Los puntos fijos no son simplemente x_j^μ sino que también $-x_j^\mu$

no solamente están estos equilibrios, sino que también hay mínimos que son estados mezcla

Definimos una métrica sobre configuraciones de actividad neuronal

Supongamos que tenemos dos configuraciones x_j, y_j

OVERLAP

$$\boxed{m = \frac{1}{N}\sum_{j=1}^{N} x_j\, y_j}$$

$\begin{cases} 1 & \text{cuando las conf. son iguales} \\ -1 & \text{cuando } x_j = -y_j \end{cases}$

Si las configuraciones son estadísticamente ind, m tendrá una distribución gaussiana con $\mu = 0$ y $\sigma = \frac{1}{\sqrt{N}}$

ejemplo estados x_j^1, x_j^2, x_j^3

$\downarrow$ construyo un estado mezcla $x_j^n = sg(x_j^1 + x_j^2 + x_j^3)$

$$m^{1,n} = \frac{1}{N}\sum_{j=1}^{N} x_j^1\, sg(x_j^1 + x_j^2 + x_j^3)$$

$$= \frac{1}{N}\sum_{j=1}^{N} sg\left(x_j^1(x_j^1 + x_j^2 + x_j^3)\right)$$

$$= \frac{1}{N}\sum_{j=1}^{N} sg\Big(1 + \underbrace{x_j^1 x_j^2}_{\pm 1} + \underbrace{x_j^1 x_j^3}_{\pm 1}\Big)$$

$$= \frac{1}{N}\left(\frac{3}{4} - \frac{1}{4}\right) \cdot N$$

$$= \frac{1}{2}$$

$1 + 1 + 1 = 3$	c/u con prob 1/4
$1 - 1 + 1 = 1$	
$1 + 1 - 1 = 1$	
$1 - 1 - 1 = -1$	

$$S_j(0) = sg(x_j^1 + x_j^2 + x_j^3)$$

$$S_i(1) = sg\left(\sum_{j=1}^{N} w_{ij}\, s_j(0)\right)$$

$$= sg\left(\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{\mu=1}^{P} x_i^\mu x_j^\mu\right) sg(x_j^1 + x_j^2 + x_j^3)\right)$$

$$= sg\left(\sum_{\mu=1,2,3}\sum_j \frac{x_i^\mu x_j^\mu}{N} sg(x_j^1 + x_j^2 + x_j^3) + \sum_{\mu>3}\sum_j \frac{x_i^\mu x_j^\mu}{N} sg(x_j^1 + x_j^2 + x_j^3)\right)$$

$$= sg\Big(\sum_{\mu=1,2,3} x_i^\mu \underbrace{m^{\mu n}}_{1/2} + \sum_{\mu>3} x_i^\mu \underbrace{m^{\mu n}}_{\mathcal{N}(0,\sigma=1/\sqrt{N})}\Big)$$

$$\Rightarrow \boxed{S_i(1) = sg\left(\frac{x_j^1 + x_j^2 + x_j^3}{2} + 2R\right)}$$

$$R = \mathcal{N}\left(0, \sigma = \sqrt{\frac{P-3}{N}}\right)$$

Los estados mezcla son más inestables

DINÁMICA ESTOCÁSTICA

$$h_i(\tau) = \sum_{j=1}^{N} W_{ij} \, S_i(\tau)$$

$S_i(\tau+1)$ depende probabilísticamente de $h_i(\tau)$

$$P(S_i(t+1)=1) = \frac{e^{\beta h_i(\tau)}}{e^{\beta h_i(t)} + e^{-\beta h_i(\tau)}}$$

$\beta \to 0$ límite estocástico $\left(P(S_i=1) = P(S_i=-1) = 1/2 \right)$

$\beta \to \infty$; $\begin{cases} S_i(t+1) = 1 & h_i(t) > 0 \\ S_i(t+1) = -1 & h_i(\tau) < 0 \end{cases}$

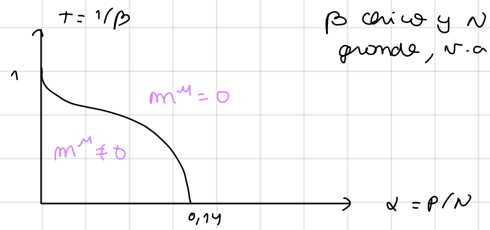
límite determinista.

$$P(S_i(t+1)=-1) = \frac{e^{-\beta h_i(\tau)}}{e^{\beta h_i(t)} + e^{-\beta h_i(t)}}$$

overlap entre el mínimo y la conf. del sist a tiempo t

$$m^\mu = \frac{1}{N} \sum_{j=1}^{N} \underbrace{\langle S_i(\tau) \rangle}_{\text{valor medio sobre la dinámica}} x_j^\mu$$

Vamos a obtener:



β chico y N grande, v.a ind

aproximación de campo medio

aproximación en modelos con v.a estocásticas. Hay una serie de problemas donde esta es exacta.

