数分三 HW 3

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1 HW 3

题目. 22. 求下列复合函数的偏导数, 其中 f 是可微函数:

(1) $z = f\left(x\mathrm{e}^y, x\mathrm{e}^{-y}\right)$;

(2) $u = f\left(\sum_{i=1}^n x_i^2, \prod_{i=1}^n x_i^2, x_3, \cdots, x_n\right)$.

解答. (1)

$$\frac{\partial z}{\partial x} = f_1(xe^y, xe^{-y})e^y + f_2(xe^y, xe^{-y})e^{-y}$$
$$\frac{\partial z}{\partial y} = f_1(xe^y, xe^{-y})xe^y - f_2(xe^y, xe^{-y})xe^{-y}$$

(2)

$$\frac{\partial u}{\partial x_i} = f_1(\sum_{i=1}^n x_i^2, \prod_{i=1}^n x_i^2, x_3, \cdots, x_n)2x_i + f_2(\sum_{i=1}^n x_i^2, \prod_{i=1}^n x_i^2, x_3, \cdots, x_n)2x_i\Pi_{j\neq i}x_j^2$$
$$+ \sum_{j=3}^n \delta_{ij} f_j(\sum_{i=1}^n x_i^2, \prod_{i=1}^n x_i^2, x_3, \cdots, x_n)$$

□

题目. 25. 若 $f(\boldsymbol{x})$ 是定义在区域 $D \subset \mathbb{R}^n (n \geqslant 2)$ 内的函数并且存在正整数 K, 使得 $f(t\boldsymbol{x}) = t^K f(\boldsymbol{x})$ 对于 $\forall t > 0, \forall \boldsymbol{x} \in D$ 成立, 则称 $f(\boldsymbol{x})$ 是 K次齐次函数. 设 K 次齐次函数 $f(\boldsymbol{x})$ 在 D 内具有各个 $k(1 \leqslant k \leqslant K)$阶连续偏导数, 证明:

$$\left(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\right)^k f(\boldsymbol{x}) = K(K-1)\cdots(K-k+1)f(\boldsymbol{x})$$

题目. 28. 设函数 $x = r\cos\alpha - t\sin\alpha, y = r\sin\alpha + t\cos\alpha$, 其中 $\alpha \in \mathbb{R}$为常数. 证明：对任何可微函数 $f(x, y)$ ，成立

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2$$

解答.

$$\frac{\partial f}{\partial r} = f_x \cos\alpha + f_y \sin\alpha$$

$$\frac{\partial f}{\partial t} = -f_x \sin\alpha + f_y \cos\alpha$$
$$\Rightarrow (\frac{\partial f}{\partial r})^2 + (\frac{\partial f}{\partial t})^2 = f_x^2 + f_y^2$$

□

题目. 31. 求下列函数的二阶偏导数, 其中函数 f 具有二阶连续导数:

(1) $z = f\left(x^2 + y^2, xy\right)$;

(2) $z = f\left(x_1 + x_2 + \cdots + x_n\right)$.

解答. (1)

$$\frac{\partial z}{\partial x} = 2xf_1 + yf_2$$
$$\frac{\partial z}{\partial y} = 2yf_1 + xf_2$$
$$\frac{\partial^2 z}{\partial x^2} = 2yf_1 + xf_2$$

□

题目. 34. 设 $f(x)$ 是一个二次可微函数, 证明 $F(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$ (其中 c 为常数) 满足偏微分方程 $\frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$.

解答.

$$\frac{\partial F}{\partial x} = \frac{1}{2}[f'(x-ct) + f'(x+ct)]$$
$$\frac{\partial F}{\partial t} = \frac{c}{2}[-f'(x-ct) + f'(x+ct)]$$
$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{2}[f''(x-ct) + f''(x+ct)]$$
$$\frac{\partial^2 F}{\partial t^2} = \frac{c^2}{2}[f''(x-ct) + f''(x+ct)]$$
$$\Rightarrow \frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$$

□

题目. 37. 设 $x = 2r - s, y = r + 2s$, 求 $\frac{\partial^2 f(x,y)}{\partial r \partial s}$, 其中函数 $f(x,y)$ 具有二阶连续偏导数.

