37. $x = 2r - s$, $y = r + 2s$, $\cancel{\frac{\partial f}{\partial s} = -f_x + 2f_y}$, $\cancel{\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial s}\right) = -[2f_{xx} + f_{xy}] + 2[-f_{yx}]}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = -f_x + 2f_y$$

$$\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial s}\right) = -\frac{\partial}{\partial r}(f_x) + 2\frac{\partial}{\partial r}(f_y) = -\left[f_{xx}\frac{\partial x}{\partial r} + f_{xy}\frac{\partial y}{\partial r}\right] + 2\left[f_{yx}\frac{\partial x}{\partial r} + f_{yy}\frac{\partial y}{\partial r}\right]$$ notice $f_{xy} = f_{yx}$

$$= -[2f_{xx} + f_{xy}] + 2[2f_{xy} + f_{yy}] = -2f_{xx} + 3f_{xy} + 2f_{yy}$$

40. (1) $\frac{1+x+y+2xy}{1+x^2+y^2}$

因为 $\frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{i=0}^{+\infty}(-t)^i$，故 $\frac{1}{1+x^2+y^2} = 1 - (\cancel{\quad} x^2+y^2) + (x^2+y^2)^2 + o((x^2+y^2)^2)$ $(\sqrt{x^2+y^2} \to 0)$

有 $\frac{1+x+y+xy}{1+x^2+y^2} = (1+x+y+xy)[1-(x^2+y^2)+(x^4+2x^2y^2+y^4)+o((x^2+y^2)^2)]$

$= (1+x+y+xy)[1-x^2-y^2-x^4-y^4-2x^2y^2+o((x^2+y^2)^2)]$

(2) $\frac{x_1^2+x_2^2+\cdots+x_n^2}{1-(x_1+\cdots+x_n)} = (x_1^2+\cdots+x_n^2)[1-(x_1+\cdots+x_n)+(x_1+\cdots+x_n)^2+o((x_1+\cdots+x_n)^2)]$

$(\sqrt{(x_1+\cdots+x_n)^2} \to 0)$

43. 存在 $U((0,0),\delta_0)$ $(\delta_0>0)$ 上连续函数，使得：

① $F(0,0)=0$

② 要么 $F_y'(0,0)$ 不存在

要么 $F_y'(0,0)$ 存在，且 $F_y'(0,0)=0$

但 $F(x,y)=0$ 在 $U((0,0),\delta_0)$ 内确定唯一连续隐函数 $y=f(x)$，$f(0)=0$，$x\in(-\delta_0,\delta_0]$ 时，$F(x,f(x))=0$

证明：(1) $\cancel{G(x,y)=|y|}$，满足①，那么有 $y=f(x)=0$

$$G(x,y)=\begin{cases} y, & y\geq 0, \forall x \\ \frac{1}{2}y, & y<0, \forall x \end{cases}$$

(2) $G(x,y)=y^3$，反函数 $y=f(x)=0$

~~45.~~ 44. $\cancel{x^2+2x}$

$x^2-2xy+z+xe^z=0$，点 $(1,1,0)$，$z=f(x,y)$

解：$F(x,y,z)=x^2-2xy+z+xe^z$

故 $F_z = 1+xe^z$，$F_z(1,1,0)=1+1=2\neq 0$，存在 $z=f(x,y)$ 在局部，$F(1,1,0) \Rightarrow f(1,1)=0$

待定系数，设 $f(x,y)=a_1(x-1)+a_2(y-1)+b_{11}(x-1)^2+b_{12}(x-1)(y-1)+b_{22}(y-1)^2+o((x-1)^2+(y-1)^2)$

