Date

Question 5

1) a) Given $a_1, a_2 \in \mathbb{Z}$ two arbitrary element, assume $f(a_1) = f(a_2)$, we now show that $a_1 = a_2$.

Since $f(a_1) = f(a_2)$, we have $a_1 - 1 = a_2 - 1$

then $a_1 = a_2$,

thus f is one-to-one

b) Take $a_1 = 1$ and $a_2 = -1$, we notice that $a_1 \neq a_2$

then $f(a_1) = f(a_2) = 2$.

thus f is not one-to-one.

c) Given $a_1, a_2 \in \mathbb{Z}$ two arbitrary element, assume $f(a_1) = f(a_2)$, we now show that $a_1 = a_2$

Since $f(a_1) = f(a_2)$, we have $a_1^3 = a_2^3$,

then $a_1 = a_2$

thus f is one-to-one

d) Take $a_1 = 0$ and $a_2 = 1$, we notice that $a_1 \neq a_2$.

Since $f(a_1) = f(a_2) = 0$,

f is not one-to-one

II) a). Given $b_0 \in \mathbb{Z}$ an arbitrary element, we show that $\exists\ x \in \mathbb{Z}$ $f(x) = b_0$

~~Let $x-1 = b_0$, then $x = b_0 + 1$~~

Take $x = b_0 + 1$,

note that $\begin{cases} ① & x \in \mathbb{Z} \text{ (since an integer plus an integer equals an integer)} \\ ② & f(x) = b_0 + 1 - 1 = b_0 \end{cases}$

thus f is onto.

b) If we take $f(n) = -1$, since n^2+1 is always positive, there is no $n \in \mathbb{Z}$, $f(n) = -1$, thus f is not onto.

c) If we take $f(n) = 2$, we show that $\exists x \in \mathbb{Z}$. $f(x) = x^3 = 2$. Since there is no integer's cube equal to 2, ~~there~~ f is not onto.

d) Given $b_0 \in \mathbb{Z}$ an arbitrary element, we show that $\exists x \in \mathbb{Z}$ $f(x) = b_0$.

take $x = 2b_0$,

note that
① $x \in \mathbb{Z}$ (since an integer ~~multiply~~ multiplies another integer equal to an integer)
② $f(x) = \lceil \frac{2b_0}{2} \rceil = \lceil b_0 \rceil = b_0$

thus f is onto.

Question 6

a) Assume $\forall a_1, a_2 \in \mathbb{R}$ and $f(a_1) = f(a_2)$,

then $-3a_1 + 4 = -3a_2 + 4$,

$-3a_1 = -3a_2$,

$a_1 = a_2$,

thus f is one to one.

Assume $\forall b_0 \in \mathbb{R}$, we show that $\exists x \in \mathbb{R}$ $f(x) = b_0$.

Take $x = \frac{4}{3} - \frac{1}{3}b_0$, then

$f(x) = (\frac{4}{3} - \frac{1}{3}b_0) \times -3 + 4 = b_0$

Since $\frac{4}{3} - \frac{1}{3}b_0 \in \mathbb{R}$, thus f is onto

Therefore, f is bijective.

Date

b) Take $x_1 = -1$ and $x_2 = 1$, notice that
$f(x_1) = f(x_2) = 4$
thus f is not one-to-one,
therefore f is not bijective

c) Take $f(x) = b_0 = 1$, notice that
$f(x) = \frac{x+1}{x+2} = \frac{x+2-1}{x+2} = 1 - \frac{1}{x+2} = 1$,
then $\frac{1}{x+2} = 0$. note that there is no solution for x makes the statement true,
thus f is not onto,
therefore f is not bijective.

d) Assume $\forall x_1, x_2 \in R$ and $f(x_1) = f(x_2)$, then
$x_1^5 + 1 = x_2^5 + 1$,
$x_1^5 = x_2^5$, as the exponent 5 is an integer,
$x_1 = x_2$,
thus f is one-to-one.
Assume $\forall b_0 \in R$ ~~ass~~, we show that $\exists x_0 \in R\ f(x_0) = b_0$
Take $x = \sqrt[5]{b_0 - 1}$, notice that since $b_0 - 1$ is a real number, $\sqrt[5]{b_0 - 1}$ is also a real number.
thus f is onto,
Therefore f is bijective

Question 7

a) $f(n) = n^2 + 1$

b) $f(n) = |n| + 1$

c) $f(n) = \begin{cases} 2n, & n > 0 \\ -2n+1, & n \leq 0 \end{cases}$

d) $f(n) = 1$

Question 8

$$f \circ g = f(g(x)) = a(cx+d)+b$$
$$= acx + ad + b$$
$$g \circ f = g(f(x)) = c(ax+b) + d$$
$$= acx + bc + \cancel{a}d$$
$$f \circ g = g \circ f \Rightarrow acx + ad + b = acx + bc + \cancel{a}d$$
$$ad + b = bc + d$$
$$d(a-1) = b(c-1)$$
$$\frac{a-1}{c-1} = \frac{b}{d}$$

Question 9

a). Since $g: A \to B$, $f: B \to C$,
then $f \circ g: A \to C$, thus
$\forall c_0 \in C \ \exists a_0 \in A \ f \circ g(a_0) = c_0$,

Assume that the cardinality of A and C is n, while the cardinality of B is $n+1$.

According to the definition of onto function, all n elements of C can be mapped to corresponding m elements of ~~B~~ A $(m \leq n)$.

Since no element in the domain can be mapped to different elements twice, thus the corresponding elements of A in B can only be less than n.

Therefore, there's always one element in B cannot be mapped to any element in A, hence $g: A \to B$ is not onto.

In conclusion, if $f \circ g$ is onto, not both f and g are necessarily onto functions.

Date

b) If both f and g are onto, note that

$\forall b_0 \in B \ \exists a_0 \in A$ ~~f(a_0)~~ $g(a_0) = b_0$

$\forall c_0 \in C \ \exists b_0 \in B \quad f(b_0) = c_0$.

thus $f(g(a_0)) = f(b_0) = c_0 = f \circ g\,(a_0)$

therefore, $\forall$ ~~a_0~~ $c_0 \in C \ \exists a_0 \in A \ f \circ g\,(a_0) = c_0$

Hence $f \circ g$ is onto.