数分三 HW 3

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1 HW 3

题目. 22. 求下列复合函数的偏导数, 其中 f 是可微函数:

(1) $z = f\left(x\mathrm{e}^{y}, x\mathrm{e}^{-y}\right)$;

(2) $u = f\left(\sum_{i=1}^{n} x_i^2, \prod_{i=1}^{n} x_i^2, x_3, \cdots, x_n\right)$.

解答. (1)

$$\frac{\partial z}{\partial x} = f_1(xe^y, xe^{-y})e^y + f_2(xe^y, xe^{-y})e^{-y}$$
$$\frac{\partial z}{\partial y} = f_1(xe^y, xe^{-y})xe^y - f_2(xe^y, xe^{-y})xe^{-y}$$

(2)

$$\frac{\partial u}{\partial x_i} = f_1(\sum_{i=1}^{n} x_i^2, \prod_{i=1}^{n} x_i^2, x_3, \cdots, x_n)2x_i + f_2(\sum_{i=1}^{n} x_i^2, \prod_{i=1}^{n} x_i^2, x_3, \cdots, x_n)2x_i\Pi_{j\neq i}x_j^2$$
$$+\sum_{j=3}^{n}\delta_{ij}f_j(\sum_{i=1}^{n} x_i^2, \prod_{i=1}^{n} x_i^2, x_3, \cdots, x_n)$$

□

题目. 25. 若 $f(\boldsymbol{x})$ 是定义在区域 $D \subset \mathbb{R}^n (n \geqslant 2)$ 内的函数并且存在正整数 K, 使得 $f(t\boldsymbol{x}) = t^K f(\boldsymbol{x})$ 对于 $\forall t > 0, \forall \boldsymbol{x} \in D$ 成立, 则称 $f(\boldsymbol{x})$ 是 K次齐次函数. 设 K 次齐次函数 $f(\boldsymbol{x})$ 在 D 内具有各个 $k(1 \leqslant k \leqslant K)$阶连续偏导数, 证明:

$$\left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right)^k f(\boldsymbol{x}) = K(K-1)\cdots(K-k+1)f(\boldsymbol{x})$$

题目. 28. 设函数 $x = r\cos\alpha - t\sin\alpha, y = r\sin\alpha + t\cos\alpha$, 其中 $\alpha \in \mathbb{R}$为常数. 证明：对任何可微函数 $f(x, y)$ ，成立

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2$$

解答.

$$\frac{\partial f}{\partial r} = f_x\cos\alpha + f_y\sin\alpha$$

$$\frac{\partial f}{\partial t} = -f_x \sin\alpha + f_y \cos\alpha$$
$$\Rightarrow (\frac{\partial f}{\partial r})^2 + (\frac{\partial f}{\partial t})^2 = f_x^2 + f_y^2$$

□

题目. 31. 求下列函数的二阶偏导数, 其中函数 f 具有二阶连续导数:
(1) $z = f\left(x^2 + y^2, xy\right)$;
(2) $z = f\left(x_1 + x_2 + \cdots + x_n\right)$.

解答. (1)

$$\frac{\partial z}{\partial x} = 2xf_1 + yf_2$$
$$\frac{\partial z}{\partial y} = 2yf_1 + xf_2$$
$$\frac{\partial^2 z}{\partial x^2} = 2yf_1 + xf_2$$

□

题目. 34. 设 $f(x)$ 是一个二次可微函数, 证明 $F(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$ (其中 c 为常数) 满足偏微分方程 $\frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$.

解答.

$$\frac{\partial F}{\partial x} = \frac{1}{2}[f'(x-ct) + f'(x+ct)]$$
$$\frac{\partial F}{\partial t} = \frac{c}{2}[-f'(x-ct) + f'(x+ct)]$$
$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{2}[f''(x-ct) + f''(x+ct)]$$
$$\frac{\partial^2 F}{\partial t^2} = \frac{c^2}{2}[f''(x-ct) + f''(x+ct)]$$
$$\Rightarrow \frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$$

□

题目. 37. 设 $x = 2r - s, y = r + 2s$, 求 $\frac{\partial^2 f(x,y)}{\partial r \partial s}$, 其中函数 $f(x,y)$ 具有二阶连续偏导数.

解答.

□