Consideramos la simplificación donde $W_{ij} = w \quad \forall \, i,j$

En ese caso:

$$\underbrace{h_i(t) = \frac{1}{N} \sum_{j=1}^{N} w \, S_j(\tau) = \frac{w}{N} \sum_{j=1}^{N} S_j(\tau)}_{\text{cada neurona ve un mismo campo } h_i(\tau) \mapsto h(\tau)}$$

$$P(S(t+1)=1) = \frac{e^{\beta h}}{e^{\beta h} + e^{-\beta h}} = 1 - P(S(\tau+1)=-1)$$

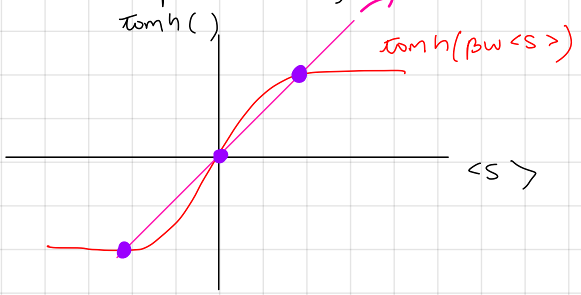
$$\langle S(\tau+1) \rangle = 1 \cdot P(S(t+1)=1) - 1 \, P(S(\tau+1)=1)$$
$$= \tanh(\beta h(\tau))$$

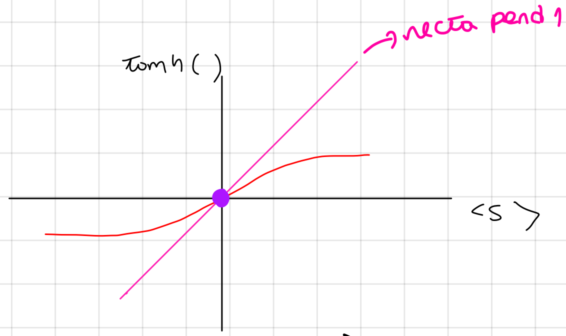
aproximación $h(\tau) \to \langle h(\tau) \rangle = w \langle S(\tau) \rangle$

en el eq $\langle S_i(\tau) \rangle = \langle S_i(\tau+1) \rangle$

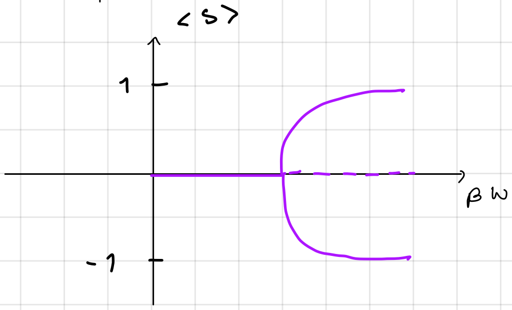
$\Rightarrow \langle S(t) \rangle = \tanh(\beta w \langle S(\tau) \rangle)$ se encuentran las sol gráficamente (abajo)

$\tanh(\beta w \langle s \rangle)$





intersecciones en función de βw



Modelo de Hopfield

$$\langle S_i(t+1)\rangle = \tanh(\beta h(t)) \qquad h_i(t) = \sum_{j=1}^{N} w_{ij} S_j(t)$$

campo medio

$$\langle S_i(t+1)\rangle = \tanh(\beta \langle h_i(t)\rangle) \qquad \langle h_i(t)\rangle = \sum_{j=1}^{N} w_{ij} \langle S_j(t)\rangle$$

$$w_{ij} = \frac{1}{N}\sum_{\mu=1}^{N} x_i^{\mu} x_j^{\mu}$$

Suponemos que estamos en el equilibrio

$$\langle S_i\rangle = \tanh\left(\frac{\beta}{N}\sum_{j,\mu} x_i^{\mu} x_j^{\mu} \langle S_j\rangle\right)$$

dependen de los x_i!!