解答.

□

37. $\begin{cases} x = 2r - s \\ y = r + 2s \end{cases}$, $\cancel{\frac{\partial f}{\partial s} = -f_x + 2f_y}$, $\cancel{\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial s}\right) = -[2f_{xx} + f_{xy}] + 2[\ldots = f_{yx}}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = -f_x + 2f_y$$

$$\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial s}\right) = -\frac{\partial}{\partial r}(f_x) + 2\frac{\partial}{\partial r}(f_y) = -\left[f_{xx}\frac{\partial x}{\partial r} + f_{xy}\frac{\partial y}{\partial r}\right] + 2\left[f_{yx}\frac{\partial x}{\partial r} + f_{yy}\frac{\partial y}{\partial r}\right]$$ notice $f_{xy} = f_{yx}$

$$= -[2f_{xx} + f_{xy}] + 2[2f_{xy} + f_{yy}] = -2f_{xx} + 3f_{xy} + 2f_{yy}$$

40. (1) $\dfrac{1+x+y+2xy}{1+x^2+y^2}$

因为 $\dfrac{1}{1+t} = \dfrac{1}{1-(-t)} = \sum_{i=0}^{+\infty}(-t)^i$，故 $\dfrac{1}{1+x^2+y^2} = 1 - (\cancel{1+}x^2+y^2) + (x^2+y^2)^2 + o((x^2+y^2)^2)$ $(\sqrt{x^2+y^2} \to 0)$

有 $\dfrac{1+x+y+xy}{1+x^2+y^2} = (1+x+y+xy)[1-(x^2+y^2)+(x^4+2x^2y^2+y^4)+o((x^2+y^2)^2)]$

$= (1+x+y+xy)[1 - x^2 - y^2 - x^4 - y^4 - 2x^2y^2 + o((x^2+y^2)^2)]$

(2) $\dfrac{x_1^2+x_2^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)} = (x_1^2+\cdots+x_n^2)[1-(x_1+\cdots+x_n)+(x_1+\cdots+x_n)^2+o((x_1+\cdots+x_n)^2)]$

$(\sqrt{(x_1+\cdots+x_n)^2} \to 0)$

43. 存在 $U((0,0), \delta_0)$ $(\delta_0 > 0)$ 上连续函数，使得：

① $F(0,0) = 0$

② 要么 $F_y'(0,0)$ 不存在

要么 $F_y'(0,0)$ 存在，且 $F_y'(0,0) = 0$

但 $F(x,y) = 0$ 在 $U((0,0), \delta_0)$ 内确定唯一连续隐函数 $y = f(x)$，$f(0) = 0$，$x \in (-\delta_0, \delta_0)$ 时，$F(x, f(x)) = 0$

证明：(1) ~~G(x,y)=|y|~~，满足①，那么有 $y = f(x) = 0$

$$G(x,y) = \begin{cases} y, & y \geq 0, \forall x \\ \frac{1}{2}y, & y < 0, \forall x \end{cases}$$

(2) $G(x,y) = y^3$，反函数 $y = f(x) = 0$

~~45.~~ 44. ~~x^2+2x~~

$x^2 - 2xy + z + xe^z = 0$，点 $(1,1,0)$，$z = f(x,y)$

解：$F(x,y,z) = x^2 - 2xy + z + xe^z$

故 $F_z = 1 + xe^z$，$F_z(1,1,0) = 1 + 1 = 2 \neq 0$，存在 $z = f(x,y)$ 在局部，$F(1,1,0) \Rightarrow f(1,1) = 0$

待定系数，设 $f(x,y) = a_1(x-1) + a_2(y-1) + b_{11}(x-1)^2 + b_{12}(x-1)(y-1) + b_{22}(y-1)^2 + o((x-1)^2+(y-1)^2)$

