Date No.

Week10 ①

习题 2.1 3, 4, 8(1), 第一值理之证明

3、ξ, η 相互独立, $\xi \sim Exp(\lambda_1)$, $\eta \sim Exp(\lambda_2)$

证明:(1) $\min\{\xi,\eta\} \sim Exp(\lambda_1+\lambda_2)$

$$P(\min\{\xi,\eta\}<k\} = \iint_{\min\{u,v\}<k} \lambda_1 e^{-\lambda_1 u} \lambda_2 e^{-\lambda_2 v} du dv$$

$\left[= \iint_{\substack{u<v \\ u<k}} \right]$ 直接算尾分布更方便

$\rightarrow$ 不好分析

$$P(\min\{\xi,\eta\} \geq k) = \iint_{k\leq u\leq v} \lambda_1\lambda_2 e^{-\lambda_1 u-\lambda_2 v} du dv + \iint_{k\leq v\leq u} \lambda_1\lambda_2 e^{-\lambda_1 u-\lambda_2 v} du dv$$

$$= \int_k^{+\infty} \lambda_2 e^{-\lambda_2 v} \left(\int_k^v \lambda_1 e^{-\lambda_1 u} du\right) dv + \int_k^{+\infty} \lambda_1 e^{-\lambda_1 u} \int_k^u \lambda_2 e^{-\lambda_2 v} dv\, du$$

$$= \int_k^{+\infty} \lambda_2 e^{-\lambda_2 v} [e^{-\lambda_1 k} - e^{-\lambda_1 v}] dv + \int_k^{+\infty} \lambda_1 e^{-\lambda_1 u} [e^{-\lambda_2 k} - e^{-\lambda_2 u}] du$$

$$= e^{-\lambda_1 k} \int_k^{+\infty} \lambda_2 e^{-\lambda_2 v} dv - \lambda_2 \int_k^{+\infty} e^{-(\lambda_1+\lambda_2) v} dv + \int_k^{+\infty} e^{-\lambda_2 k} \cdot \lambda_1 e^{-\lambda_1 u} du - \lambda_1 \int_k^{+\infty} e^{-(\lambda_1+\lambda_2) u} du$$

$$= e^{-\lambda_1 k} [e^{-\lambda_2 k}] - \lambda_2 \cdot \frac{1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)k} + e^{-\lambda_2 k} e^{-\lambda_1 k} - \lambda_1 \cdot \frac{1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)k}$$

$= e^{-(\lambda_1+\lambda_2)k}$, 因此 $\min\{\xi,\eta\} \sim Exp(\lambda_1+\lambda_2)$

(2) $$P(\xi<\eta) = \iint_{0\leq u<v} \lambda_1\lambda_2 e^{-\lambda_1 u} e^{-\lambda_2 v} du dv = \int_0^{+\infty} \lambda_2 e^{-\lambda_2 v} \left(\int_0^v \lambda_1 e^{-\lambda_1 u} du\right) dv$$

$$= \int_0^{+\infty} \lambda_2 e^{-\lambda_2 v} (1-e^{-\lambda_1 v}) dv = \int_0^{+\infty} \lambda_2 e^{-\lambda_2 v} dv - \int_0^{+\infty} \lambda_2 e^{-(\lambda_1+\lambda_2) v} dv$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

4. ~~$\xi = \zeta_1+\dots+\zeta_\nu$~~ $\nu, \zeta_1, \dots, \zeta_n$ 相互独立, $P(\nu=k) = (1-p)^{k-1} p$, $\zeta_n \sim Exp(\lambda)$

~~$P(\xi\geq k) = p$~~ $P(\xi<t) = \sum_{k=1}^{\infty} p(1-p)^{k-1} P(\zeta_1+\dots+\zeta_k<t)$ 两边同时对 t 求导

$f_\xi(t) = \sum_{k=1}^{\infty} p(1-p)^{k-1} f_{\zeta_1+\dots+\zeta_k}(t)$ 由于 $\zeta_1 \sim Exp(\lambda) = \Gamma(1,\lambda)$, 独立和 $\zeta_1+\dots+\zeta_k \sim \Gamma(k,\lambda)$

$$= \sum_{k=1}^{\infty} p(1-p)^{k-1} \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} = \lambda p e^{-\lambda t} \sum_{k=1}^{\infty} \frac{[(1-p)\lambda t]^{k-1}}{(k-1)!} = \lambda p e^{-\lambda t} e^{\lambda t(1-p)} = \lambda p e^{-\lambda p t}$$

故 $\xi \sim Exp(\lambda p)$

Poission 分布细分: ν 为几何分布, 记录了第一次分给"第一类"的时刻(第一次成功的时刻)

