c. Use the results of parts (a) and (b) to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

20. **a.** Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \ge 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

b. What happens if $x_0 < 0$?

21. Replace the assumption in Theorem 2.4 that "a positive number $k < 1$ exists with $|g'(x)| \le k$" with "g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$." (See Exercise 27, Section 1.1.) Show that the conclusions of this theorem are still valid.

22. Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $|g'(p)| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \le \delta$, then the fixed-point iteration converges.

23. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.

24. Let $g \in C^1[a,b]$ and p be in (a,b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.