代入：$F(x,y,f(x,y)) = x^2 - 2xy + f(x,y) + xe^{f(x,y)}$，之后展开 $e^{f(x,y)}$

$= x^2 - 2xy + f(x,y) + x[1 + f(x,y) + \frac{1}{2}f(x,y)^2 + o(f(x,y)^2)]$

$F_z = 2(x^2+y^2)z + \cos z$

$F_z(0,0,0) = 1 \neq 0$，原点附近存在 $z=f(x,y)$

需求 $f(x,y)$ 在 $(0,0)$ 的所有3阶偏导，考虑把 f 展开到3阶

$$f(x,y) = a_1 x + a_2 y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3 + o\left((x^2+y^3)^{\frac{3}{2}}\right)$$

$$F(x,y,f(x,y)) = x + x^2 + y^2 + (x^2+y^2)f^2(x,y) + f(x,y) - \frac{1}{6}f^3(x,y) + o(f^3(x,y))$$

$$= x + x^2 + y^2 + (x^2+y^2)[a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}xy^2 + c_{122}xy^2 + c_{222}y^3]^2$$
$$+ [a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3]$$
$$-\frac{1}{6}[a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3]^3 + o\left((x^2+y^2)^{\frac{3}{2}}\right).$$

$$= (a_1+1)x + a_2y + (1+b_{11})x^2 + (1+b_{22})y^2 + b_{12}xy + \left(c_{111} - \frac{a_1^3}{6}\right)x^3 + \left(c_{112} - \frac{a_1^2a_2}{6}\right)x^2y$$
$$+ \left(c_{122} - \frac{a_1a_2^2}{6}\right)xy^2 + \left(c_{222} - \frac{a_2^3}{6}\right)y^3 + o\left((\sqrt{x^2+y^2})^3\right)$$

$\equiv 0$

故：$a_1 = -1$，$a_2 = 0$

$b_{11} = -1$，$b_{12} = 0$，$b_{22} = -1$

$c_{111} = \frac{-1}{6}$，$c_{112} = 0$，$c_{122} = 0$，$c_{222} = 0$

又因为 ~~$f(x,y) = f_x|_{(0,0)} \cdot x + f_y|_{(0,0)} \cdot y +$~~

$f(x,y)$ 在 $(0,0)$ 处的 Taylor 展开为

$$f(x,y) = (f_x, f_y)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{3!}\cdots$$

故 $\frac{C_3^3}{3!} \cdot \frac{\partial^3 f}{\partial x^3}(0,0) = -\frac{1}{6}$，$\frac{\partial^3 f}{\partial x^3}(0,0) = -1$，$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = 0$

46. 由隐函数定理，(取交后) 存在邻域使 $y=f(x)$ 和 $x=g(y)$ 均存在

因为 $F_x dx + F_y dy = 0$，而 $\frac{dy}{dx} = -\frac{F_x}{F_y}$，$\frac{dx}{dy} = g'(y) = -\frac{F_y}{F_x}$

故有：$y'(x) \cdot g'(y) = 0$，$\frac{dy}{dx} \cdot \frac{dg}{dy} = 1$

由于 ① 对 $y=f(x)$，$f'(x)$ 不变号(邻域内)，故存在反函数，设为 $x=h(y)$

但已知 $x=g(y)$，故由唯一性知 $g=h$

49、 ~~$u=u(x,y)$~~

~~$f(x,y,z,t)=u(x)$~~

$u=f(x,y,z,t)$

$u=u(x,y)$，由 $g(y,z,t)=0$ 确定（没看懂，这里u用3两遍）

$h(z,t)=0$ 因为由后两项有 $y=\varphi(z,t)$

因为

$$\begin{cases} du = f_1dx + f_2dy + f_3dz + f_4dt \\ dg = g_1dy + g_2dz + g_3dt = 0 \\ dh = h_1dz + h_2dt = 0 \end{cases}$$

消去 dz 和 dt，用 dx 和 dy 表示 du

$0 = h_2g_1dy + h_2g_2dz + h_2g_3dt$

$0 = g_3h_1dz + g_3h_2dt$

$\Rightarrow dz = \dfrac{-h_2g_1}{h_2g_2-g_3h_1}dy$

$0 = h_1g_1dy + h_1g_2dz + h_1g_3dt$

$0 = g_2h_1dz + g_2h_2dt$

$dt = \dfrac{-h_1g_1}{h_1g_3-h_2g_2}dy$

因此：$f_1 = \dfrac{\partial u}{\partial x}$，$f_2 + \dfrac{-h_2g_1}{h_2g_2-g_3h_1} - \dfrac{h_1g_1}{h_1g_3-h_2g_2} = \dfrac{\partial u}{\partial y}$