8(1) 推论 2.1.8 (马氏性) $\forall r\geq 1$, $0\leq t_1<\cdots<t_r<t<t+s$, 单调上升非负整数 $0\leq k_1\leq\cdots\leq k_r\leq k\leq l$

有 $P(X_{t+s}=l \mid X_t=k, X_{t_1}=k_1, \cdots, X_{t_r}=k_r)$

$=P(X_{t+s}=l \mid X_t=k) = P(X_s=l-k)$

证明: $P(X_{t+s}=l \mid X_t=k, X_{t_1}=k_1, \cdots, X_{t_r}=k_r)$

$$= \frac{P(X_{t+s}=l, X_t=k, X_{t_1}=k_1, \cdots, X_{t_r}=k_r)}{P(X_t=k, X_{t_1}=k_1, \cdots, X_{t_r}=k_r)}$$

利用独立增量性质

~~分子~~分母: $P(X_t=k, X_{t_r}=k_r, \cdots, X_{t_1}=k_1) = \cancel{P(X_t=k, X_{t-t_r}=k-k_r, X_{t_r-t_{r-1}}=k_r-k_{r-1}, \cdots, X_{t_2-t_1}=k_2-k_1)}$

$= P(X_t=k)\,P(X_{t_r}=k_r, \cdots, X_{t_1}=k_1 \mid X_t=k) = P(X_t=k)\,P(X_t-X_{t_r}=k-k_r, X_{t_r}-X_{t_{r-1}}=k_r-k_{r-1}, \cdots, X_{t_2}-X_{t_1}=k_2-k_1)$

$= P(X_t=k)\,P(X_{t-t_r}=k-k_r)\,P(X_{t_r-t_{r-1}}=k_r-k_{r-1})\cdots P(X_{t_2-t_1}=k_2-k_1)$

同理有: 分子 $= P(X_{t+s}=l, X_t=k, \cdots, X_{t_r}=k_r)$

$= P(X_{t+s}=l, X_t=k)\,P(X_{t-t_r}=k-k_r)\cdots P(X_{t_2-t_1}=k_2-k_1)$

因此 分数 $= P(X_{t+s}=l \mid X_t=k) = \dfrac{P(X_{t+s}=l, X_t=k)}{P(X_t=k)} = P(X_{t+s}-X_t=l-k) = P(X_s=l-k)$

课上第一个定理的证明:

定理: 计数过程 $X=\{X_t \mid t\geq 0\}\sim PP(\lambda) \Leftrightarrow X$ 满足, $X_0=0$, 且为独立平稳增量。具有普遍性, 稀有性。

证明: ($\Leftarrow$) 即证明 $X_t\sim P(\lambda t)$ 泊松分布。记 $P_k(t)=P(X_t=k)$。

$P_0(t+h)=P(X_{t+h}=0)=P(X_t=0, X_{t+h}-X_t=0) \overset{\text{独立增量}}{=} P(X_t=0)\,P(X_{t+h}-X_t=0)$

$= P_0(t)\,[1-\lambda h+o(h)]$

$P_k(t+h)=P(X_{t+h}=k)=P(X_t=k, X_{t+h}-X_t=0)+P(X_t=k-1, X_{t+h}-X_t=1)+\cdots+P(X_t=0, X_{t+h}-X_t=k)$

$= P_k(t)\,(1-\lambda h+o(h)) + P_{k-1}(t)\,(\lambda h+o(h)) + o(h)\,[P_{k-2}(t)+\cdots+P_0(t)]$

故令 $h\to 0$ 有: $P_k'(t) = -\lambda P_k(t) + \lambda P_{k-1}(t)$, 边值条件 $P_k(0)=0$ $(\forall k)$

$\Rightarrow P_k(t)=P(X_t=k)=\dfrac{(\lambda t)^k}{k!}e^{-\lambda t}$。

~~($\Leftarrow$)~~ ($\Rightarrow$) X 为 $PP(\lambda)$, 当然满足右边条件.

Week10 ② 作业 2.1节 5,7,8(2)；定理 2.1.12(2) 叙述

5. (1) 合并后的泊松流 $\{Z_t\} \sim PP(\lambda_1+\lambda_2)$

$Z_1 - Z_0 \overset{d}{=} Z_1 \sim P(\lambda_1+\lambda_2)$

故 $P(Z_1=3) = \frac{(\lambda_1+\lambda_2)^3}{3!} e^{-(\lambda_1+\lambda_2)}$

(2) 等到第一辆甲路车的时刻 $S_1^{(X)}$，在 $[0, S_1^{(X)})$ 时间内，乙车数目 $Y_{S_1^{(X)}} \sim P(S_1^{(X)}\lambda_2)$