$$\Rightarrow \sum_i \frac{x_i^{\mu}}{N}\langle S_i\rangle = \sum_i x_i^{\mu} \tanh\left(\frac{\beta}{N}\sum_{j,\mu} x_i^{\mu} x_j^{\mu}\langle S_j\rangle\right) \quad (*)$$

$$(\pm 1)\tanh(x) = \tanh(\pm x)$$

$$\Rightarrow m^{\mu} = \frac{1}{N}\sum_{i=1}^{N} \tanh\left(\frac{\beta}{N}\sum_{i,\mu} x_i^{\mu} x_1^{\mu} x_j^{\mu}\langle S_j\rangle\right)$$

Suponemos que el estado inicial es cercano a uno de los estados

ANZATZ $\quad m^1 \sim O(1) \qquad m^{\mu} = O\left(\frac{1}{\sqrt{N}}\right) \quad \mu > 1$

$$m^{\nu} \overset{(*)}{=} \frac{1}{N}\sum_i x_i^{\nu} \tanh\left(\frac{\beta}{N}\left(x_i^1 m^1 + \sum_{\mu>1} x_i^{\mu} m^{\mu}\right)\right)$$

$$= \frac{1}{N}\sum_i x_i^{\nu} \tanh\left(\beta\left[\underbrace{x_1^1 m_1}_{O(1)} + \underbrace{x_1^{\nu} m^{\nu}}_{O\left(\frac{1}{\sqrt{N}}\right)} + \underbrace{\sum_{\mu\neq 1,\nu} x_i^{\mu} m^{\mu}}_{O\left(\sqrt{\frac{P}{N}}\right)}\right]\right)$$

único $\qquad\qquad$ grande

expansión en torno de término grande

$$m^\nu = \frac{1}{N}\sum_{i=1}^{N} x_i^\nu \left[\tanh\left(\beta \left[x_i^1 m^1 + \sum_{\mu\neq 1,\nu} x_i^\mu m^\mu \right]\right)\right] +$$

$$\left[1 - \tanh\left(\beta\left(x_i^1 m + \sum_{\mu\neq 1,\nu} x_i^\mu m^\mu\right) x_i^\nu m^\nu\right]\right.$$

$$(*)\ = \frac{1}{N}\sum_{i=1}^{N} x_i^\nu x_i^1 \tanh\left(\beta\left[m^1 + \sum_{\mu\neq 1,\nu} x_i^1 x_j^1 m^\mu\right]\right)$$

$$+ \frac{1}{N}\sum_{i=1}^{N}\left[1 - \operatorname{tgh}^2\left(\beta\left[m^1 + \sum_{\mu\neq 1,\nu} \underbrace{\overbrace{x_i^\mu x_i^1}^{\pm m^\mu} m^\mu}_{\text{dist normal con}}\right)\right] m^\nu \beta$$

dist normal con $\mu = 0$, $\sigma = \alpha R$ (TCL)

con $R = \frac{1}{\alpha}\sum_{\mu\neq 1,\nu} m^{\mu 2}$

(2)

Para N grande $\xrightarrow{\text{aproximamos}}$ $\displaystyle\int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}\left[1 - \operatorname{tgh}^2\left[\beta\left(m^1 + \sqrt{\alpha R}\, z\right]\right]\right] m^\nu \beta$

media cero y varianza 1

$$q = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \operatorname{tgh}^2[\ ] e^{-z^2/2}$$

$f(x_i)$: $x_i \sim P(x)$ $\qquad \frac{1}{N}\sum f(x_i) = \int dx\, P(x) f(x)$

reemplazamos el prom por

$$\Rightarrow m^\nu = m^\nu\beta - m^\nu \beta q = m^\nu \beta(1-q)$$

$$\Leftrightarrow m^\nu\left[1 - \beta(1-q)\right] = \frac{1}{N}\sum x_i^\nu x_i^1 \tanh\left(\beta\left[m^1 + \sum_{\mu\neq 1,\nu} x_i x_j m^\mu\right]\right)$$

de (*)

$$\Rightarrow m^\nu = \frac{1/N \sum x_i^\nu x_i^1 \tanh\left(\beta\left[m^1 + \sum_{\mu\neq 1,\nu} x_i x_j m^\mu\right)\right.}{1 - \beta(1-q)}$$

$$\boxed{R = \frac{q}{[1-\beta(1-q)]^2}}$$

$$m^1 = \frac{1}{N}\sum \tanh\left(\beta\left[m^1 + \sum_{\mu\neq 1} x_i^\mu x_i^1 m^\mu\right]\right)$$

$$m^1 = \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \tanh\left[\beta\left(m^1 + \sqrt{\alpha R}\, z\right)\right]$$

$$q = \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \tanh\left[\beta\left(m^1 + \sqrt{\alpha R}\, z\right)\right]$$

$m^1 = \tanh(\beta m^1)$