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52. $$\frac{\partial(x,y,z)}{\partial(u,v,z)} = \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix} = u^2+v^2$$

55. (1) $f(x,y) = x^3+y^3-3x-12y+1$

$f_x = 3x^2-3=0$，$x=\pm1$

$f_y = 3y^2-12=0$，$y=\pm2$

$f_{xx}=6x$，$f_{xy}=0$，$f_{yx}=0$，$f_{yy}=6y$

$H_f = 6\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}$

$(1,2)$ 正定，极小

$(-1,-2)$ ~~正定~~ 负定，极~~小~~大

$(1,-2)$，$(-1,2)$ ~~负定~~ 不定，~~极大~~ 非

(2) $f(x,y) = xy\ln(x^2+y^2)$

$f_x = y\ln(x^2+y^2) + xy\cdot\dfrac{2x}{x^2+y^2} = 0 \Leftrightarrow y=0\ (x\neq0)$ 或 $(x=0$ 且 $y=\pm1)$ 或 $\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$

$f_y = x\ln(x^2+y^2) + xy\cdot\dfrac{2y}{x^2+y^2} = 0 \Leftrightarrow x=0\ (y\neq0)$ 或 $(y=0$ 且 $x=\pm1$

驻点，$(0,\pm1)$，$(\pm1,0)$，$\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$，$\pm(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}})$

计算 H_f

$f_{xx} = \dfrac{2xy(x^2+3y^2)}{(x^2+y^2)^2}$，$f_{yy} = \dfrac{2xy(y^2+3x^2)}{(x^2+y^2)^2}$

$f_{xy}=f_{yx}=\ln(x^2+y^2)+\dfrac{2x^4+8x^2y^2+2y^4}{(x^2+y^2)^2}$

在 $(0,\pm1)$ 和 $(\pm1,0)$，H_f 不定矩阵，非极值点

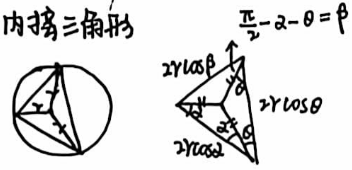
$H_f=\begin{pmatrix} 0 & 2 \\ 2 & 0\end{pmatrix}$ 不定矩阵，因为特征值为 ±2

在 $\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$，$H_f=\begin{pmatrix} 2 & 3 \\ 3 & 2\end{pmatrix}$，特征值 1，5，正定，极小

$\pm(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}})$，$H_f=\begin{pmatrix} -2 & -3 \\ -3 & -2\end{pmatrix}$，负定，极小

58.

(1) 内接三角形



$$S=\frac{1}{2}\cdot 2r\cos\theta\cdot r\sin\theta+\cdots$$

$$=\frac{r^2}{2}(\sin 2\theta+\overset{\sin 2\alpha}{\cancel{\cos 2\alpha}}+\sin(\pi-2\alpha-2\theta))$$

$$=\frac{r^2}{2}(\underset{a}{\sin 2\theta}+\underset{b}{\sin 2\alpha}+\sin(2\alpha+2\theta))$$

$\cancel{F=\sin a+\sin b}+\sin(a+b)$

$\cancel{\frac{\partial F}{\partial a}=\cos a+\cos(a+b)=\cos a+\cos a\cos b-\sin a\sin b=0}$