而 $S_1^{(X)} \overset{d}{=} \xi_1 \sim Exp(\lambda_1)$

故 $P(Y_{S_1^{(X)}}=3) = \int_0^\infty P(Y_{S_1^{(X)}}=3 \mid S_1^{(X)}=t) P(S_1^{(X)}=t)\, dt$

$= \int_0^\infty \frac{(\lambda_2 t)^3 e^{-\lambda_2 t}}{3!} \cdot \lambda_1 e^{-\lambda_1 t} dt = \frac{\lambda_1\lambda_2^3}{6} \int_0^\infty t^3 e^{-(\lambda_1+\lambda_2)t} dt$

$= \frac{\lambda_1\lambda_2^3}{6} \cdot \frac{6}{(\lambda_1+\lambda_2)^4} = \frac{\lambda_1\lambda_2^3}{(\lambda_1+\lambda_2)^3}$

7. (1) $P = \frac{800-300}{2000-300} = \frac{5}{17}$，故读到(关注)为 $\{X_t\} \sim PP(\frac{5}{17}\lambda)$

$X_{30} \sim P(\frac{150}{17}\lambda)$，$P(X_{30}=k) = \frac{1}{k!} \cdot (\frac{150\lambda}{17})^k \cdot e^{-\frac{150}{17}\lambda}$

(2) $E(e^{aY}) = E(e^{a(\xi_1+\cdots+\xi_X)}) = E(E(e^{a(\xi_1+\cdots+\xi_X)} \mid X))$

$= E(E(e^{a\xi})^X) = E\left(\left(\frac{e^{2a}-e^a}{a}\right)^X\right)$ 母函数 $E(s^X) = e^{\lambda(s-1)}$

$= e^{\frac{150}{17}\lambda\left(\frac{e^{2a}-e^a}{a}-1\right)}$

8(2) 命题 2.1.10 在 $\{X_T=k\}$ 条件下，$(S_1,\cdots,S_k)$ 的条件密度为 $\frac{k!}{T^k} 1_{\{0<S_1<\cdots<S_k<T\}}$

证明：$P(S_1=s_1,\cdots,S_k=s_k \mid X_T=k)$

$= P(\xi_1=s_1, \xi_2=s_2-s_1, \cdots, \xi_{k-1}=s_{k-1}-s_{k-2}, \xi_k=s_k-s_{k-1}, \cancel{\xi_{k+1}>T-s_k}) / P(X_T=k)$

[由独立性] $= \int \left[\int\left(\int_{\cancel{T-s_k}\ T-s_k}^{\infty} \lambda e^{-\lambda t_1} \cdots \lambda e^{-\lambda t_{k+1}} dt_{k+1}\right) dt_k\right]$ ⊗

分子 $= P(\xi_1=s_1)P(\xi_2=s_2-s_1)\cdots P(\xi_k=s_k-s_{k-1}) \cancel{P(\xi_{k+1}>T-s_k)}$

$= \lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2-s_1)} \cdots \lambda e^{-\lambda(s_k-s_{k-1})} \cancel{\int_0^{T-s_k} \lambda e^{-\lambda t} dt} \cdot 1_{\{0<s_1<\cdots<s_k<T\}}$

$= \lambda^{k+1} e^{-\lambda s_k} \cancel{(1-e^{-\lambda(T-s_k)})} \cdot 1_{\{\cdots\}}$ ⊗

分母 $P(X_T=k) = \frac{(T\lambda)^k e^{-T\lambda}}{k!}$

分子：$P(S_1=s_1,\cdots,S_k=s_k, X_T=k) = P(\xi_1=s_1, \xi_2=s_2-s_1, \cdots, \xi_{k-1}=s_{k-1}-s_{k-2}, \xi_k=s_k-s_{k-1}, \xi_{k+1}>T-s_k)$

$= \lambda^k e^{-\lambda s_k} \cdot \int_{T-s_k}^{+\infty} \lambda e^{-\lambda t} dt \cdot 1_{\{\cdots\}} = \lambda^k e^{-\lambda s_k} e^{-\lambda(T-s_k)} \cdot 1_{\{\cdots\}} = \lambda^k e^{-\lambda T} \cdot 1_{\{\cdots\}}$

分数：$\lambda^k e^{-\lambda T} \cdot 1_{\{\cdots\}} \cdot \frac{k!}{T^k \lambda^k} e^{\lambda T} = \frac{k!}{T^k} \cdot 1_{\{\cdots\}}$ □

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定理 2.1.2(2)叙述

$\{X_t\} \sim PP(\lambda_1)$，$\{Y_t\} \sim PP(\lambda_2)$，则 $\{X_t+Y_t\} \sim PP(\lambda_1+\lambda_2)$