代入：$F(x,y,f(x,y))=x^2-2xy+f(x,y)+xe^{f(x,y)}$，之后展开 $e^{f(x,y)}$

$= x^2-2xy+f(x,y)+x[1+f(x,y)+\frac{1}{2}f(x,y)^2+o(f(x,y)^2)]$

$F_z = 2(x^2+y^2)z + \cos z$

$F_z(0,0,0) = 1 \neq 0$，原点附近存在 $z=f(x,y)$

要求 $f(x,y)$ 在 $(0,0)$ 的所有3阶偏导，考虑把 f 展开到3阶

$$f(x,y) = a_1 x + a_2 y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3 + o\left((x^2+y^3)^{\frac{3}{2}}\right)$$

$$F(x,y,f(x,y)) = x + x^2 + y^2 + (x^2+y^2)f^2(x,y) + f(x,y) - \frac{1}{6}f^3(x,y) + o(f^3(x,y))$$

$$= x + x^2 + y^2 + (x^2+y^2)[a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}xy^2 + c_{122}xy^2 + c_{222}y^3]^2$$
$$+ [a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3]$$
$$- \frac{1}{6}[a_1x + a_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 + c_{111}x^3 + c_{112}x^2y + c_{122}xy^2 + c_{222}y^3]^3 + o\left((x^2+y^2)^{\frac{3}{2}}\right).$$

$$= (a_1+1)x + a_2y + (1+b_{11})x^2 + (1+b_{22})y^2 + b_{12}xy + \left(c_{111} - \frac{a_1^3}{6}\right)x^3 + \left(c_{112} - \frac{a_1^2a_2}{6}\right)x^2y$$
$$+ \left(c_{122} - \frac{a_1a_2^2}{6}\right)xy^2 + \left(c_{222} - \frac{a_2^3}{6}\right)y^3 + o\left((\sqrt{x^2+y^2})^3\right)$$

$$\equiv 0$$

故：$a_1 = -1$，$a_2 = 0$

$b_{11} = -1$，$b_{12} = 0$，$b_{22} = -1$

$c_{111} = \frac{-1}{6}$，$c_{112} = 0$，$c_{122} = 0$，$c_{222} = 0$

又因为 $\cancel{f(x,y) = f_x\big|_{(0,0)} \cdot x + f_y\big|_{(0,0)} \cdot y +}$

$f(x,y)$ 在 $(0,0)$ 处的 Taylor 展开为

$$f(x,y) = (f_x, f_y)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{3!} \cdots$$

故 $\frac{C_3^3}{3!} \cdot \frac{\partial^3 f}{\partial x^3}(0,0,0) = -\frac{1}{6}$，$\frac{\partial^3 f}{\partial x^3}(0,0) = -1$，$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = 0$

46. 由隐函数定理，(取交后) 存在邻域使 $y=f(x)$ 和 $x=g(y)$ 均存在

因为 $F_x dx + F_y dy = 0$，而 $\frac{dy}{dx} = -\cancel{\frac{F_y}{F_x}}\frac{F_x}{F_y}$，$\frac{dx}{dy} = g'(y) = -\frac{F_y}{F_x}$

故有：$y'(x) \cdot g'(y) = 0$，$\frac{dy}{dx} \cdot \frac{dg}{dy} = 1$

由于 ① 对 $y=f(x)$，$f'(x)$ 不变号 (邻域内)，故存在反函数，设为 $x=h(y)$

但已知 $x=g(y)$，故由唯一性知 $g=h$

49、~~$u=u(x,y)$~~

~~$f(x,y,z,t)=u(x)$~~

$u=f(x,y,z,t)$

$u=u(x,y)$，由 $g(y,z,t)=0$ 确定（没看懂，这里u用了两遍）
$h(z,t)=0$
↓因为由后两项有 $y=\varphi(z,t)$

因为，~~$\frac{\partial u}{\partial x}=f_1$~~

$$\begin{cases} du = f_1dx + f_2dy + f_3dz + f_4dt \\ dg = g_1dy + g_2dz + g_3dt = 0 \\ dh = h_1dz + h_2dt = 0 \end{cases}$$

消去dz和dt，用dx和dy表示du

$0 \cancel{h_2dg} = h_2g_1dy + h_2g_2dz + \underline{h_2g_3dt}$ $\Rightarrow dz = \frac{-h_2g_1}{h_2g_2-g_3h_1}dy$

$0 \cancel{g_3dh} = g_3h_1dz + \underline{g_3h_2dt}$

$0 = h_1g_1dy + \underline{h_1g_2dz} + h_1g_3dt$ $\quad dt = \frac{-h_1g_1}{h_1g_3-h_2g_2}dy$