$$S=\frac{\sin a+\sin b+\sin(a+b)}{2}$$

$$\frac{\partial S}{\partial a}=\frac{\cos a+\cos(a+b)}{2}=0$$

$$\frac{\partial S}{\partial b}=\frac{\cos b+\cos(a+b)}{2}=0$$

$(a,b)=(\frac{\pi}{3},\frac{\pi}{3})$，$H_S(\frac{\pi}{3},\frac{\pi}{3})=\begin{pmatrix}\frac{-\sin a-\sin(a+b)}{2} & \frac{-\sin(a+b)}{2}\\ \frac{-\sin(a+b)}{2} & \frac{-\sin b-\sin(a+b)}{2}\end{pmatrix}=\begin{pmatrix}\frac{-\frac{\sqrt{3}}{2}+\frac{1}{2}}{2} & \frac{+\frac{1}{2}}{2}\\ \frac{+\frac{1}{2}}{2} & \frac{-\frac{\sqrt{3}}{2}+\frac{1}{2}}{2}\end{pmatrix}$ 负定，极大

$$S_{max}=\frac{3\sqrt{3}}{4}$$

(2) 内接长方形

$S=4\cos\theta\sin\theta=2\sin 2\theta$，$S_{max}=2$

61. 点 $\vec{x_0}$ 到平面 $\sum_{i=1}^{n}(a_i x_i)=0$ 距离

$$\min_{\sum_{i=1}^{n}a_i x_i=0}\sum_{i=1}^{n}(x_i-x_i^0)^2$$

$\Leftrightarrow\begin{cases}\frac{\partial L}{\partial x_i}=2(x_i-x_i^0)+\lambda a_i=0\\ \sum_{i=1}^{n}a_i x_i=0\end{cases}$ $\Leftrightarrow\begin{cases}x_i=x_i^0-\frac{\lambda}{2}a_i\\ \sum_{i=1}^{n}a_i x_i^0=\frac{\lambda}{2}\sum_{i=1}^{n}a_i^2 \Leftrightarrow \frac{2\sum_{i=1}^{n}a_i x_i^0}{\sum_{i=1}^{n}a_i^2}=\lambda\end{cases}$

故 $\sum_{i=1}^{n}(x_i^*-x_i^0)^2=\sum_{i=1}^{n}a_i^2\cdot\frac{\lambda^2}{4}=\frac{1}{4}\cdot\sum_{i=1}^{n}a_i^2\left(\frac{2\sum_{i=1}^{n}a_i x_i^0}{\sum_{i=1}^{n}a_i^2}\right)^2=\frac{\left(\sum_{i=1}^{n}a_i x_i^0\right)^2}{\sum_{i=1}^{n}a_i^2}$

$$L_{min}=\frac{\left|\sum_{i=1}^{n}a_i x_i^0\right|}{\sqrt{\sum_{i=1}^{n}a_i^2}}$$

64.

$$\begin{cases} xyz=1 \\ y=2x \end{cases}$$ 即 $\left(x, 2x, \frac{1}{2x^2}\right)$, $d^2=5x^2+\frac{1}{4x^4}$, $\frac{\partial d^2}{\partial x}=10x+\frac{-4}{4x^5}=0$, $x^*=\left(\frac{1}{10}\right)^{\frac{1}{6}}$

$$d^*=\sqrt{5\cdot\left(\frac{1}{10}\right)^{\frac{1}{3}}+\frac{1}{4}\times(10)^{\frac{1}{4}}}$$

67. $g(a,b,c)=\int_0^1 [f(x)-ax^2-bx-c]^2 dx$

$$\frac{\partial g}{\partial a}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-x^2)dx=0 \Leftrightarrow \int_0^1 (ax^4+bx^3+cx^2)dx=\frac{1}{5}a+\frac{1}{4}b+\frac{1}{3}c=\int_0^1 x^2f(x)dx=A$$

$$\frac{\partial g}{\partial b}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-x)dx=0 \Leftrightarrow \frac{1}{4}a+\frac{1}{3}b+\frac{1}{2}c=\int_0^1 xf(x)dx=B$$

$$\frac{\partial g}{\partial c}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-1)dx=0 \Leftrightarrow \frac{1}{3}a+\frac{1}{2}b+c=\int_0^1 f(x)dx=C$$

故 $$\begin{pmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} A \\ B \\ C \end{pmatrix} \Rightarrow \begin{pmatrix} a^* \\ b^* \\ c^* \end{pmatrix}=D^{-1}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

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