$X_{s_1}, X_{s_2}-X_{s_1}, \dots, X_{s_n}-X_{s_{n-1}}$ 分独立，$P(\lambda_1)(s_k-s_{k-1})$

$Y_{s_1}, \dots, Y_{s_n}-Y_{s_{n-1}}$ $P(\lambda_2)(s_k-s_{k-1})$

$\Rightarrow X_{s_1}+Y_{s_1}, (X_{s_2}+Y_{s_2})-(X_{s_1}+Y_{s_1}), \dots$，$\dots$，也分独立，$\sim P((\lambda_1+\lambda_2)(s_k-s_{k-1}))$

$\Rightarrow \{Z_t\} \sim PP(\lambda_1+\lambda_2)$

Week11 作业

1. Komolgrov 后退方程

$S=\{1,2\}$，$Q=\begin{pmatrix}-\lambda & \lambda \\ \mu & -\mu\end{pmatrix}$

$$\begin{bmatrix} p_{11}'(t), & p_{12}'(t) \\ p_{21}'(t) & p_{22}'(t)\end{bmatrix} = \begin{bmatrix}-\lambda & \lambda \\ \mu & -\mu\end{bmatrix}\begin{bmatrix}p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t)\end{bmatrix}$$

有 $\begin{pmatrix}p_{11}'(t) \\ p_{21}'(t)\end{pmatrix} = \begin{pmatrix}-\lambda & \lambda \\ \mu & -\mu\end{pmatrix}\begin{pmatrix}p_{11}(t) \\ p_{21}(t)\end{pmatrix}$

$\uparrow$

$\lambda_1=0, V_1=\begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_2=-(\lambda+\mu), V_2=\begin{pmatrix}\lambda\\-\mu\end{pmatrix}$

$\Rightarrow c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{-(\lambda+\mu)t}\begin{pmatrix}\lambda\\-\mu\end{pmatrix}$ 且 $p_{11}(0)=1$，$p_{21}(0)=0$

有 $p_{11}(t)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$

$p_{21}(t)=\frac{\mu}{\lambda+\mu}-\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$

同理，有：

$p_{12}(t)=\frac{\lambda}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$

$p_{22}(t)=\frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$

2. 设 $X=\{X_t\}$ 有限状态 CTHMC，其转移阵标准，则 Q 保守且 $P'(t)=P(t)Q$，$P'(t)=QP(t)$

证明：$q_{jj}=p_{jj}'(0^+)=\lim\limits_{t\to0^+}\frac{p_{ii}(t)-1}{t}=\lim\limits_{t\to0^+}\frac{-\sum_{j\neq i}p_{ij}(t)}{t}\cdot(-\sum_{j\neq i}q_{ij})$

故 $q_j-q_{jj}=\sum_{j\neq i}q_{ij}<+\infty$，Q保守

$p_{ij}'(t^+)=\lim\limits_{\Delta t\to0^+}\frac{p_{ij}(t+\Delta t)-p_{ij}(t)}{\Delta t}=\lim\limits_{\Delta t\to0^+}\frac{\sum_{k\in S}p_{ik}(t)(p_{kj}(\Delta t)-\delta_{ij})}{\Delta t}$

$=(P(t)Q)_{ij}$

$p_{ij}'(t^-)=\lim\limits_{\Delta t\to0^+}\frac{p_{ij}(t-\Delta t)-p_{ij}(t)}{-\Delta t}=\lim\limits_{\Delta t\to0^+}\frac{\sum_{k\in S}p_{ij}(t-\Delta t)(\delta_{ij}-p_{kj}(\Delta t))}{-\Delta t}$

由半群性质，$P(t-\Delta t)P(\Delta t)=P(t)$

又 $\because P(t-\Delta t)P(\Delta t)=\lim\limits_{\Delta t\to0^-}P(t-\Delta t)\cdot I \Rightarrow \lim\limits_{\Delta t\to0^+}P(t-\Delta t)=P(t)$

$\therefore p_{ij}'(t^-)=\sum_{k\in S}p_{ik}(t)q_{kj}=(P(t)Q)_{ij} \Rightarrow P'(t)=P(t)Q$

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而 $P'_{ij}(t^+) = \lim_{\Delta t \to 0^+} \frac{P_{ij}(t+\Delta t) - P_{ij}(t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{\sum_{k \in S} (P_{ik}(\Delta t) - \delta_{ij}) P_{kj}(t)}{\Delta t} = \sum_{k \in S} q_{ik} P_{kj}(s)$

$P'_{ij}(t^-) = \sum_{k \in S} q_{ik} P_{kj}(t)$

故 $P'(t) = Q P(t)$

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