$0 = \underline{g_2h_1dz} + g_2h_2dt$

因此：$f_1 = \frac{\partial u}{\partial x}$，$f_2 + \frac{-h_2g_1}{h_2g_2-g_3h_1} - \frac{h_1g_1}{h_1g_3-h_2g_2} = \frac{\partial u}{\partial y}$

---

52. $\frac{\partial(x,y,z)}{\partial(u,v,z)} = \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix} = u^2+v^2$

55. (1) $f(x,y)=x^3+y^3-3x-12y+1$

$f_x=3x^2-3=0$，$x=\pm1$

$f_y=3y^2-12=0$，$y=\pm2$

$f_{xx}=6x$，$f_{xy}=0$，$f_{yx}=0$，$f_{yy}=6y$

$H_f=6\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

(1,2) 正定，极小

(-1,-2) ~~正定~~负定，极~~小~~大

(1,-2),(-1,2) 不定，~~极大~~非极

(2) $f(x,y)=xy\ln(x^2+y^2)$

$f_x = y\ln(x^2+y^2) + xy\cdot\frac{2x}{x^2+y^2} = 0 \Leftrightarrow y=0\ (x\neq0)$ 或 $(x=0$ 且 $y=\pm1)$ 或 $\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$

$f_y = x\ln(x^2+y^2) + xy\cdot\frac{2y}{x^2+y^2} = 0 \Leftrightarrow x=0\ (y\neq0)$ 或 $(y=0$ 且 $x=\pm1$

驻点，$(0,\pm1)$，$(\pm1,0)$，$\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$，$\pm(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}})$

计算Hf

$f_{xx} = \frac{2xy(x^2+3y^2)}{(x^2+y^2)^2}$，$f_{yy} = \frac{2xy(y^2+3x^2)}{(x^2+y^2)^2}$

$f_{xy}=f_{yx}=\ln(x^2+y^2)+\frac{2x^4+8x^2y^2+2y^4}{(x^2+y^2)^4}$

在 $(0,\pm1)$ 和 $(\pm1,0)$，Hf不定矩阵，非极值点

$H_f=\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ 不定矩阵，因为特征值为 ±2

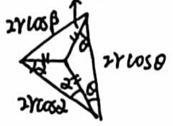
在 $\pm(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}})$，$H_f=\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$，特征值1,5，正定，极小

$\pm(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}})$，$H_f=\begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$，负定，极小

58.

(1) 内接三角形

$\frac{\pi}{2}-\alpha-\theta=\beta$



$S=\frac{1}{2}\cdot 2r\cos\theta\cdot r\sin\theta+\cdots$

$=\frac{r^2}{2}(\sin 2\theta+\overset{\sin 2\alpha}{\cancel{\cos 2\alpha}}+\sin(\pi-2\alpha-2\theta))$

$=\frac{r^2}{2}(\sin\underset{a}{\underline{\underline{2\theta}}}+\sin\underset{b}{\underline{\underline{2\alpha}}}+\sin(2\alpha+2\theta))$

~~$F=\sin a+\sin b$~~ $+\sin(a+b)$

~~$\frac{\partial F}{\partial a}=\cos a+\cos(a+b)=\cos a+\cos a\cos b-\sin a\sin b=0$~~

$$S=\frac{\sin a+\sin b+\sin(a+b)}{2}$$

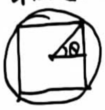
$$\frac{\partial S}{\partial a}=\frac{\cos a+\cos(a+b)}{2}=0$$

$$\frac{\partial S}{\partial b}=\frac{\cos b+\cos(a+b)}{2}=0$$

$(a,b)=(\frac{\pi}{3},\frac{\pi}{3})$, $H_S(\frac{\pi}{3},\frac{\pi}{3})=\begin{pmatrix}\frac{-\overset{\sin}{\cancel{\cos}}a-\overset{\sin}{\cancel{\cos}}(a+b)}{2} & \frac{-\overset{\sin}{\cancel{\cos}}(a+b)}{2}\\ \frac{-\overset{\sin}{\cancel{\cos}}(a+b)}{2} & \frac{-\sin b-\sin(a+b)}{2}\end{pmatrix}\Bigg|_{(\,)}=\begin{pmatrix}\frac{-\frac{\sqrt3}{2}+\frac{1}{2}}{2} & \frac{+\frac{1}{2}}{2}\\ \frac{+\frac{1}{2}}{2} & \frac{-\frac{\sqrt3}{2}+\frac{1}{2}}{2}\end{pmatrix}$ 负定，极大

