Sec:3.3

Fixed- Point Iteration

The number p is a **fixed point** for a given function g if.

$$p = g(p)$$

Example:

Determine any fixed points of the function $g(x) = x^2 - 2.$

solution

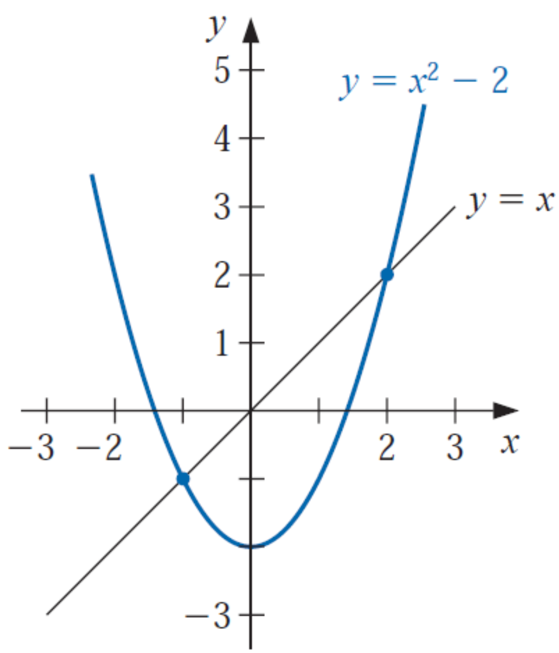
$$p = g(p)$$

$$p = p^2 - 2$$

$$0 = p^2 - p - 2$$

$$p = -1, p = 2$$

A fixed point for g occurs precisely when the graph of $y = g(x)$ intersects the graph of $y = x$



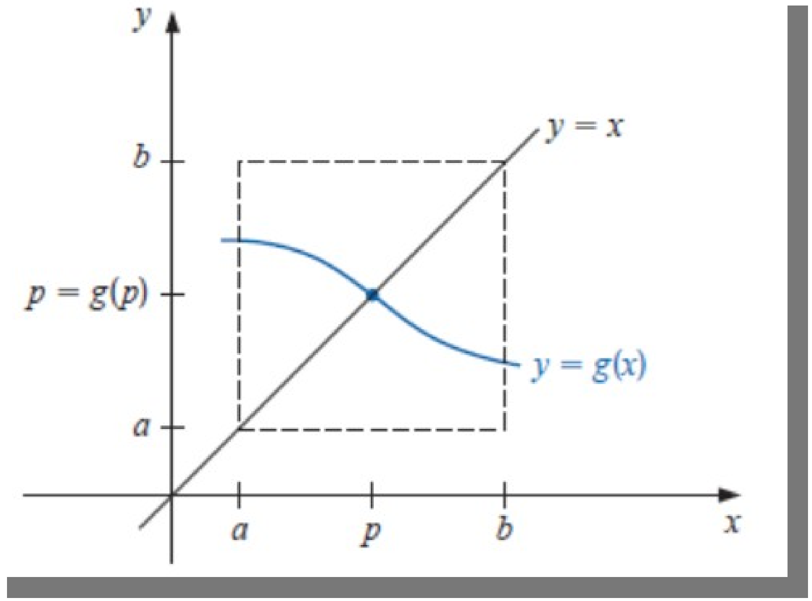
Theorem 2.3

(i) If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has **at least one** fixed point in $[a, b]$.

(ii) If, in addition, $g(x)$ exists on (a, b) and a positive constant $k <$ 1 exists with

$$|g'(x)| \leq k, \text{ for all } x \in (a, b),$$

then there is **exactly one** fixed point in $[a, b]$.



Example

Show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on the interval $[-1, 1]$.

Example

Show that $g(x) = 3^{-x}$ has at least one fixed point on the interval $[0, 1]$.

