第~~七~~十一周书面作业，第三章，3.5，7，8

3. Gauss求积公式应用

(1) $\int_0^{2h} f(x)dx \approx A_0 f(0) + A_1 f(h) + A_2 f(2h)$

$f(x)=1$，$2h = A_0 + A_1 + A_2$

$f(x)=x$，$2h^2 = 0 + A_1 h + A_2 \cdot 2h$

$f(x)=x^2$，$\frac{8}{3}h^3 = 0 + A_1 \cdot h^2 + A_2 \cdot 4h^2$

$\Rightarrow A_0 = \frac{1}{3}h$，$A_0 = \frac{4}{3}h$，$A_2 = \frac{1}{3}h$

而 $f(x)=x^3$，$\frac{1}{4}(2h)^4 = 4h^4$

$\frac{1}{3}h \cdot 0 + \frac{4}{3}h \cdot h^3 + \frac{1}{3}h \cdot 8h^3 = \frac{12}{3}h^4 = 4h^4$

而 $f(x)=x^4$，$\frac{1}{5}(2h)^5 = \frac{32}{5}h^5$

$\frac{4}{3}h \cdot h^4 + \frac{1}{3}h \cdot 16h^4 = \frac{20}{3}h^5$，代数精度为3

(2) $\int_{-h}^{h} f(x)dx \approx Af(-h) + Bf(x_1)$

$f(x)=1$，$2h = +A + B$

$f(x)=x$，$0 = -A + Bx_1$

$f(x)=x^2$，$\frac{2}{3}h^3 = Ah^2 + Bx_1^2$

$\Rightarrow \frac{3}{h^2}x_1^2 + 2x_1 - 1 = 0 \Rightarrow x_1 = -\frac{h^2}{3} + \frac{h\sqrt{3+h^2}}{3}$，$B = \frac{2h}{x_1+1}$，$A = Bx_1$

代数精度为2

(3) $\int_0^2 f(x)dx \approx A_0 f(0) + \frac{4}{3}f(x_1) + A_2 f(2)$

$f(x)=1$，$2 = A_0 + \frac{4}{3} + A_2$

$f(x)=x$，$2 = \frac{4}{3}x_1 + 2A_2$

$f(x)=x^2$，$\frac{8}{3} = \frac{4}{3}x_1^2 + 4A_2$

$\frac{8}{3} = \frac{4}{3}x_1^2 + 4(1 - \frac{2}{3}x_1)$

$\Leftrightarrow \frac{2}{3} = \frac{1}{3}x_1^2 + 1 - \frac{2}{3}x_1$

$\Leftrightarrow \frac{1}{3}x_1^2 - \frac{2}{3}x_1 + \frac{1}{3} = 0$

$A_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$A_0 = 2 - \frac{4}{3} - \frac{1}{3} = \frac{1}{3}$

$\Rightarrow$ ~~$x_1^2 - 2x_1 + 4$~~ ~~$x_1^2 - 2x_1 + 4 = 0$ (无解)~~ $\Leftrightarrow (x_1 - 1)^2 = 0 \Leftrightarrow x_1 = 1$

这里写错了导致无解

~~取 $x_1 = 1$ 使误差最小，$A_2 = (4 - \frac{4}{3}) \times \frac{1}{2} = \frac{4}{3}$，$A_0 = 2 - \frac{8}{3} = -\frac{2}{3}$~~

~~代数精度为1~~ 代数精度为3（由(1)可知）

5.