$S_{max}=\frac{3\sqrt3}{4}$

(2) 内接长方形



$S=4\cos\theta\sin\theta=2\sin 2\theta$，$S_{max}=2$

61. 点 $\vec{x_0}$ 到平面 $\sum_{i=1}^{n}(a_i x_i)=0$ 距离

$$\min_{\sum_{i=1}^{n}a_ix_i=0}\sum_{i=1}^{n}(x_i-x_i^0)^2$$

令 $\begin{cases}\frac{\partial L}{\partial x_i}=2(x_i-x_i^0)+\lambda a_i=0\\ \sum_{i=1}^{n}a_ix_i=0\end{cases}$ 令 $\begin{cases}x_i=x_i^0-\frac{\lambda}{2}a_i\\ \sum_{i=1}^{n}a_ix_i^0=\frac{\lambda}{2}\sum_{i=1}^{n}a_i^2\end{cases}$ 令 $\frac{2\sum_{i=1}^{n}a_ix_i^0}{\sum_{i=1}^{n}a_i^2}=\lambda$

故 $\sum_{i=1}^{n}(x_i^*-x_i^0)^2=\sum_{i=1}^{n}a_i^2\cdot\frac{\lambda^2}{4}=\frac{1}{4}\cdot\sum_{i=1}^{n}a_i^2\left(\frac{2\sum_{i=1}^{n}a_ix_i^0}{\sum_{i=1}^{n}a_i^2}\right)^2=\frac{\left(\sum_{i=1}^{n}a_ix_i^0\right)^2\cdot\sum_{i=1}^{n}a_i^2}{\sum_{i=1}^{n}a_i^2}$

$$L_{min}=\frac{\left|\sum_{i=1}^{n}a_ix_i^0\right|}{\sqrt{\sum_{i=1}^{n}a_i^2}}$$

64.

$$\begin{cases} xyz=1 \\ y=2x \end{cases}$$ 即 $(x, 2x, \frac{1}{2x^2})$，$d^2=5x^2+\frac{1}{4x^4}$，$\frac{\partial d^2}{\partial x}=10x+\frac{-4}{4x^5}=0$，$x^*=(\frac{1}{10})^{\frac{1}{6}}$

$$d^*=\sqrt{5\cdot(\tfrac{1}{10})^{\frac{1}{3}}+\tfrac{1}{4}\times(10)^{\frac{2}{3}}}$$

67. $g(a,b,c)=\int_0^1 [f(x)-ax^2-bx-c]^2dx$

$$\frac{\partial g}{\partial a}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-x^2)dx=0 \Leftrightarrow \int_0^1 (ax^4+bx^3+cx^2)dx=\tfrac{1}{5}a+\tfrac{1}{4}b+\tfrac{1}{3}c=\int_0^1 x^2f(x)dx=A$$

$$\frac{\partial g}{\partial b}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-x)dx=0 \Leftrightarrow \tfrac{1}{4}a+\tfrac{1}{3}b+\tfrac{1}{2}c=\int_0^1 xf(x)dx=B$$

$$\frac{\partial g}{\partial c}=\int_0^1 2[f(x)-ax^2-bx-c]\cdot(-1)dx=0 \Leftrightarrow \tfrac{1}{3}a+\tfrac{1}{2}b+c=\int_0^1 f(x)dx=C$$

故 $$\begin{pmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} A \\ B \\ C \end{pmatrix} \Rightarrow \begin{pmatrix} a^* \\ b^* \\ c^* \end{pmatrix}=D^{-1}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

记 D

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