The number p is a **fixed point** for a given function g if. $p = g(p)$

Example

Find the fixed point of the function $g(x) = e^{-x}$

$$p_{i+1} = g(p_i)$$

$$p_{i+1} = e^{-p_i}$$

Starting with an initial guess of $p_0 = 0$

$$p_1 = g(p_0) = e^{-0} = 1$$

$$p_2 = g(p_1) = e^{-1} = 0.367879$$

$$p_3 = g(p_2) = e^{-0.367879} = 0.692201$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

n	p_n
1	0.000000000000
2	1.000000000000
3	0.367879441171
4	0.692200627555
5	0.500473500563
6	0.606243535085
7	0.545395785975
8	0.579612335503
9	0.560115461361
10	0.571143115080
11	0.564879347391
12	0.568428725029

Thus, each iteration brings the estimate closer to the true fixed point: 0.56714329

The number p is a **fixed point** for a given function g if.

$$p = g(p)$$

Finding **fixed point** of $g(x)$

Root Finding of $f(x)$

p Is a **fixed point** of $g(x)$

p Is a **root** of $f(x)$

$$f(x) = x - g(x)$$

simple fixed-point iteration

Step 1

rearranging the function

$$f(x) = 0$$

so that x is on the left-hand side of the equation:

$$x = g(x) \qquad (*)$$

Step 2

given an initial guess at the root p_i, (*) can be used to compute a new estimate p_{i+1} as expressed by the iterative formula

$$p_{i+1} = g(p_i)$$

Example1

$$x^2 - 2x + 3 = 0$$

can be simply manipulated to yield

$$x = \frac{x^2 + 3}{2}$$

$$x = g(x)$$

Example2

$$\sin(x) = 0$$

can be simply manipulated to yield

$$x = x + \sin(x)$$

$$x = g(x)$$

Note

Convert the problem from root-finding to finding fixed-point

Sec:3.3 Fixed- Point Iteration

Step 1

rearranging the function

$$x = e^{-x}$$
$$x = g(x)$$

Step 2

$$p_{i+1} = g(p_i)$$
$$p_{i+1} = e^{-p_i}$$

Starting with an initial guess of $p_0 = 0$

$$x_1 = g(p_0) = e^{-0} = 1$$
$$p_2 = g(p_1) = e^{-1} = 0.367879$$
$$p_3 = g(p_2) = e^{-0.367879} = 0.692201$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Example1

Use simple fixed-point iteration to locate the root of

$$f(x) = e^{-x} - x$$

n	p_n
1	0.000000000000
2	1.000000000000
3	0.367879441171
4	0.692200627555
5	0.500473500563
6	0.606243535085
7	0.545395785975
8	0.579612335503
9	0.560115461361
10	0.571143115080
11	0.564879347391
12	0.568428725029

Thus, each iteration brings the estimate closer to the true value of the root: 0.56714329

Sec:3.3 Fixed- Point Iteration

```
clear;  clc; format long
p(1) = 0;
g = @(x) exp(-x);
true_root = 0.56714329;
for k=1:11
    p(k+1) = g( p(k) );
    rel=abs( (p(k) - true_root )/ true_root );
    fprintf('%d  %10.4f  %10.4f\n', k,p(k),rel);
end
```

$$\varepsilon_t = \left|\frac{p_n - p}{p}\right| \times 100\%$$

n	p_n	$\varepsilon_t(\%)$
1	0.0000000000000	100.00
2	1.0000000000000	76.32
3	0.3678794411714	35.13
4	0.6922006275553	22.05
5	0.5004735005636	11.75
6	0.6062435350856	6.89
7	0.5453957859750	3.83
8	0.5796123355034	2.19
9	0.5601154613611	1.23
10	0.5711431150802	0.70
11	0.5648793473910	0.39
12	0.5684287250291	0.22