Simpson 公式：$\frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$

$\int_a^b dx \int_c^d f(x,y)dy \approx \int_a^b \frac{d-c}{6}[f(x,c) + f(x,d) + 4f(x,\frac{c+d}{2})]dx$

$\approx \frac{d-c}{6} \cdot \frac{b-a}{6} \cdot \Big[[f(a,c) + f(b,c) + 4f(\frac{a+b}{2},c)] + [f(a,d) + f(b,d) + 4f(\frac{a+b}{2},d)]$

$+ 4[f(a,\frac{c+d}{2}) + f(b,\frac{c+d}{2}) + 4f(\frac{a+b}{2},\frac{c+d}{2})] \Big]$

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余项？数值积分记为 I，中间步为 $\tilde{I}$

有：$\int_a^b dx \int_a^d f(x,y)\,dy - I$

$= \int_a^b dx \int_a^d f(x,y)\,dy - \tilde{I} + \tilde{I} - I$

第一项：$\int_a^b dx \int_c^d f(x,y)\,dy - \tilde{I} = \int_a^b -\frac{(d-c)^5}{2880}\frac{\partial^4 f}{\partial y^4}(x,\xi)\,dx$

第二项：$\tilde{I} - I = -\frac{(b-a)^5}{2880}\left[\frac{\partial^4}{\partial x^4}f(\eta_1,c) + \frac{\partial^4}{\partial x^4}f(\eta_2,d) + \frac{\partial^4}{\partial x^4}f(\eta_3,d)\right]$

余项：上述之和 = （由介值性） $= -\frac{(b-a)^5}{2880}\cdot 3\cdot\frac{\partial^4}{\partial x^4}f(\eta_0,\xi_0)$

7.

~~复合中点：~~ $\cancel{\int_a^b f(x)\,dx =}$

① 复合梯形公式：$T(h)=\frac{h}{2}\sum_{i=1}^{n-1}[f(x_i)+f(x_{i+1})]$ 除左右端点外被用2次

$=\frac{h}{2}[f(a)+f(b)+2\sum_{i=1}^{n-1}f(a+ih)]$

② $S(h)=\frac{h}{6}\sum_{i=0}^{n-1}[f(x_i)+4f(x_{i+\frac{1}{2}})+f(x_{i+1})]$

$=\frac{2}{3}M(h)+\frac{h}{6}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})] = \frac{2}{3}M(h)+\frac{1}{3}T(h)$

④ ③ $T(\frac{h}{2})=\frac{h}{4}[\sum_{i=1}^{2n-1} f(a+i\cdot\frac{h}{2})+f(a)+f(b)]$ （实际上 $n^*=2n$）

$T(\frac{h}{2})$ 以 $\frac{h}{2}$ 为步长从 a 到 b

$T(\frac{h}{2})=\frac{h}{4}[\sum_{i=1}^{2n-1}f(a+i\cdot\frac{h}{2})+f(a)+f(b)]$ ， $(h=\frac{b-a}{n})$

$=\frac{M(h)}{2}+\frac{h}{4}[\sum_{k=1}^{n-1}f(a+2k\cdot\frac{h}{2})+f(a)+f(b)]$

$=\frac{M(h)}{2}+\frac{T(h)}{2}$

④ $\frac{4}{3}T(\frac{h}{2})=\frac{h}{3}[\sum_{i=1}^{2n-1}f(a+i\cdot\frac{h}{2})+f(a)+f(b)]$

$S(h)=\frac{1}{3}[T(h)+2\cdot(2T(\frac{h}{2})-T(h))] = \frac{4T(h/2)-T(h)}{3}$

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8、

① $\left|\int_a^b f(x)dx - M(x)\right| \leqslant \sum_{i=1}^{n-1} \frac{h^3}{24} \cdot |f''(\xi_i)| \cdot h \leqslant \frac{h^3}{24} M_2 \cdot n = \frac{h^3}{24} M_2 \cdot \frac{b-a}{h} = \frac{h^2}{24} M_2 (b-a)$

② $\left|\int_a^b f(x)dx - T(h)\right| \leqslant \sum_{i=1}^{n-1} \frac{h^3}{12} |f''(\xi_i)| \leqslant \frac{h^2}{12} M_2 (b-a)$

③ $\left|\int_a^b f(x)dx - S(h)\right| \leqslant \sum_{i=1}^{n-1} \frac{h^5}{2880} |f^{(4)}(\xi_i)| \leqslant \frac{h^4}{2880} M_4 (b-a)$