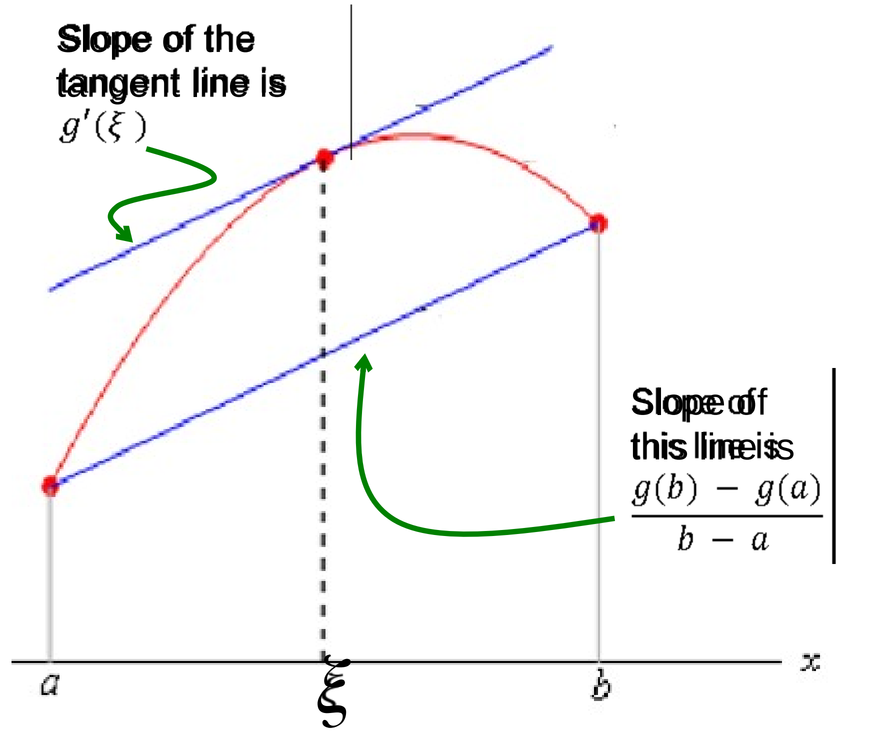
Notice that the true percent relative error for each iteration is roughly proportional (by a factor of about 0.5 to 0.6) to the error from the previous iteration. This property, called *linear convergence*

The ***derivative mean-value theorem*** **states that** if a function $g(x)$ and its first derivative are **continuous over an interval** $a \le x \le b$, then there exists at least one value of ξ within the interval such that

$$g'(\xi) = \frac{g(b) - g(a)}{b - a}$$

The right-hand side of this equation is the slope of the line joining $g(a)$ and $g(b)$.

$$g(b) - g(a) = g'(\xi)(b - a)$$



Sec:3.3 Fixed- Point Iteration

Convergence

Suppose that the true solution is

$$p = g(p)$$

the iterative equation is

$$p_{n+1} = g(p_n)$$

Subtracting these equations yields

$$p - p_{n+1} = g(p) - g(p_n)$$

The *derivative mean-value theorem gives*

$$p - p_{n+1} = g'(\xi)(p - p_n)$$

where ξ is somewhere between p and p_n

If the true error for iteration n is defined as $E_n = p - p_n$

$$E_{n+1} = g'(\xi) E_n$$

$$|E_{n+1}| = |g'(\xi)|\cdot|E_n|$$

$$|E_{n+1}| = |g'(\xi)|\cdot|E_n|$$

$$\le k|E_n| \quad (\text{If } |g'(\xi)| \le k)\ k < 1$$

the errors decrease with each iteration

$$|E_{n+1}| < k|E_n|$$

$$|E_{10}| < k|E_9| < \cdots < k^9|E_1| < k^{10}|E_0|$$

$$|E_n| < k^n|E_0|$$

Step 1

rearranging the function $f(x) = 0$
so that x is on the left-hand side
of the equation: $x = g(x)$

Select $g(x)$ *so that*
$|g'(\xi)| < 1$

Theorem 2.4 **(Fixed-Point Theorem)**
Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g(x)| \leq k, \text{ for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}),\ n \geq 1,$$

converges to the unique fixed point p in $[a, b]$.

Corollary 2.5
If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \quad \textbf{(2.5)}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1. \quad \textbf{(2.6)}$$

Two important error equation

Corollary 2.5

If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \le k^n \max\{p_0 - a,\ b - p_0\} \qquad (2.5)$$

and

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \ge 1. \qquad (2.6)$$

Remark 1

The smaller the value of k, the faster the convergence, which may be very slow if k is close to 1,

Remark 2

the rate at which $\{p_n\}$ converges

Q:2 (6+7+7) Let $f(x) = x - e^{-x}$. To find a root of $f(x) = 0$ using **Fixed-Point** method, consider two equivalent fixed point equations $x = g_1(x) = e^{-x}$ and $x = g_2(x) = -\ln(x)$.

(a) Determine for which equation, the Fixed Point Method converges in the interval $[0.2, 0.9]$.

The function g(x) is decreasing

$$g(0.2) = 0.8187$$
$$g(0.9) = 0.407$$

$$[0.407, 0.8187] \in [0.2, 0.9]$$

$g(x) \in [a, b]$, for all x in $[a, b]$

(a) $g_1'(x) = -e^{-x}$, $|g_1'(x)| = |e^{-x}| \leq e^{-0.2} = K < 1$ (2)

$g_2'(x) = \frac{-1}{x}$, $|g_2'(x)| = |\frac{1}{x}| > 1$, (2) $0.2 \leq x \leq 0.9$

(2) The fixed point method converges for $x = g_1(x)$.

Q:2 (6+7+7) Let $f(x) = x - e^{-x}$. To find a root of $f(x) = 0$ using **Fixed-Point** method, consider two equivalent fixed point equations $x = g_1(x) = e^{-x}$ and $x = g_2(x) = -\ln(x)$.

(b) Using $x_0 = 0.8$ and $g_1(x)$, estimate how many of iterations are necessary to obtain the root accurate to 10^{-2}. (accurate to two decimal places)

$$|p_n - p| \le k^n \max\{p_0 - a,\ b - p_0\} \quad (2.5)$$

$$|p_n - p| \le (e^{-0.2})^n \max\{0.8 - 0.2,\ 0.9 - 0.8\}$$

$$|p_n - p| \le (e^{-0.2})^n \{0.6\}$$

$$|p_n - p| \le (e^{-0.2})^n \{0.6\} \le 10^{-2}$$

$$e^{-0.2n} \le \frac{10^{-2}}{0.6}$$

$$-0.2n \le \ln\left(\frac{10^{-2}}{0.6}\right) \to n \ge 20.4717$$

$$\to n = 21$$

$x_1 = e^{-0.8} = 0.44933$

$$|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \ge 1. \quad (2.6)$$

$$|p_n - p| \le \frac{(e^{-0.2})^n}{1-e^{-0.2}}|0.449328964117222 - 0.8|$$

$$|p_n - p| \le \frac{(e^{-0.2})^n}{1-e^{-0.2}}\{0.350671035882778\} \le 10^{-2}$$

$$e^{-0.2n} \le \frac{10^{-2}(1-e^{-0.2})}{0.350671035882778}$$

$$-0.2n \le \ln\left(\frac{10^{-2}(1-e^{-0.2})}{0.350671035882778}\right) \to n \ge 26.3251$$

$$\to n = 27$$

Q:2 (6+7+7) Let $f(x) = x - e^{-x}$. To find a root of $f(x) = 0$ using **Fixed-Point** method, consider two equivalent fixed point equations $x = g_1(x) = e^{-x}$ and $x = g_2(x) = -\ln(x)$.

(c) Compute the root accurate to 10^{-2} and note the number of iterations actually needed.

(C) $x_1 = e^{-0.8} = 0.44933$

1 $x_2 = e^{-0.44933} = 0.63806$

1 $x_3 = 0.52832$

1 $x_4 = 0.58960$

1 $x_5 = 0.55455$

1 $x_6 = 0.57433$

1 $x_7 = 0.\underline{\underline{56}}308$

1 $x_8 = 0.\underline{\underline{56}}945$

Actual
$n = 8$

0.800000000000000	0.567129290055947
0.449328964117222	0.567151230672109
0.638056166582019	0.567138787161161
0.528318389507880	0.567145844402776
0.589595606670415	0.567141841931644
0.554551496322640	0.567144111905037
0.574329792464196	0.567142824504454
0.563082124256065	0.567143554644927
0.569451236849996	0.567143140550615
0.565835863387710	0.567143375401412
0.567885273811935	0.567143242207354
0.566722635580408	0.567143317747469
0.567381912159020	0.567143274905399
0.567007973831140	0.567143299202992
0.567220039492037	0.567143285422775
0.567099764353053	0.567143293238132
0.567167976458056	0.567143288805705
	0.567143291319526
	0.567143289893829
	0.567143290702404
	0.567143290243826
	0.567143290503906
	0.567143290356403
	0.567143290440058